Chapter 06: Function of Several Variables

Handwritten Notes of REAL ANALYSIS

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CHP # 6

Function Of Several Variables

Definition :-

If we consider a set of n - independent variables say $x, y, z, \ldots, t$ and one dependent variable u, Then the equation

$$u = f(x, y, z, \ldots, t) \longrightarrow (1)$$

denotes the functional relation b/w the dependent variable and independent variables $x, y, z, \ldots, t$.

In this case if $x, y, z, \ldots, t$ are the n - arbitrary assigned values of the independent variables, the corresponding values of the dependent variable u are determined by the functional relation (1).

The function represented by the equation is an EXPLICIT FUNCTION, but where the several variables are concerened it is nearly possible to obtain an equation expressing one of the variables explicity in terms of the others. Thus most of the functions of more than two variables are IMPLICIT FUNCTIO[N] i.e, to say, we are given a functional relation and $(x, y, z, \ldots, t) = 0$

consisting of n-variables say $x, y, z, \ldots, t$ and is not in genral possible to solve this equation to find an explicit function which expresses one of these n-variables in terms of other,

for examples,

(i) $xy + yx^2 + 3xy = 4$

(ii) $x^2y^2 + xy + 3xy^3 = 8$

(iii) $8xyz + 6x^2y^2z + 8zx^2\log xy + 6xy^2z = 291$

Here (i) and (iii) are explicit function because one variable can express in terms of the other i.e

(i) $\Rightarrow y(x + x^2 + 3x) = 4 \Rightarrow y = 4/x^2+4x$

(iii) $\Rightarrow z(8xy + 6x^2y^2 + 8x^2\log xy + 6xy^2) = 291$

$\Rightarrow z = 291/8xy + 6x^2y^2 + 8x^2\log xy + 6xy^2$,

while (ii) is implicit function becuse one variable cannot be expressed in terms of the other.

Jacobian:

Let $u_1, u_2, \ldots, u_n$ be the differentiable functions of n-variable say $x_1, x_2, \ldots x_n$ i.e

$u_1 = u_1(x_1, x_2, \ldots, x_n)$, $u_2 = u_2(x_1, x_2, \ldots, x_n)$, $u_3 = u_3(x_1, x_2, \ldots, x_n), \ldots, u_n = u_n(x_1, x_2, \ldots u_n)$

Then the determinent,

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots\cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots\cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called Jacobian and is represented by J

$$J = \frac{\partial(u_1, u_2, u_3, \ldots u_n)}{\partial(x_1, x_2, x_3, \ldots, x_n)} \text{ or } J\left(\frac{u_1, u_2, u_3, \ldots u_n}{x_1, x_2, x_3, \ldots x_n}\right)$$

Note:- A functional relation b/w $u_1, u_2, u_3, \ldots u_n$ will exist, if The Jacobian (J) is equal to zero

Question No (1)

Prove that three functions u, v, w are connected by an identical functional relation if,

(i) $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$

Solution:- we know that a functional relation b/w u, v, w exist iff

$$J\left(\frac{u, v, w}{x, y, z}\right) = 0 \text{ or } \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

or

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0 \longrightarrow (1)$$

Now

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = -1$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1, \quad \frac{\partial v}{\partial z} = 1, \text{ etc}$$

putting in (1)

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix} \text{ by } R_1+R_2$$

$$J = 2(-2z+2y-2y+2z) = 0$$

So, the functional relation b/w u, v, w exists, Now we find the functional relation

$U = x+y-z \rightarrow (1)$ $V = x-y+z \rightarrow (2)$

$w = x^2+y^2+z^2 -2yz \Rightarrow w = x^2+(y-z)^2 \rightarrow (3)$

$(1)+(2) \Rightarrow 2x = U+V \Rightarrow x = \frac{U+V}{2}$

$(1)-(2) \Rightarrow U-V = 2y-2z$

$\Rightarrow y-z = \frac{U-V}{2}$

putting these values in (3)

$w = \left(\frac{U+V}{2}\right)^2 + \left(\frac{U-V}{2}\right)^2$

$w = \frac{U^2+V^2+\cancel{2UV}+U^2+V^2-\cancel{2UV}}{4} = \frac{2U^2+2V^2}{4}$

$w = \frac{U^2+V^2}{2} \Rightarrow 2w = U^2+V^2$

is requierd relation

(Question No (2))

Prove that there exist a functional relation b/w U, V and w where

$U = x/y-z$, $V = y/z-x$, $w = z/x-y$

Solution:- we know that a functional relation b/w U, V, w exist iff

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$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

or

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

Now,

$$u = x/y-z$$

$$\frac{\partial u}{\partial x} = 1/y-z, \quad \frac{\partial u}{\partial y} = \frac{-x}{(y-z)^2}, \quad \frac{\partial u}{\partial z} = \frac{x}{(y-z)^2}$$

$$v = y/z-x$$

$$\frac{\partial v}{\partial x} = \frac{y}{(z-x)^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{z-x}, \quad \frac{\partial v}{\partial z} = -\frac{y}{(z-x)^2}$$

$$w = z/x-y$$

$$\frac{\partial w}{\partial x} = \frac{-z}{(x-y)^2}, \quad \frac{\partial w}{\partial y} = \frac{z}{(x-y)^2}, \quad \frac{\partial w}{\partial z} = \frac{1}{x-y}$$

Therefore

$$\begin{vmatrix} 1/y-z & -x/(y-z)^2 & x/(y-z)^2 \\ y/(z-x)^2 & 1/z-x & -y/(z-x)^2 \\ -z/(x-y)^2 & z/(x-y)^2 & 1/x-y \end{vmatrix}$$

$$= \begin{vmatrix} 1/y-z & -x/(y-z)^2 & 0 \\ y/(z-x)^2 & 1/z-x & z-x-y/(z-x)^2 \\ -z/(x-y)^2 & z/(x-y)^2 & z+x-y/(x-y)^2 \end{vmatrix} \quad \text{by } C_3+C_2$$

$$= \frac{z+x-y}{(y-z)(z-x)(x-y)^2} - \frac{z(z-x-y)}{(y-z)(x-y)^2(z-x)^2}$$

$$+ \frac{x}{(y-z)^2}\left[\frac{y(z+x-y)}{(z-x)^2(y-x)^2} + \frac{z(z-x-y)}{(x-y)^2(z-x)^2}\right]$$

$$= \frac{(z-x)(z+x-y)-z(z-x-y)}{(y-z)(z-x)^2(x-y)^2} + \frac{x}{(y-z)^2}\left[\frac{yz+yx-y^2-z^2-zx-yz}{(x-y)^2(z-x)^2}\right]$$

$$= \frac{z^2+zx-zy-xz-x^2+xy-z^2+xy+yz}{(y-z)(z-x)^2(x-y)^2} + \frac{x(y-z)(x-y-z)}{(y-z)^2(z-x)^2(x-y)}$$

$$= \frac{-x^2+xy+xz+x^2-xy-xz}{(y-z)(z-x)^2(x-y)^2} = 0$$

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ There fore a functional relation b/w u, v, w exist,

Now we find the functional relation

$u = \frac{x}{y-z} \Rightarrow \frac{u+1}{u-1} = \frac{x+y-z}{x-y+z} \rightarrow (1)$ by componendo dividendo.

$$v = \frac{y}{z-x} \Rightarrow \frac{v+1}{v-1} = \frac{y+z-x}{y-z+x} \rightarrow (2)$$ by dividendo and compon

$$w = \frac{z}{x-y} \Rightarrow \frac{w+1}{w-1} = \frac{z+x-y}{z-x+y} \rightarrow (3)$$ by compon-- dividendo.

$$\frac{(u+1)(v+1)(w+1)}{(u-1)(v-1)(w-1)} = \frac{(x+y-z)}{(x-y+z)} \cdot \frac{(y+z-x)}{(x+y-z)} \cdot \frac{(x-y+z)}{(y+z-x)}$$

$$\Rightarrow (u+1)(v+1)(w+1) = (u-1)(v-1)(w-1)$$

$$\Rightarrow (uv+u+v+1)(w+1) = (uv-u-v+1)(w-1)$$

$$\Rightarrow \cancel{uvw}+uw+vw+\cancel{u}+uv+\cancel{v}+\cancel{w}+1 = \cancel{uvw}-uv-uw+\cancel{u} -vw+\cancel{v}+\cancel{w}-1$$

$$\Rightarrow 2uw+2vw+2uv = -2$$

$$\Rightarrow uw+vw+uv = -1$$

is required solution.

Question No (3)

If $x_1 = \cos u_1$, $x_2 = \cos u_2 \sin u_1$

$x_3 = \cos u_3 \sin u_2 \sin u_1$

Show that

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = -\sin^3 u_1 \sin^2 u_2 \sin u_3$$

Solution:

$$\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)} = \begin{vmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 & \partial x_1/\partial u_3 \\ \partial x_2/\partial u_1 & \partial x_2/\partial u_2 & \partial x_2/\partial u_3 \\ \partial x_3/\partial u_1 & \partial x_3/\partial u_2 & \partial x_3/\partial u_3 \end{vmatrix}$$

Now,

$$\frac{\partial x_1}{\partial u_1} = -\sin u_1 , \quad \frac{\partial x_1}{\partial u_2} = 0 , \quad \frac{\partial x_1}{\partial u_3} = 0$$

$$\frac{\partial x_2}{\partial u_1} = \cos u_2 \cos u_1 , \quad \frac{\partial x_2}{\partial u_2} = -\sin u_2 \sin u_1 , \quad \frac{\partial x_2}{\partial u_3} = 0$$

$$\frac{\partial x_3}{\partial u_1} = \cos u_3 \sin u_2 \cos u_1 , \quad \frac{\partial x_3}{\partial u_2} = \cos u_3 \cos u_2 \sin u_1 ,$$

$$\frac{\partial x_3}{\partial u_3} = -\sin u_3 \sin u_2 \sin u_1$$

Therefore-

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} -\sin u_1 & 0 & 0 \\ \cos u_2 \cos u_1 & -\sin u_2 \sin u_1 & 0 \\ \cos u_3 \sin u_2 \cos u_1 & \cos u_3 \cos u_2 \sin u_1 & -\sin u_3 \sin u_2 \sin u_1 \end{vmatrix}$$

$$= -\sin u_1 (\sin u_3 \sin^2 u_2 \sin^2 u_1)$$

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = -\sin^3 u_1 \sin^2 u_2 \sin u_3$$

is Solution.

Question No (4)

Prove That there exist a functional relation b/w u, v, w where

$u = x^3 + x^2 y + x^2 z - z^2 (x+y+z)$,

$v = x + z$, $w = x^2 - z^2 + xy - zy$

Solution:- we know that a functional relation b/w u, v, w exist if and only if

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Now,

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy + 2xz - z^2, \quad \frac{\partial u}{\partial y} = x^2 - z^2$$

$$\frac{\partial u}{\partial z} = x^2 - 2z(x+y+z) - z^2$$

$$= x^2 - 2zx - 2zy - 2z^2 - z^2$$

$$= x^2 - 2zx - 2zy - 3z^2$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial z} = 1,$$

$$\frac{\partial w}{\partial x} = 2x+y, \quad \frac{\partial w}{\partial y} = x - z, \quad \frac{\partial w}{\partial z} = -2z - y.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3x^2+2xy+2xz-z^2 & x^2-z^2 & x^2-2zx-2zy-3z^2 \\ 1 & 0 & 1 \\ 2x+y & x-z & -2z-y \end{vmatrix}$$

$$= (3x^2+2xy+2xz-z^2)(z-x) - (x^2-z^2)(-2z-y-2x-y)$$
$$+ (x^2-2xz-2yz-3z^2)(x-z)$$

$$= 3x^2z + 2xyz + 2xz^2 - z^3 - 3x^3 - 2x^2y - 2x^2z + xz^2$$
$$+ 2x^2z + x^2y + 2x^3 + x^2y - 2z^3 - 4z^2 - 2xz^2 - 4z^2$$
$$+ x^3 - 2x^2z - 2xyz - 3xz^2 - x^2z + 2xz^2 + 2yz^2 + 3z^3$$

$$= 0$$

Thus the functional relation b/w u, v, w exist.

Now we try to find the functional relation.

Now

$$u = x^3 + x^2y + x^2z - xz^2 - yz^2 - z^3 \longrightarrow (1)$$

$$v = x + z \longrightarrow (2)$$

$$w = x^2 - z^2 + xy - 3y \longrightarrow (3)$$

Multiplying (2) and (3)

$$vw = x^3 - xz^2 + x^2y - xyz + x^2z - z^3 + xyz - z^2y$$

$$= x^3 + x^2y + x^2z - xz^2 - yz^2 - z^3 \longrightarrow (4)$$

comparing (1) and (4) we get,

$$vw = u$$

which is required function relation

Question No (5)

Find the condition that the expression $px + qy + rz + p'x + q'y + r'z$ are connected with the expression by a functional relation.

Solution:- Let $u = px + qy + rz$

i.e, $u = u(x, y, z)$

$v = p'x + q'y + r'z$ i.e, $v = v(x, y, z)$

and $w = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

i.e $w = w(x, y, z)$

A functional relation b/w u, v, w exist if

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

or

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0 \longrightarrow (1)$$

Now $u_x = p$, $u_y = q$, $u_z = r$
$v_x = p'$, $v_y = q'$, $v_z = r'$

$w_x = 2ax + 2gz + 2hy$, $w_y = 2by + 2fz + 2hx$
$w_z = 2cz + 2fy + 2gx$

putting in (1)

$$\begin{vmatrix} p & q & r \\ p' & q' & r' \\ 2ax+2gz+2hy & 2by+2fz+2hx & 2cz+2fy+2gx \end{vmatrix} = 0$$

$$\Rightarrow - \begin{vmatrix} p & q & r \\ 2ax+2gz+2hy & 2by+2fz+2hx & 2cz+2fy+2gx \\ p' & q' & r' \end{vmatrix} = 0$$

$$\Rightarrow 2 \begin{vmatrix} ax+gz+hy & by+fz+hx & cz+fy+gx \\ p & q & r \\ p' & q' & r' \end{vmatrix} = 0$$

$$\Rightarrow (ax+gz+hy)(qr'-rq') - (by+fz+hx)(pr'-rp') + (cz+fy+gx)(pq'-qp') = 0$$

$$\Rightarrow a(qr'-rq')x + h(qr'-rq')y + g(qr'-q'r)z$$
$$+ h(p'r-pr')x + b(p'r-pr')y + f(p'r-pr')z$$
$$+ g(pq'-qp')x + f(pq'-qp')y + c(pq'-qp')z = 0$$

$$\Rightarrow [a(qr'-rq') + h(p'r-pr') + g(pq'-qp')]x$$
$$+ [b(p'r-pr') + f(pq'-p'q) + h(qr'-rq')]y$$
$$+ [g(qr'-q'r) + f(p'r-pr') + c(pq'-qp')]z = 0$$

Since the result is true for all x, y, z. So we can write from comparing the co-efficients of both sides.

$$a(qr'-rq') + h(p'r-pr') + g(pq'-qp') = 0$$
$$h(qr'-rq') + b(p'r-pr') + f(pq'-qp') = 0$$
$$g(qr'-rq') + f(p'r-pr') + c(pq'-qp') = 0$$

Let $qr'-q'r = A$, $p'r-pr' = B$, $pq'-p'q = D$

Then we have

$$aA + hB + gD = 0$$
$$hA + bB + fD = 0$$
$$gA + fB + cD = 0$$

$$\Rightarrow \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} A \\ B \\ D \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = 0$$

which is required conditions.

{ Question No (6) }

Prove that if $f(0) = 0$ and $f'(x) = 1/{1+x^2}$, then $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

Solution :- Let $u = f(x) + f(y)$ and $v = (x+y)/(1-xy)$
we have to show that there exist a functional relation b/w u and v and that this functional relation is f.
Now a functional relation b/w u and v exist if

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \quad \text{or} \quad \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

Since $u = f(x) + f(y)$

$$\Rightarrow \frac{\partial u}{\partial x} = f'(x) = \frac{1}{1+x^2} \quad \text{(given)}$$

also $\frac{\partial u}{\partial y} = f'(y) = \frac{1}{1+y^2}$ (Just by replace x and y)

$v = \frac{x+y}{1-xy} \Rightarrow \frac{\partial v}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2}$

$\Rightarrow \frac{\partial v}{\partial x} = \frac{1+y^2}{(1-xy)^2}$ and $\frac{\partial v}{\partial y} = \frac{1+x^2}{(1-xy)^2}$

Therefore

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/(1+x^2) & 1/(1+y^2) \\ (1+y^2)/(1-xy)^2 & (1+x^2)/(1-xy)^2 \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

thus a functional relation b/w u and v exist.

Let $u = G(v)$

But $u = f(x) + f(y)$ and $v = (x+y)/(1-xy)$

So

$$f(x)+f(y) = G\left((x+y)/(1-xy)\right) \longrightarrow (1)$$

Put $y=0$ then

$$f(x) + f(0) = G(x)$$

$\Rightarrow f(x) = G(x)$ $\quad \because f(0)=0$

$\Rightarrow f = G$

So (1) becomes $f(x)+f(y) = f\left[\frac{x+y}{1-xy}\right]$ is solution

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Question No (7)

If

$$U_m = \frac{x_m}{(1 - x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2)^{1/2}}$$

Then prove that

$$\frac{\partial(U_1, U_2, U_3, \ldots, U_n)}{\partial(x_1, x_2, x_3, \ldots, x_n)} = (1 - x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2)^{-\frac{n}{2} - 1}$$

Solution:- To prove the above result let us suppose that

$$1 - x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2 = Q$$

then

$$U_m = \frac{x_m}{Q^{1/2}}$$

Now let us consider $\frac{\partial U_m}{\partial x_p}$ and discuss two cases

(i) $m = p$ (ii) $m \neq p$

case (I):- when $m = p$ Then

$$\frac{\partial U_m}{\partial x_p} = \frac{\partial}{\partial x_p}\left(\frac{x_m}{Q^{1/2}}\right) = \frac{\partial}{\partial x_p}\left(x_m Q^{-1/2}\right) = \frac{\partial}{\partial x_p}\left(x_p Q^{-1/2}\right)$$

$$= Q^{-1/2} \cdot \frac{\partial x_p}{\partial x_p} + x_p \frac{\partial}{\partial x_p}\left(Q^{-1/2}\right)$$

$$= Q^{-1/2} \cdot 1 + x_p(-1/2)\, Q^{-1/2 - 1} \frac{\partial}{\partial x_p}(Q)$$

$$= Q^{-1/2} - \frac{1}{2} x_p\, Q^{-3/2} \frac{\partial}{x_p}(1 - x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2)$$

$$= Q^{-1/2} - \frac{1}{2} x_p\, Q^{-3/2}(-2x_p)$$

$$= Q^{-1/2} + x^2 p\, Q^{-3/2}$$

$$= Q^{-1/2} + x^2 P / Q^{3/2}$$

$$= \frac{Q + x^2 \rho}{Q^{3/2}} \quad \text{as } \rho = m$$

So

$$\frac{\partial u_m}{\partial x_\rho} = \frac{Q + x_m^2}{Q^{3/2}}$$

Case (II):- when $m \neq \rho$

Then $\frac{\partial u_m}{\partial x_\rho} = \frac{\partial}{\partial x_\rho}\left(\frac{x_m}{Q^{1/2}}\right)$

$$= \frac{\partial}{\partial x_\rho}\left(x_m Q^{-1/2}\right) = x_m \frac{\partial}{\partial x_\rho}\left(Q^{-1/2}\right)$$

$$= x_m\left(-\tfrac{1}{2}\right) Q^{-3/2} \frac{\partial}{\partial x_\rho}\left(Q\right)$$

i.e $\frac{\partial u_m}{\partial x_\rho} = -\frac{1}{2} x_m Q^{-3/2} \frac{\partial}{\partial x_\rho}\left(1 - x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2\right)$

$$= -\tfrac{1}{2} x_m Q^{-3/2}\left(-2x_\rho\right) = x_m x_\rho Q^{-3/2}$$

or $\frac{\partial u_m}{\partial x_\rho} = \frac{x_m x_\rho}{Q^{3/2}}$ when $m \neq \rho$

Now

$$J = \frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{Q+x_1^2}{Q^{3/2}} & \frac{x_1x_2}{Q^{3/2}} & \frac{x_1x_3}{Q^{3/2}} & \cdots & \frac{x_1x_n}{Q^{3/2}} \\ \frac{x_2x_1}{Q^{3/2}} & \frac{Q+x_2^2}{Q^{3/2}} & \frac{x_2x_3}{Q^{3/2}} & \cdots & \frac{x_2x_n}{Q^{3/2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{x_nx_1}{Q^{3/2}} & \frac{x_nx_2}{Q^{3/2}} & \frac{x_nx_3}{Q^{3/2}} & \cdots & \frac{Q+x_n^2}{Q^{3/2}} \end{vmatrix}$$

$$J = (Q)^{-3/2} \begin{vmatrix} Q+x_1^2 & x_1x_2 & x_1x_3 & \cdots & x_1x_n \\ x_2x_1 & Q+x_2^2 & x_2x_3 & \cdots & x_2x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_nx_1 & x_nx_2 & x_nx_3 & \cdots & Q+x_n^2 \end{vmatrix}$$

let $\Delta_n = \begin{vmatrix} Q+x_1^2 & x_1x_2 & x_1x_3 & \cdots & x_1x_n \\ x_2x_1 & Q+x_2^2 & x_2x_3 & \cdots & x_2x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_nx_1 & x_nx_2 & x_nx_3 & \cdots & Q+x_n^2 \end{vmatrix}$

$$J = Q^{-3/2} \Delta n \text{ ——— } (1)$$

Now when $n=1$ The $\Delta n = \Delta_1 = Q + x_1^2$ — (2)

when $n=2$

$$\Delta n = \Delta_2 = \begin{vmatrix} Q+x_1^2 & x_1x_2 \\ x_2x_1 & Q+x_2^2 \end{vmatrix}$$

$\Rightarrow \Delta_2 = (Q+x_1^2)(Q+x_2^2) - x_1^2x_2^2$

$= Q^2 + Q(x_1^2 + x_2^2) + \cancel{x_1^2x_2^2} - \cancel{x_1^2x_2^2}$

$$= Q^2 + Q(x_1^2 + x_2^2) \longrightarrow (3)$$

For $n=3$

$$\Delta_3 = \begin{vmatrix} Q+x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & Q+x_2^2 & x_2x_3 \\ x_3x_1 & x_2x_3 & Q+x_3^2 \end{vmatrix}$$

$$\Rightarrow \Delta_3 = (Q+x_1^2)(Q^2+Qx_3^2+Qx_2^2+\cancel{x_2^2x_3^2}-\cancel{x_2^2x_3^2})$$
$$-x_1x_2(Qx_2x_1+x_1x_2x_3^2-x_1x_2x_3^2)$$
$$+x_1x_3(x_1x_2^2x_3-Qx_3x_1-x_1x_2^2x_3)$$

$$\Delta_3 = (Q+x_1^2)(Q^2+Qx_3^2+Qx_2^2)-x_1x_2(Qx_2x_1)$$
$$+x_1x_3(-Qx_3x_1).$$

$$\Delta_3 = Q^3+Q^2x_3^2+Q^2\cdot x_2^2+x_1^2Q^2+Qx_1^2x_3^2$$
$$+Qx_1^2x_2^2-Qx_1^2x_2^2-Qx_1^2x_3^2$$

$$= Q^3+Q^2(x_1^2+x_2^2+x_3^2) \longrightarrow (4)$$

Now proceeding on the same process and looking over the equation (2), (3), (4) we can be write as

$$\Delta_n = Q^n + Q^{n-1}(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)$$ putting in (1)

$$J = Q^{-\frac{3}{2}n}(Q^n+Q^{n-1}(x_1^2+x_2^2+\cdots+x_n^2))$$
$$= Q^{-\frac{3}{2}n}Q^{n-1}[Q+x_1^2+x_2^2+\cdots+x_n^2]$$
$$= Q^{-\frac{3}{2}n+n-1}[1-x_1^2-x_2^2-\cdots-x_n^2+x_1^2+x_2^2+\cdots+x_n^2]$$
$$= Q^{\frac{-3n+2n-2}{2}}[1] = Q^{-\frac{n}{2}-1}$$

Therefore $J = \dfrac{\partial(u_1,u_2,\cdots u_n)}{\partial(x_1,x_2\cdots x_n)} = (1-x_1^2-x_2^2-x_3^2-\cdots-x_n^2)^{-\frac{n}{2}-1}$ required ans

Question No (8)

Transform the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$ by the

formula of sperical polar transformation $x = r\sin\theta\cos V$,

$y = r\sin\theta\sin V$, $z = r\cos\theta$

Solution:- we know that if $v = v(x, y)$ and $x = u\cos V$, $y = u\sin V$ then the transformation of $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ becomes

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial u^2} + \frac{1}{u^2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{u}\frac{\partial v}{\partial u} \longrightarrow (A)$$

Let $u = r\sin\theta$, then $x = r\sin\theta\cos V = u\cos u$,

$y = r\sin\theta\sin V = u\sin V$, $z = r\cos\theta$.

From A, we can write.

$$\frac{\partial^2 v}{\partial x} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial u^2} + \frac{1}{u^2}\frac{\partial^2 v}{\partial u^2} + \frac{1}{u}\frac{\partial v}{\partial u} \longrightarrow (1)$$

Now we replace $x = z$, $y = u$, $u = r$ and $V = \theta$

From (A)

$$\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial u^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r} \longrightarrow (2)$$

But $\frac{\partial}{\partial y} = \sin V\frac{\partial}{\partial u} + \frac{\cos V}{u}\frac{\partial}{\partial V}$

$$\frac{\partial v}{\partial y} = \sin V\frac{\partial v}{\partial u} + \frac{u}{u} - V\frac{\partial v}{\partial V}$$

Substituting $u = y$, $r = u$, $\theta = V$ in above

$$\frac{\partial v}{\partial u} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \longrightarrow (3)$$

Now (1) + (2) $\Longrightarrow$

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\cancel{\frac{\partial^2 V}{\partial u^2}}=\cancel{\frac{\partial^2 V}{\partial u^2}}+\frac{1}{u^2}\frac{\partial^2 V}{\partial z^2}+\frac{1}{u}\frac{\partial V}{\partial u}$$
$$+\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r}\frac{\partial V}{\partial r}$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{u^2}\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$
$$+\frac{1}{r}\frac{\partial V}{\partial r}.$$
$$=\frac{1}{u^2}\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial^2 V}{\partial \theta^2}$$
$$+\frac{1}{r}\frac{\partial V}{\partial r}$$
$$=\frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial u^2}+\frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$$
$$+\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta}+\frac{1}{r}\frac{\partial V}{\partial r}.$$

By arranging.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 V}{\partial r^2}$$
$$+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial V}{\partial \theta}+\frac{1}{r}\frac{\partial V}{\partial r}$$
$$=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{u^2}\cdot\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}$$
$$+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}+\frac{1}{r}\frac{\partial V}{\partial r},$$

which the Solution

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Theorem:- If $Z = f(x,y)$ is continously differentiable and homogenous function of degree "n" in a region R. Then in R

$$xf_x(x,y) + yf_y(x,y) = nf(x,y)$$

or

$$x\frac{\partial}{\partial x}f(x,y) + y\frac{\partial}{\partial y}f(x,y) = nf(x,y).$$

Proof:- First of All we define homogenous function. Let t be a real Parameter a $f(tx, ty) = t^n f(x,y)$, then $f(x,y)$ is called Homogenous Function of degree n.

Now given that $Z = f(x,y)$ is a hom: function of degree n. So we can write $f(tx, ty) = t^n f(x,y)$, for a variable paramметor Z.

Let $u = tx$ and $v = ty$.

Then $f(u,v) = t^n f(x,y) \longrightarrow (1)$

Taking partial derivative of both side of (1) we get

$$\frac{\partial}{\partial t}f(u,v) = \frac{\partial}{\partial t}t^n f(x,y$$

$$= f(x,y)\frac{\partial}{\partial t}t^n$$

$$= nt^{n-1}f(x,y) \longrightarrow (2)$$

Now

$$f = f(u,v), \quad u = u(t), \quad v = v(t)$$

$\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial t} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial t}$, so that (2) becomes

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t}=n t^{n-1} f(x,y) \longrightarrow (3)$$

But
$\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}(x\cdot t)=x$ and $\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}(yt)=y$

so (3) becomes

$$x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}=n t^{n-1} f(x,y) \longrightarrow (4)$$

Since $u=tx$, $v=ty$
"t" is a real parameter.
Let us take $t=1$ then $u=x$, $v=y$
So equation (4) becomes

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$$

$$x f_x + y f_y = nf(x,y).$$

Extreme Values:-

Let $u=f(x,y)$ be a function of two variables x and y. Then value $f(a,b)$ is an extreme value of $f(x,y)$, if $f_x(a,b)=f_y(a,b)=0$
Let $f_{xx}(a,b)=A$, $f_{xy}(a,b)=B$ and $f_{yy}(a,b)=C$. Then

(1):- If $AC-B^2>0$ extreme value and which is maximum if $A<0$, $C<0$ and which is minimum if $A>0$, $C>0$.

(2):- If $AC-B^2>0$. Then there is no extreme value i.e $f(a,b)$ is neither max/ or minimum.

(3) If $AC - B^2 = 0$, then test fails this is called the doubt full case. In such a case the extreme value may or may not exist at (a,b).

Problem (9): Invistigate the maxima and minima. If any of the following functions
$$u = f(x,y) = 2(x-y)^2 - x^4 - y^4$$

Solution: $f(x,y) = 2(x-y)^2 - x^4 - y^4$

$$f_x = 4(x-y) - 4x^3 \quad f_y = -4(x-y) - 4y^3$$

$$f_{xx} = 4 - 12x^2 \quad f_{yy} = 4 - 12y^2$$

Now for the extreme value of the functions, $f_x = 0$, $f_y = 0$

$$4(x-y) - 4x^3 = 0 \text{ and } -4(x-y) - 4y^3 = 0$$

$\Rightarrow x - y - x^3 = 0 \longrightarrow (1)$ and $x - y + y^3 = 0 \longrightarrow (2)$

$(2) - (1) \Rightarrow x^3 + y^3 = 0$

$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$

$x + y = 0$ or $x^2 - xy + y = 0$

$y = -x$ or $x = -y$

Putting in (1)

$x + x - x^3 = 0$

$2x - x^3 = 0$

$x = 0$, $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

$y = 0 \quad , -\sqrt{2} \quad , \sqrt{2}$

possible extreme value are
$(0,0) \quad , (\sqrt{2}, -\sqrt{2}) \quad , (-\sqrt{2}, \sqrt{2})$

Now $f_{xx}(x, y) = 4 - 12x^2$
$f_{xx}(0,0) = 4 = A$

$f_{xy}(x, y) = -4 \Rightarrow f_{xy}(0,0) = -4 = B$

$f_{yy}(0,0) = 4 = C$

Now $AC - B^2 = 4.4 - (-4)^2 = 16 - 16 = 0$
it is doust full case
by
$(x, y) = \sqrt{2}, -\sqrt{2}$

$f_{xx}(\sqrt{2}, -\sqrt{2}) = 4 - 12(\sqrt{2})^2 = -20 = A$

$f_{xy}(\sqrt{2}, -\sqrt{2}) = -4 = B$

$f_{yy}(\sqrt{2}, -\sqrt{2}) = 4 - 12(-\sqrt{2})^2 = -20 = C$

$AC - B^2 = 400 - 16 = 386 > 0$

Since $AC - B^2 > 0$ and $A < 0$ and $C < 0$
$(\sqrt{2}, -\sqrt{2})$ corresponds to the maximum valed. Also we finde maximum value of the function.

$u = f(x,y) = 2(x-y)^2 - x^4 - y^4$

$f(\sqrt{2}, -\sqrt{2}) = 2(\sqrt{2}+\sqrt{2})^2 - \sqrt{2}^2 - \sqrt{2}^2$

$= 2(2\sqrt{2})^2 - 4$

$U_{max} = 16 - 4 = 12$ Ans

when $(-\sqrt{2}, \sqrt{2}) = (x,y)$

$f_{xx}(-\sqrt{2}, \sqrt{2}) = 4 - 12(-\sqrt{2})^2 = -20 = A$

$f_{xy}(-\sqrt{2}, \sqrt{2}) = -4 = B$

$f_{yy}(-\sqrt{2}, \sqrt{2}) = -4 - 12(\sqrt{2})^2 = 4 - 24 = -20 =$

Now

$AC - B^2 = 400 - 16 = 386 > 0$

Since

$AC - B^2 > 0$ and $A < 0$, $C < 0$

we find maximum value of the ftn

$\therefore U_{max} = f(-\sqrt{2}, \sqrt{2}) = 2(-\sqrt{2}-\sqrt{2})^2 - (-\sqrt{2})^2 - (\sqrt{2})^2$

$= 8$ Ans

Question No (10)

Discuss the maxima and minima of the function $u = f(x,y) = 21x - 12x^2 - 2y^2 + x^3 + xy^2$

Solution. $f_x = 21 - 24x + 3x^2 + y^2$, $f_y = -4y + 2xy$

$f_{xy} = 2y$, $f_{xx} = -24 + 6x$, $f_{yy} = -4 + 2x$

For extreme value:

$f_x = 0$

$$21 - 24x + 3x^2 + y^2 = 0 \longrightarrow (1)$$

$$-4y + 2xy = 0 \longrightarrow (2)$$

$$2y(-2+x) = 0$$

$y = 0$ or $x = 2$

when $x = 2$ put in (1)

$$21 - 24(2) + 3(2)^2 + y^2 = 0$$

$$21 - 48 + 12 + y^2 = 0$$

$$y^2 = 15$$

$$y = \pm\sqrt{15}$$

when $y = 0$ put in (1)

$$21 - 24x + 3x^2 = 0$$

$$x^2 - 8x + 7 = 0$$

$x = 1, 7$ points are:

$(2, \sqrt{15})$, $(2, -\sqrt{15})$, $(1, 0)$ and $(7, 0)$

For this: $(x, y) = (2, \sqrt{15})$ then:

$$f_{xx} = -24 + 6 \cdot 2 = -12 = A$$

$$f_{xy} = 2\sqrt{15} = B$$

$$f_{yy} = -4 + 2(2) = 0 = C$$

Now

$$AC - B^2 = -12 \cdot 0 - 60 = -60 < 0$$

So $(2, \sqrt{15})$ is neither maximum and minimum, when $(2, \sqrt{15})$, then

$f_{xx} = -24 + 6.2 = -12 = A$.

$f_{xy} = 2(-\sqrt{15}) = -2\sqrt{15} = B$

$f_{yy} = -4 + 2(2) = 0$

Since $AC - B^2 = 0 - 60 < 0$
which is neither maxamum and minimum.

when, $(x, y) = (1, 0)$. Then

$f_{xx} = -24 + 6 = -18 = A$

$f_{xy} = -4 + 2.1 = -2 = B$

$f_{yy} = 2.0 = 0 = C$

$\therefore AC - B^2 = 36 > 0$

Since $AC - B^2 > 0$ and $A < 0$ and $C < 0$
$(1, 0)$ is the point of maximum.
Maximum value of the functions

$u = f(x, y) = 21x - 12x^2 - 2y^2 + x^3 + xy^2$

$f(1, 0) = 21 - 12 - 0 + 1 + 0$

$= 10$ Ans

When $(x, y) = (7, 0)$. then
$f_{xx} = -24 + 6 \times 7 = 18 = A$

$f_{xy} = 0 = B$

$f_{yy} = -4 + 14 = 10 = C$

Since $AC - B^2 = 180 - 0 = 180 > 0$

$AC - B^2 > 0$ and A & C are +ve.

But $(7,0)$ is minimum point.
For this find minimum value of the function

$$u = f(7,0) = 21.7 - 12.49 - 0 + 7^3 + 0$$

$$= -98 \quad \text{Ans}$$

Question (11)

Find the maximum and minimum of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Solution:- $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

$f_x = 3x^2 - 3$ and $f_y = 3y^2 - 12$

$f_{xx} = 6x$ and $f_{yy} = 6y$

For extreme value

$f_x = 0$ and $f_y = 0$

$3x^2 - 3 = 0$, $3y^2 - 12 = 0$

$x^2 - 1 = 0$, $y^2 - 4 = 0$

$x = 1, -1$, $y = 2, -2$

possible extreme points are
$(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$.
when $(x,y) = (1,2)$, then

$f_{xx} = 6 = A$

$f_{xy} = 0 = B$

$f_{yy} = 12 = C$

since $AC - B^2 = 72 > 0$ and A & C are +ve
which is minimum point $(1,2)$

$$U_{min} = x^3 + y^3 - 3x - 12y + 20$$
$$= 1^3 + 2^3 - 3(1) - 12(3) + 20$$
$$= 1 + 8 - 3 - 36 + 20 = -10 \quad \text{Ans}$$

when $(x,y) = (+1,-2)$, then

$f_{xx} = 6 = A$
$f_{xy} = 0 = B$
$f_{yy} = -12 = C$

Since $AC - B^2 = -72 < 0$ and the function is neither maximum and nor minimum at $(1,-2)$.

when $(x,y) = (-1,2)$, then
$f_{xx} = 6 = A$
$f_{xy} = 0 = B$
$f_{yy} = -12 = C$

Since $AC - B^2 = -72 < 0$ and Therefore the function is neither maximum and nor mininimum at $(-1, 2)$.

When $(x, y) = (-1, -2)$, then

$f_{xx} = -6 = A$

$f_{xy} = 0 = B$

$f_{yy} = -12 = C$

Since $AC - B^2 = 72 > 0$ and A & C both -ve function value is maximum at $(-1, -2)$.

$U_{max} = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$
$= 8$ ANS

Question No (12)

Show that the function $f(x,y) = 2x^4 - 3x^2y + y^2$ has neither maximum nor a minimum at $(0,0)$

Solution:- $f(x,y) = 2x^4 - 3x^2y + y^2$

$f_x = 8x^3 - 6xy$, $f_y = -3x^2$

$f_{xx}\ 24x^2 - 6y$, $f_{yy} = 0$

$f_{xy} = -6$

$f_{xx} = 0 = A$

$f_{xy} = -6 = B$

$f_{yy} = 0 = C$

Since $AC - B^2 = -6 < 0$ and therefore the function value is neither maximum nor minimum at $(0,0)$

Question No (13)

Investigate the maxima and minima
$$f(x,y) = x^2 + 3xy + y^2 + x^3 + y^3$$

Solution:- $f(x,y) = x^2 + 3xy + y^2 + x^3 + y^3$

$f_x = 2x + 3y + 3x^2$, $f_y = 3x + 2y + 3y^2$

$f_{xx} = 2 + 6x$, $f_{yy} = 2 + 6y$

$f_{xy} = 3$

$f_x = 0$, $f_y = 0$

$2x + 3y + 3x^2 = 0 \longrightarrow$ (i)

$3x + 2y + 3y^2 = 0 \longrightarrow$ (ii)

$3(2x + 3y + 3x^2) = 0$
$-2(3x + 2y + 3y^2) = 0$

$+5y + 9x^2 - 6y^2 = 0$

or

$9x^2 - 6y^2 + 5y = 0$

in this Question wrong.

Question No (14)

Find the shortest distance from the origion to the hyperbola.

$$x^2+8xy+7y^2=225.$$

Solution: We have to find the minimum value of x^2+y^2 (The square of the distance from the origion to any point in the xy) subject to the constant

$$x^2+8xy+7y^2=225 \longrightarrow (*_1)$$

Let us consider

$$F=(x^2+y^2)+\lambda(x^2+8xy+7y^2-225) \longrightarrow (*_2)$$

Here x,y are independent variables and λ is a constant.

$$dF = 2x\,dx+2y\,dy+2x\lambda\,dx+8\lambda y\,dx+8\lambda x\,dy+14\lambda y\,dy$$

$$=(2x+2x\lambda+8\lambda y)dx+(2y+8\lambda x+14\lambda y)dy \rightarrow (A)$$

Hence x,y are independent variables

$$2x+2x\lambda+8\lambda y=0, \quad 2y+8\lambda x+14\lambda y=0$$

$$x+x\lambda+4\lambda y=0 \longrightarrow (i) \quad y+4\lambda x+7\lambda y=0 \longrightarrow (ii)$$

$$(i) \Rightarrow x=\frac{-4\lambda y}{1+\lambda} \rightarrow (iii) \text{ and } (ii) \Rightarrow x=\frac{-(1+7\lambda)y}{4\lambda} \rightarrow (iv)$$

when $x=\dfrac{-4y\lambda}{1+\lambda}$ putting in (ii)

$$y+4\left(\frac{-4y\lambda}{1+\lambda}\right)\lambda+7\lambda y=0$$

$(1+\lambda)y - 16\lambda^2 y + 7(1+\lambda)\lambda y = 0$

$1+\lambda - 16\lambda^2 + 7\lambda + 7\lambda^2 = 0$

$-9\lambda^2 + 8\lambda + 1 = 0$

$-9\lambda^2 + 9\lambda - \lambda + 1 = 0$

$-9\lambda(\lambda - 1) - 1(\lambda - 1) = 0$

$\lambda = \frac{1}{9}, 1$

$\lambda = 1$ putting in (iv)

$x = \frac{-(1+7)y}{4} = -2y$

putting $x = -2y$ in (x_1)

$(-2y)^2 + 8(-2y)y + 7y^2 = 225$

$4y^2 - 16y^2 + 7y^2 = 225$

$-5y^2 = 225$

$y^2 = -45$

which is not a real number.

when $\lambda = -1/9$ putting in (iii)

$x = \frac{-4y(-1/9)}{1 - 1/9} = y/2$

$\Rightarrow x = y/2$

$$x^2 + 8xy + 7y^2 = 225$$

$$y^2/4 + 4y^2 + 7y^2 = 225$$

$$\frac{45}{4} y^2 = 225$$

$$y^2 = 20 \longrightarrow (M)$$

$$\Rightarrow x = y/2 \Rightarrow x^2 = y^2/4$$

$$x^2 = 20/4$$

$$x^2 = 5 \longrightarrow (N)$$

Therefore $x^2 + y^2 = 20 + 5$ b/c (M)+(N)

$$x^2 + y^2 = 25$$

But Now we show that it is minimum.
Since

$$d^2F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + 2F_{xy}\,dx\,dy.$$

$$\Rightarrow F_x = 2x + \lambda 2x + 8y\lambda$$

$$F_{xx} = 2 + 2\lambda$$

$$\Rightarrow F_y = 2y + 8\lambda x + 14\lambda x$$

$$F_{yy} = 2 + 14\lambda$$

$$\Rightarrow F_{xy} = 8\lambda$$

$\Rightarrow d^2F = 2(1+\lambda)dx^2 + 2(1+7\lambda)dy^2 + 2(8)\lambda\, dx\,dy$

Putting $\lambda = -1/9$ in above

$= 2(1+(-1/9))dx^2 + 2(1+7(-1/9))dy^2 + 2(8)(-1/9)dx\,dy.$

$= \frac{16}{9}dx^2 + \frac{4}{9}dy^2 - \frac{16}{9}dx\,dy$

$= \frac{4}{9}(4dx^2 + dy^2 - 4dx\,dy).$

$= \frac{4}{9}(2dx - dy)^2 > 0$

$\Rightarrow d^2F > 0$

Find d^2F cannot vanish because $(dx, dy) \neq (0,0)$.

Therefore the function x^2+y^2 has minimum value 25.

Note:- Here F is a function of two variables and so its maximum or minimum value can be verified by the method of functions of two variables. $AC - B^2 > 0$ also.

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Extreme Values Of a function Of Several Variable.

The rule for function $f(x,y,z)$ of three independent variables are;

Sufficient condition for (a,b,c) to be an extreme point are that;

(i) $df(a,b,c) = f_x\,dx + f_y\,dy + f_z\,dz = 0$

so that

$$f_x = f_y = f_z = 0$$

and

(ii) $d^2f(a,b,c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx$

Keep the same sign for arbitrary values of dx, dy, dz. The extreme point being a maxima or minima according as the sign of d^2f is negative or positive.

The point will be neither a maxima nor a minima if d^2f does not keep the same sign and requires further investigation if d^2f keeps the same sign but vanishes at some points of a neighbourhood of (a,b,c)

The conditions that d^2f keeps the same sign may be stated in terms of matrices as follows

consider the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

d^2f will always be positive if and only if the three principle minors

$$f_{xx}, \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

are all positive, and d^2f will be always negative if and only if their signs are alternatively negative and positive.

Example (1)

Show that

$$f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$$

as a minima at $(1,1,1)$ and a maxima at $(-1,-1,-1)$.

Solution:- $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$

Diffi w.r.t x, y, z separdc

We have

$$f_x = 3(x+y+z)^2 - 3 - 24yz$$
$$f_y = 3(x+y+z)^2 - 3 - 24xz$$
$$f_z = 3(x+y+z)^2 - 3 - 24xy$$

Now the stationary points are given by

$f_x = 0 \Rightarrow 3(x+y+z)^2 - 3 - 24yz = 0$

$\Rightarrow (x+y+z)^2 - 1 - 8yz = 0$

OR

$(x+y+z)^2 - 8yz - 1 = 0 \longrightarrow (i)$

$f_y = 0 \Rightarrow (x+y+z)^2 - 8zx - 1 = 0 \longrightarrow (ii)$

$f_z = 0 \Rightarrow (x+y+z)^2 - 8xy - 1 = 0 \longrightarrow (iii)$

Now (i) − (ii)

$-8yz + 8zx = 0 \Rightarrow z(x-y) = 0$ similarly

$x(y-z) = 0$ and $y(z-x) = 0$

$\Rightarrow x=0, y=0, z=0$ or $x=y=z$

So sationary points are

$(0,0,0), (1,1,1), (-1,-1,-1)$.

Now

$f_{xx} = 6(x+y+z) = f_{yy} = f_{zz}$

$f_{xy} = 6(x+y+z) - 24z = f_{yx}$

$f_{yz} = 6(x+y+z) - 24x = f_{yz}$

$f_{zx} = 6(x+y+z) - 24y = f_{xz}$

Now at $(1,1,1)$

$f_{xx} = 6(1+1+1) = 6(3) = 18 = f_{yy} = f_{zz}$

$f_{xy} = 6(1+1+1) - 24(1) = 6(3) - 24 = -6 = f_{yz} = f_{xz}$

Now Since

$$d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dz + 2f_{yz}\,dy\,dz + 2f_{xz}\,dx\,dz$$

$$= 18(dx^2+dy^2+dz^2) + 2(-6dxdy - 6dydz - 6dxdz)$$

$$= 18(dx^2+dy^2+dz^2) - 12(dxdy + dydz + dxdz)$$

$$= 6[3(dx^2+dy^2+dz^2) - 2(dxdy + dydz + dxdz)]$$

$$= 6[3dx^2 + 3dy^2 + 3dz^2 - 2dxdy - 2dydz - 2dxdz]$$

$$= 6[dx^2+dy^2+dz^2+dx^2+dy^2+dz^2+dx^2+dy^2+dz^2 - 2dxdy - 2dydz - 2dxdz]$$

$$= 6[dx^2+dy^2+dz^2+(dx-dy)^2+(dy-dz)^2+(dz-dx)^2]$$

$$d^2f = 6[dx^2+dy^2+dz^2+(dx-dy)^2+(dy-dz)^2+(dz-dx)^2]$$

which is positive for all values of $(dx, dy, dz) \neq (0,0,0)$

Thus $(1,1,1)$ is a point of minima of the function.

Now at $(-1,-1,-1)$

$$f_{xx} = 6(-6-1-1) = 6(-3) = -18 = f_{yy} = f_{zz}$$

$$f_{xy} = 6(-1-1-1) - 24(-1) = -18 + 24$$
$$= 6 = f_{yz} = f_{zx}$$

Now

$$d^2f = -18(dx^2+dy^2+dz^2)+12(dy\,dz+dx\,dy+dx\,dz)$$

$$= -6[3dx^2+3dy^2+3dz^2-2dy\,dz-2dx\,dy-2dx\,dz]$$

$$= -6[dx^2+dy^2+dz^2+(dx-dy)^2+(dy-dz)^2+(dz-dx)^2]$$

which is negative for all dx, dy, dz and never vanishes.

Hence the function has a maximum value at $(-1,-1,-1)$

Example (2)

Show that the following function have a minima at the point indicated (i) $x^2+y^2+z^2+2xyz$ at $(0,0,0)$

Solution: $f(x,y,z) = x^2+y^2+z^2+2xyz$

$f_x = 2x+2yz$, $f_y = 2y+2xz$, $f_z = 2z+2xy$

Now for extreme values of the function let $f_x = f_y = f_z = 0$

$\Rightarrow 2x+2yz=0 \Rightarrow x+yz=0 \rightarrow (1)$

Similarly $y+xz=0 \rightarrow (2)$, $z+xy=0 \rightarrow (3)$

Now $(1) \Rightarrow x=-yz$

Then $(2) \Rightarrow y-yz^2=0$

$\Rightarrow y(1-z^2)=0 \Rightarrow y=0,\ z=\pm 1$

Similarly $x=0$, $z=0$ and

when $z = 1$ then $y = -x$ and $xy = -1$

when $y = 1$ then $x = -1$, when $y = -1$
then $x = -1$ and
when $z = -1$, then $x = y \Rightarrow xy = 1$
and $y^2 = 1 \Rightarrow y = \pm 1$ so points are
$(0,0,0), (1,-1,1), (-1,1,1), (1,1,-1), (-1,-1,-1)$

Now we required to check only $(0,0,0)$

Now
$$f_{xx} = 2 = f_{yy} = f_{zz}$$
$$f_{xy} = 2z = f_{yx}, \quad f_{yz} = 2x = f_{zy}$$
and
$$f_{xz} = 2y = f_{zx}$$

Now at $(0,0,0)$ we have
$$f_{xx} = f_{yy} = f_{zz} = 2$$ and
$$f_{xy} = f_{yx} = f_{zy} = f_{xz} = 0$$

Now
$$d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx.$$

$$= 2(dx^2 + dy^2 + dz^2) + 0$$

$$d^2f = 2(dx^2 + dy^2 + dz^2).$$
which is positive for all dx, dy, dz so
$(0,0,0)$ is point of minima.

Example No (3)

Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \quad \text{and} \quad z=x+y.$$

Solution:- Let us consider a function F of independent variables x, y, z where

$$F = x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2(x+y+z).$$

Therefore

$$dF = 2x\,dx+2y\,dy+2z\,dz+\lambda_1\left(\frac{x}{2}dx+\frac{2}{5}y\,dy+\frac{2}{25}z\,dz\right)$$
$$+\lambda_2 dx+\lambda_2 dy-\lambda_2 dz$$

$$=\left(2x+\frac{x}{2}\lambda_1+\lambda_2\right)dx+\left(2y+\frac{2y}{5}\lambda_1+\lambda_2\right)dy$$
$$+\left(2z+\frac{2z}{25}\lambda_1-\lambda_2\right)dz.$$

As x, y, z are independent variables, we get

$$2x+\frac{x}{2}\lambda_1+\lambda_2=0$$

$$2y+\frac{2}{25}y\lambda_1+\lambda_2=0$$

$$2z+\frac{2}{25}z\lambda_1-\lambda_2=0$$

$$\Rightarrow x=\frac{-2\lambda_2}{\lambda_1+4},\quad y=\frac{-5\lambda_2}{2\lambda_1+10},\quad z=\frac{25\lambda_2}{2\lambda_1+50} \longrightarrow (A)$$

Putting these values in $x+y=z$ we get

$$\frac{-2\lambda_2}{\lambda_1+4}+\frac{-5\lambda_2}{2\lambda_1+10}=\frac{25\lambda_2}{2\lambda_1+50} \longrightarrow (1)$$

$\Rightarrow \lambda_2 = 0$ or $\frac{2}{\lambda_1+4} + \frac{5}{2\lambda_1+10} + \frac{25}{2\lambda_1+50} = 0 \longrightarrow (2)$

when $\lambda_2 = 0$ Then
(A) $\Rightarrow$ $x = y = z = 0$ but $(0,0,0)$ does not satisfy the other condition of constraint.

Therefore from (2)

$$2(2\lambda_1+10)(2\lambda_1+50) + 5(\lambda_1+4)(2\lambda_1+50) + 25(\lambda_1+4)(2\lambda_1+10) = 0$$

$\Rightarrow \lambda_1 = -10, \; -75/17$

Now when $\lambda_1 = -10$
Then (A) $\Rightarrow$ $x = \frac{1}{3}\lambda_2$, $y = \frac{1}{2}\lambda_2$
and
$$z = \frac{5}{6}\lambda_2 \longrightarrow (3)$$

putting the values of x, y, z in
$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ we get

$\lambda_2^2 = 180/19$ or $\lambda_2 = \pm 6\sqrt{5/19}$
putting these values in (3) we get sationary points

$\left(2\sqrt{5/19}, 3\sqrt{5/19}, 5\sqrt{5/19}\right), \left(-2\sqrt{5/19}, -3\sqrt{5/19}, -5\sqrt{5/19}\right)$

Putting these points in $x^2+y^2+z^2$ then

$$x^2+y^2+z^2 = 4 \cdot \frac{5}{19} + 9 \cdot \frac{5}{19} + 25 \cdot \frac{5}{19}$$

$$= \frac{20}{19} + \frac{45}{19} + \frac{125}{19} = \frac{190}{19} = 10$$

and from the 2nd point we also get that $x^2+y^2+z^2=10$

Now
when $\lambda_1 = \frac{-75}{17}$ then from (A)

$$x = \frac{34}{7}\lambda_2 , \quad y = \frac{-17}{4}\lambda_2 , \quad z = \frac{17}{20}\lambda_2 \rightarrow (C)$$

putting in $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$, we get

$\lambda_2 = \pm 140/17\sqrt{646}$ putting in (C)
we get the stationary points

$$\left(\frac{40}{\sqrt{646}}, -\frac{35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right), \left(\frac{-40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, -\frac{5}{\sqrt{646}}\right)$$

putting these values in $x^2+y^2+z^2$ we get that $x^2+y^2+z^2 = 75/17$

Thus the maximum value is (10) and minimum value is (75/17).

Example No (4)

Show that the length of the dZ axes of the section of on $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c}=1$ by the plane $lx+my+nz=0$ are the roots of the quadratic in r^2,

$$\frac{l^2a^2}{r^2-a^2}+\frac{m^2b^2}{r^2-b^2}+\frac{n^2c^2}{r^2-c^2}=0.$$

Solution:- We have to find the stationary values of the function r^2 where $r^2 = x^2+y^2+z^2$, subject to the two equations of conditions

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow (1)$$

$$lx+my+nz=0 \longrightarrow (2)$$

Let us consider a function F of independent variables x, y, z, where

$$F = x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+2\lambda_2(lx+my+nz)$$

$$dF = 2x\,dx+2y\,dy+2z\,dz+\frac{2x\lambda_1}{a^2}dx+\frac{2y\lambda_1}{b^2}dy+\frac{2z\lambda_1}{c^2}dz$$
$$+2\lambda_2 l\,dx+2\lambda_2 m\,dy+2\lambda_2 n\,dz.$$

$$dF = \left(2x+\frac{2x\lambda_1}{a^2}+2\lambda_2 l\right)dx+\left(2y+\frac{2y\lambda_1}{b^2}+2\lambda_2 m\right)dy$$
$$+\left(2z+\frac{2z\lambda_1}{c^2}+2\lambda_2 n\right)dz$$

The stationary points are

$$\left.\begin{aligned} x+\frac{x}{a^2}\lambda_1+\lambda_2 l &= 0\\ y+\frac{y}{b^2}\lambda_1+\lambda_2 m &= 0\\ z+\frac{z}{c^2}\lambda_1+\lambda_2 n &= 0\end{aligned}\right\} \longrightarrow (3)$$

multiplying x, y, z and adding

$$\left(x^2+\frac{x^2}{a^2}\lambda_1+x\lambda_2 l\right)+\left(y^2+\frac{y^2}{b^2}\lambda_1+y\lambda_2 m\right)$$
$$+\left(z^2+\frac{z^2}{c^2}\lambda_1+z\lambda_2 n\right)=0$$

$$(x^2+y^2+z^2)+\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\lambda_2(lx+my+nz)=0$$

$$r^2+\lambda_1(1)+\lambda_2(0)=0$$

$$\Rightarrow \lambda_1=-r^2$$

So from (3)

$$x+\frac{x\lambda_1}{a^2}+\lambda_2 l=0$$

$$\Rightarrow x\left(1-\frac{r^2}{a^2}\right)=-\lambda_2 l$$

$$\Rightarrow x\left(\frac{a^2-r^2}{a^2}\right)=-\lambda_2 l$$

$$\Rightarrow x=\frac{a^2\lambda_2 l}{r^2-a^2}$$

Similarly $y=\dfrac{b^2m\lambda_2}{r^2-b^2}$ and $z=\dfrac{c^2n\lambda_2}{r^2-c^2}$

But $0=lx+my+nz$

putting the values

$$l\left(\frac{a^2l\lambda_2}{r^2-a^2}\right)+m\left(\frac{b^2m\lambda_2}{r^2-b^2}\right)+n\left(\frac{c^2n\lambda_2}{r^2-c^2}\right)=0$$

$$\Rightarrow \lambda_2\left[\frac{a^2l^2}{r^2-a^2}+\frac{b^2m^2}{r^2-b^2}+\frac{c^2n^2}{r^2-c^2}\right]=0$$

Since $\lambda_2\neq 0$, we get the quadratic in r^2 giving the stationary values

$$\frac{a^2 l^2}{r^2 - a^2} + \frac{b^2 m^2}{r^2 - b^2} + \frac{c^2 n^2}{r^2 - c^2} = 0$$

Example No (5)

Prove that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is } 8abc/3\sqrt{3}.$$

Solution: We have to find the greatest value of $8xyz$ subject to the conditions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \rightarrow (a), \quad x, y, z > 0$$

For this let us consider a function F of three independent variables x, y, z where

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$dF = \left(8yz + \frac{2x}{a^2}\lambda\right)dx + \left(8xz + \frac{2y}{b^2}\lambda\right)dy + \left(8xy + \frac{2z}{c^2}\lambda\right)dz.$$

Since x, y, z are independent variables so the stationary points are

$P_0 T = 0$

$$8yz + \frac{2x}{a^2}\lambda = 0 \longrightarrow (1)$$

$$8xz + \frac{2y}{b^2}\lambda = 0 \longrightarrow (2)$$

$$8xy + \frac{2z}{c^2}\lambda = 0 \longrightarrow (3)$$

$\longrightarrow$ (A)

$(1) \times x + (2) \times y + (3) \times z$ we get

$$8xyz + \frac{2x^2\lambda}{a^2} + 8xyz + \frac{2y^2\lambda}{b^2} + 8xyz + \frac{2z^2\lambda}{c^2} = 0$$

$$\Rightarrow 24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$\Rightarrow 24xyz + 2\lambda(1) = 0 \quad \text{using (a)}$$

$$\Rightarrow 12xyz + \lambda = 0$$

$$\Rightarrow \lambda = -12xyz$$

Now $(1) \Rightarrow$

$$8yz + \frac{2x}{a^2}(-12xyz) = 0$$

$$\Rightarrow 8 - \frac{24}{a^2}x^2 = 0 \Rightarrow \frac{3x^2}{a^2} = 1$$

$$\Rightarrow 3x^2 = a^2 \Rightarrow x^2 = a^2/3$$

$\Rightarrow x = \pm a/\sqrt{3}$ But we only get

$x = a/\sqrt{3}$ similarly $y = b/\sqrt{3}$, $z = c/\sqrt{3}$.

Thus $8xyz = 8\left(\frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}}\right)$

$$8xyz = 8abc/3\sqrt{3} \rightarrow (4)$$

Now we show that $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is maximum.

Since

$$d^2F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2 + 2F_{xy}\,dx\,dy + 2F_{yz}\,dy\,dz + 2F_{zx}\,dz\,dx \rightarrow (5)$$

Now $F_x = 8yz + \frac{2x\lambda}{a^2}$, $F_{xx} = \frac{2\lambda}{a^2}$

$F_y = 8xz + \frac{2y\lambda}{b^2}$, $F_{yy} = \frac{2\lambda}{b^2}$

$F_z = 8xy + \frac{2z\lambda}{c^2}$, $F_{zz} = \frac{2\lambda}{c^2}$

$F_{xy} = 8z$, $F_{xz} = 8y$, $F_{yz} = 8x$

$F_{xx} = \frac{2\lambda}{a^2} = \frac{2}{a^2}(-12xyz)$ $\because \lambda = -12xyz$

$$F_{xx} = \frac{-24}{a^2}xyz$$

$$F_{yy} = \frac{-24}{b^2}xyz$$

$$F_{zz} = \frac{-24}{c^2}xyz$$

putting in (5)

$$d^2F = -24xyz\left[\left(\frac{dx}{a}\right)^2 + \left(\frac{dy}{b}\right)^2 + \left(\frac{dz}{c}\right)^2\right] + 16\left[z\,dx\,dy + x\,dy\,dz + y\,dz\,dx\right].$$

Now if $d^2F < 0$ and

$(x > 0, \ y > 0, \ z > 0)$

then $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is

point of maxima.

And the maximum value is

$$\boxed{8abc/3\sqrt{3}}$$

The End of Real Analysis course.

End Of CHP # 6

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