CHAP # 1

calculus

REAL NUMBERS, LIMITS & CONTINUITY

Exercise 1.1

1. if $a,b \in \mathbb{R}$ and $a+b=0$, prove that $a=-b$.

Sol. Since $b \in \mathbb{R}$, $\exists$ an element $-b \in \mathbb{R}$

$\ni$ $b+(-b)=0 \longrightarrow (1)$

By given $a+b=0$

Adding $(-b)$ on both sides

$a+b+(-b)=0+(-b)$

$a+(b+(-b))=-b$ — Associative property and identity

$a+0=-b$ — from (1)

$a=-b$ — identity.

Symbols

$\ni$: such that

$\exists$: there exist

$\forall$: for all

2. Prove that $(-a)(-b)=ab$ $\forall$ $a,b \in \mathbb{R}$.

Sol. $\because$ $ab+a(-b)+(-a)(-b)=ab+[a(-b)+(-a)(-b)]$ $\because$

Associative property

$\Rightarrow$ $a[b+(-b)]+(-a)(-b)=ab+[a+(-a)](-b)$

Distributive property.

$\Rightarrow$ $a(0)+(-a)(-b)=ab+(0)(-b)$ $\because$ inverse

$\Rightarrow$ $0+(-a)(-b)=ab+0$

$\Rightarrow$ $(-a)(-b)=ab$ — Identity

symbol

$\because$: Since

$\therefore$: because

3. Prove that $||a|-|b|| \le |a-b|$ $\forall$ $a,b \in \mathbb{R}$

By Theorem:

$|a+b| \le |a|+|b|$

replace 'b' by $-b$

$|a-b| \le |a|+|-b| = |a|+|b| \to$ (A) $\quad \because |-b| = b$

Replace 'a' by 'b−a'

$|\cancel{b}-a-\cancel{b}| \le |b-a|+|b|$

$|-a| = |a| \le |b-a|+|b| \qquad \because |b-a| = |a-b|$

$\Rightarrow |a|-|b| \le |b-a|$

$\Rightarrow |a|-|b| \le |a-b| \to$ (1)

Again replace 'b' by 'a−b'.

$|\cancel{a}-\cancel{a}+b| \le |a|+|a-b|$

$|b| \le |a|+|a-b|$

$|b|-|a| \le |a-b|$

$-(|a|-|b|) \le |a-b| \Rightarrow |a|-|b| \ge -|a-b| \to$ (2)'

$\to$ (2)

from (1) and (2)'

$-|a-b| \le (|a|-|b|) \le |a-b| \qquad \because |x| < a$

$\Rightarrow ||a|-|b|| \le |a-b| \qquad \Rightarrow -a < x < a$

---

4. Express $3<x<7$ in modulus notation.

Sol. Since $|x-a| < l \Rightarrow -l < x-a < l$

$\Rightarrow a-l < x < a+l \to$ (1)

Now $3 < x < 7 \to$ (2)

Compairing (1) and (2)

$a-l = 3 \to$ (3) $, a+l = 7 \to$ (4)

adding (3) & (4)

$a-\cancel{l}+a+\cancel{l} = 7+3 \Rightarrow 2a = 10 \Rightarrow \boxed{a=5}$

put in (3)

$5-l = 3 \Rightarrow l = 5-3 = 2 \Rightarrow \boxed{l=2}$

Hence given inequality can be expresed as.

$|x-5| < 2.$

---

5. if $\delta > 0$ and $a \in \mathbb{R}$, show $a-\delta < x < a+\delta$ iff $|x-a| < \delta$

Sol. Suppose $a-\delta < x < a+\delta$

$a-\delta<x$ and $x<a+\delta$

$-\delta<x-a \rightarrow ①$ $\quad$ $x-a<\delta \rightarrow ②$

Combining ① and ②

$$-\delta < x-a < \delta$$

$$\Rightarrow |x-a| < \delta$$ proved.

Conversely suppose $|x-a|<\delta$

$$-\delta < x-a < \delta$$

$$a-\delta < x < a+\delta$$ proved

6. Give an example of a set of rational number which is bounded above but does not have rational supremum.

Sol. Consider set S of rational number defined by

$$S = \{x \in Q : x^2 < 2\}$$

The $\sup S = \sqrt{2}$, but This is not rational no.

7. Solve each of the following inequalities (7-15)

7. $|2x+5| > |2-5x|$

Associated equation

$$|2x+5| = |2-5x|$$

$$2x+5 = \pm(2-5x)$$

$2x+5 = 2-5x$

$2x+5x = 2-5$

$7x\ \cancel{8x} = -3$

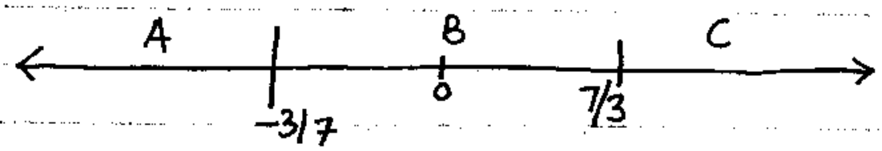
$x = -3/7$

$2x+5 = -2+5x$

$2x-5x = -2-5$

$-3x = -7$

$x = 7/3$



Region A

put $x=-1$

$|-2+5| > |2+5|$

$= |3| > |7|$

False

Region B put $x=0$

$|5| > |2|$

True

Region C put $x=3$

$|6+5| > |2-15|$

$|11| > |-13|$

$|11| > |13|$ False

So

Solution set $= \left] -\frac{3}{7}, \frac{7}{3} \right[$

8. $\left|\frac{x+8}{12}\right| < \frac{x-1}{10}$

$\pm\left(\frac{x+8}{12}\right) < \frac{x-1}{10}$

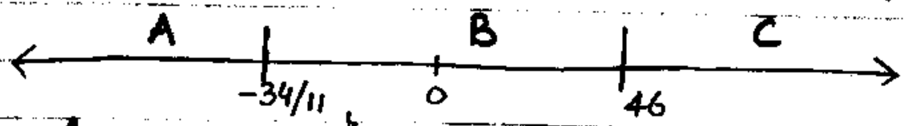
$\frac{x+8}{12} < \frac{x-1}{10}$ or $-\left(\frac{x+8}{12}\right) < \frac{x-1}{10}$

$10x+80 < 12x-12$ $-10x-80 < 12x-12$

$12x-10x > +80+12$ $12x+10x > -80+12$

$2x > \cancel{68}\ 92$ $22x > -68$

$x > 46$ $x > -\frac{34}{11}$



Region A	Region B	Region C
put $x=-4$	put $x=0$	put $x=47$
$\left\lvert\frac{-4+8}{12}\right\rvert < \frac{-4-1}{10}$	$\left\lvert\frac{8}{12}\right\rvert < \frac{-1}{10}$	$\left\lvert\frac{47+8}{12}\right\rvert < \frac{47-1}{10}$
$\left\lvert\frac{4}{12}\right\rvert < \frac{-5}{10}$	False	$\left\lvert\frac{39}{12}\right\rvert < \frac{46}{10}$
$\left\lvert\frac{1}{3}\right\rvert < -\frac{1}{2}$ False		$3.25 < 4.6$ True.

Solution set $= \left]46, \infty\right[$

9. $|x| + |x-1| > 1$ Available at www.MathCity.org

associated equation.

$|x| + |x-1| = 1$ $\pm x \pm (x-1) = 1$

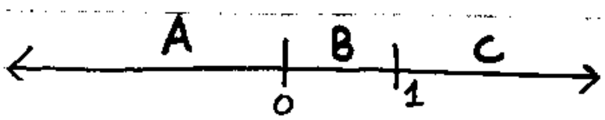
four cases.

$x + (x-1) = 1$ $-\cancel{x} + \cancel{x} - 1 = 1$ $x - x + 1 = 1$ $-x-x+1 = 1$

$2x-1 = 1$ Not possible   Not possible $-2x = 0$

$2x = 2$ $x = 0$

$\boxed{x=1}$



Region A	Region B	Region C
put $x=-1$	put $x=1/2$	put $x=2$
$\lvert-1\rvert + \lvert-1-1\rvert > 1$	$\left\lvert\frac{1}{2}\right\rvert + \left\lvert\frac{1}{2}-1\right\rvert > 1$	$\lvert 2\rvert + \lvert 2-1\rvert > 1$
$1 + \lvert-2\rvert > 1$	$\left\lvert\frac{1}{2}\right\rvert + \left\lvert\frac{1}{2}\right\rvert > 1$	$2+1 > 1$
$3 > 1$ True	$1 > 1$ False	$3 > 1$ True

Solution set $=]-\infty, 0[\cup]1, \infty[$

10. $12x^2 - 25x + 12 > 0$

Sol. associated equation

Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

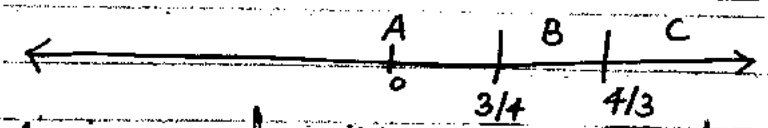
$$12x^2 - 25x + 12 = 0$$

$$x = \frac{25 \pm \sqrt{625 - 576}}{24}$$

$$x = \frac{25 \pm \sqrt{49}}{24} = \frac{25 \pm 7}{24}$$

$$x = \frac{25+7}{24} = \frac{32}{24} \quad ; \quad x = \frac{25-7}{24} = \frac{18}{24}$$

$$\boxed{x = \frac{4}{3}} \qquad \boxed{x = \frac{3}{4}}$$



Region A	Region B	Region C
put $x = 0$	put $x = 1$	put $x = 2$
$0 + 0 + 12 > 0$	$12 - 25 + 12 > 0$	$12(4) - 25(2) + 12 > 0$
$12 > 0$	$-1 > 0$	$48 - 50 + 12 > 0$
True	False	$10 > 0$ True

Solution set $=]-\infty, \frac{3}{4}[\cup]\frac{4}{3}, \infty[$

11. $\frac{x-1}{2} - \frac{1}{x} > \frac{4}{x} + 5$

here $x = 0$ is free boundry number.

Associated equation.

$$\frac{x-1}{2} - \frac{1}{x} = \frac{4}{x} + 5$$

$$x(x-1) - 2 = 4(2) + 5(2x)$$

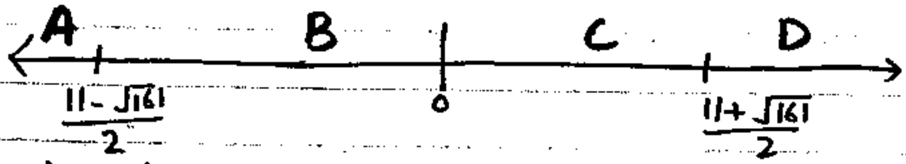
$$x^2 - x - 2 = 8 + 10x$$

$$x^2 - 10x - x - 2 - 8 = 0$$

$$x^2 - 11x - 10 = 0$$

$$x = \frac{11 \pm \sqrt{121 + 40}}{2} = \frac{11 \pm \sqrt{161}}{2}$$

Available at www.MathCity.org



Region A. put $x = -1$

$$\frac{-1-1}{2} - \frac{1}{-1} > \frac{4}{-1} + 5 \Rightarrow -1 + 1 > -4 + 5$$

$$0 > 1 \quad \text{False}$$

Region B

put $x = 0.5$

$$\frac{-0.5-1}{2} - \frac{1}{-0.5} > \frac{4}{-0.5} + 5$$
$$= -\frac{1.5}{2} + 2 > -8+5$$
$$-0.75 + 2 > -3$$
$$1.25 > -3$$
True.

Region C

put $x = 5$

$$\frac{5-1}{2} - \frac{1}{5} > \frac{4}{5} + 5$$
$$2 - \frac{1}{5} > \frac{4+25}{5}$$
$$\frac{9}{5} > \frac{29}{5}$$
False

Region D

put $x = 13$

$$\frac{13-1}{2} - \frac{1}{13} > \frac{4}{13} + 5$$
$$7 - \frac{1}{13} > \frac{4+65}{13}$$
$$\frac{90}{13} > \frac{69}{13}$$
True

Solution set $= \left]\frac{11-\sqrt{161}}{2}, 0\right[\cup \left]\frac{11+\sqrt{161}}{2}, \infty\right[$

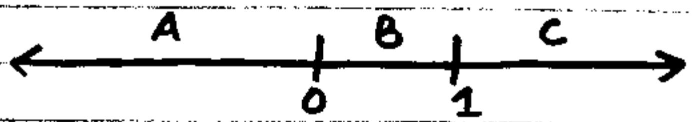
12. $|x^2 - x + 1| > 1$

associated equation

$$|x^2 = x+1| = 1$$

$$x^2 - x + 1 = 1$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0, 1$$

$$-x^2 + x - 1 = 1 \Rightarrow x^2 - x + 1 = -1$$
$$x^2 - x + 2 = 0$$
$$x = \frac{1 \pm \sqrt{1-8}}{2}$$ imaginary roots.
$$x = \frac{1 \pm \sqrt{7}i}{2}$$



Region A

put $x = -1$

$$1 + 1 + 1 > 1$$
$$3 > 1$$ True

Region B

put $x = 1/2$

$$\frac{1}{4} - \frac{1}{2} + 1 > 1$$
$$1 - 2 + 4 > 4$$
$$3 > 4$$ False

Region C

put $x = 2$

$$4 - 2 + 1 > 1$$
$$3 > 1$$
True.

Solution set. $=]-\infty, 0[\cup]1, \infty[$

13. $x^{-2} - 4x^{-1} + 4 > 0$

$$\frac{1}{x^2} - \frac{4}{x} + 4 > 0$$
$$\frac{1 - 4x + 4x^2}{x^2} > 0$$
$$\left(\frac{2x-1}{x}\right)^2 > 0$$

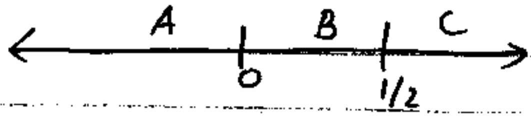
$\because 4x^2 - 4x + 1$
$= (2x)^2 - 2(2x)(1) + (1)^2$
$= (2x-1)^2$

here $x = 0$ is free boundry number.

associated equation.

$$\left(\frac{2x-1}{x}\right) = 0$$

$$2x - 1 = 0 \Rightarrow x = 1/2$$



Region A

Put $x=-1$

$\frac{(2(-1)-1)^2}{(-1)^2} \geq 0$

$\frac{(-2-1)^2}{1} > 0$

$9>0$ True

Region B

$x = 1/4$

$\frac{\left(2\left(\frac{1}{4}\right)-1\right)^2}{1/4} \geq 0$

$\frac{(-1/2)^2}{1/4} > 0$

$1>0$ True

Region C

put $x=1$

$\frac{(2(1)-1)^2}{(1)^2} \geq 0$

$(2-1)^2 > 0$

$1>0$

True

Solution set $=]-\infty, 0[\cup]0, \frac{1}{2}[\cup]\frac{1}{2}, \infty[$

14. $\frac{2x}{x+2} \geq \frac{x}{x-2}$

$x = \pm 2$ are free boundy numbers.

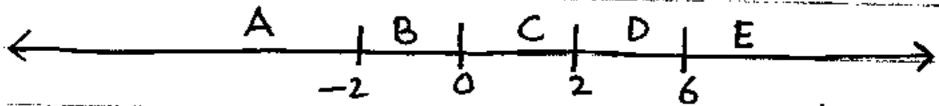
associated equation.

$\frac{2x}{x+2} = \frac{x}{x-2}$

$2x(x-2) = x(x+2)$

$2x^2 - 4x = x^2 + 2x$

$x^2 - 6x = 0 \Rightarrow x(x-6) = 0 \Rightarrow x = 0, 6$



Region A

put $x=-3$

$\frac{-6}{-1} \geqslant \frac{-3}{-5}$

$6 > 3/5$ True

Region B

put $x=-1$

$\frac{-2}{+1} \geqslant \frac{-1}{-3}$

$-2 \geqslant 1/3$

false

Region C

put $x=1$

$\frac{2}{3} \geqslant \frac{1}{-1}$

$2/3 \geqslant -1$ True.

Region D

put $x=3$

$\frac{6}{5} \geqslant \frac{3}{1}$

$1.3 \geqslant 3$ false

Region E

put $x=7$

$\frac{14}{9} \geqslant \frac{7}{5}$ true.

Solution Set $=]-\infty, -2[\cup [0, 2[\cup [6, \infty[$

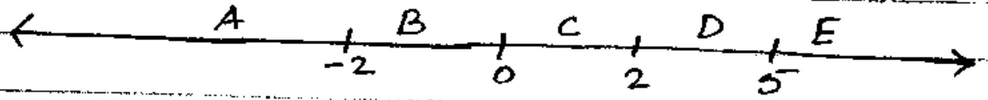
15. $x^4 - 5x^3 - 4x^2 + 20x \leq 0$

associated equation:

$x^4 - 5x^3 - 4x^2 + 20x = 0$

$$x(x^3-5x^2-4x+20)=0$$

$$x=0, \quad x^3-5x^2-4x+20=0$$
$$x^2(x-5)-4(x-5)=0$$
$$(x-5)(x^2-4)=0$$
$$(x-5)(x+2)(x-2)=0$$
$$x = 2, -2, 5$$



Region A.
Put $x=-3$.
$(-3)^4-5(3)^3-4(-3)^2+20(-3)\le 0$
$+81+135-36-60\le 0$
$120\le 0$ ~~$-42 \le 0$~~
False.

Region B
Put $x=-1$
$(-1)^4-5(-1)^3-4(-1)^2+20(-1)\le 0$
$1+5-4-20\le 0$
$-18\le 0$ True

Region C
Put $x=1$
$(1)^4-5(1)^3-4(1)+20\le$
$1-5-4+20\le 0$
$12\le 0$ False

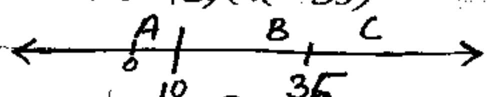
Region D
Put $x=3$
$(3)^4-5(3)^3-4(3)^2+20(3)\le 0$
$81-135-36+60$
$-30\le 0$ True

Region E
Put $x=6$
$(6)^4-5(6)^3-4(6)^2+20(6)\le 0$
$1296-1080-144+120\le 0$
$192\le 0$ False

Solution set $= [-2,0]\cup[2,5]$

16. Cost function $C(x)$ and revenue function $R(x)$ for producing x units of certain product are given by
$C(x)=5x+350$; $R(x)=50x-x^2$.
find values of x for yeild a profit.

Sol. profit is produced if revenue exeeds cost.
i.e $R(x) > C(x) \Rightarrow 50x-x^2 > 5x+350$
$$x^2+5x-50x+350<0$$
$$x^2-45x+350<0$$
associated equation
$$x^2-45x+350<0 \quad (1)$$
$$x^2-10x-35x+350<0$$
$$x(x-10)-35(x-10)$$
$$(x-10)(x-35) \Rightarrow x=10, 35$$



Region A
put $x=0$
$0-0+350<0$
$350<0$
False

Region B
put $x=15$
$225-675+350<0$
$-100<0$ true

Region C
put $x=40$
$(40)^2-45(40)+350<0$
$1600-1800+350\le 0$
$150\le 0$ False

So Solution Set for profit is $\{x : 10<x<35\}$