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Complex numbers:

The number of the form $a+ib$ where $i=\sqrt{-1}$ are called Complex numbers are denoted by "Z"

Complex numbers as an ordered pair of Real numbers:

Let C be The Set of all ordered pair of $R \times R$ which Satisfy The following Conditions

$\forall \; a, b, c, d \in R$

(i) $(a,b)+(c,d) = (a \pm c, \; b \pm d)$

(ii) $(a,b)=(c,d) \Rightarrow a=c$ and $b=d$

(iii) if $K \in R$ Then $K(a,b) = (Ka, Kb)$

(iv) $(a,b)(c,d) = (ac-bd, \; ad+bc)$ are Called Complex numbers.

Note:

$a, b \in R$

$Z = a+ib = (a,b)$ —(1)

where "a" is real part of Z denoted by $Re(Z)$ and "b" is Imagenary part of Z. denoted by $Im(Z)$

$\Rightarrow$ if $a=0$ Then (1) be Came

$Z = 0+ib = bi$ it's Called pure Imagenary

OR

if Real part of Z is zero

Then z is said to be pure Imaginary.

⇒ if Imaginary part of z is zero Then z is said to be real number

⇒ Every Real number is a complex number but every complex numbers is not real number.

Additive Identity of Complex no

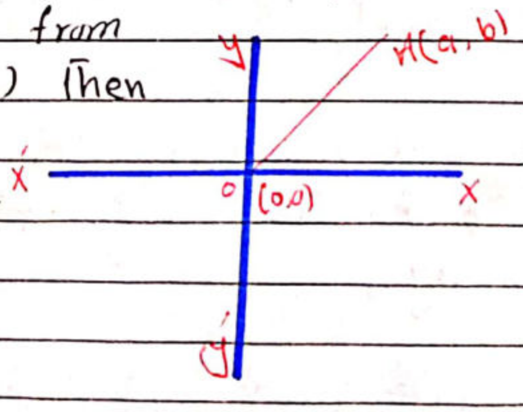
The Complex number of The form $0+0i = (0,0)$ is Called Additive Identity of complex number.

Multiplicative Identity of Complex number.

The Complex number of the form $1+0i = (1,0)$ is Called multiplicative Identity of Complex number.

Modulus of Complex number

Let $z = a+ib = (a,b)$ be a complex number Then modulus of z is a distance from orgin to $A(a,b)$ Then



$$|z| = \sqrt{(a-0)^2+(b-0)^2}$$

$$|z| = \sqrt{a^2+b^2}$$

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Prove That

$$(a,b)(c,d) = (ac-bd,\ ad+bc)$$

Taking L.H.S

$$\Rightarrow (a,b)(c,d)$$
$$= (a+ib)(c+id)$$
$$= ac + iad + ibc + i^2 bd \quad \because i^2=-1$$
$$= ac + i(ad+bc) - bd$$
$$= ac - bd + i(ad+bc)$$
$$= (ac-bd,\ ad+bc$$

L.H.S = R.H.S

Conjugate of Complex number.

Let $z = a+ib$ be a complex number The Conjugat of complex number is a new complex number obtained by chaning The Sign of i and it is denoted by $\bar{z}$

i.e $\bar{z} = a - ib$

Properties

Let $z_1, z_2 \in C$

$z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$

(i) $\bar{\bar{z}} = z$

Proof

Let $z = (x+iy)$

$$\bar{z} = \overline{(x+iy)}$$

$$\bar{z} = (x-iy)$$

Taking Conjugate again

$$\bar{\bar{z}} = \overline{(x-iy)}$$
$$= (x+iy)$$
$$\bar{\bar{z}} = z$$

(ii) $z\bar{z} = |z|^2$

Proof

Let $z = x+iy$

$$|z| = \sqrt{x^2+y^2}$$
$$|z|^2 = x^2+y^2 \quad (1)$$

Taking Conjugate of z

$$z = x+iy$$
$$\bar{z} = \overline{(x+iy)}$$
$$\bar{z} = (x-iy)$$

$$z\bar{z} = (x+iy)(x-iy)$$

using $(a+b)(a-b) = a^2 - b^2$

$$= (x^2 - (iy)^2)$$

$$= x^2 - i^2 y^2 \qquad i^2 = -1$$
$$= x^2 + y^2$$

by 1

$$z\bar{z} = |z|^2$$

Note

$$|z| = |\bar{z}| = |-\bar{z}| = |-z| = \sqrt{x^2+y^2}$$

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(iii) $\overline{Z_1+Z_2} = \bar{Z}_1 + \bar{Z}_2$

Proof

$$Z_1 = (x_1+iy_1) \Rightarrow \bar{Z}_1 = (x_1-iy_1)$$
$$Z_2 = (x_2+iy_2) \Rightarrow \bar{Z}_2 = (x_2-iy_2)$$

$$Z_1+Z_2 = (x_1+iy_1)+(x_2+iy_2)$$

$$= (x_1+x_2 + i(y_1+y_2)$$

taking Conjugate

$$\overline{Z_1+Z_2} = \overline{(x_1+x_2 + i(y_1+y_2)}$$

$$= (x_1+x_2) - i(y_1+y_2)$$

$$= (x_1+x_2 - iy_1 - iy_2)$$

$$= (x_1-iy_1 + x_2-iy_2)$$

$$\overline{Z_1+Z_2} = \bar{Z}_1 + \bar{Z}_2$$

(iv) $\overline{Z_1 Z_2} = \bar{Z}_1 \bar{Z}_2$

Proof

Let $Z_1 = x_1+iy_1$, $\bar{Z}_1 = (x_1-iy_1)$
$Z_2 = x_2+iy_2$, $\bar{Z}_2 = (x_2-iy_2)$

$$Z_1 Z_2 = (x_1+iy_1)(x_2+iy_2)$$

$$= (x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2)$$

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$\Rightarrow x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

$\overline{z_1 z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$ — (1)

Now

$\bar{z}_1\bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2)$

$= x_1x_2 - ix_1y_2 - ix_2y_1 + i^2 y_1y_2$

$= x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1)$ — 2

by 1 & 2

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

proof

$z_1 = (x_1 + iy_1)$, $\bar{z}_1 = x_1 - iy_1$

$z_2 = x_2 + iy_2$ $\bar{z}_2 = x_2 - iy_2$

Now

$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$

$= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$

$= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2 y_1y_2}{x_2^2 - i^2 y_2^2}$

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$$= \frac{x_1x_2+y_1y_2-i(x_1y_2-x_2y_1)}{x_2^2+y_2^2}$$

$$= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} - i\frac{(x_1y_2-x_2y_1)}{x_2^2+y_2^2}$$

Taking Conjugate

$$\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{x_1x_2+y_1y_2}{x^2+y^2} + i\frac{(x_1y_2-x_2y_1)}{x^2+y^2} \quad (1)$$

Now

$$\frac{\bar{Z_1}}{\bar{Z_2}} = \frac{x_1-iy_1}{x_2-iy_2}$$

$$= \frac{x_1-iy_1}{x_2-iy_2} \times \frac{x_2+iy_2}{x_2+iy_2}$$

$$= \frac{(x_1x_2+ix_1y_2-iy_1x_2-i^2y_1y_2}{x_2^2-i^2y_2^2}$$

$$= \frac{(x_1x_2+y_1y_2+i(x_1y_2-x_2y_1)}{x_2^2+y_2^2}$$

$$= \frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2} + i\frac{(x_1y_2-x_2y_1)}{x_2^2+y_2^2} \quad (2)$$

by 1 & 2

$$\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\bar{Z_1}}{\bar{Z_2}}$$

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(vi)

$$|Re(z)| \leq |z|$$

Proof

$$|Re(z)| = \sqrt{x^2} \leq \sqrt{x^2+y^2} = |z|$$

$$|Re(z)| \leq |z|$$

Properties of Complex No

(i) Commutative Law

The Commutative Law is hold in Complex number

$\forall z_1, z_2 \in \mathbb{C}$

(1) $z_1 + z_2 = z_2 + z_1$

(2) $z_1 \cdot z_2 = z_2 \cdot z_1$

(ii) Associative Law

The Associative law is hold in Complex number

$\forall z_1 z_2, z_3 \in \mathbb{C}$

(1) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(2) $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$

Distributive Law

The Distributive Law is hold in complex number

$\forall z_1 z_2, z_3 \in \mathbb{C}$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

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Note

⇒ unlike real numbers no two complex numbers are comparable by the ordering < or > so the stament that a complex number $z_1 > z_2$ or $z_1 < z_2$ is meaningless for any pair $z_2 z_1$ of complex numbers

⇒ Subtraction and Division can always be performed within complex numbers except for division by (0,0) this is evidenced by the following formulas:

if $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$

(i) $z_1 - z_2 = z_1 + (-z_2)$

$= (a_1, b_1) + (-a_2, -b_2)$

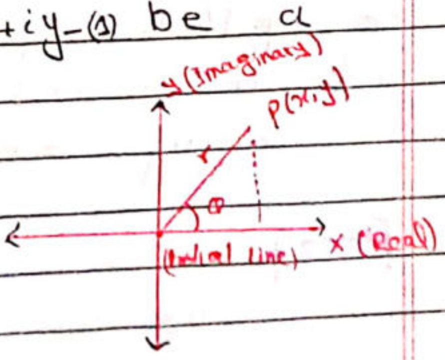
$= (a_1 - a_2, b_1 - b_2)$

(ii) $$\frac{z_1}{z_2} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}, \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right)$$

$z_2 \neq 0$

Polar form of Complex number

Let $z = x + iy$ —(1) be a complex number.



$\frac{x}{r} = \cos\varphi \Rightarrow x = r\cos\varphi$

$\frac{y}{r} = \sin\varphi \Rightarrow y = r\sin\varphi$

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As we know that the relation between rectangular and polar Coordinates

$$x = r\cos\theta \quad — ②$$
$$y = r\sin\theta \quad — ③$$

Put ② and ③ in ①

$$z = r\cos\varphi + ir\sin\varphi$$
$$z = r(\cos\theta + i\sin\theta)$$ is called The polar form of Complex number

Squaring and adding 2 and 3

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) \quad \because \sin^2\theta+\cos^2\theta = 1$$
$$x^2+y^2 = r^2(1)$$

$$x^2+y^2 = r^2$$
$$r = \sqrt{x^2+y^2}$$

we know that $|z| = \sqrt{x^2+y^2}$

so $r = \sqrt{x^2+y^2} = |z|$

$$r = |z|$$

Argument

θ is called argument or amplitude of complex number and is denoted by

$$\theta = \arg(z) = \arg(x+iy) = \tan^{-1}\left(\frac{y}{x}\right)$$

Proof

we are to prove that

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

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Diving 3 by 2

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{y}{x}$$

$$\tan\alpha = \frac{y}{x}$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

When $-\pi \leq \varphi \leq \pi$ is Called **principal argument** Then we write $\theta = \mathrm{Arg}(z)$

Argument of complex number is not Single valued function since to any $z \neq 0$ There Correspond infinitely many values of $\arg(z)$. We shal say That $\arg(z)$ is a multi valued function i.e if θ is one of the values of $\arg(z)$ we have

$$\arg(z) = \theta + 2k\pi \quad k = 0, \pm 1, \pm 2, \ldots$$

Note

⇒ if $x > 0$ and $y > 0$

$$z = x + iy$$

Then $\mathrm{Arg}(z) = \varphi = \tan^{-1}\left(\frac{y}{x}\right)$

⇒ if $x < 0$ and $y > 0$

$$z = -x + iy$$

Then $\mathrm{Arg}(z) = \varphi = \pi - \left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

⇒ if $x < 0$ and $y < 0$

$$z = -x - iy$$

Then $\mathrm{Arg}(z) = \varphi = -\pi + \tan^{-1}\left(\frac{|y|}{|x|}\right)$

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$\Rightarrow$ if $x > 0$ and $y < 0$

$$z = x - iy$$

Then $Arg(z) = \varphi = -\tan^{-1}\left(\frac{|y|}{x}\right)$

Example

Express the following Complex numbers in polar form

(i) $1 + i\sqrt{3}$ (ii) $-1 + i\sqrt{3}$

(iii) $-1 - i\sqrt{3}$ (iv) $1 - i\sqrt{3}$

Solution

$$z = 1 + i\sqrt{3}$$

where $x = 1$ and $y = \sqrt{3}$

we know that the polar form of complex numbers as:

$$z = r(\cos\varphi + i\sin\varphi)$$

also we know that $r = |z| = \sqrt{x^2 + y^2}$

$$r = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$r = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3})$$

$$\theta = \frac{\pi}{3}$$

So the polar form of $1 + i\sqrt{3}$ as $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

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Properties of Argument:

if z_1, z_2 are two complex number Then

(i) $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$

(ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$

(iii) $Arg\left(\frac{z}{\bar{z}}\right) = 2 Arg(z)$

(iv) $Arg(\bar{z}) = -Arg(z)$

DE MOIVRE.S THEOREM:

Statement:

Let $z = ((\cos\varphi + i\sin\varphi)$ be a Complex Number Then

$z^n = ((\cos\varphi + i\sin\varphi)^n$

$= (\cos n\varphi + i\sin n\varphi)$

: where n is an Integer

Proof

We are to prove That

$(\cos\varphi + i\sin\varphi)^n = (\cos n\varphi + i\sin n\varphi)$

. we will prove it by mathematical Induction methods

Step (I)

for $n=1$

$(\cos\varphi + i\sin\varphi) = ((\cos 1\varphi + i\sin 1\varphi)$

$= (\cos\varphi + i\sin\varphi$

Step (II)

for $n=K$

$((\cos\varphi + i\sin\varphi)^K = \cos K\varphi + i\sin K\varphi$

Step (III)

Now we prove for $n=K+1$

$((\cos\varphi + i\sin\varphi)^{K+1} = (\cos\varphi(K+1)\varphi + i\sin(K+1)\varphi$

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Now

$$(\cos\varphi + i\sin\varphi)^{K+1} = (\cos\varphi + i\sin\varphi)^{K}(\cos\varphi + i\sin\varphi)$$

$$\therefore (\cos\varphi + i\sin\varphi)^{K} = (\cos K\varphi + i\sin K\varphi) \because (\text{Step II})$$

$$\Rightarrow (\cos K\varphi + i\sin K\varphi)(\cos\varphi + i\sin\varphi)$$

$$= \cos K\varphi\cos\varphi + i\cos K\varphi\sin\varphi + i\sin K\varphi\cos\varphi + i^2\sin K\varphi\sin\varphi$$

$$= \cos K\varphi\cos\varphi - \sin K\varphi\sin\varphi + i(\cos K\varphi\sin\varphi + \sin K\varphi\cos\varphi)$$

Using $\Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$

$\Rightarrow \sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$

So $\cos(K+1)\varphi + i\sin(K+1)\varphi$.

$$\Rightarrow (\cos\varphi + i\sin\varphi)^{K+1} = \cos(K+1)\varphi + i\sin(K+1)\varphi$$

n^{th} Roots of a Complex Numbers:

$$(z) = (x+iy) = r(\cos\varphi + i\sin\varphi)$$

$$(z)^{1/n} = (x+iy)^{1/n} = r^{1/n}(\cos\varphi + i\sin\varphi)^{1/n}$$

$$= r^{1/n}(\cos(\varphi + 2K\pi) + i\sin(\varphi + 2K\pi))^{1/n}$$

$\because K \in \mathbb{Z}$

$$= r^{1/n}\left(\cos\frac{(\varphi + 2K\pi)}{n} + i\sin\frac{(\varphi + 2K\pi)}{n}\right)$$

by using De moivre's Theorem

$$z^{1/n} = r^{1/n}\left(\cos\frac{(\varphi + 2K\pi)}{n} + i\sin\frac{(\varphi + 2K\pi)}{n}\right)$$

Example Find The Cube Roots of $8i$

Solution

Let $z = 8i$

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$Z = 0 + 8i$

$r = |Z| = \sqrt{(0)^2 + (8)^2} = \sqrt{(8)^2} = 8$

$r = 0$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{8}{0}\right)$

$= \tan^{-1}(\infty)$

$\varphi = \frac{\pi}{2}$

As we know

$Z_k^{1/n} = r^{1/n}\left(\cos\left(\frac{\varphi + 2k\pi}{n}\right) + i\sin\left(\frac{\varphi + 2k\pi}{n}\right)\right)$

where $n = 3$, $r = 8$ and $\varphi = \frac{\pi}{2}$

$= (2^3)^{1/3}\left(\cos\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right) + i\sin\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right)\right)$

$= 2\left(\cos\left(\frac{4k\pi + \pi}{6}\right) + i\sin\left(\frac{4k\pi + \pi}{6}\right)\right)$

put $k = 0$

$= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\frac{\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$Z_0 = \sqrt{3} + i$

put $k = 1$

$Z_1 = 2\left(\cos\left(\frac{4\pi + \pi}{6}\right) + i\sin\left(\frac{4\pi + \pi}{6}\right)\right)$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

$Z_1 = -\sqrt{3} + i$

put $k = 2$

$Z_2 = 2\left(\cos\left(\frac{4\pi + \frac{\pi}{2}}{3}\right) + i\sin\left(\frac{4\pi + \frac{\pi}{2}}{3}\right)\right)$

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$$= 2\left(\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right)$$

$$= 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= 2(0 \quad -i)$$

$$Z_1 = -2i$$

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$$= 2\left(\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right)$$

$$= 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= 2(0 \quad -i)$$

$$Z_1 = -2i$$

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$$= 2\left(\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right)$$

$$= 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= 2(0 \quad - i)$$

$$Z_1 = -2i$$

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