*Open notes on*

Metric Spaces

*Dedicated to*

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**Open Notes on Metric Spaces**

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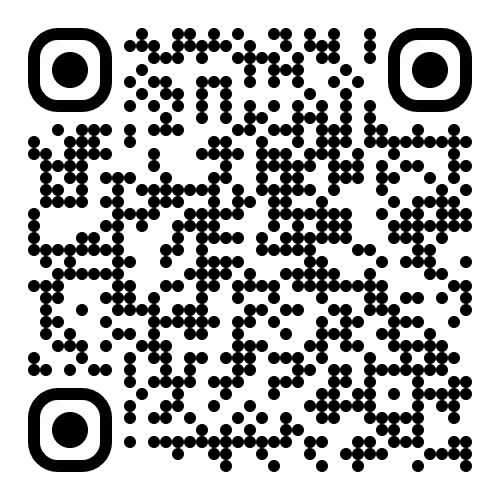
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*Open notes on*

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# ❖ Metric Spaces

Let *X* be a non-empty set and  denotes the set of real numbers. A function  is said to be metric if it satisfies the following axioms.

[M1]  i.e. *d* is finite and non-negative real valued function.

[M2]  if and only if *x* = *y*.

[M3]  (Symmetric property)

[M4]  (Triangular inequality)

The pair (*X*, *d* ) is then called *metric space*.

*d* is also called *distance function* and *d*(*x*, *y*) is the distance from *x* to *y*.

**Note:** If (*X*, *d*) be a metric space then *X* is called *underlying set*.

# ❖ Examples:

**i**) Let *X* be a non-empty set. Then  defined by



is a metric on *X* and is called *trivial metric* or *discrete metric*.

**ii**) Let  be the set of real number. Then  defined by

 is a metric on .

The space  is called *real line* and *d* is called *usual metric on* .

**iii**) Let *X* be a non-empty set and  be a metric on *X*. Then  defined by  is also a metric on *X*.

**Proof:**

[M1] Since *d* is a metric so 

as  is either 1 or  so .

[M2] If *x* = *y* then  and then  which is  will be

zero.

Conversely, suppose that  

  as *d* is metric.

[M3]  

[M4] We have 

 or 

We wish to prove 

now if ,  and 

then ,  and 

and 

therefore 

Now if ,  and 

Then ,  and 

As *d* is metric therefore 



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**iv**) Let  be a metric space. Then  defined by

 is also a metric.

**Proof.**

[M1] Since  therefore 

[M2] Let    

Now conversely suppose  then .

Then 

[M3] 

[M4] Since *d* is metric therefore 

Now by using inequality .

We get 







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**v**) The space **C**[*a*, *b*] is a metric space and the metric *d* is defined by

,

where *J* = [*a*, *b*] and *x*, *y* are continuous real valued function defined on [*a*, *b*].

**Proof.**

[M1] Since  therefore .

[M2] Let  

Conversely suppose 

Then 

[M3] 

[M4] 





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**vi**)  is a metric, where **** is the set of real number and *d* defined by



**vii**) Let  , . We define

 is a metric on ****

and called *Euclidean metric on*  or *usual metric on* .

**viii**)  is not a metric, where **** is the set of real number and *d* defined by



**Proof.**

[M1] Square is always positive therefore 

[M2] Let    

Conversely suppose that 

then 

[M3] 

[M4] Suppose that triangular inequality holds in *d*. then for any 





Since  therefore consider  and .



which is not true so triangular inequality does not hold and *d* is not metric.

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**ix**) Let  , . We define



is a metric on , called *Taxi-Cab metric* on .

**x**) Let  be the set of all real *n*-tuples. For

 and  in 

we define 

then *d* is metric on , called *Euclidean metric on*  or *usual metric on* .

**xi**) The space . As points we take bounded sequence

, also written as , of complex numbers such that



where  is fixed real number. The metric is defined as

 where 

**xii**) The space ,  is a real number, we take as member of , all sequence  of complex number such that .

The metric is defined by , where  such that .

**Proof.**

[M1] Since  therefore .

[M2] If  then



Conversely, if 

[M3] 

[M4] Let , such that 

then 



Using \*Minkowski’s Inequality



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# ❖ Pseudometric

Let *X* be a non-empty set. A function  is called pseudometric if and only if

i)  for all .

ii)  for all .

iii)  for all .

*OR*

A pseudometric satisfies all axioms of a metric except 

may not imply *x* = *y* but *x* = *y* implies .

# Example

Let  and  , 

Then  is a pseudometric on .

Let  and 

Then  but 

**Note:** Every metric is a pseudometric, but pseudometric is not metric.

***Minkowski’s Inequality***

If  and  are in  and , then



# ❖ Distance between sets

Let  be a metric space and . The distance between *A* and *B* denoted by  is defined as 

If  is a singleton subset of *X*, then  is written as  and is called distance of point *x* from the set *B*.

# ❖ Theorem

Let  be a metric space. Then for any 

.

**Proof.**

Let  then 

then 





Next







Combining equation (*i*) and (*ii*)

.

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# ❖ Diameter of a set

Let  be a metric space and , we define diameter of *A* denoted by



**Note:** For an empty set , following convention are adopted

(i) , some authors take  also as 0.

(ii)  i.e distance of a point *p* from empty set is .

(iii) , where *A* is any non-empty set.

# ❖ Bounded Set

Let  be a metric space and , we say *A* is bounded if diameter of *A* is finite i.e. .

# ❖ Theorem

The union of two bounded set is bounded.

**Proof.**

Let  be a metric space and  be bounded. We wish to prove  is bounded.

Let 

If  then since A is bounded therefore 

and hence  then  is bounded.

Similarity if  then  is bounded.

Now if  and  then

 where .

Since  and  are finite

Therefore  i.e  is bounded.

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# ❖ Open Ball

Let  be a metric space. An open ball in  is denoted by

,

where  is called centre of the ball and *r* is called radius of ball and .

# ❖ Closed Ball

Let  be a metric space. A closed ball in  is denoted by

,

where  is called centre of the ball and *r* is called radius of ball and .

# ❖ Sphere

Let  be a metric space. A sphere in  is denoted by

,

where  is called centre and *r* is called radius of sphere and .

# ❖ Examples

Consider the set of real numbers with usual metric  

then 

i.e. 

i.e. =

i.e. open ball is the real line with usual metric is an open interval.

And 

i.e. =

i.e. closed ball in a real line is a closed interval.

And =

i.e. two point  and  only.

# ❖ Open Set

Let  be a metric space and set *G* is called open in *X* if for every , there exists an open ball .

# ❖ Theorem

An open ball in metric space *X* is open.

**Proof.**

Let  be an open ball in .

Let  then 

Let , then 

Hence  is an open set.

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**Alternative:**

Let  be an open ball in .

Let  then 

Take  and consider the open ball 

we show that .

For this let  then 

and 

hence  so that . Thus  is an open.

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**Note:** Let be a metric space then

i) *X* and  are open sets.

ii) Union of any number of open sets is open.

iii) Intersection of a finite number of open sets is open.

# ❖ Limit point of a set

Let  be a metric space and , then  is called a *limit point*  or *accumulation point* of *A* if for every open ball  with centre *x*,

.

i.e. every open ball contain a point of *A* other than *x*.

# ❖ Closed Set

A subset *A* of metric space *X* is *closed* if it contains every limit point of itself.

The set of all limit points of *A* is called the *derived set of A* and denoted by .

# ❖ Theorem

A subset *A* of a metric space is closed if and only if its complement  is open.

**Proof.**

Suppose *A* is closed, we prove  is open.

Let  then .

 is not a limit point of *A*.

then by definition of a limit point there exists an open ball  such that

.

This implies . Since *x* is an arbitrary point of . So  is open.

Conversely, assume that  is an open then we prove *A* is closed.

i.e. *A* contain all of its limit points.

Let *x* be an accumulation point of *A*. and suppose .

then there exists an open ball  .

This shows that *x* is not a limit point of *A*. this is a contradiction to our assumption.

Hence . Accordingly *A* is closed.

The proof is complete.

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# ❖ Theorem

A closed ball is a closed set.

**Proof.**

Let  be a closed ball. We prove  (say) is an open ball.

Let  then .

Let  then . And take 

Consider the open ball  we prove .

For this let  then 

By the triangular inequality





 This shows that 



Hence *C* is an open set and consequently  is closed.

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# ❖ Theorem

Let  be a metric space and . If  is a limit point of *A*. Then every open ball with centre *x* contain an infinite numbers of point of *A*.

**Proof.**

Suppose contain only a finite number of points of *A*.

Let  be those points.

and let  where .

also consider 

Then the open ball  contain no point of *A* other than *x*. then *x* is not limit point of *A*. This is a contradiction therefore  must contain infinite numbers of point of *A*.

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# ❖ Closure of a Set

Let  be a metric space and . Then *closure of* *M* is denoted by  where  is the set of all limit points of *M*. It is the smallest closed superset of *M*.

# ❖ Dense Set

Let (*X*, *d*) be a metric space the a set  is called dense in *X* if .

# ❖ Countable Set

A set *A* is *countable* if it is finite or there exists a function  which is one-one and onto, where  is the set of natural numbers.

e.g.  and  are countable sets . The set of real numbers, the set of irrational numbers and any interval are not countable sets.

# ❖ Separable Space

A space *X* is said to be *separable* if it contains a countable dense subsets.

e.g. the real line  is separable since it contain the set  of rational numbers, which is dense is .

# ❖ Theorem

Let (*X*, *d*) be a metric space,  is dense if and only if *A* has non-empty intersection with any open subset of *X*.

**Proof.**

Assume that A is dense in X. then .

Suppose there is an open set  such that .

Then if  then 

which show that *x* is not a limit point of A.

This implies  but  

This is a contradiction.

Consequently  for any open .

Conversely suppose that  for any open .

We prove , for this let .

If  then  then .

If  then let  be the family of all the open subset of *X* such that  for every *i*.

Then by hypothesis  for any *i*. i.e contain point of A other then *x*.

This implies that *x* is an accumulation point of *A*. i.e. 

Accordingly  and .

The proof is complete.

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# ❖ Neighbourhood of a Point

Let (*X*, *d*) be a metric space and  and a subset  is called a *neighbourhood of*  if there exists an open ball  with centre  such that .

Shortly “*neighbourhood* ” is written as “*nhood* ”.

# ❖ Interior Point

Let (*X*, *d*) be a metric space and , a point  is called an *interior point* of *A* if there is an open ball  with centre  such that .

The set of all interior points of *A* is called *interior of* *A* and is denoted by *int(A)* or .

It is the largest open set contain in A. i.e. .

# ❖ Continuity

A function  is called continuous at a point  if for any  there is a  such that  for all *x* satisfying .

**Alternative:**

 is continuous at  if for any , there is a  such that

 .

# ❖ Theorem

 is continuous at  if and only if  is open is *X* wherever *G* is open in *Y*.

**Note :** Before proving this theorem note that if  ,  and ,  then  and .

**Proof.**

Assume that  is continuous and  is open. We will prove  is open in *X*.

Let  

When G is open, there is an open ball .

Since  is continuous, therefore for  there is a  such that

  then 

Since *y* is an arbitrary point of . Also *x* was arbitrary, this show that  is open in *X*.

Conversely, for any  we prove  is continuous.

For this let  and  be given. Now  and let  be an open ball in *Y*. then by hypothesis  is open in *X* and 

As 

 i.e. 

Consequently  is continuous.

The proof is complete.

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# ❖ Convergence of Sequence:

Let  be a sequence in a metric space , we say  converges to  if .

We write  or simply  as .

Alternatively, we say  if for every  there is an , such that

, .

# ❖ Theorem

If  is converges then limit of  is unique.

**Proof.**

Suppose  and ,

Then  as   

Hence the limit is unique. 🖸

**Alternative**

Suppose that a sequence  converges to two distinct limits *a* and *b*.

and 

Since , given any , there is a natural number  depending on 

such that

 whenever 

Also , given any , there is a natural number  depending on 

such that

 whenever 

Take  then

 and  whenever 

Since  is arbitrary, take  then



Which is a contradiction, Hence  i.e. limit is unique.

◼

# ❖ Theorem

1. A convergent sequence is bounded.
2. If  and  then .

**Proof.**

**(i)** Suppose , therefore for any  there is  such that

, 

Let  and 

Then by using triangular inequality for arbitrary 





Hence  is bounded.

**(ii)** By using triangular inequality



 as  .

Next 

 as  

From (*i*) and (*ii*)

 as 

Hence

.

◼

# ❖ Cauchy Sequence

A sequence  in a metric space  is called *Cauchy* if any  there is a  such that .

# ❖ Theorem

A convergent sequence in a metric space  is Cauchy.

**Proof.**

Let , therefore any  there is  such that

,  and .

Then by using triangular inequality





Thus every convergent sequence in a metric space is Cauchy.

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# ❖ Example

Let  be a sequence in the discrete space . If  be a Cauchy sequence, then for , there is a natural number  depending on  such that

Since in discrete space *d* is either 0 or 1 therefore  (say)

Thus a Cauchy sequence in  become constant after a finite number of terms,

i.e. 

# ❖ Subsequence

Let  be a sequence  and let  be a sequence of positive integers such that  then  is called s*ubsequence* of .

# ❖ Theorem

**(i)** Let be a Cauchy sequence in , then  converges to a point  if and only if  has a convergent subsequence  which converges to .

**(ii)** If  converges to , then every subsequence  also converges to .

**Proof.**

**(i)** Suppose  then  itself is a subsequence which converges to .

Conversely, assume that  is a subsequence of  which converges to *x*.

Then for any  there is  such that , .

Further more  is Cauchy sequence

Then for the  there is  such that , .

Suppose  then by using the triangular inequality we have



This show that .

**(ii)**  implies for any   such that 

Then in particular  

Hence.

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# ❖ Example

Let  then  is a sequence in *X*.

Then  but 0 is not a point of *X*.

# ❖ Theorem

Let  be a metric space and .

1. Then  if and only if there is a sequence  in *M* such that .
2. If for any sequence  in *M*, , then *M* is closed.

**Proof.**

**(i)** Suppose 

If , then there is a sequence  in *M* which converges to *x*.

If , then  i.e. *x* is an accumulation point of *M*, therefore each  the open ball  contain infinite number of point of *M*.

We choose  from each 

Then we obtain a sequence  of points of *M* and since  as .

Then  as .

Conversely, suppose  such that .

We prove 

If  then . 

If , then every neighbourhood of *x* contain infinite number of terms of .

Then *x* is a limit point of M i.e. 

Hence .

1. If  is in *M* and , then  then by hypothesis , then *M* is closed.

◼

# ❖ Complete Space

A metric space  is called *complete* if every Cauchy sequence in *X* converges to a point of *X*.

# ❖ Subspace

Let  be a metric space and  then *Y* is called *subspace* if *Y* is itself a metric space under the metric *d*.

# ❖ Theorem

A subspace of a complete metric space is complete if and only if *Y* is closed in *X*.

**Proof.**

Assume that *Y* is complete we prove *Y* is closed.

Let  then there is a sequence  in *Y* such that .

Since convergent sequence is a Cauchy and *Y* is complete then .

Since *x* was arbitrary point of *Y* 

Therefore  

Consequently *Y* is closed.

Conversely, suppose *Y* is closed and  is a Cauchy sequence. Then  is Cauchy in *X* and since *X* is complete so .

Also  and .

Since *Y* is closed i.e.  therefore .

Hence *Y* is complete.

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# ❖ Nested Sequence:

A sequence sets  is called *nested* if 

# ❖ Theorem (Cantor’s Intersection Theorem)

A metric space  is complete if and only if every nested sequence of non-empty closed subset of *X*, whose diameter tends to zero, has a non-empty intersection.

**Proof.**

Suppose  is complete and let  be a nested sequence of closed subsets of *X*.

Since  is non-empty we choose a point  from each . And then we will prove  is Cauchy in *X*.

Let  be given, since  then there is  such that 

Then for , .

This shows that  is Cauchy in *X*.

Since *X* is complete so (say)

We prove ,

Suppose the contrary that  then  a  such that .

Since  is closed, .

Consider the open ball  then  and  are disjoint

Now  all belong to  then all these points do not belong to 

This is a contradiction as *p* is the limit point of .

Hence .

Conversely, assume that every nested sequence of closed subset of *X* has a non-empty intersection. Let  be Cauchy in *X*, where 

Consider the sets





…………………

…………………

…………………



Then we have 

We prove 

Since  is Cauchy, therefore   such that

, , i.e. .

Now  then 

Also 

Then by hypothesis . Let 

We prove 

Since  therefore   such that 

Then for ,   

This proves that .

The proof is complete.

◼

# ❖ Complete Space (Examples)

(*i*) The discrete space is complete.

Since in discrete space a Cauchy sequence becomes constant after finite terms

i.e.  is Cauchy in discrete space if it is of the form



(*ii*) The set  of integers with usual metric is complete.

(*iii*) The set of rational numbers with usual metric is not complete.

 is a Cauchy sequence of rational numbers but its limit is , which is not rational.

(*iv*) The space of irrational number with usual metric is not complete.

We take 

We choose one irrational number from each interval and these irrational tends to zero as we goes toward infinity, as zero is a rational so space of irrational is not complete.

# ❖ Theorem

The real line is complete.

**Proof.**

Let  be any Cauchy sequence of real numbers.

We first prove that  is bounded.

Let  then  such that , 

In particular for  we have

Let  and 

then .

this shows that  is bounded with  as lower bound and  as upper bound.

Secondly we prove  has convergent subsequence .

If the range of the sequence is  is finite, then one of the term is the sequence say *b*  will repeat infinitely i.e. *b*, *b* ,*b* ,……….

Then  is a convergent subsequence which converges to *b*.

If the range is infinite then by the Bolzano Weirestrass theorem, the bounded infinite set  has a limit point, say *b*.

Then each of the open interval ,  , , … has an infinite numbers of points of the set .

i.e. there are infinite numbers of terms of the sequence  in every open interval .

We choose a point  from , then we choose a point  from  such that 

i.e. the terms  comes after  in the original sequence . Then we choose a term  such that , continuing in this manner we obtain a subsequence

.

It is always possible to choose a term because every interval contain an infinite numbers of terms of the sequence .

Since  and  as . Hence we have convergent subsequence  whose limit is *b*.

Lastly we prove that .

Since  is a Cauchy therefore for any  there is  such that

Also since  there is a natural number  such that 

Then 





Hence  and the proof is complete.

◼

# ❖ Theorem

The Euclidean space  is complete.

**Proof.**

Let  be any Cauchy sequence in .

Then for any , there is  such that 



where  and 

Squaring both sided of (*i*) we obtain





This implies  is a Cauchy sequence of real numbers for every .

Since  is complete therefore  (say)

Using these *n* limits we define

 then clearly .

We prove 

In (*i*) as ,  which show that 

And the proof is complete.

◼

**Note:** In the above theorem if we take *n* = 2 then we see complex plane  is complete. Moreover the unitary space  is complete.

# ❖ Theorem

The space  is complete.

**Proof.**

Let  be any Cauchy sequence in .

Then for any  there is  such that 



Where  and 

Then from (*i*)

  and 

It means  is a Cauchy sequence of real or complex numbers for every 

And since  and  are complete therefore  or  (say).

Using these infinitely many limits we define .

We prove  and .

In (*i*) as  we obtain  

We prove *x* is bounded.

By using the triangular inequality



Where  as  is bounded.

Hence  is bounded.

This shows that .

And the proof is complete.

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**❖ Theorem**

The space **C** of all convergent sequence of complex number is complete.

**Note:** It is subspace of .

**Proof.**

First we prove **C** is closed in .

Let , then there is a sequence  in **C** such that ,

where .

Then for any , there is  such that 

.

Then in particular for  and 

.

Now  then  is a convergent sequence therefore 

such that 

.

Then by using triangular inequality we have



.

 .

Hence *x* is Cauchy in  and *x* is convergent

Therefore  and  .

i.e. **C** is closed in  and  is complete.

Since we know that a subspace of complete space is complete if and only if it is closed in the space.

Consequently **C** is complete.

◼

**❖ Theorem**

The space ,  is a real number, is complete.

**Proof.**

Let  be any Cauchy sequence in .

Then for every , there is  such that 

 ………….. (*i*)

where  .

Then from (*i*)  ……… (*ii*)  and for any fixed *j*.

This shows that  is a Cauchy sequence of numbers for the fixed *j*.

Since  and  are complete therefore  or  (say) as .

Using these infinite many limits we define .

We prove  and  as .

From (*i*) we have

,

i.e.  …………. (*iii*)

Taking as , we get

 , *k* = 1, 2, 3, …….

Now taking, we obtain

 ………… (*iv*) 

This shows that 

Now  is a vector space and ,  then .

Also from (*iv*) we see that

i.e.  

This shows that  as .

And the proof is complete.

# ❖ Theorem

The space **C**[*a*, *b*] is complete.

**Proof.**

Let  be a Cauchy sequence in **C**[*a*, *b*].

Therefore for every , there is  such that 

 ………… (*i*) where .

Then for any fix 

It means  is a Cauchy sequence of real numbers. And since  is complete therefore  (say) as .

In this way for every , we can associate a unique real number  with .

This defines a function  on *J.*

We prove **C**[*a*, *b*] and  as .

From (*i*) we see that

 for every  and .

Letting , we obtain for all 

 .

Since the convergence is uniform and the ’s are continuous, the limit function  is continuous, as it is well known from the calculus.

Then  is continuous.

Hence , also  as 

Therefore .

The proof is complete.

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# ❖ Theorem

If  and  are complete then  is complete.

**Note:** The metric *d* (say) on  is defined as 

where ,  and , .

**Proof.**

Let  be a Cauchy sequence in .

Then for any , there is  such that 



  and  

This implies  is a Cauchy sequence in *X*.

and  is a Cauchy sequence in *Y*.

Since *X* and *Y* are complete therefore (say) and (say)

Let  then .

Also  as .

Hence .

This proves completeness of .

◼

# ❖ Theorem

 is continuous at  if and only if  implies .

**Proof.**

Assume that *f* is continuous at  then for given  there is a 

such that

 .

Let , then for our  there is  such that

, 

Then by hypothesis , 

i.e. 

Conversely, assume that   

We prove  is continuous at , suppose this is false

Then there is an  such that for every  there is an  such that

 but 

In particular when , there is  such that

 but .

This shows that  but  as .

This is a contradiction.

Consequently  is continuous at .

The proof is complete.

◼

# ❖ Rare (or nowhere dense in X )

Let *X* be a metric, a subset  is called *rare* (or *nowhere dense in* *X* ) if  has no interior point i.e. .

# ❖ Meager ( or of the first category)

Let *X* be a metric, a subset  is called *meager* (or *of the first category*) if *M* can be expressed as a union of countably many rare subset of *X*.

# ❖ Non-meager ( or of the second category)

Let *X* be a metric, a subset  is called *non-meager* (or *of the second category*) if it is not meager (of the first category) in *X*.

# ❖ Example:

Consider the set  of rationales as a subset of a real line . Let , then  because  is open. Clearly  contain no open ball. Hence  is nowhere dense in  as well as in . Also since  is countable, it is the countable union of subsets , . Thus  is of the first category.

# ❖ Bair’s Category Theorem

If  is complete then it is non-meager in itself.

***OR***

A complete metric space is of second category.

**Proof.**

Suppose that *X* is meager in itself then , where each  is rare in *X.*

Since  is rare then 

i.e.  has non-empty open subset

But *X* has a non-empty open subset ( i.e. *X* itself ) then .

This implies  is a non-empty and open.

We choose a point  and an open ball , where .

Now  is non-empty and open

Then  a point  and open ball 

( has no non-empty open subset then  is non-empty and open.)

So we have chosen a point  from the set  and an open ball  around it, where .

Proceeding in this way we obtain a sequence of balls  such that

 where  

Then the sequence of centres  is such that for 

 as .

Hence the sequence  is Cauchy.

Since *X* is complete therefore (say) as .

Also





 as .

  i.e.   

This is a contradiction.

Bair’s Theorem is proof.

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**Open Notes on Metric Spaces**

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