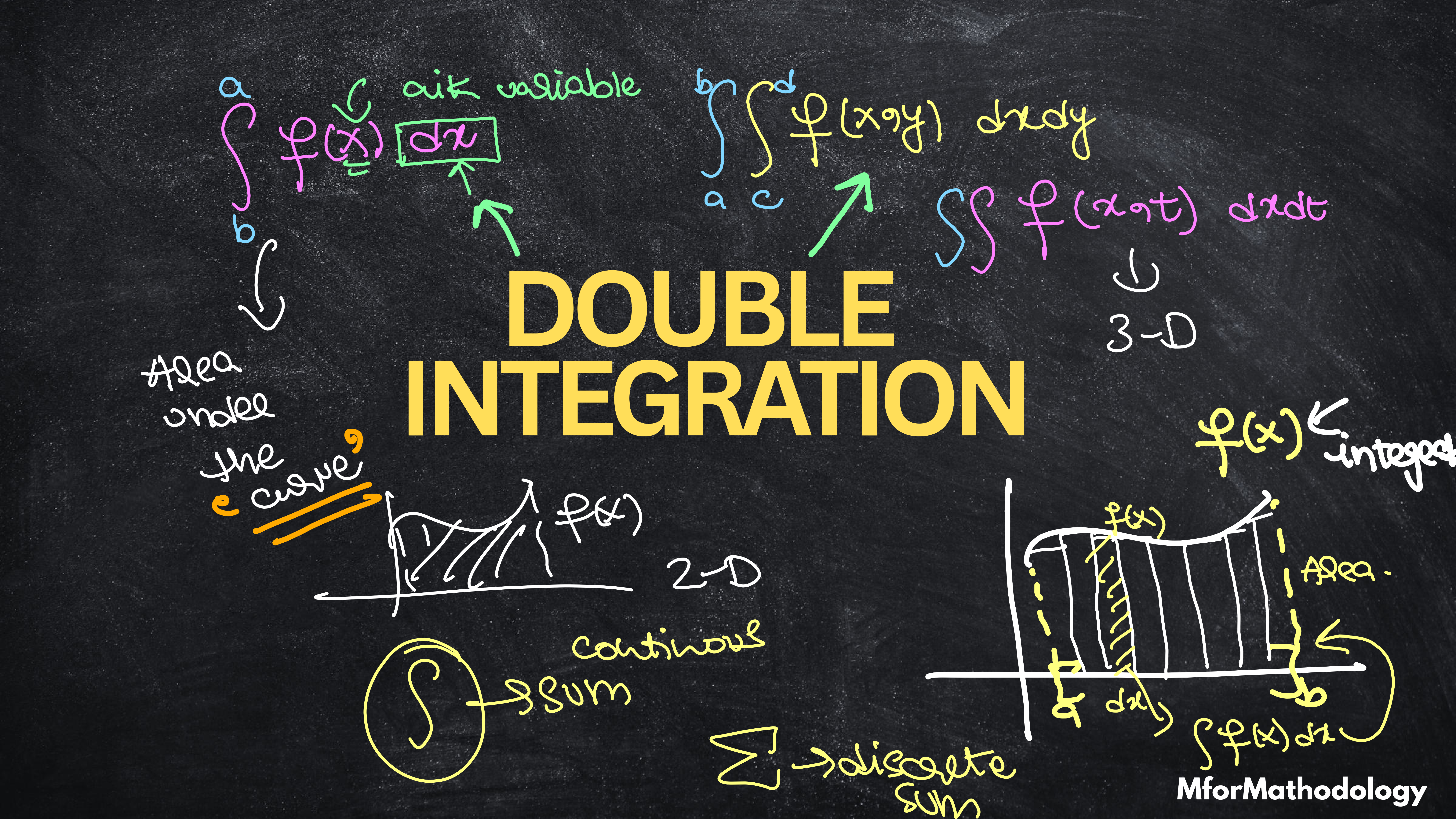


DOUBLE INTEGRATION
$\int_a^b f(x)\,\boxed{dx}$
variable
$\int_a^b \int_c^d f(x,y)\,dx\,dy$
$\iint f(x,t)\,dx\,dt$
3-D
Area under the curve
f(x)
2-D
continuous
sum
Σ → discrete sum
f(x)
Area
dx
a
b
$\int f(x)\,dx$
MforMathodology

DOUBLE INTEGRATION
KAHANI

At its core, a **double integral** is like taking the **area under a surface** — *but in 3D!* Instead of finding the area under a curve (like in single-variable calculus), we now want the **volume under a surface** over a **region in the xy-plane**.

Let's say you have a surface $z = f(x, y)$ — a sort of bumpy, rolling landscape — and a region R in the xy-plane over which this surface is defined. The **double integral** tells us:

$$\boxed{\iint_R f(x, y)\, dA}$$

This is the total volume **between the surface** $z = f(x, y)$ and the region R in the plane.

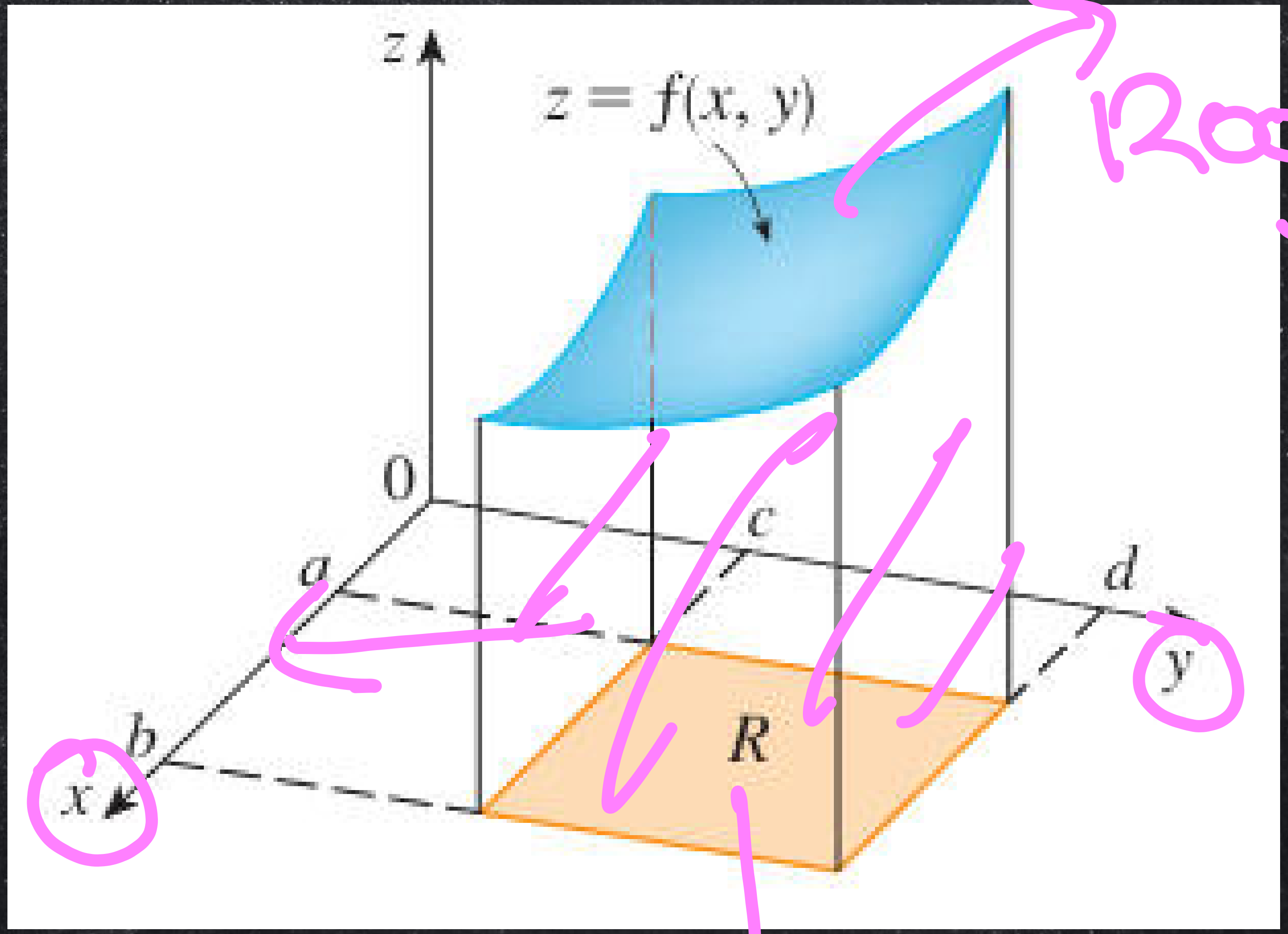
MforMathodology

DOUBLE INTEGRATION
KAHANI

Surface = the "roof"

Region = the "floor"

Double integral = the "air inside the room" between roof and floor



Roof

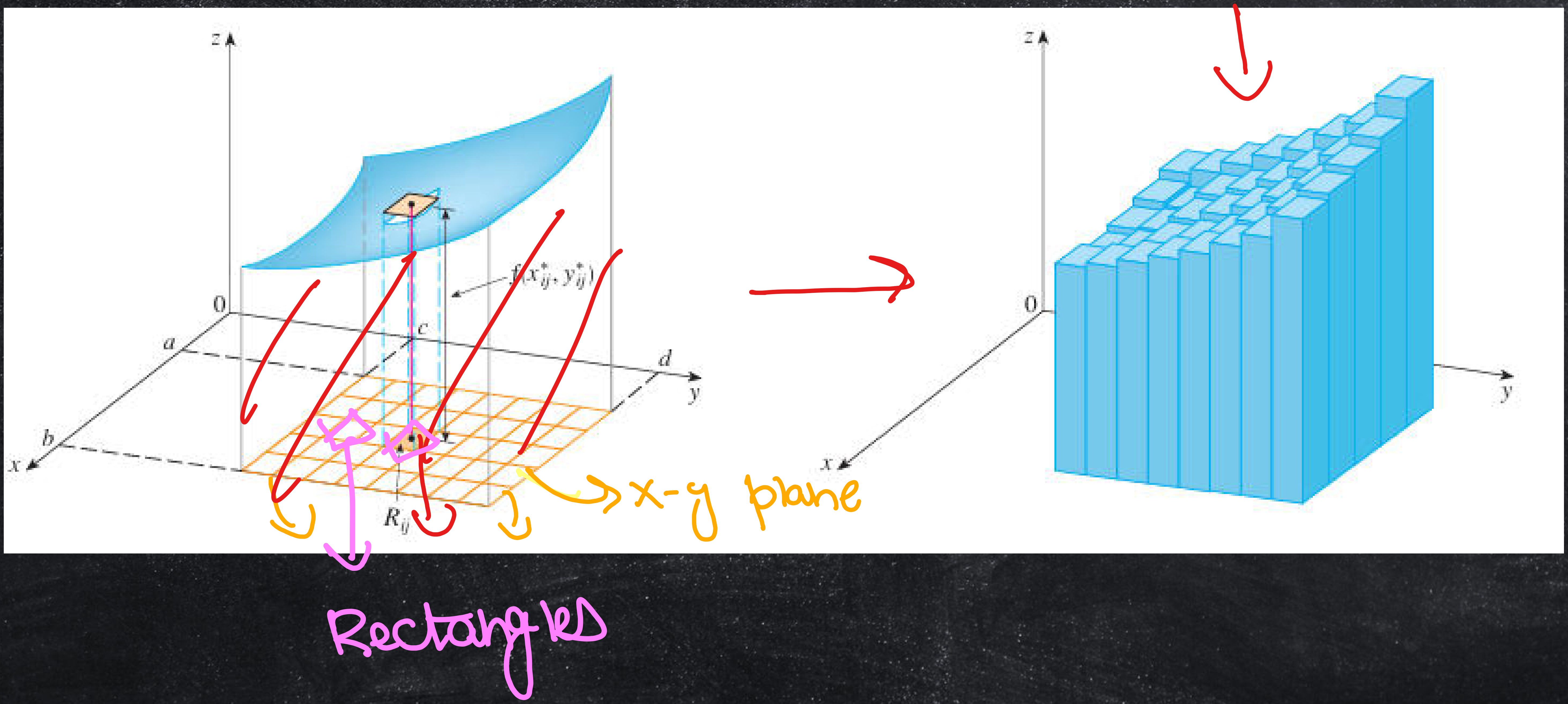
floor

$V = ?$

By

$\iint_V z\,dx\,dy$

DOUBLE INTEGRATION
KAHANI



MforMathodology

DOUBLE INTEGRATION

KAHANI

4 **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$ then

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

$$\int_0^1 \int_0^2 x+y\,dy\,dx = \int_0^1 \left[\int_0^2 x+y\,dy\right] dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^2 dx$$

$$= \int_0^1 \left[2x + \frac{4}{2} - 0 - 0\right] dx$$

$$\int_0^1 2x+2\,dx$$

$$= \left[x^2 + 2x\right]_0^1$$

$$= 1 + 2 - 0 - 0$$

$$\boxed{= 3}$$

MforMathodology

DOUBLE INTEGRATION
QUESTIONS

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx$$

$$= \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_0^3 dy$$

$$= \int_1^2 \frac{81}{3} y \, dy$$

$$= \frac{81}{3} \int_1^2 y \, dy$$

$$= \frac{81}{3} \left(\frac{y^2}{2} \right)_1^2$$

$$= \frac{81}{3} \left[\frac{4}{2} - \frac{1}{2} \right]$$

$$= \frac{81}{3} \left[2 - \frac{1}{2} \right]$$

$$= \frac{81}{\cancel{3}} \left[\frac{\cancel{4-1}}{2} \right]$$

$$= \frac{81}{2}$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: \ 0 \le x \le 2, \ \ -1 \le y \le 1.$$

Sol:

$$= \int_0^2 \int_{-1}^1 100 - 6x^2 y \, dy \, dx$$

$$= \int_0^2 \left[100y - 6x^2 \frac{y^2}{2} \right]_{-1}^1 dx$$

$$= \int_0^2 \left[100y - 3x^2 y^2 \right]_{-1}^1 dx$$

$$= \int_0^2 \left[100 - \cancel{3x^2} - 100(-1) + \cancel{3x^2}(-1)^2 \right] dx$$

$$= \int_0^2 200 \, dx = [200x]_0^2 = \boxed{400}$$

DOUBLE INTEGRATION

QUESTIONS

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1$, $0 \le y \le 2$.

$$= \int_0^2 \int_0^1 10 + x^2 + 3y^2 \, dx \, dy$$

$$= \int_0^2 \left[10x + \frac{x^3}{3} + 3xy^2 \right]_0^1 dy$$

$$= \int_0^2 \left[10 + \frac{1}{3} + 3y^2 \right] dy$$

$$= \left[\frac{31}{3} y + y^3 \right]_0^2$$

$$= 31/3(2) + 8$$

$$= \frac{62}{3} + 8$$

$$= \frac{62 + 24}{3}$$

$$= \boxed{\frac{86}{3}}$$

EXAMPLE 3 Evaluate $\iint_R y \sin(xy) \, dA$, where $R = [1, 2] \times [0, \pi]$.

$$= \int_0^\pi \int_1^2 y \sin(xy) \, dx \, dy$$

$$= \int_0^\pi \left[-\cos(xy) \right]_1^2 dy$$

$$= \int_0^\pi -\cos 2y + \cos(y) \, dy$$

$$= -\frac{1}{2} \int_0^\pi 2 \cos 2y \, dy + \int_0^\pi \cos(y) \, dy$$

$$= -\frac{1}{2} [\sin y]_0^\pi + [\sin y]_0^\pi$$

$$\boxed{= 0}$$

MforMathodology

DOUBLE INTEGRATION
QUESTION

$z = f(x, y)$

V **EXAMPLE 4** Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $\boxed{x = 2}$ and $\boxed{y = 2}$, and the three coordinate planes.

$z = 16 - x^2 - 2y^2$

So,

$$V = \int_0^2 \int_0^2 16 - x^2 - 2y^2 \, dx \, dy$$

$$V = \int_0^2 \left[16x - \frac{x^3}{3} - 2xy^2\right]_0^2 dy$$

$$V = \int_0^2 \left[32 - \frac{8}{3} - 4y^2\right] dy$$

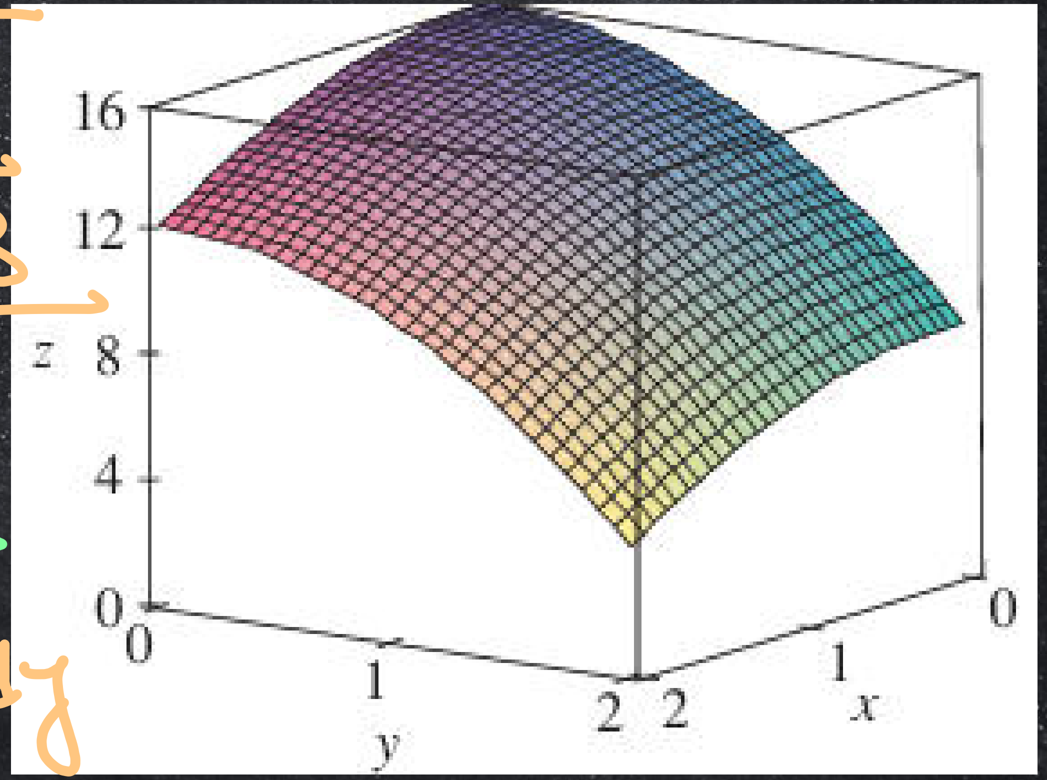
$$= \int_0^2 \frac{88}{3} - 4y^2 \, dy$$

$$= \left[\frac{88}{3} y - \frac{4y^3}{3}\right]_0^2$$

$$= \frac{88}{3}(2) - \frac{4}{3}(8)$$

$$= 48$$

$$\begin{array}{r} 32 \\ \times 3 \\ \hline 96 \\ - 8 \\ \hline 88 \end{array}$$



DOUBLE INTEGRATION

DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS

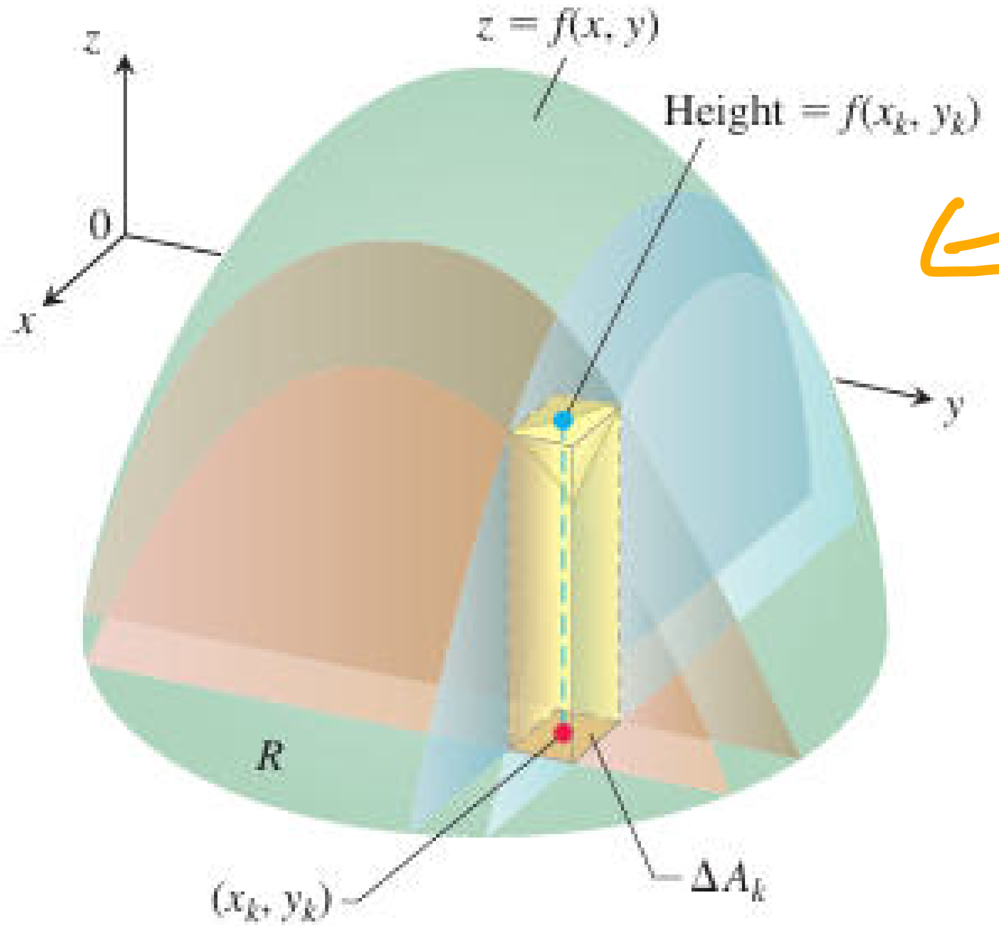
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dx\,dy$$

?

Volumes

If $f(x, y)$ is positive and continuous over R, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)\,dA$, as before (Figure 15.9).

If R is a region like the one shown in the xy-plane in Figure 15.10, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines



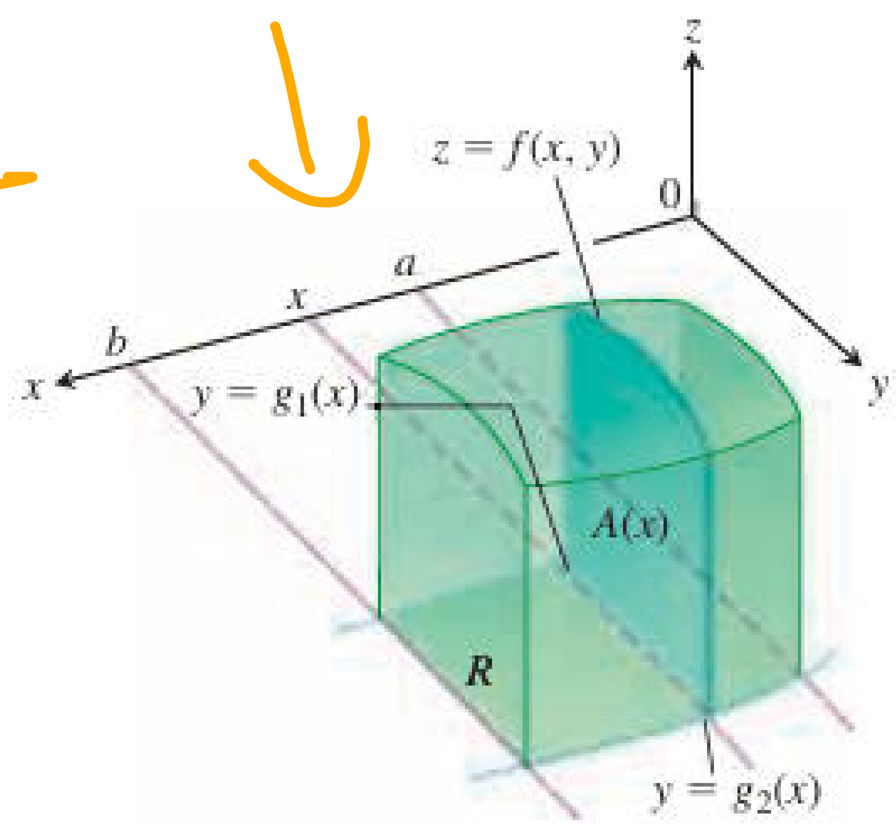


FIGURE 15.10 The area of the vertical

INDEFINITE INTEGRATION

DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS

THEOREM 2—Fubini's Theorem (Stronger Form)
Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

MforMathodology

DOUBLE INTEGRATION

FINDING LIMITS OF INTEGRATION

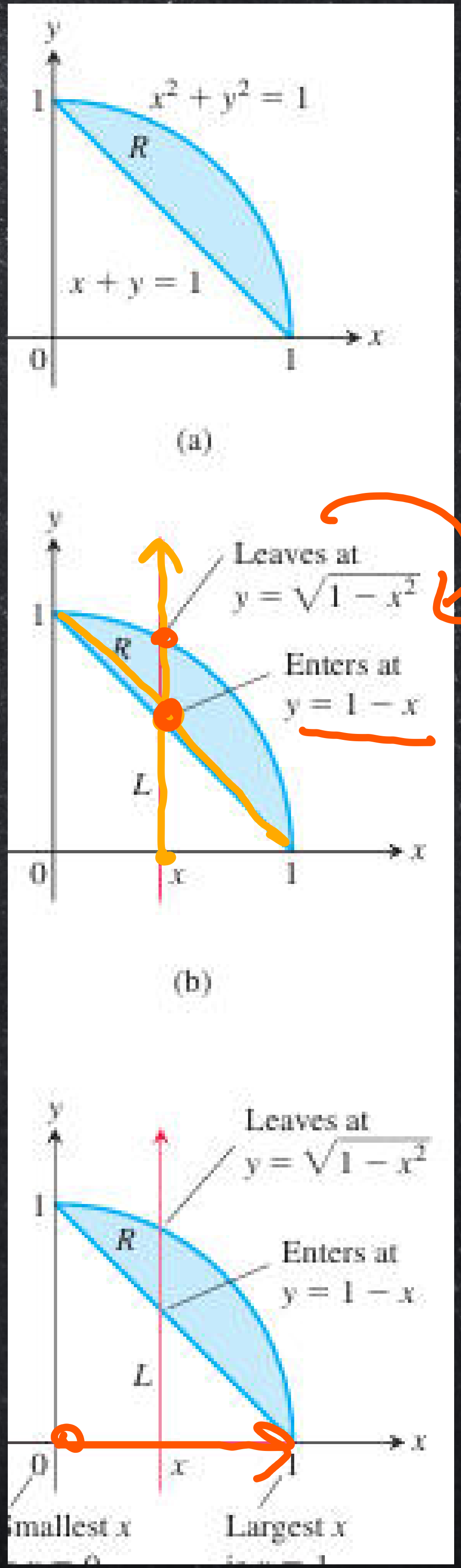
Finding Limits of Integration

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Using Vertical Cross-Sections When faced with evaluating $\iint_R f(x, y)\, dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch.* Sketch the region of integration and label the bounding curves (Figure 15.14a).
2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants) (Figure 15.14b).
3. *Find the x-limits of integration.* Choose x-limits that include all the vertical lines through R. The integral shown here (see Figure 15.14c) is

$$\iint_R f(x, y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$



MforMathodology

DOUBLE INTEGRATION

DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS

$y = x$

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

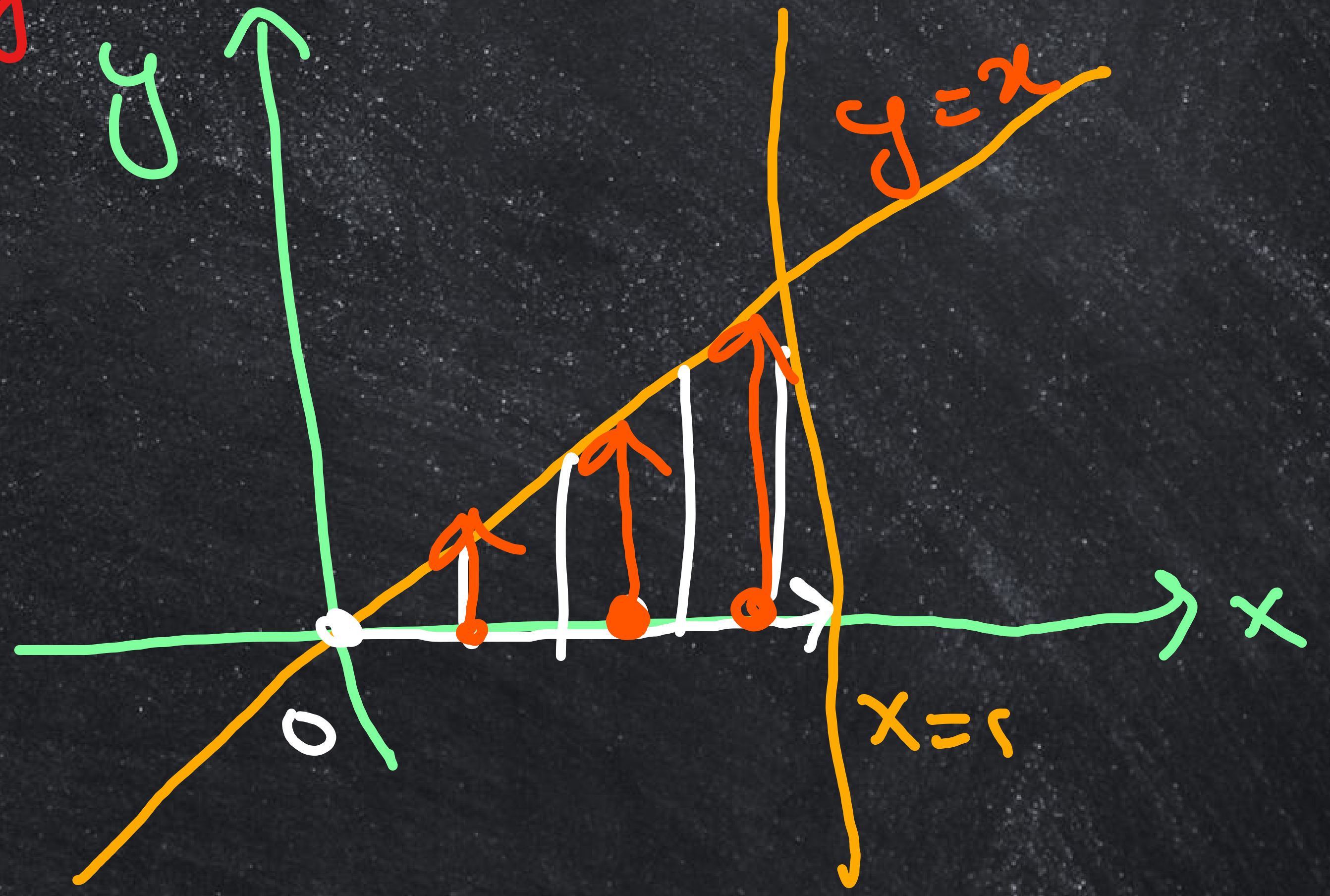
$z = f(x,y) = 3 - x - y$

$$= \int_0^1 \int_0^x 3 - x - y \; dy \, dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^x dx$$

$$= \int_0^1 \left[3x - x^2 - \frac{x^2}{2} \right] dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} - \frac{x^3}{6} \right]_0^1 = \frac{3}{2} - \frac{1}{3} - \frac{1}{6} = \frac{9-2-1}{6} = \boxed{1}$$



MforMathodology

DOUBLE INTEGRATION

DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS

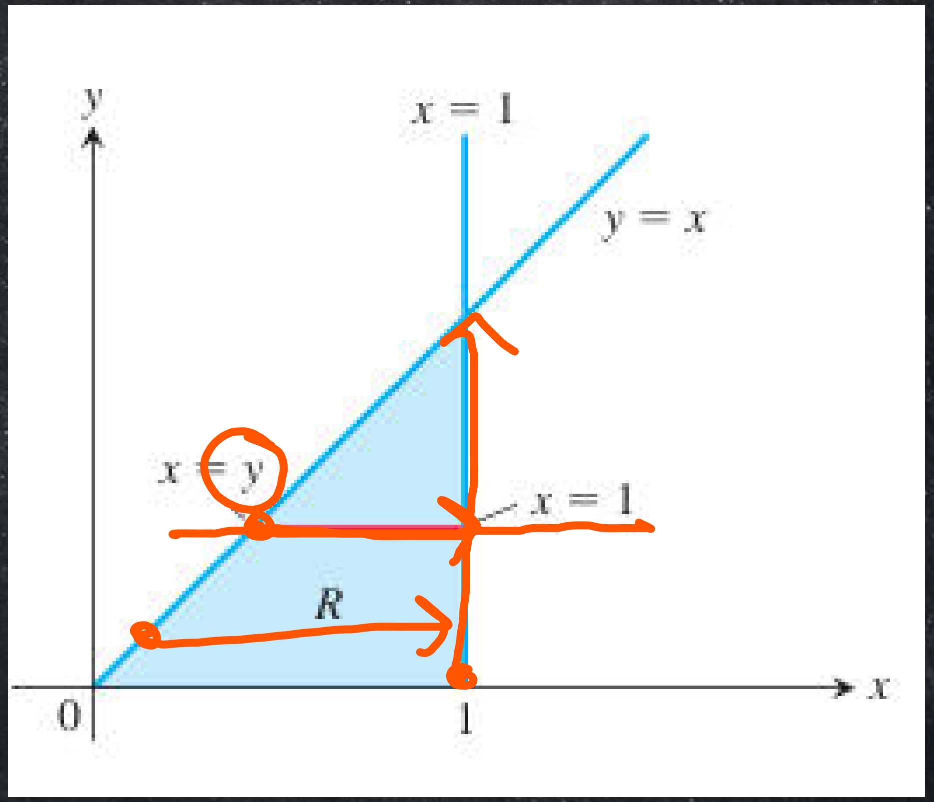
EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$



$$\int_0^1 \int_y^1 3 - x - y \, dx\, dy$$

MforMathodology

DOUBLE INTEGRATION

~~DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS~~

EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $\boxed{y = x,}$ and the line $\boxed{x = 1.}$

$$= \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \left[\frac{\sin x}{x}\, y\right]_0^x dx$$

$$= \int_0^1 \frac{\sin x}{\cancel{x}} (\cancel{x})\, dx$$

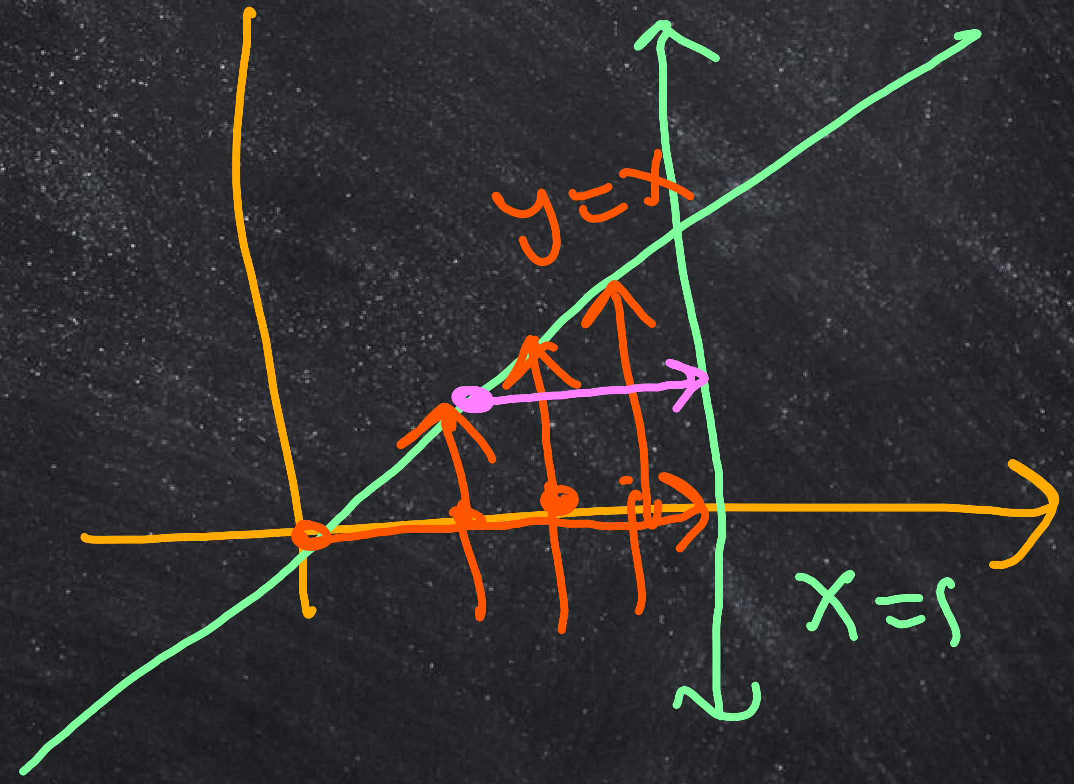
$$= \int_0^1 \sin x\, dx$$

$$= \int_0^1 \sin x\, dx$$

$$= \left[-\cos x\right]_0^1$$

$$= -\cos 1 + \cos 0$$

$$\boxed{= -\cos 1 + 1}$$



DOUBLE INTEGRATION

DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS

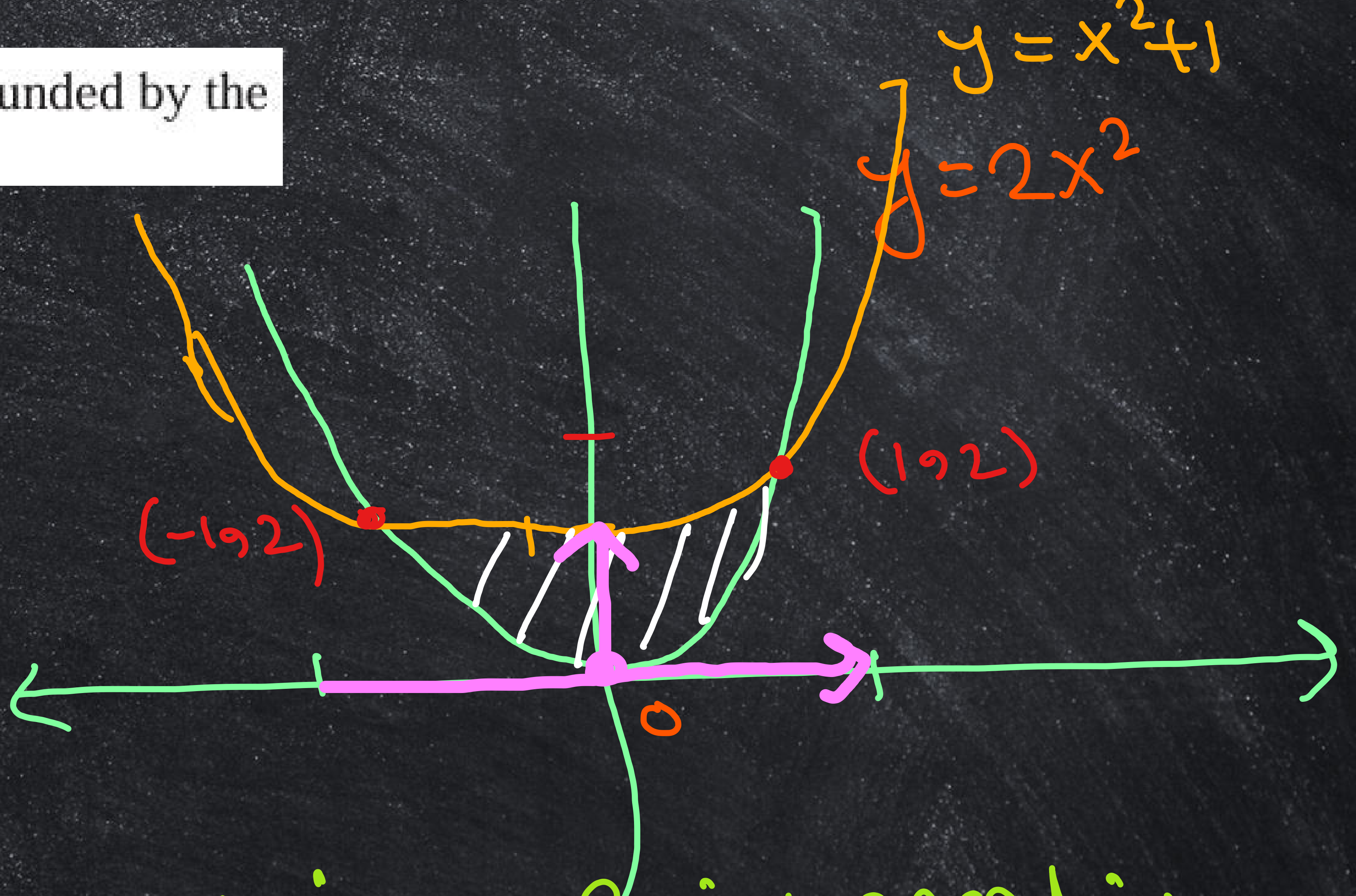
V **EXAMPLE 1** Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$$= \int_{-1}^{1} \int_{2x^2}^{x^2+1} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2 \right]_{2x^2}^{x^2+1} dx$$

$$= \int_{-1}^{1} \left[x(x^2+1) + (x^2+1)^2 - x(2x^2) - (2x^2)^2 \right] dx$$

$$= \int_{-1}^{1} (x^3 + x + x^4 + 1 + 2x^2 - 2x^3 - 4x^4)\, dx$$



point of intersection of $y = 2x^2$ and $y = 1 + x^2$

$$2x^2 = 1 + x^2$$

$$2x^2 - x^2 = 1$$

DOUBLE INTEGRATION

$x^2 = 1$

$x = \pm 1$

DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS

$y = 2$, $y = 2$

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

$$= \int_{-1}^{1} -x^3 - 3x^4 + 2x^2 + x + 1 \, dx$$

$$= \left[-\frac{x^4}{4} - \frac{3}{5}x^5 + \frac{2x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^{1}$$

$$= \boxed{\frac{32}{15}}$$

MforMathodology

DOUBLE INTEGRATION

DOUBLE INTEGRALS OVER NONRECTANGULAR, GENERAL REGIONS

V EXAMPLE 3 Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $\boxed{y = x - 1}$ and the parabola $\boxed{y^2 = 2x + 6}$.

$$\int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy \, dx \, dy$$

$$x = y+1$$

$$\frac{y^2-6}{2} = x$$

$$\frac{1}{2}y^2 - 3 = x$$

point of intersection

$$y^2 = 2x+6$$

$$y^2 = (x-1)^2$$

$$2x+6 = (x-1)^2$$

$$2x+6 = x^2+1-2x$$

$$x^2-4x-5=0$$

$$x^2-5x+x-5=0$$

$$x(x-5)+1(x-5)=0$$

$$(x+1)(x-5)=0$$

$$= \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{\frac{1}{2}y^2-3}^{y+1} dy$$

$$= \int_{-2}^{4}\frac{y}{2}\left[(y+1)^2 - \left(\frac{1}{2}y^2-3\right)^2\right] dy$$

$$= \int_{-2}^{4}\frac{y}{2}\left[y^2+1+2y-\frac{1}{4}y^4-9+3y^2\right] dy$$

MforMathodology

$$= \int_{-2}^{4} \frac{y^3}{2} + \frac{y}{2} + y^2 - \frac{y^5}{8} - \frac{9y}{2} + \frac{3y^3}{2}\, dy$$

DOUBLE INTEGRATION

AREA BY DOUBLE INTEGRATION

$x = 5y-1$

$y = x-1$

$y = 4$

$y = -2$

The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$= \int_0^1 \int_{x^2}^{x} dy\, dx$$

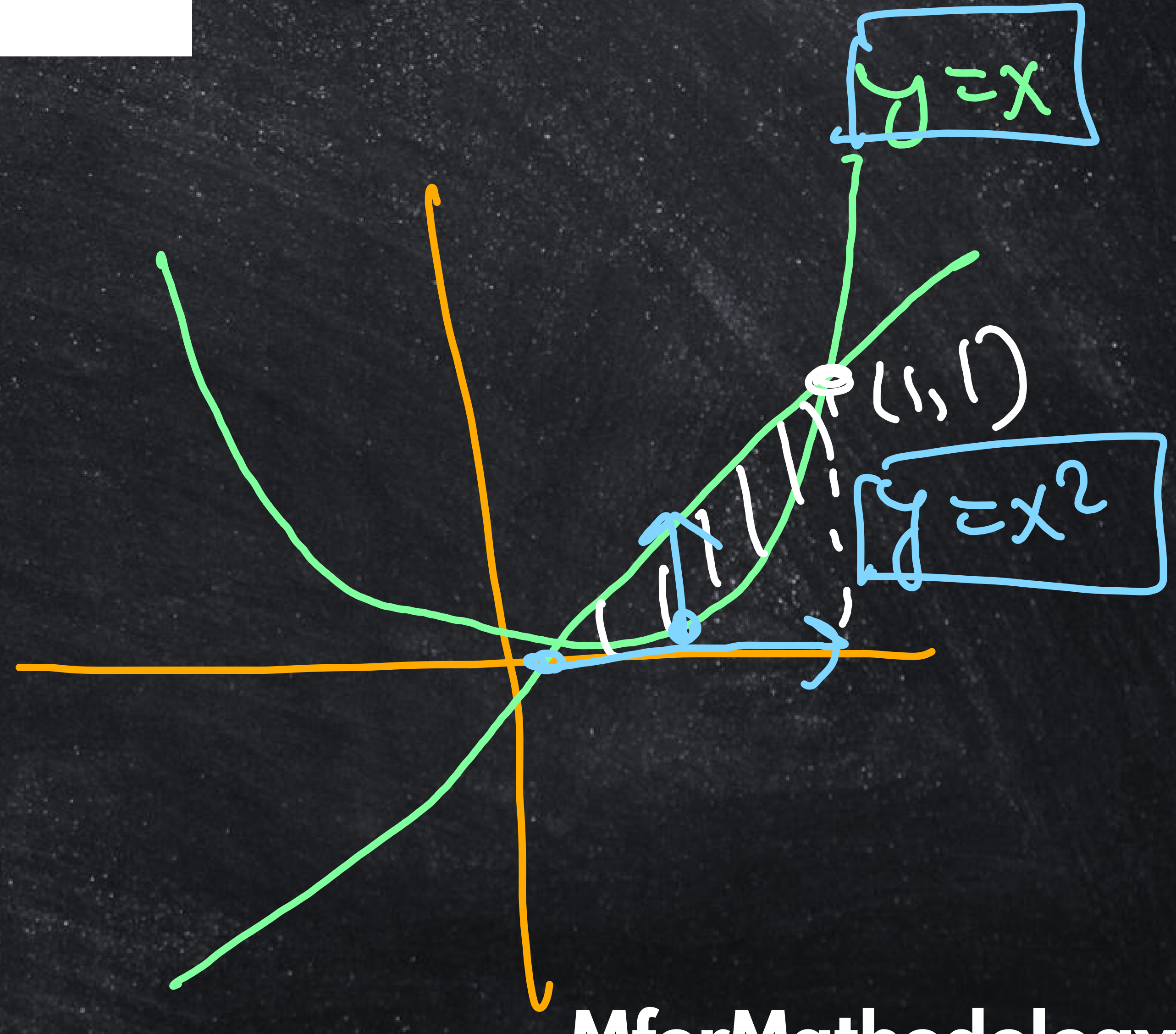
$$= \int_0^1 [y]_{x^2}^{2x^2} dx$$

$$= \int_0^1 x - x^2\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$\boxed{= 1/6}$$



MforMathodology

DOUBLE INTEGRATION

AREA BY DOUBLE INTEGRATION

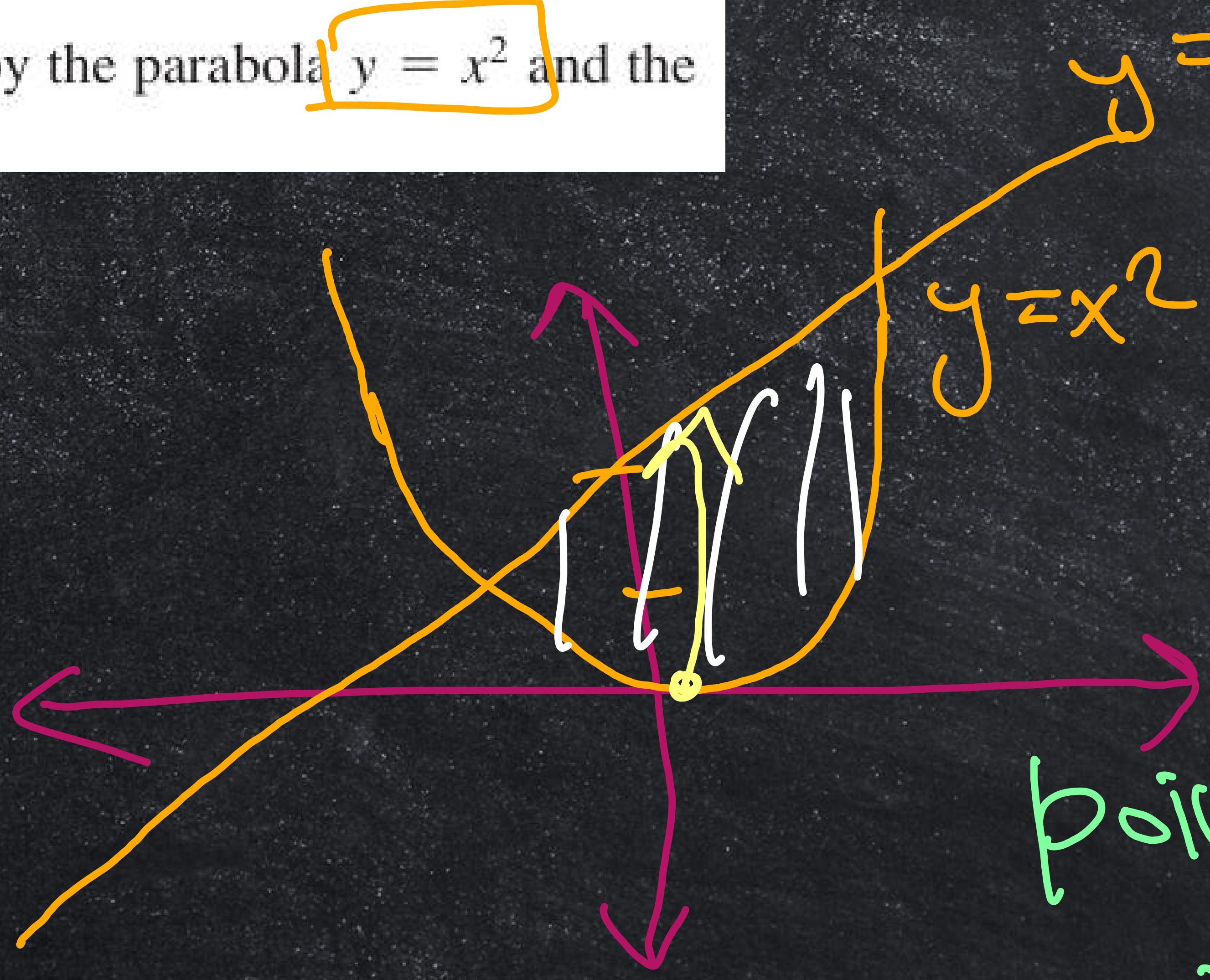
EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

$$= \int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx$$

$$= \int_{-1}^{2} \left[y \right]_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} \{x + 2 - x^2\}\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} - 2 + \frac{1}{3}$$



pt. of intersection

$x^2 = x$

$\boxed{x = 1}$

$y = x$

$\boxed{y = 1}$

point of intersection

$x^2 = x + 2$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$x = -1, 2$

DOUBLE INTEGRATION

DOUBLE INTEGRATION IN POLAR COORDINATE

$= \frac{a}{2}$

given $y = x + 2$

$y = 1 \; a \; 4$

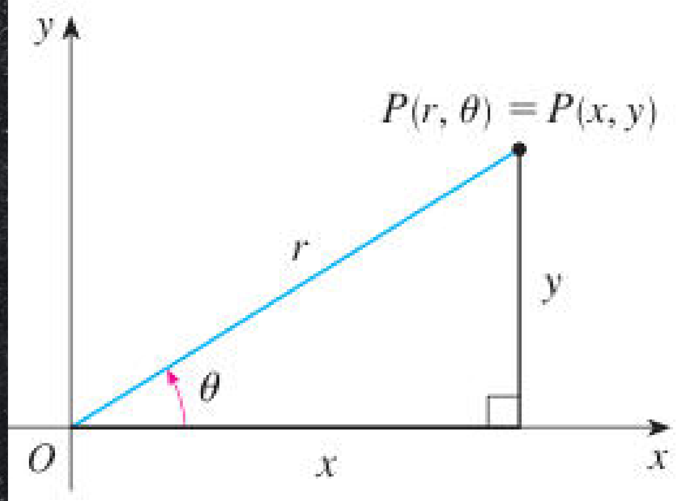


FIGURE 2

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = \boxed{x^2 + y^2} \qquad x = r\cos\theta \qquad y = r\sin\theta$$

(See Section 10.3.)

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b,\ \alpha \leqslant \theta \leqslant \beta\}$$

$\to (r, \theta)$

$\to x = r\cos\theta$

$\to y = r\sin\theta$

$\to r^2 = x^2 + y^2$

$\to \theta = \tan^{-1}\frac{y}{x}$

$$\boxed{dx\,dy \to r\,dr\,d\theta}$$

MforMathodology

DOUBLE INTEGRATION

DOUBLE INTEGRATION IN POLAR COORDINATE

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$1 \le r^2 \le 4$$

$$1 \le r \le 2$$

$$= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi \left[\int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\, dr\right] d\theta$$

$$= \int_0^\pi \left[\cancel{3}\cos\theta\, \frac{r^3}{\cancel{3}} + \cancel{4}\frac{r^4}{\cancel{4}}\sin^2\theta\right]_1^2 d\theta$$

$$= \int_0^\pi (8\cos\theta + 16\sin^2\theta - \cos\theta - \sin^2\theta)\, d\theta$$

$$= \int_0^\pi 7\cos\theta + 15\sin^2\theta\, d\theta$$

$$= 7[\sin\theta]_0^\pi + 15\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_0^\pi$$

$$= 0$$

MforMathodology