Theory of Optimization

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Dedicated

To

My Honorable Teacher

Ma'am Iqra Razzaq

&

My parents

Lecture # 01:

Recommend Books:

(i) Gottfried B.S and Weisman, J. Introduction to optimization Theory (New Jersey, 1973).

(ii) Wismar D.A and Chatterjee R. Introduction to Non-Linear optimization (North-Holland, New York, 1978)

(iii) Intriligator M.D Mathematical optimization and Economic Theory (Prentice-Hall, Inc, New Jersey,1971)

Introduction:

Optimization play an important role in daily life, business and engineering.

I. Running a business:

- To maximize profit, minimize loss.
- Maximize a (profit) efficiency, minimum risk.

II. Design:

- Minimize the weight of bridge.
- Maximize the strength within design.
- Constraints, airplane, engineering, s.t shape, design and material selection.

Planning:

- Selecting applied tools to minimize time and fuel consumption of an airplane.
- Super market pricing and easy way of communication.

Formal definition: (Min or Max)

A real function by deciding the values of free variables from within an allowed set.

Status of optimization:

In last few decades, astonishing improvements in computer hardware and software motivated optimization modeling, algorithm, designs and implementation (are good for it) solving certain optimization problems has become standard techniques and every day practice in business, science and engineering.

It is now possible to solve certain optimization problem with thousands, millions and even thousands of millions of minimums of variable optimization (along with statistics) has been foundation of mechanics learning and big data analytics.

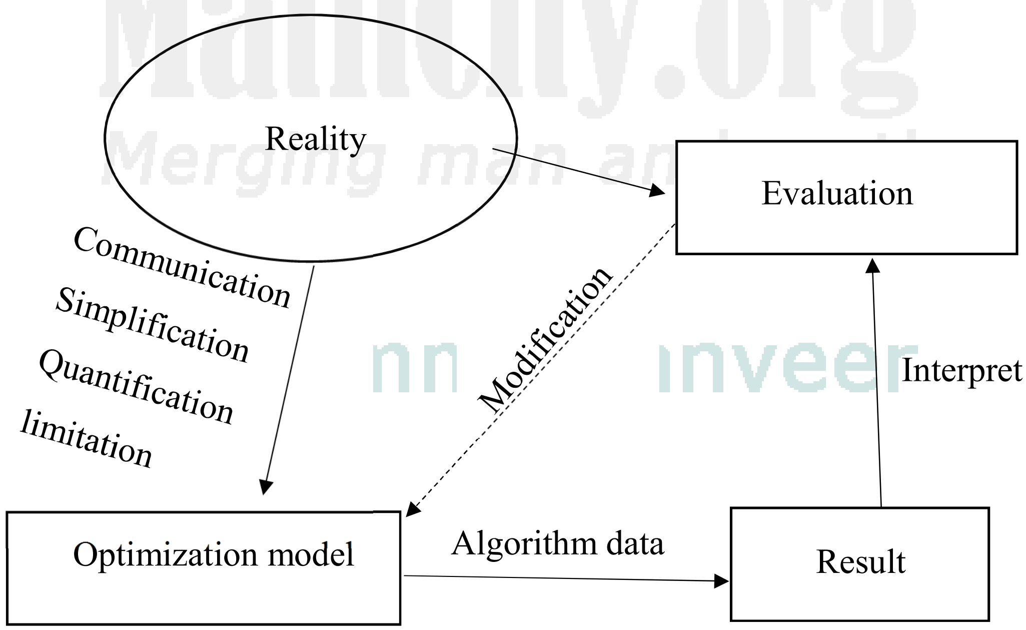
Ingredients of successful optimization

(a) **Modeling** (Turn a problem into one of the typical optimization formulation)
(b) **Algorithm** (Iterative procedure that leads you towards the solution)

Most optimization problem do not have a closed form solution.

(c) Software and hardware implementation realize the algorithm and return numerical solution.

Flow charts of Modeling process:



Requirements of Optimization Algorithm:

(a) Robustness: Perform well on a wide variety of problems.
(b) Efficiency: Reasonable computer time and storage.
(c) Accuracy: Precision not being sensitive to Norm.

Types of optimization:

Optimization is of two types

(i) Constraints optimization

(ii) Unconstraint's optimization

Constraints:

In mathematics a constraint is a condition of an optimization problem that the solution must satisfy.

There are several types of constraints like

- Equality constraints
- Inequality constraints
- Integers constraints

Example:

The following is a simple optimization problem.

Minimize $f(x) = f(x_1, x_2) = x_1^2 + x_2^2$

Subject to $x_1 \geq 1$ *and* $x_2 \geq 1$.

Where x denotes the vector (x_1, x_2)

In this example the first line defines the function to be minimized (called the objective function) and the second and third line define constraints (and subjective functions).

In this first is inequality constraint and second is equality constraint.

Lecture # 02:

Convex Function:

A convex function is a continuous function whose value at the midpoint of every interval in its domain does not exceed the arithmetic mean of its value at end of the interval.

OR

If f(x) is twice-differentiable on [a,b] , then f(x) is convex on [a,b] if and only if $f''(x) \geq 0$,$\forall x \in [a,b]$. It is concave if and only if $f''(x) \leq 0$,$\forall x \in [a,b]$.

OR

More generally, a function f(x) is convex on an interval [a,b] (finite or infinite) if for any two points x_1 and x_2 in [a,b] and for all $0 \leq \lambda \leq 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Note: If the sign of the inequality is reversed the function is called concave.

Examples:

(i) $f(x) = x^2$

The function f(x) has 2nd derivative, $f''(x) = 2 > 0$ at all point.

So, f is convex function and it is also strongly convex and convexity constant is 2.

(ii) The absolute value $f(x) = |x|$ is convex even through it does not have a derivative at the point x = 0.

(iii) The function $f(x) = |x|^P$ for P ≥ 2 is convex.

(iv) $f(x) = e^x \ is \ convex \ \forall x$, $f(x) = e^{-x} \ is \ convex \ \forall x$

Convex Optimization Problem:

Often difficult to recognize many tricks for transforming problem into convex form surprisingly many problems can be solved via convex optimization.

Example:

An office furniture manufacturer wants to maximize the daily profit decide the daily number of desks and chairs that should be manufactured such that the profit is maximum. The required material and profit of manufacturer desks and chairs is listed as followed.

Each piece	Required Material	Required Labor	Profit
Desk	10	5	20
Chair	6	2	10
Total Available	240	100	

Let the daily manufactured number of desks and chairs be x_1 and x_2.

Objective function = $20x_1 + 10x_2$

Subject to $10x_1 + 6x_2 \leq 240$

$5x_1 + 2x_2 \leq 100$

$x_1, x_2 \geq 0$

Matrix form of equations:

$$f(x) = f(x_1, x_2) = 3x_1 + 4x_2$$

In matrix form $\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$f(x_1, x_2) = 3x_1^2 + 4x_2^2 + 5x_1x_2$$

It is quadratic equation, we can write it in matrix form by using Hessian matrix method.

Hessian matrix method: $H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ *and* $\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ . & . & & . \\ . & . & & . \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$

$$f_{ij}=\frac{\partial f}{\partial i\partial j}\ ,\ \partial i\partial j=\partial j\partial i\ \ (when\ function is\ X=\begin{bmatrix}x_1\\x_2\\.\\.\\x_n\end{bmatrix}, in\ quaratic\ form=\frac{1}{2}X^t\,HX)$$

Now $f(x)=3x_1^2+4x_2^2+5x_1x_2$

$$f_{x_1}=f_1=6x_1+5x_2\ ,\ f_{11}=6\ ,\ f_{12}=5$$

$$f_{x_2}=f_2=8x_2+5x_1\ \Rightarrow f_{22}=8$$

$$H=\begin{bmatrix}6&5\\5&8\end{bmatrix}$$

In matrix form $\frac{1}{2}X^t\,HX=\frac{1}{2}\begin{bmatrix}x_1&x_2\end{bmatrix}\begin{bmatrix}6&5\\5&8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix}x_1&x_2\end{bmatrix}\begin{bmatrix}6x_1+5x_2\\5x_1+8x_2\end{bmatrix}$$

$$=\frac{1}{2}\left[6x_1^2+5x_1x_2+5x_1x_2+8x_2^2\right]$$

$$=\frac{1}{2}\left[6x_1^2+10x_1x_2+8x_2^2\right]$$

$$=3x_1^2+5x_1x_2+4x_2^2=f(x)$$

Question: $f(x)=3x_1+4x_2+3x_1^2+4x_2^2+5x_1x_2$

Solution: In matrix form

$$f(x) = g(x) + h(x)$$

Where $g(x)=3x_1+4x_2\ \ \ and\ \ \ h(x)=3x_1^2+4x_2^2+5x_1x_2$

$$g(x)=\begin{bmatrix}3&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\qquad,\qquad h(x)=\frac{1}{2}\begin{bmatrix}x_1&x_2\end{bmatrix}\begin{bmatrix}6&5\\5&8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$f(x)=\begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\frac{1}{2}\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}6 & 5\\5 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Unconstraint optimization problems:

Extreme point:

An ~ (approximate or similar function) of a function defines either a maximum or minimum of the function.

Mathematically: $X_0=\left(x_1^0,x_2^0,x_3^0,.....,x_n^0\right)$ is maximum point if $f\left(x_0+h\right)\le f\left(x_0\right)$

and minimum point if $f\left(x_0\right)\le f\left(x_0+h\right)$.

$\forall\;\; h=\left\{h_1,h_2,...,h_j,.....,h_n\right\}$ $where$ $|h_i|$ is sufficient small value.

Necessary Condition:

An ~ function for X_0 to be an extreme point of f(x) is that $\nabla f\left(X_0\right)=0$

$$\Rightarrow \nabla f=\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},...\frac{\partial f}{\partial x_n}\right)=0$$

Sufficient Condition:

An ~ function for X_0 to be an extreme point is that Hessian matrix H evaluated at X_0 is satisfy the following:

(i) H is +ve definite if X_0 is a minimum point
(ii) H is –ve definite if X_0 is a maximum point.

Question: Find extreme point of $f\left(x_1,x_2,x_3\right)=x_1+2x_3+x_2x_3-x_1^2-x_2^2-x_3^2$

And check maxima or minima of X_0.

Solution: $f\left(x_1,x_2,x_3\right)=x_1+2x_3+x_2x_3-x_1^2-x_2^2-x_3^2$

$$\frac{\partial f}{\partial x_1}=1-2x_1\;,\frac{\partial f}{\partial x_2}=x_3-2x_2\;,\frac{\partial f}{\partial x_3}=2+x_2-2x_3$$

$$\nabla f=0\;\because necessary\;\; condition$$

$$\frac{\partial f}{\partial x_1}=0\;\;\Rightarrow\;\;1-2x_1=0\;\;\Rightarrow x_1=\frac{1}{2}$$

$$\frac{\partial f}{\partial x_2} = 0 \Rightarrow x_3 - 2x_2 = 0 \Rightarrow x_3 = 2x_2$$

$$\frac{\partial f}{\partial x_3} = 0, \Rightarrow 2 + x_2 - 2x_3 = 0 \Rightarrow 2 + x_2 - 2(2x_2) = 0$$

$$\Rightarrow x_2 = \frac{2}{3}$$

$$\Rightarrow x_3 = 2\left(\frac{2}{3}\right) \Rightarrow x_3 = \frac{4}{3}$$

$$\Rightarrow X_0 = \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$$

Now sufficient condition

$H|_{X_0} = ?$

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

$$f_{11} = -2 \ , f_{12} = 0 \ , f_{13} = 0$$

$$f_{21} = 0 \ , f_{22} = -2 \ , f_{23} = 1$$

$$f_{31} = 0 \ , f_{32} = 1 \ , f_{13} = -2$$

$$H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$H|_{X_0} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

For eigen value

$$\det\left(H|_{X_0} - \lambda I\right) = 0$$

$$\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = 0$$

$$-2-\lambda\left[(-2-\lambda)^2 - 1\right] = 0$$

$$-2-\lambda = 0 \;\; and \;\; (-2-\lambda)^2 - 1 = 0$$

$$-2-\lambda = 0 \;\; and \;\; 4+\lambda^2+4\lambda-1=0$$

$$\Rightarrow \lambda^2+4\lambda+3=0$$

$$\Rightarrow \lambda^2+3\lambda+\lambda+3=0$$

$$\Rightarrow (\lambda+3)\lambda+1(\lambda+3)=0$$

$$\Rightarrow (\lambda+3)(\lambda+1)=0$$

$$\Rightarrow (\lambda+3)=0 \;\&\; (\lambda+1)=0$$

$$\Rightarrow \lambda=-1 \;\&\; \lambda=-3$$

$$\Rightarrow \lambda=-1,-2,-3$$

So, X_0 is maximum point.

Question: Find extreme point of $f(x_1,x_2)=3x_1+5x_2-4x_1^2+x_2^2-5x_1x_2$

And check maxima or minima of X_0.

Solution: $f(x_1,x_2)=3x_1+5x_2-4x_1^2+x_2^2-5x_1x_2$

$$\frac{\partial f}{\partial x_1}=3-8x_1-5x_2 \;,\; \frac{\partial f}{\partial x_2}=5+2x_2-5x_1$$

$$\because necessary \;\; condition \;\; \nabla f=0$$

$$\frac{\partial f}{\partial x_1}=0 \;\Rightarrow\; 3-8x_1-5x_2=0 \;\Rightarrow 8x_1+5x_2=3 \quad \_\_\_(i)$$

$$\frac{\partial f}{\partial x_2}=0 \;\Rightarrow\; 5+2x_2-5x_1=0 \;\Rightarrow 5x_1-2x_2=5 \quad \_\_\_\_(ii)$$

Multiplying (i) by 2 and (ii) by 5 and adding

$$16x_1 + 10x_2 = 6$$

$$25x_1 - 10x_2 = 25$$

$$41x_1 = 31 \quad \Rightarrow x_1 = \frac{31}{41}$$

Put in (i) $\quad 8\left(\frac{31}{41}\right) + 5x_2 = 3 \;\Rightarrow 5x_2 = 3 - \frac{248}{41}$

$$x_2 = -\frac{25}{41}$$

$$\Rightarrow X_0 = \left(\frac{31}{41}, \frac{-25}{41}\right)$$

Now sufficient condition $\quad H\big|_{X_0} = ? \quad \Rightarrow \quad H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$

$$f_{11} = -8 \;,\; f_{12} = -5 \;,\; f_{21} = -5,\; f_{22} = 2$$

$$H = \begin{bmatrix} -8 & -5 \\ -5 & 2 \end{bmatrix}$$

$$H\big|_{X_0} = \begin{bmatrix} -8 & -5 \\ -5 & 2 \end{bmatrix}$$

Now for eigen value $\quad \det\left(H\big|_{X_0} - \lambda I\right) = 0$

$$\begin{vmatrix} -8-\lambda & -5 \\ -5 & 2-\lambda \end{vmatrix} = 0$$

$$(-8-\lambda)(2-\lambda) - 25 = 0$$

$$-16 + 8\lambda - 2\lambda + \lambda^2 - 25 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda - 41 = 0$$

$$\Rightarrow \lambda = \frac{-6 \pm \sqrt{36+164}}{2} = \frac{-6 \pm \sqrt{200}}{2} = \frac{-6 \pm 10\sqrt{2}}{2}$$

$$\Rightarrow \lambda = -3 \pm 5\sqrt{2}$$

$$\lambda = -3 - 5\sqrt{2} < 0 \;\; \& \; \lambda = -3 + 5\sqrt{2} > 0$$

X_0 is a sedal point.

Question: Find extreme point of $f(x) = x_1^3 - x_1x_2 + x_2^2 - 2x_1 + 3x_2 - 4$

And check maxima or minima of X_0.

Solution: $$f(x) = x_1^3 - x_1x_2 + x_2^2 - 2x_1 + 3x_2 - 4$$

$$\frac{\partial f}{\partial x_1} = 3x_1^2 - x_2 - 2 \; , \frac{\partial f}{\partial x_2} = -x_1 + 2x_2 + 3$$

$$\textit{Necessary condition } \nabla f = 0$$

$$\frac{\partial f}{\partial x_1} = 0 \;\Rightarrow\; 3x_1^2 - x_2 - 2 = 0 \qquad \_\_\_(i)$$

$$\frac{\partial f}{\partial x_2} = 0 \;\Rightarrow\; -x_1 + 2x_2 + 3 = 0 \qquad \_\_\_\_(ii)$$

$$\Rightarrow x_2 = \frac{x_1 - 3}{2}$$

Put in (i) $$3x_1^2 - \left(\frac{x_1 - 3}{2}\right) - 2 = 0$$

$$\frac{6x_1^2 - x_1 + 3 - 4}{2} = 0$$

$$6x_1^2 - x_1 - 1 = 0$$

$$x_1 = \frac{1 \pm \sqrt{1 + 24}}{12} = \frac{1 \pm \sqrt{25}}{12} = \frac{1 \pm 5}{12}$$

$$x_1 = \frac{1 + 5}{12} \;,\; x_1 = \frac{1 - 5}{12}$$

$$x_1 = \frac{1}{2}, \frac{-1}{3}$$

$$\textit{When } \; x_1 = \frac{1}{2} \Rightarrow x_2 = \frac{\frac{1}{2} - 3}{2} = \frac{1 - 6}{4} = \frac{-5}{4}$$

$$When\ \ x_1 = \frac{-1}{3} \Rightarrow\ x_2 = \frac{\frac{-1}{3}-3}{2} = \frac{-1-9}{6} = \frac{-10}{4} = \frac{-5}{3}$$

$$\Rightarrow X_0 = \left(\frac{1}{2}, \frac{-5}{4}\right), \mathrm{X}_1 = \left(\frac{-1}{3}, \frac{-5}{3}\right)$$

Now sufficient condition $\quad H\big|_{X_0} = ?$

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$f_{11} = 6x_1\ ,\ f_{12} = -1\ , f_{21} = -1,\ f_{22} = 2$$

$$H = \begin{bmatrix} 6x_1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$H\big|_{X_0} = H\big|_{\left(\frac{1}{2}, \frac{-5}{4}\right)} = \begin{bmatrix} 6\left(\frac{1}{2}\right) & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

For eigen value $\quad \det\left(H\big|_{X_0} - \lambda I\right) = 0$

$$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) - 1 = 0$$

$$6 - 3\lambda - 2\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 5 = 0$$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{25-20}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

Now $\quad H\big|_{X_1} = \left(-\frac{1}{3}, \frac{-5}{3}\right) = \begin{bmatrix} 6\left(-\frac{1}{3}\right) & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$

For eigen value $\quad \det\left(H\big|_{X_1} - \lambda I\right) = 0$

$$\begin{vmatrix} -2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(2-\lambda)-1=0$$

$$-4+2\lambda-2\lambda+\lambda^2-1=0$$

$$\Rightarrow \lambda^2-5=0$$

$$\Rightarrow \lambda^2=5$$

$$\Rightarrow \lambda=\pm\sqrt{5}$$

X_1 is a sedal point.

Lecture # 03

Principle minor diagonal:

If $A = \left[a_{ij} \right]$ then Principle minor diagonals are $A_n = (-1)^{n-1} \left| a_{mn} \right|$

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A_{11} = A_1 = (-1)^{1-1} \left| a_{11} \right|$$

$$A_{22} = A_2 = (-1)^{2-1} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$A_{33} = A_3 = (-1)^{3-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example:

$$H = \begin{bmatrix} -3 & 2 & 1 \\ 8 & 5 & 2 \\ 0 & 3 & 2 \end{bmatrix}$$

$$H_1 = (-1)^{1-1} \left| -3 \right| = -3$$

$$H_2 = (-1)^{2-1} \begin{vmatrix} -3 & 2 \\ 8 & 5 \end{vmatrix} = (-1)(-15-16) = 31$$

$$H_3 = (-1)^{3-1} \begin{vmatrix} -3 & 2 & 1 \\ 8 & 5 & 2 \\ 0 & 3 & 2 \end{vmatrix}$$

$$H_3 = (1)\left[-3(10-6) - 2(16-0) + 1(24-0) \right]$$

$$H_3 = -12 - 32 + 24 = -20$$

Question: Find extreme point, also check maximum by using Principle minor diagonal. $Max \quad z=-4x_1^2-2x_1x_2-2x_2^2-x_1-x_2$

Solution: For extreme point $\nabla f=0$

$$\frac{\partial z}{\partial x_1}=-8x_1-2x_2-1=0 \;\Rightarrow\; 8x_1+2x_2=-1 \qquad \_\_\_\_\_\_(i)$$

$$\frac{\partial z}{\partial x_2}=-2x_1-4x_2-1=0 \;\Rightarrow\; 2x_1+4x_2=-1 \qquad \_\_\_\_\_\_(ii)$$

Multiplying (ii) by 4 and subtract form (i)

$$8x_1+2x_2=-1$$

$$\pm 8x_1 \pm 16x_2=\mp 4$$

$$-14x_2=3 \;\Rightarrow x_2=\frac{-3}{14}$$

Put in (i) $\quad 8x_1+2\left(\frac{-3}{14}\right)=-1$

$$8x_1=-1+\frac{6}{14}=\frac{-14+6}{14}=\frac{-8}{14} \quad\Rightarrow\; x_1=\frac{-1}{14}$$

$$\Rightarrow\; X_0=\left(\frac{-1}{14},\frac{-3}{14}\right)$$

$$H\big|_{X_0}=\begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}$$

$f_1=-8x_1+2x_2-1$, $f_2=-2x_1-4x_2-1$ $f_{11}=-8$, $f_{12}=-2$, $f_{22}=-4$

$$H\big|_{X_0}=\begin{bmatrix} -8 & -2\\ -2 & -4\end{bmatrix}$$

Now by using Principle minor diagonal we get

$$H_{11}=(-1)^{1-1}\left|-8\right|=-8$$

$$H_{22}=(-1)^{2-1}\begin{vmatrix} -8 & -2\\ -2 & -4\end{vmatrix}=(-1)(32-4)=-28$$

Question: Find extreme points and check maximum by using principle minor diagonal.

$$Max \qquad z = -\left(x_1 - \sqrt{5}\right)^2 - \left(x_2 - \pi\right)^2$$

Solution: For extreme point $\nabla f = 0$

$$\frac{\partial z}{\partial x_1} = -2\left(x_1 - \sqrt{5}\right) = 0 \;\Rightarrow\; x_1 = \sqrt{5}$$

$$\frac{\partial z}{\partial x_2} = -2\left(x_2 - \pi\right) = 0 \;\Rightarrow\; x_2 = \pi$$

So extreme points are $X_0 = \left(\sqrt{5}, \pi\right)$

$$H\big|_{X_0} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$f_1 = -2\left(x_1 - \sqrt{5}\right) \quad , \quad f_2 = -2\left(x_2 - \pi\right)$$

$$f_{11} = -2 \;,\; f_{12} = 0 \quad , \quad f_{22} = -2$$

$$H\big|_{X_0} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Now by using Principle minor diagonal we get

$$H_1 = (-1)^{1-1}\left|-2\right| = -2$$

$$H_{22} = (-1)^{2-1}\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = (-1)(4-0) = -4$$

X_0 is maximum because by Principle minor diagonal H is negative.

Newton Raphson Method:

(For unconstraint's optimization)

Choose an initial vector X_0 as in the method of steepest ascent.

Vectors $X_1, X_2, \ldots$ are determined iteratively by

$$X_{n+1} = X_n - \left(H|_{X_n}\right)^{-1} \nabla f|_{X_n}$$

If n = 0 then $$X_1 = X_0 - \left(H|_{X_0}\right)^{-1} \nabla f|_{X_0}$$

The stopping rule is the same as for Steepest ascent method (one by one increment). The Newton Raphson method will converge to local maxima if H_f is negative definite on some ε–Neighborhood around the maxima and if X_0 in that ε–Neighborhood.

Remark:

IF H_f is negative definite, H_f^{-1} exist and is negative definite. If X_0 is not chosen correctly.

The method may converge to local minima or it may not converge to all.

In either case the iterative process is terminated and then being a new with better initial approximation.

Question: Find the extreme points and also check minimum by Newton Raphson method.

$$Min \quad z = 4x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + x_2 \qquad Where \;\; \varepsilon = 0.1$$

OR Solve by Newton Raphson method

$$Min \quad z = 4x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + x_2$$

Solution:

For extreme points

$$\frac{\partial z}{\partial x_1} = 8x_1 + 2x_2 + 1 = 0 \quad \Rightarrow 8x_1 + 2x_2 = -1 \qquad \_\_\_\_(i)$$

$$\frac{\partial z}{\partial x_2} = 2x_1 + 4x_2 + 1 = 0 \quad \Rightarrow 2x_1 + 4x_2 = -1 \qquad \_\_\_\_(ii)$$

$$8x_1 + 2x_2 = -1$$

$$-8x_1 \pm 16x_2 = \mp 4$$

$$-14x_2 = 3 \quad \Rightarrow x_2 = \frac{-3}{14}$$

Put in (i) $$8x_1 + 2\left(\frac{-3}{14}\right) = -1$$

$$8x_1 = -1 + \frac{6}{14} = \frac{-14+6}{14} = \frac{-8}{14}$$

$$\Rightarrow x_1 = \frac{-1}{14}$$

$$X_0 = \left(\frac{-1}{14}, \frac{-3}{14}\right)$$

$$H\big|_{X_0} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$f_{11} = 8\,,\ f_{12} = 2\,,\ f_{22} = 4$$

$$H\big|_{X_0} = \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$$

$$H^{-1} = \frac{adjH}{|H|} \quad \Rightarrow \ AdjH = \begin{bmatrix} 4 & -2 \\ -2 & 8 \end{bmatrix}$$

$$|H| = \begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix} = 32 - 4 = 28$$

$$H^{-1} = \frac{1}{28}\begin{bmatrix} 4 & -2 \\ -2 & 8 \end{bmatrix}$$

By Newton Raphson formula

$$X_{n+1} = X_n - \left(H\big|_{X_n}\right)^{-1} \nabla f\big|_{X_n}$$

n = 0 then $$X_1 = X_0 - \left(H\big|_{X_0}\right)^{-1} \nabla f\big|_{X_0}$$

$$\because\ \nabla f\big|_{X_0} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix}\Bigg|_{X_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} \frac{-1}{14} \\ \frac{-3}{14} \end{bmatrix} - \frac{1}{28}\begin{bmatrix} 4 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} \frac{-1}{14} \\ \frac{-3}{14} \end{bmatrix}$$

$$X_1 = \left(\frac{-1}{14}, \frac{-3}{14}\right)$$

$$Now \quad f(x_1) - f(x_0) < \varepsilon$$

So, minimum point of given function $X = \left(\frac{-1}{14}, \frac{-3}{14}\right)$

Question: Find the extreme points and also check minimum by Newton Raphson method.

$$Max \quad z = -\sin x_1 x_2 + \cos(x_1 - x_2) \quad with \ \varepsilon = 0.05 \ and \ X_0 = (-0.7548, 0.5303)$$

Solution:

$$f_1 = \frac{\partial z}{\partial x_1} = -x_2 \cos x_1 x_2 - \sin(x_1 - x_2)$$

$$f_{11} = -x_2^2 \sin x_1 x_2 - \cos(x_1 - x_2)$$

$$f_2 = \frac{\partial z}{\partial x_2} = -x_1 \cos x_1 x_2 + \sin(x_1 - x_2)$$

$$f_{12} = f_{21} = -\cos x_1 x_2 + x_1 x_2 \sin x_1 x_2 + \cos(x_1 - x_2)$$

$$f_{22} = x_1^2 \sin x_1 x_2 - \cos(x_1 - x_2)$$

At $X_0 = (-0.7548 , 0.5303)$

$$f_1\big|_{X_0} = -0.5303\cos(-0.7548 \times 0.5303) - \sin(-0.7548 - 0.5303) = -0.5078$$

$$f_{11}\big|_{X_0} = (-0.5303)^2 \sin(-0.7548 \times 0.5303) - \cos(-0.7548 - 0.5303) = -1.0017$$

$$f_2\big|_{X_0} = 0.7548\cos(-0.7548\times 0.5303) + \sin(-0.7548 - 0.5303) = 0.7323$$

$$f_{22}\big|_{X_0} = (-0.7548)^2 \sin(-0.7548\times 0.5303) - \cos(-0.7548 - 0.5303) = -1.0037$$

$$f_{12}\big|_{X_0} = f_{21}\big|_{X_0} = -\cos(-0.7548\times 0.5303) + (-0.7548\times 0.5303)\sin(-0.7548\times 0.5303) + \cos(-0.7548 - 0.5303)$$

$$f_{12}\big|_{X_0} = f_{21}\big|_{X_0} = 0.0025$$

$$Now \quad \nabla f\big|_{X_0} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix}_{X_0} = \begin{bmatrix} -0.5078 \\ 0.7323 \end{bmatrix}$$

$$H\big|_{X_0} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}_{X_0} = \begin{bmatrix} -1.0017 & 0.0025 \\ 0.0025 & -1.0037 \end{bmatrix}$$

$$\left(H\big|_{X_0}\right)^{-1} = \frac{adjH}{|H|} \quad \Rightarrow \quad AdjH = \begin{bmatrix} -1.0037 & -0.0025 \\ -0.0025 & -1.0017 \end{bmatrix}$$

$$|H| = \begin{vmatrix} -1.0017 & 0.0025 \\ 0.0025 & -1.0037 \end{vmatrix} = 1.0054$$

$$H\big|_{X_0} = \frac{1}{1.0054}\begin{bmatrix} -1.0037 & 0.0025 \\ 0.0025 & -1.0017 \end{bmatrix}$$

$$H\big|_{X_0} = \begin{bmatrix} -0.9983 & -0.002486 \\ -0.002486 & -0.9963 \end{bmatrix}$$

By Newton Raphson formula

$$X_{n+1} = X_n - \left(H\big|_{X_n}\right)^{-1} \nabla f\big|_{X_n}$$

n = 0 then $\quad X_1 = X_0 - \left(H\big|_{X_0}\right)^{-1} \nabla f\big|_{X_0}$

$$X_1 = \begin{bmatrix} 0.7548 \\ 0.5303 \end{bmatrix} - \begin{bmatrix} -0.9983 & -0.002486 \\ -0.002486 & -0.9963 \end{bmatrix}\begin{bmatrix} -0.5078 \\ 0.7323 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0.7548 \\ 0.5303 \end{bmatrix} - \begin{bmatrix} 0.5051 \\ 0.7283 \end{bmatrix}$$

$$X_1 = (0.2497, 1.2586)$$

$$Now\ \ f(X_0) = -\sin(-0.7548 \times 0.5303) + \cos(-0.7548 - 0.5303) = 1.0067$$

$$f(X_1) = -\sin(0.7497 \times 1.2586) + \cos(0.2497 - 1.2586) = 0.9944$$

$$Now\ \ f(x_1) - f(x_0) = 0.9944 - 1.0067 = -0.0123 < \varepsilon$$

So, maximum point of given function is $X = (0.2497, 1.2586)$

Lecture # 04

Nonlinear programming: single-variable optimization:

A one variable, unconstrained, nonlinear program has the form

$$\text{Optimize} \quad z = f(x)$$

Where f(x) is a (nonlinear) function of the single variable x, and the search for the optimum (Max or Min) is conducted over the infinite interval $(-\infty, \infty)$. If the search is restricted to a finite subinterval [a,b], then the problem becomes

$$\text{Optimize } z = f(x)$$

$$\text{Subject to } a \le x \le b$$

Which is a one variable, constrained program.

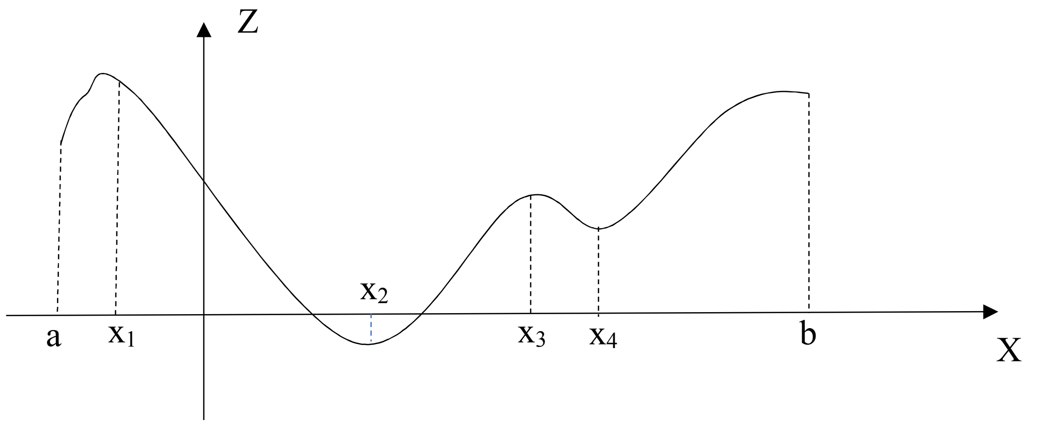
Local and Global optima:

An objective function f(x) has a local (or relative) minimum at x_0 if there exist a (small) interval centered at x_0 such that $f(x) \ge f(x_0)$ for all x in this interval at which the function is defined.

If $f(x) \ge f(x_0)$ for all x at which the function is defined, then the minimum at x_0 (besides being local) is a global (or absolute) minimum. Local and global maxima are defined similarly, in terms of the reversed inequality.

Example:

The function graphed in fig is defined only on [a,b]. It has relative minima at a, x_2, and x_4 relative maxima at x_1, x_3 and b ; a global minimum at x_2 ; and global maxima at x_1 and b.



Results from Calculus:

(i) If f(x) is continuous on the closed and bounded interval [a,b] then f(x) has global optima (both a maximum and a minimum) on this interval.

(ii) If f(x) has a local optimum at x_0 and if f(x) is differentiable on a small interval centered at x_0 then $f'(x_0)=0$.

(iii) If f(x) is twice differentiable on a small interval centered at x_0 and if $f''=(x_0)>0$, then f(x) has a local minimum at x_0. If $f''(x_0)<0$ the f(x) has a local maximum at x_0.

Example: If Max f(x) = x(5π–x) then define local maxima on [0,20].

Solution:

$$f(x) = x(5\pi - x)$$

$$f'(x)=5\pi-2x$$

$$f'(x)=0 \;\Rightarrow\; 5\pi-2x=0$$

$$\Rightarrow\; x=\frac{5\pi}{2}$$

Global maxima (at $\frac{5\pi}{2}$) Global minima (at 20)

x	0	$\frac{5\pi}{2}$	20
f(x)	0	61.685	–85.84

$$f''(x)=-2x$$

$$f''(x_0)=-2<0$$

$$\Rightarrow \text{f(x) has local maxima at } x=\frac{5\pi}{2}$$

Example: If $f(x)=x+4x^{-1}$ then define local minima on [0,20].

Solution:

$$f(x)=x+4x^{-1}$$

$$f'(x)=1+4x^{-2}(-1)$$

$$f'(x)=1-4x^{-2}=1-\frac{4}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \frac{4}{x^2} = 0$$

$$\Rightarrow 1 = \frac{4}{x^2}$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

x	0	–2	2	20
f(x)	0	–4	4	20.2

$$f''(x) = 0 - 4x^{-3}(-2) = \frac{8}{x^3}$$

$$At\ x_0 = 2 \ , f''(x_0) = \frac{8}{(2)^3} = 1 > 0$$

$\Rightarrow$ f(x) has local minima at x = 2.

Three-point interval search:

The interval under consideration is divided into quarters and the objective function evaluated at the three equally spaced interior points. The interior point yielding the best value of the objective is determined (in case of a tie, arbitrarily choose one point) and the subinterval centered at this point and made up of two quarters of the current interval replaces the current interval. Including ties there are 10 possible sampling patterns;

The three-point interval search is the most efficient equally spaced search procedure in terms of achieving a prescribed tolerance with a minimum number of functional evaluations. It is also one of the easiest sequential searches to code for computer.

Question: By using three-point interval search method to find

Max f(x) = x(5π–x) on [0,20] with ε = 0.1

Solution: Divide [0,20] into quarters (four parts).

At a = 0

$\Rightarrow$ f(a) = 0

At $x_0 = 5$

$f(x_0) = 5(5\pi-5) = 53.54$

At $x_1 = 10$

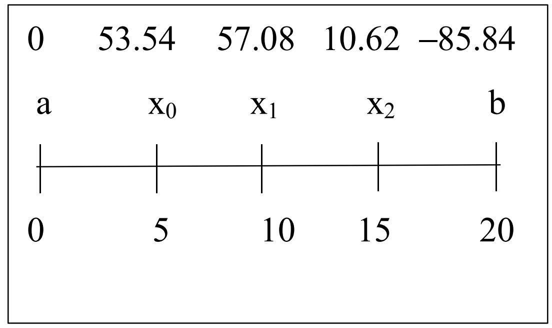
$f(x_1) = 10(5\pi-10) = 57.08$

At $x_2 = 15$

$f(x_2) = 15(5\pi-15) = 10.62$

At $b = 20$

$f(b) = 20(5\pi-20) = -85.84$



Now the centre is x_1 and divide the interval [0,20] into subinterval because $x_1 = 10$ function give maximum value.

Iteration-I:

Divide [5,15] into quarters.

At $x_0 = 5$

$f(x_0) = 5(5\pi-5) = 53.54$

At $x_3 = 7.5$

$f(x_3) = 7.5(5\pi-7.5) = 61.56$

At $x_1 = 10$

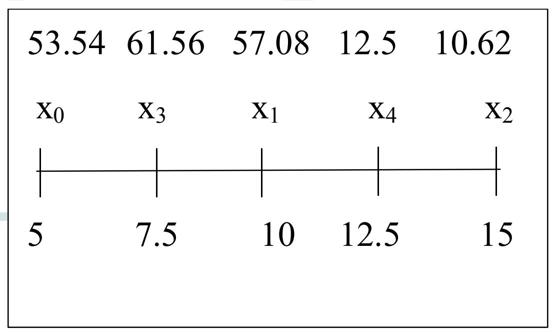
$f(x_1) = 10(5\pi-10) = 57.08$

At $x_4 = 12.5$

$f(x_4) = 12.5(5\pi-12.5) = 40.09$

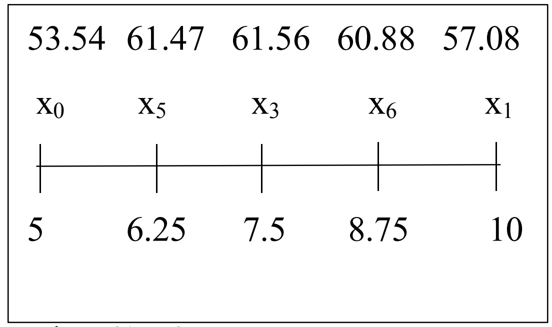
At $x_2 = 15$

$f(x_2) = 15(5\pi-15) = 10.62$



Centre at x_3 and the subinterval is [5,10] at $x_3 = 7.5$ because function has maximum value. We cannot stop the question because $f(x_2) - f(x_1) \nless \varepsilon$

Iteration-II: Divide [5,10] into quarters.



At $x_5 = 6.25$

$f(x_5) = 6.25(5\pi - 6.25) = 61.47$

At $x_6 = 8.75$

$f(x_6) = 8.75(5\pi - 8.75) = 60.88$

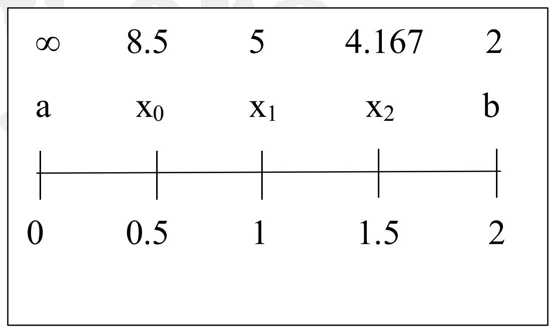
Centre at x_3 $\quad \left|f(x_3) - f(x_2)\right| = 0 < 0.1 = \varepsilon$

So, function is max at $x_3 = 7.5$ and maximum value 61.56

Question: By using three-point interval search method find

$$Max \quad f(x) = x + 4x^{-1} \quad on\ [0,2] \quad where\ \varepsilon = 0.1$$

Solution: Divide [0,2] into quarters



$f(a) = \infty$

$f(x_1) = 8.5$

$f(x_2) = 4.167$

$f(b) = 2$

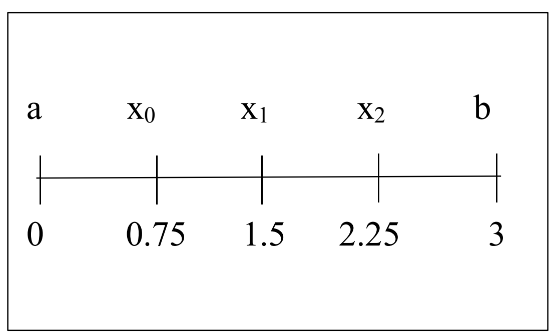
Centre at x_0 and interval is [0,1] but at 0

Function has ∞ value. So, we cannot solve it further.

Question: By using three-point interval search method find

$$Max \quad f(x) = x\sin 4x \quad on\ [0,3] \quad where\ \varepsilon = 0.002$$

Solution: Divide [0,3] into quarters



$f(a) = 0$

$f(x_0) = 0.75\sin4(0.75) = 0.03925$

$f(x_1) = 1.5\sin4(1.5) = 0.1568$

$f(x_2) = 2.25\sin4(2.25) = 0.35198$

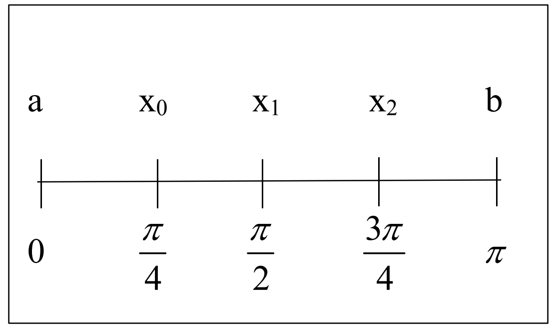
$f(b) = 3\sin4(3) = 0.6237$

Centered at ‘b’ but there does not become an interval. So, we cannot solve it further.

Question: By using three-point interval search method find

$$Max \quad f(x) = x^2 \sin x \quad on\ [0,\pi] \quad where\ \varepsilon = 0.1$$

Solution: Divide [0,π] into quarters



$$f(a) = 0$$

$$f(x_0) = \left(\frac{\pi}{4}\right)^2 \sin\frac{\pi}{4} = 0.00845$$

$$f(x_1) = \left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2} = 0.06764$$

$$f(x_2) = \left(\frac{3\pi}{2}\right)^2 \sin\frac{3\pi}{2} = 0.968$$

$$f(b) = (\pi)^2 \sin\pi = 0.5409$$

Centered at 'b' but there does not become an interval. So, we cannot solve it further.

Lecture # 05

Fibonacci Search Method:

(Fibonacci name of Italian mathematician)

The Fibonacci search sequence $F_n = \{1,1,2,3,5,8,13,21,34,55,\ldots\}$ forms the basis of the most efficient sequential search technique. Each number in the sequence is obtained by adding together the two preceding numbers, exception is the first two numbers F_0 and F_1 which are both 1.

The Fibonacci search is initialized by determining the smallest Fibonacci number that satisfies $F_n \varepsilon \geq b-a$, where ε is the prescribed tolerance and [a,b] is the original interval of interest set $\varepsilon' = \dfrac{(b-a)}{F_n}$. The first two points in the search are located $F_{n-1}\varepsilon'$ units in the form of the end points [a,b] where F_{n-1} is the Fibonacci number preceding F_n.

Successive points in the search are considered one at a time and are positioned $F_j\varepsilon'\left(j = n-2, n-3, n-4, \ldots, 2\right)$ units in the form the newest end point of the current interval. Observe that with the Fibonacci procedure we can state in advance that will be required to achieve a certain accuracy. Moreover, that number is independent of particular unimoded function.

$$Let\ F_0 = F_1 = 1$$

$$F_2 = F_0 + F_1 = 1 + 1 = 2$$

$$F_3 = F_1 + F_2 = 1 + 2 = 3$$

$$F_4 = F_2 + F_3 = 2 + 3 = 5$$

$$\begin{array}{ccc} . & . & . \\ . & . & . \\ . & . & . \end{array}$$

$$F_n = F_{n-2} + F_{n-1}$$

Question: By using Fibonacci method find

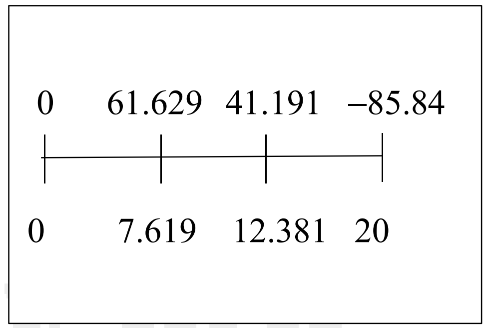
Max f(x) = x(5π−x) on [0,20] with ε = 1

Solution: Here [0,20] = [a,b] , ε = 1

f(x) = x(5π−x)

$$F_n \geq \frac{b-a}{\varepsilon} = \frac{20-0}{1} = 20$$

$$\Rightarrow F_n = 21 \; and \; n = 7$$



$$Now \; \varepsilon' \geq \frac{b-a}{F_n} = \frac{20-0}{21} = 0.952$$

$$\varepsilon' F_{n-1} = (0.952)(13) = 12.381$$

$$x_1 = a + \varepsilon' F_{n-1} = 0 + 12.381 = 12.381$$

$$x_2 = b + \varepsilon' F_{n-1} = 20 + 12.381 = 7.619$$

$$f(0) = 0$$

$$f(7.619) = 61.629$$

$$f(12.381) = 41.191$$

$$f(20) = -85.84$$

Function give maximum value at 7.619. So, new subinterval is [0,12.381]

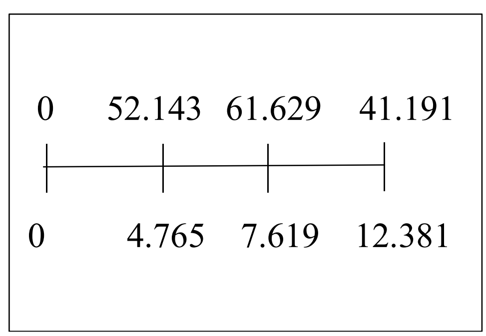
Iteration-I:

$$Now \; F_{n-2} = F_5 = 8$$

$$\varepsilon' F_5 = (0.952)(8) = 7.616$$

$$x_3 = x_1 - \varepsilon' F_5 = 12.381 - 7.616 = 4.765$$

$$f(4.765) = 52.413$$



Function give maximum value at 7.619

So, new subinterval is [4.765,12.381].

Iteration-II: *Now* $F_{n-3} = F_4 = 5$

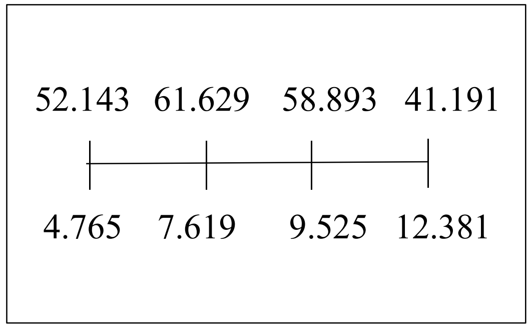
$\varepsilon' F_4 = (0.952)(5) = 4.76$

$x_4 = x_3 - \varepsilon' F_4 = 4.765 + 4.76 = 9.525$

$f(9.525) = 58.893$

Function give maximum value at 7.619

So, new subinterval is [4.765,9.525]



Iteration-III: $F_{n-4} = F_3 = 3$

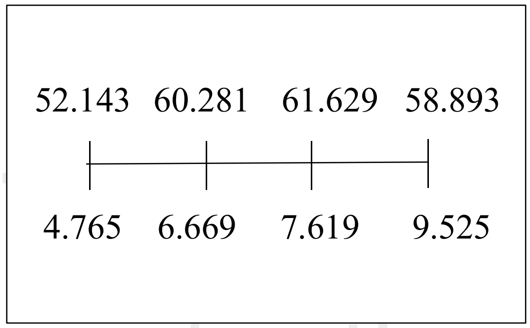
$\varepsilon' F_4 = (0.952)(3) = 2.856$

$x_5 = x_4 - \varepsilon' F_3 = 9.525 - 2.856 = 6.669$

$f(6.669) = 60.281$

Function give maximum value at 7.619

So, new subinterval is [6.669,9.525]



Iteration-IV: $F_{n-5} = F_2 = 2$

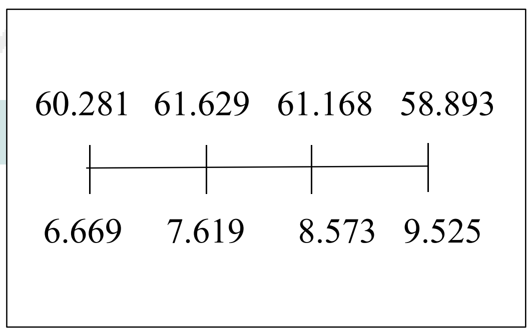
$\varepsilon' F_2 = (0.952)(2) = 1.904$

$x_6 = x_5 + \varepsilon' F_2 = 6.669 + 1.904 = 8.573$

$f(8.573) = 61.168$

Function give maximum value at 7.619

So, new subinterval is [6.669,8.573]



Iteration-V: $F_{n-6} = F_1 = 1$

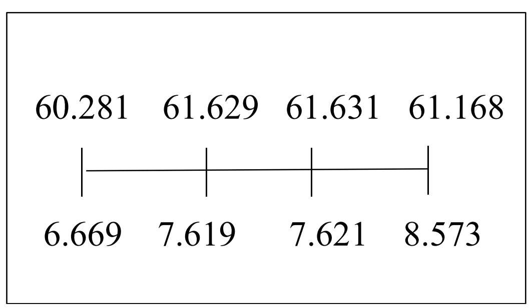
$\varepsilon' F_4 = (0.952)(1) = 0.952$

$x_7 = x_6 - \varepsilon' F_1 = 8.573 - 9.525 = 7.621$

$f(x_7) = 61.631$

Maximum point of given function

value at 7.621 and maximum value is 61.631



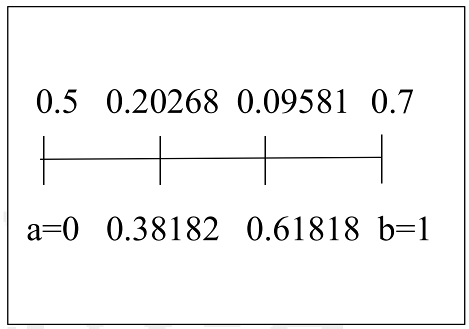
Question: Find the minimum of the $f(x)=\frac{1}{2}+x^5-\frac{4}{5}x$ on the interval [0,1] with ε = 0.02

Solution: Here [0,1] = [a,b] , ε = 0.02

$$f(x)=\frac{1}{2}+x^5-\frac{4}{5}x$$

$$F_n \geq \frac{b-a}{\varepsilon}=\frac{1-0}{0.02}=50$$

$$\Rightarrow F_n = 55 \;\; and \;\; n = 9$$



$$Now \; \varepsilon' \geq \frac{b-a}{F_n}=\frac{1-0}{55}=0.01818 \quad , \quad F_8 = 34$$

$$\varepsilon' F_8 = (0.01818)(34) = 0.61818$$

$$x_1 = a + \varepsilon' F_8 = 0 + 0.61818 = 0.61818$$

$$x_2 = b + \varepsilon' F_8 = 1 - 0.61818 = 0.38182$$

$$f(0) = 0.5$$

$$f(0.38182) = 0.20268$$

$$f(0.61818) = 0.09581$$

$$f(1) = 0.7$$

Minimum value at 0.61818 so new interval is [0.38182 ,1]

Iteration-I:

$$Now \;\; F_7 = 21$$

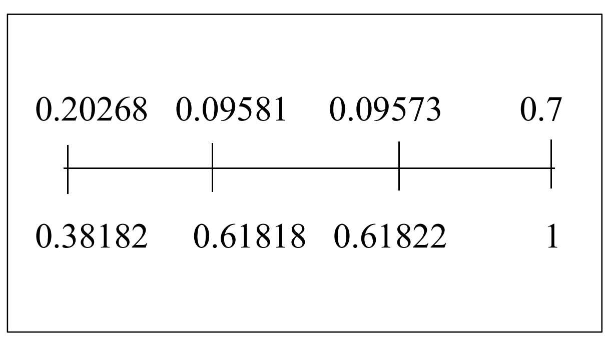
$$\varepsilon' F_7 = (0.01818)(21) = 0.38178$$

$$x_3 = 1 - 0.38178 = 0.61822$$

$$f(0.61822) = 0.09573$$

Function give minimum value at 0.61822

So, new interval is [0.61818 ,1].



Iteration-II: $F_{n-3} = F_6 = 13$

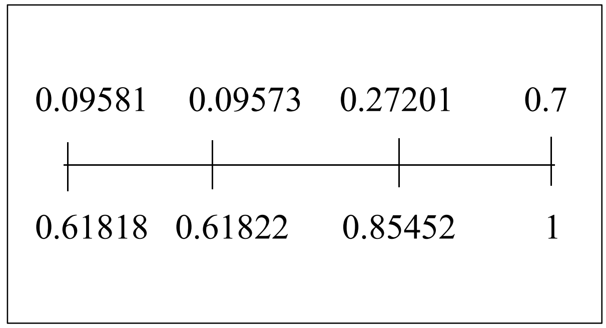
$\varepsilon' F_6 = (0.01818)(13) = 0.23634$

$x_4 = 0.61818 + 0.23634 = 0.85452$

$f(0.85452) = 0.27201$

Function give minimum value at 0.61822

So, new interval is [0.61818 ,0.85452].



Iteration-III: $F_{n-4} = F_5 = 8$

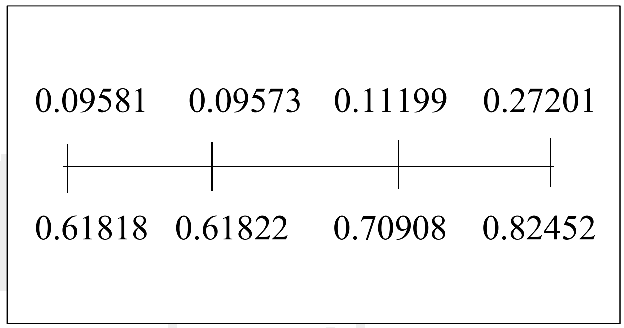
$\varepsilon' F_5 = (0.01818)(8) = 0.14544$

$x_5 = 0.85452 - 0.14544 = 0.70908$

$f(0.70908) = 0.11199$

Function give minimum value at 0.61822

So, new interval is [0.61818 ,0.70908].



Iteration-IV: $F_{n-5} = F_4 = 5$

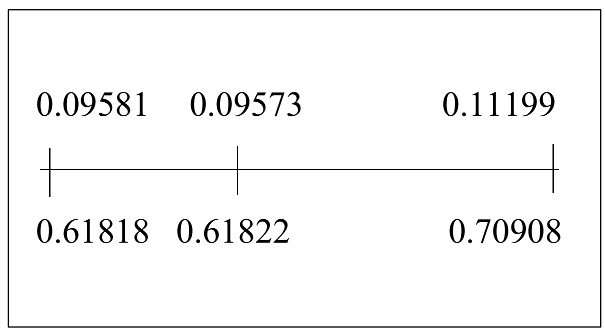
$\varepsilon' F_4 = (0.01818)(5) = 0.0909$

$x_4 = 0.61818 + 0.0909 = 0.70908$

$f(0.70908) = 0.11199$

Function give minimum value at 0.61822

So, new interval is [0.61818 ,0.70908].



Iteration-V: $F_{n-6} = F_3 = 3$

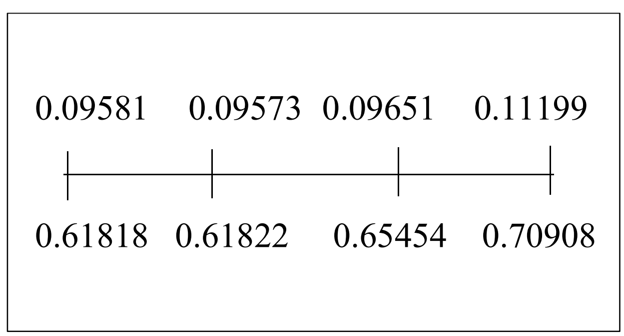
$\varepsilon' F_3 = (0.01818)(3) = 0.05454$

$x_7 = 0.70908 - 0.05454 = 0.65454$

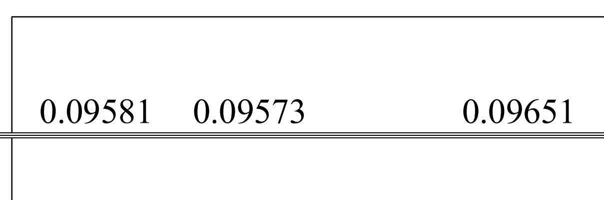
$f(0.65454) = 0.09651$

Function give minimum value at 0.61822

So, new interval is [0.61818 ,0.65454].



Iteration-VI: $F_{n-7} = F_2 = 2$



$\varepsilon' F_2 = (0.01818)(2) = 0.03636$

$x_8 = 0.61818 + 0.03636 = 0.65454$

$f(0.65454) = 0.09651$

Function give minimum value at 0.61822

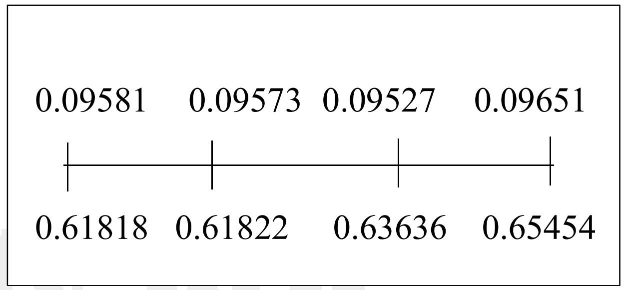
So, new interval is [0.61818 ,0.65454].

Iteration-VII: $F_{n-8} = F_1 = 1$

$\varepsilon' F_1 = (0.01818)(1) = 0.01818$

$x_8 = 0.65454 - 0.01818 = 0.63636$

$f(0.63636) = 0.09527$



Minimum point of given function value at 0.63636 and minimum value is 0.09527

Question: Find the minimum of the $f(x) = x^2 - \sin x$ on the interval [0,1] with ε = 0.01

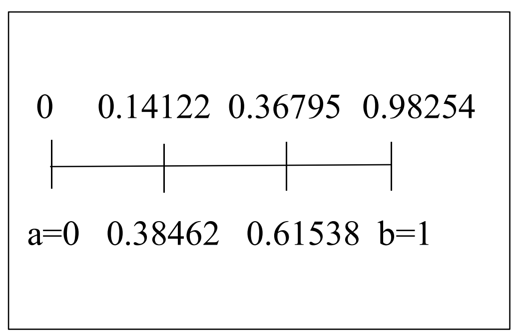
Solution: Here [0,1] = [a,b] , ε = 0.01

$$f(x) = x^2 - \sin x$$

$$F_n \geq \frac{b-a}{\varepsilon} = \frac{1-0}{0.01} = 10$$

$$\Rightarrow F_n = 13 \;\; and \;\; n = 6$$

$$Now\ \varepsilon' \geq \frac{b-a}{F_n} = \frac{1-0}{13} = 0.07692 \quad , \quad F_{n-1} = F_5 = 8$$



$\varepsilon' F_5 = (0.07692)(8) = 0.61538$

$x_1 = a + \varepsilon' F_5 = 0 + 0.61538 = 0.61538$

$x_2 = b + \varepsilon' F_5 = 1 - 0.61538 = 0.38462$

$f(0) = 0$

$f(0.38462) = 0.14122$

$f(0.61538) = 0.36795$

$f(1) = 0.98254$

Minimum value at 0.38462 so new interval is [0 , 0.61538]

Iteration-I:

$F_{n-1} = F_4 = 5$

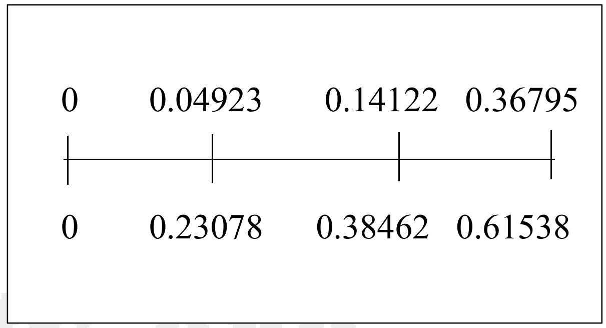
$\varepsilon' F_4 = (0.07692)(5) = 0.3846$

$x_3 = 0.61538 - 0.3846 = 0.23078$

$f(0.23078) = 0.04923$

Function give minimum value at 0.23078

So, new interval is [0 , 0.38462]



Iteration-II: $F_{n-3} = F_3 = 3$

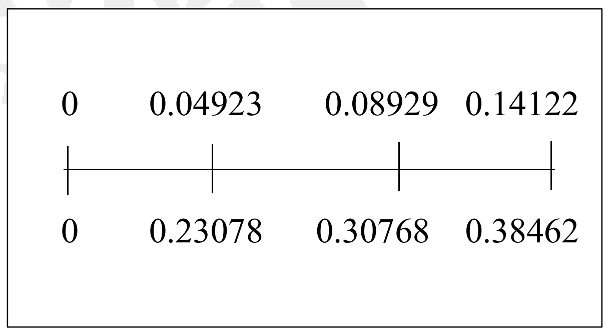
$\varepsilon' F_3 = (0.07692)(3) = 0.30768$

$x_4 = 0 + 0.30768 = 0.30768$

$f(0.30768) = 0.08929$

Function give minimum value at 0.23078

So, new interval is [0 , 0.30768]



Iteration-III: $F_{n-4} = F_2 = 2$

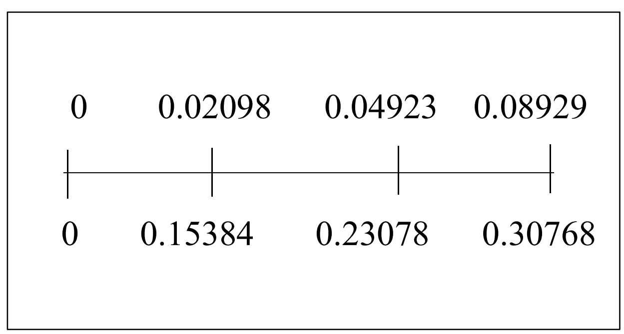
$\varepsilon' F_2 = (0.07692)(2) = 0.15384$

$x_5 = 0.30768 - 0.15384 = 0.15384$

$f(0.15384) = 0.02098$

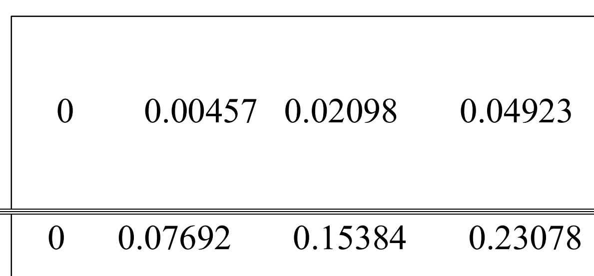
Function give minimum value at 0.61822

So, new interval is [0 , 0.23078]

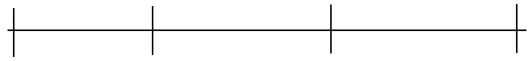


Iteration-IV: $F_{n-5} = F_1 = 1$

$\varepsilon' F_1 = (0.07692)(1) = 0.07692$



$x_4 = 0 + 0.07692 = 0.07692$



$f(0.07692) = 0.00457$

Minimum point of the given function is 0.07962 and minimum value is 0.00457

Lecture # 06

Mean Search or Golden mean search or Golden section search method:

A search namely as efficient at the Fibonacci search is based on the number

$$\varepsilon' = \frac{\sqrt{5}-1}{2} = 0.6180$$

Known as the Golden mean. The first two points on the search are located as $\varepsilon'(b-a)$ or 0.6180(b–a) units in form the end point of the initial interval [a,b] successive points are considered one at a time and are positioned (0.6180) L_i unites the form of the newest end point of the current interval, where L_i denotes the length of this interval.

Question: Solve Max $f(x) = x(5\pi - x)$ on the interval [0,20] with ε = 1

Solution: Here $\varepsilon = 1$, $\varepsilon' = 0.6180$

$$\varepsilon'(b-a) = 0.6180(20-0) = 0.6180(20) = 12.36$$

$x_1 = 0 + 12.36 = 12.36$

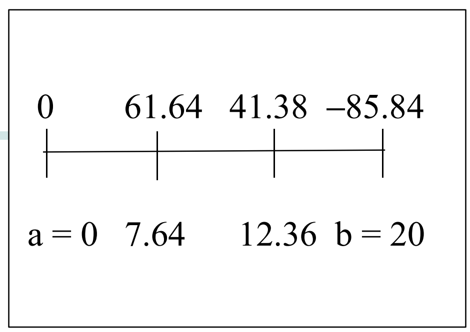
$x_2 = 20 - 12.36 = 7.64$

$f(0) = 0$

$f(7.64) = 61.64$

$f(12.36) = 41.38$

$f(20) = -85.84$



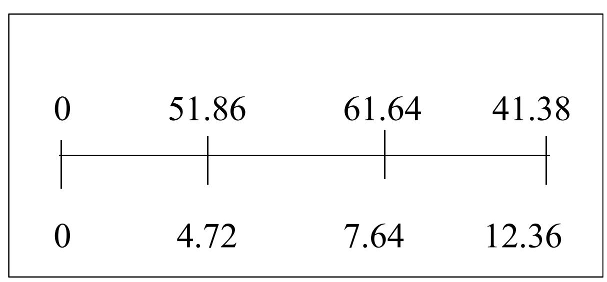
Function give maximum value 7.64 so, new subinterval is [0 , 12.36]

Iteration-I:

$\varepsilon' L_1 = (0.6180)(12.36 - 0) = 7.64$

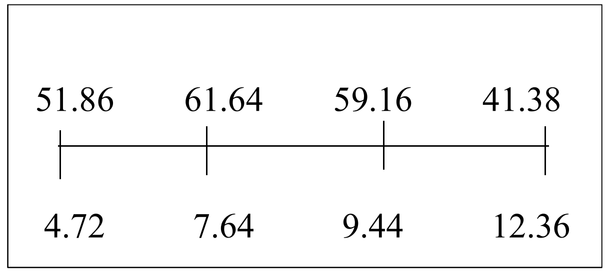
$x_3 = 12.36 - 7.64 = 4.72$

$f(x_3) = 51.86$



Function give maximum value at 7.64 so, the new sub interval is [4.72 , 12.36]

Iteration-II:



$\varepsilon' L_2 = (0.6180)(12.36 - 4.72) = 4.72$

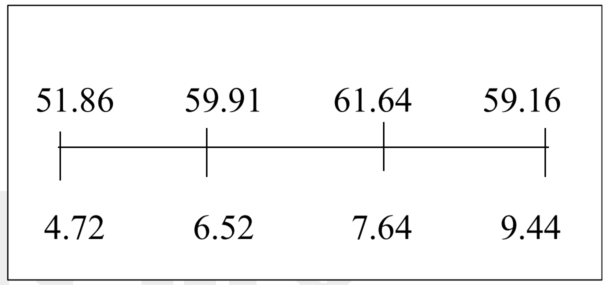
$x_4 = 4.72 + 4.72 = 9.44$

$f(x_4) = 59.16$

$\Rightarrow \ |f(x_c) - f(x_4)| \nless \varepsilon \qquad \textit{where } x_c = \textit{centered value} = 6.614$

Function give maximum value at 7.64 so, the new sub interval is [4.72 , 9.44]

Iteration-III:



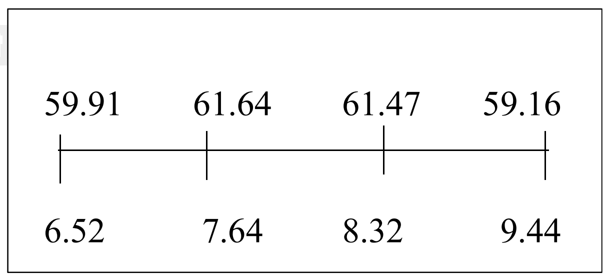
$\varepsilon' L_3 = (0.6180)(9.44 - 4.72) = 2.92$

$x_5 = 9.44 - 2.92 = 6.52$

$f(6.52) = 59.91$

Function give maximum value at 7.64 so, the new sub interval is [6.52 , 9.44]

Iteration-VI:



$\varepsilon' L_4 = (0.6180)(9.44 - 6.52) = 1.80$

$x_6 = 6.52 + 1.80 = 8.32$

$f(8.32) = 61.47$

$\Rightarrow \ |f(x_c) - f(x_4)| < \varepsilon$

So, maximum point is $x_c = 7.64$ and maximum value of given function is 61.64 on interval [6.52, 8.32]

Question: Solve Max $f(x) = 4\sin x(1 + \cos x)$ *on* $\left[0, \frac{\pi}{2}\right]$ *with* $\varepsilon = 0.05$ by using Golden mean search.

Solution: Here $\varepsilon = 0.05$, $\varepsilon' = 0.6180$

$$\varepsilon'(b - a) = 0.6180(90 - 0) = 55.62$$

$x_1 = 0 + 55.62 = 55.62$

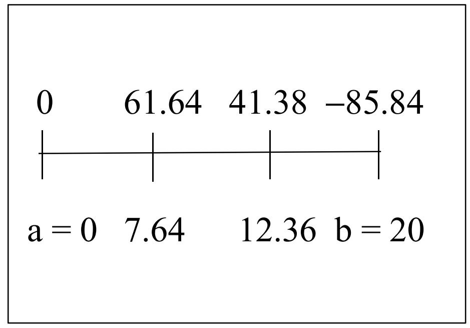
$x_2 = 90 - 55.62 = 34.38$

$f(0) = 0$

$f(34.38) = 4.12$

$f(55.62) = 5.17$

$f(90) = 4$



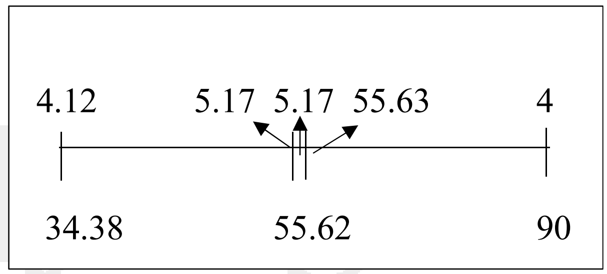
Function give maximum value 55.62 so, the new subinterval is [34.38 , 90]

Iteration-I:

$\varepsilon' L_1 = (0.6180)(90 - 34.38) = 34.37$

$x_3 = 90 - 34.37 = 55.63$

$f(55.63) = 5.17$



$\Rightarrow |f(x_c) - f(x_3)| < \varepsilon$ So, maximum point at x_c = 55.62 and maximum value is 5.17

Question: Solve Min $f(x) = \dfrac{-1}{(x-1)^2}\left[\log x - 2\dfrac{x-1}{x+1}\right]$ on [1.5,4.5] with ε = 0.2 by using Golden mean search

Solution: Here $\varepsilon = 0.2$, $\varepsilon' = 0.6180$

$$\varepsilon'(b-a) = 0.6180(4.5 - 1.5) = 1.85$$

$x_1 = 1.5 + 1.85 = 3.35$

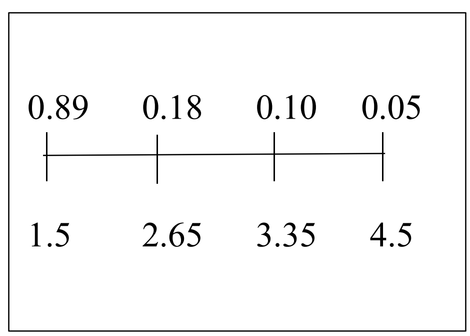
$x_2 = 4.5 - 1.85 = 2.65$

$f(1.5) = 0.89$

$f(2.65) = 0.18$

$f(3.35) = 0.10$

$f(4.5) = 0.05$



Minimum value at 4.5 but there does not become an interval.

Lecture # 07

The method of steepest Ascent (only for max):

Choose any initial vector X_0 , making use of any prior information about where the desired global maxima might be found. Then determine vector $x_1, x_2, ..., x_n, ..$ be the iterative relation

$$X_{n+1} = X_n + \lambda_n^* \nabla f\big|_{X_n}$$

Where λ_n^* is the positive scalar which maximize the $f\left(X_k + \lambda_k \nabla f\big|_{X_k}\right)$. It is best if λ_k^* represent the global maxima, however a local maxima will do the iterative process terminates if and when the difference between the values of objective functions at two successive X-vectors is smaller than a given tolerance. The last computed X-vector becomes the final approximation to X^*.

Question: Use the steepest ascent method to

$$\text{Min} \quad f(x) = \left(x_1 - \sqrt{5}\right)^2 + \left(x_2 - \pi\right)^2 + 10$$

Going over the equivalent program.

Solution: First we convert the function Min to Max because we use only Max in steepest Ascent method.

$$\text{Max} \quad f(x) = -\left(x_1 - \sqrt{5}\right)^2 - \left(x_2 - \pi\right)^2 - 10$$

we required a stating program solution which we obtain by a random sampling of the objective function.

x_1	−8.537	−0.9198	9.201	9.250	6.597	8.411	8.202
x_2	−1.099	−8.005	−2.524	7.546	5.891	−9.945	−5.709
f(x)	−144.0	−124.2	−90.61	−78.59	−36.58	−219.4	−123.9

اگر X_0 کی قیمت نہ دی گئی ہو تو یہ ٹیبل بنانا ہوتا ہے۔ اور values خود سے لینی ہوتی ہیں۔ Max 7 سے 8 value لینا ہوتی ہیں۔

Max value is −36.58. So, $X_0 = [6.597 \quad 5.891]^t$

Iteration-I: Now $X_1 = X_0 + \lambda_1 \nabla f\big|_{X_0}$

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2\left(x_1 - \sqrt{5}\right) \\ -2\left(x_2 - \pi\right) \end{bmatrix}$$

$$\nabla f\big|_{X_0} = \begin{bmatrix} -2\left(3.597 - \sqrt{5}\right) \\ -2\left(5.891 - \pi\right) \end{bmatrix} = \begin{bmatrix} -8.722 \\ -5.499 \end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix} 6.597 \\ 5.891 \end{bmatrix} + \lambda_1 \begin{bmatrix} -8.722 \\ -5.499 \end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix} 6.597 - 8.722\lambda_1 \\ 5.891 - 5.499\lambda_1 \end{bmatrix} \text{———} (i)$$

$$f\left(X_1\right) = -\left(6.597 - 8.722\lambda_1 - \sqrt{5}\right)^2 - \left(5.891 - 5.499\lambda_1 - \pi\right)^2 - 10$$

$$f\left(X_1\right) = -\left(4.361 - 8.722\lambda_1\right)^2 - \left(2.749 - 5.499\lambda_1\right)^2 - 10$$

$$f\left(X_1\right) = -\left[19.018 + 76.073\lambda_1^2 - 76.073\lambda_1\right] - \left[7.557 + 30.239\lambda_1^2 - 30.234\lambda_1\right] - 10$$

$$f\left(X_1\right) = -19.018 - 76.073\lambda_1^2 + 76.073\lambda_1 - 7.557 - 30.239\lambda_1^2 + 30.234\lambda_1 - 10$$

$$f\left(X_1\right) = -106.312\lambda_1^2 + 106.307\lambda_1 - 36.575$$

$$f'\left(X_1\right) = -2\left(106.312\right)\lambda_1 + 106.307$$

$$f'\left(X_1\right) = 0$$

$$-2\left(106.312\right)\lambda_1 + 106.307 = 0$$

$$\lambda_1 = \frac{106.307}{2\left(106.312\right)} = 0.499$$

$$\lambda_1 = 0.5$$

$$f''\left(x_1\right) = -2\left(106.312\right) < 0$$

$$f''\left(x_1\right) = f\left(X_0 + \lambda_1 \nabla f\big|_{X_0}\right) \text{ is maximum}$$

Put the value of λ_1 in (i)

$$X_1 = \begin{bmatrix} 6.597 - 8.722(0.5) \\ 5.891 - 5.499(0.5) \end{bmatrix} = \begin{bmatrix} 2.236 \\ 3.142 \end{bmatrix}$$

Now $$f(x_1) = -\left(2.236 - \sqrt{5}\right)^2 - (3.142 - \pi)^2 - 10 = -10.000$$

$$f(x_0) = -\left(6.597 - \sqrt{5}\right)^2 - (5.891 - \pi)^2 - 10 = -36.577$$

Iteration-II: $$X_1 = \begin{bmatrix} 2.236 \\ 3.142 \end{bmatrix}$$

$$\nabla f\big|_{X_1} = \begin{bmatrix} -2\left(2.236 - \sqrt{5}\right) \\ -2(3.142 - \pi) \end{bmatrix} = \begin{bmatrix} 0.0001 \\ -0.0008 \end{bmatrix}$$

$$X_2 = X_1 + \lambda_2 \nabla f\big|_{X_1}$$

$$\Rightarrow X_2 = \begin{bmatrix} 2.236 \\ 3.142 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0.0001 \\ -0.0008 \end{bmatrix}$$

$$\Rightarrow X_2 = \begin{bmatrix} 2.236 - 0.0001\lambda_2 \\ 3.142 - 0.0008\lambda_2 \end{bmatrix} \_\_\_\_\_\_ (ii)$$

$$f(X_2) = f\left(X_1 + \lambda_2 \nabla f\big|_{X_1}\right)$$

$$f(X_2) = -\left(2.236 + 0.0001\lambda_2 - \sqrt{5}\right)^2 - (3.142 - 0.0008\lambda_2 - \pi)^2 - 10$$

$$f(x_2) = -(0.000068 + 0.0001\lambda_2)^2 - (0.00041 - 0.0008\lambda_2)^2 - 10$$

$$f(x_2) = -\left[0.00000000462 + 0.00000001\lambda_2^2 - 0.0000000136\lambda_2\right]$$

$$-\left[0.0000001681 + 0.00000064\lambda_2^2 - 0.000000656\lambda_2\right] - 10$$

$$f(x_2) = -0.00000065\lambda_2^2 + 0.0000006696\lambda_2 - 0.00000017272$$

$$f'(x_2) = -2(0.00000065)\lambda_2 + 0.0000006696$$

$$\Rightarrow f'(x_2) = 0$$

$$-2(0.00000065)\lambda_2 + 0.0000006696 = 0$$

$$\lambda_2 = \frac{0.0000006696}{2(0.00000065)} = 0.5151$$

$$f''(x_2) = -2(0.00000065) < 0$$

So, $f(x_2)$ is maximum

Put the value of λ_2 in (ii)

$$X_2 = \begin{bmatrix} 2.236 + 0.0001(0.5151) \\ 3.142 - 0.0008(0.5151) \end{bmatrix} = \begin{bmatrix} 2.36 \\ 3.142 \end{bmatrix}$$

Now $$f(x_2) = -\left(2.36 - \sqrt{5}\right)^2 - (3.142 - \pi)^2 - 10 = -10.01$$

$$f(x_2) - f(x_1) = -10.01 + 10 = -0.01 < \varepsilon$$

Question:

Max $z = -\sin x_1x_2 + \cos(x_1 - x_2)$ *with* $\varepsilon = 0.05$ *and* $X_0 = [-0.7548 \;\; 0.5303]^t$

Solution: $$z = -\sin x_1x_2 + \cos(x_1 - x_2)$$

Iteration-I: $$X_0 = \begin{bmatrix} -0.7548 \\ 0.5303 \end{bmatrix}$$

Now $X_1 = X_0 + \lambda_1 \nabla f|_{X_0}$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\cos x_1x_2(x_2) - \sin(x_1 - x_2) \\ -\cos x_1x_2(x_1) + \sin(x_1 - x_2) \end{bmatrix}$$

$$\nabla f|_{X_0} = \begin{bmatrix} -(0.5303)\cos(-0.7548 \times 0.5303) - \sin(-0.7548 - 0.5303) \\ -(0.7548)\cos(-0.7548 \times 0.5303) + \sin(-0.7548 - 0.5303) \end{bmatrix} = \begin{bmatrix} 0.4711 \\ -0.2643 \end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix} -0.7548 \\ 0.5303 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0.4711 \\ -0.2643 \end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix} -0.7548 + 0.4711\lambda_1 \\ 0.5303 - 0.2643\lambda_1 \end{bmatrix}$$

$$f(X_1) = -\sin\left[(-0.7548 + 0.4711\lambda_1)(\ 0.5303 - 0.2643\lambda_1)\right] + \cos\left[-0.7548 + 0.4711\lambda_1 - 0.5303 + 0.2643\lambda_1\right]$$

$$f(X_1) = -\sin\left[0.4003 + 0.4493\lambda_1 - 0.1245\lambda_1^2\right] + \cos\left[-1.285 + 0.7345\lambda_1\right]$$

Using the golden mean search on [0,8], we determine that this function of λ_1 has a maximum at $\lambda_1 \approx 1.7$ Thus

$$X_1 = X_0 + \lambda_1 \nabla f\big|_{X_0} = \begin{bmatrix} -0.7548 + 0.4711(1.7) \\ 0.5303 - 0.2643(1.7) \end{bmatrix} = \begin{bmatrix} 0.04607 \\ 0.08099 \end{bmatrix}$$

$$f(x_1) = -\sin(0.04607 \times 0.08099) + \cos(0.04607 - 0.08099)$$

$$f(x_1) = 0.9957$$

$$f(x_0) = -\sin(-0.7548 \times 0.5303) + \cos(-0.7548 - 0.5303)$$

$$f(x_0) = 0.6715$$

$f(x_1) - f(x_0) = 0.9957 - 0.6715 = 0.3242 > 0.05$ we continue iteration.

Iteration-II: $X_1 = \begin{bmatrix} 0.04607 \\ 0.08099 \end{bmatrix}$

$$\nabla f\big|_{X_1} = \begin{bmatrix} -(0.08099)\cos(0.04607 \times 0.08099) - \sin(0.04607 - 0.08099) \\ -(0.04607)\cos(0.04607 \times 0.08099) + \sin(0.04607 - 0.08099) \end{bmatrix} = \begin{bmatrix} -0.04608 \\ -0.08098 \end{bmatrix}$$

$$X_2 = X_1 + \lambda_2 \nabla f\big|_{X_2}$$

$$\Rightarrow X_2 = \begin{bmatrix} 0.04607 \\ 0.08099 \end{bmatrix} + \lambda_2 \begin{bmatrix} -0.04608 \\ -0.08098 \end{bmatrix}$$

$$\Rightarrow X_2 = \begin{bmatrix} 0.04607 - 0.04608\lambda_2 \\ 0.08099 - 0.08098\lambda_2 \end{bmatrix}$$

$$f(X_2) = -\sin\left[(0.04607 - 0.04608\lambda_2)(0.08099 - 0.08098\lambda_2)\right]$$

$$+\cos\left[0.04607 - 0.04608\lambda_2 - 0.8099 + 0.08098\lambda_2\right]$$

$$f(x_2) = -\sin\left[0.003731 - 0.004763\lambda_2 - 0.003732\lambda_2^2\right] + \cos\left[-0.03492 + 0.0349\lambda_2\right]$$

Using the golden mean search on [0,8], we determine that this function of λ_2 has a maximum at $\lambda_{21} \approx 1$ Thus

$$X_2 = \begin{bmatrix} 0.04607 - 0.04608(1) \\ 0.08099 - 0.08098(1) \end{bmatrix} = \begin{bmatrix} 0.0000 \\ 0.0000 \end{bmatrix}$$

$$f(X_2) = -\sin(0 \times 0) + \cos(0 - 0) = 1$$

$$f(x_2) - f(x_1) = 1 - 0.9957 = 0.0043 < 0.05$$

Lecture # 08

Nelder Mead Method:

A simplex method for finding a local minimum of a function of several variables has been devised by Nelder and Mead. For two variables, a simplex is a triangle and the method are a pattern search that compares function values at the three vertices of a triangle. The worst vertex, where f(x,y) is largest, is rejected and replaced with a new vertex. A new triangle is formed and the search is continued. The process generates a sequence of triangles (which might have different shapes) for which the function values at the vertices get smaller and smaller. The size of the triangles is reduced and the coordinates of the minimum point are found.

The algorithm is stated using the term simplex (a generalized triangle in N dimensions) and will find the minimum of a function of N variables. It is effective and computationally compact.

Initial Triangles BGW:

Let f(x,y) be the function that is to be minimized. To start, we are given three vertices of a triangle $V_k = (x_k, y_k)$, k = 1,2,3. The function f(x,y) is then evaluated at each of the three points. $z_k = f(x_k, y_k)$ for k = 1,2,3. The subscripts are then reordered so that $z_1 \le z_2 \le z_3$. We use the notation

$B = (x_1, y_1), G = (x_2, y_2)$ *and* $W = (x_3, y_3)$ to help remember that B is the best vertex, G is the good (next to best) and W is the worst vertex.

Midpoint of the Good side:

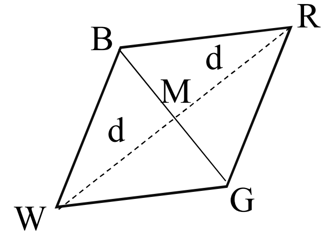
The construction process uses the midpoint of the line segment joining B and G. It is founded by averaging the coordinates:

$$M = \frac{B+G}{2} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Reflection using the point R:

The function decreases as we move along the side of the triangle from W to B and it decreases as we move along the side from W to G. Hence it is feasible that f(x,y) takes on smaller values at points that lie away from W on the opposite side of the line between B and G. We choose a test point R that is obtained by "reflecting" the triangle through the side $\overline{BG}$.

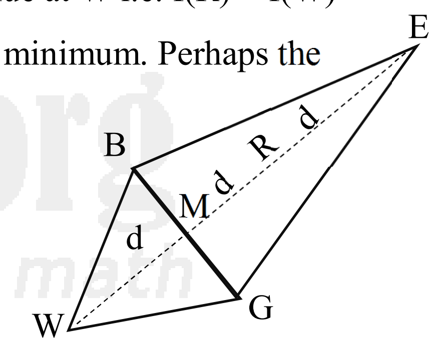
To determine R, we first find the midpoint M of the side $\overline{BG}$. Then draw the line segment from W to M and call its length d. This last segment is extended a distance d through M to locate the point R as shown in figure. The vector formula for R is



R = M+(M−W) = 2M−W

Expansion using the point E:

If the function value at R is smaller than the function value at W i.e. f(R) < f(W) Then we have moved in the correct direction toward the minimum. Perhaps the minimum is just a bit farther than the point R. So, we extend the line segment through M and R to the point E. This forms an expanded triangle BGE. The point E is found by moving an additional distance d along the line joining M and R as shown in figure. If the function value at E is less than the function value at R i.e. f(E) < f(R) , Then we have found a better vertex than R. The vector formula for E is



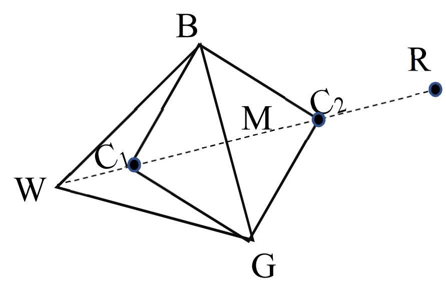
$$E = R+(R-M) = 2R-M$$

Contraction using the point C:

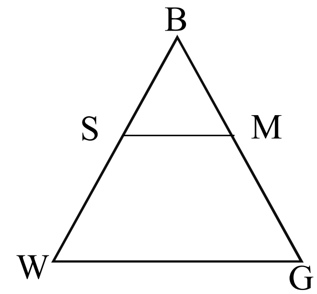
If the function value at R and W are the same i.e. f(R) = f(W) another point must be tested. Perhaps the function is smaller at M, but we cannot replace W with M because we must have a triangle. Consider the two midpoints C_1 and C_2 of the line segment $\overline{WM}$ and $\overline{MR}$, respectively as shown in figure.

The point with the smaller function value is called C and new triangle is BGC.

Note: The choice between C_1 and C_2 might seem inappropriate for the two dimensional case, but it is important in higher dimensions.



Shrink toward B:



If function value at C is not less than the value at W i.e. $f(c) \nless f(W)$ the point G and W must be shrunk toward B as shown in figure. The point G is replaced with M and W is replaced with S, which is the midpoint of the line segment joining B with W.

Logical Decision for each step:

A computationally efficient algorithm should perform function evaluation only if needed. In each step a new vertex is found, which replaces W. As soon as it is found, further investigation is not needed and iteration step is completed. The logical details for two dimensional cases are explained as below.

Logical Decisions for the Nelder-Mead Algorithm:

If f(R) < f(G), THEN perform case (i) {either reflect or extend}

ELSE perform case (ii) {either contract or shrink}

BEGIN{case(i)}
IF f(B) < f(R) THEN
replace W with R
ELSE
Compute E and f(E)
IF f(E) < f(B) THEN
replace W with E
ELSE
replace W with R
ENDIF
END{case(i)}

BEGIN{case(ii)}
IF f(R) ≤ f(W) THEN
replace W with R
Compute C = (W+M)/2
or C = (M+R)/2 and f(C)
IF f(C) < f(W) THEN
replace W with C
ELSE
Compute S and f(S)
replace W with S
replace G with M
ENDIF
END{case(ii)}

Question: Use Nelder-Mead method to find minimum of

$$f(x,y) = x^2 - 4x + y^2 - y - xy$$

Given vertices are $v_1 = (0,0)$, $v_2 = (1.2 , 0)$ and $v_3 = (0,0.8)$

Solution: Iteration-I: First find BGW triangle

$$f(v_1) = f(0,0) = 0$$

$$f(v_2) = f(1.2,0) = (1.2)^2 - 4(1.2) + (0)^2 - (0) - (1.2)(0) = -3.36$$

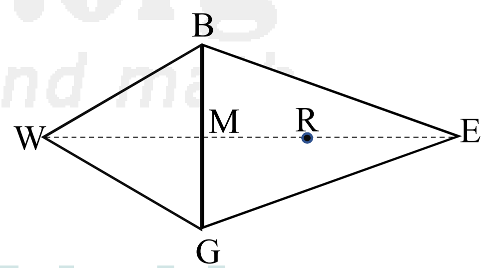
$$f(v_3) = f(0,0.8) = (0)^2 - 4(0) + (0.8)^2 - (0.8) - (0)(0.8) = -0.16$$

$$Best = B = v_2 = (1.2,0)$$

$$Good = G = v_3 = (0,0.8)$$

$$Worst = W = v_1 = (0,0)$$

Second, we find midpoint of B and G



$$M = \frac{B+G}{2} = \left(\frac{1.2+0}{2}, \frac{0+0.8}{2}\right)$$

$$M = (0.6, 0.4)$$

Now $R = 2M - W = 2(0.6,0.4) - (0,0)$

$$R = (1.2,0.8)$$

$$f(R) = (1.2)^2 - 4(1.2) + (0.8)^2 - (0.8) - (1.2)(0.8) = -4.48$$

$$\Rightarrow f(R) < f(G) \text{ i.e. } -4.48 < -0.16$$

Begins Case(i)

$$\Rightarrow f(B) \not< f(R) \text{ i.e. } -3.36 < -4.48$$

Compute E and f(E)

$$E = 2R - M = 2(1.2,0.8) - (0.6,0.4)$$

$$E = (1.8,1.2)$$

$$f(E) = (1.8)^2 - 4(1.8) + (1.8)^2 - (1.2) - (1.8)(1.2) = -5.88$$

$\Rightarrow f(E) < f(B)$ *i.e.* $-5.88 < -3.36$

Replace W with E.

Iteration-II: find BGW triangle

Now vertices are (1.2,0) , (0,0.8) , (1.8,1.2)

$$f(1.2,0) = -3.36$$

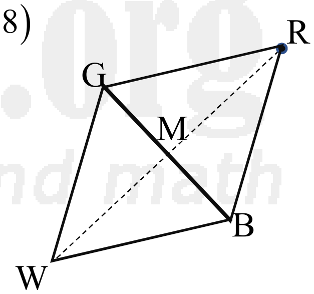
$$f(0,0.8) = -0.16$$

$$f(1.8,1.2) = -5.88$$

$$Best = B = v_2 = (1.8,1.2)$$

$$Good = G = v_3 = (1.2,0)$$

$$Worst = W = v_1 = (0,0.8)$$



Midpoint of Good and Best

$M = \frac{B+G}{2} = \left(\frac{1.8+1.2}{2}, \frac{1.2+0}{2}\right)$

$M = (1.5,0.6)$

Now $R = 2M - W = 2(1.5,0.6) - (0,0.8)$

$R = (3,0.4)$

$f(R) = (3)^2 - 4(3) + (0.4)^2 - (0.4) - (3)(0.4) = -4.44$

$\Rightarrow f(R) < f(G)$ *i.e.* $-4.44 < -3.36$

Begins Case(i)

$\Rightarrow f(B) < f(R)$ *i.e.* $-5.88 < -4.44$

Replace W with R

Iteration-III: find BGW triangle

Now vertices are (1.2,0) , (1.8,1.2) , (3,0.4)

$$f(1.2,0) = -3.36$$

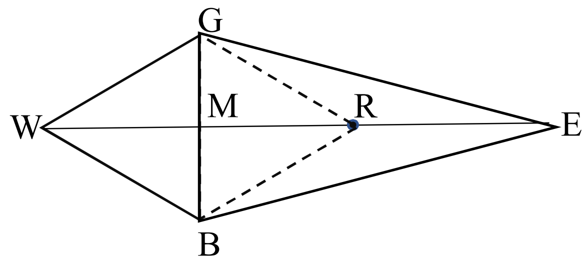
$$f(1.8,1.2) = -5.88$$

$$f(3,0.4) = -4.44$$

$$Best = B = v_2 = (1.8,1.2)$$

$$Good = G = v_3 = (3,0.4)$$

$$Worst = W = v_1 = (1.2,0)$$



Midpoint of Good and Best

$$M = \frac{B+G}{2} = \left(\frac{1.8+3}{2}, \frac{1.2+0.4}{2}\right)$$

$$M = (2.4,0.8)$$

Now $R = 2M - W = 2(2.4,0.8) - (1.2,0)$

$$R = (3.6,1.6)$$

$$f(R) = (3.6)^2 - 4(3.6) + (1.6)^2 - (1.6) - (3.6)(1.6) = -6.24$$

$\Rightarrow f(R) < f(G)$ *i.e.* $-6.24 < -4.44$

Begins Case(i)

$\Rightarrow f(B) \not< f(R)$ *i.e.* $-5.88 < -6.24$

Compute E and f(E)

$$E = 2R - M = 2(3.6,1.6) - (2.4,0.8)$$

$$E = (4.8,2.4)$$

$$f(E) = (4.8)^2 - 4(4.8) + (2.4)^2 - (2.4) - (4.8)(2.4) = -4.32$$

$\Rightarrow f(E) \not< f(B)$ *i.e.* $-4.32 \not< -5.88$

Replace W with R.

Iteration-IV: find BGW triangle

Now vertices are (1.8,1.2) , (3,0.4) , (3.6,1.6)

$$f(1.8,1.2) = -5.88$$

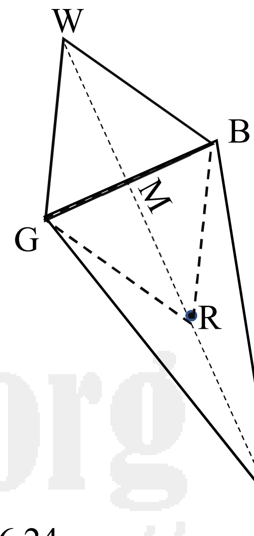
$$f(3,0.4) = -4.44$$

$$f(3.6,1.6) = -6.24$$

$$Best = B = v_2 = (3.6,1.6)$$

$$Good = G = v_3 = (1.8,1.2)$$

$$Worst = W = v_1 = (3,0.4)$$



Midpoint of Good and Best

$$M = \frac{B+G}{2} = \left(\frac{3.6+1.8}{2}, \frac{1.6+1.2}{2}\right)$$

$$M = (2.7,1.4)$$

Now $R = 2M - W = 2(2.7,1.4) - (3,0.4)$

$$R = (2.4,2.4)$$

$$f(R) = (2.4)^2 - 4(2.4) + (2.4)^2 - (2.4) - (2.4)(2.4) = -6.24$$

$\Rightarrow f(R) < f(G)$ *i.e.* $-6.24 < -5.88$

Begins Case(i)

$\Rightarrow f(B) \not< f(R)$ *i.e.* $-6.24 \not< -6.24$

Compute E and f(E)

$$E = 2R - M = 2(2.4,2.4) - (2.7,1.4)$$

$$E = (2.1,3.4)$$

$$f(E) = (2.1)^2 - 4(2.1) + (3.4)^2 - (3.4) - (2.1)(3.4) = -2.97$$

$\Rightarrow f(E) \not< f(B)$ *i.e.* $-2.97 \not< -6.24$

Replace W with R.

Iteration-V: find BGW triangle

Now vertices are (1.8,1.2), (3.6,1.6) , (2.4,2.4)

$$f(1.8,1.2) = -5.88$$

$$f(3.6,1.6) = -6.24$$

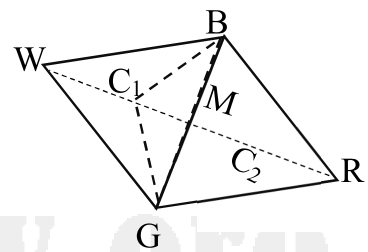
$$f(2.4,2.4) = -6.24$$

$$Best = B = (3.6,1.6)$$

$$Good = G = (2.4,2.4)$$

$$Worst = W = (1.8,1.2)$$

Midpoint of Good and Best



$$M = \frac{B+G}{2} = \left(\frac{3.6+2.4}{2}, \frac{1.6+2.4}{2}\right)$$

$$M = (3,2)$$

Now $R = 2M - W = 2(3,2) - (1.8,1.2)$

$$R = (4.2,2.8)$$

$$f(R) = (4.2)^2 - 4(4.2) + (2.8)^2 - (2.8) - (4.2)(2.8) = -5.88$$

$$\Rightarrow f(R) \not< f(G) \text{ i.e.} -5.88 \not< -6.24$$

Begins Case(ii)

$$\Rightarrow f(R) \le f(W) \text{ i.e.} -5.88 \le -5.88$$

Compute C

$$C_1 = \frac{W+M}{2} = \left(\frac{1.8+3}{2}, \frac{1.2+2}{2}\right) = (2.4,1.6)$$

$$C_1 = \frac{W+R}{2} = \left(\frac{3+4.2}{2}, \frac{2+2.8}{2}\right) = (3.6,2.4)$$

$$f(C_1) = (2.4)^2 - 4(2.4) + (1.6)^2 - (1.6) - (2.4)(1.6) = -6.72$$

$$f(C_2) = (3.6)^2 - 4(3.6) + (2.4)^2 - (2.4) - (3.6)(3.6) = -6.72$$

Both have same value. So, C = (2.4,1.6)

$$\Rightarrow f(C) = -6.72$$

$\Rightarrow f(C) < f(W)$ i.e $-6.72 < -5.88$ Replace W with C.

Iteration-VI: find BGW triangle

Now vertices are (3.6,1.6) , (2.4,2.4) , (2.4,1.6)

$$f(3.6,1.6) = -6.24$$

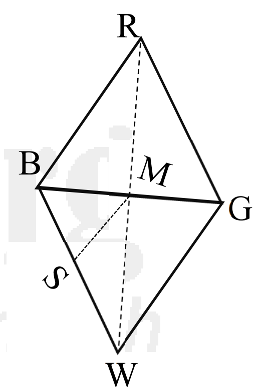
$$f(2.4,2.4) = -6.24$$

$$f(2.4,1.6) = -6.72$$

$$Best = B = (2.4,1.6)$$

$$Good = G = (3.6,1.6)$$

$$Worst = W = (2.4,2.4)$$



Midpoint of Good and Best

$$M = \frac{B+G}{2} = \left(\frac{2.4+3.6}{2}, \frac{1.6+1.6}{2}\right)$$

$$M = (3,1.6)$$

Now $R = 2M - W = 2(3,1.6) - (2.4,2.4)$

$$R = (3.6,0.8)$$

$$f(R) = (3.6)^2 - 4(3.6) + (0.8)^2 - (0.8) - (3.6)(0.8) = -4.48$$

$\Rightarrow f(R) \not< f(G)$ $i.e. -4.48 \not< -6.24$

Begins Case(ii) $\Rightarrow f(R) \le f(W)$ $i.e. -4.48 \le -6.24$

Compute S and f(S) Where S is the midpoint of B and W

$$S = \frac{B+W}{2} = \left(\frac{2.4+2.4}{2}, \frac{1.6+2.4}{2}\right) = (2.4,2)$$

$$f(S) = (2.4)^2 - 4(2.4) + (2)^2 - (2) - (2.4)(2) = -6.64$$

Replace W with S and Replace G with M and so on.

Question: Use Nelder-Mead method to find minimum of

$$Min\ f(x,y)=\left|\sin x-y^3+1\right|+x^2+\frac{1}{10}y^4$$ up to three iteration.

Given vertices are $v_1 = (1.5,0)$, $v_2 = (2 , 0)$ and $v_3 = (2,0.5)$

Solution: Iteration-I: First find BGW triangle

$$f(v_1)=f(1.5,0)=3.276$$

$$f(v_2)=f(2,0)=5.035$$

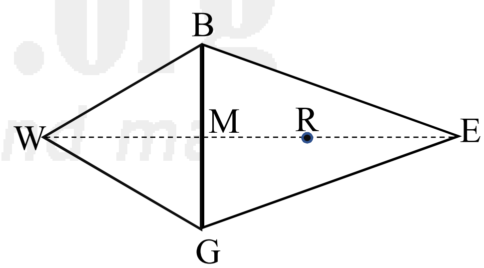
$$f(v_3)=f(2,0.5)=4.916$$

$$Best=B=(1.5,0)$$

$$Good=G=(2,0.5)$$

$$Worst=W=(2,0)$$

Second, we find midpoint of B and G



$$M=\frac{B+G}{2}=\left(\frac{1.5+2}{2},\frac{0+0.5}{2}\right)$$

$$M=(1.75,0.25)$$

Now $R=2M-W=2(1.75,0.25)-(2,0)$

$$R=(1.5,0.5)$$

$$f(R)=3.15$$

$\Rightarrow f(R)<f(G)$ *i.e.* $3.15<4.916$

Begins Case(i)

$\Rightarrow f(B)\not< f(R)$ *i.e.* $3.276<3.15$

Compute E and f(E)

$$E=2R-M=2(1.5,0.5)-(1.75,0.25)$$

$$E=(1.25,0.75)$$

$$f(E)=2.194$$

$\Rightarrow f(E)<f(B)$ *i.e.* $2.194<3.276$

Replace W with E.

Iteration-II: find BGW triangle

Now vertices are (1.5,0) , (2,0.5) , (1.25,0.75)

$$f(1.5,0)=3.276$$

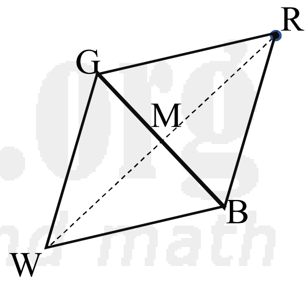
$$f(2,0.5)=4.916$$

$$f(1.25,0.75)=2.194$$

$$Best=B=(1.25,0.75)$$

$$Good=G=(1.5,0)$$

$$Worst=W=(2,0.5)$$



Midpoint of Good and Best

$$M=\frac{B+G}{2}=\left(\frac{1.25+1.5}{2},\frac{0.75+0}{2}\right)$$

$$M=(1.375,0.375)$$

Now $R=2M-W=2(1.375,0.375)-(2,0.5)$

$$R=(0.75,0.25)$$

$$f(R)=1.560$$

$$\Rightarrow f(R)<f(G)\ \ i.e.\ 1.560<3.276$$

Begins Case(i)

$$\Rightarrow f(B)\not< f(R)\ \ i.e.\ 2.194\not<1.560$$

Compute E with f(E)

$$E=2R-M=2(0.75,0.25)-(1.375,0.375)$$

$$E=(0.125,0.125)$$

$$f(E)=1.016$$

$$f(E)<f(B)\ i.e\ 1.016<2.194$$

Replace W with E

Iteration-III: find BGW triangle

Now vertices are (1.25,0.75) , (1.5,0) , (0.125,0.125)

$$f(1.25,0.75) = 2.194$$

$$f(1.5,0) = 3.276$$

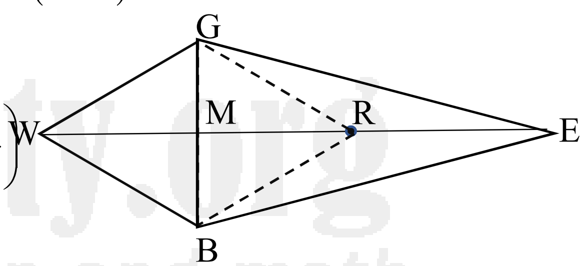
$$f(0.125,0.125) = 1.016$$

$$Best = B = (0.125,0.125)$$

$$Good = G = (1.25,0.75)$$

$$Worst = W = (1.5,0)$$

Midpoint of Good and Best



$M = \dfrac{B+G}{2} = \left(\dfrac{0.125+1.25}{2}, \dfrac{0.125+0.75}{2}\right)$

$M = (0.6875, 0.4375)$

Now $R = 2M - W = 2(0.6875, 0.4375) - (1.5, 0)$

$R = (-0.125, 0.875)$

$f(R) = 0.402$

$\Rightarrow f(R) < f(G)$ *i.e.* $0.402 < 2.194$

Begins Case(i)

$\Rightarrow f(B) \not< f(R)$ *i.e.* $1.016 \not< 0.402$

Compute E and f(E)

$E = 2R - M = 2(-0.125, 0.875) - (0.6875, 0.4375)$

$E = (-0.9375, 1.3125)$

$f(E) = 2.453$

$\Rightarrow f(E) \not< f(B)$ *i.e.* $2.453 \not< 1.016$

Replace W with R.

And so on.

Question: Use Nelder-Mead method to find minimum of

$$Min\ f(x_1, x_2) = \frac{100}{x_1 + x_2} + (x_1 - 4)^2 + (x^2 - 10)^2$$

Given vertices are $v_1 = (4,-3)$, $v_2 = (1\ ,\ 1)$ and $v_3 = (-1,3)$

Solution: Iteration-I: First find BGW triangle

$$f(v_1) = f(4,-3) = 1169$$

$$f(v_2) = f(1,1) = 590$$

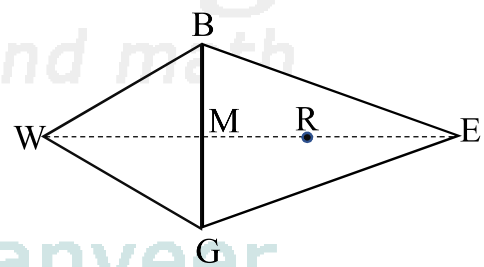
$$f(v_3) = f(-1,3) = 574$$

$$Best = B = (-1,3)$$

$$Good = G = (1,1)$$

$$Worst = W = (4,-3)$$

Second, we find midpoint of B and G



$$M = \frac{B+G}{2} = \left(\frac{-1+1}{2}, \frac{3+1}{2}\right)$$

$$M = (0,2)$$

Now $R = 2M - W = 2(0,2) - (4,-3)$

$$R = (-4,7)$$

$$f(R) = 406.33$$

$$\Rightarrow f(R) < f(G) \text{ i.e. } 406.33 < 590$$

Begins Case(i)

$$\Rightarrow f(B) \not< f(R) \text{ i.e. } 574 < 406.33$$

Compute E and f(E)

$$E = 2R - M = 2(-4,7) - (0,2)$$

$$E = (-8,12)$$

$$f(E) = 398$$

$\Rightarrow f(E) < f(B)$ *i.e.* $398 < 574$

Replace W with E.

Iteration-II: find BGW triangle

Now vertices are $(-1,3)$, $(1,1)$, $(-8,12)$

$$f(-1,3) = 574$$

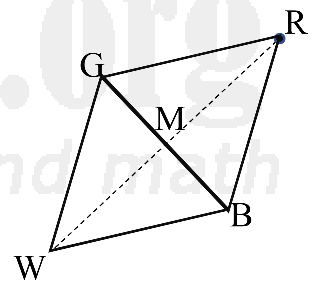
$$f(1,1) = 590$$

$$f(-8,12) = 398$$

$$Best = B = (-8,12)$$

$$Good = G = (-1,3)$$

$$Worst = W = (1,1)$$



Midpoint of Good and Best

$$M = \frac{B+G}{2} = \left(\frac{-8-1}{2}, \frac{12+3}{2} \right)$$

$$M = (-4.5, 7.5)$$

Now $R = 2M - W = 2(-4.9, 7.5) - (1,1)$

$$R = (-10,14)$$

$$f(R) = 462$$

$\Rightarrow f(R) < f(G)$ *i.e.* $462 < 574$

Begins Case(i)

$\Rightarrow f(B) < f(R)$ *i.e.* $398 < 462$

Replace W with R

Iteration-III: find BGW triangle

Now vertices are $(-8,12)$, $(-1,3)$, $(-10,14)$

$$f(-8,12) = 398$$

$$f(-1,3) = 574$$

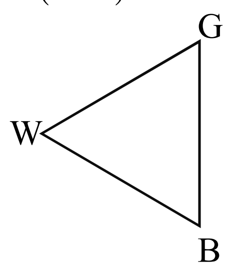
$$f(-10,14)=462$$

$$Best = B = (-8,12)$$

$$Good = G = (-10,14)$$

$$Worst = W = (-1,3)$$

Midpoint of Good and Best



$$M=\frac{B+G}{2}=\left(\frac{-8-10}{2},\frac{12+14}{2}\right)$$

$$M=(-9,13)$$

Now $R=2M-W=2(-9,13)-(-1,3)$

$$R=(-17,23)$$

$$f(R)=776.67$$

$$\Rightarrow f(R)\not< f(G)\ i.e.\ 776.67 \not< 462$$

Begins Case(ii)

$$\Rightarrow f(R)\not\leq f(W)\ i.e.\ 776.67 \not\leq 574$$

Compute S and f(S)

$$S=\frac{B+W}{2}=\left(\frac{-8-1}{2},\frac{12+3}{2}\right)=\left(\frac{-9}{2},\frac{15}{2}\right)=(-4.5,7.5)$$

$$f(S)=411.83$$

Replace W with S.

Replace G with M.

Lecture # 09

Fletcher-Powell Method:

This method is an eight-step algorithm is beginning by choosing in initial vector $\widehat{X}$ and prescribing a tolerance ε and by setting $n \times n$ matrix G equal to the identity matrix. Both $\widehat{X}$ and G are continually updated until successive value of the objective function differs by less ε whereupon the last value $\widehat{X}$ is taken as X^*.

Step-I: Evaluate $\alpha = f\left(\widehat{X}\right) and\ B = \nabla f|_{\widehat{X}}$

Step-II: Determine λ^* such that $f\left(\widehat{X} + \lambda GB\right)$ is maximized when $\lambda = \lambda^*$ set $D = \lambda^* GB$

Step-III: Designate $\widehat{X} + D$ as the update value of $\widehat{X}$

Step-IV: Calculate $\beta = f\left(\widehat{X}\right)$ for the updated value of $\widehat{X}$. If $\beta - \alpha < \varepsilon$ then go to step V; If not then go to step VI.

Step-V: Set $X^* = \widehat{X}$, $f\left(X^*\right) = \beta$ and stop.

Step-VI: Evaluate $C = \nabla f|_{\widehat{X}}$ for the updated vector $\widehat{X}$, and set Y = B−C.

Step-VII: Calculate the $n \times n$ matrix

$$L = \left(\frac{1}{D^t Y}\right) DD^t \ and\ M = \left(\frac{-1}{Y^t GY}\right) GYY^t G$$

Step-VIII: Designate G+L+M as the updated value of G. Set α equal to the current value of β , B equal to the current value of C, and return step II.

Question: Solve by Fletcher-Powell method

$Min\ f(x, y) = x - y + 2x^2 + 2xy + y^2$; starting from origin with $\varepsilon = 0.5$

Solution:

Iteration-I: Convert Min to Max

$$Max\ f(x, y) = -x + y - 2x^2 - 2xy - y^2$$

$$\text{Starting } X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-I: $$\alpha = f(X_0) = -0 + 0 - 2(0)^2 - 2(0)(0) - (0)^2$$

$$\alpha = 0$$

$$B = \nabla f\big|_{X_0} = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix}\Bigg|_{X_0} = \begin{bmatrix} -1 & -4x - 2y \\ 1 & -2x - 2y \end{bmatrix}\Bigg|_{X_0} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step-II: $\lambda_1 = \lambda^*$, $f(X_0 + \lambda_1 GB)$

$$X_0 + \lambda_1 GB = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X_0 + \lambda_1 GB = \lambda_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\lambda_1 \\ \lambda_1 \end{bmatrix}$$

$$f(X_0 + \lambda_1 GB) = -(-\lambda_1) + \lambda_1 - 2\lambda_1^2 + 2\lambda_1^2 - \lambda_1^2$$

$$f(X_0 + \lambda_1 GB) = 2\lambda_1 - \lambda_1^2$$

$$f'(X_0 + \lambda_1 GB) = 2 - 2\lambda_1$$

$$\Rightarrow f'(X_0 + \lambda_1 GB) = 0 \quad \Rightarrow 2 - 2\lambda_1 = 0$$

$$\Rightarrow \lambda_1 = 1$$

$$f''(X_0 + \lambda_1 GB) = -2 < 0$$

$f(X_0 + \lambda_1 GB)$ is maximized.

Set D = $\lambda_1 GB = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Step-III: $$X_1 = X_0 + D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step-IV: Evaluate $\beta = f(X_1)$

$$\beta = -(-1) + 1 - 2(-1)^2 - 2(-1)(1) - (1)^2 = 1$$

Now $\beta - \alpha = 1 - 0 = 1 \not< \varepsilon$

Go to step VI

Step-VI: $$C = \nabla f\big|_{X_1} = \begin{bmatrix} -1 & -4x-2y \\ 1 & -2x-2y \end{bmatrix}\Bigg|_{X_1}$$

$$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y = B - C = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Step-VII: Find L and M

$$\because D = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, D^t = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$L = \left(\frac{1}{D^tY}\right) DD^t$$

$$D^tY = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = (-1)(-2) + 0 = 2$$

$$DD^t = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M = \left(\frac{-1}{Y^tGY}\right) GYY^tG$$

$$GY = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$Y^tGY = \begin{bmatrix} -2 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = (-2)(-2) + 0 = 4$$

$$YY^t = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$GYY^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$GYY^tG = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \frac{-1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

Step-VIII: L+M+G

$$L + M + G = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$L + M + G = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix}$$

Iteration-II:

Here $G_1 = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix}$

$$\alpha = 1,\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step-II: $\lambda_2 = \lambda^*$, $f\left(X_1 + \lambda_2 G_1 B\right)$

$$X_1 + \lambda_2 G_1 B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1+\lambda_2 G_1 B=\begin{bmatrix}-1\\1\end{bmatrix}+\lambda_2\begin{bmatrix}\frac{1}{2}-\frac{1}{2}\\-\frac{1}{2}+\frac{3}{2}\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}+\lambda_2\begin{bmatrix}0\\1\end{bmatrix}$$

$$X_1+\lambda_2 G_1 B=\begin{bmatrix}-1\\1+\lambda_2\end{bmatrix}$$

$$f\left(X_1+\lambda_2 G_1 B\right)=-(-1)+(1+\lambda_2)-2(-1)^2-2(-1)(1+\lambda_2)-(1+\lambda_2)^2$$

$$f\left(X_1+\lambda_2 G_1 B\right)=1+1+\lambda_2-2+2+2\lambda_2-1-\lambda_2^2-2\lambda_2$$

$$f\left(X_1+\lambda_2 G_1 B\right)=1+\lambda_2-\lambda_2^2$$

$$f'\left(X_1+\lambda_2 G_1 B\right)=1-2\lambda_2$$

$$\Rightarrow f'\left(X_1+\lambda_2 G_1 B\right)=0 \quad \Rightarrow 1-2\lambda_2=0$$

$$\Rightarrow \lambda_2=\frac{1}{2}$$

$$f''\left(X_1+\lambda_2 G_1 B\right)=-2<0$$

$f\left(X_1+\lambda_2 G_1 B\right)$ is maximized.

Set D = $\lambda_2 G_1 B=\frac{1}{2}\begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{3}{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}-\frac{1}{2}\\-\frac{1}{2}+\frac{3}{2}\end{bmatrix}$

$$D=\frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}$$

Step-III: $X_2=X_1+D=\begin{bmatrix}-1\\1\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}-1\\ \frac{3}{2}\end{bmatrix}$

Step-IV: Evaluate $\beta=f\left(X_2\right)$

$$\beta=-(-1)+\frac{3}{2}-2(-1)^2-2(-1)\left(\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2=\frac{8+6-9}{4}=1.25$$

Now $\beta - \alpha = 1.25 - 1 = 0.25 < \varepsilon$

Step-V: Maximum vector is $X_2 = X^*$

And maximum value is $\beta = f(X_2) = 1.25$

Question: Solve by Fletcher-Powell method

$$Max\ f(x_1, x_2) = -\left(x_1 - \sqrt{5}\right)^2 - (x_2 - \pi)^2 - 10;\ and\ \varepsilon = 0.01\ and\ X_0 = \begin{bmatrix} 6.579 \\ 5.891 \end{bmatrix}$$

Solution:

Iteration-I: Here $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Step-I: $\alpha = f(X_0) = -36.420$

$$\alpha = 0$$

$$B = \nabla f\big|_{X_0} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix}_{X_0} = \begin{bmatrix} -2\left(x_1 - \sqrt{5}\right) \\ -2(x_2 - \pi) \end{bmatrix}_{X_0} = \begin{bmatrix} -8.686 \\ -5.499 \end{bmatrix}$$

Step-II: $\lambda_1 = \lambda^*,\ f(X_0 + \lambda_1 GB)$

$$X_0 + \lambda_1 GB = \begin{bmatrix} 6.579 \\ 5.891 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -8.686 \\ -5.499 \end{bmatrix}$$

$$X_0 + \lambda_1 GB = \begin{bmatrix} 6.579 \\ 5.891 \end{bmatrix} + \lambda_1 \begin{bmatrix} -8.686 \\ -5.499 \end{bmatrix} = \begin{bmatrix} 6.579 - 8.686\lambda_1 \\ 5.891 - 5.499\lambda_1 \end{bmatrix}$$

$$f(X_0 + \lambda_1 GB) = -\left(6.579 - 8.686\lambda_1 - \sqrt{5}\right)^2 - (5.891 - 5.499\lambda_1 - \pi)^2 - 10$$

$$f'(X_0 + \lambda_1 GB) = -2\left(6.579 - 8.686\lambda_1 - \sqrt{5}\right)(-8.686) - 2(5.891 - 5.499\lambda_1 - \pi)(-5.499)$$

$$f'(X_0 + \lambda_1 GB) = 17.372(4.343 - 8.686\lambda_1) - 10.998(2.749 - 5.499\lambda_1)$$

$$f'(X_0 + \lambda_1 GB) = 105.681 - 211.371\lambda_1$$

$$\Rightarrow f'(X_0 + \lambda_1 GB) = 0 \quad \Rightarrow 105.681 - 211.371\lambda_1 = 0$$

$$\Rightarrow \lambda_1 = \frac{105.681}{211.371} = 0.4999$$

$$\lambda_1 = 0.5$$

$$f''(X_0 + \lambda_1 GB) = -211.371 < 0$$

$$f(X_0 + \lambda_1 GB) \text{ is maximized.}$$

Set D = $\lambda_1 GB = (0.5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -8.686 \\ -5.499 \end{bmatrix} = \begin{bmatrix} -4.343 \\ -2.7495 \end{bmatrix}$

Step-III: $X_1 = X_0 + D = \begin{bmatrix} 6.579 \\ 5.891 \end{bmatrix} + \begin{bmatrix} -4.343 \\ -2.7495 \end{bmatrix} = \begin{bmatrix} 2.2363 \\ 3.142 \end{bmatrix}$

Step-IV: Evaluate $\beta = f(X_1)$

$$\beta = -10.0$$

Now $\beta - \alpha = -10.00 - (-36.420) = 26.420 \not< \varepsilon$

Go to step VI

Step-VI: $C = \nabla f\big|_{X_1} = \begin{bmatrix} -2(x_1 - \sqrt{5}) \\ -2(x_2 - \pi) \end{bmatrix}\Bigg|_{X_1} = \begin{bmatrix} 0.00014 \\ 0.00019 \end{bmatrix}$

$$Y = B - C = \begin{bmatrix} -8.686 \\ -5.499 \end{bmatrix} - \begin{bmatrix} 0.00014 \\ 0.00019 \end{bmatrix} = \begin{bmatrix} -8.68614 \\ -5.49919 \end{bmatrix}$$

Step-VII: Find L and M

$$L = \left(\frac{1}{D^t Y}\right) DD^t$$

$$D^t Y = \begin{bmatrix} -4.343 & -2.7495 \end{bmatrix}\begin{bmatrix} -8.686 \\ -5.499 \end{bmatrix} = 52.844$$

$$DD^t = \begin{bmatrix} -4.349 \\ -2.7495 \end{bmatrix}\begin{bmatrix} -4.343 & -2.7495 \end{bmatrix} = \begin{bmatrix} 18.862 & 11.941 \\ 11.941 & 7.559 \end{bmatrix}$$

$$L = \frac{1}{52.844}\begin{bmatrix} 18.862 & 11.941 \\ 11.941 & 7.559 \end{bmatrix} = \begin{bmatrix} 0.357 & 0.226 \\ 0.226 & 0.143 \end{bmatrix}$$

$$M=\left(\frac{-1}{Y^tGY}\right)GYY^tG$$

$$GY=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-8.68614\\ -5.49919\end{bmatrix}=\begin{bmatrix}-8.68614\\ -5.49919\end{bmatrix}$$

$$Y^tGY=\begin{bmatrix}-8.68614 & -5.49919\end{bmatrix}\begin{bmatrix}-8.68614\\ -5.49919\end{bmatrix}=105.69$$

$$YY^t=\begin{bmatrix}-8.68614\\ -5.49919\end{bmatrix}\begin{bmatrix}-8.68614 & -5.49919\end{bmatrix}=\begin{bmatrix}75.449 & 47.767\\ 47.767 & 30.241\end{bmatrix}$$

$$GYY^t=\begin{bmatrix}75.449 & 47.767\\ 47.767 & 30.241\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}75.449 & 47.767\\ 47.767 & 30.241\end{bmatrix}$$

$$GYY^tG=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}75.449 & 47.767\\ 47.767 & 30.241\end{bmatrix}=\begin{bmatrix}75.449 & 47.767\\ 47.767 & 30.241\end{bmatrix}$$

$$M=\frac{-1}{105.69}\begin{bmatrix}75.449 & 47.767\\ 47.767 & 30.241\end{bmatrix}=\begin{bmatrix}-0.714 & -0.452\\ -0.452 & -0.286\end{bmatrix}$$

Step-VIII: L+M+G

$$L+M+G=\begin{bmatrix}0.357 & 0.226\\ 0.226 & 0.143\end{bmatrix}+\begin{bmatrix}-0.714 & -0.452\\ -0.452 & -0.286\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$$L+M+G=\begin{bmatrix}0.643 & -0.226\\ -0.226 & 0.857\end{bmatrix}$$

Iteration-II:

Here $G_1=\begin{bmatrix}0.643 & -0.226\\ -0.226 & 0.857\end{bmatrix}$

$$\alpha=-10\ ,\ B=\begin{bmatrix}0.00014\\ 0.00019\end{bmatrix},\ X_1=\begin{bmatrix}2.236\\ 3.142\end{bmatrix}$$

Step-II: $\lambda_2=\lambda^*\ ,\ f\left(X_1+\lambda_2G_1B\right)$

$$X_1+\lambda_2 G_1 B=\begin{bmatrix}2.236\\3.142\end{bmatrix}+\lambda_2\begin{bmatrix}0.643 & -0.226\\-0.226 & 0.857\end{bmatrix}\begin{bmatrix}0.00014\\0.00019\end{bmatrix}$$

$$X_1+\lambda_2 G_1 B=\begin{bmatrix}2.236\\3.142\end{bmatrix}+\lambda_2\begin{bmatrix}0.00005\\0.00013\end{bmatrix}=\begin{bmatrix}2.236+0.0005\lambda_2\\3.142+0.00013\lambda_2\end{bmatrix}$$

$$f\left(X_1+\lambda_2 G_1 B\right)=-\left(2.236+0.00005\lambda_2-\sqrt{5}\right)^2-\left(3.142+0.00013\lambda_2-\pi\right)^2-10$$

$$f\left(X_1+\lambda_2 G_1 B\right)=-\left(-0.00007+0.00005\lambda_2\right)^2-\left(-0.00009+0.00013\lambda_2\right)^2-10$$

$$f'\left(X_1+\lambda_2 G_1 B\right)=0.000000007-0.000000005\lambda_2-0.000000234-0.0000000338\lambda_2$$

$$\Rightarrow f'\left(X_1+\lambda_2 G_1 B\right)=0 \quad \Rightarrow \; -0.000000227-0.0000000338\lambda_2$$

$$\Rightarrow \lambda_2=\frac{-0.000000227}{0.0000000338}=-5.851$$

$$f''\left(X_1+\lambda_2 G_1 B\right)=-0.0000000338<0$$

$f\left(X_1+\lambda_2 G_1 B\right)$ is maximized.

Set D = $\lambda_2 G_1 B=\left(-5.851\right)\begin{bmatrix}0.643 & -0.226\\-0.226 & 0.857\end{bmatrix}\begin{bmatrix}0.00014\\0.00019\end{bmatrix}=\begin{bmatrix}-0.00029\\-0.00076\end{bmatrix}$

Step-III: $X_2=X_1+D=\begin{bmatrix}2.236\\3.142\end{bmatrix}+\begin{bmatrix}-0.00029\\-0.00076\end{bmatrix}=\begin{bmatrix}2.236\\3.141\end{bmatrix}$

Step-IV: Evaluate $\beta=f\left(X_2\right)$

$$\beta=-\left(2.36-\sqrt{5}\right)^2-\left(3.141-\pi\right)^2-10=-10.0$$

Now $\beta-\alpha=-10.0+10.0=0<\varepsilon$

Go to step V.

Step-V: Maximum vector is $X_2=X^*$

And maximum value is $\beta=f\left(X_2\right)=-10.0$

Question: Solve by Fletcher-Powell method

$$Max\ f(x_1, x_2) = (1 - x_1^2) + 100(x_2 - x_1^2)^2;\ and\ \varepsilon = 0.01\ and\ X_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solution:

Iteration-I: Here $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Step-I: $\alpha = f(X_0) = (1 - (-1)^2) + 100(1 - (-1)^2)^2 = 0$

$$B = \nabla f\big|_{X_0} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix}\Bigg|_{X_0} = \begin{bmatrix} -2x_1 + 200(x_2 - x_1^2)(-2x_1) \\ 2(x_2 - x_1^2)(1) \end{bmatrix}\Bigg|_{X_0} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Step-II: $\lambda_1 = \lambda^*,\ f(X_0 + \lambda_1 GB)$

$$X_0 + \lambda_1 GB = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 + 2\lambda_1 \\ 1 \end{bmatrix}$$

$$f(X_0 + \lambda_1 GB) = (1 - (-1 + 2\lambda_1)^2) + 100(1 - (-1 + \lambda_1)^2)^2$$

$$f(X_0 + \lambda_1 GB) = (1 - (1 + 4\lambda_1^2 - 2\lambda_1)) + 100[1 - (1 + 4\lambda_1^2 - 2\lambda_1)]^2$$

$$f(X_0 + \lambda_1 GB) = (1 - 1 - 4\lambda_1^2 + 2\lambda_1) + 100[1 - 1 - 4\lambda_1^2 + 2\lambda_1]^2$$

$$f(X_0 + \lambda_1 GB) = (2\lambda_1 - 4\lambda_1^2) + 100[2(\lambda_1 - 2\lambda_1^2)]^2$$

$$f(X_0 + \lambda_1 GB) = (2\lambda_1 - 4\lambda_1^2) + 100[4(\lambda_1^2 + 4\lambda_1^4 - 4\lambda_1^3)]$$

$$f(X_0 + \lambda_1 GB) = 2\lambda_1 - 4\lambda_1^2 + 400\lambda_1^2 + 1600\lambda_1^4 - 1600\lambda_1^3$$

$$f(X_0 + \lambda_1 GB) = 2\lambda_1 + 396\lambda_1^2 - 1600\lambda_1^3 + 1600\lambda_1^4$$

$$f'(X_0 + \lambda_1 GB) = 2 + 792\lambda_1 - 4800\lambda_1^2 + 6400\lambda_1^3$$

$$\Rightarrow f'(X_0 + \lambda_1 GB) = 0 \quad \Rightarrow 2 + 792\lambda_1 - 4800\lambda_1^2 + 6400\lambda_1^3 = 0$$

$$1+396\lambda_1-2400\lambda_1^2+3200\lambda_1^3=0$$

$$\Rightarrow \lambda_1=\frac{1}{4},\frac{1}{40}\left(10-\sqrt{102}\right),\frac{1}{40}\left(10+\sqrt{102}\right) \text{ by Mathematica}$$

We select $\lambda_1=\dfrac{1}{4}=0.25$

Set $\quad D=\lambda_1 GB=(0.25)\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}0.5\\0\end{bmatrix}$

Step-III: $\quad X_1=X_0+D=\begin{bmatrix}-1\\1\end{bmatrix}+\begin{bmatrix}0.5\\0\end{bmatrix}=\begin{bmatrix}-0.5\\1\end{bmatrix}$

Step-IV: Evaluate $\beta=f\left(X_1\right)$

$$\beta=\left(1-(-0.5)\right)+100\left(1-(-0.5)\right)^2=57$$

Now $\beta-\alpha=57-0=57 \not< \varepsilon$

Go to step VI

Step-VI: $$C=\nabla f\big|_{X_1}=\left.\begin{bmatrix}-2x_1+200\left(x_2-x_1^2\right)\left(-2x_1\right)\\2\left(x_2-x_1^2\right)(1)\end{bmatrix}\right|_{X_1}=\begin{bmatrix}151\\150\end{bmatrix}$$

$$Y=B-C=\begin{bmatrix}2\\0\end{bmatrix}-\begin{bmatrix}151\\150\end{bmatrix}=\begin{bmatrix}-149\\-150\end{bmatrix}$$

Step-VII: Find L and M

$$L=\left(\frac{1}{D^tY}\right)DD^t$$

$$D^tY=\begin{bmatrix}0.5&0\end{bmatrix}\begin{bmatrix}-149\\-150\end{bmatrix}=-74.5$$

$$DD^t=\begin{bmatrix}0.5\\0\end{bmatrix}\begin{bmatrix}0.5&0\end{bmatrix}=\begin{bmatrix}0.25&0\\0&0\end{bmatrix}$$

$$L=\frac{1}{-74.5}\begin{bmatrix}0.25&0\\0&0\end{bmatrix}=\begin{bmatrix}-0.0034&0\\0&0\end{bmatrix}$$

$$M=\left(\frac{-1}{Y^tGY}\right)GYY^tG$$

$$GY=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-149\\ -150\end{bmatrix}=\begin{bmatrix}-149\\ -150\end{bmatrix}$$

$$Y^tGY=\begin{bmatrix}-149 & -150\end{bmatrix}\begin{bmatrix}-149\\ -150\end{bmatrix}=44701$$

$$YY^t=\begin{bmatrix}-149\\ -150\end{bmatrix}\begin{bmatrix}-149 & -150\end{bmatrix}=\begin{bmatrix}22201 & 22350\\ 22350 & 22500\end{bmatrix}$$

$$GYY^t=\begin{bmatrix}22201 & 22350\\ 22350 & 22500\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}22201 & 22350\\ 22350 & 22500\end{bmatrix}$$

$$GYY^tG=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}22201 & 22350\\ 22350 & 22500\end{bmatrix}=\begin{bmatrix}22201 & 22350\\ 22350 & 22500\end{bmatrix}$$

$$M=\frac{-1}{44701}\begin{bmatrix}22201 & 22350\\ 22350 & 22500\end{bmatrix}=\begin{bmatrix}-0.497 & -0.499\\ -0.499 & -0.503\end{bmatrix}$$

Step-VIII: L+M+G

$$L+M+G=\begin{bmatrix}-0.0034 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}-0.497 & -0.499\\ -0.499 & -0.503\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$L+M+G=\begin{bmatrix}0.4996 & -0.499\\ -0.499 & 0.497\end{bmatrix}$$

Iteration-II:

Here $G_1=\begin{bmatrix}0.4996 & -0.499\\ -0.499 & 0.497\end{bmatrix}$

$$\alpha=57\ ,\ B=\begin{bmatrix}151\\ 150\end{bmatrix},\ X_1=\begin{bmatrix}-0.5\\ 1\end{bmatrix}$$

Step-II: $\lambda_2=\lambda^*\ ,\ f\left(X_1+\lambda_2G_1B\right)$

$$X_1+\lambda_2 G_1 B=\begin{bmatrix}-0.5\\1\end{bmatrix}+\lambda_2\begin{bmatrix}0.4996 & -0.499\\-0.499 & 0.497\end{bmatrix}\begin{bmatrix}151\\150\end{bmatrix}$$

$$X_1+\lambda_2 G_1 B==\begin{bmatrix}-0.5\\1\end{bmatrix}+\lambda_2\begin{bmatrix}0.5816\\-0.799\end{bmatrix}$$

$$f\left(X_1+\lambda_2 G_1 B\right)=\left[1-\left(-0.5+0.5896\lambda_2\right)^2\right]+100\left[1-0.799\lambda_2-\left(-0.5+0.5896\lambda_2\right)^2\right]^2$$

$$f\left(X_1+\lambda_2 G_1 B\right)=\left[1-\left(0.25+0.3476\lambda_2^2-0.5986\lambda_2\right)\right]+100\left[1-0.799\lambda_2-\left(0.25+0.3476\lambda_2^2-0.5986\lambda_2\right)\right]^2$$

$$f\left(X_1+\lambda_2 G_1 B\right)=\left[1-0.25-0.3476\lambda_2^2+0.5986\lambda_2\right]+100\left[1-0.799\lambda_2-0.25-0.3476\lambda_2^2+0.5986\lambda_2\right]^2$$

$$f\left(X_1+\lambda_2 G_1 B\right)=\left[0.75-0.3476\lambda_2^2+0.5986\lambda_2\right]+100\left[0.75-0.3476\lambda_2^2+0.224\lambda_2\right]^2$$

$$f'\left(X_1+\lambda_2 G_1 B\right)=-2(0.3476)\lambda_2+0.5986+200\left[0.75-0.3476\lambda_2^2+0.224\lambda_2\right]\left(-2(0.3476)\lambda_2+0.5986\right)$$

$$f'\left(X_1+\lambda_2 G_1 B\right)=-2(0.3476)\lambda_2+0.5986+\left[1+150-69.52\lambda_2^2+44.8\lambda_2\right]$$

$$f'\left(X_1+\lambda_2 G_1 B\right)=-0.6952\lambda_2-104.28\lambda_2+48.33\lambda_2^3-31.14\lambda_2^2+0.5986+89.79-41.61\lambda_2^2+26.82\lambda_2$$

$$\Rightarrow f'\left(X_1+\lambda_2 G_1 B\right)=48.33\lambda_2^3-78.14\lambda_2^2+24.69\lambda_2+90.3886$$

$$\Rightarrow f'\left(X_1+\lambda_2 G_1 B\right)=0 \quad \Rightarrow 48.33\lambda_2^3-78.14\lambda_2^2+24.69\lambda_2+90.3886=0$$

$$\Rightarrow \lambda_2=-0.7827, 1.1998-0.9745i, 1.1998+0.9745i$$

We choose $\lambda_2=-0.7827$

Set D = $\lambda_2 G_1 B=(-0.7827)\begin{bmatrix}0.5896\\-0.799\end{bmatrix}=\begin{bmatrix}-0.4615\\-06254\end{bmatrix}$

Step-III: $X_2=X_1+D=\begin{bmatrix}-0.5\\1\end{bmatrix}+\begin{bmatrix}-0.4615\\0.6254\end{bmatrix}=\begin{bmatrix}-0.9615\\1.6254\end{bmatrix}$

Step-IV: Evaluate $\beta=f\left(X_2\right) \quad \Rightarrow \quad \beta=49$

Now $\beta-\alpha=49-57=-8<\varepsilon$ Go to step V.

Step-V: Maximum vector is $X_2=X^*$

And maximum value is $\beta=f\left(X_2\right)=49$

Lecture # 10

Constraint Optimization:

(i) Lagrange Multiplier method: (Applicable on equality constraint)

In mathematical optimization the method of Lagrange multiplier (named after Joseph-Louis Lagrange) is a strategy for finding local maxima and local minima of a function subject to the equality constraint i.e. subject to the condition that one or more equations have to be satisfied exactly by the chosen value of the variables. The great advantage of this method is that it allows the optimization to be solved without explicit parameterization in term of constrains. As a result, the method of Lagrange multiplier is widely used to solve challenging constraint optimization. The method can be summarized as follows:

(i) Isolate any possible singular point of the solution set of the constraining equations.
(ii) Find all the stationary points of the Lagrange function.
(iii) Establish which of the stationary points are global maxima of the objective function.

Single Constraint:

For case of only one constraint and with only two variables, consider the optimization problem

$$\text{Max } f(x,y) \quad \text{Subject to} \quad g(x,y) = 0$$

We assume that both g and g have first order partial derivatives we introduce a new variable λ (dummy variable/constraint) called Lagrange multiplier and study the Lagrangeian function or Lagrangeian expression defined by

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

$$\text{And } \nabla L(x,y,\lambda) = 0$$

$$\Rightarrow \nabla_{x,y} f(x,y) = \lambda \nabla_{x,y} g(x,y)$$

$$\text{And } g(x,y) = 0$$

Question: Max $f(x,y)=x+y$ subject to $x^2+y^2=1$

Find local maxima or minima or both by Lagrange method.

Solution:

$$f(x,y)=x+y$$

$$g(x,y)=x^2+y^2-1$$

$$L(x,y,\lambda)=f(x,y)-\lambda g(x,y)$$

$$L(x,y,\lambda)=x+y-\lambda\left(x^2+y^2-1\right)$$

$$\frac{\partial L}{\partial x}=1-2\lambda x\ ,\frac{\partial L}{\partial y}=1-2\lambda y\ ,\frac{\partial L}{\partial \lambda}=-\left(x^2+y^2-1\right)$$

Now $\frac{\partial L}{\partial x}=0 \Rightarrow 1-2\lambda x=0 \Rightarrow x=\frac{1}{2\lambda}$ ____(i)

$\frac{\partial L}{\partial y}=0 \Rightarrow 1-2\lambda y=0 \Rightarrow y=\frac{1}{2\lambda}$ ____(ii)

$\frac{\partial L}{\partial \lambda}=0 \Rightarrow x^2+y^2-1=0$ ____(iii)

Put (i) and (ii) in (iii)

$$\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}-1=0$$

$$\frac{1+1-4\lambda^2}{4\lambda^2}=0 \Rightarrow 2-4\lambda^2=0$$

$$\lambda^2=\frac{1}{2} \Rightarrow \lambda=\pm\frac{1}{\sqrt{2}}$$

When $\lambda=\frac{1}{\sqrt{2}} \Rightarrow x=\frac{1}{2.\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{2}$, $y=\frac{1}{2.\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{2}$

When $\lambda=-\frac{1}{\sqrt{2}} \Rightarrow x=-\frac{\sqrt{2}}{2}$, $y=-\frac{\sqrt{2}}{2}$

Which implies stationary points are

$$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{1}{\sqrt{2}}\right) and \left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},\frac{-1}{\sqrt{2}}\right)$$

Now $f\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\sqrt{2}$

$$f\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}=-\sqrt{2}$$

The constraint maxima is $\sqrt{2}$ at point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ and minimum is $-\sqrt{2}$ at point $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$.

Question: Max $f(x,y)=x^2y$ subject to $x^2+y^2=3$

Find local maxima or minima or both by Lagrange method.

Solution:

$$f(x,y)=x^2y$$

$$g(x,y)=x^2+y^2-3$$

$$L(x,y,\lambda)=f(x,y)-\lambda g(x,y)$$

$$L(x,y,\lambda)=xy-\lambda\left(x^2+y^2-3\right)$$

$$\frac{\partial L}{\partial x}=2xy-2\lambda x \;,\; \frac{\partial L}{\partial y}=x^2-2\lambda y \;,\; \frac{\partial L}{\partial \lambda}=-\left(x^2+y^2-3\right)$$

Now $\frac{\partial L}{\partial x}=0 \quad \Rightarrow 2xy-2\lambda x=0 \quad \_\_\_\_(i)$

$\frac{\partial L}{\partial y}=0 \quad \Rightarrow x^2-2\lambda y=0 \quad \_\_\_\_(ii)$

$\frac{\partial L}{\partial \lambda}=0 \quad \Rightarrow x^2+y^2-3=0 \quad \_\_\_\_\_(iii)$

From (i) $2x(y-\lambda)=0$

$$\Rightarrow \; x = 0 \; , \; y - \lambda = 0 \; \Rightarrow \; y = \lambda$$

Put x = 0 in (ii)

$$0 - 2\lambda y = 0$$

$$\Rightarrow y = 0 \;\; or \;\; \lambda = 0$$

Put x = 0 in (iii)

$$(0)^2 + y^2 - 3 = 0 \Rightarrow y^2 = 3 \Rightarrow y = \pm\sqrt{3}$$

$$\Rightarrow \left(0, \sqrt{3}, 0\right), \left(0, -\sqrt{3}, 0\right)$$

Put $y = \lambda$ in (ii)

$$x^2 - 2\lambda(\lambda) = 0 \;\Rightarrow x^2 = 2\lambda^2 \Rightarrow x = \sqrt{2}\lambda$$

Put $x = \sqrt{2}\lambda$, $y = \lambda$ in (iii)

$$\left(\sqrt{2}\lambda\right)^2 + (\lambda)^2 - 3 = 0$$

$$2\lambda^2 + \lambda^2 = 3$$

$$3\lambda^2 = 3 \;\Rightarrow\; \lambda^2 = 1 \quad \Rightarrow \lambda = \pm 1$$

$$x = \pm\sqrt{2} \; , \; y = \pm 1$$

Which implies stationary points are

$$\left(0, \sqrt{3}, 0\right), \left(0, -\sqrt{3}, 0\right), \left(\sqrt{2}, 1, 1\right), \left(-\sqrt{2}, -1, -1\right)$$

Now $f\left(0, \sqrt{3}\right) = (0)^2 \sqrt{3} = 0$

$$f\left(0, -\sqrt{3}\right) = (0)^2 \left(-\sqrt{3}\right) = 0$$

$$f\left(\sqrt{2}, 1\right) = \left(\sqrt{2}\right)^2 (1) = 2$$

$$f\left(-\sqrt{2}, 1\right) = \left(-\sqrt{2}\right)^2 (-1) = -2$$

The constraints have Seidel point at $\left(0,\sqrt{3}\right)$ and $\left(0,-\sqrt{3}\right)$. And Global maxima 2 at $\left(\sqrt{2},1\right)$ and Global minima at -2 at $\left(-\sqrt{2},1\right)$.

Question: Max $f\left(x,y\right)=xy$ subject to $3x^2+y^2=6$

Find local maxima or minima or both by Lagrange method.

Solution:

$$f\left(x,y\right)=xy$$

$$g\left(x,y\right)=3x^2+y^2-6$$

$$L\left(x,y,\lambda\right)=f\left(x,y\right)-\lambda g\left(x,y\right)$$

$$L\left(x,y,\lambda\right)=xy-\lambda\left(3x^2+y^2-6\right)$$

$$\frac{\partial L}{\partial x}=y-6\lambda x\ ,\ \frac{\partial L}{\partial y}=x-2\lambda y\ ,\ \frac{\partial L}{\partial \lambda}=-\left(3x^2+y^2-6\right)$$

Now $\frac{\partial L}{\partial x}=0 \quad \Rightarrow y-6\lambda x=0 \quad \_\_\_\_(i)$

$\frac{\partial L}{\partial y}=0 \ \Rightarrow\ x-2\lambda y=0 \ \_\_\_\_(ii)$

$\frac{\partial L}{\partial \lambda}=0 \Rightarrow 3x^2+y^2-6=0 \ \_\_\_\_(iii)$

From (i) $\quad y=6\lambda x$

Put in (ii) $\Rightarrow$ $\quad x-2\lambda\left(6\lambda x\right)=0$

$$x-12\lambda^2x=0$$

$$x\left(1-12\lambda^2\right)=0$$

$$x=0\ ,\ 1-12\lambda^2=0 \quad \Rightarrow \lambda^2=\frac{1}{12}$$

$$\Rightarrow\ \lambda=\frac{1}{\sqrt{12}}=\frac{1}{2\sqrt{3}}$$

Put $y=6\lambda x$ in (iii)

$$3x^2 + (6\lambda x)^2 - 6 = 0$$

$$3x^2 + 36\left(\frac{1}{12}\right)x^2 = 6$$

$$3x^2 + 3x^2 = 6$$

$$6x^2 = 6 \;\Rightarrow x^2 = 1$$

$$x = \pm 1$$

When x = 0 $\Rightarrow$ y= 0

When x = 1 , $\lambda = \frac{1}{2\sqrt{3}} \Rightarrow y = 6.\frac{1}{2\sqrt{3}}(1) = \frac{3}{\sqrt{3}}$

When x = −1 , $\lambda = \frac{1}{2\sqrt{3}} \Rightarrow y = 6.\frac{1}{2\sqrt{3}}(1) = -\frac{3}{\sqrt{3}}$

Which implies stationary points are

$$\left(0,0,\frac{1}{2\sqrt{3}}\right),\left(1,\frac{3}{\sqrt{3}},\frac{1}{2\sqrt{3}}\right),\left(-1,\frac{-3}{\sqrt{3}},\frac{1}{2\sqrt{3}}\right)$$

f(0,0) = 0

$$f\left(1,\frac{3}{\sqrt{3}}\right) = (1)\left(\frac{3}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}}$$

$$f\left(-1,\frac{-3}{\sqrt{3}}\right) = (-1)\left(\frac{-3}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}}$$

The constraint has Sedal point at (0,0) and local maxima $\frac{3}{\sqrt{3}}$ at both $\left(1,\frac{3}{\sqrt{3}}\right)$ and $\left(-1,\frac{-3}{\sqrt{3}}\right)$.

Multiple Constraints:

Lagrange function for multiple constraint is

$L(x_1, x_2, \ldots x_n, \lambda_1, \lambda_2) = f(X) - \sum_{i=1}^{n} \lambda_i g_i(X)$ Where λ_i (i=1,2,3,33n) are (unknown) constraint called Lagrange multiplier.

Question: Min $f(x_1,x_2,x_3)=x_1+x_2+x_3$

Subject to $x_1^2+x_2=3$

$x_1+3x_2+2x_3=7$

Solution: First convert minimum to maximum

$$f(x_1,x_2,x_3)=-x_1-x_2-x_3$$

$$g_1(x_1,x_2,x_3)=x_1^2+x_2-3$$

$$g_2(x_1,x_2,x_3)=x_1+3x_2+2x_3-7$$

$$L(x_1,x_2,x_3,\lambda_1,\lambda_2)=f(x_1,x_2,x_3)-\lambda_1 g_1(x_1,x_2,x_3)-\lambda_2 g_2(x_1,x_2,x_3)$$

$$L(x_1,x_2,x_3,\lambda_1,\lambda_2)=-x_1-x_2-x_3-\lambda_1\left(x_1^2+x_2-3\right)-\lambda_2\left(x_1+3x_2+2x_3-7\right)$$

$$\frac{\partial L}{\partial x_1}=-1-2\lambda_1 x_1-\lambda_2$$

$$\frac{\partial L}{\partial x_2}=-1-\lambda_1-3\lambda_2$$

$$\frac{\partial L}{\partial x_3}=-1-2\lambda_2$$

$$\frac{\partial L}{\partial \lambda_1}=-\left(x_1^2+x_2-3\right)$$

$$\frac{\partial L}{\partial \lambda_2}=-\left(x_1+3x_2+2x_3-7\right)$$

Now $$\frac{\partial L}{\partial x_1}=0 \Rightarrow -1-2\lambda_1 x_1-\lambda_2=0 \quad \_\_\_\_(i)$$

$$\frac{\partial L}{\partial x_2}=0 \Rightarrow -1-\lambda_1-3\lambda_2=0 \quad \_\_\_\_(ii)$$

$$\frac{\partial L}{\partial x_3}=0 \Rightarrow -1-2\lambda_2=0 \Rightarrow \lambda_2=\frac{-1}{2} \quad \_\_\_\_(iii)$$

$$\frac{\partial L}{\partial \lambda_1}=0 \Rightarrow x_1^2+x_2-3=0 \qquad \_\_\_\_(iv)$$

$$\frac{\partial L}{\partial \lambda_2}=0 \Rightarrow x_1+3x_2+2x_3-7=0 \qquad \_\_\_\_(v)$$

Put $\lambda_2=\frac{-1}{2}$ in (ii)

$$-1-\lambda_1-3\left(\frac{-1}{2}\right)=0$$

$$\Rightarrow \lambda_1=\frac{1}{2}$$

Put λ_1 & λ_2 in (i)

$$-1-2.\frac{1}{2}x_1-\left(\frac{-1}{2}\right)=0 \Rightarrow x_1=-1+\frac{1}{2}$$

$$\Rightarrow x_1=-\frac{1}{2}$$

Put x_1 in (iv)

$$\left(\frac{-1}{2}\right)^2+x_2-3=0 \Rightarrow x_2=3-\frac{1}{4}=\frac{11}{4}$$

Put x_1 & x_2 in (v)

$$\frac{-1}{2}+3\left(\frac{11}{4}\right)+2x_3-7=0 \Rightarrow 2x_3=7+\frac{1}{2}-\frac{33}{4}$$

$$2x_3=\frac{28+2-33}{4}=\frac{-3}{4}$$

$$x_3=\frac{-3}{8}$$

Which implies stationary point is

$$\left(\frac{-1}{2},\frac{11}{4},\frac{-3}{8},\frac{1}{2},\frac{-1}{2}\right)$$

$$\text{Now } f\left(\frac{-1}{2},\frac{11}{4},\frac{-3}{8}\right)=-\left(\frac{-1}{2}\right)-\left(\frac{11}{4}\right)-\left(\frac{-3}{8}\right)$$

$$f\left(\frac{-1}{2},\frac{11}{4},\frac{-3}{8}\right)=\frac{1}{2}-\frac{11}{4}+\frac{3}{8}=\frac{-15}{8}$$

Our question is for minimum.

$$\text{So, } f\left(\frac{-1}{2},\frac{11}{4},\frac{-3}{8}\right)=\frac{15}{8} \text{ for min.}$$

The constraint has global minimum $\frac{15}{8}$ at point $\left(\frac{-1}{2},\frac{11}{4},\frac{-3}{8}\right)$.

Question: Max $f(x,y,z)=x+y+z$

Subject to $x^2+y^2+z^2=3$

Find local maxima or minima or both by Lagrange method.

Solution:

$$f(x,y,z)=x+y+z$$

$$g_1(x,y,z)=x^2+y^2+z^2-3$$

$$L(x,y,z,\lambda)=f(x,y,z)-\lambda g(x,y,z)$$

$$L(x,y,z,\lambda)=x+y+z-\lambda\left(x^2+y^2+z^2-3\right)$$

$$\frac{\partial L}{\partial x}=1-2\lambda x$$

$$\frac{\partial L}{\partial y}=1-2\lambda y$$

$$\frac{\partial L}{\partial z}=1-2\lambda z$$

$$\frac{\partial L}{\partial \lambda}=-\left(x^2+y^2+z^2-3\right)$$

Now $\frac{\partial L}{\partial x}=0 \Rightarrow 1-2\lambda x=0 \quad \Rightarrow x=\frac{1}{2\lambda}$

$$\frac{\partial L}{\partial y}=0 \Rightarrow 1-2\lambda y=0 \quad \Rightarrow y=\frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial z}=0 \Rightarrow 1-2\lambda z=0 \quad \Rightarrow z=\frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial \lambda}=0 \quad \Rightarrow \quad x^2+y^2+z^2-3=0$$

Put above values

$$\left(\frac{1}{2\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2-3=0$$

$$\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}-3=0$$

$$\frac{1+1+1-3\left(4\lambda^2\right)}{4\lambda^2}=0$$

$$3-12\lambda^2=0 \quad \Rightarrow \lambda^2=\frac{1}{4}$$

$$\Rightarrow \lambda=\pm\frac{1}{2}$$

When $\lambda=\frac{1}{2} \quad \Rightarrow \quad x=\frac{1}{2.\frac{1}{2}}=1 \;,\; y=1\,,\; z=1$

When $\lambda=\frac{-1}{2} \quad \Rightarrow \quad x=\frac{1}{2.\left(\frac{-1}{2}\right)}=-1 \;,\; y=-1\,,\; z=-1$

The stationary points are $\left(1,1,1,\frac{1}{2}\right),\left(-1,-1,-1,\frac{-1}{2}\right)$

Now $f(1,1,1)=1+1+1=3$

$$f(-1,-1,-1)=-1-1-1=-3$$

The constraint maxima is 3 at point (1,1,1) and minima is −3 at point (−1,−1,−1)

Question: Max $f(x,y,z)=z^2$

Subject to $x^2+y^2-z=0$

Find local maxima or minima or both by Lagrange method.

Solution:

$$f(x,y,z)=z^2$$

$$g(x,y,z)=x^2+y^2-z$$

$$L(x,y,z,\lambda)=f(x,y,z)-\lambda g(x,y,z)$$

$$L(x,y,z,\lambda)=z^2-\lambda\left(x^2+y^2-z\right)$$

$$\frac{\partial L}{\partial x}=-2\lambda x$$

$$\frac{\partial L}{\partial y}=-2\lambda y$$

$$\frac{\partial L}{\partial z}=2z+\lambda$$

$$\frac{\partial L}{\partial \lambda}=-\left(x^2+y^2-z\right)$$

Now $\frac{\partial L}{\partial x}=0 \Rightarrow -2\lambda x=0 \quad \Rightarrow x=0$

$$\frac{\partial L}{\partial y}=0 \Rightarrow -2\lambda y=0 \quad \Rightarrow y=0$$

$$\frac{\partial L}{\partial z}=0 \Rightarrow 2z+\lambda=0 \quad \Rightarrow z=\frac{-\lambda}{2}$$

$$\frac{\partial L}{\partial \lambda}=0 \quad \Rightarrow \quad \left(x^2+y^2-z\right)=0$$

Put above values

$$(0)^2+(0)^2-\left(\frac{-\lambda}{2}\right)=0$$

$$\lambda=0$$

$$\Rightarrow z=0$$

Stationary point (0,0,0,0)

Now $f(0,0,0)=(0)^2=0$

$$f(-1,-1,-1)=-1-1-1=-3$$

The constraint has sedal point at (0,0,0).

Question: Max $f(x,y,z)=x^2-y^2$

Subject to $x^2+2y^2+3z^2=1$; z is constant.

Find local maxima or minima or both by Lagrange method.

Solution: $f(x,y,z)=x^2-y^2$

$$g(x,y,z)=x^2+2y^2+3z^2-1$$

$$L(x,y,z,\lambda)=f(x,y,z)-\lambda g(x,y,z)$$

$$L(x,y,z,\lambda)=x^2-y^2-\lambda\left(x^2+2y^2+3z^2-1\right)$$

$$\frac{\partial L}{\partial x}=2x-2\lambda x$$

$$\frac{\partial L}{\partial y}=-2y-4\lambda y$$

$$\frac{\partial L}{\partial \lambda}=-\left(x^2+2y^2+3z^2-1\right)$$

Now $\frac{\partial L}{\partial x}=0 \Rightarrow 2x-2\lambda x=0 \quad \Rightarrow x=0 \text{ and } \lambda=1$

$$\frac{\partial L}{\partial y}=0 \Rightarrow -2y-4\lambda y=0 \quad \Rightarrow y=0 \text{ and } \lambda=\frac{-1}{2}$$

$$\frac{\partial L}{\partial \lambda}=0 \Rightarrow \left(x^2+2y^2+3z^2-1\right)=0$$

$$\left(x^2+2y^2+3z^2-1\right)=0 \qquad \_\_\_\_\_(i)$$

Put x & y in (i)

$$(0)^2+2(0)^2+3z^2-1=0$$

$$3z^2-1=0$$

Stationary points (0,0,1) and (0,0,−1/2)

Now $f(0,0)=(0)^2-(0)^2=0$

The constraint has sedal point at (0,0).

Lecture # 11

Newton Raphson's method of constraint optimization (for equally constraints)

Min / Max z = f(X)

Subject to g(X) =0

Where $X = \{x_1, x_2, \ldots x_n\}$

$$L = f(X) - \sum_{i=1}^{n} \lambda_i g_i(X)$$

And formula for Newton Raphson is defined as

$$X_{n+1} = X_n - \left(H_L\big|_{X_n}\right)^{-1} \nabla L\big|_{X_n}$$ for constraint optimization.

Question: Use Newton Raphson method to solve

Max $z = f(x_1, x_2) = 2x_1 + x_1x_2 + 3x_2$

Subject to $x_1^2 + x_2 = 3$

Using $X_0(1,1,1)$

Solution: $f(x_1, x_2) = 2x_1 + x_1x_2 + 3x_2$

$g(x_1, x_2) = x_1^2 + x_2 - 3$

$L = f(x_1, x_2) - \lambda g(x_1, x_2)$

$L(x_1, x_2, \lambda_1) = 2x_1 + x_1x_2 + 3x_2 - \lambda_1(x_1^2 + x_2 - 3)$

Now $$\nabla L = \begin{bmatrix} \dfrac{\partial L}{\partial x_1} \\ \dfrac{\partial L}{\partial x_2} \\ \dfrac{\partial L}{\partial \lambda_1} \end{bmatrix} = \begin{bmatrix} 2 + x_2 - 2\lambda_1 x_1 \\ x_1 + 3 - \lambda_1 \\ -(x_1^2 + x_2 - 3) \end{bmatrix}$$

$$\nabla L\big|_{X_0} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Now $$H_L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

$$H_L = \begin{bmatrix} -2\lambda_1 & 1 & -2x_1 \\ 1 & 0 & -1 \\ -2x_1 & -1 & 0 \end{bmatrix}$$

$$H_L\big|_{X_0} = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

We calculate $X_1 = X_0 - \left(H_L\big|_{X_0}\right)^{-1} \nabla L\big|_{X_0}$ ______(i)

$$\left(H_L\big|_{X_0}\right)^{-1} = \frac{Adj\left(H_L\big|_{X_0}\right)}{\left|H_L\big|_{X_0}\right|} \qquad \text{\_\_\_\_\_\_}(ii)$$

$$\left|H_L\big|_{X_0}\right| = \begin{vmatrix} -2 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 0 \end{vmatrix} = -2(0-1) - 1(0-2) - 2(1-0) = 6$$

$$a_{11} = (-1)^{1+1} \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = (0-1) = -1$$

$$a_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$a_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = (-1-0) = -1$$

$$a_{21} = (-1)^{2+1} \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = -(0-2) = 2$$

$$a_{22} = (-1)^{2+2}\begin{vmatrix} -2 & -2 \\ -2 & 0 \end{vmatrix} = (0-4) = -4$$

$$a_{23} = (-1)^{2+3}\begin{vmatrix} -2 & 1 \\ -2 & -1 \end{vmatrix} = -(2+2) = -4$$

$$a_{31} = (-1)^{3+1}\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} = (-1-0) = -1$$

$$a_{32} = (-1)^{3+2}\begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix} = -(2+2) = -4$$

$$a_{33} = (-1)^{3+3}\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = (0-1) = -1$$

$$Adj\left(H_L|_{X_0}\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^t = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & -4 \\ -1 & -4 & -1 \end{bmatrix}^t$$

$$Adj\left(H_L|_{X_0}\right) = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & -4 \\ -1 & -4 & -1 \end{bmatrix}$$

$$\left(H_L|_{X_0}\right)^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & -4 \\ -1 & -4 & -1 \end{bmatrix}$$

Put in (i) $\Rightarrow$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & -4 \\ -1 & -4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1+6-1 \\ 2-12-4 \\ -1-12-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 4 \\ -14 \\ -14 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1-\frac{2}{3} \\ 1+\frac{7}{3} \\ 1+\frac{7}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{bmatrix}$$

Now $X_2 = X_1 - \left(H_L\big|_{X_1}\right)^{-1} \nabla L\big|_{X_1} \qquad \_\_\_\_\_(iii)$

$$\nabla L\big|_{X_1} = \begin{bmatrix} 2+x_2-2\lambda_1 x_1 \\ x_1+3-\lambda_1 \\ -\left(x_1^2+x_2-3\right) \end{bmatrix}\Bigg|_{X_1} = \begin{bmatrix} 2+\frac{10}{3}-2\left(\frac{10}{3}\right)\left(\frac{1}{3}\right) \\ \frac{1}{3}+3-\frac{10}{3} \\ -\left(\frac{1}{9}+\frac{10}{3}-3\right) \end{bmatrix}$$

$$\nabla L\big|_{X_1} = \begin{bmatrix} \frac{18+30-20}{9} \\ \frac{1+9-10}{3} \\ -\left(\frac{1+30-27}{9}\right) \end{bmatrix} = \begin{bmatrix} \frac{28}{9} \\ 0 \\ \frac{-4}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 28 \\ 0 \\ -4 \end{bmatrix}$$

$$H_L\big|_{X_1} = \begin{bmatrix} -2\lambda_1 & 1 & -2x_1 \\ 1 & 0 & -1 \\ -2x_1 & -1 & 0 \end{bmatrix}\Bigg|_{X_1}$$

$$H_L\big|_{X_1} = \begin{bmatrix} \frac{-20}{3} & 1 & \frac{-2}{3} \\ 1 & 0 & -1 \\ \frac{-2}{3} & -1 & 0 \end{bmatrix}$$

$$\left|H_L\right|_{X_1} = \begin{vmatrix} \frac{-20}{3} & 1 & \frac{-2}{3} \\ 1 & 0 & -1 \\ \frac{-2}{3} & -1 & 0 \end{vmatrix} = \frac{-20}{3}(0-1) - 1\left(0-\frac{2}{3}\right) - \frac{2}{3}(-1-0) = \frac{24}{3} = 8$$

$$a_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = (0-1) = -1$$

$$a_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -1 \\ \frac{-2}{3} & 0 \end{vmatrix} = -\left(0-\frac{2}{3}\right) = \frac{2}{3}$$

$$a_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ -\frac{2}{3} & -1 \end{vmatrix} = (-1-0) = -1$$

$$a_{21} = (-1)^{2+1}\begin{vmatrix} 1 & -\frac{2}{3} \\ -1 & 0 \end{vmatrix} = -\left(0-\frac{2}{3}\right) = \frac{2}{3}$$

$$a_{22} = (-1)^{2+2}\begin{vmatrix} \frac{-20}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 0 \end{vmatrix} = \left(0-\frac{4}{9}\right) = -\frac{4}{9}$$

$$a_{23} = (-1)^{2+3}\begin{vmatrix} \frac{-20}{3} & 1 \\ -\frac{2}{3} & -1 \end{vmatrix} = -\left(\frac{20}{3}+\frac{2}{3}\right) = -\frac{22}{3}$$

$$a_{31} = (-1)^{3+1}\begin{vmatrix} 1 & -\frac{2}{3} \\ 0 & -1 \end{vmatrix} = (-1-0) = -1$$

$$a_{32} = (-1)^{3+2}\begin{vmatrix} \frac{-20}{3} & -\frac{2}{3} \\ 1 & -1 \end{vmatrix} = -\left(\frac{20}{3}+\frac{2}{3}\right) = -\frac{22}{3}$$

$$a_{33}=(-1)^{3+3}\begin{vmatrix}-\frac{20}{3} & 1\\ 1 & 0\end{vmatrix}=(0-1)=-1$$

$$Adj\left(H_L|_{X_1}\right)=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}^t=\begin{bmatrix}-1 & \frac{2}{3} & -1\\ \frac{2}{3} & -\frac{4}{9} & -\frac{22}{3}\\ -1 & -\frac{22}{3} & -1\end{bmatrix}^t$$

$$Adj\left(H_L|_{X_1}\right)=\begin{bmatrix}-1 & \frac{2}{3} & -1\\ \frac{2}{3} & -\frac{4}{9} & -\frac{22}{3}\\ -1 & -\frac{22}{3} & -1\end{bmatrix}$$

$$\left(H_L|_{X_1}\right)^{-1}=\frac{3}{24}\begin{bmatrix}-1 & \frac{2}{3} & -1\\ \frac{2}{3} & -\frac{4}{9} & -\frac{22}{3}\\ -1 & -\frac{22}{3} & -1\end{bmatrix}=\frac{3}{24}.\frac{1}{3}\begin{bmatrix}-3 & 2 & -3\\ 2 & \frac{-4}{3} & -22\\ -3 & -22 & -3\end{bmatrix}$$

Put in (iii) $\Rightarrow$ $$X_2=\begin{bmatrix}\frac{1}{3}\\ \frac{10}{3}\\ \frac{10}{3}\end{bmatrix}-\frac{1}{24}.\frac{1}{9}\begin{bmatrix}-3 & 2 & -3\\ 2 & -\frac{4}{3} & -22\\ -3 & -22 & -3\end{bmatrix}\begin{bmatrix}28\\ 0\\ -4\end{bmatrix}$$

$$X_2=\begin{bmatrix}\frac{1}{3}\\ \frac{10}{3}\\ \frac{10}{3}\end{bmatrix}-\frac{1}{216}\begin{bmatrix}-84+0+12\\ 56-0+88\\ -84-0+12\end{bmatrix}=\begin{bmatrix}\frac{1}{3}\\ \frac{10}{3}\\ \frac{10}{3}\end{bmatrix}-\frac{1}{216}\begin{bmatrix}-72\\ 144\\ -72\end{bmatrix}$$

$$X_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{bmatrix} - \begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{-1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}+\frac{1}{3} \\ \frac{10}{3}-\frac{2}{3} \\ \frac{10}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{8}{3} \\ \frac{11}{3} \end{bmatrix} = \begin{bmatrix} 0.677 \\ 2.667 \\ 3.667 \end{bmatrix}$$

Now $X_3 = X_2 - \left(H_L\big|_{X_2}\right)^{-1} \nabla L\big|_{X_2}$ _____ (iv)

$$\nabla L\big|_{X_2} = \begin{bmatrix} 2+x_2-2\lambda_1 x_1 \\ x_1+3-\lambda_1 \\ -\left(x_1^2+x_2-3\right) \end{bmatrix}\Bigg|_{X_2} = \begin{bmatrix} 2+\frac{8}{3}-2\left(\frac{11}{3}\right)\left(\frac{2}{3}\right) \\ \frac{2}{3}+3-\frac{11}{3} \\ -\left(\frac{4}{9}+\frac{8}{3}-3\right) \end{bmatrix}$$

$$\nabla L\big|_{X_2} = \begin{bmatrix} \frac{18+24-44}{9} \\ \frac{2+9-11}{3} \\ -\left(\frac{4+24-27}{9}\right) \end{bmatrix} = \begin{bmatrix} \frac{-2}{9} \\ 0 \\ \frac{-1}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$$

$$H_L\big|_{X_2} = \begin{bmatrix} -2\lambda_1 & 1 & -2x_1 \\ 1 & 0 & -1 \\ -2x_1 & -1 & 0 \end{bmatrix}\Bigg|_{X_1}$$

$$H_L\big|_{X_2} = \begin{bmatrix} \frac{-16}{3} & 1 & \frac{-4}{3} \\ 1 & 0 & -1 \\ \frac{-4}{3} & -1 & 0 \end{bmatrix}$$

$$\left|H_L\right|_{X_1}=\begin{vmatrix}\frac{-16}{3} & 1 & \frac{-4}{3}\\ 1 & 0 & -1\\ \frac{-4}{3} & -1 & 0\end{vmatrix}=\frac{-16}{3}(0-1)-1\left(0-\frac{4}{3}\right)-\frac{4}{3}(-1-0)=\frac{24}{3}=8$$

$$a_{11}=(-1)^{1+1}\begin{vmatrix}0 & -1\\ -1 & 0\end{vmatrix}=(0-1)=-1$$

$$a_{12}=(-1)^{1+2}\begin{vmatrix}1 & -1\\ \frac{-4}{3} & 0\end{vmatrix}=-\left(0-\frac{4}{3}\right)=\frac{4}{3}$$

$$a_{13}=(-1)^{1+3}\begin{vmatrix}1 & 0\\ -\frac{4}{3} & -1\end{vmatrix}=(-1-0)=-1$$

$$a_{21}=(-1)^{2+1}\begin{vmatrix}1 & -\frac{4}{3}\\ -1 & 0\end{vmatrix}=-\left(0-\frac{4}{3}\right)=\frac{4}{3}$$

$$a_{22}=(-1)^{2+2}\begin{vmatrix}\frac{-16}{3} & -\frac{4}{3}\\ -\frac{4}{3} & 0\end{vmatrix}=\left(0-\frac{16}{9}\right)=-\frac{16}{9}$$

$$a_{23}=(-1)^{2+3}\begin{vmatrix}\frac{-16}{3} & 1\\ -\frac{4}{3} & -1\end{vmatrix}=-\left(\frac{16}{3}+\frac{4}{3}\right)=-\frac{20}{3}$$

$$a_{31}=(-1)^{3+1}\begin{vmatrix}1 & -\frac{4}{3}\\ 0 & -1\end{vmatrix}=(-1-0)=-1$$

$$a_{32}=(-1)^{3+2}\begin{vmatrix}\frac{-16}{3} & -\frac{4}{3}\\ 1 & -1\end{vmatrix}=-\left(\frac{16}{3}+\frac{4}{3}\right)=-\frac{20}{3}$$

$$a_{33} = (-1)^{3+3}\begin{vmatrix} -\frac{16}{3} & 1 \\ 1 & 0 \end{vmatrix} = (0-1) = -1$$

$$Adj\left(H_L|_{X_2}\right) = \begin{bmatrix} -1 & \frac{4}{3} & -1 \\ \frac{4}{3} & -\frac{16}{9} & -\frac{20}{3} \\ -1 & -\frac{20}{3} & -1 \end{bmatrix}^t = \begin{bmatrix} -1 & \frac{4}{3} & -1 \\ \frac{4}{3} & -\frac{16}{9} & -\frac{20}{3} \\ -1 & -\frac{20}{3} & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -9 & 12 & -9 \\ 12 & -16 & -60 \\ -9 & -60 & -9 \end{bmatrix}$$

$$\left(H_L|_{X_2}\right)^{-1} = \frac{1}{8}.\frac{1}{9}\begin{bmatrix} -9 & 12 & -9 \\ 12 & -16 & -60 \\ -9 & -60 & -9 \end{bmatrix} = \frac{1}{72}\begin{bmatrix} -9 & 12 & -9 \\ 12 & -16 & -60 \\ -9 & -60 & -9 \end{bmatrix}$$

Put in (iv) $\Rightarrow$ $$X_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{8}{3} \\ \frac{11}{3} \end{bmatrix} - \frac{1}{72}.\frac{1}{9}\begin{bmatrix} -9 & 12 & -9 \\ 12 & -16 & -60 \\ -9 & -60 & -9 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{8}{3} \\ \frac{11}{3} \end{bmatrix} - \frac{1}{648}\begin{bmatrix} 18+0+9 \\ -24+0+60 \\ 18+0+9 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{8}{3} \\ \frac{11}{3} \end{bmatrix} - \frac{1}{648}\begin{bmatrix} 27 \\ 36 \\ 27 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{8}{3} \\ \frac{11}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{24} \\ \frac{1}{18} \\ \frac{1}{24} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}-\frac{1}{24} \\ \frac{8}{3}-\frac{1}{18} \\ \frac{11}{3}-\frac{1}{24} \end{bmatrix} = \begin{bmatrix} \frac{15}{24} \\ \frac{47}{18} \\ \frac{87}{24} \end{bmatrix} = \begin{bmatrix} 0.625 \\ 2.61 \\ 3.625 \end{bmatrix}$$

$$X_3 - X_2 < \varepsilon$$

Hence, Max solution is X_3

Question: Use Newton Raphson method to solve

Max $f(x,y)=0.25x^2+y^2$

Subject to $5-x-y=0$

Solution: $f(x,y)=0.25x^2+y^2$

$g(x,y)=5-x-y$

$L=f(x,\mathrm{y})-\lambda g(x,y)$

$L(x,y,\lambda_1)=0.25x^2+y^2-\lambda_1(5-x-y)$

Now $\nabla L=\begin{bmatrix}\dfrac{\partial L}{\partial x}\\ \dfrac{\partial L}{\partial y}\\ \dfrac{\partial L}{\partial \lambda_1}\end{bmatrix}=\begin{bmatrix}0.5x+\lambda_1\\ 2y+\lambda_1\\ -(5-x-y)\end{bmatrix}$

Now $$H_L=\begin{bmatrix}L_{11} & L_{12} & L_{13}\\ L_{21} & L_{22} & L_{23}\\ L_{31} & L_{32} & L_{33}\end{bmatrix}$$

$$H_L=\begin{bmatrix}0.5 & 0 & 1\\ 0 & 2 & 1\\ 1 & 1 & 0\end{bmatrix}$$

We choose X_0 = (0,0,0) so that $\nabla L|_{X_0}\neq 0$

$$\nabla L|_{X_0}=\begin{bmatrix}0\\ 0\\ -5\end{bmatrix}$$

$$H_L|_{X_0}=\begin{bmatrix}0.5 & 0 & 1\\ 0 & 2 & 1\\ 1 & 1 & 0\end{bmatrix}$$

We calculate $X_1=X_0-\left(H_L|_{X_0}\right)^{-1}\nabla L|_{X_0}$ ______(i)

$$\left|H_L\big|_{X_0}\right| = \begin{vmatrix} 0.5 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0.5(0-1) - 0 + 1(0-2) = -0.5 - 2 = -2.5$$

$$a_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = (0-1) = -1$$

$$a_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = 1$$

$$a_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = (0-2) = -2$$

$$a_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = 1$$

$$a_{22} = (-1)^{2+2}\begin{vmatrix} 0.5 & 1 \\ 1 & 0 \end{vmatrix} = (0-1) = -1$$

$$a_{23} = (-1)^{2+3}\begin{vmatrix} 0.5 & 0 \\ 1 & 1 \end{vmatrix} = -(0.5-0) = -0.5$$

$$a_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = (0-2) = -2$$

$$a_{32} = (-1)^{3+2}\begin{vmatrix} 0.5 & 1 \\ 0 & 1 \end{vmatrix} = -(0.5-0) = -0.5$$

$$a_{33} = (-1)^{3+3}\begin{vmatrix} 0.5 & 0 \\ 0 & 2 \end{vmatrix} = (1-0) = 1$$

$$Adj\left(H_L\big|_{X_0}\right) = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & -0.5 \\ -2 & -0.5 & 1 \end{bmatrix}^t = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & -0.5 \\ -2 & -0.5 & 1 \end{bmatrix}$$

$$\left(H_L\big|_{X_0}\right)^{-1} = \frac{-1}{2.5}\begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & -0.5 \\ -2 & -0.5 & 1 \end{bmatrix}$$

Put in (i) $\Rightarrow$ $$X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \left(-\frac{1}{2.5}\right)\begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & -0.5 \\ -2 & -0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

$$X_1 = \frac{1}{2.5}\begin{bmatrix} 0+0+10 \\ 0+0+2.5 \\ 0+0-5 \end{bmatrix} = \frac{1}{2.5}\begin{bmatrix} 10 \\ 2.5 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

Now $$X_2 = X_1 - \left(H_L\big|_{X_1}\right)^{-1} \nabla L\big|_{X_1} \qquad \_\_\_\_\_\_(ii)$$

$$\nabla L\big|_{X_1} = \left.\begin{bmatrix} 0.5x+\lambda_1 \\ 2y+\lambda_1 \\ -(5-x-y) \end{bmatrix}\right|_{X_1} = \begin{bmatrix} 2-2 \\ 2-2 \\ -(5-4-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$H_L\big|_{X_1} = \begin{bmatrix} 0.5 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ is same as } H_L\big|_{X_0}$$

So, $$\left(H_L\big|_{X_1}\right)^{-1} = \frac{-1}{2.5}\begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & -0.5 \\ -2 & -0.5 & 1 \end{bmatrix}$$

Put in (ii) $\Rightarrow$ $$X_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} - \left(-\frac{1}{2.5}\right)\begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & -0.5 \\ -2 & -0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} + \frac{1}{2.5}\begin{bmatrix} 0+0+0 \\ 0+0+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow X_2 - X_1 < \varepsilon$$

Hence Maximum value is X_2

Lecture # 12

Karush-Kuhn-Tucker Method:

In mathematical optimization the Karush-kuhn-Tucker method also known as Kuhn-Tucker method were originally named after Harold W.Kuhn and Albert W.Tucker Who first published the conditions in 1951. Later scholars discovered that the necessary conditions for this type problem had been stated by William Karush in his master's thesis in 1939.

Any problem can be formulated as

Min f(X)

Subject to $h_i(X)=0 \quad \forall i=1,2,\ldots,\mathrm{m}$

And $g_i(X)=0 \quad \forall i=1,2,\ldots,n$

In other words, find the solution that min f(X) as long as all equalities $h_i(X)=0$ and all inequalities $g_i(X)\le 0$ holds.

- Stationarity

$$\nabla_x f(X)+\sum_{i=1}^{m}\nabla_x \lambda_i h_i(X)+\sum_{i=1}^{n}\mu_i\nabla_x g_i(X)=0 \quad \text{(minimization)}$$

$$\nabla_x f(X)+\sum_{i=1}^{m}\nabla_x \lambda_i h_i(X)-\sum_{i=1}^{n}\mu_i\nabla_x g_i(X)=0 \quad \text{(maximization)}$$

- Equality Constraints

$$\nabla_\lambda f(X)+\sum_{i=1}^{m}\nabla_\lambda \lambda_i h_i(X)-\sum_{i=1}^{n}\mu_i\nabla_\lambda g_i(X)=0$$

Given below the KKT necessary conditions, explaining each equations:

(i) Feasibility: $g_i(X)-b_i$ is feasible

(ii) $\nabla_x f(X)-\sum_{i=1}^{m}\lambda_i\nabla_x g_i(X)=0$

(iii) $\lambda_i\left(g_i(X)-b_i\right)=0$

(iv) Positive Lagrange multiples $\lambda_i\ge 0$ e.g. $x_1+x_2+x_3=1000$

The equality constraint equivalent to two constraints $x_1+x_2+x_3=1000$ and $-x_1-x_2-x_3\le -1000$

Question: Solve for the optimum to the following problem by using KKT method.

Min $f(X) = x_1^2 + 2x_2^2 + 3x_3^2$

Subject to $-5x_1 + x_2 + 3x_3 \le -3$

$2x_1 + x_2 + 2x_3 \ge 6$

Solution: Min $f(X) = x_1^2 + 2x_2^2 + 3x_3^2$

$g_1 = 5x_1 - x_2 - 3x_3 \ge 3$ $\quad\because$ multiply −ve to convert ≤ into ≥

$g_2 = 2x_1 + x_2 + 2x_3 \ge 6$

Assume both constraints are binding like $g_1 = 3$ and $g_2 = 6$. We can see the sign of the Lagrange multiplier to see it that is good assumption.

Write out the Lagrange multiplier equation

$$\nabla f(X) - \sum_{i=1}^{n} \lambda_i \nabla g_i(X) = 0$$

$$\nabla f(X) - \lambda_1 \nabla g_1(X) - \lambda_2 \nabla g_2(X) = 0$$

Or
$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_3} - \lambda_1 \frac{\partial g_1}{\partial x_3} - \lambda_2 \frac{\partial g_2}{\partial x_3} = 0$$

Then solution of above equation is

$$2x_1 - 5\lambda_1 - 2\lambda_2 = 0$$

$$4x_2 + \lambda_1 - \lambda_2 = 0$$

$$6x_3 + 3\lambda_1 - 2\lambda_2 = 0$$

And
$$5x_1 - x_2 - 3x_3 = 3$$

$$2x_1 + x_2 + 2x_3 = 6$$

In matrix form

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & -5 & -2 & 0 \\ 0 & 4 & 0 & 1 & -1 & 0 \\ 0 & 0 & 6 & 3 & -2 & 0 \\ 5 & -1 & -3 & 0 & 0 & 3 \\ 2 & 1 & 2 & 0 & 0 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{5}{2} & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 5 & -1 & -3 & 0 & 0 & 3 \\ 2 & 1 & 2 & 0 & 0 & 6 \end{array}\right] \frac{1}{2}R_1 \ , \ \frac{1}{4}R_2 \ , \frac{1}{6}R_3$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{5}{2} & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & -1 & -3 & \frac{25}{2} & 5 & 3 \\ 0 & 1 & 2 & 5 & 2 & 6 \end{array}\right] R_4 - 5R_1 \ , R_5 - 2R_1$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{5}{2} & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & -3 & \frac{51}{4} & \frac{19}{4} & 3 \\ 0 & 0 & 2 & \frac{19}{4} & \frac{9}{4} & 6 \end{array}\right] R_4 + R_2 \ , R_5 - R_2$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{5}{2} & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{57}{4} & \frac{15}{4} & 3 \\ 0 & 0 & 0 & \frac{15}{4} & \frac{35}{12} & 6 \end{array}\right] R_4+3R_3\ ,\ R_5-2R_3$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{5}{2} & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{15}{57} & \frac{4}{19} \\ 0 & 0 & 0 & \frac{15}{4} & \frac{35}{12} & 6 \end{array}\right] \frac{4}{57}R_4$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{5}{2} & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{15}{57} & \frac{4}{19} \\ 0 & 0 & 0 & 0 & 1.9298 & 5.01316 \end{array}\right] R_5-\frac{15}{4}R_4$$

$$\Rightarrow 1.9298\,\lambda_2 = 5.01316 \quad \Rightarrow \lambda_2 = 2.6$$

$$\Rightarrow \lambda_1 + \frac{15}{57}\lambda_2 = \frac{4}{19} \quad \Rightarrow \lambda_1 = \frac{4}{19} - \frac{15}{57}(2.6)$$

$$\Rightarrow \lambda_1 = -0.5$$

$$x_3 + \frac{1}{2}\lambda_1 - \frac{1}{3}\lambda_2 = 0$$

$$\Rightarrow x_3 = \frac{-1}{2}(-0.5) - \frac{1}{3}(2.6)$$

$$\Rightarrow x_3 = 1.12$$

$$x_2 + \frac{1}{4}\lambda_1 - \frac{1}{4}\lambda_2 = 0$$

$$\Rightarrow x_2 = -\frac{1}{4}(-0.5) + \frac{1}{4}(2.6)$$

$$\Rightarrow x_2 = 0.775$$

$$x_1 - \frac{5}{2}\lambda_1 - \lambda_2 = 0$$

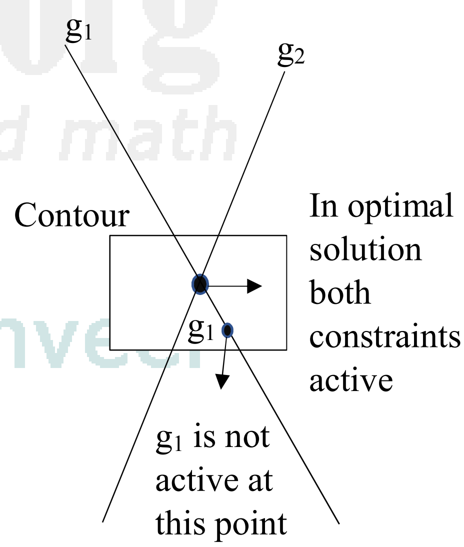
$$\Rightarrow x_1 = \frac{5}{2}(-0.5) + (2.6)$$

$$\Rightarrow x_1 = 1.35$$

Here λ_1 is negative which show g_1 is not active point. λ_2 is positive so g_2 is active point.

For optimal solution both constraints are active.

Hence, the solution of this question is not optimal.



Question: Solve for the optimum to the following problem by using KKT method.

Min $\quad f(X) = 2x_1^2 + x_2^2 + 4x_3^2$

Subject to $x_1 - 2x_2 + x_3 = 6$

$$2x_1 - 2x_2 + 3x_3 = 12$$

Solution: Min $\quad f(X) = 2x_1^2 + x_2^2 + 4x_3^2$

$$g_1 = x_1 - 2x_2 + x_3 = 6$$

$$g_2 = 2x_1 - 2x_2 + 3x_3 = 12$$

$$\nabla f(X) - \lambda_1 \nabla g_1(X) - \lambda_2 \nabla g_2(X) = 0$$

Or
$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_3} - \lambda_1 \frac{\partial g_1}{\partial x_3} - \lambda_2 \frac{\partial g_2}{\partial x_3} = 0$$

Then
$$4x_1 - \lambda_1 - 2\lambda_2 = 0$$

$$2x_2 + 2\lambda_1 + 2\lambda_2 = 0$$

$$8x_3 - \lambda_1 - 3\lambda_2 = 0$$

And
$$x_1 - 2x_2 + x_3 = 6$$

$$2x_1 - 2x_2 + 3x_3 = 12$$

In matrix form

$$\left[\begin{array}{ccccc|c} 4 & 0 & 0 & -1 & -2 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & 8 & -1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 0 & 6 \\ 2 & -2 & 3 & 0 & 0 & 12 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{8} & -\frac{3}{8} & 0 \\ 4 & 0 & 0 & -1 & -2 & 0 \\ 2 & -2 & 3 & 0 & 0 & 12 \end{array}\right] R_{14} \;,\; \frac{1}{2}R_2 \;,\; \frac{1}{8}R_3$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{8} & -\frac{3}{8} & 0 \\ 0 & 8 & -4 & -1 & -2 & -24 \\ 0 & 2 & 1 & 0 & 0 & 0 \end{array}\right] R_4 - 4R_1 \ , R_5 - 2R_1$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{8} & -\frac{3}{8} & 0 \\ 0 & 0 & -4 & -9 & -10 & -24 \\ 0 & 0 & 1 & -2 & -2 & 0 \end{array}\right] R_4 - 8R_2 \ , R_5 - 2R_2$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{8} & -\frac{3}{8} & 0 \\ 0 & 0 & 0 & -\frac{19}{2} & -\frac{23}{2} & -24 \\ 0 & 0 & 0 & -\frac{15}{8} & -\frac{13}{8} & 0 \end{array}\right] R_4 + 4R_3 \ , R_5 - R_3$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{8} & -\frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{23}{19} & \frac{48}{19} \\ 0 & 0 & 0 & -\frac{15}{8} & -\frac{13}{8} & 0 \end{array}\right] \frac{-2}{19}R_4$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{8} & -\frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{23}{19} & \frac{48}{19} \\ 0 & 0 & 0 & 0 & 0.6447 & 4.7368 \end{array}\right]$$

$$\Rightarrow 0.6447\lambda_2 = 4.7368 \quad \Rightarrow \lambda_2 = 7.347$$

$$\Rightarrow \lambda_1 + \frac{23}{19}\lambda_2 = \frac{48}{19} \quad \Rightarrow \lambda_1 = \frac{48}{19} - \frac{23}{19}(7.347)$$

$$\Rightarrow \lambda_1 = -6.367$$

Here λ_1 is negative which show g_1 is not active. So, solution of this question is not optimal.

Question: Solve for the optimum to the following problem by using KKT method.

Min $\quad f(X) = x_1^2 + x_2^2 + 60x_1$

Subject to $x_1 - 80 \geq 0$

$\qquad x_1 + x_2 - 120 \geq 0$

Solution: Min $\quad f(X) = x_1^2 + x_2^2 + 60x_1$

$$g_1 = x_1 - 80 \geq 0 \quad \Rightarrow g_1 = x_1 \geq 80$$

$$g_2 = x_1 + x_2 - 120 \geq 0 \quad \Rightarrow g_2 = x_1 + x_2 \geq 120$$

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

Then

$$2x_1 + 60 - \lambda_1 - \lambda_2 = 0 \quad \Rightarrow 2x_1 - \lambda_1 - \lambda_2 = -60 \qquad \_\_\_\_\_(i)$$

$$2x_2 - \lambda_2 = 0 \quad \_\_\_\_\_(ii)$$

$$x_1 - 80 = 0 \quad \_\_\_\_\_ (iii)$$

$$x_1 + x_2 - 120 = 0 \quad \_\_\_\_\_ (iv)$$

From (iii) $\Rightarrow$ $x_1 = 80$

Put in (iv) $\Rightarrow$ $80 + x_2 - 120 = 0$

$$x_2 = 40$$

Put in (ii) $\Rightarrow$ $2(40) - \lambda_2 = 0$

$$\lambda_2 = 80$$

Put in (i) $\Rightarrow$ $2(80) - \lambda_1 - 80 = -60$

$$160 - \lambda_1 - 80 = -60$$

$$\lambda_1 = 140$$

Both λ_1 *and* λ_2 are positive. Hence the solution of this question is optimal.

Question: Solve for the optimum to the following problem by using KKT method.

Max $f(X) = -(x-2)^2 - 2(y-1)^2$

Subject to $x + 4y \le 3$

$-x + y \le 0$

Solution: Convert Max into min

Min $f(X) = (x-2)^2 + 2(y-1)^2$

Subject to $-x - 4y \ge -3$

$x - y \ge 0$

$g_1 = -x - 4y \ge -3$

$g_2 = x - y \ge 0$

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2}-\lambda_1\frac{\partial g_1}{\partial x_2}-\lambda_2\frac{\partial g_2}{\partial x_2}=0$$

Then $\quad 2(x-2)+\lambda_1-\lambda_2=0 \quad \_\_\_\_\_(i)$

$$4(y-1)+4\lambda_1+\lambda_2=0 \quad \_\_\_\_\_(ii)$$

And $\quad -x-4y=-3 \quad \_\_\_\_\_(iii)$

$$x-y=0 \quad \_\_\_\_\_(iv)$$

Adding (ii) and (iv) $\quad \Rightarrow \quad -5y=-3 \quad \Rightarrow \quad y=\frac{3}{5}$

Put in (iv) $\quad \Rightarrow \quad x=y \quad \Rightarrow x=\frac{3}{5}$

Put the value of x and y in (i) and (ii)

(i) $\quad \Rightarrow \quad 2\left(\frac{3}{5}-2\right)+\lambda_1-\lambda_2=0$

$$2\left(\frac{3-10}{5}\right)+\lambda_1-\lambda_2=0$$

$$\frac{-14}{5}+\lambda_1-\lambda_2=0$$

$$\Rightarrow \lambda_1-\lambda_2=\frac{14}{5} \quad \_\_\_\_\_(v)$$

(ii) $\quad \Rightarrow \quad 4\left(\frac{3}{5}-1\right)+4\lambda_1+\lambda_2=0$

$$2\left(\frac{3-5}{5}\right)+4\lambda_1+\lambda_2=0$$

$$\frac{-8}{5}+4\lambda_1+\lambda_2=0$$

$$\Rightarrow 4\lambda_1+\lambda_2=\frac{8}{5} \quad \_\_\_\_\_(vi)$$

Adding (v) and (vi) $\quad \Rightarrow \quad 5\lambda_1=\frac{14}{5}+\frac{8}{5}=\frac{22}{5}$

$$\Rightarrow \quad \lambda_1 = \frac{22}{25}$$

Put in (v) $\Rightarrow$

$$\frac{22}{25} - \lambda_2 = \frac{14}{5}$$

$$\lambda_2 = \frac{22}{25} - \frac{14}{5} = \frac{22-70}{25}$$

$$\lambda_2 = \frac{-48}{25}$$

λ_2 is negative which show g_2 is not active. Hence the solution is not optimal.