Discuss the continuity of the following functions at the indicated points/set.

1. \( f(x) = |x-3| \) at \( x=3 \)

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} |x-3| = \lim_{x=3-h} |3-h-3| = \lim_{h \to 0} h = 0
\]

\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} |x-3| = \lim_{x=3+h} |3+h-3| = \lim_{h \to 0} |h| = 0
\]

\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = \lim_{x \to 3} f(x)
\]

\( f(3) = |3-3| = 0 \)

\( \lim_{x \to 3} f(x) = f(3) \)

hence function is continuous at \( x=3 \).

2. \( f(x) = \begin{cases} 
\frac{x^2-9}{x-3} & \text{if } x \neq 3 \\
0 & \text{if } x = 3 
\end{cases} \)

Sdf: \( f(3) = 0 \)

\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3} (x+3) = 3+3
\]

\( \lim_{x \to 3} f(x) = 6 \)

So \( \lim_{x \to 3} f(x) \neq f(3) \)

function is discontinuous.
3. \( f(x) = \begin{cases} 
  x - 4 & \text{if } -1 \leq x \leq 2 \\
  x^2 - 6 & \text{if } 2 < x < 5
  \end{cases} \)

Solution:
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x - 4) = 2 - 4 = -2
\]
\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 6) = 4 - 6 = -2
\]
\[
\lim_{x \to 2^-} f(x) = -2 = \lim_{x \to 2^+} f(x)
\]

Since \( \lim_{x \to 2} f(x) = -2 \) and \( f(2) = 2 - 4 = -2 \),
function is continuous.

4. \( f(x) = \begin{cases} 
  \frac{x^2 - 27}{x^2 - 9} & \text{if } x \neq 3 \\
  6 & \text{at } x = 3
  \end{cases} \)

Solution:
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 27}{x^2 - 9} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)(x + 3)} = \lim_{x \to 3} \frac{x + 3}{x + 3} = \frac{6}{2} = 3
\]

\[ f(3) = 6 \]

So \( \lim_{x \to 3} f(x) \neq f(3) \),
function is discontinuous.

5. \( f(x) = \begin{cases} 
  \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{at } x = 0
  \end{cases} \)

Solution:
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sin \left( \frac{1}{x} \right) = \sin(-\infty) = \text{any value b/w } [-1, 1]
\]
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin \left( \frac{1}{x} \right) = \sin(\infty) = \text{any value b/w } [-1, 1]
\]

Limit does not exist,
function is discontinuous.

6. \( f(x) = \sin x \quad \forall \quad x \in \mathbb{R} \)

Let \( \theta \) be an arbitrary real no.
\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} \sin x = \sin a
\]

\[
\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \sin x = \sin a
\]

\[
\sin a = \sin a
\]

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)
\]

\[
\Rightarrow f(x) \text{ is continuous at } x = a,
\]

\[
\forall a \in \mathbb{R}, \Rightarrow f(x) \text{ is continuous } \forall x \in \mathbb{R}.
\]

7. \[f(x) = \begin{cases} \frac{x^2}{a} - a & \text{if } 0 < x < a \\ 0 & \text{if } x = a \\ a - a^2 & \text{if } x > a \end{cases} \]

\[
\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \left(\frac{x^2}{a} - a\right) = \frac{a^2}{a} - a = a - a = 0
\]

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \left(\frac{x^2}{a} - a\right) = \frac{a^2}{a} = a
\]

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = 0
\]

\[
f(a) = 0
\]

\[
\lim_{x \to a} f(x) = f(a) = 0
\]

\[
f(x) \text{ function is continuous at } x = a.
\]

8. Determine all the points of continuity of the function \(f(x) = x - \lfloor x \rfloor \forall x \in \mathbb{R}.
\]

Sol. \[
\begin{align*}
\text{when } 0 &\leq x < 1 & \lfloor x \rfloor &= 0 & \text{when } -1 &< x < 0 & \lfloor x \rfloor &= -1 \\
\text{when } 1 &\leq x < 2 & \lfloor x \rfloor &= 1 & \text{when } -2 &< x < -1 & \lfloor x \rfloor &= -2 \\
\text{when } 2 &\leq x < 3 & \lfloor x \rfloor &= 2 & \text{when } -3 &< x < -2 & \lfloor x \rfloor &= -3
\end{align*}
\]

\[
\Rightarrow f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ x + 1 & \text{if } 1 < x < 2 \\ x - 2 & \text{if } 2 < x < 3 \end{cases}
\]

\[
\begin{align*}
\text{when } & -1 < x < 0 & f(x) &= x + 1 & -1 < x < 0 \\
\text{when } & -2 < x < -1 & f(x) &= x + 2 & -2 < x < -1 \\
\text{when } & -3 < x < -2 & f(x) &= x + 3 & -3 < x < -2
\end{align*}
\]
At $x = 1$

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 1
\]

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-1) = 0
\]

Therefore, the limit does not exist. The function is not continuous at $x = 1$.

Similarly, it is continuous for integral values of $x$, both positive and negative, but it is continuous at every other real value of $x$.

9. Discuss continuity at $x = 1$, $f(x) = x^2 - 1 \ |x|$

\[
\lim_{x \to 1^-} (x - |x|) = 1 - 1 = 0
\]

\[
\lim_{x \to 1^+} (x - |x|) = 1 - 1 = 0
\]

\[
\lim_{x \to 1} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1 - 1 = 0
\]

\Rightarrow \text{ function is continuous at } x = 1.

10. Show that function $f: \mathbb{R} \Rightarrow \mathbb{R}$ defined by

\[
f(x) = \begin{cases} 
\frac{x}{2} & \text{if } x \text{ is irrational.} \\
1-x & \text{if } x \text{ is rational.}
\end{cases}
\]

is continuous at $x = \frac{1}{2}$.

\[
\lim_{x \to \frac{1}{2}} f(x) = \lim_{x \to \frac{1}{2}} \left(1 - \frac{x}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}
\]

\Rightarrow \lim_{x \to \frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)

\Rightarrow \text{ function is continuous.}

11. Show that function $f: [0,1] \Rightarrow \mathbb{R}$ defined by

\[
f(x) = \frac{1}{x}
\]

is continuous on $[0,1]$. Is $f(x)$ is bounded on this interval? Explain.

Sol: $f(x)$ is defined for all real values of $x$ such that $0 < x < 1$ and its limit exists at each such $x$ and equals to its value there, so it is
Continuous on $[0, 1]$.
When $x = 1$, $f(x) = 1$ which is its lower bound.
So it is bounded below. $f(x)$ increases indefinitely
as $x$ becomes small. Thus $f(x)$ is not bounded
above. Hence $f(x)$ is not bounded on $[0, 1]$

12. Let $f(x) = \begin{cases} \cos \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, is $f$ continuous
at $x = 0$.

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \cos \left( \frac{1}{x} \right) = \text{any value } \in [-1, 1]$
Thus limit does not exist. Similarly,
$\lim_{x \to 0^+} f(x)$ does not exist.
Hence $\lim_{x \to 0} f(x)$ does not exist.
So given function is not continuous at $x = 0$.

13. Let $f(x) = \begin{cases} (x-a) \sin \left( \frac{1}{x-a} \right) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$
Discuss continuity of $f$ at $x = a$.

Sol. $\lim_{x \to a} f(x) = \lim_{x \to a} (x-a) \sin \left( \frac{1}{x-a} \right)$
$= \lim_{x \to a} (x-a) \lim_{x \to a} \sin \left( \frac{1}{x-a} \right)$
$= (a-a) \cdot \sin \left( \frac{1}{a-a} \right)$
$= 0 \cdot \sin \left( \frac{1}{1} \right)$
$= 0$ (any value from $[-1, 1]$)
$\lim_{x \to a} f(x) = 0$

$f(a) = 0$

By defined function

$\lim_{x \to a} f(x) = f(a) = 0$
So given $f$ is continuous at $x = a$.

14. Let $f(x) = \begin{cases} x \cos \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
Show $f$ is continuous at $x = 0$.
15. Let \( f(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \). Discuss continuity of \( f(x) \) at \( x = 0 \).

**Sol.**

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0 \quad \text{[any value b/w [-1,1]]}
\]

So \( f(x) \) is continuous at \( x = 0 \).

16. Let \( f(x) = \begin{cases} \frac{x}{\sin \left| \frac{x}{x} \right|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \). Discuss continuity.

**Sol.**

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x}{\sin \left| \frac{x}{x} \right|} = \lim_{x \to 0} x \lim_{x \to 0} \frac{1}{\sin \left| \frac{x}{x} \right|} = 0 \lim_{x \to 0} \frac{1}{\sin \left( \frac{x}{x} \right)} = 0 \sin(1) = 0
\]

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{\sin \left| \frac{x}{x} \right|} = \lim_{x \to 0^+} x \lim_{x \to 0^+} \frac{1}{\sin \left| \frac{x}{x} \right|} = 0 \lim_{x \to 0^+} \frac{1}{\sin \left( \frac{x}{x} \right)} = 0 \sin(1) = 0
\]

\[
\lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^-} f(x)
\]

So \( f(x) \) is continuous at \( x = 0 \).
17. Find \( c \), such that the function \( f(x) = \begin{cases} \frac{1 - \sqrt{x}}{x - 1} & \text{if } 0 \leq x < 1 \\ c & \text{if } x = 1 \end{cases} \), is continuous at \( x = 1 \) \( \text{and } x \in [0, 1] \).

So, let \( a \in [0, 1] \) be arbitrary. 

\[
\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1 - \sqrt{x}}{x - 1} = \frac{1 - \sqrt{a}}{a - 1} = f(a)
\]

Thus \( f \) is continuous at \( a \) and \( x = 1 \) is arbitrary point of \([0, 1]\). Since \( f \) is continuous on \([0, 1]\), so \( f \) is continuous on \( x = 1 \).

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1 - \sqrt{x}}{x - 1} = \frac{-1}{2}
\]

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1 - \sqrt{x}}{x - 1} = \frac{-1}{2}
\]

\[
f(1) = c
\]

Since function is continuous. So, 

\[
\lim_{x \to 1} f(x) = f(1) \Rightarrow c = \frac{1}{2}
\]

In Problem 18-20. find the points of discontinuity of the given functions.

18. \( f(x) = \begin{cases} \frac{x+4}{x} & \text{if } -6 < x < 2 \\ -4 & \text{if } 2 < x < 6 \end{cases} \)

\[
\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{x+4}{x} = -2 + 4
\]

\[
\lim_{x \to -2^+} f(x) = 2
\]

\[
\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} x = -2
\]

\[
\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)
\]

Limit does not exist. So function is discontinuous.
19. \( g(x) = \begin{cases} \frac{x^3}{6x^2 + 46} & \text{if } x < 1 \\ -x^2 & \text{if } 1 \leq x \leq 10 \\ -4 - x^3 & \text{if } x > 10 \end{cases} \)

\[
\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \frac{x^3}{6x^2 + 46} = 0
\]
\[
\lim_{x \to 1^+} g(x) = \lim_{x \to 1^-} (-4 - x^3) = -4 - 1 = -5
\]

\[
\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x)
\]

Function is discontinuous at \( x = 1 \).

\[
\lim_{x \to 10^-} g(x) = \lim_{x \to 10^-} (-4 - x^3) = -4 \times 100 = -400
\]
\[
\lim_{x \to 10^+} g(x) = \lim_{x \to 10^+} (6x^2 + 46) = 6 \times 100 + 46 = 646
\]

\[
\lim_{x \to 10^-} g(x) \neq \lim_{x \to 10^+} g(x)
\]

Function is discontinuous.

20. \( f(x) = \begin{cases} \frac{x^2}{x} & \text{if } 1 \leq x < 2 \\ x + 6 & \text{if } 2 \leq x < 3 \end{cases} \)

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2}{x} = \lim_{x \to 1^-} x = 1
\]
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2}{x} = \lim_{x \to 1^+} x = 1
\]

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 6) = 8
\]
\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + 6) = 8
\]

\[
\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)
\]

\[
\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)
\]

Function is not continuous.

21. Find constants \( a \) and \( b \) such that function \( f \) defined by

\[
\begin{cases}
3x^3 & \text{if } 0 \leq x < 1 \\
a + b & \text{if } -1 \leq x < 1 \\
x^2 + 2 & \text{if } x \geq 1
\end{cases}
\]

is continuous on all \( x \) if possible.

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax + b) = a + b
\]
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 2) = 3
\]

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = a + b = 3 \Rightarrow (a, b)
\]

\[
\lim_{x \to -1} f(x) = \lim_{x \to -1} (ax + b) = -a + b
\]
\[
\lim_{x \to -1} f(x) = \lim_{x \to -1} (x^2 + 2) = a + b
\]

\[
-a + b = -1 \Rightarrow (a, b)
\]
adding \( a \oplus b = 3 \)
\[
\begin{align*}
\frac{a^2 + b^2}{a + b} &= \frac{3}{-1} \\
\frac{2b^2}{a + b} &= \frac{3}{1} \\
\Rightarrow b &= \pm 1 \\
\Rightarrow a &= 2
\end{align*}
\]

Put value of \( b \) in \( 1 \)
\[
\begin{align*}
a + b &= 3 \\
a + 1 &= 3 \\
a &= 2 \\
\Rightarrow \frac{a}{2} &= \frac{2}{2}
\end{align*}
\]

**Find the interval on which the given function is continuous. Also find points where it is discontinuous. (Problem 22-26)**

22. \( f(x) = \frac{x^2 - 5}{x - 1} \)

1. **Function** \( f(x) = \frac{x^2 - 5}{x - 1} \) is not defined at \( x = 1 \).
2. Thus function is discontinuous at \( x = 1 \).
3. Nominator is continuous at every point of \( \mathbb{R} \) and so is the denominator is \( x - 1 \).
4. Hence \( f(x) \) is continuous at every point of \( \mathbb{R} - \{1\} \).

23. \( f(x) = \frac{x}{|x|} \)

So, \( f(x) \) is not defined at \( x = 0 \) so \( f(x) \) is discontinuous at \( x = 0 \) and continuous at \( \mathbb{R} - \{0\} \).

24. \( f(x) = \frac{\sin x}{x} \)

1. \( f(x) \) is not defined at \( x = 0 \). Hence it is discontinuous at \( x = 0 \).
2. \( f(x) \) is continuous on \( \mathbb{R} - \{0\} \).

25. \( f(x) = \frac{\sin x}{\cos x} \)

- Function is not defined on \( x = (2n+1)\pi/2 \) where \( n \) is any integer.
- Thus \( f \) is discontinuous at these points and continuous on all other points of \( \mathbb{R} \).
26. \[ f(x) = \begin{cases} \sin x & \text{if } x \leq \frac{\pi}{4} \\ \cos x & \text{if } x > \frac{\pi}{4} \end{cases} \]

Sol. \[ \lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} \sin x = \sin \left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \]

\[ \lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} \cos x = \cos \left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \]

\[ f\left(\frac{\pi}{4}\right) = \sin \left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \]

Function is continuous at \( x = \frac{\pi}{4} \). We also know that \( \sin x \) and \( \cos x \) are continuous at every point of \( \mathbb{R} \). Hence, \( f(x) \) is continuous at every point of \( \mathbb{R} \).

In Problem 27-34, examine whether the given function is continuous at \( x = 0 \).

27. \[ f(x) = \begin{cases} (1 + 3x)^{1/3} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases} \]

Sol. \[ \lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + 3x)^{1/3} = \lim_{x \to 0} \left[ (1 + 3x)^{1/3} \right]^3 \]

\[ = \left[ \lim_{x \to 0} (1 + 3x)^{1/3} \right]^3 \]

\[ \lim_{x \to 0} f(x) = e^3 \]

\[ f(0) = e^2 \]

\[ \lim_{x \to 0} f(x) \neq f(0) \quad \text{function is not continuous} \]

28. \[ f(x) = \begin{cases} (1 + x)^{1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \]

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + x)^{1/x} = e \]

\[ f(0) = 1 \]

\[ \lim_{x \to 0} f(x) \neq f(0) \quad \text{function is not continuous} \]

29. \[ f(x) = \begin{cases} (1 + 2x)^{1/2x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases} \]

Sol. \[ \lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + 2x)^{1/2x} = \lim_{x \to 0} \left[ (1 + 2x)^{1/2x} \right]^2 \]

\[ = \left[ \lim_{x \to 0} (1 + 2x)^{1/2x} \right]^2 \]

\[ = e^2 \]

\[ \lim_{x \to 0} f(x) = e^2 \]

\[ f(0) = e^2 \]

\[ \lim_{x \to 0} f(x) = f(0) \quad \text{function is continuous} \]
So \( \lim_{x \to 0} f(x) = f(0) \)

So given \( f(x) \) is continuous at \( x = 0 \).

30. \( f(x) = \begin{cases} \frac{e^{-\frac{1}{2}x}}{1 + e^{\frac{1}{4}x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

\[
\text{Sol. } \lim_{x \to 0} f(x) = \lim_{x \to 0} e^{-\frac{1}{2}x} = e^{-\frac{1}{2} \cdot 0} = e^0 = 1
\]

\[
\lim_{x \to 0} f(x) = 1
\]

\[
\lim_{x \to 0} f(x) \neq f(0) \quad \text{so given function is not continuous.}
\]

31. \( f(x) = \begin{cases} \frac{e^{\frac{1}{2}x^2}}{1 + e^{\frac{1}{2}x^2}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

\[
\text{Sol. when } x \to 0^- \quad \frac{1}{x} \to -\infty \quad \text{and so } \quad \frac{1}{e^{\frac{1}{2}x}} \to 0
\]

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0^-} e^{\frac{1}{2}x^2} = \lim_{x \to 0} \frac{1}{1 + e^{\frac{1}{2}x^2}} = \frac{1}{1 + 0} = 1
\]

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\frac{1}{2}x^2} = \lim_{x \to 0} \frac{1}{1 + e^{\frac{1}{2}x^2}} = \frac{1}{1 + \frac{1}{\infty}} = 1
\]

\[
\lim_{x \to 0^+} f(x) = 1
\]

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \quad \text{function is not continuous at } x = 0.
\]

32. \( f(x) = \begin{cases} \frac{e^{\frac{1}{2}x^2}}{e^{\frac{1}{2}x^2} - 1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

\[
\text{Sol. } \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^{\frac{1}{2}x^2}}{e^{\frac{1}{2}x^2} - 1} = \lim_{x \to 0} \frac{e^{\frac{1}{2}x^2}}{e^{\frac{1}{2}x^2} \left(1 - \frac{1}{e^{\frac{1}{2}x^2}}\right)}
\]

\[
= \lim_{x \to 0} \frac{1}{1 - \frac{1}{e^{\infty}}} = \lim_{x \to 0} \frac{1}{1 - e^{0}} = 1
\]

\[
\lim_{x \to 0} f(x) = 1 \quad \text{and } \quad f(0) = 1
\]

\[
\lim_{x \to 0} f(x) = f(0)
\]

\[
\text{function is continuous at } x = 0.
\]
33. \[ f(x) = \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \]

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x} = \frac{\sin 2 \cdot 0}{2 \cdot 0} = 1 \]

Thus, \( f(x) \) is discontinuous at \( x = 0 \).

34. \[ f(x) = \begin{cases} \frac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ 2/3 & \text{if } x = 0 \end{cases} \]

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 3x}{\sin 2x} = \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{3x} \]

\[ = \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{2} \cdot 1 = \frac{3}{2} \]

Therefore, \( f(x) \) is continuous at \( x = 0 \).

35. Let \( f(x) = x^2 \) and \( g(x) = \begin{cases} -4 & \text{if } x \leq 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \)

Determine whether \( f \circ g \) and \( g \circ f \) are continuous at \( x = 0 \).

\( f \circ g \): \( f(g(x)) = \begin{cases} g(-x)^2 & \text{if } x \leq 0 \\ f(1 - x^2) & \text{if } x > 0 \end{cases} \)

Thus \( f(g(x)) = \begin{cases} 16 & \text{if } x \leq 0 \\ (1 - x^2)^2 & \text{if } x > 0 \end{cases} \)

\( g \circ f \): \( g(f(x)) = \begin{cases} f(x)^2 - 4 & \text{if } x^2 \leq 0 \\ g(1 - x^2) & \text{if } x^2 > 0 \end{cases} \)

Thus \( g(f(x)) = \begin{cases} 0 & \text{if } x = 0 \\ 1 - x^2 - 4 & \text{if } x > 0 \end{cases} \)

\[ \lim_{x \to 0} f(x) = 16 \]

\[ \lim_{x \to 0^+} (f \circ g)(x) = \lim_{x \to 0^+} (x^2 - 4)^2 = 0^2 - 4^2 = 16 \]

\[ (g \circ f)(0) = 16 \]

Thus \( f \circ g \) is continuous at \( x = 0 \).

Again, \( (g \circ f)(x) = g[f(x)] = g(x^2) \)
\((gof)(x) = 4\) if \(x^2 \leq 0\)
\((gof)(x) = 1|x^2 - 4|\) if \(x^2 > 0\)

\[\lim_{x \to 0^-} (gof)(x) = -4\]

\[\lim_{x \to 0^+} (gof)(x) = \lim_{x \to 0^+} 1|x^2 - 4| = 10 - 4 = 6\]

\[\lim_{x \to 0^+} (gof)(x) = 4\]

\[\lim_{x \to 0^+} (gof)(x) \neq \lim_{x \to 0^-} (gof)(x)\]

Limit does not exist.

So \((gof)(x)\) is not continuous at \(x = 0\).