Vector Spaces: Handwritten notes

by

Atiq ur Rehman

http://www.MathCity.org/atiq

Partial Contents

Available at ***http://www.MathCity.org/notes***

For notes on other subject, please visit the above page.

MathCity.org is a non-profit organization, working to promote mathematics in Pakistan. If you have anything (notes, model paper, old paper etc.) to share with other peoples, you can send us to publish on MathCity.org. You may earn money by participating. For more information visit: *http://www.MathCity.org/participate*

Ring

def:- A non-empty set R is called ring if

i) R is abelian group under ~~multiplication~~ addition.

ii) R is semi-group under multiplication

iii) Distributive law holds

$$a(b+c) = a\cdot b + a\cdot c$$

$$(a+b)c = a\cdot c + b\cdot c$$

Examples

i) $(\mathbb{Z}, +, \cdot)$ is a ring
where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$

ii) $(\mathbb{Q}, +, \cdot)$, where $\mathbb{Q}$ is the set of rational numbers

iii) $(\mathbb{R}, +, \cdot)$, where $\mathbb{R}$ is set of real numbers.

iv) $(\mathbb{Z}_n, +, \cdot)$, $\mathbb{Z}_n$ = residue classes of module n.

Field

def:- A non-empty set F is called a field if

i) F is abelian group under addition.

ii) $F - \{0\}$ is abelian group under multiplication.

iii) Right distributive law holds in F.

i.e $a, b, c \in F$

$$(a+b)c = ac + bc$$

Examples

i) $(\mathbb{R}, +, \cdot)$ is a field

ii) $(\mathbb{C}, +, \cdot)$ is a field

iii) $(\mathbb{Q}, +, \cdot)$ is a field

iv) $(\mathbb{Z}, +, \cdot)$ is not a field
as $(\mathbb{Z} - \{0\}, \cdot)$ is not group under multiplication.

Vector Space

def:- Let V be a non-empty set and F is field then V is called vector space if

i) V is abelian group under addition

ii) $a(v+w) = av + aw \quad \forall\ a \in F,\ v, w \in V.$

iii) $(a+b)v = av + bv \quad \forall\ a, b \in F,\ v \in V.$

iv) $a(bv) = (ab)v \quad \forall\ a, b \in F,\ v \in V.$

v) $1 \cdot v = v \cdot 1 = v \quad , 1 \in F$ and $v \in V$.

i.e 1 is identity under multiplication

Example

i) Let V be a set of all polynomial of degree $\leq n$ then V is vector space

$$V = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in F \ \forall\ i \leq n \in \mathbb{N}\}$$

$$= \left\{\sum_{i=0}^{n} a_i x^i \mid a_i \in F \quad \forall \quad i \leq n \in \mathbb{N}\right\}.$$

addition is defined as

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i$$

and multiplication is defined as

$$r \sum a_i x^i = \sum r a_i x^i$$

$$= ra_0 + ra_1x + ra_2x^2 + \cdots + ra_nx^n$$

ii) Let F is a field then the set

$$F^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in F,\ 1 \leq i \leq n\}.$$

iii) The set M_n of all $n \times n$ matrices with entries from a field F is a vector space over F.

iv) Every field is a vector space over itself.

Subspace:-

Let V be a vector space over F and W be its non-empty subset of V.

Then W is a subspace of V if W itself is vector space under operation induced (defined) in V.

Theorem:-

A non-empty ~~subspace~~ subset W of a vector space V is subspace of V iff

i) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

ii) $\alpha \in F, w \in W \Rightarrow \alpha w \in W$.

Proof:

Let W is subspace of vector ~~field~~ space V.

then W itself is a vectorspace

i.e W is closed under addition and scalar multiplication.

Conversely, let W is a subset satisfying condition (i) and (ii).

Then for $-1 \in F$ and $w_2 \in W$.

$\Rightarrow -1 \cdot w_2 \in W$ by condition (ii).

$\Rightarrow -w_2 \in W$

i.e $w_1, -w_2 \in W$

$\Rightarrow w_1 + (-w_2) \in W$ by condition (i)

$\Rightarrow W$ is a subgroup under addition.

Since W is a subset of V and V is abelian.

So W is abelian.

Further condition $\underline{\text{II}}$ to $\overline{\text{V}}$ of the definition are satisfied in W as these are satisfied in V.

Corollary:-

W is non-empty subset of a vectorspace $V(F)$. Then W is subspace of V iff

$a, b \in F, w_1, w_2 \in W \Rightarrow aw_1 + bw_1 \in W$.

Proof. Let W is a subspace of $V(F)$. then W itself is a vector space

i.e for $a, b \in F$, $w_1, w_2 \in W$

$\Rightarrow aw_1, bw_2 \in W$.

$\Rightarrow aw_1 + bw_2 \in W$. $\because W$ is closed under addition.

Conversely,

Let for $a, b \in F$, $w_1, w_2 \in W$.

$\Rightarrow aw_1 + bw_2 \in W$.

Let $a = b = 1$

then $1 \cdot w_1 + 1 \cdot w_2 \in W$

i.e $w_1 + w_2 \in W$.

also if $b = 0 \in F$

For $aw_1 + bw_2 \in W$

$\Rightarrow aw_1 + 0 \cdot w_2 \in W$

$\Rightarrow aw_1 \in W$.

$\Rightarrow W$ is a subspace of V

Definition (Linear Sum).

Let V be a vector space over F and $W_1, W_2, \ldots, W_n$ be non-empty subset of V. then their linear sum is defined as.

$W_1 + W_2 + \ldots + W_n = \{a_1 + a_2 + \ldots + a_n : a_1 \in W_1, a_2 \in W_2$ i.e $a_i \in W_i\}$

Lemma:- Let V be a vector space and $W_1, W_2, \ldots, W_n$ be subspace prove that

$W = W_1 + W_2 + \ldots + W_n$

is also a subspace of V.

****

Available online at *http://www.MathCity.org*

****

Lemma:-

$W_1, W_2, \ldots, W_n$ are subspaces of V prove that $W = W_1 + W_2 + \cdots + W_n$ is a subspace of V.

Proof:-

$0 = 0 + 0 + 0 + \cdots + 0$, $0 \in W_i$.

$\Rightarrow 0 \in W \Rightarrow W$ is non-empty.

Let $x, y \in W$, $a, b \in F$

we have to show $ax + by \in W$.

$\because x \in W$

$\Rightarrow x = x_1 + x_2 + \cdots + x_n$ for $x_1 \in W_1, x_2 \in W_2, \ldots, x_n \in W_n$

$y = y_1 + y_2 + \cdots + y_n$ for $y_1 \in W_1, y_2 \in W_2, \ldots, y_n \in W_n$.

Now

$ax + by = a(x_1 + x_2 + \cdots + x_n) + b(y_1 + y_2 + \cdots + y_n)$.

$= ax_1 + ax_2 + \cdots + ax_n + by_1 + by_2 + \cdots + by_n$

$= (ax_1 + by_1) + (ax_2 + by_2) + \cdots + (ax_n + by_n)$.

As each W_i is a subspace

$\Rightarrow ax_i + by_i \in W_i$, $i = 1, 2, \ldots, n$.

So $\sum_{i=1}^{n} ax_i + by_i \in \sum W_i = W$.

$\Rightarrow ax + by \in W$

So W is a subspace

Lemma:-

Let V be a vector space and W_i a family of subspaces of V. Then $\cap W_i$ is also a subspace of V.

Proof

Let $v, w \in \cap W_i$

then $v, w \in W_i$ for each $i \in I$

and since each W_i is a subspace

so there must be $a, b \in F$

such that $av + bw \in W_i$ for each $i \in I$.

so $av + bw \in \cap W_i$ i.e $\cap W_i$ is a subspace.

Definition

Let U and V are two vector spaces over a field F. then τ of U into V is called homomorphism

if $\tau(u_1+u_2) = \tau(u_1)+\tau(u_2)$

$\tau(au) = a\,\tau(u)$; $a \in F$.

Definition

The kernel of homomorphism $T : U \to V$, is defined as

$\ker T = \{u : u \in U,\ T(u) = 0\}$.

Question.

Prove that $\ker \tau$ (ker. of homomorphism) is a subspace

Solution Let $u_1, u_2 \in \ker \tau$

$\Rightarrow \tau(u_1) = 0$, $\tau(u_2) = 0$.

Now let $a, b \in F$

$$\begin{aligned}\tau(au_1 + bu_2) &= \tau(au_1) + \tau(bu_2)\\ &= a\,\tau(u_1) + b\,\tau(u_2)\\ &= a(0) + b(0)\\ &= 0.\end{aligned}$$

$\Rightarrow au_1 + bu_2 \in \ker\tau$.

So $\ker\tau$ is subspace.

Linear Combination:-

Let V is a vector space

Let $v_1, v_2, \dots, v_n \in V$.

$a_1, a_2, \dots, a_n \in F$.

then an element

$a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n$ is called linear combination.

The linear combination is trivial if each $a_i = 0$.

and it is non-trivial if at least one of $a_i \neq 0$

Definition : (Linear Span)

Let S be a subset of vector space V. then the set of all linear combination of S is called linear spam denoted by $< S >$ or $L(S)$ or $[S]$

Theorem:-

Prove that $< S >$ is a subspace of V containing S. It is smallest subspace of V containing S.

Proof:-

Let $u, v \in < S >$

then $u = a_1u_1 + a_2u_2 + \cdots + a_nu_n$

$v = b_1v_1 + b_2v_2 + \cdots + b_nv_n$.

For $a, b \in F$ we have to prove $au + bv \in < S >$.

Now

$$au + bv = a(a_1u_1 + a_2u_2 + \cdots + a_nu_n) + b(b_1v_1 + b_2v_2 + \cdots + b_nv_n)$$

$$= aa_1u_1 + aa_2v_2 + \cdots + aa_nu_n + bb_1v_1 + bb_2v_2 + \cdots + bb_nv_n$$

$\Rightarrow au + bv \in < S >$

$\Rightarrow < S >$ is a subspace.

Let $u_i \in S$

then $u_i = 0u_1 + 0u_2 + \cdots + 0u_{i-1} + 1 \cdot u_i + 0 \cdot u_{i+1} \cdots + 0 \cdot u_n \in <S>$

i.e $u_i \in < S >$.

$\Rightarrow S \subseteq < S >$.

Let W be any other subspace of V containing S.

then $\sum a_iu_i \in W$.

$\because W$ is subspace containing S.

$\Rightarrow < S > \leq W$.

i.e $<S>$ is smallest subspace containing S.

Definition (Finite Dimensional Vector Space).
A vector space V is called finite dimensional if there is a subset S of V such that $\langle S \rangle = V$.

Definition:- (Linear Dependent and Independent)
Let V be a vector space then the vectors $v_1, v_2, v_3, \ldots, v_n \in V$ are linearly dependent if $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ and all $a_i \neq 0$.
If $a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n = 0$ where each $a_i = 0$ then the vectors $v_1, v_2, \ldots, v_n$ are linearly independent.

Theorem:-
Let V be a vector space and consider a set of vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent then its subset is also independent.
ii) If $\{v_1, v_2, \ldots, v_n\}$ is dependent then $\{v_1, v_2, \ldots, v_n, v\}$ is also dependent.

****
Available online at *http://www.MathCity.org*
****

- Home → http://www.mathcity.org
    - FSc → http://www.mathcity.org/fsc
    - BSc → http://www.mathcity.org/bsc
    - MSc → http://www.mathcity.org/msc
    - MPhil/PhD → http://www.mathcity.org/mphil-phd
    - Old papers → http://www.mathcity.org/papers
    - E-Books → http://www.mathcity.org/e-books

Lemma:-

Let $V(F)$ be a vector space and $S=\{v_1, v_2, \ldots, v_n\}$ a set of vectors in V. Then

i) If S is independent, then any non-empty subset of S is also independent

Proof.

Let $\{v_1, v_2, \ldots, v_i\}$ be a subset of S, $1 \le i < n$.

Consider $a_1v_1 + a_2v_2 + \cdots + a_iv_i = 0$, $a_i \in F$

then

$$a_1v_1 + a_2v_2 + \cdots + a_iv_i + 0.v_{i+1} + \cdots + 0.v_n = 0$$

Since $\{v_1, v_2, \ldots, v_n\}$ is Linearly Independent

$\Rightarrow$ each $a_k = 0$; $k = 1, 2, \ldots, n$

$\therefore$ each $a_k = 0$, $k = 1, 2, \ldots, i$

$\Rightarrow \{v_1, v_2, \ldots, v_i\}$ is L.I.

2001 (ii)

If S is dependent then

$\{v, v_1, v_2, \ldots, v_n\}$ is also dependent.

i.e $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ where all $a_i \neq 0$.

and then

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n + 0v = 0$$

where all $a_i \neq 0$.

$\Rightarrow \{v_1, v_2, \ldots, v_n, v\}$ is also dependent.

Theorem:-

A set of non-zero vectors $v_1, v_2, \ldots, v_n \in V$ is linearly dependent iff one of them is a linear combination of the other / preceding vector.

Proof:

$\{v_1, v_2, \ldots, v_n\}$ is linearly dependent

i.e $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ where all a_i's $\neq 0$.

for $a_i \in F$

Let a_k be the Last ~~non-~~ non-zero coeffients of

$\cancel{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + 0 + 0 + a_kv_k}$

$a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0 \quad \because a_{k+1} = a_{k+2} = \cdots = a_n = 0$

$\Rightarrow -a_kv_k = a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}$

$\Rightarrow v_k = -\frac{1}{a_k}(a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1})$.

Conversely, let v_k is a linear combination of the ~~prec~~ preceding vectors

$v_1, v_2, v_3, \cdots, v_{k-1}$

i.e. $v_k = a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + (-1)v_k = 0$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + (-1)v_k + 0.v_{k+1} + \cdots + 0v_n = 0$

then $\{v_1, v_2, \cdots, v_n\}$ is Linearly Dependent

$\because$ at least one co-effient of v_k is non-zero.

Basis of a Vector Space:-

Let S be a subset of a vector space $V(F)$. then S is called basis for V.

if i) S is linearly independent.

ii) S is spanning set of V.

generating

- Home → http://www.mathcity.org
    - FSc → http://www.mathcity.org/fsc
    - BSc → http://www.mathcity.org/bsc
    - MSc → http://www.mathcity.org/msc
    - MPhil/PhD → http://www.mathcity.org/mphil-phd
    - Old papers → http://www.mathcity.org/papers
    - E-Books → http://www.mathcity.org/e-books

Theorem:-

Any finite dimensional vector space contains a basis.

Proof:

Let $\{v_1, v_2, \ldots, v_r\}$ be a ~~spanning~~ spanning set of V.

If $\{v_1, v_2, \ldots, v_r\}$ is linearly independent then form a basis and there is nothing to prove.

Consider $\{v_1, v_2, \ldots, v_r\}$ is linearly dependent then one of the vectors say v_r is a linear combination of the remaining $\{v_1, v_2, \ldots, v_{r-1}\}$ we drop out this vector and obtain a set of ~~ve vec~~ $r-1$ vectors.

A ~~vector~~ linear combination of r vectors also a linear combination of $r-1$ vectors.

If this set $\{v_1, v_2, \ldots, v_{r-1}\}$ is linearly independent then form a basis

But if $\{v_1, v_2, \ldots, v_{r-1}\}$ is dependent then the above process is continued. In this way we can get a linear independent spanning set, and hence a basis

$\{v_1, v_2, \ldots, v_n\}$ $1 \le n \le r$.

Imp 2001, 11/02

Theorem:-

If $v_1, v_2, \ldots, v_n$ is a basis of $V(F)$ and if $w_1, w_2, \ldots, w_m \in V$ are linearly independent then $m \le n$.

Proof:

Since $v_1, v_2, \ldots, v_n$ is a basis of V so every element of V can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$.

In particular $w_m \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$

$w_m, v_1, v_2, \ldots, v_n$ are dependent.
therefore, a proper subset $\{w_m, v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$, $r \leq n-1$
form a basis
Similarly $\{w_{m-1}, w_m, v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ is dependent

and Its proper subset
$\{w_{m-1}, w_m, v_{j_1}, v_{j_2}, \ldots, v_{j_s}\}$, $s \leq n-2$

Repeating this procedure $(m-1)$ times, we get
a basis
$w_1, w_2, \ldots, w_{m-1}, w_m, v_{k_1}, v_{k_2}, \ldots, v_{k_t}$

$t \geq 1$ Since the vectors w_1
is not a L.C of
$w_1, w_2, \ldots, w_n$

$t \leq n-(m-1)$
$t \leq n-m+1$
$t \geq 1$

$\Rightarrow \quad 1 \leq t \leq n-m+1$
$1 \leq n-m+1$
$\Rightarrow \quad 0 \leq n-m$
$\Rightarrow \quad m \leq n$

****
Available online at *http://www.MathCity.org*
****

Question: Show that the vectors

$v_1 = (1,1,1)$, $v_2 = (1,0,1)$, $v_3 = (0,1,1)$

are linearly independent.

Solution:-

Consider $a_1v_1 + a_2v_2 + a_3v_3 = 0$

$\Rightarrow a_1(1,1,1) + a_2(1,0,1) + a_3(0,1,1) = 0$

$\Rightarrow (a_1, a_1, a_1) + (a_2, 0, a_2) + (0, a_3, a_3) = 0$

$\Rightarrow$ ~~(a_1+a_2)~~

$(a_1+a_2 \,,\, a_1+a_3 \,,\, a_1+a_2+a_3) = (0,0,0)$

$\Rightarrow a_1 + a_2 = 0$ ——— (i)

$a_1 + a_3 = 0$ ——— (ii)

$a_1 + a_2 + a_3 = 0$ ——— (iii)

$\Rightarrow a_1 + a_2 + a_3 = 0$

$-a_1 - a_2$

$a_3 = 0 \quad \Rightarrow a_1 = 0, \quad a_2 = 0.$

Since $a_1 = a_2 = a_3 = 0$

$\Rightarrow$ the vectors are L.I.

Question: Prove that the vectors

$v_1 = (3,0,-3)$, $v_2 = (-1,1,2)$, $v_3 = (1,2,-2)$

$v_4 = (2,1,1)$ are linearly dependent.

Solution

Consider

$a v_1 + b v_2 + c v_3 + d v_4 = 0.$

$\Rightarrow a(3,0,-3) + b(-1,1,2) + c(1,2,-2) + d(2,1,1) = 0$

$\Rightarrow (3a, 0, -3a) + (-b, b, 2b) + (c, 2c, -2c) + (2d, d, d) = 0$

$\Rightarrow (3a - b + c + 2d \,,\, b + 2c + d \,,\, -3a + 2b - 2c + d) = 0$

$3a - b + c + 2d = 0$

$b + 2c + d = 0$

$-3a + 2b - 2c + d = 0$

let $d = 0$

$\Rightarrow 3a + b + 4c = 0$

$b +$ [illegible]

$-3a + 2b - 2c = 0$

Using $a = -2c$, $b = -2c$, $d = 0$

into (1)

$-2cv_1 - 2cv_2 + cv_3 + 0v_4 = 0$

$2v_1 + 2v_3 - v_3 + 0v_4 = 0$

$\Rightarrow v_1, v_2, v_3, v_4$ are dependent

$\boxed{v_1, v_2 \text{ are L.I}}$ check.

****
Available online at *http://www.MathCity.org*
****

Definition: (Quotient Space)

Let V be a vector space over a field F and W be a subspace.

The set V/W of all left coset along with two operations

(i) $(v_1+W)+(v_2+W) = v_1+v_2+W$

(ii) $a(v_1+W) = av_1+W$

is called Quotient space.

2003

Lemma:

Let V be a vector space and W a subspace of V then V/W along with the operation

(i) $(v_1+W)+(v_2+W) = (v_1+v_2)+W$

(ii) $\alpha(v_1+W) = \alpha v_1 + W$ is a ~~subspace~~ vector space

Proof:

a) It is easy to show that V/W is an abelian group under addition with $0+W=W$ as it identity and $-v+W$ as an inverse of $v+W \in V/W$.

b) We see that scalar multiplication is defined in V/W.

[i.e $v+W = v'+W \Rightarrow \alpha(v+W) = \alpha(v'+W)$]

Let $v = v'+w$ for some $w \in W$.

then $\alpha(v+W) = \alpha v + W$

$= \alpha(v'+w)+W$

$= \alpha v' + \alpha w + W$

$= \alpha v' + W \quad \because \alpha w \in W$.

$= \alpha(v'+W)$.

i.e Scalar multiplication is defined

Let $v+W, v'+W \in V/W$, $a \in F$.

$a((v+W)+(v'+W)) = a(v+v'+W)$

$= a(v+v')+W$

$= av+av'+W$

$= av+W+av'+W$

$= a(v+W)+a(v'+W)$

(iv) $(a+b)(v+W) = (a+b)v + W$

$= (av + bv) + W$

$= av + W + bv + W$

$= a(v+W) + b(v+W)$.

(v) $a(b(v+W)) = a(bv + W)$

$= (ab)v + W$

$= (ab)(v + W)$

(vi) $1\cdot(v+W) = 1.v + W$

$= v + W$

Hence V/W is vector space.

- Home → http://www.mathcity.org
  - FSc → http://www.mathcity.org/fsc
  - BSc → http://www.mathcity.org/bsc
  - MSc → http://www.mathcity.org/msc
  - MPhil/PhD → http://www.mathcity.org/mphil-phd
  - Old papers → http://www.mathcity.org/papers

****
Available online at *http://www.MathCity.org*
****

3. Theorem:-

$V(F)$ is a finite dimensional vector space and if W is a subspace of V. Then

i) W is finite dimensional and $\dim W \leq \dim V$.

ii) $\dim(V/W) = \dim V - \dim W$

Proof:

Let $\dim V = n$.

and let $\{w_1, w_2, \ldots, w_m\}$ be linearly independent set of vectors of W,

then $m \leq n$

Then the set $\{w_1, w_2, w_3, \ldots, w_m, w\}$ is linearly dependent.

i.e one of these vectors in a linear combination of the preceeding vector.

however none of the vectors $w_1, w_2, \ldots, w_m$ is a linear combination of the preceeding vectors. because the vectors $w_1, w_2, \ldots, w_m$ are linearly independent.

so w can be written as a linear combination of $w_1, w_2, \ldots, w_m$.

Since $w \in W$ is an arbitrary element therefore W is finite dimensional

and $\dim W = m \leq n$.

i.e $\dim W \leq \dim V$.

ii) Let $\{w_1, w_2, \ldots, w_m\}$ be a basis of W.

and $\{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_l\}$ be a basis of V

we have to prove $\{v_1+W, v_2+W, \ldots, v_l+W\}$ is a basis of V/W.

Now

$$\alpha_1(v_1+W) + \alpha_2(v_2+W) + \cdots + \alpha_l(v_l+W) = 0.$$

$$(\alpha_1 v_1 + W) + (\alpha_2 v_2 + W) + \cdots + (\alpha_l v_l + W) = 0 + W$$

Since W is identity of V/W.

$$\Rightarrow (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_l v_l) + W = W.$$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_l v_l \in W$ $\quad \because a+H = H \Leftrightarrow a \in H$.

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_l v_l = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m$

as $\{w_1, w_2, \ldots, w_m\}$ is basis of W.

So

$\beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m - \alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_l v_l = 0$.

Since $\{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_l\}$ is a basis of V.

$\Rightarrow \beta_1 = \beta_2 = \cdots = \beta_m = \alpha_1 = \alpha_2 = \cdots = \alpha_l = 0$.

i.e $\{v_1 + W, v_2 + W, \ldots, v_l + W\}$ is linearly independent.

Set $v + W \in V/W$ for $v \in V$

then $v = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + b_1 v_1 + b_2 v_2 + \cdots + b_l v_l$.

So $v + W = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + b_1 v_1 + b_2 v_2 + \cdots + b_l v_l + W$

$\Rightarrow v + W = b_1 v_1 + b_2 v_2 + \cdots + b_l v_l + a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + W$

$= b_1 v_1 + b_2 v_2 + \cdots + b_l v_l + W$

$\because a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + W = W$

as $a_1 w_1 + a_2 w_2 + \cdots + a_m w_m \in W$.

$= (b_1 v_1 + W) + (b_2 v_2 + W) + \cdots + (b_l v_l + W)$ by def.

$= b_1 (v_1 + W) + b_2 (v_2 + W) + \cdots + b_l (v_l + W)$. by def.

i.e $\{v_1 + W, v_2 + W, \ldots, v_l + W\}$ generate V/W and hence is a basis of V/W.

$\therefore \dim(V/W) = l$

$= (m + l) - m$

$= \dim V - \dim W$

Internal Direct sum:-

def. Let $U_1, U_2, \ldots, U_n$ be subspace of a vector space V. For $v \in V$

then if v has one and only one expression of the form

$$v = u_1 + u_2 + \ldots + u_n \quad \text{for } u_i \in U_i$$

then V is called internal direct sum of subspace $U_1, U_2, \ldots, U_n$.

External Direct Sum:-

def. Let $V_1, V_2, \ldots, V_n$ be vector spaces over a field F & V be a vector space over field F. V be a vector space having n-ordered tuples $(v_1, v_2, \ldots, v_n)$, $v_i \in V_i$. then V is called external direct sum if

i) Two n-tuples $(v_1, v_2, \ldots, v_n)$ and $(v_1', v_2', \ldots, v_n')$ are equal iff $v_i = v_i'$

ii) $(v_1, v_2, \ldots, v_n) + (v_1', v_2', \ldots, v_n')$
$$= (v_1 + v_1', v_2 + v_2', \ldots, v_n + v_n')$$

iii) $\alpha(v_1, v_2, \ldots, v_n) = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$

external direct sum is denoted by

$$V_1 \oplus V_2 \oplus V_3 \oplus \ldots \oplus V_n$$

Vector Space Homomorphism:-

Let V and W are two vector spaces. A mapping $T: V \to W$ is called homomorphism if

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$
$$T(\alpha v) = \alpha T(v) \qquad \forall\ v_1, v_2 \in V \ \& \ \alpha \in F.$$

Theorem:-

If a vector space V is the internal direct sum of subspaces $U_1, U_2, \ldots, U_n$ then V is isomorphic to the external direct sum of $U_1, U_2, \ldots, U_n$.

Proof. Let $v \in V$ where $v = u_1 + u_2 + u_3 + \cdots + u_n$

Define a mapping

$T : V \to U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_n$

by $T(v) = T(u_1 + u_2 + \cdots + u_n)$

$= (u_1, u_2, \ldots, u_n)$

i) Mapping is well defined as $v \in V$

$v = u_1 + u_2 + \cdots + u_n$

has one and only one representation

(ii) T is onto because each

$(u_1, u_2, u_3, \ldots, u_n) \in U_1 \oplus U_2 \oplus \cdots \oplus U_n$

is image of $u_1 + u_2 + \cdots + u_n \in V$.

iii) T is one one

for $T(v) = T(w)$

$\Rightarrow T(u_1 + u_2 + \cdots + u_n) = T(w_1 + w_2 + \cdots + w_n)$

where $v_i, w_i \in U_i$

$\Rightarrow (u_1, u_2, \ldots, u_n) = (w_1, w_2, \ldots, w_n)$

$\Rightarrow u_1 = w_1, \; u_2 = w_2, \ldots, u_n = w_n$

$\Rightarrow u_1 + u_2 + \cdots + u_n = w_1 + w_2 + \cdots + w_n$

$\Rightarrow u = w$

(iv) $T(v + w) = T(u_1 + u_2 + u_3 + \cdots + u_n + w_1 + w_2 + \cdots + w_n)$

$= T(u_1 + w_1 + u_2 + w_2 + \cdots + u_n + w_n)$

$= (u_1 + w_1, u_2 + w_2, \ldots, u_n + w_n)$

$= (u_1, u_2, \ldots, u_n) + (w_1, w_2, \ldots, w_n)$

by def. of external direct sum,

$= T(v) + T(w)$

& $T(\alpha v) = \cancel{T(\alpha(u_1 + u_2 + \cdots + u_n)} \; T(\alpha(u_1 + u_2 + \cdots + u_n))$

$= T(\alpha u_1 + \alpha u_2 + \cdots + \alpha u_n)$

$= (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$

$= \alpha(u_1, u_2, \ldots, u_n)$

$= \alpha T(v)$ Hence T is homomorphism.

Theorem:

If A and B are finite dimensional subspace of a vector space V(F), then A+B is finite dimensional and dim(A+B) = dim A + dim B − dim(A∩B).

Proof:

Suppose $\{u_1, u_2, \ldots, u_r\}$ be a basis of $A \cap B$

$\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m\}$ be a basis of A.

$\{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_n\}$ be a basis of B

then we have to prove that

$\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\}$

is a basis of A+B.

Consider

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + \beta_1 v_1 + \cdots + \beta_m v_m + \gamma_1 w_1 + \cdots + \gamma_n w_n = 0$$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + \beta_1 v_1 + \cdots + \beta_m v_m = -\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n \quad \text{——— (i)}$$

Since L.H.S of (i) is in A so does R.H.S.

i.e $-\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n \in A$

Also

$-\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n \in B$ $\because w_1, w_2, \ldots, w_n$ is a part of basis of B.

$\therefore -\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n \in A \cap B$

$\Rightarrow -\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n = \delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_r u_r$

as $\{u_1, u_2, \ldots, u_r\}$ is a basis of $A \cap B$

& $\delta_i \in F$.

$\Rightarrow \delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_r u_r + \gamma_1 w_1 + \gamma_2 w_2 + \cdots + \gamma_n w_n = 0$.

Since $\{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_n\}$ is a basis of B (L.I)

$\Rightarrow \delta_1 = \delta_2 = \cdots = \delta_r = \gamma_1 = \gamma_2 = \cdots = \gamma_n = 0$.

so that equation (i) becomes.

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m = 0$

But $\{u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_m\}$ is a basis of A

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_r = \beta_1 = \beta_2 = \cdots = \beta_m = 0$.

i.e each $\alpha_i = \beta_i = \gamma_i = 0$.

Hence $\{u_1, u_2, \cdots, u_r, v_1, \cdots, v_m, w_1, \cdots, w_n\}$ is L.I

Let $x+y \in A+B$ i.e $x \in A$ & $y \in B$.

As basis of $A = \{u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_m\}$

then

$x = a_1 u_1 + a_2 u_2 + \cdots + a_r u_r + b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$.

Also basis of $B = \{u_1, u_2, \cdots, u_r, w_1, w_2, \cdots, w_n\}$.

so

$y = a_1' u_1 + a_2' u_2 + \cdots + a_r' u_r + b_1' w_1 + b_2' w_2 + \cdots + b_n' w_n$

By ting

$A+B = (a_1+a_1')u_1 + (a_2+a_2')u_2 + \cdots + (a_r+a_r')u_r + b_1 v_1 + b_2 v_2$
$+ \cdots + b_m v_m + b_1' w_1 + \cdots + b_n' w_n$

$\therefore \{u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_m\}$

generates A+B

and hence is a basis of A+B.

$\therefore$ A+B is a finite dimensional and

$\dim(A+B) = r+m+n$
$= (r+m) + (r+n) - r$.
$= \dim A + \dim B - \dim(A \cap B)$.

proved

Theorem:- Let V and W be vector spaces
If T is an isomorphism of V onto W.
Then T mapsps a basis of V onto a basis of W.

Proof:-

$T : V \to W$ is isomorphism defined by

$T(v) = w.$

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V.

then we have to prove

$\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis of W.

i) Consider

$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = 0$, $\alpha_i \in F$

$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \cdots + T(\alpha_n v_n) = 0$ $\because$ T is homomorphism $\therefore \alpha T(v) = T(\alpha v)$, $T(v_1 + v_2) = T(v_1) + T(v_2)$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0.$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in \ker T$

$\because$ T is isomorphism i.e one-one

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

$\because \{v_1, v_2, \ldots, v_n\}$ is basis of V.

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$

Hence $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

ii) Let $w \in W$

$\because$ T is onto there must be $v \in V$ such that

$T(v) = w.$

Now $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ for $a_i \in F$.

$\therefore w = T(v)$

$= T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$

$= T(a_1 v_1) + T(a_2 v_2) + \cdots + T(a_n v_n)$, $\because$ T is homo.

$\Rightarrow w = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n).$

i.e w can be generated by
$\{T(v_1), T(v_2), \ldots, T(v_n)\}$.

Thus $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ form a basis of W.

The proof is complete.

2002

Theorem:-

Two finite dimensional vector space are isomorphic iff they are of the same dimension.

Proof:-

Let V and W are two vector spaces of same dimension n and $\{v_1, v_2, \ldots, v_n\}$ be the basis of V and $\{w_1, w_2, \ldots, w_n\}$ be the basis of W.

Define a mapping

$T: V \to W$ by $T(v) = w$ for $v \in V$, $w \in W$.

i.e $T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$.

i) T is well defined

For $v, v' \in V$, if $v = v'$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \alpha_1' v_1 + \alpha_2' v_2 + \cdots + \alpha_n' v_n$$

$$\Rightarrow (\alpha_1 - \alpha_1') v_1 + (\alpha_2 - \alpha_2') v_2 + \cdots + (\alpha_n - \alpha_n') v_n = 0$$

Since $\{v_1, v_2, \ldots, v_n\}$ is basis of V.

$$\therefore \alpha_1 - \alpha_1' = 0 = \alpha_2 - \alpha_2' = \cdots = \alpha_n - \alpha_n'$$

$$\Rightarrow \alpha_1 = \alpha_1', \alpha_2 = \alpha_2', \ldots, \alpha_n = \alpha_n'$$

i.e
$$T(\alpha v) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$
$$= \alpha_1' w_1 + \alpha_2' w_2 + \cdots + \alpha_n' w_n$$
$$= T(v').$$

ii) T is homomorphism

$$T(v + v') = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + \alpha_1' v_1 + \cdots + \alpha_n' v_n)$$
$$= T((\alpha_1 + \alpha_1') v_1 + (\alpha_2 + \alpha_2') v_2 + \cdots + (\alpha_n + \alpha_n') v_n)$$

$= (\alpha_1+\alpha_1')w_1 + (\alpha_2+\alpha_2')w_2 + \cdots + (\alpha_n+\alpha_n')w_n$
$= (\alpha_1 w_1+\alpha_2 w_2+\cdots+\alpha_n w_n) + (\alpha_1' w_1+\alpha_2' w_2+\cdots+\alpha_n' w_n)$
$= T(v) + T(v')$

and

$T(\alpha v) = T(\alpha(\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n))$
$= T(\alpha\alpha_1 v_1+\alpha\alpha_2 v_2+\cdots+\alpha\alpha_n v_n)$
$= \alpha\alpha_1 w_1+\alpha\alpha_2 w_2+\cdots+\alpha\alpha_n w_n$
$= \alpha(\alpha_1 w_1+\alpha_2 w_2+\cdots+\alpha_n w_n)$
$= \alpha T(v)$

iii) T is one-one

Let $T(v) = T(v')$ for $v, v' \in V$.

$\Rightarrow$ ~~$\alpha_1 w_1+\alpha_2 w_2+\cdots+\alpha_n$~~

$\Rightarrow T(\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n) = T(\alpha_1' v_1+\alpha_2' v_2+\cdots+\alpha_n' v_n)$

$\Rightarrow \alpha_1 w_1+\alpha_2 w_2+\cdots+\alpha_n w_n = \alpha_1' w_1+\alpha_2' w_2+\cdots+\alpha_n' w_n$

$\Rightarrow (\alpha_1-\alpha_1')w_1+(\alpha_2-\alpha_2')w_2+\cdots+(\alpha_n-\alpha_n')w_n = 0$

$\because \{w_1, w_2, \ldots, w_n\}$ is basis of W.

$\Rightarrow \alpha_1-\alpha_1' = \alpha_2-\alpha_2' = \cdots = \alpha_n-\alpha_n' = 0$

$\Rightarrow \alpha_1=\alpha_1',\ \alpha_2=\alpha_2', \ldots, \alpha_n=\alpha_n'$

$\Rightarrow \alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n = \alpha_1' v_1+\alpha_2' v_2+\cdots+\alpha_n' v_n$

$\Rightarrow v = v'$

iv) T is onto as every element

$\alpha_1 w_1+\alpha_2 w_2+\cdots+\alpha_n w_n \in W$

is image of $\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n \in V$

Conversely, let $T: V \to W$ is isomorphism

then we have to prove

Dimension of V and W are same

Let $\{v_1, v_2, \ldots, v_n\}$ be basis of V. then we prove that

$\{T(v_1), T(v_2), T(v_3), \ldots, T(v_n)\}$ is a basis of W.

See on page 3 $\boxed{V-24}$

Vector Space Homomorphism

Let V and W are two vector spaces. The set of all homomorphism of V into W is denoted by $Hom(V, W)$

$$Hom(V, W) = \{T_1, T_2, \ldots, T_n\}$$

where each T_i is homomorphism.

Theorem:

Let $V(F)$ & $W(F)$ be two vector spaces introduce an operation in $Hom(V, W)$ and prove that $Hom(V, W)$ is a vector space under this operation.

Proof:

Let $T_1, T_2 \in Hom(V, W)$.

we define $(T_1 + T_2)(v) = T_1(v) + T_2(v)$

& $\lambda T(v) = T(\lambda v)$

to prove $Hom(V, W)$ is a vector space we proceed as follows:

Let $v_1, v_2 \in V$ & $T_1, T_2 \in Hom(V, W)$.

then

$$(T_1 + T_2)(v_1 + v_2) = T_1(v_1 + v_2) + T_2(v_1 + v_2)$$
$$= T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2)$$
$$= T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2)$$
$$= (T_1 + T_2)v_1 + (T_1 + T_2)v_2$$

Also:

$$(T_1 + T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v)$$
$$= \lambda T_1(v) + \lambda T_2(v)$$
$$\Rightarrow (T_1 + T_2)(\lambda v) = \lambda (T_1 + T_2)(v)$$
$$\Rightarrow T_1 + T_2 \in Hom(V, W).$$

i.e $Hom(V, W)$ is closed.

ii) Mapping $(T_1, T_2, \ldots, T_n)$ are associative in general. Consider a mapping T_0 which mapps an element of V into 0 (zero) i.e

$T_0(v) = 0$

then $(T + T_0)v = T(v) + T_0(v)$

$= T(v) + 0$

$= T(v)$

i.e $T_0 + T = T$

i.e T_0 is the identity of $\text{Hom}(V, W)$,

also for $T \in \text{Hom}(V, W)$

so we have

$-T \in \text{Hom}(V, W)$ such that

$\{T + (-T)\}(v) = T(v) + (-1)T(v)$

$= T(v) - T(v) = 0$.

$= T_0(v)$

$\Rightarrow$ inverse exists.

& $(T_1 + T_2)v = T_1(v) + T_2(v)$

$= T_2(v) + T_1(v)$

$= (T_2 + T_1)(v)$

$\Rightarrow$ $\text{Hom}(V, W)$ is an abelian group under '+'

(ii)

$a(T_1 + T_2) = aT_1 + aT_2$

$\cancel{a(T_1 + T_2)(v) = (T_1 + T_2)(av)}$

$\cancel{= T_1(av) + T_2(av)}$

$\cancel{= aT_1(v) + a\,T_2(v)}$

(iii) $(a+b)T = aT + bT$

$\cancel{(a+b)T(v) = T((a+b)v)}$

$\cancel{= T\{av + bv\}}$

$= aT(v) + bT(v)$

~~Hom V W is a vector space~~

(iv) $a(b)T = (ab)T$

$\cancel{a(b)T(v) = aT\{(b)v\} = T\{(a)b\,v\} = T\{(ab)v\}}$

$= ab\,T(v)$

P.T.O

v) $1 . T = T$

As $1 . T(v) = T(1 \cdot v) = T(v)$

As $v \in V$ is a vector space

Hence $Hom(V, W)$ is a vector space.

Theorem:-

If V and W are of dimension m and n resp. then $Hom(V, W)$ is of dimension mn.

Proof:

Let $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ be basis of V and W respectively.

Define a mapping

$T_{ij} : V \to W$ defined by

$$T_{ij}(v_k) = \begin{cases} \lambda_i w_j & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}, \quad \lambda_{ij} \in F$$

Let.

$$\underline{v} = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$$

$$\underline{u} = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_m v_m$$

then

$$\begin{aligned} T_{ij}(\underline{u} + \underline{v}) &= T_{ij}\big[(\mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_m v_m) \\ &\qquad + (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m)\big] \\ &= T_{ij}\big[(\mu_1 + \lambda_1) v_1 + (\mu_2 + \lambda_2) v_2 + \cdots \\ &\qquad + (\mu_m + \lambda_m) v_m\big] \\ &= (\mu_i + \lambda_i) w_j \\ &= \mu_i w_j + \lambda_i w_j \\ &= T_{ij}(\underline{u}) + T_{ij}(\underline{v}) \end{aligned}$$

And

$$\begin{aligned} T_{ij}(\alpha \underline{u}) &= T_{ij}\big(\alpha(\mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_m v_m)\big) \\ &= T_{ij}(\alpha\mu_1 v_1 + \alpha\mu_2 v_2 + \cdots + \alpha\mu_m v_m) \\ &= \alpha \mu_i w_j \\ &= \alpha T_{ij}(\underline{u}) \end{aligned}$$

Thus T_{ij} is homomorphism and $T_{ij} \in Hom(V, W)$.

Now to prove $\{T_{11}, T_{12}, \ldots, T_{ij}, \ldots, T_{mn}\}$ is a basis

Consider

$$\alpha_{11} T_{11} + \alpha_{12} T_{12} + \cdots + \alpha_{ij} T_{ij} + \cdots \alpha_{mn} T_{mn} = 0$$

Now

P.T.O

$$\begin{aligned}&(\alpha_{11}T_{11}+\alpha_{12}T_{12}+\cdots\cdots+\alpha_{1n}T_{1n}\\&+\alpha_{21}T_{21}+\alpha_{22}T_{22}+\cdots\cdots+\alpha_{2n}T_{2n}\\&+\cdots\cdots\cdots\cdots\cdots\cdots\\&+\alpha_{m1}T_{m1}+\alpha_{m2}T_{m2}+\cdots\cdots\alpha_{mn}T_{mn})\,v_1=0 \quad\text{--- (i)}\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &\alpha_{11}T_{11}(v_1)+\alpha_{12}T_{12}(v_1)+\cdots\cdots+\alpha_{1n}T_{1n}(v_1)\\&+\alpha_{21}T_{21}(v_1)+\alpha_{22}T_{22}(v_1)+\cdots\cdots+\alpha_{2n}T_{2n}(v_1)\\&+\cdots\cdots\cdots\cdots\\&+\alpha_{m1}T_{m1}(v_1)\ \alpha_{m2}T(v_1)+\cdots\cdots+\alpha_{mn}T_{mn}(v_1)=0\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &\alpha_{11}\lambda_1w_1+\alpha_{12}\lambda_1w_2+\cdots\cdots+\alpha_{1n}\lambda_1w_n\\&+0+0+\cdots\cdots+0\\&+\cdots\cdots\cdots\cdots\\&+0+0+\cdots\cdots+0=0\end{aligned}$$

$\because T_{ij}(v_k) = \lambda_i w_j,\ i=k$
$= 0,\ i\neq k$

$\Rightarrow \alpha_{11}w_1+\alpha_{12}w_2+\cdots\cdots+\alpha_{1n}w_n=0 \quad \because \lambda_1\neq 0$
and $\{w_1,w_2,\cdots,w_n\}$ is basis of W
$\Rightarrow \alpha_{11}=0=\alpha_{12}=\alpha_{13}=\cdots\cdots=\alpha_{1n}$
Similarly operating (i) on v_k we have

$$\alpha_{ij}=0\ ,\quad i=1,2,\cdots,m\quad j=1,2,\cdots,n.$$

So the set $\{T_{11},T_{12},\cdots,T_{ij},\cdots,T_{mn}\}$ is L.I.
Now consider

$$\begin{aligned}S_0=\ &a_{11}T_{11}+a_{12}T_{12}+\cdots\cdots+a_{1n}T_{1n}\\&+a_{21}T_{21}+a_{22}T_{22}+\cdots\cdots+a_{2n}T_{2n}\\&+\cdots\cdots\cdots\cdots\\&+a_{m1}T_{m1}+a_{m2}T_{m2}+\cdots\cdots+a_{mn}T_{mn}\end{aligned}$$

So

$$\begin{aligned}S_0(v_1)=\ &(a_{11}T_{11}+a_{12}T_{12}+\cdots\cdots+a_{1n}T_{1n}\\&+a_{21}T_{21}+a_{22}T_{22}+\cdots\cdots+a_{2n}T_{2n}\\&+\cdots\cdots\cdots\cdots\\&+a_{m1}T_{m1}+a_{m2}T_{m2}+\cdots\cdots+a_{mn}T_{mn})\,v_1\end{aligned}$$

$\Rightarrow\ S_0(v_1) = a_{11}T_{11}(v_1) + a_{12}T_{12}(v_1) + \cdots\cdots + a_{1n}T_{1n}(v_1)$
$+ a_{21}T_{21}(v_1) + a_{22}T_{22}(v_1) + \cdots\cdots + a_{2n}T_{2n}(v_1)$
$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$+ a_{m1}T_{m1}(v_1) + a_{m2}T_{m2}(v_1) + \cdots\cdots + a_{mn}T_{mn}(v_1)$
$= a_{11}\lambda_1 w_1 + a_{12}\lambda_1 w_2 + a_{13}\lambda_1 w_3 + \cdots\cdots + a_{1n}\lambda_1 w_n$

Similarly

$S_0(v_2) = a_{21}\lambda_2 w_1 + a_{22}\lambda_2 w_2 + a_{23}\lambda_2 w_3 + \cdots\cdots + a_{2n}\lambda_2 w_n$
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$S_0(v_k) = a_{k1}\lambda_k w_1 + a_{k2}\lambda_k w_2 + a_{k3}\lambda_k w_3 + \cdots\cdots + a_{kn}\lambda_k w_n$

Let $\quad S \in \mathrm{Hom}(V, W)$

$\Rightarrow\ S(v_1), S(v_2), \ldots\ldots, S(v_k) \in W$

so

$S(v_1) = a_{11}w_1 + a_{12}w_2 + \cdots\cdots + a_{1n}w_n$
$S(v_2) = a_{21}w_1 + a_{22}w_2 + \cdots\cdots + a_{2n}w_n$
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$S(v_k) = a_{k1}w_1 + a_{k2}w_2 + \cdots\cdots + a_{kn}w_n$

i.e $\quad S \simeq S_0 \quad$ so $\quad S_0 \in \mathrm{Hom}(V, W)$

Thus

$\{T_{11}, T_{12}, \ldots\ldots, T_{ij}, \ldots\ldots T_{mn}\}$ form a basis of $\mathrm{Hom}(V, W)$

$\Rightarrow\ \dim(\mathrm{Hom}(V, W)) = mn$

Definition (Dual Space):-

Let ~~F~~ V be a vector space over a field F. Then $\operatorname{Hom}(V, F)$ is called dual space and is denoted by V^* or $\hat{V}$. Its element are called linear functional.

Theorem:-

If V is finite dimensional vector space over F. then prove $V \cong V^*$.

Proof:

Since $\dim V = \dim V^*$

so consider $\dim V = \dim V^* = m$

Define a mapping $T : V \to V^*$ by

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m$$

i) T is homomorphism

$$\begin{aligned}
T(v + v') &= T[(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) + (\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)] \\
&= T((\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \cdots + (\alpha_m + \beta_m) v_m) \\
&= (\alpha_1 + \beta_1) f_1 + (\alpha_2 + \beta_2) f_2 + \cdots + (\alpha_m + \beta_m) f_m \\
&= (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m) + (\beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m) \\
&= T(v) + T(v')
\end{aligned}$$

and

$$\begin{aligned}
T(\alpha v) &= T(\alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m)) \\
&= T(\alpha \alpha_1 v_1 + \alpha \alpha_2 v_2 + \cdots + \alpha \alpha_m v_m) \\
&= \alpha \alpha_1 f_1 + \alpha \alpha_2 f_2 + \cdots + \alpha \alpha_m f_m \\
&= \alpha [\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m] \\
&= \alpha T(v)
\end{aligned}$$

ii) T is one-one

if $T(v) = T(v')$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = T(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)$

$\Rightarrow \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m = \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m$

$\Rightarrow (\alpha_1 - \beta_1) f_1 + (\alpha_2 - \beta_2) f_2 + \cdots + (\alpha_m - \beta_m) f_m = 0$

$\because \{f_1, f_2, \ldots, f_m\}$ is basis of V^*

$\therefore \alpha_1 - \beta_1 = 0 = \alpha_2 - \beta_2 = \cdots = \alpha_m - \beta_m$

$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_m = \beta_m$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m$

$\Rightarrow v = v'$

iii) T is onto

Since for $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m \in V^*$

we have

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \in V$$

such that

$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m$

Thus T is onto.

and hence $V \cong V^*$

Definition:

Let $T: V_1 \to V_2$ is homomorphism of a vector space $V_1(F)$ to a vector space $V_2(F)$ then $\ker T$ is called null space denoted by $N(T)$.

The dimension of $N(T)$ is called nullity.

Theorem:

Let $T: V_1 \to V_2$ be a vector space homomorphism then $\dim V_1 = \dim N(T) + \dim R(T)$.

Proof:

Let $\dim N(T) = m$

and $\dim (V_1) = n$

Let $\{v_1, v_2, \dots, v_m\}$ be basis of $N(T) = \ker T$

Since $N(T) = \ker T$ is a subspace of V_1

$\because$ we can take basis of V_1

$$\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$$

we have to prove

$\{T(v_{m+1}), T(v_{m+2}), \dots, T(v_n)\}$ form basis of $R(T)$

Let $w \in R(T)$ then there is $v \in V_1$ such that

$$T(v) = w$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \dots + \alpha_m v_m) = w$$

$$\Rightarrow \alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_m T(v_m) + \alpha_{m+1} T(v_{m+1}) + \dots + \alpha_m T(v_m) = w$$

$\because \{v_1, v_2, \dots, v_m\} \in N(T) = \ker T$

$\therefore T(v_1) = 0, T(v_2) = 0, \dots, T(v_m) = 0$

$$\Rightarrow \alpha_{m+1} T(v_{m+1}) + \alpha_{m+2} T(v_{m+2}) + \dots + \alpha_n T(v_n) = w$$

$\Rightarrow T(v_{m+1}), T(v_{m+2}), \dots, T(v_n)$ generates $R(T)$

Now consider

$$\beta_{m+1}T(v_{m+1}) + \beta_{m+2}T(v_{m+2}) + \cdots + \beta_n T(v_n) = 0$$

$$\Rightarrow T(\beta_{m+1}v_{m+1}) + T(\beta_{m+2}v_{m+2}) + \cdots + T(\beta_n v_n) = 0$$

$\because T$ is homomorphism

$$\Rightarrow T(\beta_{m+1}v_{m+1} + \beta_{m+2}v_{m+2} + \cdots + \beta_n v_n) = 0$$

$$\Rightarrow \beta_{m+1}v_{m+1} + \beta_{m+2}v_{m+2} + \cdots + \beta_n v_n \in \operatorname{Ker} T = N(T)$$

Since $\{v_1, v_2, \ldots, v_m\}$ is basis of $N(T)$

so $\exists\ \delta_1, \delta_2, \ldots, \delta_m \in F$ such that

$$\beta_{m+1}v_{m+1} + \beta_{m+2}v_{m+2} + \cdots + \beta_n v_n = \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_m v_m$$

$$\Rightarrow \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_m v_m - \beta_{m+1}v_{m+1} - \beta_{m+2}v_{m+2} - \cdots - \beta_n v_n = 0$$

As $\{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ is basis of V_1

therefore $\delta_1 = \delta_2 = \cdots = \delta_m = \beta_{m+1} = \beta_{m+2} = \cdots = \beta_n = 0$

i.e $\beta_{m+1} = \beta_{m+2} = \cdots = \beta_n = 0$

$\Rightarrow \{T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\}$ is L.I

and hence form a basis of $R(T)$.

So $\dim R(T) = n - m$

$$= \dim V_1 - \dim N(T)$$

$$\Rightarrow \dim V_1 = \dim N(T) + \dim R(T)$$

proved

Theorem

-: V is a vector space over F and $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $\varphi_1, \varphi_2, \ldots, \varphi_n \in V^* = \text{Hom}(V, F)$ are linear functional defined by

$$\varphi_i(v_j) = \delta_{ij} = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}$$

then $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a basis of V^*.

Proof.

Let $\varphi \in V^*$ be taken

$\varphi(v_1) = k_1$, $\varphi(v_2) = k_2, \ldots, \varphi(v_n) = k_n$

where $k_1, k_2, \ldots, k_n \in F$

Let

$\psi = k_1\varphi_1 + k_2\varphi_2 + k_3\varphi_3 + \cdots + k_n\varphi_n$

$$\psi(v_1) = (k_1\varphi_1 + k_2\varphi_2 + \cdots + k_n\varphi_n)v_1$$
$$= k_1\varphi_1(v_1) + k_2\varphi_2(v_1) + \cdots + k_n\varphi_n(v_1)$$
$$= k_1(1) + k_2(0) + \cdots + k_n(0)$$
$$= k_1$$

Also

$$\psi(v_2) = (k_1\varphi_1 + k_2\varphi_2 + \cdots + k_n\varphi_n)v_2$$
$$= k_1\varphi_1(v_2) + k_2\varphi_2(v_2) + \cdots + k_n\varphi_n(v_2)$$
$$= k_1(0) + k_2(1) + k_3(0) + \cdots + k_n(0)$$
$$= k_2$$

$\Rightarrow \psi(v_i) = k_i = \varphi(v_i)$

i.e $\psi = \varphi$

$\Rightarrow \varphi = \psi = k_1\varphi_1 + k_2\varphi_2 + \cdots + k_n\varphi_n$

So $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ span V^*.

To prove $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a linearly independent:

Consider

$$a_1\varphi_1 + a_2\varphi_2 + \cdots + a_n\varphi_n = 0$$

then operating it on v_1

$$(a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n)v_1 = 0 \cdot v_1$$

$$\Rightarrow a_1\phi_1(v_1) + a_2\phi_2(v_1) + \cdots + a_n\phi_n(v_1) = 0$$

$$\Rightarrow a_1(1) + a_2(0) + \cdots + a_n(0) = 0.$$

$$\Rightarrow a_1 = 0$$

Similarly for $i = 2, 3, \ldots, n$

$$(a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n)v_i = 0 \cdot v_i$$

$$\Rightarrow a_1\phi_1(v_i) + a_2\phi_2(v_i) + \cdots + a_i\phi_i(v_i) + \cdots + a_n\phi_n(v_i) = 0$$

$$\Rightarrow a_1(0) + a_2(0) + \cdots + a_i(1) + \cdots + a_n(0) = 0$$

$$\Rightarrow 0 + 0 + \cdots + a_i + \cdots + 0 = 0$$

$$\Rightarrow a_i = 0$$

i.e $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $\ldots$, $a_n = 0$

Hence $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is L.I and so is a basis of V^*

Example

Consider the basis of
$R^2 = \{v_1 = (2,1) , v_2 = (3,1)\}$
Find dual basis of $\{\phi_1, \phi_2\}$.

Solution.

$\phi_1(v_1) = 1 \quad , \quad \phi_1(v_2) = 0$
$\phi_2(v_1) = 0 \quad , \quad \phi_2(v_2) = 1$

Since ϕ_1, ϕ_2 are linear functional
$\phi_1(x,y) = ax + by$
and $\phi_2(x,y) = cx + dy$
$\phi_1(v_1) = 1$
$\Rightarrow \phi_1(2,1) = 1 \Rightarrow 2a + b = 1$ ——— (i)
$\phi_1(v_2) = 0$
$\Rightarrow \phi_1(3,1) = 0 \Rightarrow 3a + b = 0$ ——— (ii)
By (i) and (ii)
$a = -1$ and $b = 3$
Now $\phi_2(v_1) = 0$
$\phi_2(2,1) = 0 \Rightarrow 2c + d = 0$ ——— (iii)
and $\phi_2(v_2) = 1$
$\phi_2(3,1) = 1 \Rightarrow 3c + d = 1$ ——— (iv)
Solving (iii) and (iv)
$c = 1$ and $d = -2$
therefore $\phi_1 = -x + 3y$
$\phi_2 = x - 2y$

Example

Let a basis of $\mathbb{R}^3$ is $\{v_1, v_2, v_3\}$
$v_1 = \{1, -1, 3\}, \quad v_2 = \{0, 1, -1\}, \quad v_3 = \{0, 3, -2\}$
Find dual basis ϕ_1, ϕ_2 and ϕ_3
such that $\phi_i(v_j) = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}$

Do youself as above

* Question

Let $V = \{a + bt : a, b \in \mathbb{R}\}$ be a vector space of polynomial of degree ≤ 1.

Let $\phi_1, \phi_2 : V \to \mathbb{R}$ be defined by

$$\phi_1(f(t)) = \int_0^1 f(t)\,dt$$

$$\phi_2(f(t)) = \int_0^2 f(t)\,dt$$

where $\phi_1, \phi_2 \in V^*$ (dual space).

Find corresponding basis v_1, v_2 of V.

Solution:

let $v_1 = a + bt$ and $v_2 = \cancel{a+bt}\ c + dt$

By definition

$\phi_1(v_1) = 1$, $\phi_1(v_2) = 0$, $\phi_2(v_1) = 0$, $\phi_2(v_2) = 1$

$\phi_1(v_1) = 1$

$$\Rightarrow \int_0^1 v_1\,dt = 1 \Rightarrow \int_0^1 (a + bt)\,dt = 1$$

$$\Rightarrow \left| at + \frac{bt^2}{2} \right|_0^1 = 1 \Rightarrow a + \frac{b}{2} = 1$$

$$\Rightarrow 2a + b = 2 \text{ ——— (i)}$$

$\phi_2(v_1) = 0$

$$\int_0^2 (a + bt)\,dt = 0 \Rightarrow \left| at + \frac{bt^2}{2} \right|_0^2 = 0$$

$$\Rightarrow 2a + 2b = 0 \Rightarrow a + b = 0 \text{ ——— (ii)}$$

By (i) and (ii)

$$\begin{array}{r} 2a + b = 2 \\ \underline{\mp a \pm b = \mp 0} \\ a \quad = 2 \end{array} \Rightarrow b = -2$$

Further $\phi_1(v_2) = 0$

$$\Rightarrow \int_0^1 v_2\,dt = 0 \Rightarrow \int_0^1 (c + dt)\,dt = 0$$

$$\Rightarrow \left| ct + \frac{dt^2}{2} \right|_0^1 = 0$$

$$\Rightarrow \quad c + \frac{d}{2} = 0 \quad \text{or} \quad 2c + d = 0 \quad \text{--- (iii)}$$

$$\phi_2(v_2) = 1$$

$$\Rightarrow \int_0^2 v_2 \, dt = 1$$

$$\Rightarrow \int_0^2 (c + dt)\,dt = 1 \quad \Rightarrow \left| ct + \frac{dt^2}{2} \right|_0^2 = 1$$

$$\Rightarrow \quad 2c + 2d = 1 \quad \text{--- (iv)}$$

Subtracting (iii) from (iv)

$$\begin{array}{r} \cancel{2c} + 2d = 1 \\ -\cancel{2c} \pm d = -0 \\ \hline d = 1 \end{array} \quad \Rightarrow \quad c = -\tfrac{1}{2}$$

Hence

$$v_1 = 2 - 2t$$

and $v_2 = -\frac{1}{2} + t$ are basis of V corresponding to dual basis V^*

Eigen Value

def:- let 'A' be a n square matrix, then $\lambda \in F$ is eigen value of A if there exist a non-zero column vector $v \in F^n$ such that $A\cdot v = \lambda v$

here v is an eigen vector corresponding to eigen value λ.

Exercise

Find eigen values and associative eigen vector of a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Solution:-

Let $v = [x \quad y]^t = \begin{bmatrix} x \\ y \end{bmatrix}$

$\because \quad Av = \lambda v$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x + 2y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$

$\Rightarrow \quad x + 2y = \lambda x$

$3x + 2y = \lambda y$

or $\quad (1-\lambda)x + 2y = 0 \quad \text{——— (1)}$

$3x + (2-\lambda)y = 0 \quad \text{——— (2)}$

For non-trivial solution

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$

$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda-4)(\lambda+1) = 0 \quad \Rightarrow \quad \lambda = 4, -1$

hence $\lambda = 4, -1$ are eigen values

$\lambda = -1$ in eq (i) $\Rightarrow$ $2x + 2y = 0$

or $x + y = 0$

or $y = -x$

thus

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

i.e eigen vector is $[1 \ -1]^t$

and $\lambda = 4$ in eq (i) $\Rightarrow$ $-3x + 2y = 0$

$\Rightarrow$ $2y = 3x$

or $y = \frac{3}{2}x$

thus

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \frac{3}{2}x \end{pmatrix} = \frac{x}{2}\begin{pmatrix} 2 \\ 3x \end{pmatrix}$$

i.e eigen vector is $[2 \ 3]^t$

Note

$$Av = \lambda v$$
$$\Rightarrow Av - \lambda v = 0$$
$$\Rightarrow (A - \lambda I)v = 0 \quad \text{where } I \text{ is identity}$$
$$\Rightarrow |A - \lambda I| = 0$$

and

$$Av = \lambda v$$
$$A(kv) = k\lambda v$$
$$= k(\lambda v)$$
$$= \lambda(kv)$$

then λ and k are eigen values for A.

Eigen Value & Eigen Vector (~~Alternative~~)

def:- Let $T: V \to V$ be a linear operator then $\lambda \in F$ is called eigen value of T if there exist a non-zero vector v such that

$$T(v) = \lambda v$$

here v is eigen vector.

Note that kv is also eigen vector for same eigen value λ.

$$T(kv) = kT(v)$$
$$= k\lambda v = \lambda kv.$$

Theorem:-

Let λ be an eigen value of an operator $T: V \to V$. Let V_λ denotes set of all eigen vectors of T belonging to same eigen value λ. The V_λ is a subspace of V.

Proof: ~~Let λ be an eigen value of an operator~~

Let $v, w \in V_\lambda$.

then $T(v) = \lambda v$ and $T(w) = \lambda w$

Now
$$T(av+bw) = T(av) + T(bw)$$
$$= aT(v) + bT(w)$$
$$= a\lambda v + b\lambda w$$
$$= \lambda(av + bw)$$

$\Rightarrow$ $av+bw$ is also an eigen vector for λ.

hence $av + bw \in V_\lambda$

$\Rightarrow$ V_λ is a subspace

Theorem

:- Let $\{v_1, v_2, \ldots, v_n\}$ be non-zero eigen vectors of an operator T corresponding to distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively then $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

Proof.

We prove the theorem by Mathematical Induction.

Let $n=1$ so if $av_1 = 0$

$$\Rightarrow a = 0 \quad \text{as } v_1 \neq 0$$

so Condition I is true.

Let the theorem is true for $k = n-1$

i.e $v_1, v_2, \ldots, v_{n-1}$ are L.I (linearly independent)

then $a_1v_1 + a_2v_2 + \cdots + a_{n-1}v_{n-1} = 0$

$$\Rightarrow a_1 = a_2 = \cdots = a_{n-1} = 0$$

Consider

$$b_1v_1 + b_2v_2 + \cdots + b_nv_n = 0 \quad \text{--- (i)}$$

$$T(b_1v_1 + b_2v_2 + \cdots + b_nv_n) = 0$$

$$\Rightarrow T(b_1v_1) + T(b_2v_2) + \cdots + T(b_nv_n) = 0$$

$$\Rightarrow b_1T(v_1) + b_2T(v_2) + \cdots + b_nT(v_n) = 0$$

$$\Rightarrow b_1\lambda_1v_1 + b_2\lambda_2v_2 + \cdots + b_n\lambda_nv_n = 0$$

or

$$b_1\lambda_1v_1 + b_2\lambda_2v_2 + \cdots + b_{n-1}\lambda_{n-1}v_{n-1} + b_n\lambda_nv_n = 0 \quad \text{--- (ii)}$$

Multiplying eq (i) by λ_n

$$\lambda_nb_1v_1 + \lambda_nb_2v_2 + \cdots + \lambda_nb_{n-1}v_{n-1} + \lambda_nb_nv_n = 0 \quad \text{--- (iii)}$$

Subtracting (iii) from (ii)

$b_1(\lambda_1-\lambda_n)v_1 + b_2(\lambda_2-\lambda_n)v_2 + \cdots + b_{n-1}(\lambda_{n-1}-\lambda_n)v_{n-1} = 0$

Since $\{v_1, v_2, \cdots, v_{n-1}\}$ is L.I

$\Rightarrow b_1 = b_2 = b_3 = \cdots = b_{n-1}$

$\because \lambda_i - \lambda_n \neq 0 \quad ; \quad i = 1, 2, \cdots, n-1$

because if $\lambda_i - \lambda_n = 0$

$\Rightarrow \lambda_i = \lambda_n$ for $i = 1, 2, \cdots, n-1$

a contradiction as each $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct.

Now from eq (ii)

$0 + 0 + \cdots + 0 + b_n v_n = 0$

$\Rightarrow b_n v_n = 0$

$\Rightarrow b_n = 0 \quad \because v_n \neq 0$

hence the vectors $v_1, v_2, \cdots, v_n$ are Linearly independent

Characteristic Polynomial / Equation / Matrix :-

def:- Let A be a n square matrix over F. then $tI - A$ is called characteristic matrix. $|tI - A|$ is called characteristic polynomial. and $|tI - A| = 0$ is called characteristic equation.

i.e

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$tI - A = t\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} t-a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & -a_{23} & & -a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & t-a_{nn} \end{bmatrix}$$

and $\Delta_A(t) = \det(tI - A)$ is characteristic polynomial.

Also $\Delta_A(t) = 0$ or $|tI - A| = 0$ is characteristic equation.

Exercise:

Find characteristic polynomial of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$tI - A = t\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} t-1 & -3 & 0 \\ 2 & t-2 & 1 \\ -4 & 0 & t+2 \end{bmatrix}$$

$\Delta_A(t) = |tI - A|$

$$= \begin{vmatrix} t-1 & -3 & 0 \\ 2 & t-2 & 1 \\ -4 & 0 & t+2 \end{vmatrix}$$

$= t^3 - t^2 + 2t + 28$ is characteristic polynomial.

Also $\Delta_A(t) = 0$

$\Rightarrow t^3 - t^2 + 2t + 28 = 0$ is characteristic equation.

Note: Degree of eq. will be equal to the order of matrix.

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$tI - A = t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} t-2 & -3 \\ -1 & t-5 \end{bmatrix}$$

$$B(t) = \text{adj of } (tI - A) = \begin{bmatrix} t-5 & 3 \\ 1 & t-2 \end{bmatrix}$$

$$= \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} -5 & 3 \\ 1 & -2 \end{bmatrix} = B_1 t + B_0$$

If $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

then $B(t) = \text{adj}(tI - A)$

$= B_2 t^2 + B_1 t + B_0$

Calay Hamilton Theorem:

:- Every square matrix is zero of its characteristic polynomial.

OR Every square matrix satisfies its characteristic equation.

Proof:

Let A be n square metrix and $\Delta_A(t) = |tI - A|$ be its characteristic polynomial.

i.e $\Delta_A(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0$

Let $B(t)$ is adjoint of $tI - A$.

Since elements of $B(t)$ are cofactors of $tI - A$ and so are polynomial of degree not more than $n-1$. and we can write

$B(t) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1 t + B_0$

where B_i are square matrices of order n over F.

Since by definition of adjoint of a matrix

$$(tI - A)B(t) = |tI - A|\, I$$

$$\Rightarrow (tI - A)(B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1 t + B_0)$$
$$= (t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0)I$$

Comparing the co-effients.

Comparing $t^n \Rightarrow B_{n-1}I = I$

" $t^{n-1} \Rightarrow B_{n-2}I - AB_{n-1} = a_{n-1}I$

" $t^{n-2} \Rightarrow B_{n-3}I - AB_{n-2} = a_{n-2}I$

" $t^1 \Rightarrow B_0 I - AB_1 = a_1 I$

" $t^0 \Rightarrow -AB = a_0 I$

Multiplying above equations by first to last by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$ respectively. we have.

$$A^n B_{n-1} I = A^n I$$

$$A^{n-1} B_{n-2} I - A^n B_{n-1} I = a_{n-1} A^{n-1} I$$

$$A^{n-2} B_{n-3} I - A^{n-1} B_{n-2} I = a_{n-2} A^{n-2} I$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$A B_0 I - A^2 B_1 = a_1 A I$$

$$-A B_0 I = a_0 I$$

Adding both sides of above equations

$$0 = A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \cdots + a_1 A + a_0$$

As required.

Minimum Polynomial

A polynomial $m(t)$ is called minimum polynomial if

i) $m(t)$ divides $\Delta(t)$

ii) Each irreducible factor of $\Delta(t)$ divides $m(t)$

iii) $m(A) = 0$.

Question.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$tI - A = \begin{bmatrix} t-2 & -1 & 0 & 0 \\ 0 & t-2 & 0 & 0 \\ 0 & 0 & t-1 & -1 \\ 0 & 0 & 2 & t-4 \end{bmatrix}$$

$$|tI - A| = \begin{vmatrix} t-2 & -1 & 0 & 0 \\ 0 & t-2 & 0 & 0 \\ 0 & 0 & t-2 & -1 \\ 0 & 0 & -2 & t-4 \end{vmatrix}$$

expanding by c_1

$$= (t-2) \begin{vmatrix} \cancel{t-2} & \cancel{0} & \cancel{0} \\ \cancel{0} & t-2 & -1 \\ 0 & -2 & t-4 \end{vmatrix}$$

$$= (t-2)(t-2) \begin{vmatrix} \cancel{t-2} & -1 \\ 2 & t-4 \end{vmatrix}$$

$$= (t-2)^3(t-3) \quad = (t-2)^2\big((t-2)(t-4)+2\big)$$

$$= (t^2-4t+4)(t^2-6t+8+2)$$

$$= t^4 - 10t^3 - 4t^2 + 40t + 64 \quad \text{(after solving)}$$

is characteristic polynomial

Possible minimum polynomial are

i) $(t-2)(t-3) = f(t)$

ii) $(t-2)^2(t-3) = g(t)$

iii) $(t-2)^3(t-3) = h(t)$.

$$f(A) = (A-2)(A-3)$$
$$= (A-2I)(A-3I)$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \neq 0$$

$\Rightarrow f(t)$ is not minimum polynomial.

Now $g(t) = (t-2)^2(t-3)$

$$\Rightarrow g(A) = (A-2)^2(A-3)$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$=$ solve $= 0$

$\Rightarrow g(t) = (t-2)^2(t-3)$ is minimum polynomial

iii) $h(A) = (A-2)^2(A-3)$

Do yourself

Theorem:-

Prove that the minimum polynomial $m(t)$ divides every polynomial which has A as a zero. In particular $m(t)$ divides the characteristic polynomial $\Delta(t)$ of A.

Proof:

Let $f(t)$ be a polynomial for which $f(A)=0$. then by division algorithm, there are polynomial $q(t)$ and $r(t)$ such that

$$f(t) = q(t).m(t) + r(t) \quad \text{------ (i)}$$

where $r(t)=0$ or degree of $r(t)$ is less then that of $m(t)$.

From (i) $\quad f(A) = q(A)\,m(A) + r(A) \quad$ by $t=A$.

$$\Rightarrow \quad 0 = q(A)\times 0 + r(A)$$

$$\Rightarrow \quad r(A) = 0$$

then $r(t)$ is a polynomial of degree less than that of $m(t)$, which has A as a zero which contradict the definition of $m(t)$.

hence $r(t) = 0$

$$\Rightarrow \quad f(t) = g(t).m(t)$$

i.e $m(t)$ divides $f(t)$

Also then $m(t)$ divides $\Delta(t)$

Theorem

:- Let $m(t)$ be the minimum polynomial of an n-square matrix A. Then show that characteristic polynomial of A divides $(m(t))^n$.

Proof.

Let $m(t) = t^r + c_1 t^{r-1} + c_2 t^{r-2} + \cdots + c_{r-1} t + c_r$

Consider

$$B_0 = I \quad \text{———} \quad (i)$$

$$B_1 = A + c_1 I \quad \text{———} \quad (2)$$

$$B_2 = A^2 + c_1 A + c_2 I \quad \text{———} \quad (3)$$

$$B_3 = A^3 + c_1 A^2 + c_2 A + c_3 I \quad \text{———} \quad (4)$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$B_{r-1} = A^{r-1} + c_1 A^{r-2} + \cdots + c_{r-1} I \quad \text{———} \quad (r).$$

Take

$$B(t) = t^{r-1} B_0 + t^{r-2} B_1 + t^{r-3} B_2 + \cdots + t B_{r-2} + B_{r-1}$$

Now

$$(tI - A) B(t) = (tI - A)\big(t^{r-1} B_0 + t^{r-2} B_1 + \cdots\cdots + t B_{r-2} + B_{r-1}\big)$$

$$= t^r B_0 I + t^{r-1} B_1 I + t^{r-2} B_2 I + \cdots + t^2 B_{r-2} I + t B_{r-1} I - \big(t^{r-1} A B_0 + t^{r-2} A B_1 + \cdots\cdots + t A B_{r-2} + A B_{r-1}\big)$$

$$= t^r B_0 + t^{r-1}(B_1 - A B_0) + t^{r-2}(B_2 - A B_1) + \cdots + t(B_{r-1} - A B_{r-2}) - A B_{r-1}$$

$$\text{———} \quad (a)$$

Now from eqs (i) to (r) gives

$$B_1 - A B_0 = c_1 I$$

$$B_2 - A B_1 = c_2 I$$

$$\cdots\cdots\cdots\cdots$$

$\because \quad B_{r-1} - AB_{r-2} = c_{r-1} I$

Also from rth equation

$AB_{r-1} = A^r + c_1 A^{r-1} + \cdots\cdots + c_{r-1} AI$

$= A^r + c_1 A^{r-1} + \cdots\cdots + c_{r-1} AI + c_r I - c_r I$

$= m(A) - c_r I$

$\Rightarrow AB_{r-1} = -c_r I \qquad \because m(A) = 0$

Using all these values in eq. (2).

$(tI - A)\cdot B(t) = t^r I + t^{r-1} c_1 I + t^{r-2} c_2 I + \cdots\cdots$

$\cdots\cdots + t c_{r-1} I + c_r I$

$= (t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + t c_{r-1} + c_r) I$

taking determinant to both sides.

$|(tI - A)\, B(t)| = |(t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + c_r) I|$

$\Rightarrow |tI - A|\,|B(t)| = (t^r + c_1 t^{r-1} + c_2 t^{r-2} + \cdots + c_r)^n$

$= (m(t))^n$

$\Rightarrow |tI - A|$ divides $(m(t))^n$

i.e characteristic polynomial divide $(m(t))^n$

Similar Matrix

def:– A matrix B is similar to a matrix A if there is non-singular matrix P such that

$$B = P^{-1}AP \quad \text{or} \quad PB = AP.$$

Diagonalization of Matrix:

def:– A matrix A is said to be diagonalizable if there is a matrix such that

$$B = P^{-1}AP$$

In this case column of P are eigen vectors of A and diagonal element of B are corresponding eigen values of A.

Question If $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ then diagonalize this matrix

Solution.

To find eigen values

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 4 & -2 \\ -3 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 5, -2$$

i) $\lambda = 5$ then for eigen vectors

$$MX = 0$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow x_1 - 2x_2 = 0$$

$$-3x_1 + 6x_2 = 0$$

One of its solution is $x_2 = 1 \Rightarrow x_1 = 2$

eigen vector $(2, 1)^t$

ii) $\lambda = -2$

$$\Rightarrow M \cdot X = 0 \Rightarrow \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow -6x-2y=0$
$-3x-y=0$

$\Rightarrow$ if $x=1 \Rightarrow y=-3$

eigen vector $=(1,-3)^t$

Now
$$P=\begin{pmatrix}2 & 1\\ 1 & -3\end{pmatrix}$$

$$|P|=-6-1=-7$$

$$P^{-1}=-\frac{1}{7}\begin{bmatrix}-3 & -1\\ -1 & 2\end{bmatrix}=\begin{pmatrix}3/7 & 1/7\\ 1/7 & -2/7\end{pmatrix}$$

Now
$$P^{-1}AP=\begin{pmatrix}3/7 & 1/7\\ 1/7 & -2/7\end{pmatrix}\begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & -3\end{pmatrix}$$

$$=\begin{pmatrix}3/7 & 1/7\\ 1/7 & -2/7\end{pmatrix}\begin{pmatrix}10 & -2\\ 5 & 6\end{pmatrix}$$

$$=\begin{pmatrix}5 & 0\\ 0 & -2\end{pmatrix}$$

is diagonal where diagonal ~~val~~ elements are eigen values of A.

Question: Find A^{10} for $A=\begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}$

$B=P^{-1}AP$

$PBP^{-1}=A$

So $A^{10}=(PBP^{-1})^{10}$

$=PB^{10}P^{-1}$

$$=\begin{bmatrix}2 & 1\\ 1 & -3\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}^{10}\begin{bmatrix}3/7 & 1/7\\ 1/7 & -2/7\end{bmatrix}$$

Simplify yourself

$\because (PBP^{-1})^2$
$=(PBP^{-1})(PBP^{-1})$
$=PBP^{-1}PBP^{-1}$
$=PBIBP^{-1}$
$=PB^2P^{-1}$

Theorem:

Similar matrix A and $P^{-1}AP$ have the same characteristic polynomial.

Proof:

Let A and B are similar matrices then $B = P^{-1}AP$

Using

$$tI = P^{-1}tIP$$

$$|tI - B| = |tI - P^{-1}AP|$$

$$= |P^{-1}tIP - P^{-1}AP|$$

$$= |P^{-1}(tI - A)P|$$

$$= |P^{-1}|\,|tI - A|\,|P|$$

$$= |tI - A|\,|P^{-1}|\,|P|$$

$$= |tI - A|$$

As required