Chapter 04: Derivative

Handwritten Notes of REAL ANALYSIS

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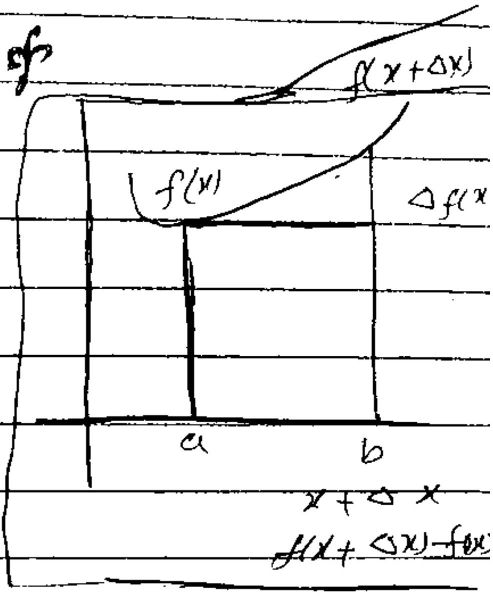
CHP # 4

Derivative



$y = f(x) \rightarrow$ function

$\Delta f(x) = f(x+\Delta x) - f(x)$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Averenge change

If we take $\Delta x \rightarrow 0$ Then

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$$

$f(x)$, provided that the limit exist.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{h}$$

If $h > 0$ Then

$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$ is called right hand derivative of $f(x)$ and it is denoted by $R f'(x) = \lim_{x \to 0^+} \frac{f(x+h) - f(x)}{h}$

If $h < 0$ Then

$\lim_{x \to 0^-} \frac{f(x+h) - f(x)}{h}$ is called ~~right~~ left hand derivative of $f(x)$ and it is denoted by $L f'(x) = \lim_{x \to 0^-} \frac{f(x+h) - f(x)}{h}$

$f'(x)$ slope of the tangent line passing through $(x, f(x))$.

Theorem: If "f" is differentiable function defined on I. Then "f" is continuous function. But the converse is not true in general.

Proof: Let $a \in I$ we show that "f" is continous at "a"

i.e we show that $\lim_{x \to a} f(x) = f(a)$

Now

$$(f(a+h) - f(a)) = \left(\frac{f(a+h) - f(a)}{h} \right)$$

Taking Limit $h \to 0$

$$\lim_{h \to 0} (f(a+h) - f(a)) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right) \cdot \lim_{h \to 0}$$

$$\lim_{h \to 0} f(a+h) - f(a) = 0$$

Now Let $a+h = x$ if $h \to 0$ Then $x \to a$

Then $\lim_{h \to 0} f(x) - f(a) = 0$

$\Rightarrow \lim_{h \to 0} f(x) = f(a)$

$\Rightarrow$ "f" is continous on $a \in I$

$\therefore$ "f" is continous.

For the converse we give the example:

Let $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Clearly "f" is continous on $\mathbb{R}$,

But "f" is is not differentiable at

$$Rf'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$Rf'(0) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{|h| - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

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$$Lf'(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-1}{1} = -1$$

So the converse is not true.

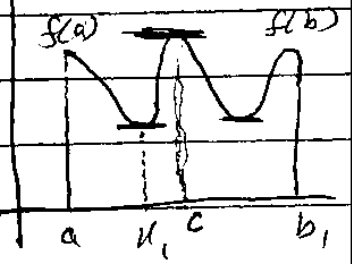
Roll's Theorem:-

Statment:- Let "f" be a function such that

(i) $f(x)$ is continous on $[a,b]$

(ii) $f(x)$ is differentiable on (a,b)

(iii) $f(a) = f(b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$



Proof: Since "f" is continous on $[a,b]$

So that exist $c, d \in [a,b]$

Such that $f(c) = \text{lub } f(x) = M$

$f(d) = \text{glb } f(x) = m$

we discuss two cases

case (i): if $m = M \Rightarrow$ Then f is constant

$\Rightarrow f(x) = K \quad \forall\ x \in [a,b]$

$\Rightarrow f'(x) \quad \forall\ x \in [a,b]$

case (ii): If $m \neq M$

Then $f(a)$ and $f(b)$ must be different from atleast m or M.

Let $f(c) \neq f(a)$ and $f(c) \neq f(b)$

$\Rightarrow c \in (a,b)$

we prove that $f'(c) = 0$

Since $f(c) = \text{lub } f(x)$

$\Rightarrow f(c+h) \leq f(c) \quad , h>0$

$\Rightarrow f(c+h) - f(c) \leq 0$

$$\Rightarrow \frac{f(c+h)-f(c)}{h} \leq 0$$

Taking Limit $h \to 0$

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \leq 0$$

$\Rightarrow Rf'(c) \leq 0 \longrightarrow (1)$

Also $f(c+h) \leq f(c) \quad , \forall\ h<0$

$\Rightarrow f(c+h) - f(c) \leq 0$

$$\Rightarrow \frac{f(c+h)-f(c)}{h} \geq 0 \quad \text{as } h<0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \geq 0$$

$\Rightarrow Lf'(c) \geq 0 \longrightarrow 0$

since "f" is differentiable

So from (1) and (2) we have

$$\boxed{f'(c) = 0}$$

Similarly:- Note that:- If $f(d) \neq f(a)$, $f(d) \neq f(b)$

$\boxed{\text{Same Proof}} \rightarrow$ Roll's Theorem

Theorem:- **Lagrange's Mean Value Theorem: (LM-V. Theorem).**

Statement:- If f is a function defined on $[a,b]$ such that

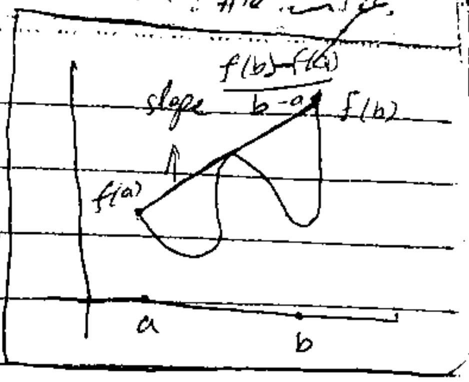
(i) f is continous on $[a,b]$

(ii) f is differentiable on (a,b) Then There exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof: $y - y_1 = m(x - x_1)$

$(x_1, f(x_1)) = (a, f(a))$

$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$



$y(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$

Now consider $g(x) = f(x) - y(x)$

$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$

$g(a) = f(a) - f(a) + \frac{f(b) - f(a)}{b - a}(a - a)$

$g(a) = 0$

$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{\cancel{b - a}}(\cancel{b - a})$

$g(b) = 0$

g is continous on $[a, b]$

g is differentiable on (a, b)

$g(a) = g(b)$ So by Rolle's Theorem

$\exists$ $c \in (a, b)$ such that $g'(c) = 0$

Now $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$

$g'(c) = 0 \implies f'(c) - \frac{f(b) - f(a)}{b - a} = 0$

So $\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$ proved

Caushy Mean Value Theorem:

Statement:- Let f and g be two functions defined on $[a, b]$ such that

(i) f and g are continous on $[a, b]$

(ii) f and g are differentiable on (a, b)

then $\exists$ $c \in (a, b)$ such that

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$$\frac{f'(c)}{g(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \quad g(c) \neq 0$$

Proof:- let $y(x) = f(a) + \frac{f(b) - f(a)}{g(b) - g(a)} \left(g(x) - g(a) \right)$

let $h(x) = f(a) + \frac{f(b) - f(a)}{g(b) - g(a)} \left(g(x) - g(a) \right) - f(x)$

$$h(a) = \cancel{f(a)} + \frac{f(b) - f(a)}{g(b) - g(a)} \left(\overset{0}{\cancel{g(a) - g(a)}} \right) - \cancel{f(a)}$$

$$\boxed{h(a) = 0}$$

$$h(b) = \cancel{f(a)} + \frac{\cancel{f(b) - f(a)}}{\cancel{g(b) - g(a)}} \left(\cancel{g(b) - g(a)} \right) - \cancel{f(b)} \Rightarrow \boxed{h(b) = 0}$$

Apply Rolle's Theorem $\exists$ $c \in (a, b)$ such that

$$h'(c) = 0 \Rightarrow \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) - f'(c) = 0$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Hence $\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$

Q: Verify Rolle's Theorem on the following functions.

(i) $f(x) = x^2 - 4x + 3, \quad x \in [1, 3]$

Solution:

$$f(1) = (1)^2 - 4(1) + 3$$
$$= 1 - 4 + 3$$
$$f(1) = 0$$
$$f(3) = (3)^2 - 4(3) + 3$$
$$= 9 - 12 + 3$$

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$\Rightarrow f(3) = 0$

So $f(1) = f(3)$

$f'(x) = 2x - 4$

$c = 2 \in [1,3]$.

X —— X —— X —— X —— X

(ii) $f(x) = \sin^2 x \quad , x \in [0, \pi]$

solution:- Put $x = 0$

$f(0) = \sin^2(0)$

$f(0) = 0$

Now $x = \pi$

$f(\pi) = \sin^2(\pi)$

$f(\pi) = 0$

So $f(0) = f(\pi)$

So $f(x) = \sin^2 x$ is continous on $x \in [0, \pi]$.

X —— X —— X —— X —— X —— X

(iii) $f(x) = x^3 - x \quad , \quad x \in [-1, 1]$

solution:- put $x = -1$

$f(-1) = (-1)^3 - (-1)$

$= -1 + 1$

$f(-1) = 0$

Now put $x = 1$

$f(1) = (1)^3 - (1)$

$= 1 - 1$

$f(1) = 0$

So $f(-1) = f(1)$

So $f(x) = x^3 - x$ is continous on $x \in [-1, 1]$.

$f'(x) = 3x^2 - 1$

$f'(c) = 3c^2 - 1$

$0 = 3c^2 - 1 \Rightarrow 3c^2 = 1 \Rightarrow c^2 = \frac{1}{3}$

$\boxed{c = \sqrt{\frac{1}{3}}}$

$\boxed{\text{So } \sqrt{\frac{1}{3}} \in [-1, 1]}$

A-C

(iv) :- $x^2 - 4x - 3y + 13 = 0$, $x \in [-1, 5]$

Solution $x^2 - 4x + 13 = 3y$

$$y = \frac{x^2}{3} - \frac{4x}{3} + \frac{13}{3}$$

$$f(x) = y = \frac{x^2}{3} - \frac{4x}{3} + \frac{13}{3}$$

put $x = -1$

$$f(-1) = \frac{(-1)^2}{3} - \frac{4(-1)}{3} + \frac{13}{3}$$

$$= \frac{1}{3} + \frac{4}{3} + \frac{13}{3} = \frac{18}{3} = 6$$

$$\boxed{f(-1) = 6}$$

put $x = 5$

$$f(5) = \frac{(5)^2}{3} - \frac{4(5)}{3} + \frac{13}{3}$$

$$= \frac{25}{3} - \frac{20}{3} + \frac{13}{3} = \frac{25 - 20 + 13}{3} = \frac{18}{3}$$

$$f(5) = 6$$

So $f(-1) = f(5)$

$f(x) = \frac{x^2}{3} - \frac{4x}{3} + \frac{13}{3}$ is continous on $x \in [-1, 5]$

$$f'(x) = \frac{2x}{3} - \frac{4}{3}$$

$$f'(c) = \frac{2c}{3} - \frac{4}{3}$$

$$0 = \frac{2}{3}c - \frac{4}{3} \Rightarrow \frac{2}{3}c - \frac{4}{3} = 0$$

$$\frac{2}{3}c = \frac{4}{3}$$

$$\Rightarrow c = \frac{\cancel{4}^2}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}}$$

$$c = 2$$

So $2 \in [-1, 5]$.

Theorem :- If a function f is differentiable on (a,b) such that its derivative vanishes at each point. Then f is continous.
OR. If f is differentiable on (a,b) and $f'(x)=0$ Then f is constant.

Proof :- Let $x_1, x_2 \in (a,b)$ we show that f is constant i.e we show that
$f(x_1) = f(x_2)$
Since f is defferentiable on (a,b)
So f is continous on $[x_1, x_2]$,
assume $x_1 < x_2$
$\Rightarrow$ f is differentiable on (x_1, x_2)
So by mean value Theorem There exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x - x_1} \quad \text{But } f'(x) = 0 \ \forall x \in (a,b)$$

$$\Rightarrow f'(c) = 0$$

Hence $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$

$\Rightarrow f(x_2) - f(x_1) = 0$

$\Rightarrow f(x_2) = f(x_1) \Rightarrow f$ is constant.

Q :- Let $x_0, x_1, x_2, \ldots, x_n$ be real numbers such that $\dfrac{x_0}{n+1} + \dfrac{x_1}{n} + \dfrac{x_2}{n-1} + \cdots + \dfrac{x_{n-1}}{2} + x_n = 0$

Show that $\exists\, c \in (0,1)$ such that
$$x_0 c^n + x_1 c^{n-1} + x_2 c^{n-2} + \cdots + x_{n-1} c + x_n = 0.$$

Solution. Let $f(x) = \dfrac{x_0 x^{n+1}}{n+1} + \dfrac{x_1 x^n}{n} + \cdots + \dfrac{x_{n-1} x^2}{2} + x_n x.$

Clearly f is continous on $[0,1]$

f is differentiable on $(0,1)$

and $f(0)=0$, $f(1)=\frac{x_0}{n+1}+\frac{x_1}{n}+\frac{x_2}{n-1}+\cdots+x_n=0$

Hence $f(0)=f(1)$

Apply Rolle's Theorem $\exists$ $c\in(0,1)$ s.t

$f'(c)=0$

$\Rightarrow x_0c^n+x_1c^{n-1}+\cdots x_{n-1}c+x_n=0$

Q:- Suppose f'' is continous on $[a,b]$ and f has three roots in $[a,b]$ show f'' has at least one root in (a,b)

Solution:- Since f has three roots in $[a,b]$

let us suppose $x_1,x_2,x_3\in[a,b]$

such that $x_1<x_2<x_3$ and $f(x_1)=f(x_2)=f(x_3)$

consider the interval $[x_1,x_2]$ Then f is continous on $[x_1,x_2]$

f is differentiable on (x_1,x_2)

and $f(x_1)=f(x_2)$

By Rolle's Theorem $\exists$ $c\in(x_1,x_2)$ such that

$f'(c)=0$

Similarly consider the interval $[x_2,x_3]$ and by the same proceedure $\exists$ $d\in(x_2,x_3)$

such that $f'(d)=0$

Now clearly f' is continous on $[c,d]$

f' is differentiable on (c,d)

$f'(c)=f'(d)$

So by Rolle's Theorem

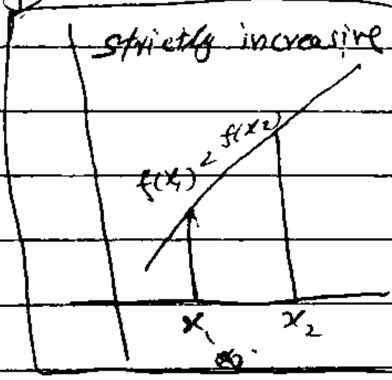
There exist $e\in(c,d)$ such that

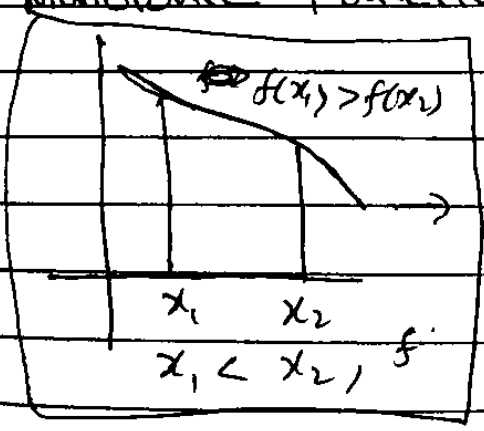
$f''(c) = 0$

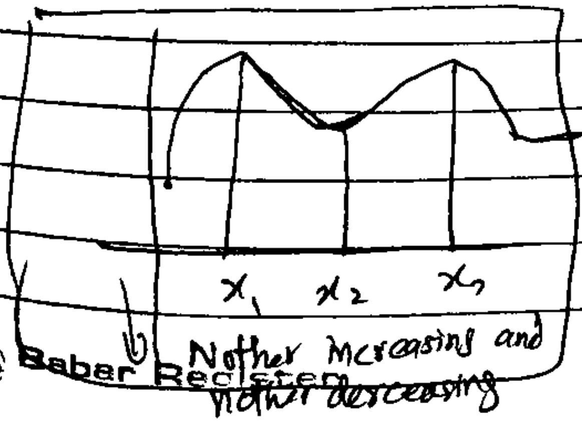
Hence f'' has at least one root in (a,b).

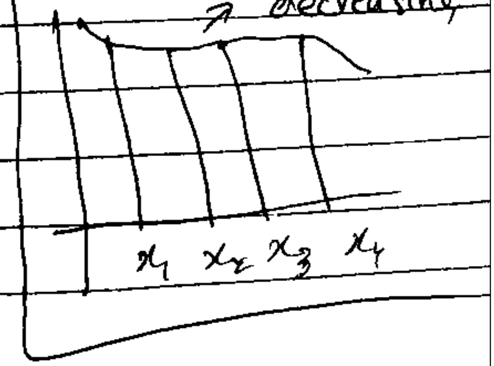
Monotonic functions:

A functions f is said to be monotonically increasing if for $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$ and f is said to strictly increasing if for $x_1 < x_2$ we have $f(x_1) < f(x_2)$
and A function f is said to be decreasing (monotonically decreasing) if for $x_1 < x_2$ we have $f(x_1) \geq f(x_2)$
Also f is said to be strictly decreasing if for $x_1 < x_2$ we have $f(x_1) > f(x_2)$. If f is decreasing or increasing Then f is said to be Monotonic Function.









Theorem :- Let f be differentiable function defined on (a,b)

(i) If $f'(x) > 0$ Then f is strictly increasing f
(ii) If $f'(x) \geq 0$ Then f is increasing function
(iii) If $f'(x) < 0$ Then f is strictly decreasing function
(iv) If $f'(x) \leq 0$ Then f is decreasing function.

Proof :- (i) we prove f is strictly increasing function
i.e let $x_1 < x_2$ we show that $f(x_1) < f(x_2)$
Since f is differentiable on (a,b)
$\Rightarrow$ f is differentiable on (x_1, x_2)
f is continuous on $[x_1, x_2]$
by Mean Value Theorem (M.V.T) there exists
$c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \rightarrow (1)$$

Since $f'(x) > 0 \quad \forall\, x \in (a,b)$

$$\Rightarrow f'(c) > 0$$

$$(1) \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

$$\Rightarrow f(x_2) - f(x_1) > 0$$

$$f(x_2) > f(x_1)$$

$\Rightarrow$ f is strictly increasing function.

Part (ii) we prove that f is increasing
i.e let $x_1 < x_2$ we show that $f(x_1) \leq f(x_2)$
since f is differentiable on (a,b)
$\Rightarrow$ f is differentiable on (x_1, x_2)

f is continuous on $[x_1, x_2]$
by (M.V.T) $\exists$ $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \rightarrow (1)$$

Since $f'(x) \geq 0 \;\forall,\; x \in (a,b)$
$\Rightarrow f'(c) \geq 0$

$$(1) \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$$

$\Rightarrow f(x_2) - f(x_1) \geq 0$

$f(x_2) \geq f(x_1)$
$\Rightarrow f$ is increasing function

Here
(iii) Part:- we prove that f is strictly decreasing
let $x_1 < x_2$ we show that $f(x_1) > f(x_2)$
since f is differentiable on (a,b)
f is differentiable on (x_1, x_2)
f is continuous on $[x_1, x_2]$
by Mean Value Theorem there exists
$c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \rightarrow (1)$$

Since $f'(x) < 0 \quad \forall\, x \in (a,b)$
$\Rightarrow f'(x) < 0$

$$(1) \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$$

$\Rightarrow f(x_2) - f(x_1) < 0$
$f(x_2) < f(x_1)$

$\Rightarrow$ f is strictly decreasing function

Part (ii): we prove that f is decreasing
i.e let $x_1 < x_2$ we show that $f(x_1) \geq f(x_2)$
since f is differentiable on (a,b)
$\Rightarrow$ f is differentiable on (x_1, x_2)
f is continous on $[x_1, x_2]$
by M.V.T There exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \longrightarrow (1)$$

since $f'(x) \leq 0, \forall x \in (a,b)$

$\Rightarrow f'(x) \leq 0$,

$$(1) \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0$$

$$\Rightarrow f(x_2) - f(x_1) \leq 0$$

$$\Rightarrow f(x_2) \leq f(x_1)$$

$\Rightarrow$ f is decreasing function.

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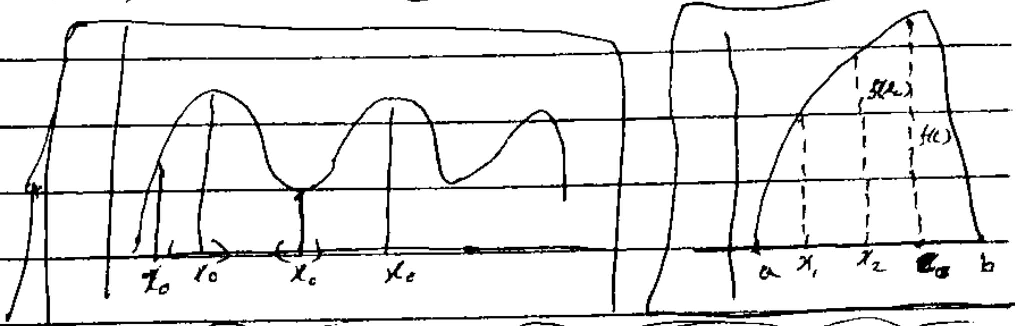
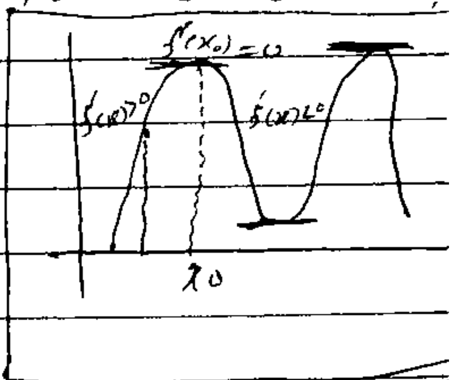
Local minimum and Local maximum of a Function

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function. Then a function f is said to has local minimum at $x_0 \in [a,b]$ if there exists $\delta > 0$ such that $f(x) \geq f(x_0)$; $\forall\, x \in (x_0 - \delta, x_0 + \delta)$ and f is said to has local maximum at $x' \in [a,b]$ if there exists a $\delta > 0$ such that $f(x) \leq f(x')$; $\forall\, x \in (x' - \delta, x' + \delta)$



Critical Point:-

Let $f : I \longrightarrow \mathbb{R}$ be a function. Then a point is said to be critical point if at that point the function have local minimum or local Maximum.



Theorem:- If a function f has a local minimum at the point 'c' Then $f'(c) = 0$

Proof: Let f has local minimum at "c" we show that $f'(c) = 0$

Since f is differentiable So

$\therefore \; Rf'(c) = Lf'(c) = f'(c) \longrightarrow (*)$

by def: of local minimum ther exist $\delta > 0$ Such that

$f(c) \le f(x) \; ; \forall \; x \in (c-\delta, c+\delta)$

$\Rightarrow \; f(c) \le f(x) \; ; \forall \; x \in (c-\delta, c)$

If $x \to \bar{c}$ Then clearly $x \in (c-\delta, c)$

i.e $x < c \Rightarrow x - c < 0$

Now $f(x) - f(c) \ge 0 \; ; \forall \; x \in (c-\delta, c)$

$\Rightarrow \; \dfrac{f(x) - f(c)}{x - c} \le 0 \; ; \forall \; x \in (c-\delta, c)$

$\Rightarrow \; \lim\limits_{x \to \bar{c}} \dfrac{f(x) - f(c)}{x - c} \le 0$

$\Rightarrow \; Lf'(c) \le 0 \longrightarrow (1)$

Now $f(c) \le f(x) \; ; \forall \; x \in (c-\delta, c+\delta)$

$\Rightarrow \; f(c) \le f(x) \; ; \forall \; x \in (c, c+\delta)$

$\Rightarrow \; f(x) - f(c) \ge 0 \; ; \forall \; x \in (c, c+\delta)$

$\Rightarrow \; \dfrac{f(x) - f(c)}{x - c} \ge 0 \; ; \forall \; x \in (c, c+\delta)$

$\Rightarrow \; \lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \ge 0$

$\Rightarrow \; Rf'(c) \ge 0$

using (1) and (2) in (*) we have

$$\boxed{f'(c) = 0}$$