Open notes on

Real Analysis I

Dedicated to
Prof. Syed Gul Shah
Ex Chairman, Department of Mathematics,
University of Sargodha, Sargodha, Pakistan.

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Authors:

- Dr. Atiq ur Rehman (COMSATS University Islamabad, Pakistan)
- Prof. Syed Gul Shah (University of Sargodha, Sargodha, Pakistan)
- Dr. Khuram Ali Khan (University of Sargodha, Sargodha, Pakistan)

Authors are individuals who make significant contributions to the content. Any new author must be approved by the first author or by at least three existing authors.

Open Notes Auditor:

- Ms. Marruim Izhar (Attock, Pakistan)

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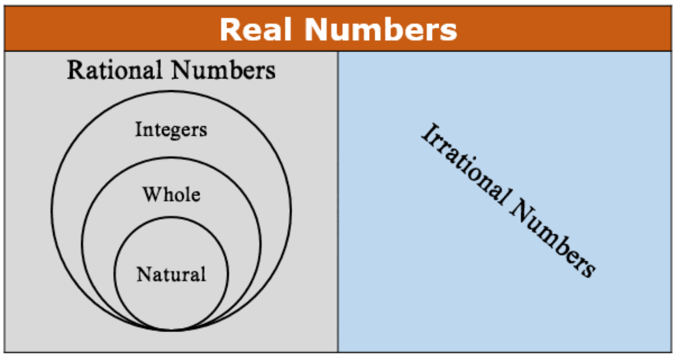
Table of Contents

Chapter 01

Real Number System

You don't have to be a mathematician to have a feel for numbers. ***John Forbes Nash, Jr.***

Historical Note: Numbers are like blood cells in the body of mathematics. Just as the understanding of anatomy and physiology of an organic system depends much on the knowledge of blood cells, so does the understanding of mathematics depend on the knowledge of numbers. In fact, a major part of mathematics bases its development on numbers and their multifarious properties.



It is very difficult, if not impossible, to spell out as to when did the concept of numbers came to human civilization. History, however, reveals that a formal study of numbers started almost five thousand years ago and that too by the Hindus who studied numbers purely as abstract symbols and were very proficient not only in discovering very large and very small numbers but also in using them effectively. Evidence are there that the Greek studied numbers purely on geometric conceptualization as they were very proficient in geometry and as a result had a relatively retarded progress. The greatest contribution of the Hindus is the discovery of zero, negative numbers and the decimal scale of representing numbers. In fact, they showed commendable mastery over rational numbers as early as the 5th century after Christ. The formal rigorous study of numbers, however, began even much later when mathematics faced several foundational crises. All these started in the 17th century but reached a climax after George Cantor (1845-1925) in 18th and 19th century. The contribution of 20th century in this regard is, on the one hand, stunning remarkable but on the other hand, devastating from the foundation point of view. The work and criticism by Russell (1872-1970), Lowenheim (1887-1940), Skolem (1887-1963) and Church (1903-1995) have been instrumental in bringing about a drastic change in our attitude and approach towards mathematics in general. In our modern approach, we start directly from real numbers defined axiomatically and then pass on to the related concept. (for more details see [4]). Many authors have different approach to define set of real numbers. Here we use the idea of Rudin introduced in [1].

❖ Preliminaries

In this section, we give some basic definitions and facts. These will help to learn and understand our main topic.

Definition: The set $\{1,2,3,\ldots\}$, which is usually denoted by $\mathbb{N}$ is called set of natural numbers.

Definition: The set $\{\ldots,-2,-1,0,1,2,\ldots\}$, which is usually denoted by $\mathbb{Z}$ is called set of integers.

❖ **Remarks:**

a. A set $\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$ can also be written as $\mathbb{Z}=\{0,\pm1,\pm2,\pm3,\ldots\}$.

b. A set of positive integers is denoted by $\mathbb{Z}^{+}=\{1,2,3,\ldots\}$ and set of negative integers is denoted by $\mathbb{Z}^{-}=\{-1,-2,-3,\ldots\}$

c. $\mathbb{Z}=\mathbb{Z}^{-}\cup\{0\}\cup\mathbb{Z}^{+}$, that is, a number 0 is neither positive nor negative.

Definition: Given two integers $a,b\in\mathbb{Z}$, $a\neq 0$, we say a divides b if there exists some integer q such that $b=a\cdot q$.

Notation: If a divides b, then we write $a\,|\,b$ and if a doesn't divides b, then we write $a\nmid b$.

Examples: (i) 2 divides 6, i.e. $2\,|\,6$ because if $a=2$ and $b=6$, then $q=3$.

(ii) -2 divides 6, i.e. $-2\,|\,6$ because if $a=-2$ and $b=6$, then $q=-3$.

(iii) -1, 1, -*a* and *a* divide every integer *a*.

(iv) Every non-zero integer divides 0.

Definition: An integer is called even if it is divisible by 2, otherwise it is called odd.

Note: A set $E:=\{0,\pm2,\pm4,\ldots\}$ represents set of all even integers and a set of odd integers is represented as $O:=\{\pm1,\pm3,\pm5,\ldots\}$.

Definition: A positive integer *p* is called prime if it has exactly four divisors (or two positive divisors).

Examples: 2, 3, 11, 29 are prime numbers.

Definition: A set $\left\{\frac{p}{q}\mid p,q\in\mathbb{Z}\wedge q\neq 0\right\}$ is called set of rational numbers and it is usually denoted by $\mathbb{Q}$.

❖ **Remarks:**

a. All the integers are rational number but there are numbers which are rational but not integer.

b. One rational number can be written as infinitely many ways e.g. $\dfrac{1}{3}$ can be written as $0.333\ldots$ or $\dfrac{2}{6}$ or $\dfrac{-4}{-12}$.

c. Between any two rational numbers there exist a rational number, that is, there are infinity many rational between any two rational numbers.

d. There are operations of addition (+) and multiplication ($\cdot$) on $\mathbb{N},\mathbb{Z}$ and $\mathbb{Q}$, which has nice properties.

e. The set of integers is exclusively the point of interest in Number Theory.

❖ **Preparation to Define Set of Real Numbers**

It is not easy to define set of real numbers as we define $\mathbb{N},\mathbb{Z}$ or $\mathbb{Q}$. The real number system can be described as a "complete ordered field". Therefore, let's discusses and understand these notions first.

❖ **Order or Ordered Set**

Definition: Let *S* be a non-empty set. An *order* on a set *S* is a relation denoted by "$<$" with the following two properties

(*i*) If $x,y\in S$, then one and only one of the statements

$x<y$, $x=y$, $y<x$ is true.

(ii) If $x, y, z \in S$ and if $x < y$, $y < z$ then $x < z$.

Examples: Consider the following sets:

- $A = \{1, 2, 3, ..., 50\}$
- $B = \{a, e, i, o, u\}$
- $C = \{x : x \in \mathbb{Z} \wedge x^2 \leq 19\}$

There is an order on *A* and *C* but there is no order on *B* (we can define order on *B*).

Definition: A non-empty set *S* is said to be *ordered set* if an order is defined on *S*.

Examples: (i) The set $\{2, 4, 6, 7, 8, 9\}$, $\mathbb{Z}$ and $\mathbb{Q}$ are examples of ordered set with standard order relation.

(ii) The set $\{a, b, c, d\}$ and $\{\alpha, \beta, \chi, \vartheta\}$ are examples of set with no order.

❖ Bounded & Unbounded Set

Definition: Let *S* be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, then we say that *E* is *bounded above*. The number β is known as *upper bound* of *E*.

Definition: Let *S* be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \geq \beta$ for all $x \in E$, then we say that *E* is *bounded below*. The number β is known as *lower bound* of *E*.

Definition: Let *S* be an ordered set and $E \subset S$. A set *E* is said to be bounded if it has both upper and lower bounds. Otherwise it is said to be an unbounded.

Examples: (i) Consider $S = \{1, 2, 3, ..., 50\}$ and $E = \{5, 10, 15, 20\}$.

Set of all lower bounds of $E = \{1, 2, 3, 4, 5\}$.

Set of all upper bounds of $E = \{20, 21, 22, ..., 50\}$.

(ii) Consider $S = \mathbb{N}$, $E = \{1, 2, 3, ..., 100\}$ and $F = \{10, 20, 30, ...\}$.

Set of lower bounds of $E = \{1\}$.

Set of lower bounds of $F = \{1, 2, 3, ..., 10\}$.

Set of upper bounds of $E = \{100, 101, 102, ...\}$.

Set of upper bounds of $F = \varphi$.

❖ Least Upper Bound (Supremum) and Greatest Lower Bound (Infimum)

Definition: Suppose *S* is an ordered set, $E \subset S$ and *E* is bounded above. Suppose there exists an $\alpha \in S$ such that

(i) α is an upper bound of *E*.

(ii) If $\gamma < \alpha$ for $\gamma \in S$, then γ is not an upper bound of *E*.

Then α is called *least upper bound* of *E* or *supremum* of *E* and written as $\sup E = \alpha$.

Example: Consider $S = \{1, 2, 3, ..., 50\}$ and $E = \{5, 10, 15, 20\}$.

(i) It is clear that 20 is upper bound of E.

(ii) For $\gamma \in S$ if $\gamma < 20$ then clearly γ is not an upper bound of E. Hence $\sup E = 20$.

Definition: Suppose *S* is an ordered set, $E \subset S$ and *E* is bounded below. Suppose there exists a $\beta \in S$ such that

(i) β is a lower bound of *E*.

(*ii*) If $\beta < \gamma$ for $\gamma \in S$, then γ is not a lower bound of *E*.

Then β is called *greatest lower bound* or *infimum* of *E* and written as $\inf E = \beta$.

Example: Consider $S = \{1,2,3,...,50\}$ and $E = \{5,10,15,20\}$.

(i) It is clear that 5 is lower bound of E.

(ii) For $\gamma \in S$ if $5 < \gamma$, then clearly γ is not lower bound of E. Hence $\inf E = 5$.

❖ Remarks

- A set is unbounded if either its set of upper bounds or set of lower bounds is empty.
- Supremum is the least member of the set of upper bound of the given set.
- Infimum is the greatest member of the set of lower bound of the given set.
- If α is supremum or infimum of *E,* then α may or may not belong to *E*.
  - Let $E_1 = \{r : r \in \mathbb{Q} \ \wedge \ r < 0\}$ and $E_2 = \{r : r \in \mathbb{Q} \ \wedge \ r \geq 0\}$. Then $\sup E_1 = \inf E_2 = 0$ but $0 \notin E_1$ and $0 \in E_2$.
  - Let $E \subset \mathbb{Q}$ be the set of all numbers of the form $1/n$, where *n* is the natural numbers, that is,

$$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}.$$

Then $\sup E = 1$ which is in *E*, but $\inf E = 0$ which is not in *E*.

❖ Least Upper Bound Property and Greatest Lower Bound Property

Definition: A set *S* is said to have the *least upper bound property* if the followings is true

(*i*) *S* is non-empty and ordered.

(*ii*) If $E \subset S$ and *E* is non-empty and bounded above then sup*E* exists in *S*.

Definition: A set *S* is said to have the *greatest lower bound property* if the followings is true

(*i*) *S* is non-empty and ordered.

(*ii*) If $E \subset S$ and *E* is non-empty and bounded below then inf*E* exists in *S*.

Examples: (i) The sets $\mathbb{N}$ and $\mathbb{Z}$ satisfies least upper bound property.

(ii) The set of rational numbers $\mathbb{Q}$ doesn't satisfy completeness axiom. Consider a set $E = \{x : x \in \mathbb{Q} \wedge x^2 \leq 2\}$. One can prove that supremum of E doesn't exist in $\mathbb{Q}$ even *E* is a bounded set.

If *U* and *L* denotes the set of upper and lower bounds of E respectively, then

$U = \{x : x \in \mathbb{Q} \wedge x^2 \geq 2 \wedge x > 0\}$ and $L = \{x : x \in \mathbb{Q} \wedge x^2 \geq 2 \wedge x < 0\}$.

If r is the supremum of E, then clearly $r^2 = 2$.

Here, we prove there is no rational p such that $p^2 = 2$.

Let us suppose that there exists a rational *p* such that $p^2 = 2$.

This implies we can write

$$p = \frac{m}{n} \qquad \text{where } m, n \in \mathbb{Z},\ n \neq 0 \ \& \ m, n \text{ have no common factor.}$$

Then $p^2 = 2 \quad \Rightarrow \quad \dfrac{m^2}{n^2} = 2 \quad \Rightarrow \quad m^2 = 2n^2$

$\Rightarrow$ m^2 is even $\Rightarrow$ m is even

$\Rightarrow$ m is divisible by 2 and so m^2 is divisible by 4.

$\Rightarrow$ $2n^2$ is divisible by 4 and so n^2 is divisible by 2. $\because$ $m^2 = 2n^2$.

i.e. n^2 is even $\Rightarrow$ n is an even

$\Rightarrow$ *m* and *n* both have common factor 2.

which is contradiction because *m* and *n* have no common factor.

Hence $p^2 = 2$ is impossible for rational *p*.

Finally, we conclude that the set *E*, which is bounded in $\mathbb{Q}$ doesn't have supremun and infimum in $\mathbb{Q}$, hence set of rational $\mathbb{Q}$ doesn't satisfy the least upper bound property.

❖ Remark

The above property is known as *completeness axiom* or *LUB axiom* or *continuity axiom* or *order completeness axiom*.

❖ Theorem

Suppose *S* is an ordered set with least upper bound property, $B \subset S$, *B* is non-empty and is bounded below. Let *L* be set of all lower bound of *B*. Then

$$\alpha := \sup L$$

exists in *S* and $\alpha = \inf B$.

Proof

Since *B* is bounded below therefore *L* is non-empty.

Since *L* consists of exactly those $y \in S$ which satisfy the inequality.

$$y \leq x \qquad \forall\ x \in B.$$

We see that every $x \in B$ is an upper bound of *L*.

This implies *L* is bounded above.

Since *S* is ordered and non-empty with least upper bound property therefore *L* has a supremum in *S*, that is, $\alpha := \sup L$ exists in *S*.

If $\gamma < \alpha$, then (by definition of supremum) γ is not upper bound of *L*.

$\Rightarrow$ $\gamma \notin B$.

It follows that $\alpha \leq x \ \ \forall\ x \in B$.

Thus α is lower bound of B.

Now if $\alpha < \beta$, then $\beta \notin L$ because $\alpha = \sup L$, that is, β is not lower bound of B.

this means (by definition of infumum) $\alpha = \inf B$.

❖ Remark

Above theorem can be stated as follows:

An ordered set which has the least upper bound property has also the greatest lower bound property.

❖ Field

A set *F* with two operations called addition and multiplication satisfying the following axioms is known to be field.

Axioms for Addition:

(*i*) If $x, y \in F$ then $x + y \in F$. *Closure Law* (*ii*) $x+y=y+x, \quad \forall\ x,y\in F$. *Commutative Law*

(*iii*) $x+(y+z)=(x+y)+z \quad \forall\ x,y,z\in F$. *Associative Law*

(*iv*) For any $x\in F$, $\exists\ 0\in F$ such that $x+0=0+x=x$ *Additive Identity*

(*v*) For any $x\in F$, $\exists\ -x\in F$ such that $x+(-x)=(-x)+x=0$ *+tive Inverse*

Axioms for Multiplication:

(*i*) If $x,\ y\in F$ then $x\,y\in F$. *Closure Law*

(*ii*) $x\,y=y\,x, \quad \forall\ x,y\in F$ *Commutative Law*

(*iii*) $x(y\,z)=(x\,y)z \quad \forall\ x,y,z\in F$

(*iv*) For any $x\in F$, $\exists\ 1\in F$ such that $x\cdot 1=1\cdot x=x$ *Multiplicative Identity*

(*v*) For any $x\in F$, $x\neq 0$, $\exists\ \frac{1}{x}\in F$, such that $x\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)x=1$ *×tive Inverse.*

Distributive Law

For any $x,y,z\in F$, (*i*) $x(y+z)=xy+xz$

(*ii*) $(x+y)z=xz+yz$

❖ Existence of Real Field

It is worth mentioning that $\mathbb{N}$ and $\mathbb{Z}$ are completely order sets but not a field. While $\mathbb{Q}$ is ordered field but not satisfy completeness axiom. What about a set which satisfy all three properties, that is, *i*. ordered *ii*. field and *iii*. satisfy completeness axiom. Amazingly, $\mathbb{R}$ (set of real numbers) is the only set which satisfy all these properties.

❖ Theorem:

There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover $\mathbb{R}$ contains $\mathbb{Q}$ (set of rational numbers) as a subfield.

Proof

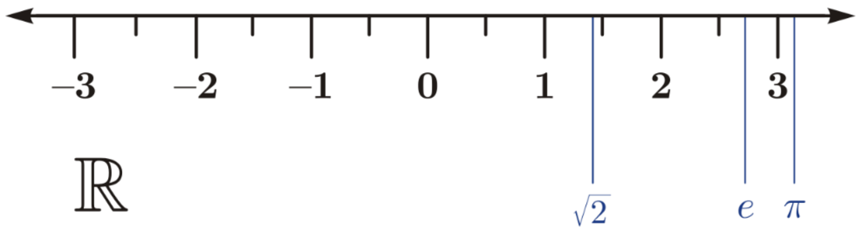
The proof of the theorem is rather long and a bit tedious. So, we are skipping the proof, one can see it at [1, Page 17]. □

Definition: The members of $\mathbb{R}$ are called *real numbers*.

Definition: Real numbers which are not rational are called *irrational* numbers.

❖ Explanation about $\mathbb{R}$

The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as $\sqrt{2}$ (1.41421356…, the square root of two, an irrational algebraic number) and π (3.14159265…, a transcendental number). Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one. Or a real number is a value that represents any quantity along a number line. Because



they lie on a number line, their size can be compared. You can say one is greater or less than another and do arithmetic with them.
By using the fact that $\mathbb{R}$, the set of real numbers, is a completely order field, one can prove the following theorem.

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then axioms for addition imply the following.

(***a***) If $x+y=x+z$ then $y=z$

(***b***) If $x+y=x$ then $y=0$

(***c***) If $x+y=0$ then $y=-x$.

(***d***) $-(-x)=x$

Proof

(Note: *We have given the proofs here just to show that the things which looks simple must have valid analytical proofs under some consistence theory of mathematics*)

(*a*) Suppose $x+y=x+z$.

$$\begin{aligned}
\text{Since}\quad y &= 0+y \\
&= (-x+x)+y && \text{since } -x+x=0. \\
&= -x+(x+y) && \text{by associative law.} \\
&= -x+(x+z) && \text{by supposition.} \\
&= (-x+x)+z && \text{by associative law.} \\
&= 0+z && \text{since } -x+x=0. \\
&= z
\end{aligned}$$

(*b*) Take $z=0$ in (*a*)

$$x+y=x+0 \quad \Rightarrow \quad y=0$$

(*c*) Take $z=-x$ in (*a*)

$$x+y=x+(-x) \quad \Rightarrow \quad y=-x$$

(*d*) Since $(-x)+x=0$,

then (*c*) gives $x=-(-x)$. □

We are skipping the proofs of following three theorems as these may be the part of the mathematics of FSc.

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then axioms of multiplication imply the following.

(***a***) If $x \neq 0$ and $xy=xz$ then $y=z$.

(***b***) If $x \neq 0$ and $xy=x$ then $y=1$.

(***c***) If $x \neq 0$ and $xy=1$ then $y=\dfrac{1}{x}$.

(***d***) If $x \neq 0$, then $\dfrac{1}{1/x}=x$.

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then field axioms imply the following.

(***i***) $0 \cdot x = 0$. (***ii***) if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

(***iii***) $(-x)y = -(xy) = x(-y)$. (***iv***) $(-x)(-y) = xy$.

❖ Theorem

Let $x, y, z \in \mathbb{R}$. Then the following statements are true:

i) If $x > 0$ then $-x < 0$ and vice versa.

ii) If $x > 0$ and $y < z$ then $xy < xz$.

iii) If $x < 0$ and $y < z$ then $xy > xz$.

iv) If $x \neq 0$ then $x^2 > 0$ in particular $1 > 0$.

v) If $0 < x < y$ then $0 < \dfrac{1}{y} < \dfrac{1}{x}$.

❖ Theorem (Archimedean Property)

If $x, y \in \mathbb{R}$ and $x > 0$ then there exists a positive integer *n* such that $nx > y$.

Proof

Let $A = \left\{ nx : n \in \mathbb{Z}^+ \wedge x > 0, x \in \mathbb{R} \right\}$

Suppose the given statement is false i.e. $nx \leq y$.

This implies *y* is an upper bound of *A*, that is, *A* is bounded above.

Since we are dealing with a set of real and it satisfies the least upper bound property,

Therefore supremum of *A* exists in $\mathbb{R}$.

Assume that $\alpha = \sup A$.

As $x > 0$ so we have $\alpha - x < \alpha$.

This gives $\alpha - x$ is not an upper bound of *A*.

Hence $\alpha - x < mx$, where $mx \in A$ for some positive integer *m*.

So, we have $\alpha < (m+1)x$, where $m + 1$ is integer.

This implies $(m+1)x \in A$.

This is impossible because α is least upper bound of A i.e. $\alpha = \sup A$.

Hence, we conclude that our supposition is wrong and the given statement is true. □

❖ Theorem

The set $\mathbb{N}$ of natural numbers is not bounded above.

Proof.

By Archimedean property in real number, for each positive real numbers x, there exist $n \in \mathbb{N}$ such that $n \cdot 1 > x$, that is, $n > x$.

This implies, there is no positive real number x such that $n \leq x$ for all $n \in \mathbb{N}$.

This implies no real number is an upper bound of $\mathbb{N}$.

Hence $\mathbb{N}$ is not bounded above. □

❖ The Density Theorem

If $x, y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{Q}$ such that $x < p < y$.

i.e., between any two real numbers there is a rational number *or* $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof

Let us assume that $x, y \in \mathbb{R}$ with $x < y$. Then $y - x > 0$.

By Archimedean property, for $y - x, 1 \in \mathbb{R}$, $y - x > 0$, there exists positive integer n such that

$$n(y - x) > 1,$$

$$\Rightarrow \; 1 + nx < ny \; \ldots\ldots\ldots\ldots\ldots (i)$$

Again, we use Archimedean property, for $1, nx \in \mathbb{R}$ and $1, -nx \in \mathbb{R}$, $1 > 0$, to obtain two positive integers m_1 and m_2 such that

$$m_1 \cdot 1 > nx \quad \text{and} \quad m_2 \cdot 1 > -nx,$$

that is,

$$nx < m_1 \quad \text{and} \quad -m_2 < nx,$$

$$\Rightarrow \; -m_2 < nx < m_1.$$

Then there is an integer $m(\text{with } -m_2 \le m \le m_1)$ such that

$$m - 1 \le nx < m,$$

$$\Rightarrow \; nx < m \quad \text{and} \quad m \le 1 + nx,$$

$$\Rightarrow \; nx < m < 1 + nx.$$

Using (i) in the above inequality, we get

$$nx < m < ny$$

Since $n > 0$, it follows that

$$x < \frac{m}{n} < y$$

$$\Rightarrow \; x < p < y, \quad \text{where} \quad p = \frac{m}{n} \quad \text{is a rational.}$$

This completes the proof. □

❖ **Relatively Prime**

Definition: For $a, b \in \mathbb{Z}$, the numbers a and b are said to be *relatively prime* or *co-prime* if a and b don't have common factor other than 1. If a and b are relatively prime, then we write $(a, b) = 1$.

❖ **Theorem**

(i) If r is rational and x is irrational, then $r + x$ is irrational.

(ii) If r is non-zero rational and x is irrational, then $r\,x$ is irrational.

Proof

(i) Suppose the contrary that $r + x$ is rational. Then

$$r + x = \frac{a}{b}, \qquad \text{where } a, b \in \mathbb{Z},\ b \ne 0 \text{ such that } (a, b) = 1,$$

$$\Rightarrow \; x = \frac{a}{b} - r. \quad \ldots\ldots. (1)$$

Since r is rational, there exists $c, d \in \mathbb{Z}$, $d \ne 0$ and $(c, d) = 1$ such that

$$r = \frac{c}{d}.$$

Using it in (1) to get

$$x=\frac{a}{b}-\frac{c}{d} \quad \Rightarrow x=\frac{ad-bc}{bd}, \text{ where } bd\neq 0.$$

As $ad-bc, bd \in \mathbb{Z}$, we get x is rational.

This cannot happen because x is given to be irrational, hence we conclude that $r+x$ is irrational. □

(ii) Let us suppose the contrary that $r\,x$ is rational. Then

$$r\,x=\frac{a}{b} \qquad \text{for some } a,b\in\mathbb{Z},\ b\neq 0 \text{ such that } (a,b)=1.$$

$$\Rightarrow\ x=\frac{a}{b}\cdot\frac{1}{r} \quad \ldots\ldots (2)$$

Since r is non-zero rational, there exists $c,d\in\mathbb{Z},\ c,d\neq 0$ and $(c,d)=1$ such that

$$r=\frac{c}{d}.$$

Using it in (2) to get

$$x=\frac{a}{b}\cdot\frac{1}{c/d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}, \text{ where } bc\neq 0.$$

This shows that x is rational, which is again contradiction; hence we conclude that $r\,x$ is irrational. □

❖ **Theorem**

Given two real numbers x and y, $x<y$ there is an irrational number u such that $x<u<y$.

Proof

We have given $x<y$, therefore $\frac{x}{\sqrt{2}}<\frac{y}{\sqrt{2}}$.

By density theorem, for real numbers $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, we can obtain a rational number $r\neq 0$ such that

$$\frac{x}{\sqrt{2}}<r<\frac{y}{\sqrt{2}}$$

$$\Rightarrow\ x<r\sqrt{2}<y$$

$$\Rightarrow\ x<u<y,$$

where $u=r\sqrt{2}$ is an irrational as product of non-zero rational and irrational is irrational. □

❖ **Theorem**

For every real number x there is a set E of rational number such that $x=\sup E$.

Proof

Take $E=\{q\in\mathbb{Q}: q<x\}$, where x is a real.

Then E is bounded above. Since $E \subset \mathbb{R}$ therefore supremum of E exists in $\mathbb{R}$.
Suppose $\sup E = \lambda$.
It is clear that $\lambda \leq x$.
If $\lambda = x$ then there is nothing to prove.
If $\lambda < x$ then $\exists\ q \in \mathbb{Q}$ such that $\lambda < q < x$,
which cannot happen as λ is the upper bound of E.
Hence, we conclude that real x is sup$E$. □

❖ Question

Let E be a non-empty subset of an ordered set, suppose α is a lower bound of E and β is an upper bound then prove that $\alpha \leq \beta$.

Proof

Since E is a subset of an ordered set S i.e. $E \subseteq S$.
Also α is a lower bound of E therefore by definition of lower bound

$$\alpha \leq x \quad \forall \quad x \in E \quad \ldots\ldots\ldots\ldots\ldots (i)$$

Since β is an upper bound of E therefore by the definition of upper bound

$$x \leq \beta \quad \forall \quad x \in E \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Combining (i) and (ii)

$$\alpha \leq x \leq \beta \quad \Rightarrow \quad \alpha \leq \beta \quad \text{as required.}$$ □

❖ Question

Show that for any two real numbers a and b.

(i) $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$ (ii) $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$.

Note: Above question is proposed to know the difference between supremum & maximum.

❖ The Extended Real Numbers

Definition: The extended real number system consists of real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty \quad \forall\ x \in \mathbb{R}.$$

❖ Remarks

It is clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. If, for example, E is a nonempty set of real numbers which is not bounded above in $\mathbb{R}$, then sup$E = +\infty$ in the extended real number system.

The same observations apply to lower bounds.

❖ Extension of Operation in Extended Real Numbers

The extended real number system does not form a field. But it is customary to make the following conventions:

a) If x is real, then

$$x + \infty = +\infty, \qquad x - \infty = -\infty, \qquad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

b) If $x > 0$ then $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$.

c) If $x < 0$ then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$.

Note: (*i*) Mostly we write $+\infty = \infty$.

(*ii*) The above operations hold in extended real number system not in $\mathbb{R}$.

❖ Euclidean Space

Definitions: For each positive integer *k*, let $\mathbb{R}^k$ be the set of all ordered *k*-tuples

$$\underline{x} = (x_1, x_2, ..., x_k)$$

where $x_1, x_2, ..., x_k$ are real numbers, called the *coordinates* of $\underline{x}$.

The elements of $\mathbb{R}^k$ are called *points* or *vectors*, especially when $k > 1$.

If $\underline{y} = (y_1, y_2, ..., y_n)$ and α is a real number, we define

$$\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, ..., x_k + y_k)$$

and $\alpha \underline{x} = (\alpha x_1, \alpha x_2, ..., \alpha x_k)$.

Observation: It is clear that $\underline{x} + \underline{y} \in \mathbb{R}^k$ and $\alpha \underline{x} \in \mathbb{R}^k$. This defines addition of vectors, as well as multiplication of a vector by a real number (a scalar). These two operations satisfy the commutative, associative, and distributive laws and make $\mathbb{R}^k$ into a vector space over the real field. The *zero element* of $\mathbb{R}^k$ (sometimes called the *origin* or the *null vector*) is the point $\underline{0}$ (or we simply write 0), all of whose coordinates are 0. These operations make $\mathbb{R}^k$ into a *vector space over the real field.*

Definitions: The *inner product* or *scalar product* of $\underline{x}$ and $\underline{y}$ from $\mathbb{R}^k$ is defined as

$$\underline{x} . \underline{y} = \sum_{i=1}^{k} x_i y_i = x_1 y_1 + x_2 y_2 + ... + x_k y_k$$

and the norm of $\underline{x}$ is defined by

$$\| \underline{x} \| = (x \cdot x)^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}.$$

Definition: The vector space $\mathbb{R}^k$ with the above inner product and norm is called *Euclidean k-space* or *Euclidean space*.

❖ Theorem

Let $\underline{x}, \underline{y} \in \mathbb{R}^k$ then

i) $\| \underline{x} \|^2 = \underline{x} \cdot \underline{x}$,

ii) $| \underline{x} \cdot \underline{y} | \leq \| \underline{x} \| \| \underline{y} \|$. (*Cauchy-Schwarz's inequality*)

Proof

i) Since $\| \underline{x} \| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}}$ therefore $\| \underline{x} \|^2 = \underline{x} \cdot \underline{x}$

ii) If $\underline{x}=0$ or $\underline{y}=0$, then Cauchy-Schwarz's inequality holds with equality.

If $\underline{x}\neq 0$ and $\underline{y}\neq 0$, then for $\lambda\in\mathbb{R}$, we have

$$0\leq\left\|\underline{x}-\lambda\underline{y}\right\|^2 =\left(\underline{x}-\lambda\underline{y}\right)\cdot\left(\underline{x}-\lambda\underline{y}\right)$$
$$=\underline{x}\cdot\left(\underline{x}-\lambda\underline{y}\right)+\left(-\lambda\underline{y}\right)\cdot\left(\underline{x}-\lambda\underline{y}\right)$$
$$=\underline{x}\cdot\underline{x}+\underline{x}\cdot(-\lambda\underline{y})+(-\lambda\underline{y})\cdot\underline{x}+(-\lambda\underline{y})\cdot(-\lambda\underline{y})$$
$$=\left\|\underline{x}\right\|^2-2\lambda(\underline{x}\cdot\underline{y})+\lambda^2\left\|\underline{y}\right\|^2$$

Now put $\lambda=\dfrac{\underline{x}\cdot\underline{y}}{\left\|\underline{y}\right\|^2}$ (*certain real number*)

$$\Rightarrow\ 0\leq\left\|\underline{x}\right\|^2-2\frac{\left(\underline{x}\cdot\underline{y}\right)\left(\underline{x}\cdot\underline{y}\right)}{\left\|\underline{y}\right\|^2}+\frac{\left(\underline{x}\cdot\underline{y}\right)^2}{\left\|\underline{y}\right\|^4}\left\|\underline{y}\right\|^2 \quad \Rightarrow\ 0\leq\left\|\underline{x}\right\|^2-\frac{\left(\underline{x}\cdot\underline{y}\right)^2}{\left\|\underline{y}\right\|^2}$$

$$\Rightarrow\ 0\leq\left\|\underline{x}\right\|^2\left\|\underline{y}\right\|^2-\left|\underline{x}\cdot\underline{y}\right|^2 \qquad \because\ a^2=|a|^2\ \forall\ a\in\mathbb{R},$$

$$\Rightarrow\ 0\leq\left(\left\|\underline{x}\right\|\left\|\underline{y}\right\|+\left|\underline{x}\cdot\underline{y}\right|\right)\left(\left\|\underline{x}\right\|\left\|\underline{y}\right\|-\left|\underline{x}\cdot\underline{y}\right|\right).$$

This holds if and only if

$$0\leq\left\|\underline{x}\right\|\left\|\underline{y}\right\|-\left|\underline{x}\cdot\underline{y}\right| \quad \text{i.e., } \left|\underline{x}\cdot\underline{y}\right|\leq\left\|\underline{x}\right\|\left\|\underline{y}\right\|.$$ □

❖ **Question**

Suppose $\underline{x},\underline{y},\underline{z}\in\mathbb{R}^k$, then prove that

a) $\left\|\underline{x}+\underline{y}\right\|\leq\left\|\underline{x}\right\|+\left\|\underline{y}\right\|$.

b) $\left\|\underline{x}-\underline{z}\right\|\leq\left\|\underline{x}-\underline{y}\right\|+\left\|\underline{y}-\underline{z}\right\|$.

Solution

a) Consider $\left\|\underline{x}+\underline{y}\right\|^2=\left(\underline{x}+\underline{y}\right)\cdot\left(\underline{x}+\underline{y}\right)$

$$=\underline{x}\cdot\underline{x}+\underline{x}\cdot\underline{y}+\underline{y}\cdot\underline{x}+\underline{y}\cdot\underline{y} =\left\|\underline{x}\right\|^2+2\left(\underline{x}\cdot\underline{y}\right)+\left\|\underline{y}\right\|^2$$
$$\leq\left\|\underline{x}\right\|^2+2\left|\underline{x}\cdot\underline{y}\right|+\left\|\underline{y}\right\|^2 \qquad \because\ |a|\geq a\ \forall\ a\in\mathbb{R}.$$
$$\leq\left\|\underline{x}\right\|^2+2\left\|\underline{x}\right\|\left\|\underline{y}\right\|+\left\|\underline{y}\right\|^2 \qquad \because\ \left\|\underline{x}\right\|\left\|\underline{y}\right\|\geq\left|\underline{x}\cdot\underline{y}\right|$$
$$=\left(\left\|\underline{x}\right\|+\left\|\underline{y}\right\|\right)^2$$
$$\Rightarrow\ \left\|\underline{x}+\underline{y}\right\|\leq\left\|\underline{x}\right\|+\left\|\underline{y}\right\| \quad \ldots\ldots\ldots\ldots (i)$$

b) We have $\left\|\underline{x}-\underline{z}\right\|=\left\|\underline{x}-\underline{y}+\underline{y}-\underline{z}\right\|$

$$\leq\left\|\underline{x}-\underline{y}\right\|+\left\|\underline{y}-\underline{z}\right\| \qquad \text{from } (i)$$ □

Chapter 02

Sequences

*A sequence is a story told by numbers, each taking its turn in a pre-determined order. --- **Prof. Anya Sharma, Analyst***

Sequences form an important component of Mathematical Analysis and arise in many situations. The first rigorous treatment of sequences was made by A. Cauchy (1789-1857) and George Cantor (1845-1918). A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence (some authors considered it finite sequence); a sequence must continue without interruption. Formally it is defined as follows:

Sequence

A function whose domain is the set of natural numbers and range is a subset of real numbers is called *real sequence*.

Since in this chapter, we shall be concerned with *real sequences* only, we shall refer to them as just *sequences*.

Notation:

A sequence is usually denoted as

$$\{s_n\}_{n=1}^{\infty} \text{ or } \{s_n : n \in \mathbb{N}\} \text{ or } \{s_1, s_2, s_3, \ldots\} \text{ or simply as } \{s_n\} \text{ or by } (s_n).$$

But it is not limited to above notations only.

The values s_n are called the *terms* or the *elements* of the sequence $\{s_n\}$.

e.g. i) $\{n\} = \{1, 2, 3, \ldots\}$.

ii) $\left\{\dfrac{1}{n}\right\} = \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\right\}$.

iii) $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \ldots\}$.

iv) $\{2, 3, 5, 7, 11, \ldots\}$, a sequence of positive prime numbers.

v) $\{s_n\}$ such that $s_1 = 1$, $s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n$.

Range of a sequence

The set of all distinct terms of a sequence is called its range.

Remark:

In a sequence $\{s_n\}$, since $n \in \mathbb{N}$ and $\mathbb{N}$ is an infinite set, the number of the terms of a sequence is always infinite. However, the range of the sequence may be finite.

Subsequence

It is a sequence whose terms are contained in given sequence.

A subsequence of $\{s_n\}$ is usually written as $\{s_{n_k}\}$.

Examples:

1. $\{2,4,6,...\}$ is subsequence of $\{1,2,3,...\}$
2. $\left\{\dfrac{1}{2n}\right\}$ and $\left\{\dfrac{1}{n+1}\right\}$ is subsequence of $\left\{\dfrac{1}{n}\right\}$.

Increasing sequence

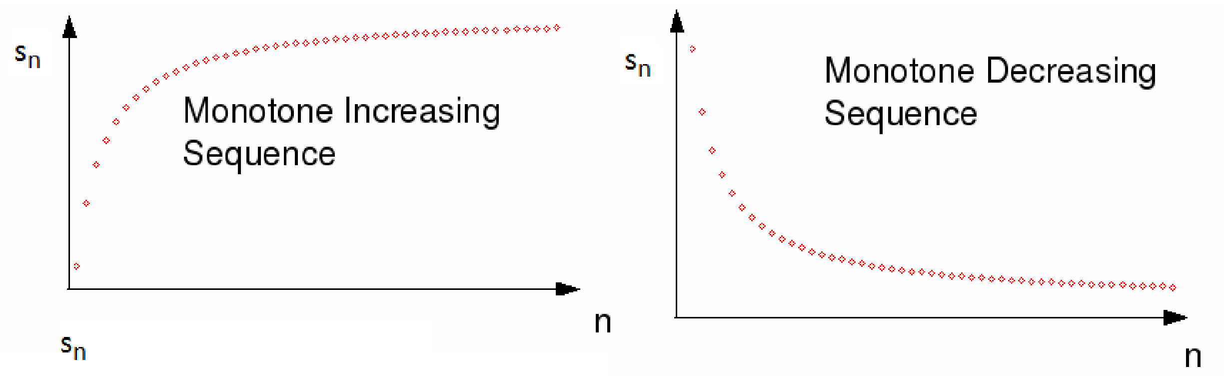
A sequence $\{s_n\}$ is said to be an increasing sequence if $s_{n+1} \geq s_n \quad \forall \; n \geq 1$.

Decreasing sequence

A sequence $\{s_n\}$ is said to be a decreasing sequence if $s_{n+1} \leq s_n \quad \forall \; n \geq 1$.

Monotonic sequence

A sequence $\{s_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.



Remarks:

- A sequence $\{s_n\}$ is monotonically increasing if $s_{n+1} - s_n \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically increasing if $\dfrac{s_{n+1}}{s_n} \geq 1, \; \forall \; n \geq 1$.
- A sequence $\{s_n\}$ is monotonically decreasing if $s_n - s_{n+1} \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically decreasing if $\dfrac{s_n}{s_{n+1}} \geq 1$, $\forall \; n \geq 1$.

Strictly Increasing or Decreasing

A sequence $\{s_n\}$ is called strictly increasing or decreasing according as

$$s_{n+1} > s_n \quad \text{or} \quad s_{n+1} < s_n \quad \forall \; n \geq 1.$$

Examples:

- $\{n\} = \{1,2,3,...\}$ is an increasing sequence (also it is strictly increasing).
- $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence. (also it is strictly decreasing).
- $\{1,1,2,2,3,3,...\}$ is increasing sequence but it is not strictly increasing.
- $\{\cos n\pi\} = \{-1,1,-1,1,...\}$ is neither increasing nor decreasing.

Questions:

1) Prove that $\left\{1+\frac{1}{n}\right\}$ is a decreasing sequence.

2) Is $\left\{\frac{n+1}{n+2}\right\}$ is increasing or decreasing sequence?

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there is a positive number λ such that

$$|s_n| \leq \lambda \quad \forall \; n \in \mathbb{N}.$$

For such a sequence, every term belongs to the interval $[-\lambda, \lambda]$. Also inequality in the above definition can be replaced with strict inequality. Alternatively, a sequence is bounded if its range is a bounded set.

It can be noted that if the sequence is bounded then its supremum and infimum exist. If S and s are the supremum and infimum of the bounded sequence $\{s_n\}$, then we write $S = \sup s_n$ and $s = \inf s_n$.

Remarks:

It is easy to conclude that if $\{s_n\}$ is bounded sequence and n_0 is positive integer then there exists $\lambda > 0$ such that

$$|s_n| \leq \lambda \text{ whenever } n \geq n_0.$$

Examples:

(i) $\{u_n\} = \left\{\frac{(-1)^n}{n}\right\}$ is a bounded sequence

(ii) $\{v_n\} = \{\sin n\}$ is also bounded sequence. Its supremum is 1 and infimum is -1.

(iii) The geometric sequence $\{ar^{n-1}\}$, $r > 1$ is an unbounded above sequence. It is bounded below by a.

(iv) $\{\exp(n)\}$ is an unbounded sequence.

Convergence of the sequence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

is getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L.

However, this is too vague and too weak to serve as definition but a rough guide for the intuition, this is misleading in other respects. What about the sequence

0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, 0.00001, 0.00002, ...?

Surely this should converge to 0 but the terms do not get steadily "closer and closer" but back off a bit at each second step.

The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of "arbitrarily close" using inequalities.

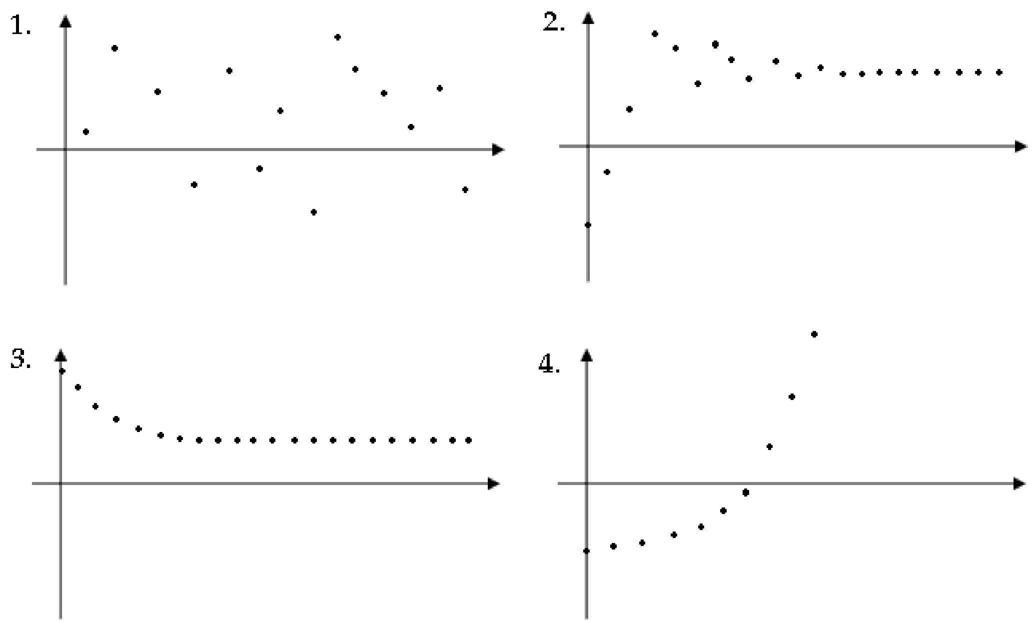
Definition

A sequence $\{s_n\}$ of real numbers is said to convergent to limit 's' as $n \to \infty$, if for every real number $\varepsilon > 0$, there exists a positive integer n_0, depending on ε, so that

$$|s_n - s| < \varepsilon \quad \text{whenever } n > n_0.$$

A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to *divergent* (it will be discussed later).

We will try to understand it by graph of some sequence. Graphs of any four sequences is drawn in the picture below.



Examples

a) Prove that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ (or $\left\{\dfrac{1}{n}\right\}$ converges to 0).

Solution: Let $\varepsilon > 0$ be given. By the Archimedean Property, there is a positive integer $n_0 = n_0(\varepsilon)$ such that $n_0 \cdot \varepsilon > 1$, that is, $\dfrac{1}{n_0} < \varepsilon$. Then, if $n > n_0$, we have

$$\frac{1}{n} < \frac{1}{n_0} < \varepsilon.$$

Thus we proved that for all $\varepsilon > 0$, there exists n_0, depending upon ε, such that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \quad \text{whenever } n > n_0.$$

Hence $\left\{\dfrac{1}{n}\right\}$ converges to point '0'.

b) Prove that $\lim\limits_{n\to\infty} \dfrac{1}{n^2+1} = 0$ (by definition).

Solution: Let $\varepsilon > 0$ be given. Now consider

$$\left|\frac{1}{n^2+1}-0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} < \frac{1}{n}. \quad (\text{Since } n^2+1 > n^2 > 0)$$

Now if we choose n_0 such that $\frac{1}{n_0} < \varepsilon$ (or $n_0 > \frac{1}{\varepsilon}$), then the above expression gives us

$$\left|\frac{1}{n^2+1}-0\right| < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon \quad \text{whenever} \quad n \geq n_0 > \frac{1}{\varepsilon}.$$

Hence, we conclude that, $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$.

c) Prove that $\lim_{n\to\infty}\frac{3n+2}{n+1} = 3$ (by definition).

Solution: Let $\varepsilon > 0$ be given. Now consider

$$\begin{aligned}\left|\frac{3n+2}{n+1}-3\right| &= \left|\frac{3n+2-3n-3}{n+1}\right| \\ &= \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} \qquad (\because\ n+1 > n > 0)\end{aligned}$$

Now if we take n_0 such that $\frac{1}{n_0} < \varepsilon$ (or $n_0 > \frac{1}{\varepsilon}$), then the above expression gives us

$$\left|\frac{3n+2}{n+1}-3\right| < \varepsilon \quad \text{whenever} \quad n \geq n_0.$$

Hence, we conclude that $\lim_{n\to\infty}\frac{3n+2}{n+1} = 3$.

Questions:

Use definition of the limits to prove the followings:

a) $\lim_{n\to\infty}\frac{2n}{n+1} = 2$. b) $\lim_{n\to\infty}\frac{n^2-1}{2n^2+3} = \frac{1}{2}$ c) $\lim_{n\to\infty}\frac{1}{3^n} = 0$

Definitions

i. A bounded sequence which does not converge is said to *oscillate finitely*.

ii. A sequence $\{s_n\}$ is said to be *divergent to* ∞, if to each given positive number Δ, there correspond an integer m such that

$$s_n > \Delta \quad \text{for all } n \geq m.$$

iii. A sequence $\{s_n\}$ is said to be *divergent to* $-\infty$, if to each given positive number Δ, there correspond an integer m such that

$$s_n < -\Delta \quad \text{for all } n \geq m.$$

iv. A sequence $\{s_n\}$ is said to *oscillate infinitely*, if it is unbounded and is divergent neither to ∞ nor to $-\infty$.

Examples

a. $\{1+(-1)^n\}$ oscillates finitely.

b. $\{(-1)^n n\}$ oscillates infinitely.

c. $\{2^n\}$ diverges to ∞.

d. $\{-2n\}$ diverges to $-\infty$.

Question

Prove that $\{-e^n\}$ diverges to $-\infty$ (by definition)

Solution.

Suppose $\Delta > 0$ be given and $s_n = -e^n$.

Take $s_n < -\Delta$, i.e. $-e^n < -\Delta \quad \Rightarrow \; e^n > \Delta \quad \Rightarrow \; n > \log\Delta$.

Now if m is positive integer such that $m > \log\Delta$, then

$s_n < -\Delta$ for all $n > m$.

This implies $\{-e^n\}$ is diverges to $-\infty$.

Question

Prove that $\{5^n\}$ diverges to ∞ (by definition).

Prove that $\{n^2\}$ diverges to ∞ (by definition).

Review

- Triangular inequality: If $a,b \in \mathbb{R}$, then $\big||a|-|b|\big| \leq \big|a \pm b\big| \leq |a|+|b|$.
- If $0 \leq a < \varepsilon$ for all $\varepsilon > 0$, then $a = 0$.

Theorem

A convergent sequence of real number has one and only one limit (i.e. limit of the sequence is unique.)

Proof:

Suppose $\{s_n\}$ converges to two limits s and t, where $s \neq t$.

Then for all $\varepsilon > 0$, there exists two positive integers n_1 and n_2 such that

$$|s_n - s| < \frac{\varepsilon}{2} \qquad \forall \; n > n_1 \; \ldots\ldots\ldots\ldots\ldots\ldots \; (1)$$

and $$|s_n - t| < \frac{\varepsilon}{2} \qquad \forall \; n > n_2. \; \ldots\ldots\ldots\ldots\ldots\ldots \; (2)$$

As (1) and (2) hold simultaneously for all $n > \max\{n_1, n_2\}$.

Thus, for all $n > \max\{n_1, n_2\}$ we have

$$\begin{aligned} 0 \leq |s-t| &= |s - s_n + s_n - t| \\ &\leq |s_n - s| + |s_n - t| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As ε is arbitrary, we get $|s-t| = 0$, this gives $s = t$, that is, the limit of the sequence is unique.

❑

Theorem

If the sequence $\{s_n\}$ converges to s, where $s \neq 0$, then there exists a positive integer n_1 such that $|s_n| > \frac{1}{2}|s|$ for all $n > n_1$.

Proof:

Since $\{s_n\}$ converges to s, therefore for all real $\varepsilon > 0$, there exists positive integer n_1 such that

$$|s_n - s| < \varepsilon \quad \text{for } n > n_1.$$

We fix $\varepsilon = \frac{1}{2}|s| > 0$ to get

$$|s_n - s| < \frac{1}{2}|s| \quad \text{for } n > n_1,$$

that is,

$$-\frac{1}{2}|s| < -|s_n - s| \quad \text{for } n > n_1. \quad \ldots\ldots\ldots\ldots (1)$$

Now

$$\begin{aligned}
\frac{1}{2}|s| &= |s| - \frac{1}{2}|s| \\
&< |s| - |s_n - s| \quad \text{for } n > n_1 \qquad \text{(by using (1))} \\
&\le |s + (s_n - s)| \quad \text{for } n > n_1
\end{aligned}$$

This ultimately gives us

$$|s_n| > \frac{1}{2}|s| \text{ for all } n > n_1.$$ ❑

Theorem

Let a and b be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, then

(*i*) $\{as_n + bt_n\}$ converges to $as + bt$.

(*ii*) $\{s_n t_n\}$ converges to st.

(*iii*) $\left\{\dfrac{s_n}{t_n}\right\}$ converges to $\dfrac{s}{t}$, provided $t_n \ne 0$ for all n and $t \ne 0$.

Proof:

Since $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, therefore

$$|s_n - s| < \varepsilon \qquad \forall\ n > n_1 \in \mathbb{N}$$

$$|t_n - t| < \varepsilon \qquad \forall\ n > n_2 \in \mathbb{N}$$

Also $\exists\ \lambda > 0$ such that $|s_n| < \lambda \quad \forall\ n > 1$ ($\because$ $\{s_n\}$ is bounded)

(*i*) We have

$$\begin{aligned}
|(as_n + bt_n) - (as + bt)| &= |a(s_n - s) + b(t_n - t)| \\
&\le |a(s_n - s)| + |b(t_n - t)| \\
&< |a|\varepsilon + |b|\varepsilon \qquad \forall\ n > \max\{n_1, n_2\} \\
&= \varepsilon_1,
\end{aligned}$$

where $\varepsilon_1 = |a|\varepsilon + |b|\varepsilon$ a certain number.

This implies $\{as_n + bt_n\}$ converges to $as + bt$.

(*ii*) $$\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&= |s_n(t_n - t) + t(s_n - s)|
\end{aligned}$$

$$\leq |s_n|\cdot|(t_n - t)| + |t|\cdot|(s_n - s)|$$
$$< \lambda\varepsilon + |t|\varepsilon \qquad \forall\ n > \max\{n_1, n_2\}$$
$$= \varepsilon_2, \qquad \text{where } \varepsilon_2 = \lambda\varepsilon + |t|\varepsilon \text{ a certain number.}$$

This implies $\{s_n t_n\}$ converges to *st.*

(*iii*) $$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t_n t}\right|$$
$$= \frac{|t_n - t|}{|t_n||t|} < \frac{\varepsilon}{\frac{1}{2}|t||t|} \qquad \forall\ n > \max\{n_1, n_2\}$$
$$\because\ |t_n| > \frac{1}{2}|t|$$
$$= \frac{\varepsilon}{\frac{1}{2}|t|^2} = \varepsilon_3, \qquad \text{where } \varepsilon_3 = \frac{\varepsilon}{\frac{1}{2}|t|^2} \text{ a certain number.}$$

This implies $\left\{\frac{1}{t_n}\right\}$ converges to $\frac{1}{t}$.

Hence $\left\{\frac{s_n}{t_n}\right\} = \left\{s_n \cdot \frac{1}{t_n}\right\}$ converges to $s\cdot\frac{1}{t} = \frac{s}{t}$. (from (ii)) ❑

Question

Prove that if $\lim\limits_{n\to\infty} s_n = t$, then $\lim\limits_{n\to\infty}|s_n| = |t|$ but converse is not true in general.

Question

Prove that every convergent sequence is bounded.

Solution:

Consider a sequence $\{s_n\}$ converges to limit l, that is, for all $\varepsilon > 0$, there exists positive integer n_0 such that

$$|s_n - l| < \varepsilon \quad \text{for all } n > n_0.$$

For $\varepsilon = 1$, we have

$$|s_n - l| < 1 \quad \text{for all } n > n_0 \quad \ldots\ldots.. (i)$$

Now $$|s_n| < |s_n - l + l| \leq |s_n - l| + |l|$$

Using (*i*), in above expression, we get

$$|s_n| < 1 + |l| \quad \text{for all } n > n_0.$$

Now take $\lambda = \max\left\{|s_1|, |s_2|, \ldots, |s_{n_0}|, 1 + |l|\right\}$, then we have

$$|s_n| \leq \lambda \quad \text{for all } n \in \mathbb{N}.$$

This implies $\{s_n\}$ is bounded. ❑

Review:

- For all $a, b, c \in \mathbb{R}$, $|a - b| < c \iff b - c < a < b + c$ or $a - c < b < a + c$.

Theorem (Sandwich Theorem or Squeeze Theorem)
Suppose that $\{s_n\}$ and $\{t_n\}$ be two convergent sequences such that $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = s$. If $s_n < u_n < t_n$ for all $n \geq n_0$, then the sequence $\{u_n\}$ also converges to *s*.

Proof:

Since the sequence $\{s_n\}$ and $\{t_n\}$ converge to the same limit *s* (say), therefore for given $\varepsilon > 0$ there exists two positive integers n_1 and n_2 such that

$$|s_n - s| < \varepsilon \qquad \forall\ n > n_1,$$
$$|t_n - s| < \varepsilon \qquad \forall\ n > n_2.$$

i.e.
$$s - \varepsilon < s_n < s + \varepsilon \qquad \forall\ n > n_1,$$
$$s - \varepsilon < t_n < s + \varepsilon \qquad \forall\ n > n_2.$$

Also, we have given
$$s_n < u_n < t_n \qquad \forall\ n > n_0.$$

Consider $n_3 = \max\{n_0, n_1, n_2\}$, then we have
$$s - \varepsilon < s_n < u_n < t_n < s + \varepsilon \qquad \forall\ n > n_3$$
$$\Rightarrow\ s - \varepsilon < u_n < s + \varepsilon \qquad \forall\ n > n_3$$

i.e. $|u_n - s| < \varepsilon \quad \forall\ n > n_3$

i.e. $\lim\limits_{n\to\infty} u_n = s$. ❑

Example

Show that $\lim\limits_{n\to\infty}\left(\dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \ldots + \dfrac{1}{(2n)^2}\right) = 0$.

Solution.

Consider
$$s_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}$$

As
$$\underbrace{\frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \ldots + \frac{1}{(2n)^2}}_{n\ times} \leq s_n < \underbrace{\frac{1}{n^2} + \frac{1}{n^2} + \ldots + \frac{1}{n^2}}_{n\ times},$$

that is,
$$n \cdot \frac{1}{(2n)^2} \leq s_n < n \cdot \frac{1}{n^2} \qquad \Rightarrow \frac{1}{4n} \leq s_n < \frac{1}{n}$$
$$\Rightarrow \lim_{n\to\infty} \frac{1}{4n} \leq \lim_{n\to\infty} s_n < \lim_{n\to\infty} \frac{1}{n} \qquad \Rightarrow 0 \leq \lim_{n\to\infty} s_n < 0$$
$$\Rightarrow \lim_{n\to\infty} s_n = 0.$$ ❑

Theorem

For each irrational number *x*, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim\limits_{n\to\infty} r_n = x$.

Proof:

Since x and $x+1$ are two different real numbers, so there exist a rational number r_1 such that

$$x < r_1 < x+1$$

Similarly there exists a rational number $r_2 \neq r_1$ such that

$$x < r_2 < \min\left\{r_1, x+\frac{1}{2}\right\} < x+1$$

Continuing in this manner we have

$$x < r_3 < \min\left\{r_2, x+\frac{1}{3}\right\} < x+1$$

$$x < r_4 < \min\left\{r_3, x+\frac{1}{4}\right\} < x+1$$

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$$x < r_n < \min\left\{r_{n-1}, x+\frac{1}{n}\right\} < x+1$$

This implies that there is a sequence $\{r_n\}$ of the distinct rational number such that

$$x < r_n < x+\frac{1}{n}.$$

Since $\lim\limits_{n\to\infty}(x) = \lim\limits_{n\to\infty}\left(x+\frac{1}{n}\right) = x.$

Therefore

$$\lim_{n\to\infty} r_n = x.$$ ❑

Theorem

Let a sequence $\{s_n\}$ be a bounded sequence.

(i) If $\{s_n\}$ is monotonically increasing then it converges to its supremum.

(ii) If $\{s_n\}$ is monotonically decreasing then it converges to its infimum.

Proof

(i) Let $S = \sup s_n$ and take $\varepsilon > 0$.

Since there exists s_{n_0} such that $S - \varepsilon < s_{n_0}$

Since $\{s_n\}$ is monotonically increasing,

therefore

$$S-\varepsilon < s_{n_0} < s_n < S < S+\varepsilon \qquad \text{for } n > n_0$$

$$\Rightarrow\ S-\varepsilon < s_n < S+\varepsilon \qquad \text{for } n > n_0$$

$$\Rightarrow\ |s_n - S| < \varepsilon \qquad \text{for } n > n_0$$

$$\Rightarrow\ \lim_{n\to\infty} s_n = S$$

(ii) Let $s = \inf s_n$ and take $\varepsilon > 0$.

Since there exists s_{n_1} such that $s_{n_1} < s + \varepsilon$

Since $\{s_n\}$ is monotonically decreasing, therefore

$$s-\varepsilon < s < s_n < s_{n_1} < s+\varepsilon \qquad \text{for } n > n_1$$

$$\Rightarrow\ s-\varepsilon < s_n < s+\varepsilon \qquad \text{for } n > n_1$$

$$\Rightarrow\ \left|s_n - s\right| < \varepsilon \qquad \text{for } n > n_1$$

Thus $\lim\limits_{n\to\infty} s_n = s$ ❑

Questions:

1. Let $\{s_n\}$ be a sequence and $\lim\limits_{n\to\infty} s_n = s$. Then prove that $\lim\limits_{n\to\infty} s_{n+1} = s$.
2. Prove that a bounded increasing sequence converges to its supremum.
3. Prove that a bounded decreasing sequence converges to its infimum.
4. Prove that if a sequence $\{s_n\}$ converges to l, then every subsequence of $\{s_n\}$ converges to l.
5. If the subsequence $\{s_{2n}\}$ and $\{s_{2n-1}\}$ of sequence $\{s_n\}$ converges to the same limit l then $\{s_n\}$ converges to l.

Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

Example:

Let $t_1 > 1$ and let $\{t_n\}$ be defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$ for $n \geq 1$.

(i) Show that $\{t_n\}$ is decreasing sequence.
(ii) It is bounded below.
(iii) Find the limit of the sequence.

Since $t_1 > 1$ and $\{t_n\}$ is defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$; $n \geq 1$

$$\Rightarrow t_n > 0 \qquad \forall\ n \geq 1$$

Also $\quad t_n - t_{n+1} = t_n - 2 + \dfrac{1}{t_n}$

$$= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n - 1)^2}{t_n} > 0.$$

$$\Rightarrow t_n > t_{n+1} \qquad \forall\ n \geq 1.$$

This implies that t_n is monotonically decreasing.

Since $t_n > 1 \qquad \forall\ n \geq 1$,

$\Rightarrow$ t_n is bounded below.

Since t_n is decreasing and bounded below therefore t_n is convergent.

Let us suppose $\lim\limits_{n\to\infty} t_n = t$.

Then $\quad \lim\limits_{n\to\infty} t_{n+1} = \lim\limits_{n\to\infty} t_n \Rightarrow \lim\limits_{n\to\infty}\left(2 - \dfrac{1}{t_n}\right) = \lim\limits_{n\to\infty} t_n$

$$\Rightarrow\ 2-\frac{1}{t}=t \qquad \Rightarrow\ \frac{2t-1}{t}=t \qquad \Rightarrow\ 2t-1=t^2 \quad \Rightarrow\ t^2-2t+1=0$$

$$\Rightarrow\ (t-1)^2=0 \qquad \Rightarrow\ t=1.$$ ❑

Question:

- Let $\{t_n\}$ be a positive term sequence. Find the limit of the sequence if $4t_{n+1}=\dfrac{2}{5}-3t_n$ for all $n\geq 1$.
- Let $\{u_n\}$ be a sequence of positive numbers. Then find the limit of the sequence if $u_{n+1}=\dfrac{1}{u_n}+\dfrac{1}{4}u_{n-1}$ for $n\geq 1$.
- The Fibonacci numbers are: $F_1=F_2=1$, and for every $n\geq 3$, F_n is defined by the recurrence relation $F_n=F_{n-1}+F_{n-2}$. Find the $\lim\limits_{n\to\infty}\dfrac{F_n}{F_{n-1}}$ (this limit is known as golden number)

Cauchy Sequence

A sequence $\{s_n\}$ of real number is said to be a *Cauchy sequence* if for given number $\varepsilon>0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$|s_n-s_m|<\varepsilon \qquad \forall\ m,n>n_0$$

Example

The sequence $\left\{\dfrac{1}{n}\right\}$ is a Cauchy sequence.

Suppose $s_n=\dfrac{1}{n}$ and $\varepsilon>0$ be given. We choose a positive integer $n_0=n_0(\varepsilon)$ such that $n_0>\dfrac{2}{\varepsilon}$.

Then if $m,n>n_0$, we have $\dfrac{1}{n}<\dfrac{1}{n_0}<\dfrac{\varepsilon}{2}$ and similarly $\dfrac{1}{m}<\dfrac{1}{n_0}<\dfrac{\varepsilon}{2}$. Therefore, it follows that if $m,n>n_0$, then

$$|s_n-s_m| = \left|\frac{1}{n}-\frac{1}{m}\right| \leq \frac{1}{n}+\frac{1}{m} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude that $\left\{\dfrac{1}{n}\right\}$ is Cauchy sequence.

Theorem

A Cauchy sequence of real numbers is bounded.

Proof:

Let $\{s_n\}$ be a Cauchy sequence. Then for given number $\varepsilon>0$, there exists a positive integer n_0 such that

$$|s_n-s_m|<\varepsilon \qquad \forall\ m,n>n_0.$$

Take $\varepsilon=1$, then we have

$$|s_n - s_m| < 1 \ \forall \ m,n > n_0.$$

Fix $m = n_0 + 1$ then

$$\begin{aligned} |s_n| &= |s_n - s_{n_0+1} + s_{n_0+1}| \\ &\le |s_n - s_{n_0+1}| + |s_{n_0+1}| \\ &< 1 + |s_{n_0+1}| \qquad \forall \ n > n_0. \end{aligned}$$

Now take $\lambda = \max\left\{|s_1|,|s_2|,\ldots,|s_{n_0}|,1+|s_{n_0+1}|\right\}$, then we have

$$|s_n| \le \lambda \ \text{ for all } n \in \mathbb{N}.$$

Hence we conclude that $\{s_n\}$ is a Cauchy sequence, which is bounded one. ❑

Remarks:

The converse of the above theorem does not hold, that is, every bounded sequence is not Cauchy.

Consider the sequence $\{s_n\}$, where $s_n = (-1)^n$, $n \ge 1$. It is bounded sequence because

$$|(-1)^n| = 1 < 2 \qquad \forall \ n \ge 1.$$

But it is not a Cauchy sequence if it is then for $\varepsilon = 1$ we should be able to find a positive integer n_0 such that $|s_n - s_m| < 1$ for all $m,n > n_0$.

But with $m = 2k+1$, $n = 2k+2$ when $2k+1 > n_0$, we arrive at

$$\begin{aligned} |s_n - s_m| &= |(-1)^{2n+2} - (-1)^{2k+1}| \\ &= |1+1| = 2 < 1 \qquad \text{is absurd.} \end{aligned}$$

Hence $\{s_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence.

Questions:

a. Prove that every Cauchy sequence of real number is bounded but converse is not true.
b. Prove that every convergent sequence is bounded but converse is not true.

Theorem

Every Cauchy sequence of real numbers has a convergent subsequence.

Proof:

Suppose $\{s_n\}$ is a Cauchy sequence, therefore it is bounded.

First, we assume that $\{s_n : n \in \mathbb{N}\}$ has maximum value, then set

$$s_{n_1} = \max\{s_n : n \ge 1\}$$
$$s_{n_2} = \max\{s_n : n > n_1\}$$
$$s_{n_3} = \max\{s_n : n > n_2\} \text{ and so on}$$

Then clearly $\{s_{n_k}\}$ is subsequence of $\{s_n\}$ and it is decreasing and bounded.

Hence it is convergent.

On the other hand, if $\{s_n : n \in \mathbb{N}\}$ has no maximum value, then there exist some positive integer N such that $\{s_n : n > N\}$ has no maximum value.

Now for $m > N$, we can find some s_m such that $s_m > s_N$, otherwise one of the $s_{N+1}, s_{N+2}, \ldots, s_m$ will be the maximum value of $.\{s_n : n > N\}$.

So assume $s_{n_1} = s_{N+1}$.

Now s_{n_2} can be the first term after s_{n_1} such that $s_{n_2} > s_{n_1}$.

Then s_{n_3} can be the first term after s_{n_2} such that $s_{n_3} > s_{n_2}$.

Continuing in this way, we get $\{s_{n_k}\}$ be a subsequence of $\{s_n\}$ such that it is increasing and bounded. Thus it is convergent. ❑

Question:

Prove that every bounded sequence has convergent subsequence.

Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

Proof:

Let $\{s_n\}$ be a convergent sequence, which converges to s.

Then for given $\varepsilon > 0$ $\exists$ a positive integer n_0, such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \forall \;\; n > n_0$$

Now for $n > m > n_0$

$$\begin{aligned} |s_n - s_m| &= |s_n - s + s - s_m| \\ &\leq |s_n - s| + |s - s_m| = |s_n - s| + |s_m - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \end{aligned}$$

This shows that $\{s_n\}$ is a Cauchy sequence.

Conversely, suppose that $\{s_n\}$ is a Cauchy sequence then for $\varepsilon > 0$, there exists a positive integer m_1 such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \;\; \forall \;\; n, m > m_1 \;\; \ldots\ldots\ldots\ldots (i)$$

Since $\{s_n\}$ is a Cauchy sequence,

therefore it has a subsequence $\{s_{n_k}\}$ converging to s (say).

This implies there exists a positive integer m_2 such that

$$|s_{n_k} - s| < \frac{\varepsilon}{2} \qquad \forall \;\; n_k > m_2 \;\; \ldots\ldots\ldots\ldots (ii)$$

Now

$$\begin{aligned} |s_n - s| &= |s_n - s_{n_k} + s_{n_k} - s| \\ &\leq |s_n - s_{n_k}| + |s_{n_k} - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall \;\; n > \max\{m_1, m_2\}, \end{aligned}$$

this shows that $\{s_n\}$ is a convergent sequence. ❑

Example

Prove that $\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right\}$ is divergent sequence.

Let $\{t_n\}$ be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

For $m, n \in \mathbb{N}$, $n > m$ we have

$$\begin{aligned} \left| t_n - t_m \right| &= \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} \\ &> \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \quad (n-m \text{ times}) \\ &= (n-m)\frac{1}{n} = 1 - \frac{m}{n}. \end{aligned}$$

In particular if $n = 2m$ then

$$\left| t_n - t_m \right| > \frac{1}{2}.$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent. ❑

Theorem (Nested intervals)

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n$ $\forall\ n \geq 1$, and $(b_n - a_n) \to 0$ as $n \to \infty$ then $\bigcap I_n$ contains one and only one point.

Proof:

Since $I_{n+1} \subset I_n$, therefore

$$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n < b_n < b_{n-1} < \ldots < b_3 < b_2 < b_1.$$

Note that $\{a_n\}$ is increasing sequence, bounded above by b_1 and bounded below by a_1. Also note that $\{b_n\}$ is decreasing sequence bounded below by a_1 and bounded above by b_1.

This implies both $\{a_n\}$ and $\{b_n\}$ are monotone and bounded sequences and hence convergent.

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b.

But $$\begin{aligned} \left| a - b \right| &= \left| a - a_n + a_n - b_n + b_n - b \right| \\ &\leq \left| a_n - a \right| + \left| a_n - b_n \right| + \left| b_n - b \right| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

$$\Rightarrow a = b$$

and $\quad a_n < a < b_n \qquad \forall\ n \geq 1.$

This given $\bigcap I_n = \{a\}$, that is, $\bigcap I_n$ contains only one point. ❑

Limit inferior of the sequence

Suppose $\{s_n\}$ is bounded below then we define limit inferior of $\{s_n\}$ as follow

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} u_n, \quad \text{where} \quad u_n = \inf\{s_k : k \geq n\}.$$

If s_n is not bounded below then we define

$$\liminf_{n\to\infty} s_n = -\infty.$$

Limit superior of the sequence

Suppose $\{s_n\}$ is bounded above then we define limit superior of $\{s_n\}$ as follow

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n, \text{ where } v_n = \sup\{s_k : k \geq n\}$$

If s_n is not bounded above then we define

$$\limsup_{n\to\infty} s_n = +\infty.$$

Remarks:

i. *Limit inferior* is also known as *lower limit* and *limit superior* is also known sas *upper limit* of the sequence in the literature with the notations $\underline{\lim}$ and $\overline{\lim}$ respectively.

ii. A bounded sequence has unique limit inferior and superior.

iii. It is easy to prove that limit inferior is less than or equal to limit superior.

Examples

(i) Let $s_n = (-1)^n$, then limit superior of $\{s_n\}$ is 1 and limit inferior of $\{s_n\}$ is -1.

(*ii*) Let $s_n = (-1)^n\left(1+\dfrac{1}{n}\right)$

then limit superior of $\{s_n\}$ is 1 and limit inferior of $\{s_n\}$ is -1.

(*iii*) Let $s_n = \left(1+\dfrac{1}{n}\right)\cos n\pi$.

Then $u_n = \inf\{s_k : k \geq n\}$

$$= \inf\left\{\left(1+\frac{1}{n}\right)\cos n\pi, \left(1+\frac{1}{n+1}\right)\cos(n+1)\pi, \left(1+\frac{1}{n+2}\right)\cos(n+2)\pi, \ldots\right\}$$

$$= \begin{cases} \left(1+\dfrac{1}{n}\right)\cos n\pi & \textit{if } n \textit{ is odd} \\ \left(1+\dfrac{1}{n+1}\right)\cos(n+1)\pi & \textit{if } n \textit{ is even} \end{cases}$$

$$\Rightarrow \liminf_{n\to\infty} s_n s_n = \lim_{n\to\infty} u_n = -1.$$

Also $v_n = \sup\{s_k : k \geq n\}$

$$= \begin{cases} \left(1+\dfrac{1}{n+1}\right)\cos(n+1)\pi & \textit{if } n \textit{ is odd} \\ \left(1+\dfrac{1}{n}\right)\cos n\pi & \textit{if } n \textit{ is even} \end{cases}$$

$$\Rightarrow \limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n = 1.$$ ❑

Theorem

If $\{s_n\}$ is a convergent sequence, then

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$$

Proof:

Let $\lim_{n\to\infty} s_n = s$ then for a real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$|s_n - s| < \varepsilon \quad \text{whenever } n \geq n_0.$$

i.e. $\quad s-\varepsilon < s_n < s+\varepsilon$ whenever $n \ge n_0$. …………… (i)

If we take $u_n = \inf\{s_k : k \ge n\}$ and $v_n = \sup\{s_k : k \ge n\}$, then (i) gives us

$$s-\varepsilon < u_n \le v_n < s+\varepsilon \text{ whenever } n \ge n_0.$$

This gives $\lim\limits_{n\to\infty} u_n = s$ and $\lim\limits_{n\to\infty} v_n = s$

that is, $\liminf\limits_{n\to\infty} s_n = \limsup\limits_{n\to\infty} s_n = s$. ❑

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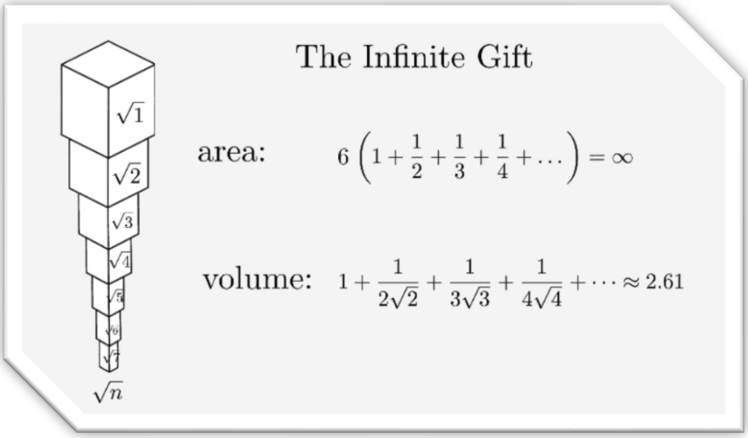
Chapter 03

Series

*It's the ultimate outcome when a pattern of numbers keeps adding itself, revealing a hidden sum. --- **Dr. Elias Vance, Applied Mathematician***



I'm standing 5 m from a wall. I jump half the distance (2.5 m) towards the wall. I halve the distance again (1.25 m) and continue getting closer to the wall by stepping half the remaining distance each time. Do I ever reach the wall? Zeno, the 5th century BCE Greek philosopher, proposed a similar question in his famous Paradoxes (search for Zeno's paradox).
The first known example of an infinite sum was when Greek mathematician Archimedes showed in the 3rd century BCE that the area of a segment of a parabola is 4/3 the area of a triangle with the same base. The notation he used was different, of course, and some of the approach was more geometric than algebraic, but his approach of summing infinitely small quantities was quite remarkable for the time.
Mathematicians Madhava from Kerala, India studied infinite series around 1350 CE. Among his many contributions, he discovered the infinite series for the trigonometric functions of sine, cosine, tangent and arctangent, and many methods for calculating the circumference of a circle
In the 17th century, James Gregory (1638-1675) worked in the new decimal system on infinite series and published several Maclaurin series. In 1715, a general method for constructing the Taylor series for all functions for which they exist was provided by Brook Taylor (1685-1731). Leonhard Euler (1707-1783) derived series for sine, cosine, exp, log, etc., and he also discovered relationships between them. He also introduced sigma notation (Σ) for sums of series.

Infinite Series

Let $\{a_n\}$ be a given sequence. Then a sum of the form

$$a_1+a_2+a_3+\dots$$

is called an infinite series.

Another way of writing this infinite series is $\sum\limits_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.

Convergence and divergence of the series

A series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_k$, is convergent. If the sequence $\{s_n\}$ diverges then the series is said to be diverge.

Remarks:

For a series $\sum_{n=1}^{\infty} a_n$, the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_k$, is called the sequence of partial sum of the series. The numbers a_n are called terms and s_n are called partial sums. One can note that

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3 \text{ and}$$
$$s_n = a_1 + a_2 + \ldots + a_n \quad \text{or} \quad s_n = s_{n-1} + a_n .$$

If the sequence $\{s_n\}$ converges to *s,* we say that the series converges and write $\sum_{n=1}^{\infty} a_n = s$, the number *s* is called the sum or value of the series but it should be clearly understood that the '*s*' is the limit of the sequence of sums and is not obtained simply by addition.

Also note that the behaviors of the series remain unchanged by addition or deletion of the first finite terms. Just as a sequence may be indexed such that its first element is not a_n, but is a_0, or a_5 or a_{99}, we will denote the series having these numbers as their first element by the symbols

$$\sum_{n=0}^{\infty} a_n \quad \text{or} \quad \sum_{n=5}^{\infty} a_n \quad \text{or} \quad \sum_{n=99}^{\infty} a_n .$$

Review:

- Let $\{a_n\}$ be a convergent sequence, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1}$.

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof

Assume that $s_n = a_1 + a_2 + a_3 + \ldots + a_n$.

As $\sum_{n=1}^{\infty} a_n$ is convergent, therefore $\{s_n\}$ is convergent.

Suppose $\lim_{n\to\infty} s_n = s$, then we have $\lim_{n\to\infty} s_{n-1} = s$.

Now we have $s_n = s_{n-1} + a_n$ for $n > 1$,

or $a_n = s_n - s_{n-1}$ for $n > 1$.

Therefore
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(s_n - s_{n-1}\right)$$
$$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$$
$$= s - s = 0 .$$

❑

Remark:

(i) The converse of the above theorem is false. For example, consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$. We know that the sequence $\{s_n\}$, where $s_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, is divergent therefore $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent series, although $\lim_{n\to\infty} a_n = 0$.

(ii) The above theorem shows that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent (This is called basic divergent test).

Examples:

(i) Is the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is convergent or divergent?

Solution.

Assume $a_n = 1+\frac{1}{n}$.

Now we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$.

Hence $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is divergent (by basic divergent test)

(ii) Show that the series $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots$ is divergent.

Solution.

The above series can be written as $\sum_{n=1}^{\infty}\sqrt{\frac{n}{2(n+1)}}$.

Then take

$$a_n = \sqrt{\frac{n}{2(n+1)}},$$

As we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}} \neq 0$.

Hence the given series is divergent by basic comparison test.

(iii) Is the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is convergent or divergent?

Solution.

Assume that $a_n = \frac{n^n}{n!}$.

As
$$a_n = \frac{n^n}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1}$$

$$= \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \ldots \cdot \frac{n}{2} \cdot \frac{n}{1} \geq 1 \text{ for all } n \geq 1.$$

We conclude $\lim_{n\to\infty} a_n$ cannot be zero.

Hence $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent (by basic divergent test).

Questions:

(i) Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$ but converse is not true.

(ii) Prove that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.

Review:

- A series $\sum a_n$ is convergent if and only if it sequence of partial sum $\left\{s_n := \sum_{k=1}^{n} a_k\right\}$ is convergent.
- A sequence in $\mathbb{R}$ is convergent iff it is Cauchy sequence.
- A sequence $\{s_n\}$ is Cauchy sequence if and only if for all $\varepsilon > 0$ there exists positive integer n_0 such $|s_n - s_m| < \varepsilon$ for all $n, m > n_0$ (or $n \geq m > n_0$).

Theorem (General Principle of Convergence or Cauchy Criterion for Series)

A series $\sum a_n$ is convergent if and only if for any real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left| \sum_{i=m+1}^{n} a_i \right| < \varepsilon \qquad \forall \ n \geq m > n_0$$

Proof

Assume that $s_n = a_1 + a_2 + a_3 + \ldots + a_n$.

Then $\sum a_n$ is convergent if and only if $\{s_n\}$ is convergent.

Now $\{s_n\}$ is convergent if and only $\{s_n\}$ is Cauchy sequence,

that is, for all real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$| \ s_n - s_m \ | < \varepsilon \qquad \forall \ n \geq m > n_0 \ \ldots\ldots\ldots (i)$$

As $n > m$, therefore

$$s_n = s_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

$$\Rightarrow \ s_n - s_m = a_{m+1} + a_{m+2} + \ldots + a_n.$$

So by using (i), we have

$$| s_n - s_m | = | a_{m+1} + a_{m+2} + \ldots + a_n | < \varepsilon \qquad \forall \ n \geq m > n_0.$$

This gives

$$\left| \sum_{i=m+1}^{n} a_i \right| < \varepsilon \quad \forall \ n \geq m > n_0. \qquad \square$$

Review:

- A bounded and monotone sequence, then it is convergent.

- An unbounded sequence is divergent.

Theorem

Let $\sum a_n$ be an infinite series of non-negative terms and let $\{s_n\}$ be a sequence of its partial sums. Then $\sum a_n$ is convergent if $\{s_n\}$ is bounded and it diverges if $\{s_n\}$ is unbounded.

Proof

We have $s_n = a_1 + a_2 + a_3 + ... + a_n$, this give $s_{n+1} = s_n + a_{n+1}$.

As we have given $a_n \geq 0$ for all $n \geq 1$ and $s_{n+1} = s_n + a_{n+1} \geq s_n$ for all $n \geq 1$.

Therefore, the sequence $\{s_n\}$ is monotonic increasing.

Now if $\{s_n\}$ is bounded then we concluded that $\{s_n\}$ is convergent.

Now if $\{s_n\}$ is unbounded, then it is divergent.

Hence we conclude that $\sum a_n$ is convergent if $\{s_n\}$ is bounded and it divergent if $\{s_n\}$ is unbounded. ❑

Review:

- A series $\sum a_n$ is divergent if and only if there exists real number $\varepsilon > 0$, such that for all positive integer n_0,

$$\left| \sum_{i=m+1}^{n} a_i \right| > \varepsilon \qquad \text{whenever } n > m > n_0$$

Theorem (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0$ for all n. Also suppose that for a fixed positive number λ and positive integer k,

$$a_n < \lambda b_n \quad \forall\, n \geq k.$$

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Proof

(i) Suppose $\sum b_n$ is convergent and

$$a_n < \lambda b_n \quad \forall\, n \geq k,\ \lambda > 0, \ \ldots\ldots\ldots\ldots (i)$$

By Cauchy criterion; for any positive number $\varepsilon > 0$ there exists n_0 such that

$$\sum_{i=m+1}^{n} b_i < \frac{\varepsilon}{\lambda} \qquad n > m > n_0$$

from (i)

$$\sum_{i=m+1}^{n} a_i < \lambda \sum_{i=m+1}^{n} b_i < \varepsilon\ , \quad n > m > n_0 \quad \Rightarrow \quad \sum a_n \text{ is convergent.}$$

(ii) Now suppose $\sum a_n$ is divergent then there exists a real number $\beta > 0$, such that

$$\sum_{i=m+1}^{n} a_i > \lambda \beta\ , \quad n > m.$$

From (i)

$$\sum_{i=m+1}^{n} b_i > \frac{1}{\lambda}\sum_{i=m+1}^{n} a_i > \beta, \qquad n > m$$

$\Rightarrow \quad \sum b_n$ is divergent. ❑

Example

Prove that $\sum \frac{1}{\sqrt{n}}$ is divergent.

Since $n \geq \sqrt{n} > 0 \quad \forall \quad n \geq 1$.

$$\Rightarrow \quad \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

$\Rightarrow \quad \sum \frac{1}{\sqrt{n}}$ is divergent as $\sum \frac{1}{n}$ is divergent. ❑

Example

The series $\sum \frac{1}{n^\alpha}$ is convergent if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Let $\quad s_n = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha}$.

If $\alpha > 1$ then

$$s_n < s_{2n} \quad \text{and} \quad \frac{1}{n^\alpha} < \frac{1}{(n-1)^\alpha}.$$

Now $\quad s_{2n} = \left[1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \frac{1}{4^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right]$

$$= \left[1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + \ldots + \frac{1}{(2n-1)^\alpha}\right] + \left[\frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \frac{1}{6^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right]$$

$$= \left[1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + \ldots + \frac{1}{(2n-1)^\alpha}\right] + \frac{1}{2^\alpha}\left[1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{(n)^\alpha}\right]$$

$$< \left[1 + \frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \ldots + \frac{1}{(2n-2)^\alpha}\right] + \frac{1}{2^\alpha}s_n \qquad \text{(replacing 3 by 2, 5 by 4 and so on.)}$$

$$= 1 + \frac{1}{2^\alpha}\left[1 + \frac{1}{2^\alpha} + \ldots + \frac{1}{(n-1)^\alpha}\right] + \frac{1}{2^\alpha}s_n$$

$$= 1 + \frac{1}{2^\alpha}s_{n-1} + \frac{1}{2^\alpha}s_n \quad < 1 + \frac{1}{2^\alpha}s_{2n} + \frac{1}{2^\alpha}s_{2n} \qquad \because \; s_{n-1} < s_n < s_{2n}$$

$$= 1 + \frac{2}{2^\alpha}s_{2n}$$

$$\Rightarrow \quad s_{2n} < 1 + \frac{1}{2^{\alpha-1}}s_{2n}.$$

$$\Rightarrow \left(1-\frac{1}{2^{\alpha-1}}\right)s_{2n} < 1 \quad \Rightarrow \left(\frac{2^{\alpha-1}-1}{2^{\alpha-1}}\right)s_{2n} < 1 \quad \Rightarrow s_{2n} < \frac{2^{\alpha-1}}{2^{\alpha-1}-1},$$

i.e. $s_n < s_{2n} < \dfrac{2^{\alpha-1}}{2^{\alpha-1}-1}$

$\Rightarrow \{s_n\}$ is bounded and also monotonic. Hence, we conclude that $\sum \dfrac{1}{n^\alpha}$ is convergent when $\alpha > 1$.

If $\alpha \leq 1$ then

$$n^\alpha \leq n \quad \forall \ n \geq 1$$

$$\Rightarrow \frac{1}{n^\alpha} \geq \frac{1}{n} \quad \forall \ n \geq 1$$

Since $\sum \dfrac{1}{n}$ is divergent therefore $\sum \dfrac{1}{n^\alpha}$ is divergent when $\alpha \leq 1$. ❑

Theorem (Limit Comparison Test)

Let $a_n > 0$, $b_n > 0$ and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lambda$, where $\lambda \geq 0$.

(i) If $\lambda \neq 0$, then the series $\sum a_n$ and $\sum b_n$ behave alike.

(ii) If $\lambda = 0$ and if $\sum b_n$ is convergent, then $\sum a_n$ is convergent. If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

Proof

Since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lambda$, therefore for $\varepsilon > 0$, there exists positive integer n_0 such that

$$\left|\frac{a_n}{b_n} - \lambda\right| < \varepsilon \quad \forall \ n \geq n_0. \ \ldots\ldots\ldots\ldots\ldots (*)$$

(i) If $\lambda \neq 0$, then take $\varepsilon = \dfrac{\lambda}{2}$ (as λ will be positive)

$$\Rightarrow \left|\frac{a_n}{b_n} - \lambda\right| < \frac{\lambda}{2} \quad \forall \ n \geq n_0.$$

$$\Rightarrow -\frac{\lambda}{2} < \frac{a_n}{b_n} - \lambda < \frac{\lambda}{2} \quad \forall \ n \geq n_0.$$

$$\Rightarrow \lambda - \frac{\lambda}{2} < \frac{a_n}{b_n} < \lambda + \frac{\lambda}{2} \quad \forall \ n \geq n_0.$$

$$\Rightarrow \frac{\lambda}{2} < \frac{a_n}{b_n} < \frac{3\lambda}{2} \quad \forall \ n \geq n_0.$$

Then we got

$$a_n < \frac{3\lambda}{2} b_n \quad \text{and} \quad b_n < \frac{2}{\lambda} a_n \ \text{ for } n \geq n_0.$$

Hence by comparison test we conclude that $\sum a_n$ and $\sum b_n$ converge or diverge together.

(ii) If $\lambda = 0$, then (*) implies $a_n < \varepsilon b_n$

Hence by comparison test we conclude that $\sum a_n$ is convergent if $\sum b_n$ converges. Also $\sum b_n$ is divergent if $\sum a_n$ diverges. ❑

Example

Is the series $\sum \frac{1}{n}\sin^2\frac{x}{n}$ is convergent or divergent for real x ?

Consider $a_n = \frac{1}{n}\sin^2\frac{x}{n}$ and take $b_n = \frac{1}{n^3}$.

Then $$\frac{a_n}{b_n} = n^2\sin^2\frac{x}{n} = \frac{\sin^2\frac{x}{n}}{\frac{1}{n^2}}$$

$$= x^2\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2$$

Applying limit as $n \to \infty$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} x^2\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 = x^2\left(\lim_{n\to\infty}\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 = x^2(1) = x^2.$$

$\Rightarrow \sum a_n$ and $\sum b_n$ have the similar behavior for all finite values of x except $x = 0$.

Since $\sum \frac{1}{n^3}$ is convergent series therefore the given series is also convergent for finite values of x except $x = 0$.

If $x = 0$, then the given series is also convergent because it is just zero. ❑

Theorem (Cauchy Condensation Test)

Let $a_n \geq 0$, $a_n > a_{n+1}$ for all $n \geq 1$ (i.e. $\{a_n\}$ is positive term decreasing sequence). Then the series $\sum a_n$ and $\sum 2^{n-1}a_{2^{n-1}}$ converges or diverges together.

Proof

The condensation test follows from noting that if we collect the terms of the series into groups of lengths 2^n, each of these groups will be less than $2^n a_{2^n}$ by monotonicity. Observe,

$$\sum_{n=1}^{\infty} a_n = a_1 + \underbrace{a_2 + a_3}_{\leq a_2+a_2} + \underbrace{a_4 + a_5 + a_6 + a_7}_{\leq a_4+a_4+a_4+a_4} + \cdots + \underbrace{a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1}}_{\leq a_{2^n}+a_{2^n}+\cdots+a_{2^n}} + \cdots$$

$$\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} + \cdots = \sum_{n=0}^{\infty} 2^n a_{2^n}.$$

We have use the fact that a_n is decreasing sequence. The convergence of the original series now follows from direct comparison to this "condensed" series. To see that convergence of the original series implies the convergence of this last series, we similarly put,

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \underbrace{a_1 + a_2}_{\le a_1 + a_1} + \underbrace{a_2 + a_4 + a_4 + a_4}_{\le a_2 + a_2 + a_3 + a_3} + \cdots + \underbrace{a_{2^n} + a_{2^{n+1}} + \cdots + a_{2^{n+1}}}_{\le a_{2^n} + a_{2^n} + a_{(2^n+1)} + a_{(2^n+1)} + \cdots + a_{(2^{n+1}-1)}} + \cdots$$

$$\le a_1 + a_1 + a_2 + a_2 + a_3 + a_3 + \cdots + a_n + a_n + \cdots = 2\sum_{n=1}^{\infty} a_n.$$

And we have convergence, again by direct comparison. And we are done. Note that we have obtained the estimate

$$\sum_{n=1}^{\infty} a_n \le \sum_{n=0}^{\infty} 2^n a_{2^n} \le 2\sum_{n=1}^{\infty} a_n.$$ ❑

Example

Find value of p for which $\sum \frac{1}{n^p}$ is convergent or divergent.

If $p \le 0$ then $\lim\limits_{n\to\infty} \frac{1}{n^p} \ne 0$, therefore the series diverges when $p \le 0$.

If $p > 0$ then the condensation test is applicable, and we are lead to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{(p-1)k}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{(p-1)}}\right)^k$$

$$= \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

Now $2^{1-p} < 1$ iff $1-p < 0$ i.e. when $p > 1$.

And the result follows by comparing this series with the geometric series having common ratio less than one.

The series diverges when $2^{1-p} = 1$ (i.e. when $p = 1$).

The series is also divergent if $0 < p < 1$. ❑

Example

Prove that if $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges and if $p \le 1$ the series is divergent.

Since $\{\ln n\}$ is increasing, therefore $\left\{\frac{1}{n \ln n}\right\}$ decreases

and we can use the condensation test to the above series.

We have $a_n = \frac{1}{n(\ln n)^p}$

$$\Rightarrow \quad a_{2^n} = \frac{1}{2^n\left(\ln 2^n\right)^p} \qquad \Rightarrow \quad 2^n a_{2^n} = \frac{1}{\left(n\ln 2\right)^p}$$

Now $\sum 2^n a_{2^n} = \sum \frac{1}{(n\ln 2)^p} = \frac{1}{\left(\ln 2\right)^p}\sum \frac{1}{n^p}$.

This converges when $p>1$ and diverges when $p \le 1$. ❑

Example

Prove that $\sum \frac{1}{\ln n}$ is divergent.

Since $\{\ln n\}$ is increasing there $\left\{\frac{1}{\ln n}\right\}$ decreases.

We can apply the condensation test to check the behavior of the series.

Take $a_n = \frac{1}{\ln n}$, then $a_{2^n} = \frac{1}{\ln 2^n}$.

So $2^n a_{2^n} = \frac{2^n}{\ln 2^n} \quad \Rightarrow \quad 2^n a_{2^n} = \frac{2^n}{n\ln 2}$.

Since $\frac{2^n}{n} > \frac{1}{n} \qquad \forall \ n \ge 1$

and $\sum \frac{1}{n}$ is diverges therefore the given series is also diverges. ❑

Alternating Series

A series in which successive terms have opposite signs is called an alternating series.

Example:

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{ is an alternating series.}$$

Review:

- If $\{s_n\}$ is convergent to s, then every subsequence of $\{s_n\}$ converges to s.
- If $\sum a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.
- If a sequence is decreasing and bounded below then it is convergent.

Theorem (Alternating Series Test or Leibniz Test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim\limits_{n\to\infty} a_n = 0$ then the alternating series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$ converges.

Proof

Looking at the odd numbered partial sums of this series we find that

$$s_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \ldots + (a_{2n-1} - a_{2n}) + a_{2n+1}.$$

Since $\{a_n\}$ is decreasing therefore all the terms in the parenthesis are non-negative

$$\Rightarrow s_{2n+1} > 0 \quad \forall\ n.$$

Moreover

$$\begin{aligned} s_{2n+3} &= s_{2n+1} - a_{2n+2} + a_{2n+3} \\ &= s_{2n+1} - (a_{2n+2} - a_{2n+3}) \end{aligned}$$

Since $a_{2n+2} - a_{2n+3} \geq 0$ therefore $s_{2n+3} \leq s_{2n+1}$.

Hence the sequence of odd numbered partial sum is decreasing and is bounded below by zero. (as it has +ive terms)

It is therefore convergent.

Thus s_{2n+1} converges to some limit l (say).

Now consider the even numbered partial sum. We find that

$$s_{2n+2} = s_{2n+1} - a_{2n+2}$$

and

$$\begin{aligned} \lim_{n\to\infty} s_{2n+2} &= \lim_{n\to\infty}(s_{2n+1} - a_{2n+2}) \\ &= \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} a_{2n+2} = l - 0 = l \qquad \because \lim_{n\to\infty} a_n = 0. \end{aligned}$$

so that the even partial sum is also convergent to l.

$\Rightarrow$ both sequences of odd and even partial sums converge to the same limit.

Hence, we conclude that the corresponding series is convergent. ❑

Absolute Convergence

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

Review:

- A series $\sum a_n$ is convergent if and only if for any real number $\varepsilon > 0$, there exists a positive integer n_0 such that $\left|\sum_{i=m+1}^{n} a_i\right| < \varepsilon$ for all $n > m > n_0$.
- For all $a_i \in \mathbb{R}$, $i = 1, 2, ..., n$; $\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$.

Theorem

An absolutely convergent series is convergent.

Proof:

If $\sum |a_n|$ is convergent then by Cauchy criterion for convergence; for a real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left|\sum_{i=m+1}^{n} |a_i|\right| = \sum_{i=m+1}^{n} |a_i| < \varepsilon \quad \forall\ n, m > n_0. \ \ldots\ldots\ldots\ldots\text{(i)}$$

Also, we have

$$\left|\sum_{i=m+1}^{n} a_i\right| < \sum_{i=m+1}^{n} |a_i| \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

By using (i) and (ii), one has

$$\left|\sum_{i=m+1}^{n} a_i\right| < \varepsilon \quad \forall\ n,m > n_0 .$$

This implies the series $\sum a_n$ is convergent.

Note:

The converse of the above theorem does not hold.

e.g. $\sum \frac{(-1)^{n+1}}{n}$ is convergent but $\sum \frac{1}{n}$ is divergent. ❑

Question:

- Prove that every absolute convergent series is convergent, but convers is not true in general.

Review

- Let $x, y \in \mathbb{R}$ and $x < y$. Then there exist number r such that $x < r < y$.
- If $\lim\limits_{n\to\infty} a_n$ exists and $\lim\limits_{n\to\infty} a_n < l$, $l > 0$, then there exist positive integer n_0 such that $a_n < l$ for $n \geq n_0$.

Theorem (The Root Test)

Let $\lim\limits_{n\to\infty} |a_n|^{1/n} = p$.

Then $\sum a_n$ converges absolutely if $p < 1$ and it diverges if $p > 1$.

Proof

Let $p < 1$ then there exist real number r such that $p < r < 1$.

As we have $\lim\limits_{n\to\infty} |a_n|^{1/n} = p$, there is some n_0 so that

$$|a_n|^{1/n} < r \quad \forall\ n > n_0$$

$$\Rightarrow |a_n| < r^n < 1 \qquad \forall\ n > n_0 .$$

Since $\sum r^n$ is convergent because it is a geometric series with $|r| < 1$, therefore $\sum |a_n|$ is convergent.

$\Rightarrow \sum a_n$ converges absolutely.

Now let $p > 1$. Also we have $\lim\limits_{n\to\infty} |a_n|^{1/n} = p$, there is some n_0 so that

$\Rightarrow |a_n|^{1/n} > 1$ for $n \geq n_0$.

$\Rightarrow |a_n| > 1$ for $n \geq n_0$.

$\Rightarrow \lim\limits_{n\to\infty} |a_n| \neq 0 \Rightarrow \lim\limits_{n\to\infty} a_n \neq 0.$

$\Rightarrow \sum a_n$ is divergent. ❑

Note:

The above test gives no information when $p = 1$.

e.g. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

For each of these series; $p=1$, but $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent.

Theorem (Ratio Test)

The series $\sum a_n$

(i) Converges if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$. (ii) Diverges if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$.

Proof

If (i) holds we can find $\beta<1$ and integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for } n\geq N$$

In particular

$$\left|\frac{a_{N+1}}{a_N}\right| < \beta$$

$$\Rightarrow \left|a_{N+1}\right| < \beta\left|a_N\right|$$

$$\Rightarrow \left|a_{N+2}\right| < \beta\left|a_{N+1}\right| < \beta^2\left|a_N\right|$$

$$\Rightarrow \left|a_{N+3}\right| < \beta^3\left|a_N\right|$$

....

....

....

$$\Rightarrow \left|a_{N+p}\right| < \beta^p\left|a_N\right|$$

$$\Rightarrow \left|a_n\right| < \beta^{n-N}\left|a_N\right| \qquad \text{we put } N+p=n.$$

i.e. $\left|a_n\right| < \left|a_N\right|\beta^{-N}\beta^n$ for $n\geq N$.

Sine $\sum \beta^n$ is convergent because it is geometric series with common ratio less than 1, therefore $\sum a_n$ is convergent (by comparison test).

If (ii) holds, then we can find integer n_0 such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{for } n\geq n_0.$$

This gives

$$\left|a_{n+1}\right|\geq\left|a_n\right| \quad \text{for } n\geq n_0,$$

that is, the terms are getting larger and guaranteed to not be negative, therefore $\lim\limits_{n\to\infty}\left|a_n\right|\neq 0$. This provide us $\lim\limits_{n\to\infty}a_n\neq 0$.

$$\Rightarrow \sum a_n \text{ is divergent.}$$ ❑

Note:

The knowledge $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$ implies nothing about the convergent or divergent of series.

Example

Prove that series $\sum a_n$ with $a_n = \left[\dfrac{n}{n+1} - \left(\dfrac{n}{n+1}\right)^{n+1}\right]^{-n}$ is divergent.

Since $\dfrac{n}{n+1} < 1$, therefore $a_n > 0 \ \forall \ n$.

Also $(a_n)^{\frac{1}{n}} = \left[\dfrac{n}{n+1} - \left(\dfrac{n}{n+1}\right)^{n+1}\right]^{-1}$

$$= \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n}{n+1}\right)^{n}\right]^{-1} = \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n+1}{n}\right)^{-n}\right]^{-1}$$

$$= \left(1 + \frac{1}{n}\right)\left[1 - \left(1 + \frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)\left[1 - \left(1 + \frac{1}{n}\right)^{-n}\right]^{-1} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) \lim_{n\to\infty}\left[1 - \left(1 + \frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= 1\cdot\left[1 - e^{-1}\right]^{-1} = \left[1 - \frac{1}{e}\right]^{-1} = \left[\frac{e-1}{e}\right]^{-1} = \frac{e}{e-1} > 1.$$

This implies the given series is divergent. ❑

Dirichlet's Theorem

Suppose that

(i) $\{s_n\}$, $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ is bounded and

(ii) $\{b_n\}$ be positive term decreasing sequence such that $\lim\limits_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ is convergent.

Proof

Since $\{s_n\}$ is bounded, therefore, there exists a positive number λ such that

$$|s_n| \le \lambda \qquad \forall \ n \ge 1.$$

Then $a_i b_i = (s_i - s_{i-1}) b_i$ for $i \ge 2$

$$= s_i b_i - s_{i-1} b_i$$
$$= s_i b_i - s_{i-1} b_i + s_i b_{i+1} - s_i b_{i+1}$$
$$= s_i (b_i - b_{i+1}) - s_{i-1} b_i + s_i b_{i+1}$$

$$\Rightarrow \sum_{i=m+1}^{n} a_i b_i = \sum_{i=m+1}^{n} s_i (b_i - b_{i+1}) - (s_m b_{m+1} - s_n b_{n+1})$$

Since $\{b_n\}$ is positive term decreasing,

therefore $\left|\sum_{i=m+1}^{n} a_i b_i\right| = \left|\sum_{i=m+1}^{n} s_i (b_i - b_{i+1}) - s_m b_{m+1} + s_n b_{n+1}\right|$

$$\le \sum_{i=m+1}^{n} \{|s_i|(b_i - b_{i+1})\} + |s_m| b_{m+1} + |s_n| b_{n+1}$$

$$\le \sum_{i=m+1}^{n} \{\lambda(b_i - b_{i+1})\} + \lambda b_{m+1} + \lambda b_{n+1} \qquad \because\ |s_i| \le \lambda$$

$$= \lambda\left(\sum_{i=m+1}^{n} (b_i - b_{i+1}) + b_{m+1} + b_{n+1}\right)$$

$$= \lambda\big((b_{m+1} - b_{n+1}) + b_{m+1} + b_{n+1}\big) = 2\lambda b_{m+1} < 2\lambda b_{m+1} + 1.$$

$\Rightarrow \left|\sum_{i=m+1}^{n} a_i b_i\right| < \varepsilon$, where $\varepsilon = 2\lambda b_{m+1} + 1$ a certain number

$\Rightarrow$ The $\sum a_n b_n$ is convergent. (We have use Cauchy criterion here.) ❑

Theorem

Suppose that

(i) $\sum a_n$ is convergent and

(ii) $\{b_n\}$ is monotonic convergent sequence,

then $\sum a_n b_n$ is also convergent.

Proof

Suppose $\{b_n\}$ is decreasing and it converges to b.

Put $c_n = b_n - b$ for all n.

$\Rightarrow c_n \ge 0$ and $\lim\limits_{n\to\infty} c_n = 0$.

Since $\sum a_n$ is convergent,

therefore $\{s_n\}$, $s_n = a_1 + a_2 + \ldots + a_n$ is convergent, that is, $\{s_n\}$ is bounded.

By Dirichlet's theorem, we have $\sum a_n c_n$ is convergent.

Since $a_n b_n = a_n c_n + a_n b$ and $\sum a_n c_n$ and $\sum a_n b$ are convergent,

therefore $\sum a_n b_n$ is convergent.

Now if $\{b_n\}$ is increasing and converges to *b* then we shall put $c_n = b - b_n$. ❑

Example

A series $\sum \frac{1}{(n\ln n)^\alpha}$ is convergent if $\alpha > 1$ and divergent if $\alpha \le 1$.

To see this we proceed as follows

$$a_n = \frac{1}{(n\ln n)^\alpha}$$

Take $b_n = 2^n a_{2^n} = \dfrac{2^n}{\left(2^n \ln 2^n\right)^\alpha} = \dfrac{2^n}{\left(2^n\ n \ln 2\right)^\alpha}$

$$= \frac{2^n}{2^{n\alpha} n^\alpha (\ln 2)^\alpha} = \frac{1}{2^{n\alpha - n} n^\alpha (\ln 2)^\alpha}$$

$$= \frac{1}{(\ln 2)^\alpha} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n^\alpha}$$

Since $\sum \frac{1}{n^\alpha}$ is convergent when $\alpha > 1$ and $\left(\frac{1}{2}\right)^{(\alpha-1)n}$ is decreasing for $\alpha > 1$ and it converges to 0.

Therefore $\sum b_n$ is convergent

$\Rightarrow \sum a_n$ is also convergent.

Now $\sum b_n$ is divergent for $\alpha \le 1$ therefore $\sum a_n$ diverges for $\alpha \le 1$. ❑

Example

To check $\sum \frac{1}{n^\alpha \ln n}$ is convergent or divergent.

We have $a_n = \frac{1}{n^\alpha \ln n}$

Take $$b_n = 2^n a_{2^n} = \frac{2^n}{(2^n)^\alpha (\ln 2^n)} = \frac{2^n}{2^{n\alpha}(n \ln 2)}$$

$$= \frac{1}{\ln 2} \cdot \frac{2^{(1-\alpha)n}}{n} = \frac{1}{\ln 2} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n}$$

$\because \sum \frac{1}{n}$ is divergent although $\left\{\left(\frac{1}{2}\right)^{n(\alpha-1)}\right\}$ is decreasing, tending to zero for $\alpha > 1$ therefore $\sum b_n$ is divergent.

$\Rightarrow \sum a_n$ is divergent.

The series also divergent if $\alpha \le 1$.
i.e. it is always divergent. ❑

References:

[1] Principles of Mathematical Analysis by Walter Rudin (McGraw-Hill, Inc.)
[2] Introduction to Real Analysis by R.G.Bartle, and D.R. Sherbert (John Wiley & Sons, Inc.)
[3] Mathematical Analysis by Tom M. Apostol, (Pearson; 2nd edition.)
[4] Real Analysis by Dipak Chatterjee (PHI Learning, 2nd edition.)

Chapter 04

Limit & Continuity

*Limits define where we're he422ed, and continuity ensures the journey there is smooth --- **Dr. Ben Carter, Data Scientist***

In this chapter we will introduce the important notion of the limit of a function. The intuitive idea of the function f having a limit L at the point a is that the values $f(x)$ are close to L when x is close to (but different from) a. But it is necessary to have a technical way of working with the idea of "close to" and this is accomplished in the $\varepsilon-\delta$ definition given below.
In order for the idea of the limit of a function f at a point a to be meaningful, it is necessary that f be defined at points near a. It need not be defined at the point a, but it should be defined at enough points close to a to make the study interesting. This is the reason for the following definition.

❖ LIMIT OF THE FUNCTION

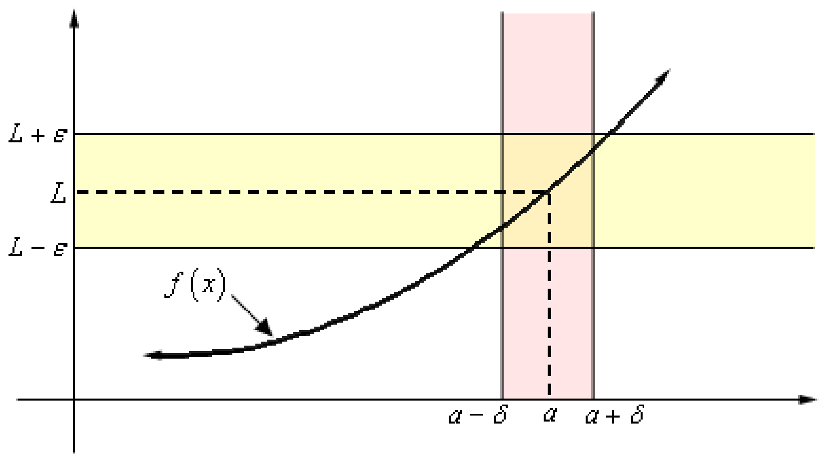
Definition: Suppose $E\subseteq\mathbb{R}$ and $f:E\to\mathbb{R}$ be a function. A number L is called the limit of f when x approaches to a if for all $\varepsilon>0$, there exists $\delta>0$ (depending upon ε) such that

$$|f(x)-L|<\varepsilon \text{ whenever } 0<|x-a|<\delta.$$

Notation: It is written as $\lim\limits_{x\to a}f(x)=L$.

Note: *i*) It is to be noted that $a\in\mathbb{R}$ but that a need not a point of E in the above definition (a is a limit point of E which may or may not belong to E.)
ii) Even if $a\in E$, we may have $f(a)\neq\lim\limits_{x\to a}f(x)$.

Example:
In the following diagram we have illustrated $\lim\limits_{x\to a}f(x)=L$.



What the definition is telling us is that for any number $\varepsilon>0$ that we pick we can go to our graph and sketch two horizontal lines at $L+\varepsilon$ and $L-\varepsilon$ as shown on the graph above. Then somewhere out there in the world is another number $\delta>0$, which we will need to determine, that will allow us to add in two vertical lines to our graph at $a+\delta$ and $a-\delta$.

❖ ***Example***

(i) Consider the function $f(x)=\dfrac{x^2-1}{x-1}$, $x\neq 1$.

It is to be noted that f is not defined at $x=1$ but if $x\neq 1$ and is very close to 1, then $f(x)$ is close to 2.

To check limit of $f(x)\to 2$ as $x\to 1$, let's start off by letting $\varepsilon>0$ be any number then we need to find a number $\delta>0$ so that the following will be true.

$$\left|\frac{x^2-1}{x-1}-2\right|<\varepsilon \text{ whenever } 0<|x-1|<\delta.$$

We'll start by simplifying the left inequality in an attempt to get a guess for δ. Doing this gives,

$$\left|\frac{x^2-1}{x-1}-2\right|=|x+1-2|=|x-1|<\varepsilon \quad \text{implies } 0<|x-1|<\delta=\varepsilon. \qquad \square$$

(ii) Lets see by definition: $\lim\limits_{x\to 2}(5x-4)=6$.

Let's start off by letting $\varepsilon>0$ be any number then we need to find a number $\delta>0$ so that the following will be true.

$$|(5x-4)-6|<\varepsilon \text{ whenever } 0<|x-2|<\delta.$$

We'll start by simplifying the left inequality in an attempt to get a guess for δ. Doing this gives,

$$|(5x-4)-6|=|5x-4-6|=|5x-10|=5|x-2|<\varepsilon \quad \text{implies } 0<|x-2|<\delta=\frac{\varepsilon}{5}. \qquad \square$$

Note: Today, we have developed lot of tools to find the limit of functions without using the definition (even without knowing the limit). Here our aim is to understand the limit by definition.

If the definition of limit is violated or leads to something absurd even by choosing one value of ε, then we say limit doesn't exist.

❖ ***Example***

$$\lim_{x\to 0}\sin\frac{1}{x} \quad \text{does not exist.}$$

Suppose that $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ exists and take it to be l, then there exist a positive real number δ such that

$$\left|\sin\frac{1}{x}-l\right|<1 \quad \text{when} \quad 0<|x-0|<\delta \quad (\text{we take here } \varepsilon=1>0)$$

We can find a positive integer n such that

$$\frac{2}{n\pi}<\delta \quad \text{then} \quad \frac{2}{(4n+1)\pi}<\delta \quad \text{and} \quad \frac{2}{(4n+3)\pi}<\delta.$$

It thus follows

$$\left|\sin\frac{(4n+1)\pi}{2}-l\right|<1 \quad \Rightarrow \quad |1-l|<1$$

$$\text{and} \quad \left|\sin\frac{(4n+3)\pi}{2}-l\right|<1 \quad \Rightarrow \quad |-1-l|<1 \quad \text{or} \quad |1+l|<1.$$

So that

$$2 = |1+l+1-l| \le |1+l| + |1-l| < 1+1$$
$$\Rightarrow\ 2 < 2.$$

This is impossible; hence limit of the function does not exist. ❑

❖ ***Example***

Consider the function $f:[0,1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irratioanl} \end{cases}$$

Show that $\lim\limits_{x\to p} f(x)$, where $p \in [0,1]$ does not exist.

Solution

On the contrary, suppose that $\lim\limits_{x\to p} f(x) = q$.

Then for given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|f(x) - q| < \varepsilon \quad \text{whenever} \quad 0 < |x - p| < \delta.$$

Consider two points r and s from interval $(p-\delta, p+\delta) \subset [0,1]$ such that r is rational and s is irrational.

Then $f(r) = 0$ & $f(s) = 1$.

Now

$$\begin{aligned} 1 = |f(s)| &= |f(s) - q + q| \\ &= |(f(s) - q + q - 0| \\ &= |f(s) - q + q - f(r)| \qquad (\text{since } 0 = f(r)). \\ &\le |f(s) - q| + |f(r) - q| < \varepsilon + \varepsilon. \end{aligned}$$

i.e. $1 < 2\varepsilon$

In particular, if we take $\varepsilon = \dfrac{1}{4}$, then $1 < \dfrac{1}{2}$.

This is absurd.

Hence the limit of the function does not exist. ❑

❖ ***Theorem***

If $\lim\limits_{x\to c} f(x)$ exists, then it is unique.

Proof

Suppose $\lim\limits_{x\to c} f(x)$ is not unique.

Take $\lim\limits_{x\to c} f(x) = l_1$ and $\lim\limits_{x\to c} f(x) = l_2$, where $l_1 \ne l_2$.

So for $\varepsilon > 0$, there exists real numbers δ_1 and δ_2 such that

$$|f(x) - l_1| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x - c| < \delta_1$$

$$\& \quad |f(x) - l_2| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x - c| < \delta_2.$$

Now $$\begin{aligned} |l_1 - l_2| &= |(f(x) - l_1) - (f(x) - l_2)| \\ &\le |f(x) - l_1| + |f(x) - l_2| \end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \quad \text{whenever} \quad |x-c| < \min(\delta_1, \delta_2).$$

That is, $0 \le |l_1 - l_2| < \varepsilon$ for all $\varepsilon > 0$.

$$\Rightarrow \; l_1 - l_2 = 0 \quad \text{or} \quad l_1 = l_2 .$$ ❑

❖ RIGHT HAND LIMIT OF THE FUNCTION

Definition: Suppose $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function. If for all $\varepsilon > 0$, there exists $\delta > 0$ (depending upon ε) such that

$$|f(x) - L| < \varepsilon \text{ whenever } a < x < a + \delta,$$

Then *L* is called right hand limit of function *f* at *a*.

Notation: It is written as $\lim\limits_{x \to a+} f(x) = L$.

❖ LEFT HAND LIMIT OF THE FUNCTION

Definition: Suppose $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function. If for all $\varepsilon > 0$, there exists $\delta > 0$ (depending upon ε) such that

$$|f(x) - L| < \varepsilon \text{ whenever } a - \delta < x < a,$$

Then *L* is called left hand limit of function *f* at *a*.

Notation: It is written as $\lim\limits_{x \to a-} f(x) = L$.

Remark: One can easily prove that if the right hand limit or left hand limit of the function exists then it is unique.

Examples:

(i) Consider a function $f(x) = \dfrac{|\sin x|}{\sin x}$ for $x \in \mathbb{R}$.

It is easy to see that $\lim\limits_{x\to 0+} \dfrac{|\sin x|}{\sin x} = 1$, but $\lim\limits_{x \to 0-} \dfrac{|\sin x|}{\sin x} = -1$.

(ii) Suppose

$$f(x) = \begin{cases} 2x+1, & x < 1; \\ 5 & x = 1; \\ 7x^2 - 4 & x > 1. \end{cases}$$

To compute $\lim\limits_{x \to 1+} f(x)$, we use the part of the definition for *f* which applies to $x > 1$, so

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2x+1) = 3.$$

To compute $\lim\limits_{x \to 1-} f(x)$, we use the part of the definition for *f* which applies to $x < 1$, so

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (7x^2 - 4) = 3.$$

Note that $\lim\limits_{x \to 1+} f(x) = \lim\limits_{x \to 1-} f(x) = 3$, but $f(1) = 5$. ❑

The proof of the following theorem can be seen in FSc or BSc mathematics book.

❖ *Theorem*

Suppose f is a function define on E may not containing point a. Then

$$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = \lim_{x\to a} f(x).$$

❖ LIMIT AS A INFINITY

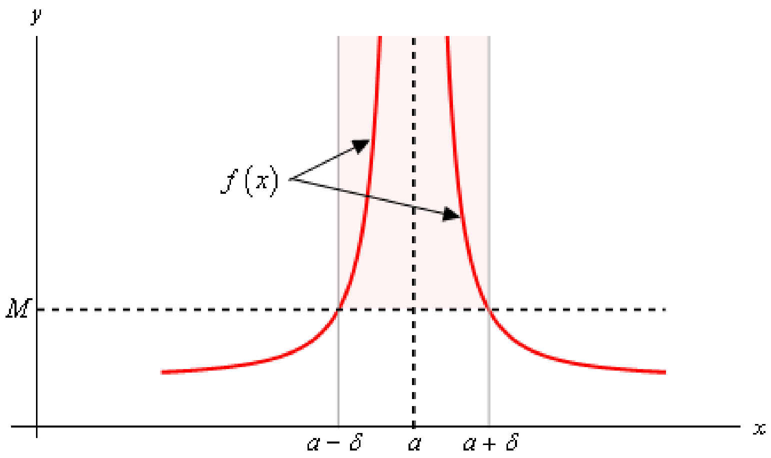
Definition: Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that

$$\lim_{x\to a} f(x) = \infty$$

if for every number $M > 0$, there is some number $\delta > 0$ such that

$$f(x) > M \text{ whenever } 0 < |x - a| < \delta.$$

Above definitions is telling us that no matter how large we choose M to be we can always find an interval around $x = a$, given by $0 < |x - a| < \delta$ for some number $\delta > 0$, so that as long as we stay within that interval the graph of the function will be above the line $y = M$ as shown in the graph.



Similarly, one can define limit as negative infinity.

❖ LIMIT AS NEGATIVE INFINITY:

Definition: Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that

$$\lim_{x\to a} f(x) = -\infty$$

if for every number $N < 0$, there is some number $\delta > 0$ such that

$$f(x) < N \text{ whenever } 0 < |x - a| < \delta.$$

❖ *Example*

Use the definition of the limit to prove the following limit.

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Solution:

Let $M > 0$ be any number and we'll need to choose a δ so that,

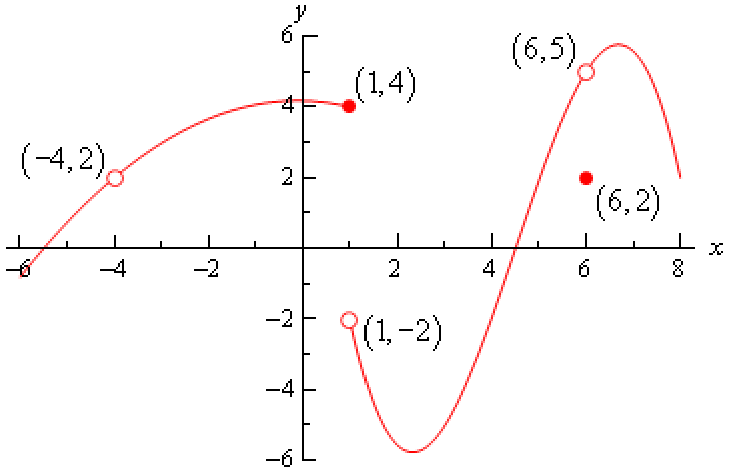
$$\frac{1}{x^2} > M \quad \text{whenever} \quad 0 < |x - 0| = |x| < \delta.$$

We take

$$\frac{1}{x^2} > M \quad \Rightarrow \quad x^2 < \frac{1}{M}$$

$$\Rightarrow \quad |x| < \frac{1}{\sqrt{M}} = \delta. \qquad \square$$

Exercise: Given the following graph of function f :



(a) $f(-4)$ (b) $\lim\limits_{x\to-4-} f(x)$ (c) $\lim\limits_{x\to-4+} f(x)$ (d) $\lim\limits_{x\to-4} f(x)$

(e) $f(1)$ (f) $\lim\limits_{x\to1-} f(x)$ (g) $\lim\limits_{x\to1+} f(x)$ (h) $\lim\limits_{x\to1} f(x)$

(i) $f(6)$ (j) $\lim\limits_{x\to6-} f(x)$ (k) $\lim\limits_{x\to6+} f(x)$ (l) $\lim\limits_{x\to6} f(x)$

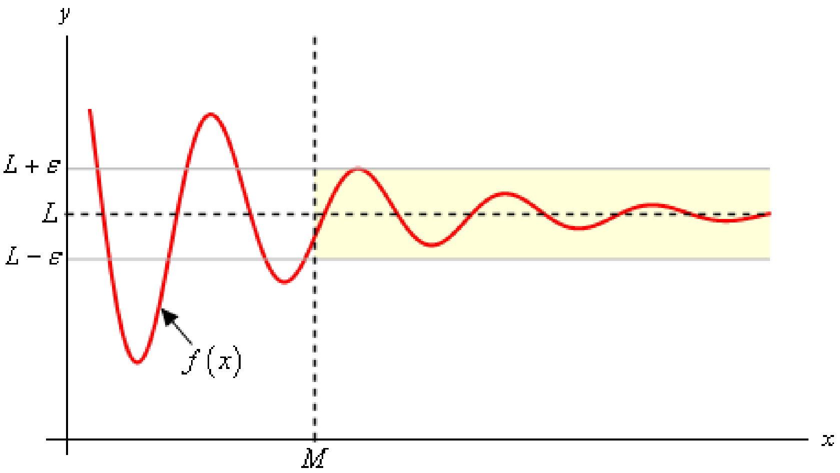
❖ **LIMIT AT INFINITY**

Definition: Let X and Y be subsets of $\mathbb{R}$. A function $f : X \to Y$ is said to tend to limit L as $x \to \infty$, if for a real number $\varepsilon > 0$ however small, there exists a positive number M which depends upon ε such that distance

$$|f(x) - L| < \varepsilon \quad \text{when } x > M.$$

Notation: This is written as $\lim\limits_{x\to\infty} f(x) = L$.

Above definition tells us that no matter how close to L we want to get, mathematically this is given by $|f(x) - L| < \varepsilon$ for any chosen $\varepsilon > 0$, we can find another number M such that provided we take any x bigger than M, then the graph of the function for that x will be closer to L than $L-\varepsilon$ and $L+\varepsilon$.



Similarly, one can define limit at negative infinity.

❖ LIMIT AT NEGATIVE INFINITY

Definition: Let X and Y be subsets of $\mathbb{R}$. A function $f: X \to Y$ is said to tend to limit L as $x \to -\infty$, if for a real number $\varepsilon > 0$ however small, there exists a positive number N which depends upon ε such that distance

$$\left| f(x) - L \right| < \varepsilon \quad \text{when } x < N.$$

Notation: This is written as $\lim\limits_{x\to\infty} f(x) = L$.

❖ *Example*

By definition, prove that $\lim\limits_{x\to\infty} \dfrac{2x}{1+x} = 2$.

We have $\left| \dfrac{2x}{1+x} - 2 \right| = \left| \dfrac{2x - 2 - 2x}{1+x} \right| = \left| \dfrac{-2}{1+x} \right| < \dfrac{2}{x}$.

Now if $\varepsilon > 0$ is given we can find $M = \dfrac{2}{\varepsilon}$ so that

$$\left| \frac{2x}{1+x} - 2 \right| < \varepsilon \quad \text{whenever} \quad x > M = \frac{2}{\varepsilon}.$$

❑

The following theorem is very useful to find the limit of different function. Here we are not giving the proof as one can found it in the mathematics book of FSc.

❖ *Theorem*

Let $f: E \to \mathbb{R}$ and $g: E \to \mathbb{R}$ be real valued functions. If $\lim\limits_{x\to p} f(x) = A$ and $\lim\limits_{x\to p} g(x) = B$ then

i- $\lim\limits_{x\to p} \big(f(x) \pm g(x) \big) = A \pm B$,

ii- $\lim\limits_{x\to p} (fg)(x) = AB$, iii- $\lim_{x\to p}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{A}{B}$, provided $B\neq 0$.

❖ **CONTINUITY**

***Definition*:** Suppose $E\subseteq\mathbb{R}$ and $f:E\to\mathbb{R}$ be a function. Then f is said to be continuous at p if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\left|f(x)-f(p)\right|<\varepsilon \text{ for all points } x\in E \text{ for which } 0<\left|x-p\right|<\delta.$$

***Definition*:** If f is continuous at every point of E, then f is said to be continuous on E.

Note: Comparing the definition of continuity with the definition of the limit, It is to be noted that f has to be continuous at p iff $\lim\limits_{x\to p}f(x)=f(p)$.

❖ ***Examples***

A function $f(x)=x^2$ is continuous for all $x\in\mathbb{R}$.

Here $f(x)=x^2$. Take $p\in\mathbb{R}$ and $\varepsilon>0$.

Then we have to show

$$\left|f(x)-f(p)\right|<\varepsilon \;\Rightarrow\; \left|x^2-p^2\right|<\varepsilon \text{ whenever } \left|x-p\right|<\delta.$$

Now
$$\begin{aligned}\left|x^2-p^2\right| &=\left|(x-p)(x+p)\right|\\ &=\left|(x-p)(x-p+2p)\right|\\ &\leq\left|x-p\right|\left(\left|x-p\right|+2\left|p\right|\right)\end{aligned}$$

Now if $\left|x-p\right|<\delta$, then we have

$$\begin{aligned}\left|x^2-p^2\right| &\leq \left|x-p\right|\left(\left|x-p\right|+2\left|p\right|\right)\\ &<\delta\left(\delta+2\left|p\right|\right)=\varepsilon.\end{aligned}$$

Since p is arbitrary real number,

therefore, the function $f(x)$ is continuous for all real numbers. ❑

❖ ***Example***

A function $f(x)=\sqrt{x}$ is continuous on $\left[0,\infty\right[$.

Let c be an arbitrary point such that $0<c<\infty$

For $\varepsilon>0$, we have

$$\begin{aligned}\left|f(x)-f(c)\right| &=\left|\sqrt{x}-\sqrt{c}\right|\\ &=\frac{\left|x-c\right|}{\sqrt{x}+\sqrt{c}}<\frac{\left|x-c\right|}{\sqrt{c}}\end{aligned}$$

$$\Rightarrow \left|f(x)-f(c)\right|<\varepsilon \quad\text{whenever}\quad \frac{\left|x-c\right|}{\sqrt{c}}<\varepsilon$$

i.e. $\left|x-c\right|<\sqrt{c}\,\varepsilon=\delta$

$\Rightarrow$ f is continuous for $x=c$.

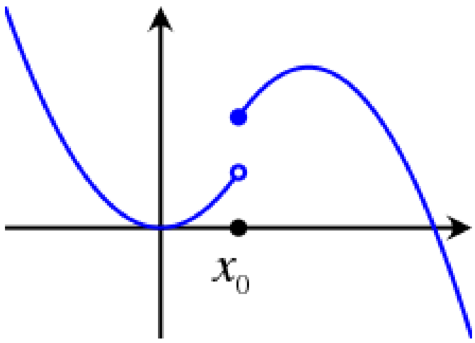
$\because$ c is an arbitrary point lying in $\left[0,\infty\right[$

$\therefore \ f(x)=\sqrt{x}$ is continuous on $[0,\infty[$ ❑

❖ **RIGHT CONTINUOUS AND LEFT CONTINUOUS**

***Definition*:** Let f be a real valued function. It is said to be right continuous at point a if $\lim\limits_{x\to a+} f(x)=f(a)$ and it is said to be left continuous at point a if $\lim\limits_{x\to a-} f(x)=f(a)$.

❖ ***Example***



Consider a function given in above graph. We see f is not continuous at point x_0. It is right continuous at point x_0 but not left continuous at point x_0.

❖ ***Example***

Let

$$f(x)=\begin{cases} x^2+1 \ \ if \ \ x\le 2, \\ \dfrac{x^2-4}{x-2} \ \ if \ \ x>2. \end{cases}$$

Then f is left continuous at 2 but it is not right continuous at 2.

❖ **RIGHT CONTINUOUS AND LEFT CONTINUOUS**

***Definition*:** A function $f:[a,b]\to\mathbb{R}$ is said to be continuous on closed interval $[a,b]$ if

f is continuous on (a,b)

f is right continuous at a.

f is left continuous at b.

❖ ***Theorem (The intermediate value theorem)***

Suppose f is continuous on $[a,b]$ and $f(a)\neq f(b)$, then given a number λ that lies between $f(a)$ and $f(b)$, there exist a point $c\in(a,b)$ with $f(c)=\lambda$.

Proof

Without loss of generality, we can consider $f(a)<f(b)$ and $f(a)<\lambda<f(b)$.

Also let $S=\{x\in[a,b] \mid f(x)<\lambda\}$. Then S is non-empty as $a\in S$ and b is an upper bound of S. Since we are dealing with the set of real numbers, therefore supremum of S exist in $\mathbb{R}$, say $c=\sup S$.

Since f is continuous on $[a,b]$, in particular at $x=c$, therefore for all $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|f(x)-f(c)\right|<\varepsilon \text{ whenever } 0<\left|x-c\right|<\delta.$$

This means that

$$f(x) - \varepsilon < f(c) < f(x) + \varepsilon \quad \text{for all } x \text{ between } c-\delta \text{ and } c+\delta.$$

By the properties of the supremum, there exist x_1 between $c - \delta$ and c that is contained in S, so that

$$f(c) < f(x_1) + \varepsilon < \lambda + \varepsilon. \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Choose x_2 between c and $c + \delta$. Then $x_2 \notin S$, so we have

$$f(c) > f(x_2) - \varepsilon \geq \lambda - \varepsilon. \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

From (i) and (ii), we have for all $\varepsilon > 0$,

$$\lambda - \varepsilon < f(c) < \lambda + \varepsilon.$$

$$\Rightarrow \quad |f(c) - \lambda| < \varepsilon$$

So ultimately, we have $\quad f(c) = \lambda$. ❑

❖ **UNIFORM CONTINUITY**

Definition: Suppose $f : E \to \mathbb{R}$ is a real valued function. We say that f is uniformly continuous on E if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(p) - f(q)| < \varepsilon \quad \forall \quad p, q \in E \quad \text{for which} \quad |p - q| < \delta.$$

The uniform continuity is a property of a function on a set, that is, it is a global property but continuity can be defined at a single point i.e. it is a local property.

Uniform continuity of a function at a point has no meaning.

It is evident that every uniformly continuous function is continuous.

To emphasize a difference between continuity and uniform continuity on set S, we consider the following examples.

❖ ***Example***

Let S be a half open interval $0 < x \leq 1$ and let $f(x)$ be defined for each x in S by the formula $f(x) = x^2$. It is uniformly continuous on S. To prove this, assume $x, y \in (0,1]$ and take

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x - y||x + y| \\ &< 2|x - y| \end{aligned}$$

If $|x - y| < \delta$ then $|f(x) - f(y)| < 2\delta = \varepsilon$

Hence if ε is given we need only to take $\delta = \dfrac{\varepsilon}{2}$ to guarantee that

$$|f(x) - f(y)| < \varepsilon \text{ for every pair } x, y \text{ with } |x - y| < \delta$$

Thus f is uniformly continuous on the set S. ❑

❖ ***Example***

Let S be the half open interval $0 < x \leq 1$ and let a function f be defined for each x in S by the formula $f(x) = \dfrac{1}{x}$. This function is continuous on the set S, however we shall prove that this function is not uniformly continuous on S.

Solution

Let suppose $\varepsilon = 10$ and suppose we can find a δ, $0 < \delta < 1$, to satisfy the condition of the definition.

Taking $x = \delta$, $y = \dfrac{\delta}{11}$, we obtain

$$|x - y| = \frac{10\delta}{11} < \delta$$

and

$$|f(x) - f(y)| = \left|\frac{1}{\delta} - \frac{11}{\delta}\right| = \frac{10}{\delta} > 10$$

Hence for these two points we have $|f(x) - f(y)| > 10$.

This contradict the definition of uniform continuity.

Hence the given function being continuous on a set S is not uniformly continuous on S.

❑

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Chapter 05

Differentiation

*Differentiation: the art of knowing the next step, precisely --- **Dr. Ben Carter, Data Scientist***

Differentiation allows us to find rates of change. For example, it allows us to find the rate of change of velocity with respect to time (which is acceleration). Calculus courses succeed in conveying an idea of what a derivative is, and the students develop many technical skills in computations of derivatives or applications of them. We shall return to the subject of derivatives but with a different objective.
Now we wish to see a little deeper and to understand the basis on which that theory develops.
Let f be defined and real valued on (a,b). For any point $c\in(a,b)$, form the quotient

$$\frac{f(x)-f(c)}{x-c}.$$

We fix point c and study the behaviour of this quotient as $x\to c$.

❖ DERIVATIVE OF A FUNCTION

Definition: Let f be defined on an open interval (a,b), and assume that $c\in(a,b)$. Then f is said to be differentiable at c whenever the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists. This limit is denoted by $f'(c)$ and is called the derivative of f at point c.

Definition: If f is differentiable at each point of (a,b), then we say f is differentiable on (a,b).

❖ *Remarks*

- There are so many notations to represents the derivative of the function in the literature.
- If $x-c=h$, then we have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}.$$

❖ *Example*

(i) A function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ;\ x\neq 0\\ 0 & ;\ x=0\end{cases}$$

This function is differentiable at $x=0$ because

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}$$
$$=\lim_{x\to 0}x\sin\tfrac{1}{x}=0.$$

(ii) Let $f(x)=x^n$; $n\geq 0$ (n is integer), $x\in\mathbb{R}$.
Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to c}\frac{x^n-c^n}{x-c}\\ &= \lim_{x\to c}\frac{(x-c)(x^{n-1}+cx^{n-2}+\ldots+c^{n-2}x+c^{n-1})}{x-c}\\ &= \lim_{x\to c}(x^{n-1}+cx^{n-2}+\ldots+c^{n-2}x+c^{n-1})\\ &= nc^{n-1}.\end{aligned}$$

This implies that f is differentiable every where and $f'(x)=nx^{n-1}$.

❖ ***Theorem*** (Differentiability implies continuity)

Let f be defined on (a,b), if f is differentiable at a point $x\in(a,b)$, then f is continuous at x.

Proof

We know that

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=f'(x), \text{ where } t\neq x \quad \text{and} \quad a<t<b.$$

Now

$$\begin{aligned}\lim_{t\to x}\big(f(t)-f(x)\big) &= \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}(t-x)\\ &= f'(x)\cdot 0\\ &= 0\end{aligned}$$

$$\Rightarrow \lim_{t\to x}f(t)=f(x).$$

This show that f is continuous at x. ❑

❖ ***Remarks***

(i) The converse of the above theorem does not hold.

Consider $f(x)=|x|=\begin{cases} x & \text{if } x\geq 0,\\ -x & \text{if } x<0.\end{cases}$

Then $f'(0)$ does not exists but $f(x)$ is continuous at $x=0$.

(ii) If f is discontinuous at some point c of the domain of the function then $f'(c)$ does not exist. e.g.

$$f(x)=\begin{cases} 1 & \text{if } x\geq 0,\\ 0 & \text{if } x<0.\end{cases}$$

A function f is discontinuous at $x=0$ therefore it is not differentiable at $x=0$.

❖ ***Question***

Prove that a differentiable function is continuous, but the converse is not true.

❖ ***Theorem***

Suppose f and g are defined on (a,b) and are differentiable at a point $x\in(a,b)$, then $f+g$, fg and $\dfrac{f}{g}$ are differentiable at x and

(*i*) $(f+g)'(x)=f'(x)+g'(x)$,

(*ii*) $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$,

(*iii*) $\left(\dfrac{f}{g}\right)'(x)=\dfrac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$, proved $g(x)\neq 0$.

The proof of this theorem can be get from any F.Sc or B.Sc textbook.

❖ ***Remark***

As we know $\lim\limits_{t\to x}\dfrac{f(t)-f(x)}{t-x}=f'(x)$, this gives

$$\frac{f(t)-f(x)}{t-x}=f'(x)+u(t),$$

where $u(t)$ is a function such that $u(t)\to 0$ as $t\to x$.

This gives us $f(t)-f(x)=(t-x)\left[f'(x)+u(t)\right]$, where $u(t)\to 0$ as $t\to x$, as an alternative definition of derivative.

❖ ***Theorem (Chain Rule)***

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x\in(a,b)$. A function g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t)=g\left(f(t)\right)$; $a\le t\le b$, then h is differentiable at x and

$$h'(x)=g'\left(f(x)\right)\cdot f'(x).$$

Proof

Let $y=f(x)$.

By the definition of the derivative, we have

$$f(t)-f(x)=(t-x)\left[f'(x)+u(t)\right] \quad \ldots\ldots\ldots..(i)$$

and $$g(s)-g(y)=(s-y)\left[g'(y)+v(s)\right] \quad \ldots\ldots\ldots..(ii)$$

where $t\in[a,b]$, $s\in I$ and $u(t)\to 0$ as $t\to x$ and $v(s)\to 0$ as $s\to y$.

Let us suppose $s=f(t)$. Then

$$\begin{aligned}h(t)-h(x)&=g\left(f(t)\right)-g\left(f(x)\right)=g(s)-g(y)\\&=\left[s-y\right]\left[g'(y)+v(s)\right] \qquad \text{by } (ii)\\&=\left[f(t)-f(x)\right]\left[g'(y)+v(s)\right]\\&=(t-x)\left[f'(x)+u(t)\right]\left[g'(y)+v(s)\right] \qquad \text{by } (i)\end{aligned}$$

or if $t\neq x$

$$\frac{h(t)-h(x)}{t-x}=\left[f'(x)+u(t)\right]\left[g'(y)+v(s)\right],$$

taking the limit as $t\to x$ we have

$$\begin{aligned}h'(x)&=\left[f'(x)+0\right]\left[g'(y)+0\right]\\&=g'\left(f(x)\right)\cdot f'(x), \qquad \because\ y=f(x)\end{aligned}$$

This is the required result. ❑

❖ ***Example***

Let us find the derivative of $\sin(2x)$, One way to do that is through some trigonometric identities. Indeed, we have

$$\sin(2x) = 2\sin(x)\cos(x)\cdot$$

So we will use the product formula to get

$$\left(\sin(2x)\right)' = 2\left(\sin'(x)\cos(x) + \sin(x)\cos'(x)\right)$$

which implies

$$\left(\sin(2x)\right)' = 2\left(\cos^2(x) - \sin^2(x)\right)\cdot$$

Using the trigonometric formula $\cos(2x) = \cos^2(x) - \sin^2(x)$, we have

$$\left(\sin(2x)\right)' = 2\cos(2x)\cdot$$

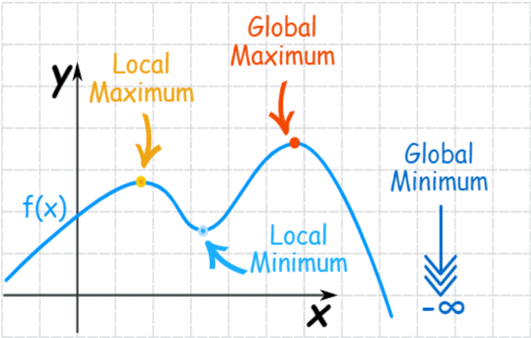
Once this is done, you may ask about the derivative of $\sin(5x)$? The answer can be found using similar trigonometric identities, but the calculations are not as easy as before. We will see how the Chain Rule formula will answer this question in an elegant way.
Let us find the derivative of $\sin(5x)\cdot$

We have $h(x) = f\left(g(x)\right)$, where $g\left(x\right) = 5x$ and $f(x) = \sin x$. Then the Chain rule implies that $h'(x)$ exists and

$$h'(x) = 5\cdot\left[\cos(5x)\right] = 5\cos(5x)\cdot$$

❖ ***Maxima and Minima of Functions***

Maxima and minima of a function are the largest and smallest value of the function respectively either within a given range or on the entire domain. Collectively they are also known as extrema of the function. The maxima and minima are the respective plurals of maximum and minimum of a function. Before understanding maxima and minima in detail, let's understand the local maximum and minimum value of the function first.



❖ **LOCAL MAXIMUM**

Definition: Let f be a real valued function defined on a set $E \subseteq \mathbb{R}$, we say that f has a local maximum at a point $p \in E$ if there exist $\delta > 0$ such that $f(x) \le f(p)$ for all $x \in E$ with $\left|x - p\right| < \delta$.

Local minimum is defined likewise.

❖ **GLOBAL (OR ABSOLUTE) MAXIMUM AND MINIMUM**

Definition: The maximum or minimum over the entire domain of the function is called an "global" or "absolute" maximum or minimum.

❖ ***Remark:*** There might be only one global maximum (and one global minimum) but there can be more than one local maximum or minimum.

❖ ***Theorem***

Let f be defined on $[a,b]$ and it is differentiable on (a,b). If f has a local maximum at a point $x \in (a,b)$ and if $f'(x)$ exist, then $f'(x) = 0$.

Proof

Choose a $\delta > 0$ such that

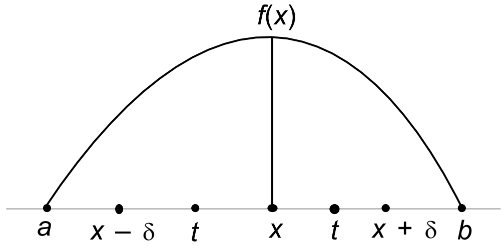
$$a < x - \delta < x < x + \delta < b$$

Now if $x - \delta < t < x$ then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Taking limit as $t \to x$ we get

$$f'(x) \geq 0 \quad \ldots\ldots\ldots\ldots (i)$$



If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0$$

Again, taking limit when $t \to x$ we get

$$f'(x) \leq 0 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Combining (i) and (ii) we have $f'(x) = 0$. ❑

❖ ***Theorem***

Let f be defined on $[a,b]$ and it is differentiable on (a,b). If f has a local minimum at a point $x \in (a,b)$ and if $f'(x)$ exist then $f'(x) = 0$.

The proof of this theorem is like the proof of above theorem.

❖ ***Lagrange's Mean Value Theorem.***

Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

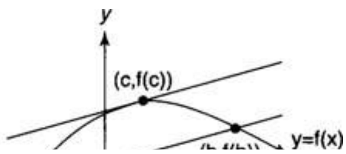
Proof.

Let us design a new function

$$h(t) = [f(b) - f(a)]t - (b - a)f(t) \qquad ,(a \leq t \leq b)$$

then clearly $h(a) = h(b)$.

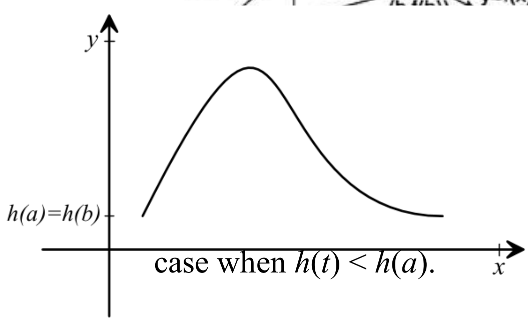
Since $h(t)$ depends upon t and $f(t)$ therefore it possesses all the properties of f.



Now there are two cases:

i) h is a constant.
implies that $h'(x) = 0 \quad \forall \quad x \in (a,b)$.

ii) h is not a constant, then
if $h(t) > h(a) = h(b)$ for some $t \in (a,b)$,
then there exists a point $c \in (a,b)$ at which h attains its maximum implies that $h'(c) = 0$.
and if $h(t) < h(a) = h(b)$
then there exists a point $c \in (a,b)$ at which h attain its minimum implies that $h'(c) = 0$.



case when $h(t) < h(a)$.

Since $h(t)=[f(b)-f(a)]t-(b-a)f(t)$,
therefore $h'(c)=[f(b)-f(a)]-(b-a)f'(c)$.

This gives that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$ as desired. ❑

❖ *Generalized Mean Value Theorem*

If f and g are continuous real valued functions on closed interval $[a,b]$ and f and g are differentiable on (a,b), then there is a point $c\in(a,b)$ at which

$$[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c),$$

Proof.

Let

$$h(t)=[f(b)-f(a)]g(t)-[g(b)-g(a)]f(t) \qquad (a\le t\le b)$$

Since h involves f and g therefore h is

i) continuous on close interval $[a,b]$.
ii) differentiable on open interval (a,b).
iii) and $h(a)=h(b)$.

To prove the theorem, we have to show that $h'(c)=0$ for some $c\in(a,b)$.

There are two cases to be discussed:

(i) If h is constant function, then $h'(x)=0 \quad \forall \quad x\in(a,b)$.

(ii) If h is not constant, then
if $h(t)>h(a)=h(b)$ for some $t\in(a,b)$,
then there exists a point $c\in(a,b)$ at which h attains its maximum,
this implies that $h'(c)=0$,
and if $h(t)<h(a)=h(b)$ for some $t\in(a,b)$,
then there exists a point $c\in(a,b)$ at which h attain its minimum,
this implies that $h'(c)=0$.

Hence

$$h'(c)=[f(b)-f(a)]g'(c)-[g(b)-g(a)]f'(c)=0$$

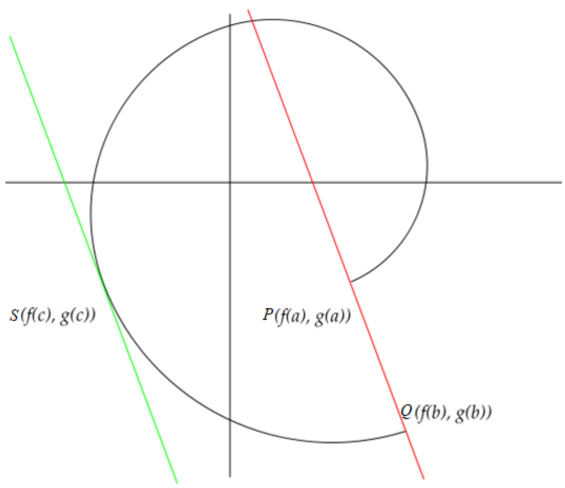
This gives the desire result. ❑

❖ *Geometric interpretation of generalized MVT*

Consider a plane curve C represented by

$$x=f(t),\ y=g(t).$$

Then generalized mean value theorem (MVT) states that there is a point S on C between two points $P(f(a),g(a))$ and $Q(f(b),g(b))$ of C such that the tangent at S to the curve C is parallel to the chord PQ.



❖ *Theorem (Darboux's Theorem)*

Suppose f is a real differentiable function on some interval I with $a,b \in I, a<b$ and suppose λ is a number between $f'(a)$ and $f'(b)$ then there exist a point $x \in (a,b)$ such that $f'(x)=\lambda$.

Proof

Without loss of generality assume that $f'(a)<\lambda<f'(b)$.

Also assume that $g(t)=f(t)-\lambda t$ for $t \in I$.

Then $g'(t)=f'(t)-\lambda$

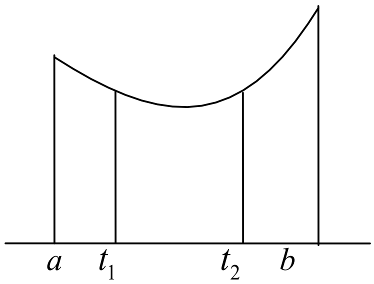
If $t=a$ we have

$$g'(a)=f'(a)-\lambda.$$

Since $f'(a)-\lambda<0$, therefore $g'(a)<0$.

This implies that g is monotonically decreasing at a.

So there exists a point $t_1 \in (a,b)$ such that $g(a)>g(t_1)$.



Similarly,

$$g'(b)=f'(b)-\lambda$$

Since $f'(b)-\lambda>0$, therefore $g'(b)>0$.

This implies that g is monotonically increasing at b.

So there exists a point $t_2 \in (a,b)$ such that $g(t_2)<g(b)$

This implies the function attain its minimum on (a,b) at a point x (say)

such that $g'(x)=0 \quad \Rightarrow \quad f'(x)-\lambda=0$

$$\Rightarrow \quad f'(x)=\lambda.$$ ❑

❖ ***Question***

Let f be defined for all real x and suppose that $|f(x)-f(y)| \le (x-y)^2$ for all real x and y. Then prove that f is constant.

Solution

Since $\quad |f(x)-f(y)| \le (x-y)^2$,

Therefore

$$-(x-y)^2 \le f(x)-f(y) \le (x-y)^2.$$

Dividing throughout by $x-y$ for $x \ne y$, we get

$$-(x-y) \le \frac{f(x)-f(y)}{x-y} \le (x-y) \quad \text{when} \quad x>y$$

and

$$-(x-y) \ge \frac{f(x)-f(y)}{x-y} \ge (x-y) \quad \text{when} \quad x<y$$

Taking limit as $x \to y$, we get

$$\left.\begin{array}{l} 0 \le f'(y) \le 0 \\ 0 \ge f'(y) \ge 0 \end{array}\right] \quad \Rightarrow f'(y)=0$$

This shows that function is constant.

❖ ***Question (L'Hospital Rule)***

Suppose $f'(x), g'(x)$ exist, $g'(x) \ne 0$ and $f(x)=g(x)=0$.

Prove that $\lim\limits_{t\to x}\dfrac{f(t)}{g(t)}=\dfrac{f'(x)}{g'(x)}$.

Proof

$$\lim_{t\to x}\frac{f(t)}{g(t)}=\lim_{t\to x}\frac{f(t)-0}{g(t)-0}=\lim_{t\to x}\frac{f(t)-f(x)}{g(t)-(x)} \qquad \because\ f(x)=g(x)=0$$

$$=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot\frac{t-x}{g(t)-(x)}$$

$$=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot\lim_{t\to x}\frac{1}{\dfrac{g(t)-(x)}{t-x}}$$

$$=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot\frac{1}{\lim\limits_{t\to x}\dfrac{g(t)-(x)}{t-x}}=f'(x)\cdot\frac{1}{g'(x)}=\frac{f'(x)}{g'(x)}.$$

……………………………

Chapter 06

Riemann Integrals

It is through logic that we prove, but through intuition that we discover. The Riemann integral, at its heart, is an intuitive leap in quantifying accumulation.

We assume that the reader is familiar at least informally with the integral from a calculus course (FSc or BSc). In addition, they know about integrating a function on an interval $[a,b]$ and know few of its interpretation as the "area under the graph", or its many applications to physics, engineering, economics, etc. Here our aim is to focus on the purely mathematical aspects of the integral. However, we first recall some basic terms that will be frequently used (see [1]).

Partition

Let $[a,b]$ be a given interval. By a partition P of $[a,b]$, we mean a finite set of points $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

The points of P are used to divide $[a,b]$ into n non-overlapping subintervals

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3], \ldots, [x_{n-1}, x_n].$$

Each sub-interval is called a *component* of the partition.

Obviously, corresponding to different choices of the points x_i we shall have different partition.

The maximum of the length of the components is defined as the *norm* of the partition and it is denoted by $\|P\|$, that is,

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}.$$

Examples

Consider an interval $[1,10]$ and following partitions of this interval.

$P_1 = \{1,2,3,10\}$,

$P_2 = \{1,2,3,6,9,10\}$,

$$P_3 = \left\{1, 1+\frac{9}{100}, 1+2\left(\frac{9}{100}\right), 1+3\left(\frac{9}{100}\right), \ldots, 1+99\left(\frac{9}{100}\right), 10\right\}$$

and more generally for any positive integer n, we can write

$$P_4 = \left\{1, 1+\frac{9}{n}, 1+2\left(\frac{9}{n}\right), 1+3\left(\frac{9}{n}\right), \ldots, 1+(n-1)\left(\frac{9}{n}\right), 1+n\left(\frac{9}{n}\right)=10\right\}.$$

Also note that $\|P_1\| = 7$, $\|P_2\| = 3$, $\|P_3\| = \dfrac{9}{100}$, $\|P_4\| = \dfrac{9}{n}$.

Refinement of a Partition

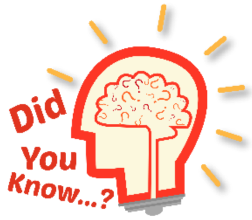
Let P and P^* be two partitions of an interval $[a,b]$ such that $P \subset P^*$ i.e. P^* contains all the points of P and possibly some other points as well. Then P^* is said to be a *refinement* of P.

Example

Note that P_2 is refinement of P_1.

Remark

Note that if $P_1 \subseteq P_2$ implies $\|P_1\| \geq \|P_2\|$, that is, refinement of a partition decreases its norm but the convers does not necessarily hold.

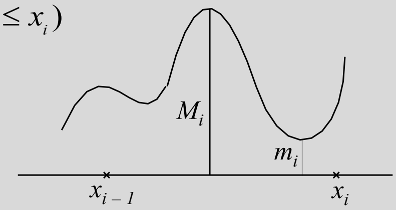


- How many partition can be made for any closed interval $[a,b]$?
- Can you write two different partitions of $[1,3]$ with same norm?
- Can you write two partitions P_1 and P_2 of $[0,5]$ such that $\|P_1\| < \|P_2\|$ but $P_1 \not\supset P_2$.

Riemann Integral

Let f be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we put

$$M_i = \sup f(x) \qquad (x_{i-1} \leq x \leq x_i)$$
$$m_i = \inf f(x) \qquad (x_{i-1} \leq x \leq x_i)$$



We define upper and lower sums as

$$U(P,f) = \sum_{i=1}^{n} M_i \,\Delta x_i$$

and $$L(P,f) = \sum_{i=1}^{n} m_i \,\Delta x_i ,$$

where $\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, \ldots, n)$.

Now we define $$\overline{\int_a^b} f\,dx = \inf U(P,f), \ldots\ldots\ldots\ldots\ldots (i)$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f), \ldots\ldots\ldots\ldots\ldots (ii)$$

where the infimum and the supremum are taken over all partitions P of $[a,b]$. Then $\int_a^{\overline{b}} f(x)dx$ and $\int_{\underline{a}}^{b} f(x)dx$ are called the upper and lower Riemann integrals of f over $[a,b]$ respectively.

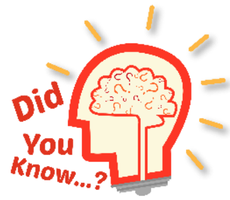
In case the upper and lower integrals are equal, we say that f is Riemann integrable on $[a,b]$ and we write $f \in \mathcal{R}[a,b]$, where $\mathcal{R}[a,b]$ denotes the set of Riemann integrable functions over $[a,b]$.

The common value of (i) and (ii) is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Which is known as the Riemann integral of f over $[a,b]$.

Exercises

1. Let $P_1 = \{1,2,3,4,5\}$ be partition of $[1,5]$ and $f:[1,5] \to \mathbb{R}$ be function defined by $f(x) = x^2$. Find $U(P_1, f)$ and $L(P_1, f)$.
2. Let $P_2 = \{0, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\}$ be partition of $[0,\pi]$ and $f:[0,\pi] \to \mathbb{R}$ be function defined by $f(x) = \sin x$. Find $U(P_1, f)$ and $L(P_1, f)$.



- If a function f is increasing on $[a,b]$, then $\max\limits_{x\in[a,b]} f(x) = f(b)$ and $\min\limits_{x\in[a,b]} f(x) = f(a)$.
- If a function f is decreasing on $[a,b]$, then what about its maximum and minimum value over interval $[a,b]$.
- Let f be bounded on interval $[a,b]$. Can you guess its maximum and minimum value over interval $[a,b]$.

Theorem
The upper and lower integrals are defined for every bounded function f over interval $[a,b]$.

Proof
Since f is bounded on $[a,b]$, so its supremum and infimum values exist over $[a,b]$.
Take M and m to be the maximum and minimum value of f in $[a,b]$ respectively, that is,

$$m \le f(x) \le M \qquad (a \le x \le b)$$

Let M_i and m_i denote the supremum and infimum of f in $[x_{i-1}, x_i]$ for certain partition P of $[a,b]$ respectively. Then

$$M_i \le M \quad \text{and} \quad m_i \ge m \quad (i = 1, 2, \ldots\ldots, n).$$

This gives

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \ge \sum_{i=1}^{n} m \Delta x_i \qquad (\Delta x_i = x_i - x_{i-1})$$

$$\Rightarrow L(P,f) \ge m \sum_{i=1}^{n} \Delta x_i$$

But $$\sum_{i=1}^{n} \Delta x_i = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \ldots. + (x_n - x_{n-1}),$$

$$= x_n - x_0 = b - a.$$

This gives

$$L(P,f) \ge m(b-a). \quad \ldots\ldots. \text{(i)}$$

Similarity one can have

$$U(P,f) \le M(b-a). \quad \ldots\ldots. \text{(ii)}$$

Also we have $L(P,f) \le U(P,f)$ ……. (iii)
Combining (i), (ii) and (iii), we have

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

This shows that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set over all the partitions P of $[a,b]$.
This gives the upper and lower integrals are defined for every function f over interval.

Remark: In mathematics, different author approached to Riemann integral with the same ideas but slightly different than above e.g. see [2] and [3].

Theorem
If P^* is a refinement of P, then following holds:

(i) $L(P,f) \le L(P^*,f)$,

(ii) $U(P,f) \ge U(P^*,f)$.

Theorem

Let f be a real and bounded function defined on $[a,b]$. Then

$$\sup L(P,f) \;\le\; \inf U(P,f) \qquad \text{i.e.} \quad \underline{\int_a^b} f\,dx \;\le\; \overline{\int_a^b} f\,dx\,.$$

Theorem (Condition of Integrability or Cauchy's Criterion for Integrability.)

A function $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$. there exists a partition P such that $U(P,f) - L(P,f) < \varepsilon$.

Theorem

If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$ and

$$\left| \int_a^b f\,dx \right| \;\le\; \int_a^b |\,f\,|\,dx\,.$$

Theorem (Fundamental Theorem of Calculus)

If $f \in \mathcal{R}[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Theorem

Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$. Then $f \in \mathcal{R}[a,b]$.

References:

1. Walter Rudin, Principles of mathematical analysis. Vol. 3. New York: McGraw-hill, 1964.
2. Bartle, Robert Gardner, and Donald R. Sherbert. Introduction to real analysis. Vol. 2. New York: Wiley, 2000.
3. Tom M. Apostol, Mathematical Analysis, 2nd Edition, MA: Addison-Wesley, 1974.