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EXERCISES ON PAGE 100

Q(1) Evaluate $\int_c \frac{z+2}{z}\,dz$ where C is the Semi circle $z = 2e^{it}$, $0 \le t \le \pi$

Consider $\int_0^\pi \left(1+\frac{2}{z}\right) dz$ $\qquad dz = 2ie^{it}\,dt$

$= \int_0^\pi dz + 2\int_0^\pi \frac{1}{z}\,dz$

$= \int_0^\pi 2ie^{it}\,dt + 2\int_0^\pi \frac{2ie^{it}}{2e^{it}}\,dt$

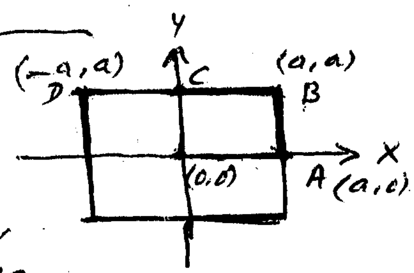
$= 2i\left|\frac{e^{it}}{i}\right|_0^\pi + 2i\int_0^\pi dt$

$= 2\left(e^{i\pi} - e^{0}\right) + 2i\,|t|_0^\pi$

$= -4 + 2i\pi$

Q(2) $\int_c \frac{dz}{z}$.

C is square described in +ve Sense with sides $\parallel$ to axes length $2a$



$\cos\theta = x$
$\sin\theta = y$
$-a$, to a

$z = e^{i\theta}$ $\quad dz = ie^{i\theta}$

$\int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = i(2\pi)$ $\quad$ Ahead

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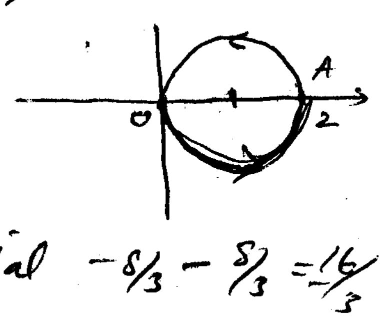
Q(3) $\int_C |z|^2\,dz$ $\qquad$ C is circle $|z-1| = 1$ +ve Sense.

$\int_2^0 (x^2+y^2)(dx+idy)$ $\qquad z-1 = \pm 1$
$\qquad z = 2,\ z = 0$

$y = 0,\ x = t$

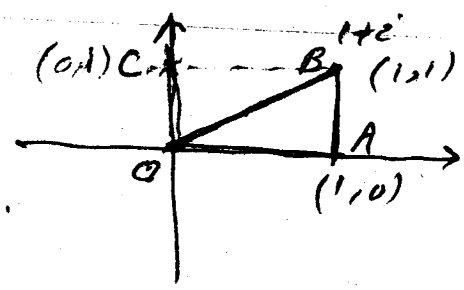
$= \int_2^0 t^2\,dt = \left|t^3/3\right|_2^0$



above x axis $= -8/3$

below x axis $\int_0^2 t^2\,dt = 8/3$ $\qquad$ Total $-8/3 - 8/3 = -16/3$

Q (4) $\int_0^{1+i} (x - y + i x^2)\, dz$



(i) $z = 0 \quad z = 1+i$

Eq OB $\qquad y = x$

$dz = d(x+iy)$
$= dx + i\,dy$
$= (1+i)\,dx$

$$\int_0^1 (0 + i x^2)(1+i)\,dx$$

$$= i(1+i) \int_0^1 x^2\,dx$$

$$= (i-1) \left| \frac{x^3}{3} \right|_0^1 = \frac{1}{3}(i-1)$$

(ii) Along the Imaginary axis OC from $z = 0$, $z = i$

Put $x = 0 \quad y = t$

$dz = dx + i\,dy$
$= 0 + i\,dt$

$$\int_0^1 (-t)\, i\,dt$$

$$= -i \left| \frac{t^2}{2} \right|_0^1 = -\frac{i}{2} \quad \text{along OC}$$

iii) Integral from $z = i$ to $z = 1+i$

$y = 1, \quad x = t \qquad 0 \le t \le 1$

$z = x + i$
$dz = dx + 0$
$= dt$

$$\int_0^1 \left[(t-1) + i t^2 \right] dt$$

$$\left| \frac{t^2}{2} - t + i\frac{t^3}{3} \right|_0^1$$

$$= \frac{1}{2} - 1 + i\frac{1}{3}$$

$$= -\frac{1}{2} + \frac{i}{3} = \frac{-3 + 2i}{6}$$

Total (ii) & (iii) $\quad \frac{-3 + 2i}{6} - \frac{i}{2}$

$$= \frac{-3 - i}{6}$$

Ans

Q(6) $\int_0^{1+i} (z-1)\,dz$ on the curve $y = x^2$

Let $z = x+iy$, $dz = dx+i\,dy$ | $z=0$, $z=1+i$ along $y=x^2$

let $x=t$, $y=t^2$; $dx=dt$, $dy=2t\,dt$; $0\le t\le 0$

$$\int_0^1 [x+iy-1]\,(dx+i\,dy)$$

$$\Rightarrow \int_0^1 (t+it^2-1)\,(dt+i\,2t\,dt)$$

$$\Rightarrow \int_0^1 (t-1+it^2)\,(1+2it)\,dt$$

$$\Rightarrow \int_0^1 (t-2t^3-1+3it^2-2it)\,dt$$

$$\Rightarrow \left[\, t^2/2 - t^4/2 - t + it^3 - it^2 \right|_0^1$$

$$= \cancel{1/2} - \cancel{1/2} - 1 + \cancel{i(1)^3} - \cancel{i(1)^2} - 0$$

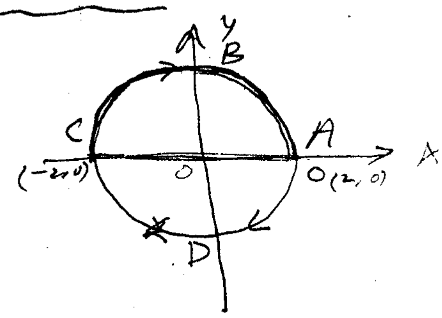
$$= -1 \quad (\text{Answer})$$

Q(7) $\int_c \frac{1}{z^2-1}\,dz$

where c is $|z|=2$

$\sqrt{x^2+y^2}=2$

$x^2+y^2=4$



$$\int_{-2}^{2} \frac{1}{z^2-1}\,dz$$

$y=0$, $-2\le x\le 2$

$y=0$, $dx=dt \leftarrow x=t$

$$\int_{-2}^{2} \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|_{-2}^{2} = \tfrac12\left(\ln \tfrac13 - \ln(+3)\right)$$

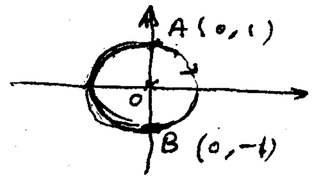
$$\& \int_{2}^{-2} \frac{dx}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|_{2}^{-2} = \tfrac12\left(\ln 3 - \ln \tfrac13\right)$$

Sum is zero.

(8) Prove That $\left| \int (x^2 + iy^2)\, dz \right| \le \pi$ where C is a semi circle $z = \pm i$

$f(z) = x^2 + iy^2$

$|f(z)| = \sqrt{x^4 + y^4}$



For semi circle $x = \cos\theta$, $y = \sin\theta$

$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$L = \int_{-\pi/2}^{\pi/2} 1 \cdot d\theta = \left| \theta \right|_{-\pi/2}^{\pi/2} = \pi$

$M = |f(z)| = \sqrt{x^4 + y^4}$ $\quad x = 0,\ y = 1$

$= 1$

Since $|f(z)\, dz| \le ML$

$\le 1 \cdot \pi$

Hence $\left| \int (x^2 + iy^2)\, dz \right| \le \pi$

(9) $\int_C |z-1|\, dz$ C is unit circle ie $|z| = 1$

$z = \cos t + i \sin t = e^{it}$

$z - 1 = \cos t - 1 + i \sin t$

$|z-1| = \sqrt{(\cos t - 1)^2 + \sin^2 t}$

$= \sqrt{1 + \cos^2 t + \sin^2 t - 2\cos t}$

$= \sqrt{2 - 2\cos t} = \sqrt{2}\sqrt{2\sin^2 t/2}$

$= 2 \sin t/2$

$z = e^{it}$ $\quad dz = \cancel{i e^{it}}$ $|dz| = dt$

$0 \le t \le 2\pi$ $\quad \cancel{-\pi \le t \le \pi}$

$\cancel{i \int 2 \sin t/2\, e^{it}\, dt}$

$= \cancel{2i \left[\frac{e^{it}}{i} \sin t/2 - \int \cos t/2 \cdot \tfrac{1}{2} \frac{e^{it}}{i}\, dt \right.}$

$= \cancel{2 \left[e^{it} \sin t/2 - 1/2 \right.}$

$\int |z-1|\, |dz| = 2\int_0^{2\pi} \sin t/2\, dt = -4 \cos t/2 \Big|_0^{2\pi} = -4[-1-1]$

$= 8$

Q(5) Find $\int_C R(z)\,dz$ where C denotes the unit circle

$|z| = 1$

$z = \pm 1$

$z = +1$ & $z = -1$

$0 \le \theta \le 2\pi$

let $z = e^{i\theta}$

$R(z) = \cos\theta$

$dz = i e^{i\theta} d\theta$

$$\int_{-1}^{1} \cos\theta \; i e^{i\theta} d\theta$$

$$i \int_0^{2\pi} \cos\theta \; e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} \cos\theta (\cos\theta + i\sin\theta)\, d\theta$$

$$= i \int_0^{2\pi} (\cos^2\theta + i \sin\theta\cos\theta)\, d\theta$$

$$i \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta + \frac{i}{2}\int_0^{2\pi} i\, 2\sin\theta \cos\theta\, d\theta$$

$$= \cancel{i \left[\theta/2\right]_0^{2\pi} + \left|\frac{\sin 2\theta}{2}\right|_0^{2\pi}}$$

$$= \cancel{\pi i \;\; 0}$$

$$= i \left| \theta/2 + \frac{\sin 2\theta}{4} \right|_0^{2\pi} + \frac{i^2}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= i\,(2\pi/2) + \frac{1}{4}\left[\cos 2\pi - \cos 0\right]$$

$$= i\pi + 0$$

$i/2\,(-\pi - \pi)$

$= -i\pi$

Q(6)

$$\int_0^{1+i} (z-1)\, dz$$

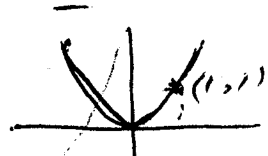
$y = x^2$

$dy = 2x\,dx$

$dz = dx + i\,dy$

$= dx + i\,2x\,dx$

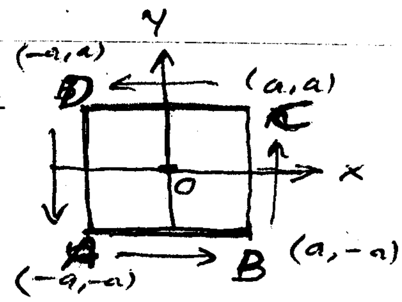
$= (1 + 2ix)\,dx$



$$= \int_0^1 (x - 1 + ix^2)(1 + 2ix)\,dx$$

$$\int ((x-1) + ix^2)(1 + 2ix)\,dx$$

Q (2) $\int_c \frac{1}{z}\,dz$ C is Square | Centre at O, each length 2a.



Step I Integ. along AB $x=t$, $dx=dt$, $dy=0$ | $y=-a$, $-a\le t\le a$

$z = x+iy$
$= t+ia$

$$\int_{AB}\frac{1}{z}\,dz = \int_{-a}^{+a}\frac{1}{t-ai}\,dt \quad \text{——— (I)}$$

Step II Integ along ~~AD~~ BC $x=a$, $y=t$, $dx=0$, $dy=dt$, $-a\le t\le a$

$$\int_{BC}\frac{1}{z}\,dz = \int_{-a}^{a}\frac{i\,dt}{a+it} \quad \text{——— (II)}$$

Step III

$$\int_{CD}\frac{1}{z}\,dz = -\int_{-a}^{a}\frac{1}{t+ai}\,dt \quad \text{——— (III)}$$

$y=+a$, $x=t$ $\quad dz = dx+i\,dy = dt$

Step IV

$$\int_{DA}\frac{dz}{z} = \int_{-a}^{a}\frac{i\,dt}{-a+it} \quad \text{(IV)}$$

$x=-a$, $y=t$, $dx=0$, $dy=dt$, $-a+it$

$$\int_{ABCDA}\frac{1}{z}\,dz = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$= \int_{-a}^{a}\frac{1}{t-ai}\,dt + i\int_{-a}^{a}\frac{dt}{a+it} - \int_{-a}^{a}\frac{dt}{t+ai} - \int_{-a}^{a}\frac{i\,dt}{-a+it}$$

$$= \int_{-a}^{a}\left(\frac{1}{t-ai} - \frac{1}{t+ai}\right)dt + i\int_{-a}^{a}\left(\frac{1}{a+it} - \frac{1}{it-a}\right)dt$$

$$= \int_{-a}^{a}\frac{2ai}{t^2+a^2}\,dt + (-2ai)\int_{-a}^{a}\frac{dt}{-(t^2+a^2)}$$

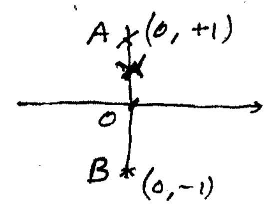
$$= 2ai\int_{-a}^{a}\frac{2}{a^2+t^2}\,dt = \frac{4ai}{a}\,\text{Tan}^{-1}(t/a)$$

$$= 4i\left(\pi/4 - (-\pi/4)\right)$$

$$= 4i\,\pi/2 = 2i\pi$$

(0) $Z = x + iy$

Find the value of $\int_C |z|\, dz$ C is St. line from $z = -i$ to $z = i$



$|z| = \sqrt{x^2+y^2}$

Integral along OA $x = 0,\ y = t$
$0 < t < 1$

$dz = x + iy$

$x = 0 \quad y = t \qquad dz = i\,dt$

$|z| = y = t$

$$\int_0^1 y\ i\,dy = i\left|\frac{y^2}{2}\right|_0^1 = \frac{i}{2}$$

Integral along BO $\displaystyle\int_{-1}^{0} iy\,dy = i\left|\frac{y^2}{2}\right|_{-1}^{0} = \frac{i}{-2}$

Integral along $AB = AO + OB$

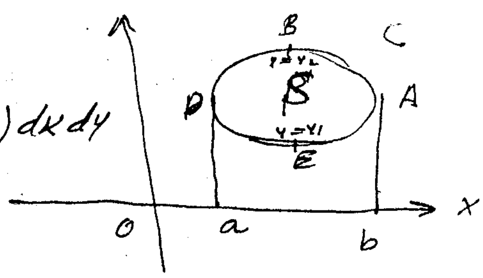
$= i/2 - (-i/2) = i$

GREEN'S Theorem. (In the plane)

Statement: If C is a simple closed curve enclosing the area S, then

$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Where P & Q are fns of x and y



Proof Consider $\displaystyle\iint_S \frac{\partial P}{\partial y}\, dx\,dy$ Insert the limits.

So that area S is covered by the Integral

Thus $$\iint_S \frac{\partial P}{\partial y}\, dx\,dy = \int_a^b dx \int_{y_1}^{y_2} \frac{\partial P}{\partial y}\, dy$$

$$= \int_a^b [P]_{y_1}^{y_2}\, dx \longrightarrow (1)$$

as P is a fn of x & y both

$\therefore$

$$[P]_{y_1}^{y_2} = P(x, y_2) - P(x, y_1) \longrightarrow (2)$$

Now $$\int_a^b P(x, y_2)\,dx = \int_{DBA} P(x, y)\,dx$$ as $y = y_2$ along the path DBA