Ex. on Page (63)

Q(1) Prove that the fn. $f(z) = e^{-z^{-4}}$ $z \neq 0$
$= 0$ $z = 0$
is not analytic at $z=0$ although C-R Eqs are satisfied.

Solution: In Polar form.

$$f(z) = e^{-1/z^4} = e^{-\left(\frac{1}{r^4}e^{-4i\theta}\right)} = e^{-\frac{1}{r^4}e^{-4i\theta}}$$

$$= e^{-1/r^4}(\cos 4\theta + i\sin 4\theta)$$

$$= e^{-\frac{1}{r^4}\cos 4\theta}\left[\cos\left(\frac{\sin 4\theta}{r^4}\right) + i\sin\left(\frac{\sin 4\theta}{r^4}\right)\right]$$

$$U(x,y) + iV(x,y) = e^{-\frac{\cos 4\theta}{r^4}}\left(\cos\left(\frac{\sin 4\theta}{r^4}\right) + i\sin\left(\frac{\sin 4\theta}{r^4}\right)\right)$$

Where

$$U = e^{-\frac{\cos 4\theta}{r^4}}\left(\cos\left(\frac{\sin 4\theta}{r^4}\right)\right), \quad V = e^{-\frac{\cos 4\theta}{r^4}} \cdot \sin\left(\frac{\sin 4\theta}{r^4}\right).$$

$$\frac{\partial u}{\partial r} = e^{-\frac{\cos 4\theta}{r^4}}\left[-\sin\left(\frac{\sin 4\theta}{r^4}\right)(-4r^{-5}\sin 4\theta)\right] + \left[e^{-\frac{\cos 4\theta}{r^4}} \cdot \left(\frac{4\cos 4\theta}{r^5}\right) \cdot \cos\left(\frac{\sin 4\theta}{r^4}\right)\right]$$

$$= e^{-\frac{\cos 4\theta}{r^4}}\left(\frac{4\cos 4\theta}{r^5}\cos\left(\frac{\sin 4\theta}{r^4}\right) + \frac{4}{r^5}\sin 4\theta \cdot \sin\left(\frac{\sin 4\theta}{r^4}\right)\right) \quad \text{--- (i)}$$

$$\frac{\partial v}{\partial \theta} = e^{-\frac{\cos 4\theta}{r^4}}\left(\frac{4\sin 4\theta}{r^4}\right) \cdot \sin\left(\frac{\sin 4\theta}{r^4}\right) + e^{-\frac{\cos 4\theta}{r^4}}\left[\cos\left(\frac{\sin 4\theta}{r^4}\right) \cdot \frac{4\cos 4\theta}{r^4}\right]$$

$$= e^{-\frac{\cos 4\theta}{r^4}}\left[4\frac{\sin 4\theta}{r^4}\sin\left(\frac{\sin 4\theta}{r^4}\right) + 4\frac{\cos 4\theta}{r^4} \cdot \cos\left(\frac{\sin 4\theta}{r^4}\right)\right] \quad \text{--- (ii)}$$

For C-R Eqs. Clearly $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ From (i) & (ii)

& $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$$\text{Now } \frac{\partial u}{\partial \theta} = e^{-\frac{\cos 4\theta}{r^4}}\left(\frac{4\sin 4\theta}{r^4} \cdot \cos\left(\frac{\sin 4\theta}{r^4}\right)\right) + e^{-\frac{\cos 4\theta}{r^4}}\left(-\sin\left(\frac{\sin 4\theta}{r^4}\right) \cdot \frac{4\cos 4\theta}{r^4}\right)$$

$$= e^{-\frac{\cos 4\theta}{r^4}} \cdot \frac{4}{r^4}\left(\sin 4\theta \cdot \cos\left(\frac{\sin 4\theta}{r^4}\right) - \cos 4\theta\,\sin\left(\frac{\sin 4\theta}{r^4}\right)\right)$$

$$\frac{\partial v}{\partial r} = e^{-\frac{\cos 4\theta}{r^4}}(+4r^{-5}\cos 4\theta)\sin\left(\frac{\sin 4\theta}{r^4}\right) + e^{-\frac{\cos 4\theta}{r^4}}\cos\left(\frac{\sin 4\theta}{r^4}\right) \cdot \sin 4\theta\left(\frac{-4}{r^5}\right)$$

$$= e^{-\frac{\cos 4\theta}{r^4}}\frac{4}{r^5}\left[+\cos 4\theta \cdot \sin\left(\frac{\sin 4\theta}{r^4}\right) - \sin 4\theta\,\cos\left(\frac{\sin 4\theta}{r^4}\right)\right]$$

Therefore, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Step II $f'(z)$ does not exist at $z=0$

$f(z) = e^{-1/z^4}$ $\quad f'(z) = \underbrace{e^{-1/z^4} \cdot -\left(-4/z^5\right)}$ does not exist at $z=0$

Q(2) The function $w = u + iv$ and $z = e^{-v}(-\cos u + i \sin u)$ Calculate for what value of $z$ the function ceases to be analytic function.

Sol If $w = u + iv$ & $z = e^{-v}(-\cos u + i \sin u)$, then To show $\frac{dw}{dz}$ is undefined for some $z$.

For $z = e^{-v}(-\cos u + i \sin u)$

$$\frac{\partial z}{\partial u} = e^{-v}[\sin u + i \cos u]$$

Now
$$z^2 = e^{-2v}[\cos^2 u - \sin^2 u - 2i \cos u \sin u]$$
$$= -e^{-2v}[\sin^2 u - \cos^2 u + i2 \sin u \cos u]$$
$$= -e^{-2v}[(\sin u)^2 + (i\cos u)^2 + 2i \cos u \sin u]$$
$$= -[e^{-v}(\sin u + i \cos u)]^2$$
$$z^2 = -\left(\frac{\partial z}{du}\right)^2$$

$$\left(\frac{\partial z}{\partial u}\right)^2 = -z^2 = (iz)^2$$
$$\frac{\partial z}{\partial u} = \pm iz \quad \text{—— (ii)}$$

Since $w = u + iv \longrightarrow \frac{\partial w}{\partial u} = 1$

$$\frac{dz}{dw} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial w}$$
$$\frac{\partial z}{\partial u} = \frac{dz}{\partial w}$$

$$\frac{dw}{dz} = \frac{\partial u}{\partial z} = \frac{1}{\pm iz} = \infty \quad \text{at } z = 0$$

$\frac{dw}{dz}$ is undefined at $z = 0$

Q(3) Find analytic function $w = u + iv$ where $u(x,y) = \frac{\sin 2x}{\cosh 2y + \cos 2x}$

(i)
$$\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y + \cos 2x) - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$
$$= \frac{2\cos 2x \cosh 2y + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x \cosh 2y + 2}{(\cosh 2y + \cos 2x)^2}$$

(ii) $$\frac{\partial u}{\partial y} = \sin 2x \left[ -1 (\cosh 2y + \cos 2x)^{-2} \right] 2\sinh 2y$$

$$= \frac{-2\sinh 2y \sin 2x}{(\cosh 2y + \cos 2x)^2}$$

Put $x = z$, $y = 0$

$$U_x(z,0) = \frac{2\cos 2z + 2}{(1+\cos 2z)^2} = \frac{2}{1+\cos 2z}$$

$U_y(z,0) = 0$ Now $f'(z) = U_x(z,0) - iU_y(z,0)$

$$f'(z) = \frac{2}{1+\cos 2z} - i(0) = \frac{2}{1+\cos 2z}$$

$$= \frac{2}{2\cos^2 z} = \sec^2 z$$

On Integ.

$$f(z) = \tan z + c$$

$$= \tan(x+iy) + c$$

Q(4) $$f(z) = \frac{xy^2(x+iy)}{x^2+y^4} \quad \text{when } z \neq 0$$

$$= 0 \quad \text{when } z = 0$$

Sol

$$f'(z) = \lim_{z\to 0} \left( \frac{f(z) - f(0)}{z - 0} \right)$$

$$= \lim_{z\to 0} \left( \frac{\frac{xy^2(x+iy)}{x^2+y^4} - 0}{x+iy} \right) = \lim_{z\to 0} \left( \frac{xy^2}{x^2+y^4} \right) \quad (A)$$

i) Suppose $z \to 0$ along $y = x$.

$$f'(0) = \lim_{x\to 0} \left( \frac{x^3}{x^2+x^4} \right) = \lim_{x\to 0} \left( \frac{x}{1+x^2} \right) = 0$$

ii) Suppose $z \to 0$ along $y^2 = x$

$$= \lim_{x\to 0} \left( \frac{x^2}{x^2+x^2} \right) = 1/2$$

Since both the limits are different

$f'(0)$ does not exist

For C-R Eqs.

$$f(z) = u + iv = \left( \frac{x^2y^2 + ixy^3}{x^2+y^4} \right)$$

Where $U = \frac{x^2y^2}{x^2+y^4}$ & $V = \frac{xy^3}{x^2+y^4}$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\left(\frac{U(x,0)-U(0,0)}{x-0}\right) = \lim_{x\to 0}\left(\frac{0-0}{x}\right) = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\left(\frac{U(0,y)-U(0,0)}{y-0}\right) = \lim_{y\to 0}\frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\left(\frac{V(x,0)-V(0,0)}{x-0}\right) = \lim_{x\to 0}\left(\frac{0-0}{x}\right) = 0$$

$$\frac{\partial V}{\partial y} = \lim_{y\to 0}\left(\frac{V(0,y)-V(0,0)}{y-0}\right) = \lim_{y\to 0}\left(\frac{0-0}{y}\right) = 0$$

$\Rightarrow$ C-R eqs are satisfied at $(0,0)$.

Q(5) If $n$ is real, prove that the function $f(z) = r^n(\cos n\theta + i\sin n\theta)$ is an analytic fn except when $r=0$, also calculate $f'(z)$.

Sol Let

$$f(z) = r^n(\cos n\theta + i\sin n\theta)$$

$U(r,\theta) = r^n\cos n\theta$

$V(r,\theta) = r^n\sin n\theta$

$\frac{\partial u}{\partial r} = n r^{n-1}\cos n\theta$ & $\frac{\partial v}{\partial r} = n r^{n-1}\sin n\theta$

$\frac{\partial u}{\partial \theta} = -r^n n\sin n\theta$ $\qquad \frac{\partial v}{\partial \theta} = r^n n\cos n\theta$

For C-R Eq. (i) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$n r^{n-1}\cos n\theta = \frac{1}{r}(n r^n\cos n\theta) = n r^{n-1}\cos n\theta$

satisfied

also $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$n r^{n-1}\sin n\theta = -\frac{1}{r}\left(-r^n n\sin n\theta\right)$

$= n r^{n-1}\sin n\theta$ LHS = RHS

C-R eqs are satisfied

Now $f(z) = r^n(\cos n\theta + i\sin n\theta) = z^n$

$f'(z) = n z^{n-1}$

$= n r^{n-1}\left(\cos(n-1)\theta + i\sin(n-1)\theta\right)$

Q(6) For what value of $z$ does the function $f(z) = u + iv = w$, where $z = \sin u\cosh v + i\cos u\sinh v$ ceases to be analytic.

Sol Let $f(z) = u + iv = w$

Given $z = \sin u\cosh v + i\cos u\sinh v$

To prove $\frac{dw}{dz}$ does not defined at some $z$.

Since $w = u + iv$ $\quad \frac{\partial w}{\partial u} = 1 \Rightarrow dw = du$

$\Rightarrow \frac{dz}{dw} = \frac{\partial z}{\partial u}$

Now $\frac{\partial z}{\partial u} = \cos u\cosh v - i\sin u\sinh v$

$z^2 = (\sin u\cosh v + i\cos u\sinh v)^2$

$= \sin^2 u\cosh^2 v - \cos^2 u\sinh^2 v + 2i\sin u\cos u\cosh v\sinh v$

$= (1-\cos^2 u)\cosh^2 v - (1-\sin^2 u)\sinh^2 v + 2i\sin u\cos u\cosh v\sinh v$

$= (\cosh^2 v - \sinh^2 v) - \cos^2 u\cosh^2 v + \sin^2 u\sinh^2 v$
$\quad + 2i\sin u\cos u\cosh v\sinh v$

$= \cancel{(\cosh^2 v - \sinh^2 v) - \cos^2 u\cosh^2 v + \sin^2 u}$

$= -\left[\cosh^2 v\cos^2 u + i^2\sin^2 u\sinh^2 v - 2i\sin u\cos u\sinh v\cosh v\right] + 1$

$= -\left(\cosh v\cos u - i\sin u\sinh v\right)^2 + 1$

$z^2 = -\left(\frac{dz}{du}\right)^2 + 1$

$\left(\frac{dz}{du}\right)^2 = 1 - z^2 \qquad \frac{dz}{du} = \pm\sqrt{1-z^2}$

$\frac{du}{dz} = \frac{dw}{dz} = \frac{\pm 1}{\sqrt{1-z^2}}$

It is not defined when $1 - z^2 = 0$, $z = \pm 1$

$\therefore f(z)$ ceased to be analytic when $z = \pm 1$

Q(7) If $f(z)$ is an analytic fn. of $z$, Prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|R\, f(z)|^2 = 2|f'(z)|^2$$

Sol

Let $S = |R\, f(z)|^2 = U^2$ & $U$ is fn of $x$ & $y$

$$\frac{\partial S}{\partial x} = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 S}{\partial x^2} = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2}$$

& $$\frac{\partial S}{\partial y} = 2u\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 S}{\partial y^2} = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$$

Now $$\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2} = 2u\cancel{\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)} + 2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right) \quad \text{①}$$

$\because$ $U_{xx}+U_{yy}=0$, $\nabla^2 U = 0$ Laplace Eq

& $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \quad \text{(ii)}$$

from ① & (ii)

$$\left(\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}\right) = 2|f'(z)|^2$$

i.e. $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|R\, f(z)|^2 = 2|f'(z)|^2$$

Q(8): Construct the analytic fn $f(z)$ whose real part is

$$U(x,y) = y^3 - 3x^2y$$

$$\frac{\partial u}{\partial x} = -6xy = \frac{\partial v}{\partial y} \quad (\text{by CR Eq})$$

Hence $$\frac{\partial v}{\partial y} = -6xy$$

Integ wrt $y$

$$V = -3y^2x + g(x)$$

$$\therefore \frac{\partial v}{\partial x} = -3y^2 + g'(x) = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = -(-3y^2 - 3x^2) = -3y^2 + g'(x)$$

$$g'(x) = 3x^2$$

Integ w.r.t $x$ $\quad g(x) = x^3 + c$

$$\therefore V = -3xy^2 + x^3 + c$$

$$\begin{aligned} f(z) &= U + iV \\ &= y^3 - 3x^2y + i(-3xy^2 + x^3 + c) \\ &= i[x^3 - 3xy^2 + 3ix^2y - iy^3] + ic \\ &= i[(x+iy)^3] + ic \\ &= i(z^3 + c) \end{aligned}$$

Q(9) Without verifying C-R Eqs, prove that $f(z) = \cos(7x + 5iy)$ is not analytic.

Sol Let $\left.\begin{aligned} z &= x+iy \\ \bar{z} &= x-iy \end{aligned}\right] \Rightarrow x = \frac{z+\bar{z}}{2}$

$$y = \frac{z-\bar{z}}{2i}$$

$$\therefore f(z) = \cos\left[7\left(\frac{z+\bar{z}}{2}\right) + 5i\left(\frac{z-\bar{z}}{2i}\right)\right]$$

$$f(z) = \cos(6z + \bar{z})$$

Since $f(z)$ involves $\bar{z}$, Therefore, It is not analytic.

Q(10) If $f(z)$ is analytic & $f'(z) = 0$, then prove that $f(z)$ is constant.

Sol Given $f(z)$ is analytic, then C-R Eqs are satisfied

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Let $f(z) = u + iv$ where $u$ & $v$ are fn of $x$ & $y$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 0 \quad \& \quad \frac{\partial v}{\partial x} = 0$$

On solving $u = g(x)$ & $v = h(x)$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow u = c_1$$

$$\frac{\partial v}{\partial y} = 0 \Rightarrow v = c_2$$

Hence $f(z) = c_1 + ic_2 = c'$

$\Rightarrow f(z)$ is constant

Q (11) Prove that $|e^z - 1| \leq |z| e^{|z|}$

Since $e^z = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots$

$$\therefore e^z - 1 = \left[\cancel{1} + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots - \cancel{1}\right]$$

$$= z\left[1 + \frac{z}{\lfloor 2} + \frac{z^2}{\lfloor 3} + \cdots\right]$$

$$\therefore |e^z - 1| \leq |z| + \left|\frac{z^2}{2}\right| + \left|\frac{z^2}{\lfloor 3}\right| + \cdots \quad (i)$$

Also $|z| e^{|z|} = |z| \left\{1 + |z| + \frac{|z|^2}{\lfloor 2} + \frac{|z|^3}{\lfloor 3} + \cdots\right\}$

$$= |z| + \frac{|z|^2}{\ } + \frac{|z|^3}{\lfloor 2} \cdots \quad (ii)$$

from (i) & (ii)

$$|z| e^{|z|} \geq |e^z - 1|$$

or $|e^z - 1| \leq |z| e^{|z|}$

Q (1) Show that the fn

$$f(z) = \begin{cases} \dfrac{(\bar z)^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is not diff at z=0 However C.R eqs are satisfied at z=0

Sol

$$f(z) = \frac{(\bar z)^2}{z} = u + iv$$

$$u + iv = \frac{x^2 - y^2 - 2ixy}{x+iy} \cdot \frac{x-iy}{x-iy}$$

$$= \frac{x^3 - xy^2 - 2ix^2y - ix^2y + iy^3 - 2xy^2}{x^2+y^2}$$

$$= \frac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}, \quad v(x,y) = +\frac{(y^3 - 3x^2y)}{x^2+y^2}$$

$y=0$, $x \to 0$

$$\frac{\partial u}{\partial x} \text{ at } x \to 0 = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x \to 0} \frac{x^3/x^2}{x} = 1$$

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 1$$

C R Satisfied

&

$$f'(z) = \lim_{z \to 0} \left( \frac{f(z) - f(0)}{z - 0} \right)$$

$$= \lim_{z \to 0} \frac{\dfrac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2+y^2} - 0}{x+iy}$$

(i) Put $y = x$

$$= \lim_{x \to 0} \left( \quad \right) = \frac{i-1}{i+1}$$

(ii) along x axis $y = 0$ & $x \to 0$

$$f'(0) = \lim_{x \to 0} \left( \frac{\dfrac{x^3}{x^2} - 0}{x - 0} \right) = \lim_{x \to 0} \frac{x^3}{x^3} = 1$$

$\therefore$ $f'(0)$ does not exist.