Chapter 05: Riemann Theory of Integration

Handwritten Notes of REAL ANALYSIS

Written By

Asim Marwat

MSc Mathematics (UOP)

asimmarwat41@gmail.com

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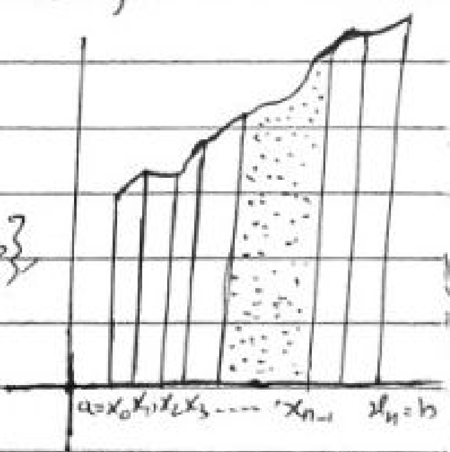
CHP # 5
"Rieman Theory of Integration"

Partition of Domain of the function:

Let $I = [a,b]$ is the domain of any function (real valued function).
The partition say 'P' of I is a finite order set of points $x_0, x_1, x_2, \ldots, x_n$ and is denoted by $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$, where $x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n$



upper Sum:-

Let $f(x)$ be any real valued function (bounded) on $I = [a,b]$ and let "P" is any partition of I
i.e $P = \{x_0, x_1, x_2, \ldots, x_n\}$
let

$$M_1 = \sup\{f(x) \; ; \; x \in [x_0, x_1]\}$$

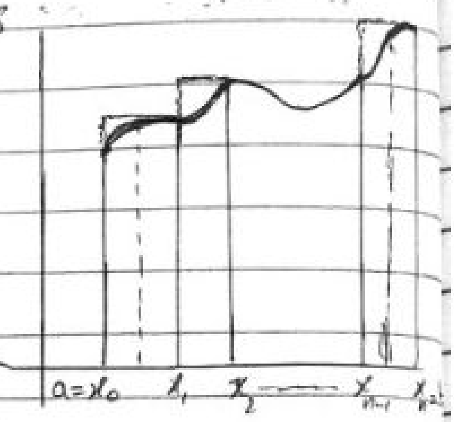
$$M_2 = \sup\{f(x) \; ; \; x \in [x_1, x_2]\}$$

$$M_3 = \sup\{f(x) \; ; \; x \in [x_2, x_3]\}$$

$$\vdots$$

$$M_n = \sup\{f(x) \; ; \; x \in [x_{n-1}, x_n]\}.$$

Then the area of the first rectangle in fig: is i.e on $[x_0, x_1]$ is



$$A_1 = M_1(x_1 - x_0) = M_1 \Delta x_1$$

similarly $A_2 = M_2(x_2 - x_1) = M_2 \Delta x_2$

$$A_3 = M_3(x_3 - x_2) = M_3 \Delta x_3$$

$$\vdots$$

$$A_n = M_n(x_n - x_{n-1}) = M_n \Delta x_n$$

Thus $A_1 + A_2 + A_3 + \cdots + A_n = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$

$$\Rightarrow A_1 + A_2 + A_3 + \cdots A_n = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} M_k \Delta x_k$$

Let us denote $A_1 + A_2 + A_3 + \cdots A_n$ by: $S(f, P)$

i.e $S(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \longrightarrow (*)$

Thus the summation $(*)$ is called the upper sum of $f(x)$ corresponding to partition P.

Sometimes it,s denoted by $S = S(f, P) = U(f, P)$

X——X——X——X——X——

Lower Sum:-

Let $f(x)$ be any real valued and bounded function on $I = [a, b]$ and let "P" be any arbitrary partition of I. i.e $P = \{x_0, x_1, x_2, \cdots, x_n\}$, Then

Consider $m_1 = \text{Inf}\{f(x) ; x \in [x_0, x_1]\}$

$m_2 = \text{Inf}\{f(x) ; x \in [x_1, x_2]\}$

$\vdots$

$m_n = \text{Inf}\{f(x) ; x \in [x_{n-1}, x_n]\}$

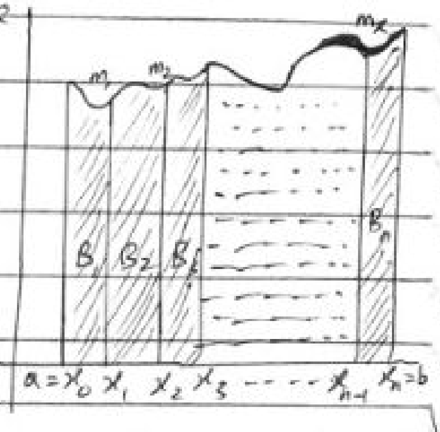
Then, area of first rectangle i.e on interval $[x_0, x_1]$ is

$B_1 = m_1(x_1 - x_0)$ similarly

$B_2 = m_2(x_2 - x_1)$

$\vdots$

$B_n = m_n(x_n - x_{n-1})$



Thus;

$B_1 + B_2 + B_3 + \cdots B_n = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$

$\Rightarrow B_1 + B_2 + B_3 + \cdots B_n = \sum_{K=1}^{n} m_K(x_K - x_{K-1}) = \sum_{K=1}^{n} m_K \Delta x_K$

Let us denote $B_1 + B_2 + B_3 + \cdots + B_n$ by $\mathcal{S}(f, P)$

then

$$\mathcal{S}(f, P) = \sum_{K=1}^{n} m_K(x_K - x_{K-1}) \longrightarrow (**)$$

The summation $(**)$ is called the lower Sum of '$f(x)$' corresponding to the partition P

Sometimes it's denoted by $\mathcal{S} = \mathcal{S}(f, P)$

$\times$ ——— $\times$ ——— $\times$ ——— $\times$ ——— $\times$

Since for given partition the upper and lower sums S and $\mathcal{S}$ are given by

$$S = \sum_{K=1}^{n} M_K \Delta x_K \quad \text{and}$$

$$\mathcal{S} = \sum_{K=1}^{n} m_K \Delta x_K,$$

where M_K and m_K are the values of lub $f(x)$ and glb $f(x)$ in $[x_{K-1}, x_K]$. ~~Now~~

Now

$$\lim_{n\to\infty} S = \lim_{\substack{n\to\infty \\ \Delta x_K \to 0}} \sum_{K=1}^{n} M_K \Delta x_K = J = \int_a^{\bar{b}} f(x)\,dx$$

and

$$\lim_{n\to\infty} s = \lim_{\substack{n\to\infty \\ \Delta x_K \to 0}} \sum_{K=1}^{n} m_K \Delta x_K = I = \int_{\underline{a}}^{b} f(x)\,dx$$

i.e $J = \text{glb}\{S\}$ and $I = \text{lub}\{s\}$ $\forall$ possible partition.

Definition:-

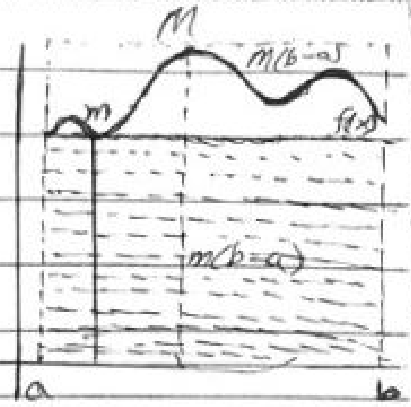
A function $f(x)$ is Riemann integrable iff $I = J$

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x_K \to 0}} \sum_{K=1}^{n} f(x_K^*)\Delta x_K$$

where $x_{K-1} \le x_K^* \le x_K$ and

$$\sum_{K=1}^{n} \Delta x_K = b - a$$ where $[a,b]$ is the given interval.

Theorem:- If $S(f,P)$ and $s(f,P)$ are the upper and lower sums of real valued function f, corresponding to partition P and let M, m are supremum and Infimum of f on $[a,b]$ respectively, then prove that $m(b-a) \le s(f,P) \le S(f,P) \le M(b-a)$



Proof: Let M_K and m_K be the lub and glb of f on K^{th} sub-interval $[x_{K-1}, x_K]$.
Also given that M and m are the lub and glb of f on $[a,b]$ respectively.
Then obviously $m \leq m_K \leq M_K \leq M$

$$\Rightarrow m(x_K - x_{K-1}) \leq m_K(x_K - x_{K-1}) \leq M_K(x_K - x_{K-1}) \leq M(x_K - x_{K-1})$$

$$\Rightarrow m\Delta x_K \leq m_K \Delta x_K \leq M_K \Delta x_K \leq M \Delta x_K$$

Taking summation $\forall$ $K = 1, 2, 3, \ldots, n$.

$$\Rightarrow \sum_{K=1}^{n} m\Delta x_K \leq \sum_{K=1}^{n} m_K \Delta x_K \leq \sum_{K=1}^{n} M_K \Delta x_K \leq \sum_{K=1}^{n} M \Delta x_K$$

$$\Rightarrow m\sum_{K=1}^{n} \Delta x_K \leq s(f,P) \leq S(f,P) \leq M \sum_{K=1}^{n} \Delta x_K \longrightarrow (*)$$

Since

$$\sum_{K=1}^{n} \Delta x_K = \sum_{K=1}^{n} (x_K - x_{K-1}) = \cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cancel{x_3} - \cancel{x_2} + \cdots x_n - \cancel{x_{n-1}}$$

$$= -x_0 + x_n = x_n - x_0$$

$$= b - a \quad \text{where } x_0 = a \text{ and } x_n = b$$

Put in $(*)$, we get:

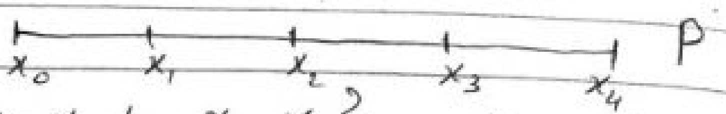
$$(*) \Rightarrow m(b-a) \leq s(f,P) \leq S(f,P) \leq M(b-a)$$

Proved.

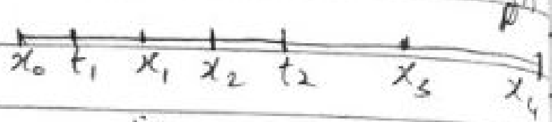
Refinement of a partition:-

By the refinement of partition 'P' we mean to add (introduce) one or more points to the given partition 'P'.

Let $P = \{x_0, x_1, x_2, x_3, x_4\}$ be a partition



Then $P^{(1)} = \{x_0, t_1, x_1, x_2, t_2, x_3, x_4\}$ is the refinement of partition P.



X ——— X ——— X ——— X ——— X ——— X

Theorem:- If $S = S(f,P)$ and $\mathcal{S} = \mathcal{S}(f,P)$ are the values of upper and lower sums of function f corresponding to partition P, and $P^{(1)}$ is the refinement of P and $S^{(1)}, \mathcal{S}^{(1)}$ are the upper and lower sums of 'f' corresponding $P^{(1)}$, then
$\mathcal{S} \le \mathcal{S}^{(1)} \le S^{(1)} \le S$.

i.e A refinement of a partition increases the lower sum and decreases the upper sum. **OR** A refinement of a partition does not increases the upper sum and does not decreases the lower sum.

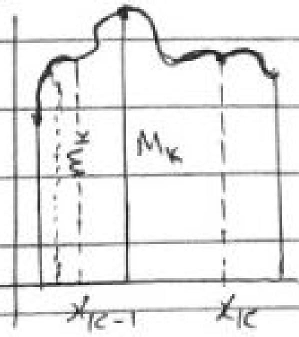
Proof, Let $P = \{a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n = b\}$ is a partition of $[a,b]$ and suppose that
$P^{(1)} = \{x_0, x_1, x_2, \ldots, x_{k-1}, u, x_k, x_{k+1}, \ldots, x_n\}$

is the refinement of 'P' and let
$S = S(f,P)$
$\mathcal{S} = \mathcal{S}(f,P)$

$\mathcal{S}^{(1)} = \mathcal{S}^{(1)}(f, P^{(1)})$

$s^{(1)} = s^{(1)}(f, P^{(1)})$

with M_k is the lub of "f" on $[x_{k-1}, x_k]$ and m_k is the glb of "f" on $[x_{k-1}, x_k] \longrightarrow$

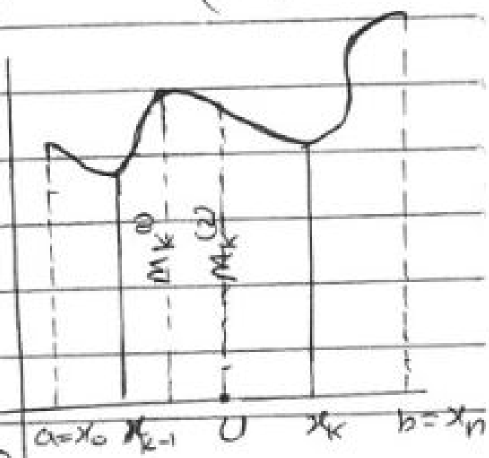


clearly $M_k(x_k - x_{k-1}) = M_k(x_k - U + U - x_{k-1})$

and $m_k(x_k - x_{k-1}) = m_k(x_k - U + U - x_{k-1})$

Since 'U' is a point introduced in $[x_{k-1}, x_k]$

Thus $[x_{k-1}, x_k] = [x_{k-1}, U] \overset{+}{\cup} [U, x_k]$ $(\cup + = \cup)$



Let $M_k^{(1)}$ is lub of f on $[x_{k-1}, U]$

and $M_k^{(2)}$ is lub of f on $[U, x_k]$

Since $M_k = \text{lub}\{f(x);\ x \in [x_{k-1}, x_k]\}$

Then obviously $M_k^{(1)} \le M_k$ and $M_k^{(2)} \le M_k$

$\Rightarrow M_k^{(1)}(U - x_{k-1}) \le M_k(U - x_{k-1}) \longrightarrow$ (i)

and $M_k^{(2)}(x_k - U) \le M_k(x_k - U) \longrightarrow$ (ii)

adding (i) and (ii) we get

$$M_k^{(1)}(U - x_{k-1}) + M_k^{(2)}(x_k - U) \le M_k(U - x_{k-1}) + M_k(x_k - U)$$
$$= M_k(\cancel{U} - x_{k-1} + x_k - \cancel{U})$$
$$= M_k(x_k - x_{k-1})$$

$$\Rightarrow M_k^{(1)}(U - x_{k-1}) + M_k^{(2)}(x_k - U) \le M_k(x_k - x_{k-1})$$

The similar result (argument) can be extended for $K = 1, 2, 3, \ldots, n$.

Thus $S_1 \leq S$ for any arbitrary refiement of P

Now we show that $s \leq s_1$

Since u is a point introduced in $[x_{k-1}, x_k]$.

Thus $[x_{k-1}, x_k] = [x_{k-1}, u] \overset{+}{\cup} [u, x_k]$ $[+ = u$

let $m_k^{(1)}$ is the Inf of 'f' on $[x_{k-1}, u]$

and $m_k^{(2)}$ is the Inf of 'f' on $[u, x_k]$

Since $m_k = \text{Inf}\{f(x) ; x \in [x_{k-1}, x_k]\}$

Then obviously $m_k^{(1)} \geq m_k$ and $m_k^{(2)} \geq m_k$

$\Rightarrow m_k^{(1)}(u - x_{k-1}) \geq m_k(u - x_{k-1}) \longrightarrow$ (iii)

and $m_k^{(2)}(x_k - u) \geq m_k(x_k - u) \longrightarrow$ (iv)

Adding (iii) and (iv)

$\Rightarrow m_k^{(1)}(u - x_{k-1}) + m_k^{(2)}(x_k - u) \geq m_k(u - x_{k-1} + x_k - u)$

$\Rightarrow m_k^{(1)}(u - x_{k-1}) + m_k^{(2)}(x_k - u) \geq m_k(x_k - x_{k-1})$

The similar result can be extended for $K = 1, 2, 3, \ldots, n$.

Thus $s_1 \geq s$ for any arbitrary refinement of P. $s \leq s_1$, Since we know from definition that $s \leq S$ and $s_1 = S_1$

$\Rightarrow s \leq s_1 \leq S_1 \leq S$, proved

Example: (1) Define $f:[0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Solo.

Then $\int_0^1 1/x \, dx$

is not defined as a Riemann integral becuse f is unbounded.

$$0 < x_1 < x_2 < \dots < x_{n-1} < 1$$

is a partition of $[0,1]$, then

$$\sup_{[0,x_1]} f = \infty$$

So the upper Riemann sums of f are not well-defined.

Example(2):- The constant function $f(x)=1$ on $[0,1]$ is Riemann integrable, and

$$\int_0^1 1 \, dx = 1$$

Soli.

To show this, let $P = \{I_1, I_2, \dots I_n\}$ be an partition of $[0,1]$ with endpoints

$$\{0, x_1, x_2, \dots, x_{n-1}, 1\}$$

Since f is constant.

$M_k = \sup_{I_k} f = 1$, $m_k = \inf_{I_k} f = 1$ for $k=1,\dots,n$

and therefore

$$U(f;P) = L(f;P) = \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = 1$$

Geometrically, this equation is the obvious fact that the sum of the areas of the rectangles over (or, equivalently, under) the graph of a constant function is exactly equal to the area under the graph. Thus, every upper and lower sum of f on $[0,1]$ is equal to 1, which implies that the upper and lower integrals

$$U(f) = \inf_{P\in\Pi} U(f;P) = \operatorname{Inf}\{1\} = 1$$

$$L(f) = \sup_{P\in\Pi} L(f;P) = \sup\{1\} = 1$$

are equal, and the integral of f is 1.

Example 3:- The function
$$f(x) = \begin{cases} 0 & \text{if } 0 < x \le 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is Riemann integrable and $\int_0^1 f\,dx = 0.$

Sol: To Show this let $P = \{I_1, I_2, \ldots I_n\}$ be a partition of $[0,1]$. Then, Since $f(x) = 0$ for $x > 0$

$$M_k = \sup_{I_k} f = 0, \quad m_k = \inf_{I_k} f = 0 \quad \text{for } k = 2, \ldots n$$

The first interval in the partition is $I_1 = [0, x_1]$, where $0 < x_1 \le 1$ and

$M_1 = 1$, $m_1 = 0$

Since $f(0) = 1$ and $f(x) = 0$ for $0 < x \le x_1$, it follows that

$$U(f;P) = x_1 \quad , \quad L(f;P) = 0.$$

Thus, $L(f) = 0$ and

$$U(f) = \inf\{x_1 : 0 < x_1 \le 1\} = 0$$

Example: 4: The Dirichlet function $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Solution:

That is, f is one at every rational number and zero at every irrational numbers. This function is not Riemann integrable. If $P = \{I_1, I_2, \dots I_n\}$ is a partition of $[0,1]$, then

$$M_k = \sup_{I_k} f = 1 \quad , \quad m_k = \inf_{I_k} f = 0$$

Since every interval of non-zero length contains both rational and irrational numbers. It follows that

$$U(f;P) = 1 \quad , \quad L(f;P) = 0$$

for every partition P of $[0,1]$, so $U(f) = 1$ and $L(f) = 0$ are not equal. The Dirichlet function is discontinous at every point of $[0,1]$, and the moral of the last example is that the Riemann integral of a highly discontinous function need not exist.

Theorem:- If $\overline{S}_2 = \overline{S}(f;P_2)$ and $\underline{S}_2 = \underline{S}(f;P_2)$ are upper and lower sums of f on $[a,b]$ corresponding to partition P_2 and $\overline{S}_3 = \overline{S}(f;P_3)$, $\underline{S}_3 = \underline{S}(f;P_3)$ are upper and lower sums of f on $[a,b]$ corresponding to partition P_3. then

$$\underline{S}_3 \le \overline{S}_2 \quad \text{and} \quad \underline{S}_2 \le \overline{S}_3$$

OR :- Any lower sum is never greater than any upper sum regardless of partition used.

Proof:- Since $\overline{S}_2, \underline{S}_2$ are the upper and lower sums of 'f' corresponding to the partition P_2 and $\overline{S}_3, \underline{S}_3$ are upper and lower sums of 'f' corresponding to partition P_3 we prove that $\underline{S}_3 \le \overline{S}_2$ and $\underline{S}_2 \le \overline{S}_3$

There are two cases.

Case (I):-

when the partitions are same i.e $P_2 = P_3 = P$ the the result is obvious.
i.e $\underline{S}_2 = \underline{S}_3 = \underline{S}$ and $\overline{S}_2 = \overline{S}_3 = \overline{S}$
but from definition we know that

$$\underline{S} \le \overline{S}.$$

Hence we can write $\underline{S}_2 \le \overline{S}_3$ and $\underline{S}_3 \le \overline{S}_2$

Case (II):-

when partitions P_2 and P_3 are different partitions on $I = [a,b]$
Let us construct a new partition P_4 from P_2 and P_3, then P_4 is the refinement of P_2 and P_3
Let $\overline{S}_4 = \overline{S}(f;P_4)$ and $\underline{S}_4 = \underline{S}(f;P_4)$

are the upper and lower sums of 'f' corresponding to partition P_4.

Then $\left.\begin{array}{l} s_3 \le s_4 \\ S_4 \le S_3 \end{array}\right\} \longrightarrow$ (i)

and $\left.\begin{array}{l} s_2 \le s_4 \\ S_4 \le S_2 \end{array}\right\} \longrightarrow$ (ii)

by previous theorem

Since from (i) $s_3 \le s_4$ but $s_4 \le S_4$ (obvious)

$\Rightarrow s_3 \le s_4 \le S_4 \longrightarrow$ (iii)

but by (ii) $S_4 \le S_2 \longrightarrow$ (iv)

So (iii), (iv) $\Rightarrow s_3 \le S_4 \le S_2 \Rightarrow \boxed{s_3 \le S_2}$

Also from (ii) $s_2 \le s_4$ but $s_4 \le S_4$

$\Rightarrow s_2 \le s_4 \le S_4$

$\Rightarrow s_2 \le S_4 \longrightarrow$ (v)

but from (i) $S_4 \le S_3 \longrightarrow$ (vi)

by (v), (vi) $\Rightarrow s_2 \le S_4 \le S_3$

$\Rightarrow \boxed{s_2 \le S_2}$

This completes the proof.

Riemann Integration :-

Let P is the set of all possible partitions of $[a,b]$

i.e. $P = \{P_i$; where P_i is a partition of $[a,b]$; i belongs to some indexing familly.

$= \{P_1, P_2, P_3, \ldots, P_n, \ldots\}$

let $I = \sup\{S_i = S(f;P_i)\ ;\ P_i \in P\}$

$J = \inf\{s_i = s(f;P_i)\ ;\ P_i \in P\}$.

Definition:-

The function f is said to be Riemann integrable on $[a,b]$ if $I = J$ and is denoted as

$$I = J = \int_a^b f(x)\,dx.$$

Theorem:- A necessary sufficient condition for a bounded function $f(x)$ to be Riemann integrable on $[a,b]$ is that for every $\epsilon > 0$ there exists a partition say P_ϵ with the corresponding upper and lower sums $S = S(f;P_\epsilon)$ and $\mathcal{S} = \mathcal{S}(f;P_\epsilon)$ such that $S - \mathcal{S} < \epsilon$.

Proof:- Necessary condition:-

Let "f" is Riemann integrable on $[a,b]$ i.e $I = J$.

we show that for any $\epsilon > 0$, there exists partition P_ϵ such that

$$S - \mathcal{S} < \epsilon.$$

where $S = S(f;P_\epsilon)$, $\mathcal{S} = \mathcal{S}(f;P_\epsilon)$

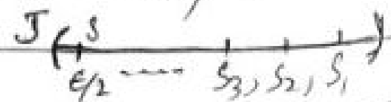
Since we know that

$$J = \text{Inf}\{S_i = S(f;P_i)\ ;\ P_i \in P\}$$

or $J = \text{Inf}(glb)\{S_1, S_2, S_3, \ldots\}$

Now by definition of Infimum (glb) $\exists$ some $S \in \{S_1, S_2, S_3, \ldots, S_n, \ldots\}$ such that

$$J + \epsilon/2 > S \rightarrow (i)$$



similarly, for $I = \sup\{\mathcal{S}_i = \mathcal{S}(f;P);\ P_i \in P\}$

by definition of supremum $\exists$
Some $s \in \{s_1, s_2, s_3, \ldots\}$
such that $I - \epsilon/2 < s$

$$\Rightarrow -I + \epsilon/2 > -s \longrightarrow (ii)$$

Adding (i) and (ii); we get

$$J + \epsilon/2 - I + \epsilon/2 > S - s$$

$$\Rightarrow J - I + \cancel{2}\epsilon/\cancel{2} > S - s$$

$$\Rightarrow 0 + \epsilon > S - s \quad \because J = I \Rightarrow J - I = 0$$

or $\boxed{S - s < \epsilon}$

This completes half of the Theorem.

Suffecient condition:-

Let for any $\epsilon > 0$ there exists some partition say P_ϵ such that

$$S - s < \epsilon \;;\; S = S(f, P_\epsilon),\; s = s(f, P_\epsilon)$$

we prove ~~p~~ that $I = J$.

Since by definition of I and J, we have;

$$J \le S \longrightarrow (*) \quad s \le I \Rightarrow -s \ge -I \longrightarrow (**)$$

adding $(*)$ and $(**)$; we get

$$J - I \le S - s$$

but $S - s < \epsilon$ (given)

so $J - I \le S - s < \epsilon$

$$\Rightarrow J - I < \epsilon \longrightarrow (R)$$

we know that $J \ge I$ (always true)

$$\Rightarrow J = I$$

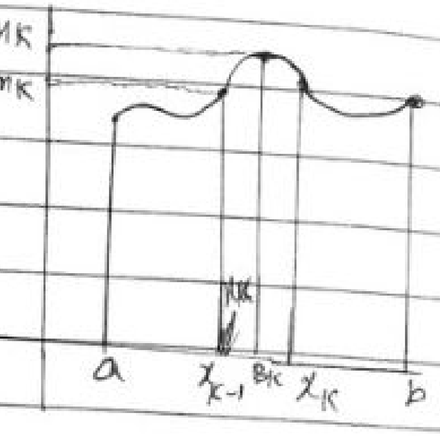
$\therefore$ f is Riemann integrable on $[a, b]$.

Asim Marwat.
university of peshawar
MSc Mathematics (Previous)
cell 03151944572 / 03321944772.

Theorem: If f is continuous on $[a,b]$, Then f is Riemann integrable on $[a,b]$.

Proof:
Given that f is continous on $[a,b]$. we prove that f is Riemann integrable on $[a,b]$.
For This let $\epsilon > 0$ and 'P' is any arbitary partition of $[a,b]$.
i.e $P = \{ x_0 = a, x_1, x_2, \dots, x_n = b \}$.

Let $[x_{K-1}, x_K]$ is the K^{th} arbitrary sub-interval of $[a,b]$.



Since f is continous on $[a,b]$
So f will be continous on $[x_{K-1}, x_K]$; $K = 1, 2, 3, \dots n$
since $[x_{K-1}, x_K]$ is closed interval.
So 'f' must attains it,s bounds on $[x_{K-1}, x_K]$
i.e There must exists $\alpha_K, \beta_K \in [x_{K-1}, x_K]$ such that $f(\alpha_K) = m_K$ and $f(\beta_K) = M_K$
where m_K and M_K are the glb and lub of 'f' on $[x_{K-1}, x_K]$ respectively.
Also, every continuous function on a closed interval is always uniformly continous on $[a,b]$.
So 'f' must be uniformly continuous on $[x_{K-1}, x_K]$, $K = 1, 2, 3, \dots, n$.
Therefore for $\epsilon' = \epsilon/_{b-a}$ as $\epsilon > 0$, $b-a>$

$$\Rightarrow \frac{\epsilon}{b-a} = \epsilon' > 0$$

There exists $\delta > 0$ such that

$|f(\beta_K) - f(\alpha_K)| < \epsilon'$ $\forall$ $|\beta_K - \alpha_K| < \delta$ $\because |f(x)-f(y)| < \epsilon$ when ever $|x-y| < \delta$

$\Rightarrow |M_K - m_K| < \epsilon'$ $\forall$ $|\beta_K - \alpha_K| < \delta$

$$\Rightarrow M_K - m_K < \epsilon'$$

$$\Rightarrow (M_K - m_K)(x_K - x_{K-1}) < \epsilon'(x_K - x_{K-1})$$

$$\Rightarrow \sum_{K=1}^{n} (M_K - m_K)(x_K - x_{K-1}) < \sum_{K=1}^{n} \epsilon'(x_K - x_{K-1})$$

$$\Rightarrow \sum_{K=1}^{n} (M_K - m_K)\Delta x_K < \epsilon' \sum_{K=1}^{n} \Delta x_K$$

$$\Rightarrow \sum_{K=1}^{n} M_K \Delta x_K - \sum_{K=1}^{n} m_K \Delta x_K < \frac{\epsilon}{b-a} \times (b-a)$$

$\Rightarrow S - s < \epsilon$, where $S = S(f,P)$, $s = s(f,P)$

Thus by previous Theorem,
'f' is Riemann integrable on $[a,b]$.
Thus every continous function is Riemann integrable

Theorem:- If 'f' and 'g' are Riemann integrable functions on $[a,b]$, then $f+g$ is also Riemann-integrable on $[a,b]$, and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof:- Given that f and g are

Riemann - integrable on $[a,b]$

$\Rightarrow$ f and g are bounded on $[a,b]$

$\Rightarrow$ $f+g$ is also bounded on $[a,b]$

Let m_k, M_k are the lower and upper bounds of 'f' on $[x_{k-1}, x_k]$, where $[x_{k-1}, x_k]$ is the k^{th} sub-interval of $[a,b]$ corresponding to partition $P = \{x_0 = a, x_1, x_2, \dots, x_n = b\}$ and let $m_k^{(1)}, M_k^{(1)}$ are the lower and upper bounds of 'g' on $[x_{k-1} - x_k]$ respectively and $m_k^{(2)}, M_k^{(2)}$ are lower and upper bounds of $f+g$ on $[x_{k-1}, x_k]$ respectively.

By definition $m_k \le f(x) \le M_k$; $\forall x \in [x_{k-1}, x_k]$ $\to$ (i) , $k = 1, 2, \dots n$

$m_k^{(1)} \le g(x) \le M_k^{(1)} \to$ (ii) ; $\forall x \in [x_{k-1}, x_k]$, $k = 1, 2, 3, \dots n$

$m_k^{(2)} \le f(x) + g(x) \le M_k^{(2)}$; $\forall x \in [x_{k-1}, x_k]$ $k = 1, 2, 3, \dots n$

Now (i) + (ii) ; $m_k + m_k^{(1)} \le f(x) + g(x) \le M_k + M_k^{(1)}$ $\to$ (iv)

By (iii), (iv) ; $m_k + m_k^{(1)} \le m_k^{(2)} \to$ (v) ; $m_k^{(2)}$ = glb of $f+g$

and $M_k^{(2)} \le M_k + M_k^{(1)} \to$ (vi)

(v) $\Rightarrow$ $\sum_{k=1}^{n} (m_k + m_k^{(1)}) \Delta x_k \le \sum_{k=1}^{n} m_k^{(2)} \Delta x_k$

$$\Rightarrow \sum_{k=1}^{n} m_k \Delta x_k + \sum_{k=1}^{n} m_k^{(1)} \Delta x_k \le \sum_{k=1}^{n} m_k^{(2)} \Delta x_k$$

$$\Rightarrow s + s^{(1)} \le s^{(2)} \longrightarrow (*)$$

where $s = s(f, P)$

$s^{(1)} = s(g, P)$

$s^{(2)} = s(f+g, P)$

Similarly (vi) $\Rightarrow S^{(2)} \le S + S^{(1)} \longrightarrow (**)$

where $S^{(2)} = S(f+g, P)$

$S = S(f, P)$

$S^{(1)} = S(g, P)$

Since P is arbitrary thus $P = P_i$; where i belongs to some indexing family

$(*)\,(**) \Rightarrow s_i + s^{(1)} \le s_i^{(2)}$
and $S_i^{(2)} \le S_i + S_i^{(1)}$ $\Big\} \longrightarrow (A)$

let $I = \sup\{s_i = s(f, P_i);\ P_i \in P\}$

$J = \inf\{S_i = S(f, P_i);\ P_i \in P\}$

$I^{(1)} = \sup\{s_i = s(g, P_i);\ P_i \in P\}$

$J^{(1)} = \inf\{S_i = S(g, P_i);\ P_i \in P\}$

$I^{(2)} = \sup\{s_i = s(f+g, P_i);\ P_i \in P\}$

$J^{(2)} = \inf\{S_i = S(f+g, P_i);\ P_i \in P\}$

(A) $\Rightarrow I + I^{(1)} \leq I^{(2)}$ and $J^{(2)} \leq J + J^{(1)} \rightarrow$ (B)

but f, g are Reimann integrable

(B) $\Rightarrow J + J^{(1)} \leq I^{(2)}$ and $J^{(2)} \leq J + J^{(1)}$

$\Rightarrow J^{(2)} \leq J + J^{(1)} \leq I^{(2)} \longrightarrow (***)$

$\Rightarrow J^{(2)} \leq I^{(2)}$

and we know that $I^{(2)} \leq J^{(2)}$

Thus $J^{(2)} = I^{(2)}$

$\Rightarrow$ $f+g$ is Riemann integrable on $[a,b]$ putting $I^{(2)} = J^{(2)}$ in $(***)$

$(***) \Rightarrow J^{(2)} \leq J + J^{(1)} \leq J^{(2)}$

$\Rightarrow J^{(2)} = J + J^{(1)}$

$$\Rightarrow \int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

proved

Theorem:- Every bounded monotonic function is Riemann integrable.

Proof:- Let f is a bounded increasing (monotonic) function on $[a,b]$. let $\epsilon > 0$ and P is partition of choose in such that

$$l([x_{k-1}, x_k]) < \frac{\epsilon}{f(b) - f(a)} \longrightarrow (*)$$

Since f is inc.
So $f(b) = M$ as
$f(a) = m$
$\because f(b) - f(a)$

where $[x_{k-1}, x_k]$ is the k^{th} sub-interval of $[a,b]$ corresponding to partition P.

i.e $P = \{x_0 = a, x_1, x_2, \ldots x_n = b\}$; $k=1,2,3,\ldots,n$

let m_k and M_k are the glb and lub of 'f' on $[x_{k-1}, x_k]$; $k = 1, 2, 3, \ldots, n$, and S, s are the upper and lower sums of 'f' corresponding to partition P.

i.e $S = S(f,P)$; $s = s(f,P)$

$$\Rightarrow S = \sum_{k=1}^{n} M_k \Delta x_k, \quad s = \sum_{k=1}^{n} m_k \Delta x_k$$

Since f is increasing function on $[a,b]$ so f is increasing on $[x_{k-1}, x_k]$

Thus

$f(x_{k-1}) = m_k$, $f(x_k) = M_k$

Now consider

$$S - s = \sum_{k=1}^{n} M_k \Delta x_k - \sum_{k=1}^{n} m_k \Delta x_k$$

$$= \sum_{k=1}^{n} (M_k - m_k) \Delta x_k$$

$$= \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right) \cdot \Delta x_k$$

$$< \sum_{k=1}^{n} \left(f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})\right) \times \frac{\epsilon}{f(b)-f(a)}$$

$$\Rightarrow S - s < \left(f(x_n) - f(x_0)\right) \times \frac{\epsilon}{f(b) - f(a)}$$

$\Rightarrow \cancel{\rho}\, S - s < \frac{f(b)-f(a)}{f(b)-f(a)} \times \epsilon$

$\Rightarrow S - s < \epsilon$

So by previous theorem "f" is R-integrable on [a,b].

Theorem:- If 'f' and 'g' are R-integrable on [a,b] and $f(x) \leq g(x)$; $\forall\ x \in [a,b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)\,dx.$$

Proof:-

Since given that "f" and "g" are R-integrable on [a,b], Such that $f(x) \leq g(x)$ we need to prove that

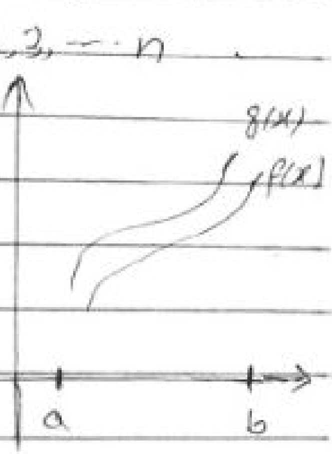
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

For this let P be any arbitarary partition of [a,b] such that

$P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$

and let $[x_{k-1}, x_k]$ is the K^{th} sub-interval of [a,b] corresponding to partition 'P' and let m_K, M_K are the glb & lub of f on $[x_{K-1}, x_K]$; $K = 1,2,3,\ldots,n$

and $m_k^{(1)}, M_k^{(1)}$ are the glb & lub of 'f' on $[x_{k-1}, x_k]$; $k=1,2,3,\dots n$.



Since $f(x) \le g(x)$; $\forall\ x \in [a,b]$

so $f(x) \le g(x)$; $\forall\ x \in [x_{k-1}, x_k]$ (obvious) ; $k=1,2,3,\dots,n$

$\Rightarrow m_k \le m_k^{(1)}$ and $M_k \le M_k^{(1)}$; $k=1,2,\dots,n$

$\Rightarrow m_k \Delta x_k \le m_k^{(1)} \Delta x_k$ and $M_k \Delta x_k \le M_k^{(1)} \Delta x_k$

$\Rightarrow \sum\limits_{k=1}^{n} m_k \Delta x_k \le \sum\limits_{k=1}^{n} m_k^{(1)} \Delta x_k$ and $\sum\limits_{k=1}^{n} M_k \Delta x_k \le \sum\limits_{k=1}^{n} M_k^{(1)} \Delta x_k$

$\Rightarrow s(f,P) \le s(g,P)$ and $S(f,P) \le S(g,P) \rightarrow (*)$

As for any arbitrary partition P (*) holds. Thus from (*) we conclude that $I \le I^{(1)}$ and $J \le J^{(1)}$

where

$I = \sup\{ s = s(f,P)$; where P is arbitrary partition of $[a,b]\}$

$I^{(1)} = \sup\{ s = s(g,P)$; " " " $\}$

$J = \inf\{ S = S(f,P)$; " " " $\}$

$J^{(1)} = \inf\{ S = S(g,P)$; " " " $\}$

Since it is given that f and g are R-integrable, Thus

$I^{(1)} = \int_a^b g(x)\,dx = J^{(1)}$ and $I = \int_a^b f(x)\,dx = J$

Thus ⊛ $\Rightarrow I \le I^{(1)} \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

or $J \le J^{(1)} \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Theorem:- If 'f' is R-integrable on $[a,b]$ and has upper and lower bounds M and m in $[a,b]$, Then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

Proof:- Since "f" is Riemann integrable on $[a,b]$ and M, m are upper and lower bounds of 'f' on $[a,b]$.

we prove that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Since M, m are the upper and lower bounds of 'f' on $[a,b]$,

So by definition of upper and lower bounds

$$m \le f(x) \le M \rightarrow ① \quad \forall x \in [a,b]$$

let P is any arbitrary partition of $[a,b]$

i.e $P = (x_0 = a, x_1, x_2, \ldots, x_n = b)$.

let $[x_{k-1}, x_k]$ be any k^{th} sub-interval of $[a,b]$ corresponding to the partition P.

Let m_k, M_k are the glb and lub

of 'f' on $[x_{k-1}, x_k]$; $k = 1, 2, 3, \dots, n$.
So by definition of glb and lub

$$m_k \le M_k \longrightarrow (2)$$

So from (1) and (2)

$$\Rightarrow m \le m_k \le M_k \le M$$

Now taking Summation.

$$\Rightarrow \sum_{k=1}^{n} m \Delta x_k \le \sum_{k=1}^{n} m_k \Delta x_k \le \sum_{k=1}^{n} M_k \Delta x_k \le \sum_{k=1}^{n} M \Delta x_k$$

$$\Rightarrow m \sum_{k=1}^{n} \Delta x_k \le \sum_{k=1}^{n} m_k \Delta x_k \le \sum_{k=1}^{n} M_k \Delta x_k \, M \sum_{k=1}^{n} \Delta x_k$$

$$\Rightarrow m(b-a) \le \underline{S} = \underline{S}(P_i f) \le \overline{S} = \overline{S}(P_i f) \le M(b-a)$$

$$\Rightarrow m(b-a) \le \underline{S} \le \overline{S} \le M(b-a)$$

Since P is any arbitrary partition on $[a,b]$
Thus for any $P_i \in P$; where P is the set of partition of $[a,b]$, So to generalize the above, we have

$$m(b-a) \le \underline{S}_i \le \overline{S}_i \le M(b-a) \rightarrow (3)$$

Since we have:

$$I = \sup \{ \underline{S}_i = \underline{S}_i(P_i f) \, ; P_i \in P \}$$

$$J = \inf \{ \overline{S}_i = \overline{S}_i(P_i f) \, ; P_i \in P \}$$

So (3) $\Rightarrow m(b-a) \le I \le J \le M(b-a)$

Since 'f' is Riemann integrable on $[a,b]$, So $I = J$

So $m(b-a) \leq I = J \leq M(b-a)$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \quad \because I = J = \int_a^b f(x)dx$$

This representing R.R

Theorem:- If 'f' is Riemann integrable on $[a,b]$, then cf is also Riemann integrable on $[a,b]$; where c is any real Number.

Proof:- Since $c \in \mathbb{R}$, So we have three possibilities

(i) $c = 0$, (ii) $c > 0$, (iii) $c < 0$

Case (i) if $c = 0$

then $cf = 0$

So in this case cf is obviously R-integrable because $I = J = 0$

Case (ii) if $c > 0$

Let P is any partition of $[a,b]$ i.e $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ and let $[x_{k-1}, x_k]$ is the k^{th} sub-interval of $[a,b]$ corresponding to partition P.

let m_k, M_k are the glb & lub of 'f' on $[a,b]$

$$\because S = S(f,P) = \sum_{k=1}^{n} M_k \Delta x_k \text{ and } s = s(f,P) = \sum_{k=1}^{n} m_k \Delta x_k$$

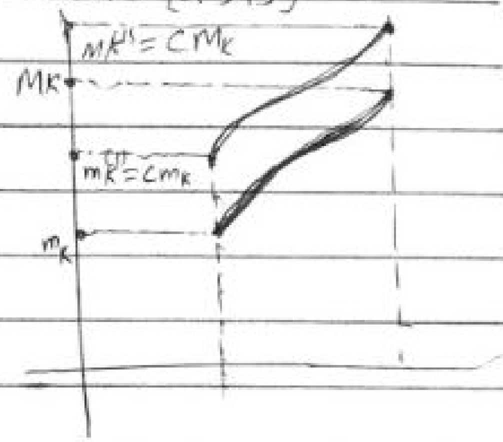
are the lower & upper sums of f corresponding to P.
and let $m_K^{(1)}, M_K^{(1)}$ are the glb & lub of Cf on $[x_{K-1}, x_K]$ and

$S^{(1)} = S^{(1)}(cf, P) = \sum_{K=1}^{n} M_K^{(1)} \Delta x_K$ and

$s^{(1)} = s^{(1)}(cf, P) = \sum_{K=1}^{n} m_K^{(1)} \Delta x_K$ are the upper and lower sums of Cf on $[a, b]$

As $C \geq 0$, the clearly

$M_K^{(1)} = CM_K$ and $m_K^{(1)} = Cm_K$



$\Rightarrow \sum_{K=1}^{n} M_K^{(1)} \Delta x_K = C \sum_{K=1}^{n} M_K \Delta x_K$

and $\sum_{K=1}^{n} m_K^{(1)} \Delta x_K = C \sum_{K=1}^{n} m_K \Delta x_K$

$\Rightarrow s^{(1)} = Cs$ and $S^{(1)} = CS \rightarrow (*)$

but P is any arbitrary partition,

Thus $(*) \Rightarrow I^{(1)} = CI$ and $J^{(1)} = CJ \rightarrow (**)$

where

$I = \sup\{ s : s = s(f, P) ; P$ is arbitrary partition$\}$

$I^{(1)} = \sup\{ s : s = s(cf, P) ;$ " " " $\}$

$J = \inf\{ S : S = S(f, P) ;$ " " " $\}$

$J^{(1)} = \inf\{ S : S = S(cf, P) ;$ " " " $\}$

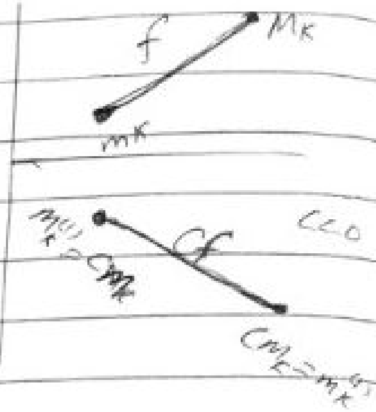
Since f is R-integrable.

So $I = J$

$\Rightarrow I^{(1)} = CI$ and $J^{(1)} = CI$

$\Rightarrow I^{(1)} = J^{(1)} \Rightarrow Cf$ is R-integrable.

Case iii $C < 0$, Then, from figure:



$M_k^{(1)} = C m_k$ and $m_k^{(1)} = C M_k$

$$\Rightarrow \sum_{k=1}^{n} M_k^{(1)} \Delta x_k = C \sum_{k=1}^{n} m_k \Delta x_k$$

and $\sum_{k=1}^{n} m_k^{(1)} \Delta x_k = C \sum_{k=1}^{n} M_k \Delta x_k$

$\Rightarrow S^{(1)} = C s$ and $s^{(1)} = C S \rightarrow (1)$

As P is any arbitrary partition of $[a,b]$
Thus for any $P_i \in P$ (1) can be written as:
$S_i^{(1)} = C s_i$ and $s_i^{(1)} = C S_i \rightarrow (2)$

$\Rightarrow J^{(1)} = C I$ and $I^{(1)} = C J \rightarrow (3)$

Since 'f' is R-integrable on $[a,b]$,
so $I = J$
Thus from (3) $\Rightarrow J^{(1)} = I^{(1)}$
$\Rightarrow CI = CJ$; $C < 0$
$\Rightarrow$ 'Cf' is R-integrable on $[a,b]$

Theorem:- If $f(x)$ is R-integrable on $[a,b]$, then $|f(x)|$ is also R-integrable on $[a,b]$, and

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

Proof:- Given that $f(x)$ is R-integrable on $[a,b]$. we have prove that $|f(x)|$ is also R-integrable on $[a,b]$

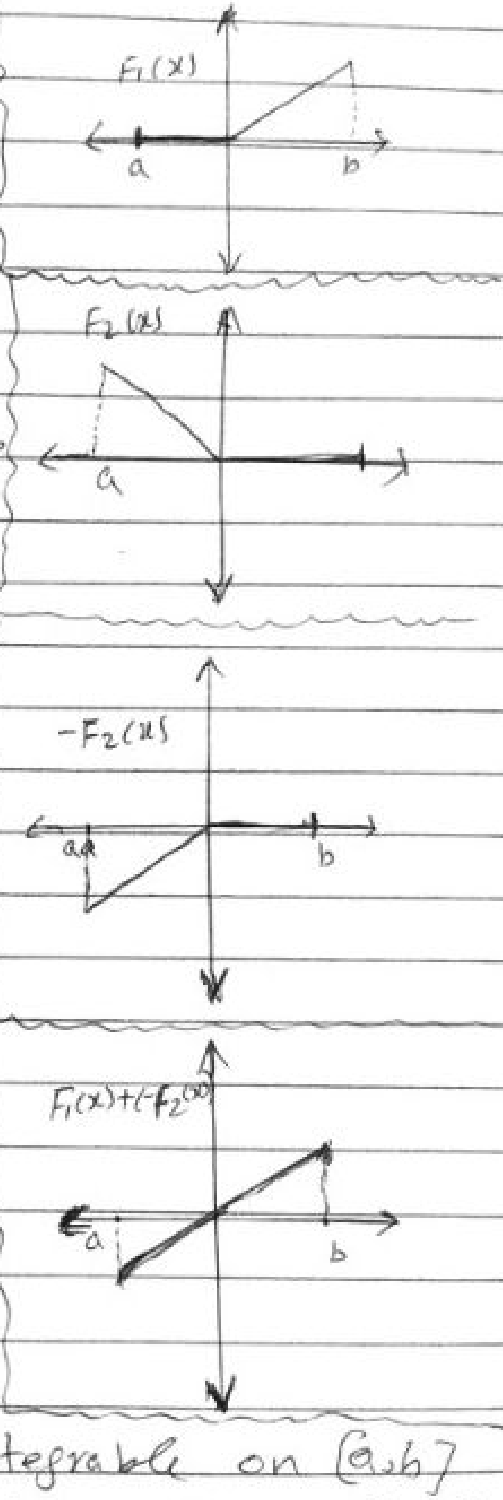
Date: 19/6/20 (151) Day

Now to prove the required result, let us define the function $F_1(x)$ & $F_2(x)$ as under

$$F_1(x) = \begin{cases} f(x) ; & f(x) \geq 0 \\ 0 ; & f(x) < 0 \end{cases}$$

and

$$F_2(x) = \begin{cases} -f(x) ; & f(x) \leq 0 \\ 0 ; & f(x) \geq 0 \end{cases}$$



we can clearly observe from the figure that

$F_1(x) - F_2(x) = f(x) \rightarrow$ (i)

and $F_1(x) + F_2(x) = |f(x)| \rightarrow$ (ii)

Since $f(x)$ is R-integrable on $[a,b]$,

So $F_1(x)$ and $F_2(x)$ are also R-integrable on $[a,b]$.

Now by theorem "which states that if f and g R-integrable on $[a,b]$, then '$f+g$' is also R-integrable on $[a,b]$"

Thus using this theorem, we can say that

$F_1(x) + F_2(x)$ is R-integrable on $[a,b]$

So (ii) $\Rightarrow |f(x)|$ is R-integrable on $[a,b]$

Next we prove that $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

Since

Babar Register

$$|f(x)| = F_1(x) + F_2(x)$$

$$\Rightarrow \int_a^b |f(x)|\,dx = \int_a^b (F_1(x) + F_2(x))\,dx$$

$$\Rightarrow \int_a^b |f(x)|\,dx = \int_a^b F_1(x)\,dx + \int_a^b F_2(x)\,dx \rightarrow (*)$$

Now (i) $\Rightarrow \int_a^b f(x)\,dx = \int_a^b (F_1(x) - F_2(x))\,dx$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| = \left|\int_a^b (F_1(x) - F_2(x))\,dx\right|$$

$\because |A-B| \leq |A|+|B|$

$$\leq \left|\int_a^b F_1(x)\,dx\right| + \left|\int_a^b F_2(x)\,dx\right|$$

Since $F_1(x)$ and $F_2(x)$ both are non-negative functions, So

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b F_1(x)\,dx + \int_a^b F_2(x)\,dx$$

$$= \int_a^b |f(x)|\,dx \quad \because \text{using } (*)$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

ASIM

Fundamental Theorem of Calculus :-

Statement:-

Let 'f' is a R - integrable function on [a, b] and suppose that there exists a function F(x) continuous on [a, b] such that $F'(x) = f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

if $b = x$, then

$$\int_a^x f(u)\,du = F(x) - F(a)$$

Proof:- Given that f is R - integrable function on [a, b].

Let $P = \{x_0 = a, x_1, x_2, x_3, \ldots, x_n = b\}$ is a partition of [a, b] and $[x_{k-1}, x_k]$ is the K^{th} - sub - interval of [a, b] corresponding to P and let m_k, M_k are the glb and lub of f on $[x_{k-1}, x_k]$. where $k = 1, 2, \ldots, n$

Since 'F(x)' is continous on [a, b],

Therefore 'F(x)' is continuous on $[x_{k-1}, x_k]$.

Also, it is given that $F'(x) = f(x)$

i.e $F'(x)$ exists on [a, b]

$\Rightarrow$ $F'(x)$ exist on (a, b).

$\Rightarrow$ F(x) is differentiable on (x_{k-1}, x_k)

Thus F(x) satisfies all the conditions of Lagrange's Mean value Theorem on $[x_{k-1}, x_k]$

Thus by L.M.V.T there exists $C_k \in (x_{k-1}, x_k)$

Such that

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(c_k) \longrightarrow (*)$$

$$\Rightarrow F(x_k) - F(x_{k-1}) = F'(c_k)(x_k - x_{k-1})$$

Also $F'(x) = f(x) \Rightarrow F'(c_k) = f(c_k)$

So $F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$; $k = 1, 2, \dots, n$.

Taking summation

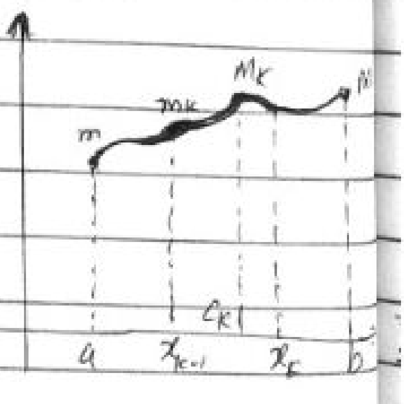
$$\Rightarrow \sum_{k=1}^{n} \left(F(x_k) - F(x_{k-1})\right) = \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$$

$$\Rightarrow F(\cancel{x_1}) - F(x_0) + F(\cancel{x_2}) - F(\cancel{x_1}) + \dots + F(x_n) - F(\cancel{x_{n-1}}) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

$$\Rightarrow -F(x_0) + F(x_n) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

$$\Rightarrow F(b) - F(a) = \sum_{k=1}^{n} f(c_k)\Delta x_k \longrightarrow (**)$$

since $c_k \in (x_{k-1}, x_k)$



$$\Rightarrow m_k \le f(c_k) \le M_k$$

$$\Rightarrow \sum_{k=1}^{n} m_k \Delta x_k \le \sum_{k=1}^{n} f(c_k)\Delta x_k \le \sum_{k=1}^{n} M_k \Delta x_k$$

$$\Rightarrow \underline{S} = \underline{S}(f, P) \le \sum_{k=1}^{n} f(c_k)\Delta x_k \le \overline{S} = \overline{S}(f, P)$$

$$\Rightarrow \underline{S} \le \sum_{k=1}^{n} f(c_k)\Delta x_k \le \overline{S} \longrightarrow (***)$$

let $I = \operatorname{Sup}\{\underline{S} = \underline{S}(f, P)$; P is any arbitrary partition of $[a$

$J = \inf\{\, S = S(f,P) \;;\; \text{" " " "} \,\}$

As for arbitrary partition we obtaine $(\ast\ast\ast)$

Thus from $(\ast\ast\ast)$ we conclude that

$$I \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq J \longrightarrow \boxed{\ast\ast\ast\ast}$$

Since 'f' is R-integrable on $[a,b]$

So $I = J = \int_a^b f(x)\,dx$

Thus $(\ast\ast\ast\ast) \Rightarrow I \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq I$

$$\Rightarrow \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

$$\Rightarrow \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx$$

Thus $(\ast\ast) \Rightarrow F(b) - F(a) = \int_a^b f(x)\,dx$

By putting $b = x$; we get

$$F(x) - F(b) = \int_a^x f(x)\,dx$$ Complete.

Theorem: prove that every constant function is Rieman integrable.

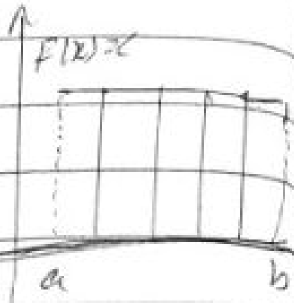
Proof:-

Let $f(x) = C$, where C is constant No.

Now for all partitions

$$S = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} C\Delta x_k = C(b-a)$$

and $S = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} C \Delta x_k = C(b-a)$



$\Rightarrow J = C(b-a)$ & $I = C(b-a)$

Thus $I = J = \int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a)$

Hence constant function is R- integrable.

Assinments:-

Q.No:(1)

Show that $f(x) = x$ is R- Integrable on $[0,1]$

Proof:- Here we have give that
$f(x) = x \; ; \; \forall \; x \in [0,1]$.
we are going to prove that $f(x)$ is R- Integrable on $[0,1]$
For this,
Let "P" be the arbitrary partition of $[0,1]$ such that the partitioning points of "P" are
i.e $P = (0 = x_0, x_1, x_2, \ldots, x_n = 1)$
Now Let $[x_{k-1}, x_k]$ be the k^{th} Sub-interval of $[0,1]$ due to partition P such that
$$(x_k - x_{k-1}) < \epsilon \; ; \; \epsilon > 0$$
Now let
$$M_k = \sup\{f(x) \; ; \; x \in [x_{k-1}, x_k]\}$$

and $m_K = \text{Inf}\{f(x) ; x \in [x_{K-1}, x_K]\}$

Let "S" & "s" are the upper & lower sums of "f(x)" corresponding to the partition "P" then

$$S = S(f, P) = \sum_{K=1}^{n} M_K \Delta x_K$$

and

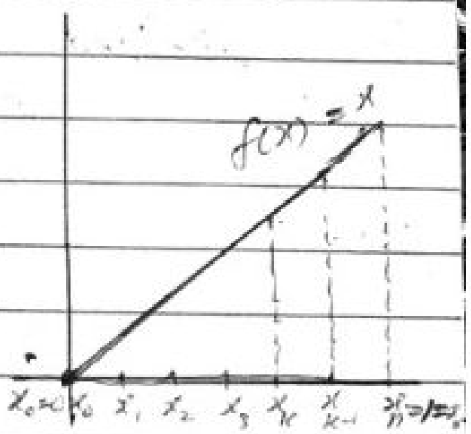
$$s = s(f, P) = \sum_{K=1}^{n} m_K \Delta x_K$$

are the upper and lower.
Now since the given function is defined on closed interval, so it is bounded and will its bounds, we also know that the given function is an increasing function.
So we can write

$M_K = x_K \longrightarrow (*_1)$

and

$m_K = x_{K-1} \longrightarrow (*_2)$



Now

$(*_1) - (*_2)$ we get

$M_K - m_K = x_K - x_{K-1}$

Taking Summation we have;

$$\sum_{K=1}^{n} (M_K - m_K) \Delta x_K = \sum_{K=1}^{n} (x_K - x_{K-1}) \Delta x_K$$

$$\Rightarrow \sum_{K=1}^{n} M_K \Delta x_K - \sum_{K=1}^{n} m_K \Delta x_K = (x_K - x_{K-1}) \sum_{K=1}^{n} \Delta x_K$$

$\Rightarrow S - s = (x_k - x_{k-1})(x_n - x_0)$

$\Rightarrow S - s < \epsilon(1-0)$

$\Rightarrow S - s < \epsilon$

$\because P$ is any arbitrary partition. so the given function $f(x) = x$ is R - Integrable on $[0, 1]$.

$\because \sum_{k=1}^{n} \Delta x_k = (b-a)$

Here $a = x_0 = 0$

& $b = x_n = 1$

Son $\sum_{k=1}^{n} \Delta x_k = (1-0)$

$\Rightarrow \sum_{k=1}^{n} \Delta x_k = 1$

Also

$x_k - x_{k-1} < \epsilon$

Q. No: 2

prove that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is defined on the interval $[a,b]$ is not R - Integrable.

Proof:- Here it is given that

$$f(x) = \begin{cases} 1 & \text{if } x \in [a,b] \text{ \& } x \in Q \\ 0 & \text{if } x \in [a,b] \text{ \& } x \in Q' \end{cases}$$

we have required to prove that $f(x)$ is not R - Integrable on $[a,b]$

For this,

Let P is any arbitrary partition of $[a,b]$ and $[x_{k-1}, x_k]$ is the

K^{th} sub-interval of $[a,b]$ corresponding to the partition "P".
Now since Q & Q' are dense in $\mathbb{R}$ so it will be dense in $[x_{k-1}, x_k]$. So,

$M_k = 1$ and $m_k = 0$ where "M_k" & "m_k" are the $Sup(f(x))$ & $Inf(f(x))$
Now let S & s be the upper & lower sums of $f(x)$ corresponding to the partition P.
Then
$$S = \sum_{k=1}^{n} M_k \Delta x_k \quad \text{and}$$
$$s = \sum_{k=1}^{n} m_k \Delta x_k.$$

$$\Rightarrow S = \sum_{k=1}^{n} 1 \cdot \Delta x_k \quad \text{and} \quad s = \sum_{k=1}^{n} 0 \Delta x_k$$

$$\Rightarrow S = b - a \quad \text{and} \quad s = 0$$

$$\Rightarrow \lim_{n\to\infty} S = \lim_{n\to\infty} (b-a) \text{ and } \lim_{n\to\infty}(s) = \lim_{n\to\infty}(0)$$

$$\Rightarrow J = b - a \quad \text{and} \quad I = 0$$

which shows that $I \neq J$
So, we can say that $f(x)$ is not R-integrable on $[a,b]$.
which ~~is~~ we wished to prove

Q : NO # (3)

Given an example to show that $|f(x)|$ can be R-I even though $f(x)$ is not R.I on $[a,b]$

~~Solution~~:-

Proof:- Here we have to given an example of a to show that $|f(x)|$ can be R-I on $[a,b]$ while $f(x)$ is not itself to be R.I on $[a,b]$.

For this,

Let us concider a function

$$f(x) = \begin{cases} 1 & \text{if } x \in [a,b] \text{ where } x \in Q \\ -1 & \text{if } x \in [a,b] \text{ where } x \in Q' \end{cases}$$

Now 1^{st} we are to show that $f(x)$ is not R.I

For this.

Let "P" is any arbitrary partition of $[a,b]$ and $[x_{k-1}, x_k]$ be the k^{th} Sub-interval of $[a,b]$ due to the partition P.

Now since we know that Q & Q' are dense in R, So it will be dense in $[x_{k-1}, x_k]$. So,

$M_k = 1$ and $m_k = -1$

Now let "S" & "s" be the upper & lower sums of "f(x)" corresponding to the partition "P".

Then we have;

$$S = \sum_{K=1}^{n} M_K \Delta x_K \quad \text{and} \quad s = \sum_{K=1}^{n} m_K \Delta x_K$$

$$\Rightarrow S = \sum_{K=1}^{n} 1 \Delta x_K \quad \text{and} \quad s = \sum_{K=1}^{n} (-1) \Delta x_K$$

$$\Rightarrow S = 1(b-a) \quad \text{and} \quad s = -(b-a)$$

$$\Rightarrow S = b-a \quad \text{and} \quad s = -(b-a)$$

$$\Rightarrow \lim_{n\to\infty} S = \lim_{n\to\infty} (b-a) \quad \text{and} \quad \lim_{n\to\infty} s = \lim_{n\to\infty} (-(b-a))$$

$$\Rightarrow J = b-a \quad \text{and} \quad I = -(b-a)$$

$$\Rightarrow J \neq I$$

Hence the function $f(x)$ is not R.I on $[a,b]$.

Now $|f(x)| = 1$, which is constant.

So $|f(x)|$ is constant & we know that constant function is R.I.

So $|f(x)|$ is R-Integrable.

Thus $f(x)$ is not Reimann Integrable, but $|f(x)|$ is Reimann Integrable.

Hence there are functions which are not Reimann Integrable, but its absolute is Reimann Integrable.

Theorem:- If 'f' is R-Integrable on $[a,b]$, then f^2 is also R-integrable on $[a,b]$.

Proof:- Given that f is R-integrable on $[a,b]$. So f is bounded on $[a,b]$ so by definition of bounded, $\exists$ $M>0$ such that $|f(x)| \leq M$; $\forall$ $x \in [a,b]$
Also, As f is R-integrable, so for any $\epsilon' > 0$ $\exists$ partition 'P' such that

$$S - s < \epsilon' \rightarrow (1)$$

$$\text{where } S = S(f,P) = \sum_{k=1}^{n} M_k \Delta x_k$$

$$\text{and } s = s(f,P) = \sum_{k=1}^{n} m_k \Delta x_k$$

where M_k and m_k are lub and glb of f on $[x_{k-1}, x_k]$; $k = 1, 2, \ldots n$.
k^{th} sub-interval correspond. to P.

consider

$$|f^2(x) - f^2(y)| = |(f(x)+f(y))(f(x)-f(y))| \; ; \forall \; x,y \in [a,b]$$

$$\Rightarrow |f^2(x) - f^2(y)| \leq (|f(x)| + |f(y)|)\,|f(x) - f(y)| \rightarrow (*) \; ; \; " \; "$$

Since $|f(x)| \leq M$; $\forall$ $x \in [a,b]$.

So $|f(x)| \leq M$; $\forall$ $y \in [a,b]$

$$\Rightarrow |f(x)| + |f(y)| \leq 2M \; ; \; \forall \; x,y \in [a,b].$$

(*) $\Rightarrow |f^2(x) - f^2(x)| \leq 2M|f(x) - f(y)|$; $\forall x, y \in [a,b]$

$\Rightarrow |M_K^{(1)} - m_K^{(1)}| \leq 2M|M_K - m_K|$

$\Rightarrow \sum_{K=1}^{n} |M_K^{(1)} - m_K^{(1)}| \Delta x_K \leq 2M \sum_{K=1}^{n} |M_K - m_K| \Delta x_K$

$\Rightarrow \sum_{K=1}^{n} (M_K^{(1)} - m_K^{(1)}) \Delta x_K \leq 2M \sum_{K=1}^{n} (M_K - m_K) \Delta x_K$

$\Rightarrow S^{(1)} - \mathcal{S}^{(1)} \leq 2M(S - \mathcal{S}) \longrightarrow (\ast\ast)$

where $S^{(1)} = S(f^2, P)$, $\mathcal{S}^{(1)} = (f^2, P)$

$(\ast\ast) \Rightarrow S^{(1)} - \mathcal{S}^{(1)} < 2M\epsilon'$ $\because$ by (1)

$\Rightarrow S^{(1)} - \mathcal{S}^{(1)} < 2M \cdot \frac{\epsilon}{2M}$ $\because$ as ϵ' is arbitrary

$\Rightarrow S^{(1)} - \mathcal{S}^{(1)} < \epsilon$

$\Rightarrow f^2$ is R-integrable on $[a,b]$.

Theorem:- If f, g are R.I functions on $[a,b]$ then $f.g$ is also R.I on $[a,b]$.

Proof:- Since $f.g = \frac{1}{2}\{(g+f)^2 - (f^2 + g^2)\} \longrightarrow$ (A)

Since f, g are R.I functions on $[a,b]$

$\Rightarrow f+g$ is R.I on $[a,b]$.

$\Rightarrow (f+g)^2$ is R.I on $[a,b] \rightarrow (1)$

and again, since f, g are R.I on $[a,b]$

$\Rightarrow f^2$ & g^2 are R.I on $[a,b]$

$\Rightarrow f^2 + g^2$ is R.I on $[a,b]$

$\Rightarrow (-1)(f^2+g^2)$ is R.I on $[a,b] \rightarrow (2)$

from (1) and (2)

$(f+g)^2 + (-1)(f^2+g^2)$ is R.I on $[a,b]$

$\Rightarrow \frac{1}{2}\{(f+g)^2 + (-1)(f^2+g^2)\}$ is R.I on $[a,b]$.

So from (A)
we get
$f.g$ is R.I on $[a,b]$.

Name: Asim Marwat

MSC. Mathematics (Previous)

University of Peshawar.

Cell No: 0315 19 49 572.

Mean value theorem for Integration:

Statment: Let 'f' be continuous function on [a,b] and h(x) is integrable function on [a,b], with $h(x) > 0$, then $\exists$ a point $C \in [a,b]$ such that

$$\int_a^b f(x)\,h(x)\,dx = f(c)\int_a^b h(x)\,dx$$

Proof: Given that f is a continuous function on [a,b], and we know that every continuous function is R-integrable, so f is R-integrable on [a,b]

$\Rightarrow$ f is bounded on [a,b]

Let m, M are the glb and lub of f on [a,b].

Then $m \leq f(x) \leq M$; $\forall\ x \in [a,b]$

$\Rightarrow m.h(x) \leq f(x).h(x) \leq M.h(x) \rightarrow (1)$ since h(x) is also integrable on (a,b).

$\Rightarrow$ So $m.h(x)$, $f(x).h(x)$, and $M.h(x)$ are integrable.

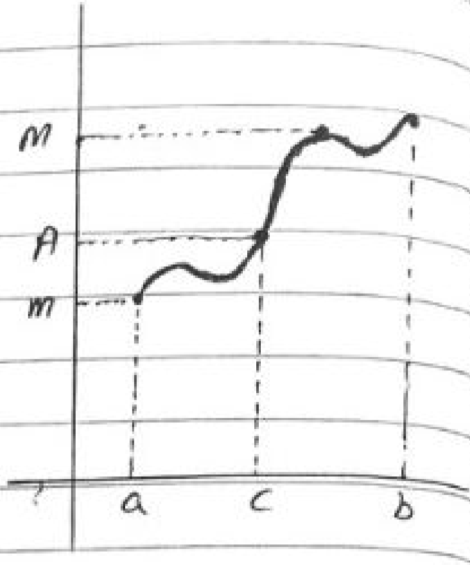
Thus (1) $\Rightarrow \int_a^b m\,h(x)\,dx \leq \int_a^b f(x)\,h(x)\,dx \leq \int_a^b M.h(x)\,dx$

$$\Rightarrow m\int_a^b h(x)\,dx \leq \int_a^b f(x)\,h(x)\,dx \leq M\int_a^b h(x)\,dx$$

$$\Rightarrow m \leq \frac{\int_a^b f(x)\,h(x)\,dx}{\int_a^b h(x)\,dx} \leq M \rightarrow (2)$$

Let

$$A = \frac{\int_a^b f(x)\, h(x)\, dx}{\int_a^b h(x)\, dx}$$



(2) $\Rightarrow$ $m \leq A \leq M$

As 'f' is continuous function on $[a,b]$ with $f(a) \neq f(b)$

and $A \in$ Rang f

Thus by intermediate value theorem there exists $c \in (a,b)$ such that $f(c) = A$

$$\Rightarrow f(c) = \frac{\int_a^b f(x)\, h(x)\, dx}{\int_a^b h(x)\, dx}$$

$$\Rightarrow \int_a^b f(x)\, h(x)\, dx = f(c) \int_a^b h(x)\, dx$$ — prove

Remark or Corollary.

If $h(x) = 1$ and $f(c) = \mu$

then $\int_a^b f(x) \cdot 1\, dx = \mu \int_a^b dx$

$$\Rightarrow \int_a^b f(x)\, dx = \mu\, |x|_a^b$$

$$\Rightarrow \int_a^b f(x)\,dx = \mu\,(b-a)$$

Representing Mean value theorem.

Note:- where the point 'c' is not arbitrary, but it is a particular point.

Q:-

Prove that if "f" is R.I, then f^5 is also R-I.

Solution:- Since f is R-I,

So f^2 is also R-I

$\Rightarrow f \cdot f^2 = f^3$ is also R-I

$\Rightarrow f^2 \cdot f^3 = f^5$ is also R.I

Generally f^n is R-I

Just by Theorem

Prepared By:-

Asim Marwat

Msc - Mathematics (Previous) UOP

Session 2019 - 2021

cell No: 0315 19 49 572

Theorem :- If $f(x)$ is R-integrable on $[a,b]$ and $c \in [a,b]$, Then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

proof:- Given that $f(x)$ is R-integrable on $[a,b]$ and $c \in [a,b]$, Then we need to prove that
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

For this let 'P' is any arbitrary partition of $[a,b]$ which does not contain the point c.
Let S and s are the upper and lower sums of 'f' on $[a,b]$ corresponding to P

i.e $S = S(f,P)$, $s = s(f,P)$
By adding 'c' to P, we get a new partition P_1 which is the refinement of P and let S^*, s^* are the upper and lower sums of f corresponding to P_1, Then obviously

$$s \le s^* \text{ and } S^* \le S \rightarrow (1)$$
Now consider the intervals $[a,c]$, $[c,b]$ with partitioning points of P_1.
Since $f(x)$ is R-integrable on $[a,b]$

So $f(x)$ is also R-integrable on $[a,c]$ and $[c,b]$.

Let S_1, s_1 are the upper and lower sums of 'f' on $[a,c]$ corresponding to P_1 and

Let S_2, s_2 are the upper and lower sums of 'f' on $[c,b]$ corresponding to P_1, Then obiously

$$S^* = S_1 + S_2$$

and

$$s^* = s_1 + s_2$$

Thus (1) $\Rightarrow$ $s \leq s_1 + s_2$ and $S_1 + S_2 \leq S \rightarrow (2)$

$$(2) \Rightarrow s \leq s_1 + s_2 \leq S_1 + S_2 \leq S \longrightarrow (3)$$

Let I, J, I_1, J_1 and I_2, J_2 are the lower and upper integrables of f on $[a,b]$, $[a,c]$ respectively. Then from (3) we get

$$I \leq I_1 + I_2 \leq J_1 + J_2 \leq J \longrightarrow (4)$$

Since f is R-integrable on $[a,b]$, $[a,c]$ and $[c,b]$, So we have

$$I = J = \int_a^b f(x)\, dx$$

$$I_1 = J_1 = \int_a^c f(x)\, dx \qquad \text{and}$$

$$I_2 = J_2 = \int_c^b f(x)\, dx$$

Thus (4) $\Rightarrow$ $I \leq I_1 + I_2 \leq I_1 + I_2 \leq I$

$\Rightarrow$ $I \leq I_1 + I_2 \leq I$

$\Rightarrow$ $I = I_1 + I_2$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Representing required result.

Theorem:- A bounded function whose discontinuities can be enclosed in a finite number of sub-interval whose total length is less than a positive real number ϵ is integerable on $[a, b]$.

Proof:-

To prove the required result, let $c_1, c_2, \ldots, c_p$ be the points of discontinuities of bounded function f in $[a,b]$ and let $[a_1, b_1]$ and $[a_2, b_2], \ldots, [a_p, b_p]$ are the non overlapping subinterval of $[a,b]$ such that $c_1 \in [a_1, b_1]$, $c_2 \in [a_2, b_2], \ldots, c_p \in$ [with sum of their is less than ϵ ie

$$\sum_{r=1}^{p} (b_r - a_r) < \epsilon \quad \text{where } \epsilon > 0 \longrightarrow (1)$$

As given that f is bounded on $[a,b]$ and all dicontinuities of f lies in $[a_r, b_r]$ where $r = 1, 2, \ldots, p$.

Thus f is continous on $[a, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_p, b]$.

So f is R-I on each of the $(p+1)^{th}$ sub-interval on $[a, a_1], [b_1, a_2], [a_2, a_3], \ldots, [b_p, b]$ because every continuous function on $[a,b]$ is R-integrable.

Thus as f is R-I on $[a, a_1]$.
So there exists partition P_1 such that

$$S_1 - s_1 < \epsilon/2(p+1) \longrightarrow (2)$$

where $S_1 = S_1(P_1, f[a, a_1])$ and $s_1 = s_1(P_1, f[a, a_1])$.

Similarly as f is R-I on $[b_1, a_2]$.
So there exists partition P_2 such that

$$S_2 - s_2 < \epsilon/2(p+1) \longrightarrow (3)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

Similarly as f is R-I on $[b_p, b]$.
So there exists partition P_p such that

$$S_p - s_p < \epsilon/2(p+1) \longrightarrow (4)$$

Adding from (2) (3) upto (4) we get

$$(S_1+S_2+S_3+\cdots+S_p)-(s_1+s_2+s_3+\cdots+s_p) < \frac{\epsilon}{2(p+1)} + \frac{\epsilon}{2(p+1)} + \frac{\epsilon}{2(p+1)} + \cdots + \frac{\epsilon}{2(p+1)}$$

$$= \frac{\epsilon(p+1)}{2(p+1)}$$

$$= \frac{\epsilon}{2}$$

$$\Rightarrow (S_1+S_2+S_3+\cdots+S_p)-(s_1+s_2+s_3+\cdots+s_p) < \frac{\epsilon}{2} \rightarrow (5)$$

Let the upper sum is S^c i.e

$$S_1+S_2+S_3+\cdots+S_p = S^c$$

and the lower sum is s^c i.e

$$s_1+s_2+s_3+\cdots s_p = s^c$$

So (5) can be written as

$$S^c - s^c < \epsilon/2 \longrightarrow (6)$$

Now for remaining part the total upper sums corresponding to the sub-interval $[a_1,b_1]$, $[a_2,b_2]$, $[a_3,b_3]$, ..., $[a_p,b_p]$ can be calculated as follow.

Note that on the iterval $[a_1,b_1]$, f is bounded.

Let $P_1^{(d)}$ be the partition of $[a_1, b_1]$ then

$$S_1^{(d)} = S(P_1^{(d)}; f[a_1, b_1]) = \sum_{i=1}^{n} M_i' \Delta x_i \rightarrow (7)$$

where M_i' is the lub of f in each sub-interval of $[a_1, b_1]$.

Let M_1 be the lub of f on $[a_1, b_1]$, then

$$S_1^{(d)} = \sum_{i=1}^{n} M_i' \Delta x_i \leq \sum_{i=1}^{n} M_1 \Delta x_i$$

$$= M_1 \sum_{i=1}^{n} \Delta x_i$$

$$= M \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= M_1 (x_n - x_0)$$

$$= M_1 (b - a)$$

$$S_1^{(d)} \leq M_1 (b_1 - a_1)$$

Similarly for lower sum

$$s_1^{(d)} = \sum_{i=1}^{n} m_i' \Delta x_i \longrightarrow (8)$$

Let m_1 be the glb of f on $[a_1, b_1]$, then

$$s_1^{(d)} = \sum_{i=1}^{n} m_i' \Delta x_i \geq \sum_{i=1}^{n} m_1 \Delta x_i$$

$$= m_1 \sum_{i=1}^{n} \Delta x_i$$

$$= m_1 (b_i - a_i)$$

$$\Rightarrow S_1^{(d)} \geq m_1(b-a)$$

$$\Rightarrow -S_1^{(d)} \leq -m_1(b-a).$$

Thus for $[a_1, b_1]$ we have the upper and lower sum are

$$\int_1^{(d)} \leq M_1(b_1-a_1) \text{ and } -\int_1^{d} \leq -m_1(b_1-a_1)$$

Adding

$$\int_1^{(d)} - S_1^{(d)} \leq M_1(b_1-a_1) - m_1(b_1-a_1)$$

$$\int_1^{(d)} - S_1^{(d)} \leq (M_1-m_1)(b_1-a_1) \rightarrow (9)$$

Similarly for $[a_2, b_2]$ we have

$$\int_2^{(d)} - S_2^{(d)} \leq (M_2-m_2)(b_2-a_2) \rightarrow (10)$$

Similarly for $[a_p, b_p]$ we have

$$\int_p^{(d)} - S_p^{(d)} \leq (M_p-m_p)(b_p-a_p) \rightarrow (11)$$

Adding from (9) and (10) upto (11) we have

$$\left(\int_1^{(d)} + \int_2^{(d)} + \int_3^{(d)} + \dots S_p^{(d)}\right) - \left(S_1^{(d)} + S_2^{(d)} + S_3^{(d)} + \dots + S_p^{(d)}\right)$$
$$\leq (M_1-m_1)(b_1-a_1) + (M_2-m_2)(b_2-a_2) + (M_3-m_3)(b_3-a_3) + \dots + (M_p-m_p)(b_p-a_p)$$

$$\Rightarrow \left(\int_1^{(d)} + \int_2^{(d)} + \int_3^{(d)} + \dots + \int_p^{(d)}\right) - \left(S_1^{(d)} + S_2^{(d)} + S_3^{(d)} + \dots + S_p^{(d)}\right)$$
$$\leq \sum_{r=1}^{p} (M_r - m_r)(b_r - a_r) \rightarrow (12)$$

Let M, m are the lub and glb of f on $[a,b]$ then

$$m \le m_r \le M_r \le M$$

$$\Rightarrow m \le m_r \quad \text{and} \quad M_r \le M$$

$$\Rightarrow -m \ge -m_r \Rightarrow -m_r \le -m$$

$$M_r - m_r \le M - m.$$

So that (12) becomes

$$\Rightarrow \left(S_1^{(d)} + S_2^{(d)} + S_3^{(d)} + \dots S_p^{(d)}\right) - \left(s_1^{(d)} + s_2^{(d)} + s_3^{(d)} + \dots s_p^{(d)}\right)$$

$$\le \sum_{r=1}^{p} (M_r - m_r)(b_r - a_r)$$

$$= (M - m) \sum_{r=1}^{p} (b_r - a_r)$$

$$S^{(d)} - s^{(d)} < (M - m)\epsilon \longrightarrow (13)$$

Let p be the partition of $[a,b]$ and $S = S(P;f)$, $s = s(P;f)$. Then the total upper sum is

$$S = S^c + S^d$$

and the total lower sum is

$$s = s^c + s^d$$

$$S - s = (S^c + S^d) - (s^c + s^d)$$

$$< \frac{\epsilon}{2} + (M - m)\epsilon$$

$$< \epsilon(1/2 + M - m)$$

$$< \epsilon K$$

where $\frac{1}{2} + M - m = K =$ constant

The End of This CHP # 5