Chapter 02: Sequence & Series

Handwritten Notes of REAL ANALYSIS

Written By

Asim Marwat

MSc Mathematics (UOP)

asimmarwat41@gmail.com

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CHP # 2 : [Sequence and Series]

Sequence:-

A Sequence is the value of a function whose doman is the set of Natural number.

$f(n) = n+1 \quad , \quad n = 1, 2, 3 \ldots$

$f(1) = 2 \qquad \{2, 3, 4, 5, \ldots\}$

$f(2) = 3$

$f(3) = 4$

and so on.

$f(n) = n^2$

$f\{1, 4, 8 \ldots\}$

$f(n) = 5 \quad , \quad \{5, 5, 5, \ldots\}$

$f(x), f(y), \ldots \quad x \in R\}.$

$a_n, b_n, c_n, a'_n, b'_n, c'_n, \ldots$

$\{a_1, a_2, a_3, \ldots\} \; \{b_1, b_2, b_3, \ldots\}$

$\langle a_n \rangle$ or $\langle a_n : n \in N \rangle$

Convergent of a Sequence.

we say that a sequence $\langle a_n \rangle$ converges to a point a

If $\lim_{n\to\infty} a_n = a$, $a_n \to a$, $a_n \xrightarrow{n\to\infty} a$

$(\; (\; (\; a \;) \;) \;).$

$(a-\epsilon, a+\epsilon)$

$a_1, a_2, a_3, \ldots \in (a-\epsilon, a+\epsilon)$

or $a_n \in (a-\epsilon, a+\epsilon) \quad \forall \; n \geq 1$

$a_2, a_3, \ldots \in (a-\epsilon, a+\epsilon)$

or $a_n \in (a-\epsilon, a+\epsilon) \quad \forall\ n \geq 2$

$a_n \in (a-\epsilon, a+\epsilon) \quad \forall\ n \geq 10$

for every $\epsilon > 0$ we have

$a_n \in (a-\epsilon, a+\epsilon) \quad \forall\ n \geq m$

or

$|a_n - a| < \epsilon \quad \forall\ n \geq m$

$-\epsilon < a_n - a < \epsilon$

$a - \epsilon < a_n < a + \epsilon$

$\Rightarrow a_n \in (a-\epsilon, a+\epsilon)$

$\lvert x\rvert \leq a$
$-a < x < a$
$\lvert x\rvert < 4$
$\Rightarrow x \in (-4, 4)$

we say that a sequence $\langle a_n \rangle$ converges to "a"
for every $\epsilon > 0$ there exist a +ve integer m such that
$|a_n - a| < \epsilon \quad \forall\ n \geq m$.
theere exists lot of Point of $\langle a_n \rangle$ in $(a-\epsilon, a+\epsilon)$.

Example- $\lim_{n\to\infty} \frac{1}{n} = 0 \qquad n \geq 0$

let $\epsilon = 0.01$

$\left|\frac{1}{n} - 0\right| < 0.01$

$\left|\frac{1}{n}\right| < 0.01 \Rightarrow \frac{1}{n} < \frac{1}{100}$

$n > 100$

it is mean if $n > 100$ then the element of a_n "y_n" $\in (a-\epsilon, a+\epsilon)$.

If $a_n \to a$ then for every $\epsilon > 0$ there exists a number "K" such that $n \geq K$ $|a_n - a| < \epsilon \quad \forall\ n \geq K$.

Theorem:- if $<a_n>$ is convergent sequence then prove that the limit of $<a_n>$ is unique.

proof:- Let us suppose that $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = m$ $a_n \to a$

assume that $l \neq m$

$\Rightarrow l - m \neq 0$

$|l-m| > 0$

$\Rightarrow \frac{|l-m|}{5} > 0$

as $\frac{|l-m|}{5}$

$\frac{|l-m|}{2}$

Now since $\lim_{n\to\infty} a_n = l$

So by def: for every $\epsilon = \frac{|l-m|}{5}$

$\exists$ $K_1 \in N$ such that

$|a_n - l| < \frac{|l-m|}{5} \rightarrow (1), \forall n \geq K_1$

Similarly: $\lim_{n\to\infty} a_n = m$ so by def: $\exists$ $K_2 \in N$ such that $|a_n - m| < \frac{|l-m|}{5} \rightarrow (2); \forall n \geq K_2$

let $\max\{K_1, K_2\} = K$

Then from (1) we have

$|a_n - l| < \frac{|l-m|}{2} \rightarrow (3)\ \forall n \geq K$

$|a_n - m| < \frac{|l-m|}{2} \rightarrow (4)\ n \geq K$

let $|l-m| = |l - a_n + a_n - m|$

$\leq |a_n - l| + |a_n - m|$

$< \frac{|l-m|}{5} + \frac{|l-m|}{5}$

$|l-m| < 2\frac{|l-m|}{5}$

$1 < 2/5 \Rightarrow 5 < 2$ which is contradiction Hence $l = m$

if $|a_n - l| < \epsilon; \forall n \geq 1000$

$|a_n - m| < \epsilon; \forall n \geq 1000$

$|a_n - l| < \epsilon; \forall n \geq 1000$

$|l-m| = l - a_n + a_n - m|$

$\leq |l - a_n| + |a_n - m|$

$< \frac{|l-m|}{5} + \frac{|l-m|}{5}$

$\Rightarrow |l-m| < 2\frac{|l-m|}{5}$

$1 < 2/5$

$5 < 2$

contradiction

Bounded Sequence

If $\langle a_n \rangle$ is a sequence and if there exists some +ve real number "K" such that $|a_n| \le K$

$\Rightarrow -K \le a_n \le K$

also if there exist $m, M \in \mathbb{R}$ such that

$m \le a_n \le M$

$\sup = M$, $\inf = m$

If all the interval of $\langle a_n \rangle$ lies in bounded interval then Sequence is bounded.

Theorem:- If $\langle a_n \rangle$ is a sequence such that $\langle a_n \rangle$ is convergent then prove that $\langle a_n \rangle$ is bounded but the converse is not true in general.

Proof:- Since $\langle a_n \rangle$ is convergent

let $\lim_{n\to\infty} a_n = a$ then by def:

for every $\epsilon > 0$ $\exists$ a +ve integer "K"

such that $|a_n - a| < \epsilon$ $\forall n \ge K$

in particular let $\epsilon = 1$

Then $|a_n - a| < 1$; $\forall n \ge K$

$\Rightarrow |a_n| = |a_n - a + a|$

$\le |a_n - a| + |a| < 1 + a$; $\forall n \ge K$

$\Rightarrow |a_n| < 1 + |a|$ $\forall n \ge K$

i.e $|a_K| < 1 + |a|$

$|a_{K+1}| < 1 + |a|$

$|a_k + 2| < 1 + |a|$

let $M = \max\{|a_1|, |a_2|, \ldots, |a_{k-1}|, 1+|a|\}$

$\Rightarrow |a_1| \le M$

$|a_2| \le M$

$|a_n| \le M$

$\vdots$

$|a_{k-1}| \le M$

$1 + |a| \le M \longrightarrow (1)$

Now $|a_k| < 1 + |a|$

$|a_k| \le M$

$\Rightarrow |a_n| \le M \quad \forall\, n \in N$

converse:- let $\langle a_n \rangle = (-1)^n, \ n = 1, 2, 3, \ldots$

So "a_n" is bounded but not converge.

Theorem:- If $a_n \to a$ and $b_n \to b$ then

$a_n + b_n \to a + b$ or $a_n - b_n \to a - b$.

Proof: since $a_n \to a$

$\Rightarrow |a_n - a| < \epsilon/2$ whenever $n \ge k_1$

Also $b_n \to b$

$\Rightarrow |b_n - b| < \epsilon/2$ when ever $n \ge k_2$

let $\max\{k_1, k_2\} = K$ then

$|a_n - a| < \epsilon/2 \to (1)$ when ever $n \ge k_1$

$|b_n - b| < \epsilon/2 \to (2)$ when ever $n \ge k_2$

let $|a_n + b_n - (a+b)|$

$$= |a_n - a + b_n - b|$$
$$\leq |a_n - a| + |b_n - b|$$
$$< \epsilon/2 + \epsilon/2 = 2\epsilon/2 \quad \forall \; n \geq k$$
$$\Rightarrow |a_n + b_n - (a+b)| < \epsilon$$
$$\Rightarrow a_n + b_n \to a + b$$
similarly $a_n - b_n \to a - b$

Theorem: If $a_n \to a$ and $b_n \to b$ then prove that $a_n b_n \to an$.

Proof:- since $a_n \to a$ and $b_n \to b$
$$\Rightarrow |a_n - a| < \epsilon_1 \text{ and } |b_n - b| < \epsilon_2, \forall n \geq k$$

consider $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab|$
$$= |b_n(a_n - a) + a(b_n - b)|$$
$$\leq |b_n||a_n - a| + |a||b_n - b|$$
$$|a_n b_n - ab| < |b_n|\epsilon_1 + |a|\epsilon_2 \to (1) \; \forall \; n \geq k$$
Since $\langle b_n \rangle$ is convergent so
$\langle b_n \rangle$ is bounded
$$\Rightarrow |b_n| \leq M \quad \forall n$$
$$(1) \Rightarrow |a_n b_n - ab| < |M|\epsilon_1 + |a|\epsilon_2 = \epsilon \; ; \; n \geq k$$
$$\Rightarrow |a_n b_n - ab| < \epsilon \quad \forall \; n \geq k$$
$\Rightarrow a_n b_n$ converge to "ab"
Hence $a_n b_n \to ab$

Asim Marwat UOS

Theorem: If $a_n \to a$ and $a_n \neq 0$ $\forall n$ then prove that $\frac{1}{a_n} \to \frac{1}{a}$

Proof: since $a_n \to a \Rightarrow |a_n - a| < \epsilon$; whenever $n \geq k$

consider $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n a|}$

$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|} < \frac{\epsilon}{|a_n||a|} \to (1)$ $\forall n \geq k$

as $\big||a_n| - |a|\big| \leq |a_n - a| < \epsilon$ $\forall n \geq k$

$\Rightarrow \big||a_n| - |a|\big| < \epsilon$ $\forall n \geq k$

$\Rightarrow -\epsilon < |a_n| - |a| < \epsilon$

$\Rightarrow |a| - \epsilon < |a_n| < |a| + \epsilon$

$\Rightarrow a_n > |a| - \epsilon$ $\forall n \geq k$

Assum that let $\epsilon = \frac{|a|}{2}$

$|a_n| > |a| - \frac{|a|}{2}$

$|a_n| > \frac{|a|}{2}$; $\forall n \geq k$

$\frac{1}{|a_n|} < \frac{2}{|a|}$ whenever $n \geq k$

(1) $\Rightarrow \left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{\epsilon}{|a|} \frac{2}{|a|}$; $n \geq k$

$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{1}{\cancel{|a|}} \cdot \frac{\cancel{|a|}}{2}$ $\forall n \geq k$

$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{1}{2} = \epsilon$

$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{a}\right| < \epsilon$ $\boxed{1/a_n \to 1/a}$

So ~~$1/a_n \to 1/a$~~

Theorem:- If $a_n \to a$ and $b_n \to b$ and $b_n \neq 0$ then prove that $a_n/b_n \to a/b$

Proof:- Now we show that if $a_n \to a$ and $b_n \to b$

with $b \neq 0$ the $a_n/b_n \to a/b$

since $b_n \to b$ and $b \neq 0$
so $1/b_n \to 1/b$ prove

Now $a_n \to a$ and $1/b_n \to 1/b$

$$a_n \cdot 1/b_n \to a \cdot 1/b$$

i.e $\boxed{a_n/b_n \to a/b}$

Asim Marwat Zangi Khel

MSc Maths (previous) UOP

Cell No > 0315 19 49 572

Date 9/1/ 2020

Theorem :- If $\langle a_n \rangle$ is a sequence such that $a_n \geq 0$ $\forall n$ and $\lim_{n\to\infty} a_n = a$ Then prove that $a \geq 0$.

Proof :- Since $\lim_{n\to\infty} a_n = a$, we prove that $a \geq 0$

Assume that $a < 0 \Rightarrow -a > 0$

since $a_n \to a \Rightarrow$ for every $\epsilon > 0$,

in particular

$\epsilon = -a$, we have

$|a_n - a| < -a$ whenever $n \geq K$

$\Rightarrow a < a_n - a < -a$

$\Rightarrow a + a < a_n < -a + a$

$\Rightarrow 2a < a_n < 0 \quad \forall \; n \geq K$

ie

$a_K < 0$

$a_{K+1} < 0$

$a_{K+2} < 0$

$\vdots \quad \vdots$

which is contradiction to the fact that

$a_n \geq 0$

Hence $a \geq 0$.

Theorem :- If $\langle x_n \rangle$ and $\langle y_n \rangle$ are two convergent sequence and $x_n \leq y_n$ $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$

Proof :- Since $x_n \leq y_n \Rightarrow y_n - x_n \geq 0 \;\forall n$.

since $\langle x_n \rangle$ and $\langle y_n \rangle$ are two convergent sequence.

But $\langle y_n - x_n \rangle$ is non-negative

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by provius Theorem

$$\lim_{n\to\infty} (y_n - x_n) \geq 0$$

$$\Rightarrow \lim_{n\to\infty} y_n - \lim_{n\to\infty} x_n \geq 0$$

$$\Rightarrow \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n .$$

H-W

Theorem:- If $a \leq x_n \leq b$ Then prove that

$$a \leq \lim_{n\to\infty} x_n \leq b .$$

Proof:- Let x_n is a convergent sequence. and $a \leq x_n \leq b$, Then we have to show that $a \leq \lim_{n\to\infty} x_n \leq b$.

For this let us take $a \leq \lim_{n\to\infty} x_n \to ① \ \forall n \in \mathbb{N}$

and $\lim_{n\to\infty} \geq b \to ② \ ; \forall n \in \mathbb{N}$

As x_n is convergent the $\lim_{n\to\infty} x_n = x$

let a and b are also convergent sequence

After $n \to \infty$ the $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$

Taking ①, $a \leq \lim_{n\to\infty} x_n$.

As a and x_n both are convergent Seq

Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} x_n \to ③$

Now taking ②

As both b and x_n are convergent sequence

Then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} b_n \to ④$

From ③ and ④, we have

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} b_n .$$

As $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$

So $\boxed{a \leq \lim_{n\to\infty} x_n \leq b .}$

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Sandwich Theorem:-

If $\langle x_n \rangle$, $\langle y_n \rangle$ and $\langle z_n \rangle$ are sequence of real numbers such that $x_n \le y_n \le z_n \ \forall \ n \in N$ and $\lim\limits_{n\to\infty} x_n = \lim z_n$ Then prove that $\boxed{\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n}$

Proof:- let $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = l$

by def: convergent sequence

$|x_n - l| < \epsilon$ when ever $n \ge k_1$

and $|z_n - l| < \epsilon$ " " $n \ge k_2$

let $k = \max\{k_1, k_2\}$

Then $|x_n - l| < \epsilon \quad \forall \ n \ge k$

$|z_n - l| < \epsilon \quad \forall \ n \ge k$

$|x_n - l| < \epsilon \quad \forall \ n \ge k$

$\Rightarrow -\epsilon < x_n - l < \epsilon$

$\Rightarrow l - \epsilon < x_n < l + \epsilon \rightarrow (1) \ \forall \ n \ge k$

Similarly $l - \epsilon < z_n < l + \epsilon \rightarrow (2) \ \forall n \ge k$

Since $x_n \le y_n \le z_n$ (given)

$l - \epsilon < x_n \le y_n \le z_n < l + \epsilon \quad \forall \ n \ge k$ (1) and (2)

$l - \epsilon < y_n < l + \epsilon \ ; \ \forall \ n \ge k$

$\Rightarrow |y_n - l| < \epsilon \ ; \ \forall \ n \ge k$

$\Rightarrow \lim\limits_{n\to\infty} y_n = l$

$\Rightarrow y_n \rightarrow l$.

Thus $\lim\limits_{n\to\infty} x_n = l$, $\lim\limits_{n\to\infty} y_n = l$, $\lim\limits_{n\to\infty} z_n = l$

Hence $\boxed{\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n}$

H.W Date: / / (42) Day

Q: Let $\langle x_n \rangle$ be a sequence with

$$x_n = \frac{(\cos \pi n)(\sin^2 n)}{e\sqrt{n}}$$

Discuss the convergence

Ans:- Given $x_n = \frac{(\cos \pi n)(\sin^2 n)}{e\sqrt{n}} = \frac{\cos \pi n \sin^2 n}{n^{1/e}}$

Since $\cos \pi n = (-1)^n$ and $\sin^2 n \le 1$

we can write

$$-1 \le \cos \pi n \le 1$$

$$\Rightarrow \quad -1(\sin^2 n) \le \cos \pi n \sin^2 n \le (1)\sin^2 n$$

since $\sin^2 n \le 1$

$$\Rightarrow -1(\sin^2 n) \ge -1$$

i.e $-1 \le -\sin^2 n$, So form above

$$-1 \le (-1)\sin^2 n \le \cos \pi n \sin^2 n \le 1(\sin^2 n) \le 1$$

and $-1 \le \cos \pi n \sin^2 n \le 1$

But $(n)^{1/e} > 0$

$$-\frac{1}{n^{1/e}} \le \frac{\cos \pi n \sin^2 n}{n^{1/e}} \le \frac{1}{n^{1/e}}$$

let $a_n = -\frac{1}{n^{1/e}}$ and $b_n = \frac{1}{n^{1/e}}$ Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{1}{n^{1/e}} = 0$$

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n^{1/e} = 0$

So by ~~Sandwich~~ sandwich Theorem

$$\lim_{n\to\infty} \frac{\cos \pi n \sin^2 n}{n^{1/e}} = 0.$$

ie $\lim_{n\to\infty} x_n = 0$.

Note

Q:- Let $\langle x_n \rangle$ be a sequence with $x_1 = 2$ and $x_n = \sqrt{5x_{n-1}+6}$ for $n \geq 2$. suppose $\lim_{n\to\infty} x_n$ exists. find the limit.

Ans:- Given that $\langle x_n \rangle$ is a sequence with $x_1 = 2$ and $x_n = \sqrt{5x_{n-1}+6}$. Also given that $\lim_{n\to\infty} x_n$ exists which is x

$$x_n^2 = 5x_{n-1} + 6 \Rightarrow \lim_{n\to\infty} x_n^2 = \lim_{n\to\infty}(5x_{n-1}+6)$$

$$\Rightarrow \lim_{n\to\infty} x_n^2 = 5\lim_{n\to\infty} x_{n-1} + 6 \Rightarrow x^2 - 5x + 6$$

$$\Rightarrow x^2 - 5x - 6 = 0 \Rightarrow x^2 - 6x + x - 6 = 0$$

$$\Rightarrow x(x-6) + 1(x-6) = 0$$

$$\Rightarrow (x-6)(x+1) = 0$$

$$\Rightarrow x-6 = 0,\ x+1 = 0$$

$$\Rightarrow x = 6,\ x = -1$$

But from the def: of the sequence i.e $x_1 = 2$ and $x_n = \sqrt{5x_{n-1}+6}$, $x_n > 0$

so $\lim_{n\to\infty} x_n \geq 0$.

$$\therefore \boxed{\lim_{n\to\infty} x_n = 6} \text{ Ans}$$

Asim Marwat Zangi Khel

universety of peshawer

Msc : Mathematics (Previ)

0315 19 49 572.

Monotonic Sequence :-

A Sequence $\langle a_n \rangle$ is said to be monotonic increasing if $a_1 \le a_2 \le a_3 \le \cdots$ i.e $a_n \le a_{n+1}$; $\forall n \in \mathbb{N}$.
and a sequence $\langle a_n \rangle$ is said to be monotonic decreasing if $a_1 \ge a_2 \ge a_3 \ge \cdots$ i.e $a_n \ge a_{n+1}$
If $a_1 < a_2 < a_3 < \cdots$ Then $\langle a_n \rangle$ is said to strictly increasing Sequence.
If $a_1 > a_2 > a_3 > \cdots$ Then $\langle a_n \rangle$ is said to strictly decreasing sequence.

Theorem:- If $\langle a_n \rangle$ is increasing and bounded (bounded above) sequence Then prove that $\langle a_n \rangle$ converges to its lub.

Proof:- Since $\langle a_n \rangle$ is bounded (bounded above) Therefore there must exists lub of $\langle a_n \rangle$
let lub of $\langle a_n \rangle$ is x^* we prove that $a_n \to x^*$.
since lub of $\langle a_n \rangle = x^*$, So for any $\epsilon > 0$
$x^* - \epsilon$ can not be upper bound of $\langle a_n \rangle$
$\Rightarrow$ There exist some k such that $x^* - \epsilon < x_k$
But $x_k \le x_{k+1} \le x_{k+2} \le \cdots$
Therefore $x^* - \epsilon < x_k \le x_{k+1} \le x_{k+2} \cdots$
$\Rightarrow x^* - \epsilon < a_n \quad \forall n \ge k$
But $a_n \le x^* < x^* + \epsilon$
Therefore $x^* - \epsilon < a_n < x^* + \epsilon \quad ; \forall n \ge k$
$a_n \in (x^* - \epsilon, x^* + \epsilon) \quad ; \forall n \ge k$
$\Rightarrow |a_n - x^*| < \epsilon \quad , \forall n \ge k$

$\boxed{a_n \to x^*}$

Theorem:- If $\langle a_n \rangle$ is decreasing and bounded (bounded below) Then prove that $\langle a_n \rangle$ converges to its glb.

Proof:- Since $\langle a_n \rangle$ is bounded (bounded below) Therefore there must exists glb of $\langle a_n \rangle$

let glb of $\langle a_n \rangle$ is y^* we prove that

$$a_n \to y^*$$

since glb of $\langle a_n \rangle = y^*$, So for any $\epsilon > 0$

$y^* + \epsilon$ can not be lower bound of $\langle a_n \rangle \Rightarrow$ There exist some k such that

$$y^* + \epsilon > a_k$$

~~But Therefore~~ $a_k \geq a_{k+1} \geq a_{k+2} \geq$ ---

therefore $y^* + \epsilon > a_k \geq a_{k+2} \geq$ ---

$\Rightarrow$ $y^* + \epsilon > a_n \quad ; \forall \; n \geq k$

But $y^* - \epsilon < y^* \leq a_n \quad \forall \; n$

So $y^* - \epsilon < a_n < y^* + \epsilon \quad ; \forall \; n \geq k$

$a_n \in (y^* - \epsilon, y^* + \epsilon) \quad ; \forall \; n \geq k$

$|a_n - y^*| < \epsilon \quad \forall \; n \geq k$

Hence $\boxed{a_n \to y^*}$

Monotonic convergence Theorem:-

Statement:- Every monotonic sequence of real numbers is convergent $\Leftrightarrow$ it is bounded.

Proof. If $\langle a_n \rangle$ is monotonic and convergent the $\langle a_n \rangle$ bounded

(we proved every convergent is bounded)

If $\langle a_n \rangle$ is bounded and monotonic. Then $\langle a_n \rangle$ converges to it lub if it increasing.
and $\langle a_n \rangle$ converges to it glb if it is decreasing.

Q:- Let $a_n = \sqrt{n+1} - \sqrt{n}$ Show that $\langle a_n \rangle$ is convergent.

Ans:- $\dfrac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \times \sqrt{n+1} + \sqrt{n}$

$$= \frac{\cancel{n}+1 - \cancel{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$a_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$.

since $\sqrt{n+1} + \sqrt{n} \leq \sqrt{n+2} + \sqrt{n+1}$

$$\Rightarrow \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

$\Rightarrow a_n \geq a_{n+1}$

$\Rightarrow a_n$ is decreasing sequence.

since $a_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < 1$

$\Rightarrow 0 \leq a_n \leq 1$

$\Rightarrow \langle a_n \rangle$ is decreasing and bounded

$\Rightarrow \lim\limits_{n \to \infty} a_n = \text{glb}\{a_n\}$

$\lim\limits_{n \to \infty} a_n = 0$.

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interval

Nested intervals.-

If $I_1, I_2, \ldots$ are intervals such that $I_1 \supseteq I_2 \supseteq \ldots$

Theorem:- If $(I_n)_{n \in \mathbb{N}}$ is a sequence of closed intervals i.e $I_n = \{a_n, b_n\}$ such that (i) $I_1 \supseteq I_2 \supseteq \ldots$
(ii) $\lim_{n\to\infty} l(I_n) = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ containing only one point.

Proof:- Let $I_1 = [a_1, b_1]$, $I_2 = [a, b_2], \ldots$
since $I_1 \supseteq I_2 \supseteq \ldots$ therefore
$a_1 \leq a_2 \leq a_3 \leq \ldots$ and $b_1 \geq b_2 \geq b_3 \geq \ldots$
i.e $\langle a_n \rangle$ is increasing and $\langle b_n \rangle$ is decreasing sequence by $a_n \in [a_1, b_1]$ $\forall n$ and $b_n \in [a_1, b_1]$
There fore $\langle a_n \rangle$ and $\langle b_n \rangle$ are monotonic and bounded sequence.

$\langle a_n \rangle$ is increasing and bounded, So $\langle a_n \rangle$ converges to it lub x^* (say).
and $\langle b_n \rangle$ is decreasing and bounded. SO $\langle b_n \rangle$ converges to it glb y^* (say).
ie $\lim_{n\to\infty} a_n = x^*$ and $\lim_{n\to\infty} b_n = y^*$

But $\lim_{n\to\infty} l(I_n) = 0 \implies \lim_{n\to\infty} (b_n - a_n) = 0$

$\implies \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$

$\implies y^* - x^* = 0$ $\quad y^* = x^*$

since lub$\{a_n\} = x^*$ and glb$\{b_n\} = y^*$ $\forall n$.

$\implies x^* \geq a_n$ and $y^* \geq b_n$

③ Babar Register

$\Rightarrow a_n \leq x^* = y^* \leq b_n \quad \forall n \in \mathbb{N}$

$\Rightarrow x^* \in [a_n, b_n] \quad \forall n$

$\Rightarrow x^* \in \bigcap_{n=1}^{\infty} [a_n, b_n]$

$\Rightarrow x^* \in \bigcap_{n=1}^{\infty} I_n$

Next we show that x^* is unique. For this assume that $z^* \in \bigcap_{n=1}^{\infty} I_n$ and $x^* \neq z^*$

Let $z^* > x^* \Rightarrow z^* - x^* > 0$

$\Rightarrow \delta = z^* - x^*$

Since $\lim_{n \to \infty} l(I_n) = 0$

Therefore we can find and interval say I_k such that $l(I_k) < \delta$.

But $z^* \in I_k$, $x^* \in I_k$

and $z^* - x^* = \delta$

which is contradiction and hence $x^* = z^*$.

Sub-sequence:-

If $(x_n)_{n \in \mathbb{N}}$ is a sequence and $n_1, n_2, n_3, \ldots \in \mathbb{N}$ such that $n_1 < n_2 < n_3 \ldots$ Then $(x_{n_k})_{k \in \mathbb{N}}$ is called sub-sequence of x_n.

Theorem:- If $\langle x_n \rangle$ is convergent sequence Then it must have convergent sub-sequence.

Proof:- Since $\langle x_n \rangle$ is convergent so let $x_n \to l$.

by $|x_n - l| < \epsilon \quad \forall, \ n \geq m$

If $\langle x_{n_k} \rangle$ is sub-sequence $(x_n)_{n \in N}$ Then

$$|x_{n_k} - l| < \epsilon \qquad \forall \ n_k \geq m$$

$$\Rightarrow x_{n_k} \to l.$$

Theorem:- Every bounded and infinite sequence of real number has a convergent sub-sequence.

OR

(Bolzano-Weistress Theorem for Seq)

Proof:- Let $\langle x_n \rangle$ be bounded and infinite sequence of real number.

we show that $\langle x_n \rangle$ has a convergent sub-sequence

Since $\langle x_n \rangle$ is bounded

$\Rightarrow$ There exists $M \in \mathbb{R}$ such that

$|x_n| \leq M \quad \forall \ n \in N$

$\Rightarrow -M \leq x_n \leq M \qquad \forall \ n$

$\Rightarrow x_n \in [-M, M] \qquad \forall \ n \in N$

Let $I = [-M, M]$. Bisect I Then we have $[-M, 0]$, $[0, M]$

One of These intervals must contains infinite points of $\langle x_n \rangle$ and let I_1 be one of Them which contain infinite point.

Then $\ell(I_1) = M$.
by similar way we can find a sub-interval I_2 of I_1 such that I_2 contain infinite points of I_1

$\ell(I_2) = M/2$, $I_1 \supseteq I_2$

$\vdots \qquad \vdots$

$\ell(I_n) = M/2^{n-1}$

Hence we a sequence of nested intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$
and $\ell(I_n) = M/2^{n-1}$

$\Rightarrow \lim_{n\to\infty} \ell(I_n) = 0$

By Nested interval then we can find exactly one point x^* such that $x^* \in I_n \forall n$
Choose sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ as
$x_{n_1} \in I_1$, $x_{n_2} \in I_2$, ——
Now,

$\because x_{n_k} \in I_k$
$x^* \in I_k$
$\ell(I_k) = \frac{M}{2^{k-1}}$

$$\lim_{k\to\infty} |x_{n_k} - x^*| \le \lim_{k\to\infty} \frac{M}{2^{k-1}}$$

$$\left[\lim_{k\to\infty} x_{n_k} - x^*\right] \le 0$$

$$\lim_{k\to\infty} x_{n_k} - x^* = 0$$

$$\boxed{\lim_{k\to\infty} x_{n_k} = x^*}$$

Cauchy Sequence :-

A sequence $\langle x_n \rangle$ is said to be cauchy sequence if for every $\epsilon > 0$. There exits positive integer K such that $|x_n - x_m| < \epsilon ; \forall n,m \geq k$
or if $\langle x_n \rangle$ is cauchy sequence then

$$\lim_{\substack{n\to\infty \\ m\to\infty}} |x_n - x_m| = 0$$

Theorem :- prove that every convergent sequence is cauchy sequence.

Proof :- Let $\langle x_n \rangle \longrightarrow x$, we prove that x_n is cauchy seq.

$|x_n - x| < \epsilon_1$ — whenever $n \geq k_1$
$|x_m - x| < \epsilon_2$ — " $m \geq k_2$

$$\left.\begin{aligned} |x_n - x| &< \epsilon_1 \\ |x_m - x| &< \epsilon_2 \end{aligned}\right\} \begin{aligned} n &\geq k_1 \\ m &\geq k_2 \end{aligned} \Big| \; K = Max\{k_1, k_2\}$$

Consider $|x_n - x_m| = x_n - x + x - x_m$
$\leq |x_n - x| + |x_m - x| < \epsilon_1 + \epsilon_2$

$\Rightarrow$
$|x_n - x_m| < \epsilon_1 + \epsilon_2 \quad ; \quad n, m \geq k$
let $\epsilon_1 + \epsilon_2 = \epsilon$

$|x_n - x_m| < \epsilon$ whenever $n, m \geq k$

Theorem:- Every Cauchy sequence is bounded

Proof:- Let $\langle x_n \rangle$ be a cauchy sequence of numbers we show that $\langle x_n \rangle$ is bounded.

since $\langle x_n \rangle$ is cauchy sequence So for every $\epsilon > 0$. There exist +ve integer k such that

$$|x_n - x_m| < \epsilon \quad ; \text{where } n, m \geq k$$

In particular $m = k$, we have

$$|x_n - x_k| < \epsilon \quad ; \text{where } n \geq k$$

Now $|x_n| = |x_n - x_k + x_k| \leq |x_n - x_k| + |x_k|$

$$|x_n| \leq |x_n - x_k| + |x_k| < \epsilon + |x_k| ; \forall n \geq k$$

$$|x_n| < \epsilon + |x_k| \quad ; \forall n \geq k$$

$M = \text{Max} \{|x_1|, |x_2|, \ldots |x_{k-1}|, \epsilon + |x_k|\}$

$\Rightarrow |x_1| \leq M, |x_2| \leq M, \ldots, \epsilon + |x_k| < M$

~~But~~

But $|x_n| < \epsilon + |x_k| \quad , \forall n \geq k$

$$|x_n| < M \quad , \forall n \geq k$$

$$\Rightarrow |x_n| \leq M \quad , \forall n \in N$$

$\Rightarrow \langle x_n \rangle$ is bounded.

Theorem:- If $\langle x_n \rangle$ is a cauchy sequence of real numbers and $\langle x_{n_k} \rangle$ is its sub-sequence such that $\langle x_{n_k} \rangle$ converges to x^* Then prove That $\langle x_n \rangle$ converges to x^*.

Proof:- Since $\langle x_n \rangle$ is cauchy sequence

$|x_n - x_m| < \epsilon_1 \rightarrow (1); \; \forall \; n, m \geq K_1$

Also $x_{n_k} \rightarrow x^*$, So

$|x_{n_k} - x^*| < \epsilon_2 \rightarrow (2); \;$ when $n_k \geq K_2$

From (1) we can also write

$|x_n - x_{n_k}| < \epsilon_1 \rightarrow (3)$; where $n, n_k \geq K_1$

let $Max\{K_1, K_2\} = K_3$

so (2) and (3) can be written as:

$|x_{n_k} - x^*| < \epsilon_2$; whenever $n \geq K_3$

$|x_n - x_{n_k}| < \epsilon_1$; // $n, n_k \geq K_3$

Now $|x_n - x^*| = |x_n - x_{n_k} + x_{n_k} - x^*|$; $n, n_k \geq K_3$

$\leq |x_n - x_{n_k}| + |x_{n_k} - x^*| < \epsilon_1 + \epsilon_2$

$|x_n - x^*| < \epsilon_1 + \epsilon_2$; $n \geq K_3$

let $\epsilon_1 = \epsilon/2$, $\epsilon_2 = \epsilon/2$

$|x_n - x^*| < \epsilon/2 + \epsilon/2$; whenever $n \geq K_3$

$|x_n - x^*| < \epsilon$ // //

$\Rightarrow \langle x_n \rangle$ converges to x^*.

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Theorem: Every infinite sequence of real numbers is chauchy iff it is convergent.

Proof: let $\langle x_n \rangle \to x$, we prove that x_n is chauch sequence.

$|x_n - x| < \epsilon_1$ whenever $n \geq k_1$

$|x_m - x| < \epsilon_2$ " " $n \geq k_2$

$$\left.\begin{array}{l} |x_n - x| < \epsilon_1 \ ; n \geq k_1 \\ |x_m - x| < \epsilon_2 \ ; n \geq k_2 \end{array}\right] \to K = Max\{K_1, K_2\}$$

consider $|x_n - x_m| = x_n - x + x - x_m$

$$\leq |x_n - x| + |x_m - x|$$

$$< \epsilon_1 + \epsilon_2$$

$\Rightarrow |x_n - x_m| < \epsilon_1 + \epsilon_2 \ ; n, m \geq k$

let $\epsilon_1 + \epsilon_2 = \epsilon$

$|x_n - x_m| < \epsilon \ ; n, m \geq k$

Conv: let $\langle x_n \rangle$ be cauchy sequence.

$\Rightarrow \langle x_n \rangle$ is bounded.

$\Rightarrow \langle x_n \rangle$ is infinite and bounded sequence

$\Rightarrow \langle x_n \rangle$ has convergent sub-sequence $\langle x_{n_i} \rangle$.

$\Rightarrow \langle x_n \rangle$ is convergent.

Q:- If $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$; $n = 1, 2, \ldots$
prove that $\langle a_n \rangle$ is convergent sequence.

Solution:- Let $\langle a_n \rangle$ is converges to l.

$$\Rightarrow \lim_{n \to \infty} a_n = l$$

Since $a_{n+1} = \sqrt{2 + a_n}$

$$\lim_{n \to \infty} a_{n+1} = \sqrt{2 + \lim_{n \to \infty} a_n}$$

$$l = \sqrt{2 + l}$$

$$\Rightarrow l^2 = 2 + l$$

$$l^2 - l - 2 = 0$$

$$l^2 - 2l + l - 2 = 0$$

$$l(l-2) + (l-2) = 0$$

$$(l+1)(l-2) = 0$$

$$l = -1, \quad l = 2$$

Since $a_n \geq 0 \ \forall n$

there $l = -1$ can not be limit of $\langle a_n \rangle$

Hence $\boxed{\lim_{n \to \infty} a_n = 2}$

Q:- If $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5 + a_n}$, $n = 1, 2, \ldots$
prove that $\langle a_n \rangle$ is convergent

Solution:- Let $\langle a_n \rangle$ is converges to l.

$$\Rightarrow \lim_{n \to \infty} a_n = l$$

Since $a_{n+1} = \sqrt{5 + a_n}$

$$\lim_{n \to \infty} a_{n+1} = \sqrt{5 + \lim_{n \to \infty} a_n}$$

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$l = \sqrt{5 + l}$

$\Rightarrow l^2 = 5 + l$

$\Rightarrow l^2 - l - 5 = 0$

$a = 1, \; b = -1, \; c = -5$

$l = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$l = \frac{1 \pm \sqrt{1 + 20}}{2}$

$l = \frac{1 \pm \sqrt{21}}{2}$

$l = \frac{1 + \sqrt{21}}{2}, \quad l = \frac{1 - \sqrt{21}}{2}$

$l = 3, \quad l = -1$

since $a_n \geq 0 \quad \forall n$

There $l = -1$ can not be limt of $\langle a_n \rangle$

Hence $\boxed{\lim_{n \to \infty} a_n = 3}$

Series:-

If $\langle a_n \rangle$ is a sequence Then

$\sum_{n=1}^{\infty} a_n$ is called Series.

ie: $S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots \qquad \vdots$

$S_n = a_1 + a_2 + a_3 \cdots a_n$

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{k=1}^{\infty} a_k.$$

$\langle a_n \rangle = \langle n \rangle$, $n = 1, 2, \cdots$

$\Rightarrow$ show that $\sum_{n=1}^{\infty} a_n$ is divergent

$S_1 = 1$

$S_2 = 1+2$

$S_3 = 1+2+3$

$\vdots$

$S_n = 1+2+3+ \cdots n$

General

$$S_n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n(n+1)/2 = \infty$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

Q: Show that $\sum_{K=1}^{\infty} \frac{1}{K(K+1)}$ is convergent.

Solution: Hence

$$a_K = \frac{1}{K(K+1)}$$

$$a_K = \frac{1}{K} - \frac{1}{K+1} \quad \text{(Partial fraction)}$$

$$S_n = a_1 + a_2 + a_3 \text{------} a_n$$

$$S_n = \left\{1 - \cancel{\tfrac{1}{2}}\right\} + \left\{\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right\} + \left\{\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right\} \text{------} \cancel{\tfrac{1}{n}} - \frac{1}{n+1}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} S_n = 1 - 0$$

$$\Rightarrow \lim_{n\to\infty} S_n = 1$$

So $\boxed{\sum_{K=1}^{\infty} \frac{1}{K(K+1)} = 1}$

~~Q: Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{10^n}$ is convergent.~~

Asim Marwat Zanji Khel
University of Peshawer

Msc Mathematics (Prev)

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Q:- Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof:- Given $\sum_{n=1}^{\infty} \frac{1}{n}$

Now Let $a_n = \frac{1}{n}$ and $f(x) = \frac{1}{x}$

(i) $f(x) = \frac{1}{x}$ is continous ; $\forall\ x \geq 1$

(ii) $f(x) = \frac{1}{x}$ is non-negative ; $\forall\ x \geq 1$

(iii) $f(x) = \frac{1}{x}$ is decreasing function.

Now by using integral test.

we have,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^{\infty}$$

$$= \ln(\infty) - \ln(1)$$

$$= \infty - 0$$

$$\int_1^{\infty} f(x)\,dx = \infty \quad \text{(undefined)}$$

So by integral test we say that the series

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Theorem:- If $\sum_{n=1}^{\infty} a_n$ is convergent Series Then $\lim_{n\to\infty} a_n = 0$

Proof:- Given that $\sum_{n=1}^{\infty} a_n$ is convergent

let $S_n = \sum_{i=1}^{n} a_i$ Then the sequence $\langle S_n \rangle$ is convergent.

$$S_n = a_1 + a_2 + \dots + a_n$$
$$S_{n+1} = a_1 + a_2 + \dots a_n + a_{n+1}$$
$$S_{n+1} - S_n = \cancel{a_1} + \cancel{a_2} + \dots \cancel{a_n} + a_{n+1} - \cancel{a_1} - \cancel{a_2} - \dots - a_n$$

$$S_{n+1} - S_n = a_{n+1}$$
$$\Rightarrow \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = \lim_{n\to\infty} a_{n+1}$$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = 0$$

But the converse is not true

let $a_n = \frac{1}{n}$

Then clearly $\lim_{n\to\infty} \frac{1}{n} = 0$

i.e $a_n \longrightarrow 0$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem:- The Series $\sum_{n=1}^{\infty} a_n$ is convergent $\Longleftrightarrow$ for every $\epsilon > 0$ There exists a natural number m such that $|S_n - S_m| = |a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon$, where $n \geq m \geq k$

Proof:- If $\langle S_n \rangle$ is the sequence of partial sum of $\langle a_n \rangle$ Then

$\langle S_n \rangle$ is convergent $\Leftrightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

Since $\langle S_n \rangle$ is convergent

$\Rightarrow \langle S_n \rangle$ is ~~a~~ cauchy Sequence.

by def: of Cauchy Sequence

$|S_n - S_m| < \epsilon \rightarrow (1)$ where $n, m \geq K$

If $n = m$ then clearly one holds

let $n > m$

$|S_n - S_m| = |\cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_m} + a_{m+1} + \dots + a_n - \cancel{a_1} - \cancel{a_2} - \dots - \cancel{a_m}| < \epsilon$

$= |a_{m+1} + a_{m+2} + \dots a_n| < \epsilon \quad , n, m \geq K$

Theorem:- Let $\langle x_n \rangle$ be a Sequence of non-negative real ~~n~~ numbers, Then the Series $\sum_{n=1}^{\infty} x_n$ is convergent $\Leftrightarrow$ the Sequence of its partial sum is bounded.

Proof:- Let $\langle S_n \rangle$ the Sequence of partial sum and If $\sum_{n=1}^{\infty} x_n$ is convergent Then the Sequence $\langle S_n \rangle$ is convergent

$\Rightarrow \langle S_n \rangle$ is bounded.

~~then~~ Let $\langle S_n \rangle$ is bounded. we show that the Series $\sum_{n=1}^{\infty} x_n$ is convergent

$S_1 = x_1$

$S_2 = x_1 + x_2$

$S_3 = x_1 + x_2 + x_3$

$\vdots \quad \vdots \quad \vdots$

since $x_1 \le x_1 + x_2 \le x_1 + x_2 + x_3 \le \cdots$

$s_1 \le s_2 \le s_3 \le \cdots$

$\Rightarrow \langle s_n \rangle$ is increasing sequence

But $\langle s_n \rangle$ is bounded.

$\Rightarrow \langle s_n \rangle$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent

"Comparison Test"

Theorem: Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series of +ve terms such that $a_k \le b_k \ \forall \ k = 1, 2, \cdots$

Then (i) If $\sum_{k=1}^{\infty} b_k$ is convergent Then $\sum_{k=1}^{\infty} a_k$ convergent.

(ii) If $\sum_{k=1}^{\infty} a_k$ is divergent then $\sum_{k=1}^{\infty} b_k$ is divergent

Proof: (i) If $\sum_{k=1}^{\infty} b_k$ is convergent we prove that $\sum_{k=1}^{\infty} a_k$ is convergent.

since $a_k \le b_k \quad \forall \quad k = 1, 2, \cdots$

$a_1 \le b_1$

$a_2 \le b_2$

$\vdots \quad \vdots$

$a_n \le b_n$

$a_1 + a_2 + \cdots a_n \le b_1 + b_2 + \cdots + b_n$

$\Rightarrow s_n \le s'_n$

But $\sum_{k=1}^{\infty} b_k$ is convergent

$\Rightarrow \langle s'_n \rangle$ is convergent $\Rightarrow s'_n$ is bounded

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So $|S_n| \le |S'_n|$

$\Rightarrow \langle S_n \rangle$ is bounded

$\Rightarrow \langle S_n \rangle$ is convergent

$\Rightarrow \sum_{k\ge1}^{\infty} a_k$ is convergent.

(ii) If $\langle S_n \rangle$ and $\langle S'_n \rangle$ are Sequences of partial Sum of $\sum_{k\ge1}^{\infty} a_k$ and $\sum_{k\ge1}^{\infty} b_k$ respectively.

If $\sum_{k\ge1}^{\infty} a_k$ is divergent

$\Rightarrow \langle S_n \rangle$ is divergent

But $S_n \le S'_n$

$\Rightarrow \lim_{n\to\infty} S_n \le \lim_{n\to\infty} S'_n$

$\Rightarrow \langle S'_n \rangle$ is divergent

$\Rightarrow \sum_{k\ge1}^{\infty} b_k$ is divergent.

(ii) 2nd Method:

suppose $\sum_{k\ge1}^{\infty} b_k$ is convergent

by (i) $\Rightarrow \sum_{k\ge1}^{\infty} a_k$ is convergent

which is contradiction

Hence $\sum_{k\ge1}^{\infty} b_k$ is convergent

" Limit Comparison Test "

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be positive terms Series such that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ (say) Then

(i) If L is positive then both the Series behave like.

(ii) If $L=0$ and $\sum_{k=1}^{\infty} b_k$ is convergent then $\sum_{k=1}^{\infty} a_k$ also convergent.

(iii) If $L=\infty$ and $\sum_{k=1}^{\infty} b_k$ divergent, then $\sum_{k=1}^{\infty} a_k$ also divergent.

Proof: (i) $\lim_{n\to\infty} \frac{a_n}{b_n} = L \Rightarrow \left|\frac{a_n}{b_n} - L\right| < \epsilon$ where $n \geq m$

$\Rightarrow -\epsilon < \frac{a_n}{b_n} - L < \epsilon$

$\Rightarrow L - \epsilon < \frac{a_n}{b_n} < \epsilon + L$

$\Rightarrow (L-\epsilon)b_n < a_n < (\epsilon+L)b_n$, where $n \geq m$

$\Rightarrow (L-\epsilon)\sum_{n=m} b_n \leq \sum_{n=m} a_n < (L+\epsilon)\sum_{n=m} b_n$

If $\sum_{k=1}^{\infty} b_k$ is convergent

Then $\sum_{k=m}^{\infty} b_k$ is also convergent.

$(L+\epsilon)\sum_{k=m}^{\infty} b_k$

By comparison test

we have $\sum_{n=m}^{\infty} a_n$ is convergent

$\Rightarrow \sum_{n=m}^{\infty} a_n$ is convergent.

Similarly:- if $\sum_{n=1}^{\infty} a_n$ is convergent then

$\sum_{k=1}^{\infty} b_k$ is convergent.

If $\sum_{k=1}^{\infty} a_k$ is divergent.

$\Rightarrow \sum_{k\geq 1}^{\infty} b_k$ is divergent

If $\sum_{k\geq 1}^{\infty} b_k$ is divergent
Then
$\sum_{k\geq 1}^{\infty} a_k$ is divergent

(ii) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ for every $\epsilon > 0$ then $\exists$ +ve integer "m" such that
$\left|\frac{a_n}{b_n} - 0\right| < \epsilon$, where $n \geq m$

$\Rightarrow -\epsilon < \frac{a_n}{b_n} < \epsilon$
$\Rightarrow -\epsilon b_n < a_n < \epsilon \cdot b_n \;;\; \forall\, n \geq m$
$\Rightarrow -\epsilon \sum_{n\geq m}^{\infty} b_n < \sum_{n\geq m}^{\infty} a_n < \epsilon \sum_{n\geq m}^{\infty} b_n$.

If $\sum_{k\geq 1}^{\infty} b_k$ is convergent
Then
$\sum_{k\geq m}^{\infty} b_k$ is convergent.
also $\epsilon \sum_{k\geq m}^{\infty} b_k$ is convergent.
by comparison test from (i)
we have $\Rightarrow \sum_{n\geq m}^{\infty} a_n$ is convergent

$\Rightarrow \sum_{n\geq 1}^{\infty} a_n$ is convergent.

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then we choose sufficient large number G
and we can find positive integer n, such that $\frac{a_n}{b_n} > G \;;\; \forall\, n \geq m$

$\Rightarrow a_n > b_n G \quad \forall\, n \geq m$
$\Rightarrow \sum_{n\geq m}^{\infty} a_n > G \sum_{n\geq m}^{\infty} b_n$

But $\sum_{k=1}^{\infty} b_k$ is divergent

$\Rightarrow \sum_{k=1}^{\infty} b_k$ is also divergent.

$\Rightarrow \sum_{n=m}^{\infty} a_n$ is divergent

then

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is also divergent.

" D. Alembert Test (Ratio Test) "

Suppose $\sum_{k=1}^{\infty} a_k$ is +ve terms series such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

(i) If $L < 1$ then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $L > 1$ Then $\sum_{k=1}^{\infty} b_k$ diverges.

(iii) If $L = 1$, Test fails.

Proof (i) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ and $L < 1$

So we can find $r \in \mathbb{R}$ such That

$L < r < 1$

Let $\epsilon = r - L$

For $\epsilon = r - L$ we can find +ve integer "m" such that

$$\left| \frac{a_{n+1}}{a_n} - L \right| < \epsilon = r - L \text{ ; where } n \geq m$$

$$\Rightarrow -r + L + L < \frac{a_{n+1}}{a_n} < r - \cancel{L} + \cancel{L} \text{ ; } n \geq m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} < r \text{ ; } n \geq m$$

$\Rightarrow a_{n+1} < r a_n \quad ; \quad n \geq m$

$a_{m+1} < r a_m \rightarrow (i)$

and

$a_{m+2} < r a_{m+1} < r(r a_m)$

$a_{m+2} < r^2 a_m \rightarrow (ii)$

similarly,

$a_{m+3} < r^3 a_m \rightarrow (iii)$

$a_{m+4} < r^4 a_m \rightarrow (iv)$

adding above ~~eq~~ Eq/s

$\sum_{k>m}^{\infty} a_k < a_m[r + r^2 + r^3 + \cdots] \Rightarrow a_m \sum_{i\geq 1}^{\infty} r^i$

$\Rightarrow \sum_{k=m}^{\infty} a_k$ is convergent

$\Rightarrow \sum_{k\geq 1}^{\infty} a_k$ is convergent

(ii) Now let $L > 1$. i.e $1 < L$.

let we chose r, such that

$1 < r < L \quad , \quad L - r > 0$.

let we take $\epsilon = L - r$ that $\epsilon > 0$

and hence $L - \epsilon = r$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. So for the above $\epsilon > 0$

we have a natural number "m_1"

such that $\left|\frac{a_{n+1}}{a_n} - L\right| < \epsilon$ for $n \geq m_1$

$\Rightarrow -\epsilon < \frac{a_{n+1}}{a_n} - L < \epsilon$ for $n \geq m_1$

$\Rightarrow L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon$ for $n \geq m$

Taking $L - \epsilon < \frac{a_{n+1}}{a_n}$ for $n \geq m_1$

$\Rightarrow \quad r < \frac{a_{n+1}}{a_n}$ for $n \geq m_1$

$\Rightarrow \quad r a_n < a_{n+1}$ for $n \geq m_1$

$\Rightarrow \quad a_{m_1+1} > r a_{m_1}$

$$a_{m_1+2} > r a_{m_1+1} > r \cdot r a_{m_1} = r^2 a_{m_1}$$
$$a_{m_1+3} > r a_{m_1+2} > r \cdot r^2 a_{m_1} = r^3 a_{m_1}$$
$$\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots$$

Adding these inequalites. we get

$$a_{m_1+1} + a_{m_1+2} + a_{m_1+3} + \cdots > r a_{m_1} + r^2 a_{m_1} + r^3 a_{m_1} + \cdots$$

$$\sum_{k=m_1+1}^{\infty} a_k > r a_{m_1} (1 + r + r^2 + \cdots) \longrightarrow (*)$$

Now $1 + r + r^2 + \cdots$ is an infinite, geometric series with common ratio $r > 1$ and so it divergent so $(1 + r + r^2 + \cdots)$ is inifinite

So $(*) \Rightarrow \sum_{k=m_1+1}^{\infty}$ is infinite

$\Rightarrow \quad \sum_{k=1}^{\infty} a_k$ is infinite and

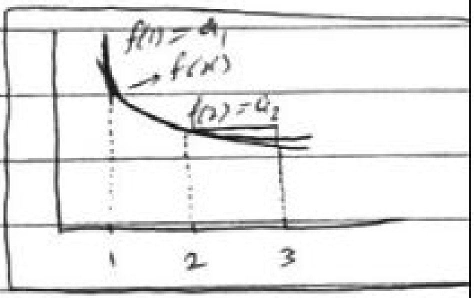
Thus $\sum_{k=1}^{\infty} a_k$ is divergent

Integral Test :-

suppose f is continous, +ve and decreasing fuction on $[1, \infty]$ and let $a_n = f(n)$, Then

(a) If $\int_1^\infty f(x)dx$ is convergent the $\sum_{n=1}^\infty a_n$ is convergent.

(b) If $\int_1^\infty f(x)dx$ is divergent Then $\sum_{n=1}^\infty a_n$ is divergent.



Proof:- (b) $\int_1^2 f(x)\,dx \le a_1$

$$\int_2^3 f(x)\,dx \le a_2$$

$$\vdots$$

$$\int_{n-1}^{n} f(x)\,dx \le a_{n-1}$$

$$\int_n^{n+1} f(x)\,dx \le a_n$$

adding all These

$$\int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots + \int_n^{n+1} f(x)dx \le a_1 + a_2 + \cdots a_n$$

$$\Rightarrow \int_1^n f(x)dx \le \sum_{i=1}^n a_i$$

Taking $\lim_{n\to\infty}$, we have

$$\int_1^\infty f(x)dx \le \sum_{i=1}^\infty a_i \longrightarrow (1)$$

If $\int_1^\infty f(x)dx$ is divergent Then

From (1) we have

$\sum_{i=1}^\infty a_i$ is divergent.

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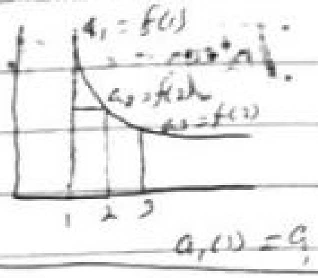


Part (a)

$$\int_1^2 f(x)\,dx \geq a_2$$

$$\int_1^2 f(x)\,dx \geq a_3$$

$$\vdots \qquad \vdots$$

$$\int_{n-1}^{n} f(x)\,dx \geq a_n$$

adding all these, we have

$$\int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \cdots \int_{n-1}^{n} f(x)\,dx \geq a_2 + a_3 + \cdots a_n$$

$$\int_1^n f(x)\,dx \geq \sum_{i=2}^{n} a_i$$

Taking $\lim_{n\to\infty}$ we get

$$\int_1^\infty f(x)\,dx \geq \sum_{i=2}^{\infty} a_i \longrightarrow (1)$$

If $\int_1^\infty f(x)\,dx$ is finite then from (1) we have

$\sum_{i=2}^{\infty} a_i$ is convergent.

Absolute Convergent :-

A series $\sum_{k=1}^{\infty} a_k$ is said to be absolute convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent

* Theorem :- Show that every absolute convergent series is convergent but the converse is true always.

Proof :- Let $\sum_{k=1}^{\infty} a_k$ is absolute convergent

i.e $\sum_{k=1}^{\infty} |a_k|$ is convergent.

we prove that $\sum_{k=1}^{\infty} a_k$ is convergent.

since $a_k \le |a_k|$

adding $|a_k|$ to both sides

$$a_k + |a_k| \le 2|a_k| \rightarrow (1)$$

since $\sum_{k=1}^{\infty} |a_k|$ is convergent

$\Rightarrow 2\sum_{k=1}^{\infty} |a_k|$ is convergent

by comparison test

$\sum_{k=1}^{\infty} (a_k + |a_k|)$ is convergent.

but $a_k = (a_k + |a_k|) - |a_k|$

$$\Rightarrow \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k| \rightarrow (2)$$

since $\sum_{k=1}^{\infty} (a_k + |a_k|)$ and $\sum_{k=1}^{\infty} |a_k|$ are converge

$\Rightarrow \sum_{k=1}^{\infty} a_k$ is convergent

Root Test :-

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that

$$\lim_{n\to\infty} a_n^{1/n} = \ell$$

(i) If $\ell < 1$ then $\sum_{n=1}^{\infty} a_n$ is converges.

(ii) If $\ell > 1$ then $\sum_{n=1}^{\infty} a_n$ is diverges.

(iii) If $\ell = 1$ the Test fail.

Proof :- Given $\sum_{n=1}^{\infty} a_n$, a series of +ve terms such that

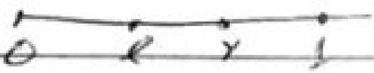
$$\lim_{n\to\infty} a_n^{1/n} = \ell$$

(i) let $\ell < 1$ and $\ell > 0$.

let us chosse a real number "r" such that $0 < \ell < r < 1$

since $r > \ell$

$\Rightarrow$ $r - \ell > 0$



let $r - \ell = \epsilon$ then $\epsilon > 0$.

now since $\lim_{n\to\infty} a_n^{1/n} = \ell$

So by the def: of convergence, for the above ϵ we can find a natural number "m" such that

$$|a_n^{1/n} - \ell| < \epsilon \quad \text{for } n \geq m,$$

$$\Rightarrow -\epsilon < a_n^{1/n} - \ell < \epsilon \quad \text{for } n \geq m,$$

$$\Rightarrow \ell - \epsilon < a_n^{1/n} < \ell + \epsilon \rightarrow (1) \quad \text{for } n \geq m,$$

But $r - \ell = \epsilon$

$\Rightarrow$ $r = \epsilon + \ell$, so that we have from the last result

$a_n^{1/n} < l+\epsilon$ for $n \geq m$

$\Rightarrow a_n^{1/n} < r$ " "

$\Rightarrow a_n < r^n$ " "

$\Rightarrow \sum_{n=m}^{\infty} a_n < \sum_{n=m}^{\infty} r^n < \sum_{n=m}^{\infty} r^n$

i.e $\sum_{n=m}^{\infty} a_n < \sum_{n=1}^{\infty} r^n \longrightarrow (*)$

now $0<r<1$ and $\sum_{n=1}^{\infty} r^n$ is an infinite geometric series with $r<1$,

so $\sum_{n=1}^{\infty} r^n$ is convergent.

so $\sum_{n=1}^{\infty} r^n$ has a finite sum S_1 (say)

i.e $\sum_{n=1}^{\infty} r^n = S_1$ (finite). So from $(*)$

$\sum_{n=m}^{\infty} a_n < \sum_{n=1}^{\infty} r^n = S_1$ which is finite

$\Rightarrow \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n < \sum_{n=1}^{m-1} a_n + S_1 = S$ (say)

$\Rightarrow \sum_{n=1}^{\infty} a_n < S$ which is finite as

$\sum_{n=1}^{\infty} a_n$ and S_1 are finite

i.e $\sum_{n=1}^{\infty}$ is finite. so by def:

$\sum_{n=1}^{\infty} a_n$ is convergent.

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(ii) Now let $l > 1$
let us choose "r" $\quad 1 \quad r \quad l$
such that $1 < r < l$
since $l > r \Rightarrow l - r > 0$.
let $l - r = \epsilon$ then $\epsilon > 0$.
Then since $\lim_{n\to\infty} a_n^{1/n} = l$,
so for the above $\epsilon > 0$
we can find a natural number
"m" such that
$$|a_n^{1/n} - l| < \epsilon \quad \text{for } n \geq m$$

$\Rightarrow -\epsilon < a_n^{1/n} - l < \epsilon$ " "
$\Rightarrow l - \epsilon < a_n^{1/n} < l + \epsilon$ " "
so $a_n^{1/n} > l - \epsilon$ " "
But $l - r = \epsilon \Rightarrow l - \epsilon = r$

$\Rightarrow a_n^{1/n} > r$ for $n \geq m$
$\Rightarrow a_n > r^n$ " "
$$\Rightarrow \sum_{n=m}^{\infty} a_n > \sum_{n=m}^{\infty} r^n \longrightarrow (A)$$

Now $\sum_{n=m}^{\infty} r^n = r^m + r^{m+1} + r^{m+2} + \cdots$

$$\sum_{n=m}^{\infty} r^n = r^m [1 + r + r^2 + r^3 + \cdots] \longrightarrow (B)$$
since $1 + r + r^2 + r^3 \cdots$ is an
infinite geometric series with $r > 1$, so
$1 + r + r^2 + r^3 + \cdots$ is divergent.
$\therefore \sum 1 + r + r^2 + r^3 + \cdots$ has an infinite sum.

So from (B)

$\sum_{n=m}^{\infty} r^n$ has infinite sum.

i.e $\sum_{n=m}^{\infty} r^n = \infty$

and so (A) $\Rightarrow \sum_{n=m}^{\infty} a_n = \infty$ and thus

$\sum_{n=1}^{\infty} a_n = \infty$ proving,

there by $\boxed{\text{that } \sum_{n=1}^{\infty} a_n \text{ is divergent}}$

(iii) If $\lim_{n\to\infty} a_n^{1/n} = l = 1$. let $a_n^{1/n} = \left(\frac{1}{n}\right)^{1/n}$
$= \frac{1}{n^{1/n}}$

then $\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{1} = 1$

i.e if we choose $a_n = \frac{1}{n}$ then $\lim_{n\to\infty} a_n^{1/n} = 1$

But $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

But again we take $a_n = \frac{1}{n^2}$ then
$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = 1$

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Then we observe that if $\lim_{n\to\infty} a_n^{1/n} = 1$ then sometimes $\sum_{n=1}^{\infty} a_n$ is convergent and sometimes it is divergent.

$\therefore$ if $\lim_{n\to\infty} a_n^{1/n} = l = 1$ then we can draw no conclusion, So we cannot apply this test for the case $l = 1$

So this Test is fail $\boxed{l=1}$

Q: Test for convergence of $\sum_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{k}}\right)^{-k^{3/2}}$

Solution :- Here $\sum_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{k}}\right)^{-k^{3/2}}$ and we need to find whether the series in convergent or divergent. For this, we use "Root Test".

since

$$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$$= \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$$

$$\Rightarrow a_n^{1/n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{\frac{3}{2}\times\frac{1}{n}}}}$$

$$= \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}\times n^{-1}}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2-1}}}$$

$$= \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{1/2}}}$$

$$\Rightarrow (a_n)^{1/n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\Rightarrow \lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\Rightarrow \lim_{n\to\infty}(a_n)^{1/n} = \frac{1}{e} < 1 \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right.$$

Hence by "Root test" the given series $\sum_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{k}}\right)^{-k^{3/2}}$ is convergent.

CHP # 2 END