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Ex. on Page 226

Q(1)

$$\int_{C_1} \frac{2z+3}{z(z^2+1)(z+1)^2} \qquad |z| = 3$$

Poles are $z=0$, $z=\pm i$, $z=-1$ (is a pole of order 2)

$$R(f,0) = \lim_{z\to 0} \frac{z\,(2z+3)}{z\,(z^2+1)(z+1)^2} = 3$$

$$R(f,i) = \lim_{z\to i} \frac{(z-i)(2z+3)}{z(z-i)(z+i)(z+1)^2} = -\tfrac{1}{2} + \tfrac{3i}{4}$$

$$R(f,-i) = -\tfrac{1}{2} - \tfrac{3i}{4} \quad (\text{replace } i \text{ by } -i)$$

$$R(f,-1) = \lim_{z\to -1} \frac{d}{dz}\left(\frac{(z+1)^2(2z+3)}{z(z^2+1)(z+1)^2}\right)$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{2z+3}{z^3+z}\right) = \lim_{z\to -1} \frac{(z^3+z)2-(2z+3)(3z^2+1)}{(z^3+z)^2}$$
$$= -2$$

$$\sum R_i = 3 - \tfrac{1}{2} + \tfrac{3i}{4} - \tfrac{1}{2} - \tfrac{3i}{4} - 2 = 0$$

2). Find $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$ with the help of $\int_c \frac{dz}{z+2}$, $|z|=1$

Since $z=-2$ is a pole not in circle $|z|=1$

$\therefore \int_c \frac{dz}{z+2} = 0$

So $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$

$$= \tfrac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta \qquad \text{let } z = e^{i\theta}$$

$dz = i e^{i\theta} d\theta$

$d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z + \frac{1}{z}}{2}$

$$\frac{1}{2i}\int_0^{2\pi} \frac{\frac{z+z^2+1}{z}}{z\left(5+2\frac{(z^2+1)}{z}\right)}\, dz$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{z^2+z+1}{z(z+2)(2z+1)}\, dz \qquad 0, -\tfrac{1}{2} \text{ lie inside}$$

$$R(f,0) = \lim_{z\to 0}\left(\frac{z(z^2+z+1)}{z(2z+1)(z+2)}\right) = \tfrac{1}{2}$$

$$R(f,-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}} \frac{z^2+z+1}{2z(z+2)} = -\tfrac{1}{2}$$

$$\sum R_i = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

ch # 6 (2)

Q(3) $\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$ (i) C is $|z| = 2$
(ii) $|z+i| = \sqrt{3}$

Sol (i) $f(z) = \frac{12z-7}{(z-1)^2(2z+3)}$ Poles
for $|z| = 2$
$z = 1$ is of order 2
$z = -3/2$ simple pole

$R(f,1) = \lim_{z\to 1} \frac{d}{dz} \frac{\cancel{(z-1)^2}(12z-7)}{\cancel{(z-1)^2}(2z+3)} = \lim_{z\to 1} \frac{(2z+3)12-(12z-7)2}{(2z+3)^2}$

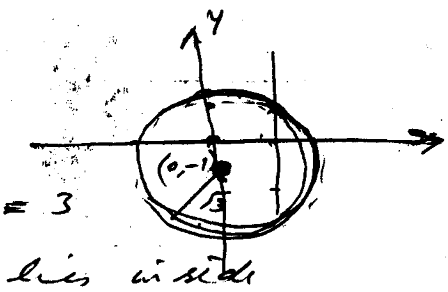
$= \frac{60-10}{25} = 2$

$R(f,-3/2) = \lim_{z\to -3/2} \left(\frac{(2z+3)\ (12z-7)}{(2z+3)\ (z-1)^2}\right) = -2$

$\Sigma R_i = 2-2 = 0$

(ii) $|z+i| = \sqrt{3}$

$\sqrt{x^2+(y+1)^2} = \sqrt{3}$ $x^2+(y+1)^2 = 3$



$z = 1$ lies inside

$R(f,1) = 2$

Using Cauchy Residue Th $\int_C f(z)\,dz = 2(2\pi i)$
$= 4\pi i$

Q(4) $\int_C \frac{dz}{z^2(z-a)^n}$ n is +ve C is a countour enclosing origin
& 'a' inside C
& (ii) a outside C

Pole $z = 0$ order 2
$z = a$ ord n

$R(f,0) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{\cancel{z^2}}{\cancel{z^2}(z-a)^n}\right)$

$= \lim_{z\to 0} \frac{d}{dz}(z-a)^{-n} = \lim_{z\to 0}(-n)(z-a)^{-n-1} = \frac{-n}{(0-a)^{n+1}}$

$= \frac{-n}{(-1)^{n+1}a^{n+1}} = \frac{n(-1)^n}{a^{n+1}}$

let $y = z^{-2}$
$y_1 = -2z^{-3}$
$y_2 = (-2)(-3)z^{-4}$
$y_{n-1} = (-1)^{n-1}\lfloor n \; z^{-(n-1+2)}$
$= \frac{(-1)^{n-1}\lfloor n}{z^{n+1}}$

$R(f,a) = \lim_{z\to a} \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dz^{n-1}}\left(\frac{\cancel{(z-a)^n}}{z^2\cancel{(z-a)^n}}\right)$

$= \lim_{z\to a} \frac{1}{\lfloor n-1} \frac{(-1)^{n-1}\lfloor n}{z^{n+1}} = \frac{(-1)^{n-1}\lfloor n}{a^{n+1}\lfloor n-1}$

$\Sigma R_i = \frac{n(-1)^n}{a^{n+1}} + \frac{(-1)^{n-1}n}{a^{n+1}} = \frac{-n(-1)^{n-1}}{a^{n+1}} + \frac{n(-1)^{n-1}}{a^{n+1}} = 0$

(ii) 'a' is outside C
0 is inside

$R(f,0) = \frac{n(-1)^n}{a^{n+1}}$ $\int_C f(z)\,dz = \frac{2\pi i(-1)^n n}{a^{n+1}}$

ch # 6 (3)

Q (5) $\int_c \frac{\sin z\, dz}{(1+z^2)^2}$ C is $|z-i| = r$

$x^2 + (y-1)^2 = r^2$ $C = (0,1)$ radius $= r$

$$R(f, i) = \frac{d}{dz}\left[\frac{\cancel{(z-i)^2} \sin z}{\cancel{(z-i)^2}(z+i)^2}\right]_{z=i}$$

$$= \frac{(z+i)^2 \cos z - \sin z(2)(z+i)}{(z+i)^4} = \left|\frac{(z+i)\cos z - 2\sin z}{(z+i)^3}\right|_{z=i}$$

$$= \frac{2i\cosh(1) - 2i\sinh(1)}{-8i}$$

$R(f, i) = \frac{1}{4}[\sinh 1 - \cosh 1]$

$R(f, -i) = \frac{1}{4}[\sinh(1) - \cosh 1]$

$\Sigma R_i = \frac{1}{2}[\sinh(1) - \cosh(1)]$

$$\int_c \frac{\sin z\, dz}{(1+z^2)^2} = 2\pi i\,[\tfrac{1}{2}\sinh(1) - \cosh(1)]$$

$$= \pi i\,[\sinh(1) - \cosh(1)]$$

(ii) for $|z-i| > 2$ Only pole $z = i$

$$\int_c \frac{\sin z}{(z^2+1)^2} dz = \frac{\pi i}{2}[\sinh(1) - \cosh(1)]$$

Q (6) $\int_c \frac{dz}{z^2 \sin z}$ $|z| = 1$

$z = 0$ is a pole $\sin z = 0$ $z = \sin^{-1}(0) = 0$, 3rd order

Consider $f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^2[z - z^3/\lfloor 3 + z^5/\lfloor 5 - \dots]}$

$$= \frac{1}{z^3[1 - z^2/6 + \frac{z^4}{\lfloor 5} - + \dots]}$$

$$= \frac{1}{z^3}\left[1 - (z^2/6 - \frac{z^4}{\lfloor 5} + \dots)\right]^{-1}$$

$$= \frac{1}{z^3}\left(1 + z^2/6 + z^4/120 \dots\right)$$

$R(f, 0) = 1/6$ Hence $\int_c f(z)dz = 2\pi i(1/6)$

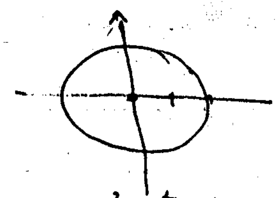
$= \pi i/3$

Q (7) Correct Q. is $\int_c \frac{e^{iz}}{z^4 + 2z^2 + 1} dz$ $|z| = 2$

$$f(z) = \frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z+i)^2(z-i)^2}$$

$$R(f, i) = \lim_{z\to i} \frac{d}{dz} \frac{\cancel{(z-i)^2} e^{iz}}{\cancel{(z-i)^2}(z+i)^2} = \lim_{z\to i} \frac{(z+i)^2 i e^{iz} - e^{iz}(2)(z+i)}{(z+i)^4}$$

$$= \frac{e^{ii}[(2i)i - 2]}{-8i} = \frac{1}{2ie}$$



$$\int_C \frac{e^{2z}\,dz}{(z^2+1)^2} = 2\pi i \frac{1}{2ei} = \pi/e \qquad x^2+y^2=4$$

along the curve

Q(8) $\int_c \frac{z\,dz}{(z^2+1)(z^2+2z+2)}$ (i) $|z| = 3/2$. (ii) $|z-i| = 3/2$

Poles $z = \pm i$ $\quad z = -1 \pm i$

$R(f, i) = \lim_{z\to i} \frac{z\cancel{(z-i)}}{(z+i)\cancel{(z-i)}(z^2+2z+2)} = \frac{1}{2(1+2i)} = \frac{1}{10}(1-2i)$

$R(f, -i) = \lim_{z\to -i} f(z).(z+i) = \frac{1}{10}(1+2i)$

$R(f, -1+i) = \lim \frac{z\,(z+1-i)}{(z^2+1)(z+1-i)(z+1+i)} = \frac{i-1}{(1-2i)\,2i} = \frac{(i-1)(4-2i)}{20} = \frac{6i-2}{20}$

$= -1/10 + \frac{3i}{10}$

$R(f, -1-i) = \frac{-6i-2}{20} = -1/10 - \frac{3i}{10}$

$\Sigma R_i = \frac{1}{10}(1-2i) + \frac{1}{10}(1+2i) + \left(\frac{-1}{10}+\frac{3i}{10}\right) + \left(-\frac{1}{10} - \frac{3i}{10}\right)$

$= 0$

(ii) $|z-i| = 3/2$ $\quad x^2+(y-1)^2 = 9/4$ $\quad z = \pm i$, $z = (0,1)\ (0,-1)$ $\quad z=(0,1)$ lies inside

other pts $(-1,1)\ (-1,-1)$ $\quad (-1,1)$ lies ins

$R(f, i) = \frac{1}{10}(1-2i)$

$R(f, -1+i) = -\frac{1}{10}(1-3i)$

$\Sigma R_i = \frac{1}{10} - \frac{2i}{10} - \frac{1}{10} + \frac{3i}{10} = \frac{i}{10}$

$\int_c f(z)dz = 2\pi i\,(i/10) = -\pi/5$

Q(4) $\int_c \frac{e^z\,dz}{(z-1)^n}$ $\quad |z| = 2$, $\quad z=1$ is pole of order n

$\lim_{z\to 1} \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dz^{n-1}} \frac{\cancel{(z-1)^n}\,e^z}{\cancel{(z-1)^n}} = \frac{1}{\lfloor n-1}(e^1) \quad \therefore \int_c f(z)dz = \frac{2\pi i\, e}{\lfloor n-1}$

Q 10) $\int_c \frac{dz}{z^4+1}$ $\quad |z-1| = 1 \Rightarrow (x-1)^2 + y^2 = 1$

Centre $(1,0)$ rad $= 1$

$z_k = cis\left(\frac{\pi+2\pi k}{n}\right)$, $k = 0, 1, 2, 3$

$z_0 = \frac{1}{\sqrt2} + i\frac{1}{\sqrt2}$, $z_1 = -1/\sqrt2 + i/\sqrt2$, $z_2 = -\frac{1}{\sqrt2} - \frac{i}{\sqrt2}$ $\quad z_3 = \frac{1}{\sqrt2} - \frac{i}{\sqrt2}$

z_0 & z_3 lies inside

$R\left(f, \frac{1+i}{\sqrt2}\right) = \lim_{z\to} \left[\frac{\cancel{\left(z-\frac{1+i}{\sqrt2}\right)}}{\cancel{\left(z-\frac{1+i}{\sqrt2}\right)}\left(z-\left(\frac{i-1}{\sqrt2}\right)\right)\left(z+\frac{1+i}{\sqrt2}\right)\left(z+\left(\frac{i-1}{\sqrt2}\right)\right)}\right]$

$= \frac{1}{\left(\frac{2}{\sqrt2}\right)(\sqrt2(1+i))\frac{2i}{\sqrt2}} = \frac{1}{2\sqrt2\, i(1+i)} = \frac{1}{2\sqrt2(-1+i)} = \frac{1+i}{-4\sqrt2}$

$R\left(f, \frac{1-i}{\sqrt2}\right) = \frac{1-i}{-4\sqrt2} \Rightarrow \Sigma R_i = -\frac{2}{4\sqrt2} = -\frac{1}{2\sqrt2}$

$\int_c f(z)dz = 2\pi i\left(\frac{-1}{2\sqrt2}\right) = \frac{-\pi i}{\sqrt2}$ Ans

Q(11) Easy

Q(5) $\int_c \frac{\sin z\, dz}{(1+z^2)^2}$ $\quad c$ is $|z-i| = 2$

$x^2 + (y-1)^2 = 2^2$ $\quad c = (0,1)$ radius $= 2$

$R(f, i) = \frac{d}{dz}\left[\frac{\cancel{(z-i)^2} \sin z}{\cancel{(z-i)^2}(z+i)^2}\right]_{z=i}$

$= \frac{(z+i)^2 \cos z - \sin z (2)(z+i)}{(z+i)^4} = \left|\frac{(z+i)\cos z - 2\sin z}{(z+i)^3}\right|_{z=i}$

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$\int_c \frac{\sin z}{(z^2+1)^2} dz = \frac{\pi i}{2}\left[\sinh(1) - \cosh(1)\right]$

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$z = 0$, is a pole of ord 2 $\quad \sin z = 0 \quad z = \sin^{-1}(0) = 0,$

consider $f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^2\left[z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} - \cdots\right]}$

$= \frac{1}{z^3\left[1 - \frac{z^2}{6} + \frac{z^4}{\underline{|5}} + \cdots\right]}$

$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{6} - \frac{z^4}{\underline{|5}} + \cdots\right)^{-1}\right]$

$= \frac{1}{z^3}\left[1 + \frac{z^2}{6} + \frac{z^4}{120} - \cdots\right]$

$R(f, 0) = \frac{1}{6}$ $\quad$ Hence $\int_c f(z) dz = 2\pi i\left(\frac{1}{6}\right)$

$= \frac{\pi i}{3}$

Q(7) Correct Q. is $\int_c \frac{e^{iz}}{z^4 + 2z^2 + 1} dz$ $\quad |z| = 2$

$f(z) = \frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z+i)^2(z-i)^2}$

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$= e^{i i}\frac{\left[(2i)i - 2\right]}{-8i} = \frac{1}{2ie}$

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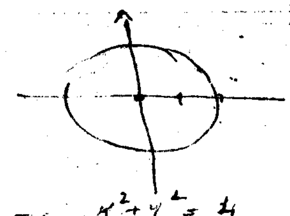
~~R(f,i) = 1/2ei~~



$x^2+y^2=4$

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$$R(f,-1-i) = \frac{-6i-2}{20} = -\frac{1}{10} - \frac{3i}{10}$$

$$\Sigma R_i = \frac{1}{10}(1-2i) + \frac{1}{10}(1+2i) + \left(\frac{-1}{10} + \frac{3i}{10}\right) + \left(\frac{-1}{10} - \frac{3i}{10}\right) = 0$$

(ii) $|z-i| = 3/2$ $x^2+(y-1)^2 = 9/4$

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Q(9) $\displaystyle\int_c \frac{e^z\,dz}{(z-1)^n}$ $|z| = 2$, $z=1$ is pole of order n

$$\lim_{z\to 1} \frac{1}{\lfloor n-1}\frac{d^{n-1}}{dz^{n-1}} \frac{\cancel{(z-1)^n}\,e^z}{\cancel{(z-1)^n}} = \frac{1}{\lfloor n-1}(e^1) \quad \therefore \int_c f(z)dz = \frac{2\pi i\,e}{\lfloor n-1}$$

Q 10) $\displaystyle\int_c \frac{dz}{z^4+1}$ $|z-1| = 1 \Rightarrow (x-1)^2+y^2 = 1$

Centre $(1,0)$ rad $= 1$

$$z_k = \text{cis}\left(\frac{\pi+2\pi k}{n}\right),\ k = 0,1,2,3$$

$$z_0 = \frac{1}{\sqrt2} + i\frac{1}{\sqrt2},\ z_1 = -1/\sqrt2 + i/\sqrt2,\ z_2 = \frac{-1}{\sqrt2} - \frac{i}{\sqrt2}\ \ z_3 = \frac{1}{\sqrt2} - \frac{i}{\sqrt2}$$

z_0 & z_3 lies inside

$$R\left(f, \frac{1+i}{\sqrt2}\right) = \lim_{z\to}\left[\frac{\left(z-\left(\frac{1+i}{\sqrt2}\right)\right)}{\left(z-\frac{1+i}{\sqrt2}\right)\left(z-\left(\frac{i-1}{\sqrt2}\right)\right)\left(z+\frac{1+i}{\sqrt2}\right)\left(z+\left(\frac{i-1}{\sqrt2}\right)\right)}\right]$$

$$= \frac{1}{\left(\frac{2}{\sqrt2}\right)(\sqrt2(1+i))\frac{2i}{\sqrt2}} = \frac{1}{2\sqrt2\,i(1+i)} = \frac{1}{2\sqrt2(-1+i)} = \frac{1+i}{-4\sqrt2}$$

$$R\left(f, \frac{1-i}{\sqrt2}\right) = \frac{1-i}{-4\sqrt2} \Rightarrow \Sigma R_i = -\frac{2}{4\sqrt2} = \frac{-1}{2\sqrt2}$$

$$\int_c f(z)dz = 2\pi i\left(\frac{-1}{2\sqrt2}\right) = \frac{-\pi i}{\sqrt2}\ \text{Ans}$$

Q(11) Easy