Chapter 03: Limit of a Function

Handwritten Notes of REAL ANALYSIS

Written By

Asim Marwat

MSc Mathematics (UOP)

asimmarwat41@gmail.com

Special Thanks to Dr. Adil Khan (UOP)

Date: 03/02/2020 Day Mon

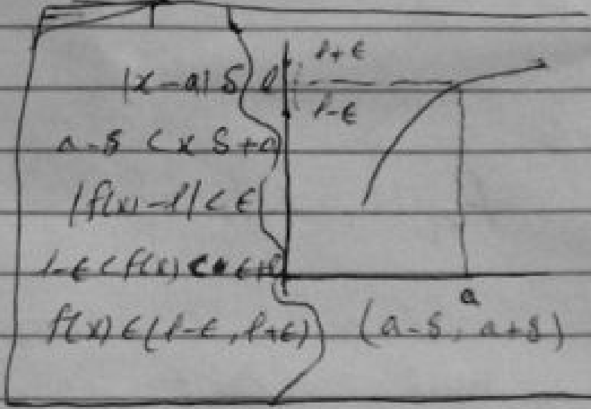
:- CHP # 3 :-

Limit of a Function :

Definition:- If f is a function defined on $\mathbb{R}$ and $a \in \mathbb{R}$ Then the limit of the function $f(x)$ is denoted by $\lim_{x \to a} f(x)$ and is defined by
$\lim_{x \to a} f(x) = l$ if for every $\epsilon > 0$, There exists a $\delta > 0$ such that
$|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$

$\Rightarrow f(x) \in (l-\epsilon, l+\epsilon)$

$\lim_{x \to a} f(x) = l$



Example:- Show that $\lim_{x \to 0} x \sin 1/x = 0$

Solution:- $|f(x) - l| < \epsilon$ if $|x - a| < \delta$

Here $f(x) = x \sin 1/x$, $l = 0$

$|x \sin 1/x - 0| = |x \sin 1/x| = |x| |\sin 1/x|$

Since $|\sin 1/x| \leq 1$

So $|x \sin 1/x - 0| \leq |x| = |x - 0|$

If $|x - 0| < \delta$

$|x \sin 1/x - 0| < \delta$ if $|x - 0| < \delta$

$\lim_{x \to 0} x \sin 1/x = 0$

Example: Show that $\lim_{x\to 0} x^2 \sin 1/x = 0$

Solution: consider $|x^2 \sin 1/x - 0| = |x^2| |\sin 1/x|$

$\Rightarrow |x^2 \sin 1/x - 0| \leq |x^2|$ as $|\sin 1/x| \leq 1$

$\Rightarrow |x^2 \sin 1/x - 0| < \epsilon$ if $|x^2| < \epsilon$

if $|x| < \sqrt{\epsilon}$

$|x^2 \sin 1/x - 0| < \epsilon$ if $|x - 0| < \delta$

where $\delta = \sqrt{\epsilon}$

Hence $\lim_{x\to 0} x^2 \sin 1/x = 0$

Right hand limit:

If f is a function defined on $\mathbb{R}$ and $a \in \mathbb{R}$, Then right hand limit of f at a is denoted by $\lim_{x\to a^+} f(x) = l$ and defind by

If for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$|f(x) - l| < \epsilon$ if $x \in (a, a+\delta)$

and similarly

The left hand limit is denoted by

$\lim_{x\to a^-} f(x) = l$

$|f(x) - l| < \epsilon$ if $x \in (a-\delta, a)$

Theorem:- $\lim_{x \to a} f(x) = l$ iff $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ exist and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.

Proof:- Let $\lim_{x\to a} f(x) = l$. Then by def: for every $\epsilon > 0$ there exist $\delta > 0$ such that.

$|f(x) - l| < \epsilon$ whenever $x \in (a-\delta, a+\delta) \to (1)$

$\Rightarrow |f(x) - l| < \epsilon$ if $x \in (a, a+\delta)$

$\Rightarrow \lim_{x\to a^+} f(x) = l$

Also from (1) we have

$|f(x) - l| < \epsilon$ if $x \in (a-\delta, a)$

$\Rightarrow \lim_{x\to a^-} f(x) = l$

Conversely:-

Let $\lim_{x\to a^+} f(x) = l$ and

$\lim_{x\to a^-} f(x) = l$ we prove

that $\lim_{x\to a} f(x) = l$

since $\lim_{x\to a^+} f(x) = l$. so by def: for every $\epsilon > 0$, there exist $\delta > 0$ such that

$|f(x) - l| < \epsilon$ if $x \in (a, a+\delta_1) \to (1)$

Also $\lim_{x\to a^-} f(x) = l$, so by def: for every $\epsilon > 0$, there exist $\delta > 0$

$|f(x) - l| < \epsilon$ if $x \in (a-\delta_2, a) \to (2)$

Let $\delta = \min(\delta_1, \delta_2)$

from (1) and (2) we have

$|f(x) - l| < \epsilon$ if $x \in (a, a+\delta)$

$|f(x) - l| < \epsilon$ if $x \in (a-\delta, a)$

$\Rightarrow |f(x) - l| < \epsilon$ if $x \in (a-\delta, a+\delta)$

$|f(x) - l| < \epsilon$ if $|x-a| < \delta$

$\Rightarrow \lim_{x \to a} f(x) = l$.

Q:- If $[x]$ is bracket function. Then $\lim_{x \to 2^+} [x]$ and $\lim_{x \to 2^-} [x]$.

Solution:- $\lim_{x \to 2^+} [x] = 2$

$x > 2$

$[2.0] = 2$

and $\lim_{x \to 2^-} [x] = 1$

Since $\lim_{x \to 2^+} [x] = 2 \neq \lim_{x \to 2^-} [x] = 1$

and $\lim_{x \to 2} [x]$ is does not exist.

Q:- Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution:- $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$

$\boxed{|x| = x \text{ if } x \geq 0 \\ = -x \text{ if } x < 0}$

Also $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

since $\lim_{x \to 0^+} \frac{|x|}{x} \neq \lim_{x \to 0^-} \frac{|x|}{x}$

so $\lim_{x \to 0} \frac{|x|}{x}$ is dose not exist.

Theorem:- Let f be a real valued function and $\lim_{x\to a} f(x) = l$, Then that $\lim_{x\to a} |f(x)| = |l|$. But the converse is not true in general

Proof:- Since $\lim_{x\to a} f(x) = l$ So by def:
for every $\epsilon > 0$ there exists a $\delta > 0$ such that
$|f(x) - l| < \epsilon$ when $|x-a| < \delta \rightarrow$ (1)
Now since $\big||f(x)| - |l|\big| \leq |f(x) - l| < \epsilon$ when $|x-a|<\delta$

$\Rightarrow$ $\big||f(x)| - |l|\big| < \epsilon$ when $|x-a| < \delta$
$\Rightarrow$ $\lim_{x\to a} |f(x)| = |l|$

for converse we give Example:-

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } x \leq a \\ 1 & \text{if } x \geq a \end{cases}$$

Then Left hand Limit $= \lim_{x\to a^-} (-1) = -1$
and Right hand Limit $= \lim_{x\to a^+} (1) = 1$
Here $L.H.L \neq R.H.L$
So $\lim_{x\to a} f(x)$ does not exist
But $|f(x)| = 1 \quad \forall\, x$
$\therefore \lim_{x\to a} |f(x)| = \lim_{x\to a} (1) = 1$, which exists.
From above discussion, we have concluded that the absolute value of function i.e. Function is convergent there it is not necessary that without absolute value the function $f(x)$ is convergent
So the converse of the above theorem is not true generally.

Theorem: Let f be a real valued function and $\langle x_n \rangle$ be a sequence in the domain of $f(x)$ such that $x_n \neq a$ and $\lim_{n \to \infty} x_n = a$

Then $\lim_{x \to a} f(x) = l \iff \lim_{n \to \infty} (x_n) = l$.

Proof: Given $\langle x_n \rangle$ is a sequence from Domain of f, such that

$\lim_{n \to \infty} x_n = a$ and $\lim_{x \to a} f(x) = l$

we prove that $\lim_{n \to \infty} f(x_n) = l$

Since $x_n \to a$,

$|x_n - a| < \epsilon_1$ when $n \geq k \to (1)$

Also $\lim_{x \to a} f(x) = l$

$\Rightarrow |f(x) - l| < \epsilon_2$ when $|x - a| < \delta$

Since $x_n \in$ Dom of f, so

$|f(x_n) - l| < \epsilon_2$ whenever $|x_n - a| < \delta \to (2)$

Since ϵ_1 is arbitrary we so take $\epsilon_1 = \delta$

From (1) we have

$|x_n - a| < \delta$ when $n \geq k$

Therefore (2)

$\Rightarrow |f(x) - l| < \epsilon_2$ whenever $|x_n - a| < \delta$; $n \geq k$

$|f(x) - l| < \epsilon_2$ when $n \geq k$

$$\boxed{\lim_{n \to \infty} f(x) = l}$$

Conversely: Let $\lim_{x \to \infty} x_n = a$

and $\lim_{n \to \infty} f(x_n) = \ell$

we prove that

$$\lim_{x \to a} f(x) = \ell$$

Assume that $\lim_{x \to a} f(x) \neq \ell$, Then by def:
for every $\epsilon > 0$ then $\delta > 0$, Such that

$|f(x) - \ell| \geq \epsilon$ when $|x - a| < \delta$

$\Rightarrow |f(x_n) - \ell| \geq \epsilon$ when $|x_n - a| < \delta$

But $|x_n - a| < \delta$ whenevere $n \geq K$

$|f(x_n) - \ell| \geq \epsilon$ whenevere $n \geq K$

$\Rightarrow \lim_{n \to \infty} f(x) \neq \ell$

which is contradoction

Hence $\boxed{\lim_{x \to a} f(x) = \ell}$

وصي فتي شركيا

Theorem: If $\lim_{x \to a} f(x) = 0$ and $g(x)$ is bounded on some deleted nbhd of a then prove that $\lim_{x \to a} f(x) \cdot g(x) = 0$

Proof:- Since $\lim_{x \to a} f(x) = 0$ and $g(x)$ is bounded on some deleted nbhd

$N_\delta(a) \setminus \{a\} = S$

Then $|g(x)| \leq M$, $\forall x \in S$

Also, as $\lim_{x \to a} f(x) = 0$

$$\Rightarrow |f(x) - 0| < \frac{\epsilon}{|M|} \text{ when } |x - a| < \delta$$

Now $|f(x) \cdot g(x) - 0| = |f(x)||g(x)| < \frac{\epsilon}{|M|} \cdot |g(x)|$; $|x-a| < \delta$

$|f(x).g(x) - 0| < \frac{\epsilon}{|M|}|g(x)| \le \frac{\epsilon}{|M|}|M|$; when $|x-a| < \delta$

$|f(x).g(x) - 0| < \epsilon$ when $|x-a| < \delta$

Hence $\boxed{\lim_{x \to a} f(x).g(x) = 0}$

Def: If $\lim_{x \to a} f(x) = \infty \Rightarrow$ If M is any +ve number, Then there exist $\delta > 0$, such That $|f(x)| > M$ for $|x-a| < \delta$

Def: If $\lim_{x \to a} f(x) = -\infty \Rightarrow$ If M is any +ve number, Then there exist $\delta > 0$, such tha $|f(x)| < -M$ for $|x-a| < \delta$.

Q:- Let $f(x) = \log|x|$, prove that $\lim_{x \to 0} \log|x| = -\infty$

Sol:- Let $M > 0 \Rightarrow -M < 0$

if $f(x) < -M$

$\Rightarrow \log|x| < -M$

$\Rightarrow |x| < e^{-M} = \delta$

$f(x) < -M$ if $|x| < \delta$

$\lim_{x \to 0} f(x) = -\infty$

Q(1): $\lim_{x\to 0} e^{-x}$

Solution:- we find left hand and right sides limits.

Left side limit : As $x<0$, Then $x\to 0^-$

since $x<0$, Then $x = 0-h$ where $h>0$

As $x\to 0^-$, Then $h\to 0$

Now

$$\lim_{x\to 0^-} e^{-x} = \lim_{h\to 0} e^{-(-h)}$$

$$\Rightarrow \lim_{h\to 0} e^{h} = e^{0}$$

$$\Rightarrow \lim_{x\to 0^-} e^{-x} = 1 \to (i)$$

Right side limit : As $x>0$, Then $x\to 0$

since $x>0$, The $x = 0+h$ where $h<0$

As $x\to 0^+$, Then $h\to 0^-$

Now

$$\lim_{x\to 0} e^{-x} = \lim_{h\to 0^-} e^{-h} = e^{0} = 1 \to (ii)$$

As from (i) and (ii)

$$\lim_{x\to 0^-} e^{-x} = \lim_{x\to 0^+} e^{-x}$$

Thus $\lim_{x\to 0} e^{-x}$ is exists.

Q(2): $\lim_{x\to 0} x\cos 1/x$

Solution:- $\lim_{x\to 0} x\cos \frac{1}{x} = 0.\cos(1/0) = 0.\cos\infty = 0$

$$\Rightarrow \boxed{\lim_{x\to 0} x\cos \frac{1}{x} = 0}$$

Q(3): $\lim_{x\to\infty} \frac{a^x-1}{x}$

Solution:- Let $a^x - 1 = y$

$\Rightarrow a^x = 1+y$ (taking Log)

$\Rightarrow \log a^x = \log(1+y)$

$\Rightarrow x = \frac{\log(1+y)}{\log a}$

$a^x - 1 = y$ As $x\to\infty$ then $y\to\infty$

Now As $x\to\infty$ then $y\to\infty$

Then

$$\lim_{x\to\infty}\frac{a^x-1}{x} = \lim_{y\to\infty}\frac{y}{\frac{\log(1+y)}{\log a}}$$

$$= \lim_{y\to\infty}\frac{\log a}{\frac{1}{y}\log(1+y)}$$

$$= \lim_{y\to\infty}\frac{\log a}{\log(1+y)^{1/y}}$$

$$= \frac{\log a}{\log\left[\lim_{y\to\infty}(1+y)^{1/y}\right]}$$ As $\lim_{x\to\infty}(1+x)^{1/x} = e$

$$= \frac{\log a}{\log(1)} = \frac{\log a}{0} = \infty$$

So $\boxed{\lim_{x\to\infty}\frac{a^x-1}{x} = \infty}$

Q(4):- $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x$

Solution:- $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x = \lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x}$

$$= \lim_{x\to\infty}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x}$$

$$= \lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{-x}$$

$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-1}$$

$$\boxed{\text{As } \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{n} = e}$$

$$= e^{-1}$$

So $\boxed{\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x} = \frac{1}{e}}$

Q: $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x}$

Solution: $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x} = \left[\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{-x}\right]^{-1}$

$$= e^{-1}$$

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x} = \frac{1}{e}$$

$$\boxed{\text{As } \lim_{x\to\infty}\left(1-\frac{1}{n}\right)^{-n} = e}$$

Q6: $\lim_{x\to 0}\frac{1}{x}\tan\frac{x}{2}$

Solution: $\lim_{x\to 0}\frac{1}{x}\tan\frac{x}{2} = \lim_{x\to 0}\frac{\sin\frac{x}{2}}{x\cos\frac{x}{2}}$

$$= \lim_{x\to 0}\frac{\sin\frac{x}{2}}{2\frac{x}{2}\cdot\cos\frac{x}{2}}$$

$$= \lim_{x\to 0}\frac{\sin\frac{x}{2}}{2\,x/2}\cdot\frac{1}{\cos x/2}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\sin x/2}{x/2}\cdot\frac{1}{\cos x/2}$$

$$= \frac{1}{2}\cdot 1\cdot\frac{1}{\cos 0} \qquad \text{As } \cos 0 = 1$$

$$= \frac{1}{2}\cdot 1\cdot\frac{1}{1} = \frac{1}{2}$$

$\boxed{\text{So} \Rightarrow \lim_{x\to 0}\frac{1}{x}\tan x/2 = \frac{1}{2}}$

Date: 14/2/20 (88) Day 2

Q7: $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Solution: $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

$$= \lim_{x\to\infty}\left\{\left(1+\frac{2}{x}\right)^{x/2}\right\}^2$$

$$= \left|\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x/2}\right|^2 = e^2$$

So $\boxed{\lim_{x\to 0}\left(1+\frac{2}{x}\right)^x = e^2}$

Q8: Prove that $\lim_{x\to a}\cos x = \cos a$

Solution: $|f(x)-\cos a| = |\cos x - \cos a|$

$$= \left|-2\sin\frac{x+a}{2}\sin\frac{x-a}{2}\right|$$

$$= |-2|\left|\sin\frac{x+a}{2}\right|\left|\sin\frac{x-a}{2}\right|$$

$$= 2\left|\sin\frac{x+a}{2}\right|\left|\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cdot\frac{x-a}{2}\right|$$

$$= 2\left|\sin\frac{x+a}{2}\right|\cdot 1\cdot\left|\frac{x-a}{2}\right|$$

$$< 2\cdot 1\cdot\left|\frac{x-a}{2}\right| \quad \text{As } |\sin x| < x$$

$$= |x-a| \quad \text{Let } \delta = \epsilon$$

Hence $|f(x)-\cos a| < \epsilon$ if $0<|x-a|<\epsilon=\delta$

$\therefore \lim_{x\to a} f(x) = \lim_{x\to a}\cos x = \cos a$

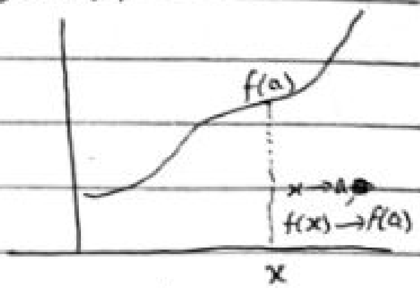
Continuity :-

A function f is said to be continues at "a" if the following conditions are satisfied.

(1) $f(a)$ is defnd ($a \in Dom f$)

(2) $\lim_{x \to a} f(x)$ exists.

(3) $\lim_{x \to a} f(x) = f(x)$.



If $f: I \longrightarrow \mathbb{R}$ and $a \in I$ then the function f is continoues at $a \in I$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ when $|x - a| < \delta$. If f is continuoes at every point of I then we can say that f is continues on I.

Q1:- $f(x) = \dfrac{x^2 - x - 2}{x - 2}$.

Solution:- Given $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

$\Rightarrow$ $Dom f = \mathbb{R} \setminus \{2\}$

$\Rightarrow$ f is not continues at $x = 2$

$f(x) = \dfrac{x^2 - x - 2}{x - 2}$, $x \neq 2$

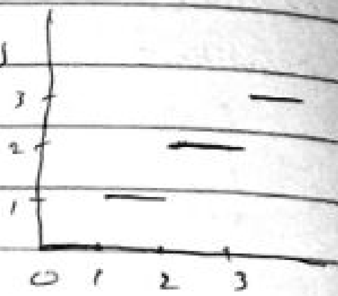
$\boxed{f(x) = 5 \text{ if } x = 2}$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^2 - 2x + x - 2}{x - 2}$

$= \lim_{x \to 2} \dfrac{x(x-2) + 1(x-2)}{x - 2} = \lim_{x \to 2} (x+1) = 3$

$\Rightarrow \lim_{x \to 2} f(x) = 3 \Rightarrow \boxed{f(2) = 5}$

Q : $f(x) = [x]$ is a bracket function so $f(x) = [x]$ is continoues Funct.



Solution : From the graph $f(x)$ is discontinoes on
$Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
we check continous at $x = 0$

$f(0) = [0] = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} [x] = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} [x] = -1$

$\lim_{x \to 0} f(x)$ does not exists

So $f(x)$ is not continoues at $x = 0$

Def:
If f is a functions and $a \in \text{Dom} f$
Also, $\lim_{x \to a^+} f(x) = f(a)$
Then we say f is right continous at a.

If $\lim_{x \to a^-} f(x) = f(a)$
Then we say f is left continoues at a.

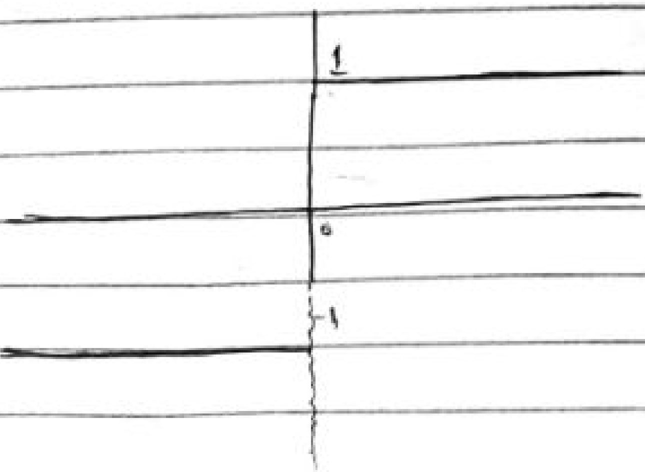
Def: If $f : [a, b] \longrightarrow \mathbb{R}$ is a function defined on $[a, b]$.
For continuity at "a" we chek only ~~Left~~ Right continuity and for continuity at b. we chek only left continuity at b.

Date: 17/12/2020 (91) Day Mon

Signum Function:- (Sgn Function):-

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = 1 \quad \text{If } x > 0$$
$$= 0 \quad \text{if } x = 0$$
$$= -1 \quad \text{if } x < 0$$

This ~~con~~ function is discontinous only at $x = 0$



$f(0) = 0$

$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = -1$

So $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\lim_{x \to 0} f(x)$ does not exists so the function or $f(x)$ is not continoues.

H-w

Theorem:- If $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$

Then,

(i) $\lim_{x \to a} (f(x) \pm g(x)) = f(a) \pm g(a)$

(ii) $\lim_{x \to a} f(x)\, g(x) = f(a)\, g(a)$

(iii) $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{f(a)}$; $f(a) \neq 0$.

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$; $g(a) \neq 0$

Proof:- (i) As given that $\lim_{x \to a} f(x) = f(a)$

and $\lim_{x \to a} g(x) = g(a)$.

So by def. for every $\epsilon/2 > 0$ $\exists$ $\delta_1 > 0$ such that

$|f(x) - g(a)| < \epsilon$; $|x - a| < \delta_1 \longrightarrow (1)$

Date: 17/2/2020 (92) Day

Also $\lim_{x\to a} g(x) = g(a)$

So by def: for every $\epsilon/2 > 0$, $\exists\ \delta_2 > 0$, such that

$$|g(x) - g(a)| < \epsilon \ ; \quad |x-a| < \delta_2 \to (2)$$

Now

$$|f(x)+g(x) - (f(a)+g(a))| = |f(x)-f(a) + (g(x)-g(a))|$$
$$\leq |f(x)-f(a)| + |g(x)-g(a)|$$
$$= \epsilon/2 + \epsilon/2$$

whenever $|x-a| < \delta_1$ and $|x-a| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$

So,

$$|f(x)+g(x) - (f(a)+g(a))| < \epsilon \ ; \quad |x-a| < \delta$$

$\Rightarrow$ So $\lim_{x\to a} (f(x)+g(x)) = f(a)+g(a)$.

X ——— X ——— X ——— X

Proof (ii) Now

$$~~Given~~\ |f(x)g(x) - f(a)g(a)| = |f(x)g(x) - f(a)g(a) + f(x)g(a) - f(a)g(a)|$$

$$|f(x)g(x) - f(a)g(a)| \leq |f(x) - g(a)||g(x)| + |f(a)||g(x) - g(a)|$$
$$\leq \epsilon/2\ |g(x)| + f(a)\ \epsilon/2$$

Now

$$|g(x)| = |g(x) - g(a) + g(a)| \leq |g(x) - g(a)| + |g(a)|$$
$$\leq \epsilon/2 + |g(a)|$$
$$= M$$

$\Rightarrow |g(x)| < M$

So Let $\delta = \min\{\delta_1, \delta_2\}$

So, $|f(x)g(x) - g(a)f(a)| \leq \epsilon/2 \cdot M + |f(a)| \cdot \epsilon/2$

$$= \frac{\epsilon}{2M} M + |f(a)| \cdot \frac{\epsilon}{2|f(a)|}$$

$|f(x)g(x) - f(a)g(a)| = \epsilon \ ; \quad |x-a| < \delta$

$\Rightarrow \lim_{x\to a} f(x)g(x) = f(a)g(a)$

X ——— X ——— X ———

Proof: (iii) Let $\frac{1}{f(x)} = hx$

$$f(x)\,h(x) = 1$$

$$\lim_{x\to a}\left(f(x)\cdot h(x)\right) = \lim_{x\to a} 1 = 1$$

$$\lim_{x\to a} f(x) \cdot \lim_{x\to a} h(x) = 1$$

$$\lim_{x\to a} h(x) = \frac{1}{\lim_{x\to a} f(x)}$$

$$\boxed{\lim_{x\to a}\left(\frac{1}{f(x)}\right) = \frac{1}{f(a)}}$$

Proof: (iv) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$

As above Theorem ⟹ proof (iii) we have $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{f(a)}$

So we can write as;

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \cdot \lim_{x\to a} \frac{1}{g(x)}$$

$$= f(a) \cdot \frac{1}{g(a)}$$

$$= \frac{f(a)}{g(a)}$$

Hence

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}}$$

Theorem: Let $\langle x_n \rangle$ be convergent Sequence which converges to x_0 i.e $\lim_{x\to\infty} x_n = x_0$ then $\lim_{x\to x_0} f(x) = f(x_0)$ iff $\lim_{x\to\infty} f(x_n) = f(x_0)$.

Proof: since $\lim_{x\to\infty} x_n = x_0$, and

let $\lim_{x\to x_0} f(x) = f(x_0)$

we prove that $\lim_{x\to\infty} f(x_n) = f(x_0)$

since $\lim_{x\to\infty} x_n = x_0$. it means that for every $\epsilon > 0$ there exists some positive real number "K" such that

$$|x_n - x_0| < \epsilon_1 \text{ ; whenever } n \geq K \longrightarrow (1)$$

Now since $\lim_{x\to\infty} f(x) = f(x_0)$

it means that for every $\epsilon > 0$ there exist some $\delta > 0$ (depending on ϵ) such that

$|f(x) - f(x_0)| < \epsilon$ whenevere $|x - x_0| < \delta$

$|f(x_n) - f(x_0)| < \epsilon$ whenevere $|x_n - x_0| < \delta \longrightarrow (2)$

$\Rightarrow |f(x_n) - f(x_0)| < \epsilon$; $n \geq K$

$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$

Conversely: Given that $\lim_{x\to\infty} x_n = x_0$, So by def:

$$|x_n - x_0| < \delta \text{ ; } n \geq K \longrightarrow (1)$$

Let us taking $\lim_{x\to\infty} f(x_n) = f(x_0)$

So $|f(x_n) - f(x_0)| < \epsilon$; $n \geq K$ $\Big\} \longrightarrow (2)$

we need to show that $\lim_{x\to x_0} f(x) = f(x_0)$

suppose on contrary that $\lim_{x\to x_0} f(x) \neq f(x_0)$

So by def: for every $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

$|f(x) - f(x_0)| \geq \epsilon$; $|x - x_0| < \delta$.

replacing x by x_n, we have

$|f(x_n) - f(x_0)| \geq \epsilon$; $(x - x_0) < \delta$ and $n \geq K$

$\Rightarrow |f(x_n) - f(x_0)| \geq \epsilon$; $n \geq K$

$\Rightarrow |f(x_k) - f(x_0)| \geq \epsilon \quad ; \quad n \geq k$

$\Rightarrow |f(x_{k+1}) - f(x_0)| \geq \epsilon \quad ; \quad$ "

$\vdots \qquad \vdots \qquad \vdots$

Thus we proved an $\epsilon > 0$ s.t $|f(x_n) - f(x_0)| \geq 0$; $n \geq k$

This our supposition is wrong. Hence $\lim\limits_{x \to \infty} f(x_n) = f(x_0)$.

Theorem:- Let $f : A \to B$ and $g : B \to \mathbb{R}$ are functions such that f is continous at $x_0 \in A$ and g is condions at $f(x_0)$ Then prove that ~~g~~ $g \circ f$ is continous at x_0.

Proof:- we show that $g \circ f : A \to \mathbb{R}$ is continous at x_0

since g is continous at $f(x_0)$

$\epsilon_1 > 0, \quad |g(y) - g(f(x_0))| < \epsilon_1$ when $|y - f(x_0)| < \delta$

since $y \in B$ and $f : A \to B$

$\Rightarrow \exists \; x \in A$ such that $f(x) = y$.

So, $|g(f(x)) - g(f(x_0))| < \epsilon_1$ when $|f(x) - f(x_0)| < \delta_1 \to$ (1)

$\Rightarrow |g\circ f(x) - g \circ f(x_0)| < \epsilon_1$ " $|f(x) - f(x_0)| < \delta \to$ (1) OR

since f is continous at x_0 so for every $\delta_1 > 0$ we can find $\delta_2 > 0$

such that $|f(x) - f(x_0)| < \delta_1$ when $|x - x_0| < \delta_2 \to$ (2)

Using (2) in (1) we obtain

$|g\circ f(x) - g\circ f(x_0)| < \epsilon_1$ whenever $|x - x_0| < \delta_2$

So $g \circ f$ is continous at x_0

Theorem:- If $f : (a, b) \to \mathbb{R}$ is continous function

Then (i) If $x_0 \in (a, b)$ and $f(x_0) > 0$ then There exists $\delta > 0$, such that $f(x) > 0 \; \forall x \in (x_0 - \delta, x_0 + \delta)$

(ii) If $x_0 \in (a, b)$ and $f(x_0) < 0$ then there exits $\delta > 0$ such that $f(x) < 0 \quad \forall \; x \in (x_0 - \delta, x_0 + \delta)$.

OR

OB

If f is continous on (a,b) and $x_0 \in (a,b)$ such that $f(x_0) \neq 0$. then there exists nbhd $(x_0 - \delta, x_0 + \delta)$ of x_0 such that $f(x_0)$ and $f(x)$ have the same sign on $x \in (x_0 - \delta, x_0 + \delta)$

Proof:-(i) since $x_0 \in (a,b)$ and $f(x_0) > 0$

Also f is continous at x_0.

So by $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$.

Let $\epsilon = \frac{1}{2} f(x_0)$

$\Rightarrow |f(x) - f(x_0)| < \frac{1}{2} f(x_0)$ when $|x - x_0| < \delta$

$\Rightarrow -\frac{1}{2} f(x_0) + f(x_0) < f(x) < \frac{1}{2} f(x_0) + f(x_0)$; $|x - x_0|$

$\Rightarrow \frac{1}{2} f(x_0) < f(x) < \frac{3}{2} f(x_0)$ when $|x - x_0| < \delta$

$\Rightarrow f(x) > \frac{1}{2} f(x_0)$ when $|x - x_0| < \delta$

But $\frac{1}{2} f(x_0) > 0$

$\Rightarrow f(x) > 0$ when $|x - x_0| < \delta$.

part(ii):- $x_0 \in (a,b)$, $f(x_0) < 0$

Also f is continous at x_0

Let $= -\frac{1}{2} f(x_0)$

$|f(x) - f(x_0)| < -\frac{1}{2} f(x_0)$ wherever $|x - x_0| < \delta$

$\Rightarrow$ ~~$\frac{1}{2} f(x_0) + f(x_0)$~~

$\frac{1}{2} f(x_0) + f(x_0) < f(x) < -\frac{1}{2} f(x_0) + f(x_0)$; $|x - x_0| < \delta$

$\Rightarrow \frac{3}{2} f(x_0) < f(x) < \frac{1}{2} f(x_0)$, $|x - x_0| < \delta$

$\Rightarrow f(x) < \frac{1}{2} f(x_0)$ when $|x - x_0| < \delta$

But $\frac{1}{2} f(x_0) < 0$

$\Rightarrow f(x) < 0$ when $|x - x_0| < \delta$.

Theorem: Let $f : [a,b] \to \mathbb{R}$ be continous function. If $f(a) < 0$ and $f(b) > 0$. Then there exist $x_0 \in [a,b]$ such that $f(x_0) = 0$

OR

If $f : [a,b] \to \mathbb{R}$ is continous and $f(a)$ and $f(b)$ have opposite sign then $f(x)$ vanishes for at least one point of $[a,b]$.

Proof: Let $A = \{ x \in [a,b] : f(x) < 0 \}$

$\Rightarrow A \subseteq [a,b]$

$\Rightarrow A$ is bounded $\Rightarrow A$ is bounded above.

Let us suppose $Sup(A) = x_0$

we prove that $f(x_0) = 0$

Assume that $f(x_0) \neq 0$

Case (i) If $f(x_0) > 0$ then By previous Theorem there exists $\delta > 0$ such that $f(x) > 0 \quad \forall x \in (x_0 - \delta, x_0 + \delta)$

Case (ii) If $f(x_0) < 0$ then by previous Theorem there exists $\delta > 0$ such that $f(x) < 0 \; \forall x \in (x_0 - \delta, x_0 + \delta)$

$f(x) < 0 \quad \forall \; x_0 - \delta < x < x_0 + \delta$

$\Rightarrow f(x) < 0 \; \forall \; x \in (x_0 - \delta_1, x_0 + \delta_1)$

If $x_0 < y < x_0 + \delta$

$x_0 < y$.

which is contradiction

So we accept that $f(x_0) < 0$.

To prove case (i) suppose on contrary that $f(x_0) > 0$

As f is continous function and $f(x_0) > 0$ then exist a number $\delta > 0$ such that

$f(x_0) > 0 \quad ; \forall \quad x \in (x_0 - \delta, x_0 + \delta)$

As $A = \{x \in [a,b] : f(x_0) \geq 0\}$

$\Rightarrow (x_0 - \delta, x_0) \in A^c \quad ; \quad y \in A^c$

so $\quad y \in (x_0 - \delta, x_0) \subset A^c$

$\Rightarrow \quad x_0 - \delta \; y < x_0 \quad ; \quad y \in A^c$

$\Rightarrow \quad y < x_0 \quad ; \quad y \in A^c$

which is contradiction.

Thus we accepted that $f(x_0) > 0$

Hence from the two "cases"

we have $\quad f(x_0) = 0 \quad$ proved.

Intermediat value Theorem:

Let f be a continuous function on $[a,b]$ with $f(a) \neq f(b)$. suppose K is any number lying b/w $f(a)$ and $f(b)$. Then there exist some $c \in (a,b)$ such that $f(c) = K$.

Proof: Given that f is continuous function on $[a,b]$.

Since K lies b/w $f(a)$ and $f(b)$

Let $f(a) < f(b) \Rightarrow K \in (f(a), f(b))$

Consider $g(x) = f(x) - K$

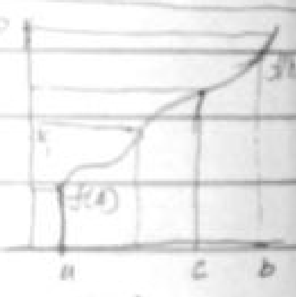
$g(a) = f(a) - K < 0$

and $\quad g(b) = f(b) - K < 0$

By previous Theorem there exists $c \in (a,b)$ such that

$$g(c) = 0 \Rightarrow f(c) - K = 0$$

$$\boxed{f(c) = K}$$



$f(c) = K$

$f(c) - K = 0$

$f(x) - K > 0$

Theorem:- Every continous function defined on $[a,b]$ is bounded.

(Every continous Function defiend on compact set is bounded.)

Proof:- Given that f is continous on $[a,b]$, we prove that f is bounded.

Let us suppose that f is not bounded. Then we can find $x_1, x_2, x_3, \ldots \in [a,b]$ such that

$$f(x_1) > 1, \; f(x_2) > 2, \; f(x_3) > 3, \ldots$$

Then clearly the sequence $\langle f(x_n) \rangle$ is unbounded.

Now Since $x_i \in [a,b]$ $\forall$ $i = 1, 2, 3, \ldots$

$\Rightarrow$ There exists a sub-sequence x_{n_k} such that $x_{n_k} \longrightarrow \alpha$. $\Rightarrow$ $\langle f(x_{n_k}) \rangle$ is convergent.

So which is contradiction and Hence f is bounded.

Theorem:- Every continous function on closed interval $[a,b]$ attain its bounds on $[a,b]$.

Proof:- Let $\text{Lub}\, f(x) = M$ and $\text{glb}\, f(x) = m$

we prove that There exists $c, d \in [a,b]$ such that $f(c) = M = \text{lub}\{f(x)\}$

and $f(d) = m = \text{glb}\{f(x)\}$.

Let $f(x) < M$ $\forall$ $x \in [a,b]$.

$\Rightarrow f(x) - M < 0 \Rightarrow M - f(x) > 0$

$\Rightarrow M - f(x) \neq 0$ $\forall$ $x \in [a,b]$

Let $g(x) = \dfrac{1}{M - f(x)}$ $\Rightarrow$ g is continous $[a,b]$

$\Rightarrow$ g is bounded on $[a,b]$.

$\Rightarrow$ There exists K such that

Date: 20/2/2020 (100) Day

$|g(x)| < K \implies \frac{1}{M-f(x)} < K \quad \forall \; x \in [a,b]$

$\implies M - f(x) > \frac{1}{K}$

$\implies f(x) < M - 1/K \quad \forall \quad x \in [a,b]$

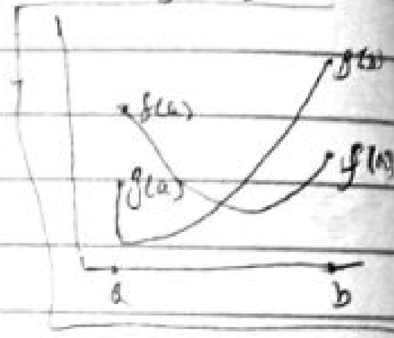
$f(x) < M - 1/K < M$

which is contradiction

$\implies f(x) - M = 0$ at point "c"

$\implies f(c) = M \quad ; \quad c \in [a,b]$

<u>Theorem</u>:- If $f(x)$ and $g(x)$ are continous function $[a,b]$ such that $f(a) > g(a)$ and $f(b) < g(b)$. Then there exist $c \in (a,b)$ such that $g(c) = f(c)$



<u>Proof</u> Let $h(x) = f(x) - g(x)$

since $f(a) > g(a)$

$\implies f(a) - g(a) > 0$

$\implies h(a) > 0$

Also $f(b) < g(b) \implies f(b) - g(b) < 0$

$\implies h(b) < 0$

h is continous and $h(a) > 0$ and $h(b) < 0$

By previous Theorem there exists $c \in (a,b)$ such that $h(c) = 0$

$\implies f(c) - g(c) = 0$

$\implies \boxed{f(c) = g(c)}$

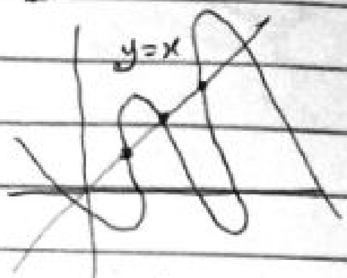
Date: 27/2/2020 (101) Day: جمعرات

Fixed Point :-

Let "f" be a real valued function Then a point $x_0 \in$ Dom'f' is called fixed point of "f" if $f(x_0) = x_0$

Let $f(x) = x$, each point in the Domf is fixed point

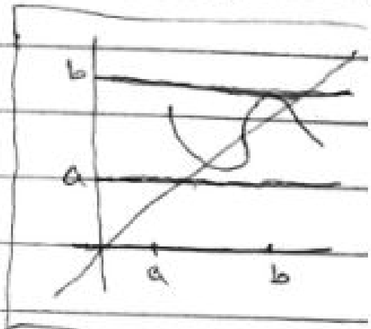
$f(x) = x^2$, $f(0) = 0$, $f(1) = 1$



$f(x) = x^2 - x + 1$

$x^2 - x + 1 = x$

Theorem:- If $f : [a,b] \longrightarrow [a,b]$ is continous function. Then $f(x)$ has at least one fixed point.



Proof:- If $f(a) = a$ or $f(b) = b$. Then a or b is a fixed point of "f".

Let $f(a) \neq a$ and $f(b) \neq b$

Then $f(a) > a$ and $f(b) < b$.

Now let $g(x) = f(x) - x$ Then g is continous.

$g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$

$\Rightarrow$ There exists $x_0 \in [a, b]$ such that

$g(x_0) = 0$

$f(x_0) - x_0 = 0$

$f(x_0) = x_0$

Theorem: Let $f(x)$ be a continous function on $[a,b]$. Then for every $\epsilon>0$, the interval $[a,b]$ can be divided into finite number of sub-intervales such that $|f(x')-f(x'')|<\epsilon$ whenevere x', x'' belong to same sub-intervals.

Proof: On contrary let us suppose that for every $\epsilon>0$ we can not divided $[a,b]$ into sub-interval such that

$|f(x')-f(x'')|<\epsilon$ whenever x', x'' belong to same sub-interval $\rightarrow (1)$

Let bisect $[a,b]$ as $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$ under our assumptions (1) will not holds in one of the interval $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$

Let that interval is $I_1=[a_1, b_1]$

$$\Rightarrow l(I_1)=\frac{b-a}{2}$$

similarly proceed for $[a_1, b_1]$ and we get $I_2=[a_2, b_2]$ such that (1) does not hold in I_2 and $l(I_2)=\frac{b-a}{2^2}$, $I \supset I_2$

we continous this process and we obtain a sequence of intervals

$$I_1 \supset I_2 \supset I_3 \supset \ldots \quad \lim_{n\to\infty} l(I_n)=0$$

So

$$\bigcap_{n=1}^{\infty} I_n=\{x_0\}$$

since f is continous on x_0

$$\Rightarrow |f(x)-f(x_0)|<\epsilon/2 \quad \text{when } |x-x_0|<\delta$$

$$\Rightarrow |f(x)-f(x_0)|<\epsilon/2 \quad \text{when } x\in(x_0-\delta, x_0+\delta)$$

$|f(x') - f(x_0)| < \epsilon/2$ when $x' \in (x_0 - \delta, x_0 + \delta)$

$|f(x'') - f(x_0)| < \epsilon/2$ when $x'' \in (x_0 - \delta, x_0 + \delta)$

Let us choose I_{n_0} such that

$\ell(I_{n_0}) = \delta/2$ as $x_0 \in I_{n_0}$

$\Rightarrow |f(x') - f(x_0)| < \epsilon/2$ when $x' \in I_{n_0}$

$|f(x'') - f(x_0)| < \epsilon/2$ " $x'' \in I_{n_0}$

$$|f(x') - f(x'')| = f(x') - f(x_0) + f(x_0) - f(x'')$$
$$\leq |f(x') - f(x_0)| + |f(x_0) - f(x'')|$$
$$< \epsilon/2 + \epsilon/2 \text{ when } x', x'' \in I_{n_0}$$

$|f(x') - f(x'')| < \epsilon$ when $x', x'' \in I_{n_0}$

Asim Marwat Zangi Khel

University of Peshawar

MSc Mathematics (Previous)

Date 27/2/2020

نہ تیرے آنے کی خوشی نہ تیرے جانے کا غم

اب وہ وقت ہی گزر گیا جب تھے تیرے دیوانے ہم

03151949572

Uniform Continuity:-

A function $f : I \to \mathbb{R}$ is said to be uniform continous on I. If for every $\epsilon > 0$ there exist $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x-y| < \delta \quad x, y \in I.$$

Theorem:- If $f : I \to \mathbb{R}$ is uniformly continous function, Then "f" is continous function.

Proof:- since f is uniformly continous so for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x-y| < \delta$, $\forall x, y \in$

Let $y = a \in I$ Then from above we have

$|f(x) - f(a)| < \epsilon$ when $|x-a| < \delta$

$\Rightarrow$ "f" is continous at "a" but "a" is arbitrary.

So "f" is continous on I.

Theorem:- Every continous function "f" defined on $[a, b]$ is uniformly continous

Proof:- Given "f" is continous on $[a, b]$ we show that "f" is uniformly continous by that theorem for every $\epsilon > 0$, the interval $[a, b]$ can be ~~also~~ devided into sub-intervals say

$I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ..., $I_n = [a_n, b_n]$

such that

$|f(x') - f(x'')| < \epsilon$ where x', x'' lies in same sub-intervele.

Let $\delta = \min\{l(I_1), (I_2), \ldots l(I_n)\}$

Let $x', x'' \in [a,b]$ such that $|x' - x''| < \delta$

Then there are two cases:

case(i):- x', x'' belong to the same interval.

Then $f(x') - f(x'')| < \epsilon$ whenever $x', x'' \in$ same sub-interval.

case(ii):- Both x', x'' lies in the adjust sub-interval, then there exists $x_r \in [a,b]$ such that $|x' - x''| = |x' - x_r| + |x_r - x''|$

Then x', x_r belong to one interval from I_i, $i = 1, 2, 3, \ldots n$.

and x'', x_r belong to ~~another~~ another interval from I_i, $i = 1, 2, 3, \ldots n$.

So by previous Theorem we have

$|f(x') - f(x_r)| < \epsilon/2$ for x', x'' belong to the same interval.

and $|f(x'') - f(x_r)| < \epsilon/2$ for " "

Now

$$|f(x') - f(x'')| = |f(x') - f(x_r) + f(x_r) - f(x'')|$$
$$\leq |f(x') - f(x_r)| + |f(x_r) - f(x'')|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$|f(x') - f(x'')| < \epsilon$ for $|x' - x''| < \delta$

Hence f is uniformly continous.

Example: Show that $f(x) = x^2$ is not uniformly continous on $(0, \infty)$.

Solution: let $\epsilon = 1$ and $\delta > 0$

let $y = \frac{1}{\delta}$ and $x = \frac{1}{\delta} + \frac{\delta}{2}$

Now

$$|x - y| = \left|\cancel{\tfrac{1}{\delta}} + \frac{\delta}{2} - \cancel{\tfrac{1}{\delta}}\right| = \left|\frac{\delta}{2}\right| < \delta$$

$\Rightarrow |x - y| < \delta$

Now

$$|f(x) - f(y)| = \left|\left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \frac{1}{\delta^2}\right|$$

$$= \left|\cancel{\tfrac{1}{\delta^2}} + 1 + \frac{\delta^2}{4} - \cancel{\tfrac{1}{\delta^2}}\right|$$

$$= \left|1 + \frac{\delta^2}{4}\right| > 1 = \epsilon$$

$\Rightarrow |f(x) - f(y)| > 1 = \epsilon$

$\Rightarrow$ "f" is not uniformly continous.

Q:- Show That $f(x) = 1/x$ is not uniformly continous on $(0, \infty)$.

solution. $f(x) = 1/x$, choos $\epsilon = 1$ and $\delta > 0$.

let $y = \min(\delta, 1)$, $x = y/2$

Now

$$|x - y| = \left|\frac{y}{2} - y\right| = \left|-\frac{y}{2}\right| = \frac{y}{2} \le \frac{\delta}{2} < \delta$$

$y \le 1$, $\frac{1}{y} \ge 1$

i.e $x, y \in (0, \infty)$ such that $|x - y| < \delta$

Now

$$|f(x) - f(y)| = \left|\frac{2}{y} - \frac{1}{y}\right| = \frac{1}{y} \ge 1$$

$\Rightarrow$ "f" is not uniformly continous