Example (7) For the function
$$f(z) = \frac{2z^3+1}{z^2+z}$$
Find (i) a Taylor Series valid in the nbhd. of the point $z=i$ (ii) a Laurent series valid within the annulus of which centre is the origin.

Sol

$$f(z) = \frac{2z^3+1}{z^2+z} = 2(z-1) + \frac{1}{z} + \frac{1}{z+1}, \longrightarrow (1)$$

We split $f(z)$ into 3 parts

$$f_1(z) = 2(z-1), \quad f_2(z) = \frac{1}{z} \quad f_3(z) = \frac{1}{z+1}$$

so that $f(z) = f_1(z) + f_2(z) + f_3(z)$. $\longrightarrow$ (2)

For $f_1(z) = 2(z-1)$

$f_1(i) = 2(i-1)$

$f_1'(z) = 2 \Rightarrow f_1'(i) = 2 \quad f_1''(z) = 0 = f'''(z) = f_1^n(z)$

Taylors Expansion for $f_1(z)$ about $z=i$ is given by $f_1(z) = 2(z-1) = \sum_{n=0}^{\infty} \frac{f_1^n(i)(z-i)^n}{\lfloor n}$

Taylor Series $f_1(z) = 2(z-1) \equiv 2(i-1) + 2(z-i)$ for $n = 0, 1$ — (3)

$f_2(z) = \frac{1}{z} \qquad f_2(i) = 1/i = -i$

$f_2'(z) = -1/z^2, \quad f_2'(i) = -1/i^2$

$f_2^n(z) = \frac{(-1)^n \lfloor n}{z^{n+1}}, \quad f_2^n(i) = \frac{(-1)^n \lfloor n}{(i)^{n+1}}$

Taylors Series is $f_2(z) = 1/i + \sum_{n=1}^{\infty} \frac{(-1)^n}{(i)^{n+1}} (z-i)^n$

or $f_2(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(i)^{n+1}}$ $\longrightarrow$ (4)

for $f_3(z) = \frac{1}{z+1} \qquad f_3^n(z) = \frac{(-1)^n \lfloor n}{(z+1)^{n+1}}$

$f_3^n(i) = \frac{(-1)^n \lfloor n}{(i+1)^{n+1}}$

Taylors Series for $f_3(z)$ is $= \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(i+1)^{n+1}}$ $\longrightarrow$ (5)

From (3) (4) + 5,
We write (2) as $f(z) = f_1(z) + f_2(z) + f_3(z)$

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$$f(z) = 2(z-1) + 2(z+i) + \sum_{n=0}^{\infty}\left\{\left[\frac{(-1)^n}{(i)^{n+1}} + \frac{(-1)^n}{(i+1)^{n+1}}\right](z-i)^n\right\}$$

Step II Laurent Series for $f(z)$ is

$$f(z) = 2(z-1) + \frac{1}{z} + \frac{1}{z+1}$$
$$= 2(z-1) + \frac{1}{z} + (z+1)^{-1}$$
$$= 2(z-1) + \frac{1}{z} + (1+z)^{-1}$$
$$= 2z - 2 + \frac{1}{z} + [1 - z + z^2 - z^3 + z^4 - \cdots]$$
$$= \frac{1}{z} - 1 + z + z^2 - z^3 + z^4 - + \cdots$$

Example (8) Laurent Series for

$$f(z) = \frac{1}{(z+2)(1+z^2)} \quad \text{for (i) } |z| < 1$$
(ii) $1 < |z| < 2$
(iii) $|z| > 2$

Step I for $|z| < 1$

$$f(z) = \frac{1}{(z+2)(1+z^2)} \quad \text{for P.F} = \frac{A}{z+2} + \frac{Bz+C}{1+z^2}$$

$$1 \equiv A(1+z^2) + (Bz+C)(z+2)$$

Put $z=-2$ $\quad 1 = A(1+4)$ $\quad \boxed{A = 1/5}$

Comp z^2, $0 = A + B$ $\quad B = -1/5$

Comparing constant $\quad 1 = A + 2C$, $\quad 1 = 1/5 + 2C$ $\quad 2C = 4/5$, $\quad C = 2/5$

$$f(z) = 1/5 \left[\frac{1}{z+2} - \frac{z-2}{1+z^2}\right] \quad \text{——— (A)}$$

for $|z| < 1$, $\quad f(z) = -1/5 \dfrac{1}{2(1+z/2)} - 1/5\,(z-2)(1+z^2)^{-1}$

$$= \frac{1}{10}(1+z/2)^{-1} - 1/5\,(z-2)(1+z^2)^{-1}$$

$$= \frac{1}{10}\left[1 - z/2 + (z/2)^2 - z^3/8 + \cdots\right] - 1/5(z-2)\left[1 - z^2 + z^4 - \cdots\right]$$

$$= \frac{1}{5}\left[\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{2^{n+1}} - (z-2)\sum_{n=0}^{\infty}(-1)^n z^{2n}\right]$$

Step II for $1<|z|<2$

In (A) f(z) can be written as

$$f(z) = 1/5 \cdot \frac{1}{2(1+z/2)} - \frac{z-2}{5} \cdot \frac{1}{z^2(1+1/z^2)}$$

$$= \frac{1}{10}\left(1+z/2\right)^{-1} - \frac{z-2}{5z^2}\left(1+1/z^2\right)^{-1}$$

$$= \frac{1}{10}\left[1 - z/2 + (z/2)^2 + (z/2)^3 \cdots\right] - \frac{z-2}{5z^2}\left(1+1/z^2\right)^{-1}$$

$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n (z/2)^n - \frac{z-2}{5z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}}$$

For $|z|>2$, A can be written as

$$f(z) = \frac{1}{5z(1+2/z)} - \frac{1}{5}(z-2)\frac{-1}{z^2(1+1/z^2)}$$

$$= \frac{1}{5z}\left[1 - 2/z + (2/z)^2 - + \cdots\right] - 1/5\left[\frac{1}{z} - \frac{2}{z^2}\right]\left[1+1/z^2\right]^{-1}$$

$$= \frac{1}{5z}\left[\sum_{n=0}^{\infty}\left(\frac{2^n}{z^n}\right)\right] - 1/5\left[\frac{1}{z} - 2/z^2\right]\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}}$$

Example(9) Given the Series expansion

$$\sin z = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!}, \quad |z|<\infty$$

Find first 3 non-zero terms of the Laurent Series expansion of cosec z about the pt $z=0$

Solution

Step I, $\sin z = 0$, $z = n\pi$ ..... $n = 0, \pm1, \pm2 \cdots$

Hence $\operatorname{cosec} z = \frac{1}{\sin z}$ is analytic in the region $0<|z|<\pi$

Therefore, we can write

$$\operatorname{cosec} z = \frac{1}{\sin z} = \frac{1}{z - z^3/\underline{3} + z^5/\underline{5} - + \cdots}$$

$$= \frac{1}{z\left[1 - z^2/6 + z^4/120 + \cdots\right]}$$

$$= \frac{1}{z}\left[1 - z^2/6 + z^4/120 - \cdots\right]^{-1}$$

$$= \frac{1}{z}\left[1 - \left(z^2/6 - \frac{z^4}{120} + \cdots\right)\right]^{-1}$$

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Let $y = \frac{z^2}{6} - \frac{z^4}{120} + \text{——}$

$\operatorname{Cosec} z = \frac{1}{z}[1-y]^{-1}$

$= \frac{1}{z}[1+y+y^2+y^3+ \cdots]$

$= \frac{1}{z}\left(1+\left(\frac{z^2}{6}-\frac{z^4}{120}+\cdots\right)+\left(\frac{z^2}{6}-\frac{z^4}{120}+\cdots\right)^2+\cdots\right)$

$= \frac{1}{z}\left[1+\frac{z^2}{6}-\frac{z^4}{120}+\frac{z^4}{36}+\text{——}\right]$

$= \frac{1}{z}+\frac{z}{6}+z^3\left(\frac{1}{36}-\frac{1}{120}\right)+\text{——}$

$= \frac{1}{z}+\frac{1}{6}z+z^3\left(\frac{10-3}{360}\right)$

$= \frac{1}{z}+\frac{1}{6}z+\frac{7z^3}{360}+\text{——}$ for $0<|z|<\pi$

Exercise on Page 176

① $f(z) = \frac{1}{(z^2+1)(z^2+2)}$ in power of z $\quad y = z^2$

For P.F $\quad \frac{1}{(y+1)(y+2)} = \frac{A}{y+1}+\frac{B}{y+2}$

$1 \equiv A(y+2)+B(y+1)$

Put $y=-1$, $\quad 1 = A \qquad A = 1$

$y=-2 \qquad 1 = -B \qquad B = -1$

$f(z) = \frac{1}{z^2+1} - \frac{1}{z^2+2}$

for $|z|<1$ $\quad = (1+z^2)^{-1} - \frac{1}{2}\left(1+\frac{z^2}{2}\right)^{-1}$

$= 1-z^2+z^4-z^6+\cdots - \frac{1}{2}\left\{1-\frac{z^2}{2}+\left(\frac{z^2}{2}\right)^2+\cdots\right\}$

$= \left(1-\frac{1}{2}\right)-z^2+\frac{1}{4}z^2+z^4-\frac{z^4}{8}+\text{——}$

$= \frac{1}{2}+z^2\left(-\frac{3}{4}\right)+\frac{7}{8}z^4+\text{——}$

$= \sum_{n=0}^{\infty}(-1)^n\left(1-\frac{1}{2^{n+1}}\right)z^{2n} = \sum_{n=0}^{\infty}(-1)^n\left(1-\frac{1}{2^{n+1}}\right)z^{2n}$

And $1<|z|<\sqrt{2}$

⑤ ch # 5

$$= (1-1/2) - (1-1/4)z^2 + (1-1/8)z^4 - \cdots$$

(ii) $1 < |z| < \sqrt{2}$

$$= \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^{2n}$$

$$f(z) = \frac{1}{1+z^2} - \frac{1}{z^2+2}$$

$$= \frac{1}{z^2(1+1/z^2)} - \frac{1}{2(1+z^2/2)} = \frac{1}{z^2}(1+1/z^2)^{-1} - \frac{1}{2}(1+z^2/2)^{-1}$$

$$= \frac{1}{z^2}\left[1 - 1/z^2 + 1/z^4 - 1/z^6 + \cdots\right] - 1/2\left[1 - z^2/2 + (z^2/2)^2 - (z^2/2)^3 - \cdots\right]$$

$$= \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \cdots - 1/2 + \frac{1}{2^2}(z^2) - 1/2\, z^4/4 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n+2}} + \sum_{n=0} (-1)^{n+1} \frac{z^{2n}}{2^{n+1}}$$

(iii) $f(z) = \dfrac{1}{1+z^2} - \dfrac{1}{z^2+2}$

$$= \frac{1}{z^2[1+1/z^2]} - \frac{1}{z^2[1+2/z^2]}$$

$$= \frac{1}{z^2}\left[(1+1/z^2)^{-1} - (1+2/z^2)^{-1}\right]$$

$$= \frac{1}{z^2}\left[(1 - 1/z^2 + 1/z^4 - \cdots) - (1 - 2/z^2 + (2/z^2)^2 - + \cdots)\right]$$

$$= \frac{1}{z^2}\left[(1-1) + 1/z^2 + (1-4)\frac{1}{z^4} + \cdots\right]$$

$$= 0(1/z^2) + 1/z^4 + (1-2^2)\frac{1}{z^6} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(1-2^n)}{z^{2n+2}}$$

---

② $f(z) = \dfrac{1}{z^2 - z + 2} = \dfrac{1}{(z+1)(z-2)} = \dfrac{A}{z+1} + \dfrac{B}{z-2}$

$$1 \equiv A(z-2) + B(z+1)$$

Put $z = 2$ $\quad 1 = 3B \quad B = 1/3$

$z = -1$ $\quad 1 = -3A \quad A = -1/3$

$$f(z) = \frac{1}{3}\left[\frac{1}{z-2} - \frac{1}{z+1}\right]$$

$1 < |z| < 2$

$$= -1/3\left[\frac{1}{z(1+1/z)}\right] - 1/3\,(1/2)\left[\frac{1}{(1-z/2)}\right]$$

$$= -1/3\left(\frac{1}{z}\right)\left[(1+1/z)^{-1}\right] - \frac{1}{6}\left((1-z/2)^{-1}\right)$$

$$= -\frac{1}{3z}\left[1 - 1/z + 1/z^2 + \cdots\right] - 1/6\left[1 + z/2 + z^2/4 + \cdots\right]$$

(6) Ch#5

$= -\frac{1}{3}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} \mp \cdots\right) - \frac{1}{6}\left(1 + \frac{z}{2} + \frac{z^2}{4} \cdots\right)$

$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{z^{n+1}} - \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n$

Q3) (i) $f(z) = \sinh z$ about $z = \pi i = a$

$f'(z) = \cosh z$ $\quad f(a) = 0$
$f''(z) = \sinh z$ $\quad f'(a) = -1$
$f'''(z) = \cosh z$ $\quad f''(a) =$

For alternate term $(-1)^{2n-1} = -1$

Taylors Series $f(z) = \sum_{n=0}^{\infty}\frac{(z-a)^n}{\lfloor n}f^n(a)$

$= -\sum_{n=1}^{\infty}\frac{(z-\pi i)^{2n-1}}{\lfloor 2n-1}$

(ii) $f(z) = \frac{1}{z-4}$ about $z = 3$

$f'(z) = (-1)(z-4)^{-2}$
$f''(z) = (-1)(-2)(z-4)^{-3}$
$f^n(z) = (-1)^n\lfloor n/(z-4)^{n+1} = \frac{(-1)^n\lfloor n}{(z-4)^{n+1}}$

Taylors Theorem at $z = 3$

$f(z) = \sum\frac{(z-3)^n}{\lfloor n}\frac{(-1)^n\lfloor n}{(3-4)^{n+1}} = -\sum_{0}^{\infty}(z-3)^n$

---

(4) Mac Th

$\frac{e^0 + e^{-0}}{2}$

(i) $f(z) = \cosh z$ $\quad f(0) = \cosh 0 = 1$
$f'(z) = \sinh z$ $\quad f'(0) = 0$
$f''(z) = \cosh z$ $\quad f''(0) = 1$
$f'''(z) = \sinh z$ $\quad f'''(0) = 0$

$f(z) = \sum_{n=0}^{\infty}\frac{z^{2n}}{\lfloor 2n}(1)$ $\quad$ When $f^{2n}(0) = 1$

(7) Ch # 5

Q4 (ii) $f(z) = \frac{z}{1-z} = -\left(\frac{z-1+1}{z-1}\right) = -\left[1+\frac{1}{z-1}\right]$

$f(z) = -1 + \frac{1}{1-z}$

$f'(z) = (-1)(1-z)^{-2}(-1)$

$f''(z) = (-2)(1-z)^{-3}(-1) = 2(1-z)^{-3}$

$f'''(z) = -6(1-z)^{-4}(-1) = 6(1-z)^{-4}$

$= \frac{\lfloor 3}{(1-z)^{3+1}}$

$f^{n}(z) = \frac{\lfloor n}{(1-z)^{n+1}}$ $\qquad f^{n}(0) = \lfloor n$

Mac. formula

$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\lfloor n} f^{n}(0) = \sum_{0}^{\infty} \frac{z^n}{\cancel{\lfloor n}} \cancel{\lfloor n} = \sum_{1}^{\infty} z^n$

$\because f(0) = 0$

Q (5) $\cosh\{c(z+1/z)\} = \sum_{0}^{\infty} a_n z^n$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2c\cos\theta)\cos n\theta \, d\theta$

Sol Since for $z = 1/z$ f^n is unchanged $\therefore a_n = b_n$

$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^{n+1}}$ $\qquad a = 0$

$f(z) = \cosh(cz + c/z)$ $\qquad$ Put $z = e^{i\theta}$
$dz = ie^{i\theta}$

$\therefore a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cosh(ce^{i\theta} + c/e^{i\theta})\, ie^{i\theta}\, d\theta}{(e^{i\theta})^{n+1}}$ $\quad 0 \le \theta \le 2\pi$

$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(c\,2\cos\theta)\cdot[\cos n\theta - i\sin n\theta]\, d\theta$

$\frac{1}{2\pi}\int_0^{2\pi} \cosh(2c\cos\theta)\cos n\theta \, d\theta - \frac{i}{2\pi}\int_0^{2\pi} \cosh(2c\cos\theta)\sin n\theta \, d\theta$

$\Downarrow I$ zero as in example

Put $\theta = 2\pi - \varphi$
& proceed in example

$I = -I$
$2I = 0$ $\longrightarrow$ $I = 0$ & so on

(8) Ch#5

Q(6) Prove that $\exp\left\{\frac{k}{2}\left(z-\frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} a_n z^n$

When $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - k\sin\theta)\, d\theta$

Sol:

The fn $f(z)$ is not analytic at $z=0$, we can expand it for $|z|>0$, $|z|\neq 0$

$$\operatorname{Exp}\left\{\tfrac{1}{2}k\left(z-\tfrac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} a_n z^n$$

when $a_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z^{n+1}}$ $\because a=0$

$$= \frac{1}{2\pi i}\int_c \frac{e^{\frac{k}{2}\left(z-\frac{1}{z}\right)}\,dz}{z^{n+1}}$$

Put $z = e^{i\theta}$

$dz = i e^{i\theta}$

$0 \le \theta \le 2\pi$ $a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\frac{k}{2}\left(e^{i\theta}-e^{-i\theta}\right)}}{e^{i(n+1)\theta}}\, i e^{i\theta} d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ik\sin\theta}\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(k\sin\theta - n\theta)\, d\theta + \frac{i}{2\pi}\int_0^{2\pi} \sin(k\sin\theta - n\theta)\, d\theta$$

Taking $\theta = 2\pi - \varphi$ $d\theta = -d\varphi$

$$\int_0^{2\pi} \sin(k\sin\theta - n\theta)\, d\theta = -\int_{2\pi}^{0} \sin(-k\sin\varphi - 2n\pi + n\varphi)\, d\varphi$$

$$= -\int_0^{2\pi} \sin(k\sin\theta - n\theta)\, d\theta$$

So that $\int_0^{2\pi} \sin[k(\sin\theta) - n\theta]\, d\theta = 0$

Hence $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(k\sin\theta - n\theta)\, d\theta$

Or $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - k\sin\theta)\, d\theta$

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