# Chapter 01: Real Numbers

## Handwritten Notes of REAL ANALYSIS

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# CH :- 01 = Real Number



**Bounded above Set :-**

A set $S$ is said to be bounded above if there exist a real number $M \in \mathbb{R}$ Such that $M \geq x \ \forall \ x \in S$ and the number $M$ is called the upper bound of $S$

$S = \{2, 4, 8, 10\}$

upper bound $15 \in \mathbb{R}$

$20 \geq x \ \forall \ x \in S$

$10 \geq x \ \forall \ x \in S$

$\mathbb{N} = \{1, 2, 3, 4 \dots\}$

$\mathbb{N}$ is not bounded above

**Lub (least upper bound) (Supremum):-**

Let, $S \subseteq \mathbb{R}$ such that $S$ has upper bound $M$, Then $M$ is said to Lub or Supremum if $M$ is smaller among the other upper bound.

**Bounded Below Set :-**

A set $S \subseteq \mathbb{R}$ is said to be bounded below if there exists an $m \in \mathbb{R}$ Such that $m \leq x \ \forall \ x \in S$ and the number $m$ is said to be lower bound of $S$.

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$

~~$1 \in \mathbb{R}$~~ $1 \in \mathbb{R}, \ 1 \leq 1, 2, 3, \dots$

$0.5 \in \mathbb{R}, \ 0.5 \leq 1, 2, 3, \dots$

$-1, -2, \dots$ are lower bound of $S$.

$S = \{-1, -2, 3, \dots\}$

S has upper bound S has no lower bounded.
A lower bound m of a S is said to be infimum or Greatest lower bound (glb) if it is greater than $\forall$ lower bound.
Let
$A = \{-1, 5\}$
upper bounds of A are 5
$5 \geq x \quad \forall x \in A$
$5.1 \geq x \quad \forall x \in A$
$6 \geq x \quad \forall x \in A$
$\vdots$

Lower bounds of A are $-1, -2, -3$
Lub of A or Sup (A) = 5
inf or glb (A) = $-1$

$B = (2, 10)$
upper bounds of B 10, 10.1, 10.2, 10.3
lower bound of B Are 2, 1.9, 1.7
Sup (B) = 10
inf (B) = 2

$C = \{5, 10)$
~~Sup~~ lub (C) = $10 \notin C$
glb (C) = $5 \in C$

$D = (1, \infty)$
$glb\ (D) = 1$

If a set S has upper and lower bound then S is said to be bounded

Empty Set :- { } :
All the real number are the lower as well as upper bound of empty set
$\inf(\{\}) = \infty$
$\sup(\{\}) = -\infty$

Theorem: If $A \subseteq \mathbb{R}$ and lub (A) exists then prove that lub is unique:

proof:-
Let us assume that $K_1$ and $K_2$ are the lub of A.
we prove that $K_1 = K_2$
Since $K_1$ is lub of A
$\Rightarrow$ $K_1$ is upper bound of A
But $K_2$ is lub of A
$\Rightarrow K_2 \leq K_1 \longrightarrow$ (i)
similarly $K_2$ is lub of A
So $K_2$ is upper bound of A
But $K_1$ is lub of A
$\Rightarrow K_1 \leq K_2 \longrightarrow$ (ii)

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From (i) and (ii)

$K_1 = K_2$

lub of a set is always unique.

Theorem:- If a set $A \subseteq \mathbb{R}$ has glb/infimum then show that glb/infimum is unique.

Proof:- Let us assume that $K_1$ and $K_2$ are the glb of A

we prove that $K_1 = K_2$

since $K_1$ is glb of A

$\Rightarrow$ $K_1$ is lower bound of A.

But $K_2$ is glb of A

$\Rightarrow$ $K_2 \leq K_1 \longrightarrow$ (i)

similarly $K_2$ is glb of A

So $K_2$ is lower bound of A

But $K_1$ is glb of A

$\Rightarrow$ $K_1 \leq K_2 \rightarrow$ (ii)

From (i) and (ii)

$K_1 = K_2$

glb of a set is always unique.

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**Proposition:-** If $A \subseteq \mathbb{R}$ and $Sup(A)$ and $inf(A)$ exist then prove that $inf(A) \leq Sup(A)$

**Proof:-** Let $Sup(A) = M$ and $inf(A) = m$

But difinition of Sup

$M \geq x \; \forall \; x \in A$

and by difinition of Inf

$m \leq x \; \forall \; x \in A$

$m \leq x \leq M \; \forall \; x \in A$

$\Rightarrow m \leq M$

$Inf(A) \leq Sup(A)$

**Theorem:-** Let $A \subseteq \mathbb{R}$, then a real number ~~M~~ $M$ is Sup of $A$ if and only if the following condition are satisfied.

(i) $M$ is an upper bound of $A$

(ii) for every real number $M'$ such that $M' < M$, There exists $x \in A$ such that $M' < x$

**Proof:-** Let $Sup(A) = M$ we prove (i) and (ii) By definition Sup $M$ is an bound so (i) holds

$M' \in \mathbb{R}$ such that $M' < M$

difinition of Sup

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~~cannot ... the upper bound and so ... exist ... $\in A$ such that $> \dots$~~

Conversly:

Let (i) and (ii) are satisfied we prove that M is sup.

it is already given that M is the upper bound of A.

Also from (ii) we have no real number which is smaller than M and which is upper bound.

Hence M is lub of A.

Theorem: Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ $(A \neq \phi, B \neq \phi)$

(i) If $\sup(A)$ and $\sup(B)$ exist then $\sup(A) \leq \sup(B)$ if $A \subseteq B$

(ii) Also if $\inf(A)$ and $\inf(B)$ exist then $\inf(A) \geq \inf(B)$ if $A \subseteq B$

Proof: (i) Let $\sup(B) = M$

$M \geq x \quad \forall \, x \in B$

$M \geq x \quad \forall \, x \in A$

$\Rightarrow M$ is upper bound of A

and by def of $\sup(A)$, we have

$\sup(A) \leq M = \sup(B)$

$\sup(A) \leq \sup(B)$

... $\inf (B) = m$

... $m$ is also lower bound of B.

$\Rightarrow m \leq x \quad \forall \quad x \in B$

$\Rightarrow m \leq x \quad \forall \quad x \in A$ as $A \subseteq B$

$\Rightarrow m$ is lower bound of $A$.

by def; of inf(A), we have

$\inf(A) \geq m = \inf(B)$

$\Rightarrow \inf(A) \geq \inf(B)$

proposition: An upper bound $U$ of a set $S$ is supremum iff for every $\epsilon > 0$ There exist $s_\epsilon \in S$ such that $U - \epsilon < s_\epsilon$

Proof:- Let the upper bound $U$ is supremum Since $U - \epsilon < U$

Since $U$ is supremum and $U - \epsilon < U$ So by previous There exist some $s_\epsilon \in S$ such that

$U - \epsilon < s_\epsilon$

Conversely:-

Let $U$ be an upper bound such that for every $\epsilon > 0$ There exist $s_\epsilon \in S$ Such that

$U - \epsilon < s_\epsilon$

we have to Show that $U$ is supremum

Let $V < U$ such $V$ is upper bound of $S$. There as

$V < U \Rightarrow U - V > 0$

Let $\epsilon = U - V$

from hypothess there exist

Some $S_\epsilon \in S$ such that
$U - \epsilon < S_\epsilon$
$U - (U - V) < S_\epsilon$
$V < S_\epsilon$
$\Rightarrow$ $V$ is not upper bound of $S$.
which is contradoction to the fact
that $V$ is upper bound.
Hence there is no smaller number
which is ~~less~~ than $U$ and
which is upper bound
$\Rightarrow$ $U$ is supremum

**Theorem:-** If $A \subseteq \mathbb{R}$ such that $\sup(A)$
exists then ~~Sup(A)~~
$\sup(cA) = c\,\sup(A)$
for any $c \geq 0$

**Proof:-**
If $c = 0$ the case is trivial
If $c > 0$ the let $\sup(A) = M$
and $\sup(cA) = M'$
we show that $M' = cM$
Since $M = \sup(A) \Rightarrow M \geq x ; \forall x$
$\Rightarrow cM \geq cx ; \forall x \in A$
$\Rightarrow$ $cM$ is upper of $cA$.
But $\sup(cA) = M'$
So $cM \geq M' \rightarrow (i)$
Since $\sup(cA) = M'$
$\Rightarrow M' \geq cx ; \forall x \in A$
$\Rightarrow \frac{M'}{c} \geq x ; \forall x \in A$

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$\Rightarrow \frac{M'}{c}$ is upper bound of A.

But $Sup(A) = M$

$\Rightarrow M \le \frac{M'}{c}$

$\Rightarrow cM \le M' \rightarrow$ (ii)

From (i) and (ii)

$$cM = M'$$

So $c \, Sup(A) = Sup(cA)$

Theorem:- If $A \subseteq \mathbb{R}$ such that $inf(A)$ exists then

$$inf(cA) = c \, inf(A)$$

Proof:- If $c=0$ the case is trivial
If $c>0$ then let $inf(A) = m$
and $inf(cA) = m'$
we prove that $m = m'$
since $inf(A) = m$
$\Rightarrow m$ is lower bound of A
$\Rightarrow m \le x \; ; \forall x \in A$
$\Rightarrow cm \le cx \; ; \forall x \in A$ $\quad \because inf(cA) = m'$
$\Rightarrow m' \ge cm \rightarrow$ (i)
Since $inf(cA) = m'$

$m' \le cx \; ; \forall x \in A$

$\frac{m'}{c} \le x \; ; \forall x \in A$

$\Rightarrow \frac{m'}{c}$ is lower bound of A
But $inf(A) = m$

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$\Rightarrow m \geq \frac{m'}{c}$

$\Rightarrow mc \geq m' \longrightarrow$ (ii)

From (i) and (ii)

$cm = m'$

$c \inf(A) = \inf(cA)$

Theorem:- If $A \subseteq \mathbb{R}$ such that $\inf(A)$ and $\sup(A)$ exist then
(i) $\sup(cA) = c\inf(A)$ if $c<0$
(ii) $\inf(cA) = c\sup(A)$ if $c<0$

Proof:- Let $\sup(cA) = M'$
and $\inf(A) = m$
Since $\inf(A) = m$
$\Rightarrow m$ is lower bound of $A$.
$\Rightarrow m \leq x \; ; \forall x \in A$
$\Rightarrow cm \geq cx \; ; \forall x \in A$
$\Rightarrow cm$ is an upper bound of $cA$
But $\sup(cA) = M'$
$M' \leq cm \longrightarrow$ (i)
Now since $\sup(cA) = M'$
$M' \geq cx \; ; \forall x \in A$
$\frac{M'}{c} \leq x \; ; \forall x \in A$ as $c<0$

$\Rightarrow \frac{M'}{c}$ is lower bound $A$
But $\inf(A) = m$
$m \geq \frac{M'}{c} \Rightarrow cm \leq M' \longrightarrow$ (ii)
From (i) and (ii)
$M' = cm \Rightarrow \sup(cA) = c\inf(A)$ if $c<0$

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(ii) Let $\inf(cA) = M'$

and $\sup(A) = m$

Since $\sup(A) = m$

$\Rightarrow$ $m$ is upper bound of $A$.

$\Rightarrow$ $m \leq x \quad ; \forall\, x \in A$

$\Rightarrow$ $cm \leq cx \quad ; \forall\, x \in A$

$\Rightarrow$ $cm$ is lower bound of $cA$

But $\inf(cA) = M'$

$M' \leq cm \rightarrow (i)$

Now since $\inf(cA) = M'$

$M' \geq cx \quad ; \quad \forall x \in A$

$\frac{M'}{c} \geq x \quad ; \quad \forall\, x \in A$

$\Rightarrow$ $\frac{M'}{c}$ is upper bound of $A$.

But $\sup(A) = m$

$m \geq \frac{M'}{c} \Rightarrow cm \leq M' \rightarrow (ii)$

from (i) and (ii)

$M' = cm$

$\Rightarrow \inf(cA) = c\,\sup(A)$.

Corollary: (i) $\inf(-A) = -\sup(A)$

(ii) $\sup(-A) = -\inf(A)$

proof: (i) we know that

$\inf(cA) = c\,\sup(A)$ if $c < 0$

[illegible] putting $c = -1$

$\inf(-A) = -\sup(A)$

Now ) [illegible] that

$\sup(cA) = c\inf(A)$ if $c < 0$

[illegible] putting $c = -1$

$\sup(-A) = -\inf(A)$

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Theorem:- If $A$ and $B$ are two sets and $\sup(A)$ and $\sup(B)$ exist then $\sup(A+B) = \sup(A) + \sup(B)$

Proof:- Let $\sup(A) = M_1$, and $\sup(B) = M$

and $\sup(A+B) = M_3$

Now $\sup(A) = M_1 \Rightarrow M_1 \geq a \; ; \forall a \in A$

and $\sup(B) = M_2 \Rightarrow M_2 \geq b \; ; \forall b \in B$

$M_1 + M_2 \geq a + b \; ; \forall, a \in A, b \in B$

But $\sup(A+B) = M_3$

$\Rightarrow M_3 \leq M_1 + M_2 \rightarrow$ (i)

Since $\sup(A+B) = M_3$

$\Rightarrow M_3 \geq a+b \;, \forall a \in A, b \in B$

$\Rightarrow M_3 - a \geq b \; ; \forall b \in B$

But $\sup(B) = M_2$

$M_2 \leq M_3 - a$

$M_2 + a \leq M_3 \quad ; \forall a \in A$

$M_3 \geq M_2 + M_1 \rightarrow$ (ii)

From (i) and (ii)

$M_1 + M_2 = M_3$

So $\sup(A+B) = \sup(A) + \sup(B)$

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**Theorem:** If $A, B \subseteq \mathbb{R}$ and $\text{Inf}(A)$ and $\text{Inf}(B)$ exists

$\text{Inf}(A+B) = \inf(A) + \inf(B)$

**Proof:-** Let $\inf(A) = m_1$ and $\inf(B) = m_2$ and $\inf(A+B) = m_3$

since $\inf(A) = m_1 \Rightarrow m_1 \leq a \quad \forall\, a \in A$

similarly $m_2 \leq b \quad \forall\, b \in B$

$m_1 + m_2 \leq a + b \quad \forall\, a \in A,\ b \in B$

But $\inf(A+B) = m_3$

So $m_3 \geq m_1 + m_2 \longrightarrow (i)$

~~⇒ $m_3$ is lower bound~~

since $\inf(A+B) = m_3$

$\Rightarrow m_3$ is the lower bound of $A+B$

$\Rightarrow m_3 \leq a+b \quad \forall\, a \in A,\ b \in B$

$\Rightarrow m_3 - b \leq a \quad \forall\, a \in A$

$\Rightarrow m_3 - b$ is the lower bound of $A$

But $\inf(A) = m_1$

So $m_1 \geq m_3 - b \Rightarrow m_1 + b \geq m_3$

$\Rightarrow$

$\Rightarrow m_1 + m_2 \geq m_3$

$\Rightarrow m_3 \leq m_1 + m_2 \longrightarrow (ii)$

From (i) and (ii)

$$m_3 = m_1 + m_2$$

So $\text{Inf}(A+B) = \inf(A) + \inf(B)$

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**Theorem**:- If A is bounded the prove that

(i) $\inf(A-B) = \inf(A) - \sup(B)$

(ii) $\sup(A-B) = \sup(A) - \inf(B)$

Proof:-

(ii). We know that by the result of supremum of sets that

$\sup(A+B) = \sup(A) + \sup(B)$

Replace "B" by $-B$ in the above we have $\sup(A+(-B)) = \sup(A) + \sup(-B)$

$\Rightarrow \sup(A-B) = \sup(A) + \sup(-1 \cdot B) \Rightarrow (*)$

Now we know that if "A" is bounded and $c \in \mathbb{R} \ni c<0$, then

$\sup(cA) = c\,\inf(A)$

So in (*) we can write that

$\sup(-1B) = -1\,\inf(B)$

Therefore (*) $\Rightarrow \sup(A-B) = \sup(A) - \inf(B)$

which is the required proof.

(i) $\inf(A-B) = \inf(A) - \sup(B)$

We know that by the result of Infimum of sets that

$\inf(A+B) = \inf(A) + \inf(B)$

Replace "B" by $-B$ in the above we have

$\inf(A+(-B)) = \inf(A) + \inf(-B)$

$\Rightarrow \inf(A-B) = \inf(A) + \inf(-1 \cdot B) \rightarrow (**)$

Now we know that if $A$ is bounded and $c \in \mathbb{R} \ni c<0$ then

$$\mathrm{Inf}(cA) = c\ \sup(A)$$

So in (*) we can write that

$$\inf(-1 \cdot B) = -1\ \sup(B)$$

Therefore

$$(*) \Rightarrow \mathrm{Inf}(A-B) = \inf(A) - \sup(B)$$

which is the required proof.

Example for proof No (ii)

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$

Also $-B = \{-4, -5, -6\}$

Now $A-B = \{1-4, 1-5, 1-6, 2-4, 2-5, 2-6, 3-4, 3-5, 3-6\}$

$\Rightarrow A-B = \{-3, -4, -5, -2, -1\}$

$\Rightarrow A-B = \{-5, -4, -3, -2, -1\}$

Then $\sup(A) = 3$, $\sup(A-B) = -1$

and $\inf(B) = 4$

Now using the above result we have

$$\sup(A-B) = -1$$

and $\sup(A) - \inf(B) = 3 - 4 = -1$ So

$$\sup(A-B) = \sup(A) - \inf(B) = -1$$

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Theorem (Infimum or completeness property of Real Number)

Every non-empty subset of real numbers which is bounded below must have glb. (Inf)

Proof:- we show that
(i) $p^*$ is lower bound of A.
(ii) If $p^* < v$ then $v$ is not the lower bound of A.

(i) let $p^*$ is not a lower bound then there exist $s \in A$ such that
$p^* > s$.
$$p^* - s > 0 \Rightarrow \delta = p^* - s$$
let $[a_k, b_k]$ be the intervals such that
Let $l(I_k) < \delta$
$$a_k \leq p^* \leq b_k$$
$$\Rightarrow p^* \leq b_k$$
$$\Rightarrow p^* - a_k \leq b_k - a_k \longrightarrow (i)$$
and $l(I_k) < \delta$
$$\Rightarrow b_k - a_k < p^* - s \longrightarrow (ii)$$
$$\cancel{p^*} - a_k < \cancel{p^*} - s$$

$$\Rightarrow a_k > s$$
contradiction becaus $a_k$ is lower bound.

(ii) $v > p^* \Rightarrow v - p^* > 0$
$$\delta = v - p^*$$

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choose intervals $\{a_m, b_m\}$.

$\Rightarrow b_m - a_m < \delta$

$\Rightarrow b_m - a_m < v - p^* \longrightarrow$ (i)

$p^* \in \{a_m, b_m\}$

$\Rightarrow a_m \leq p^* \leq b_m \longrightarrow$ (ii)

$a_m \leq p^* \longrightarrow$ (iii)

$\therefore$ (i) + (iii)

$b_m < v$

$\Rightarrow v$ is not a lower bound.

**Theorem:** If $A \subseteq \mathbb{R}$ and $A$ has upper bound then $A$ has lub (sup).

**proof:** — we show that $p$ is a supremum.

$\frac{a+b}{2}$ $\frac{a+b}{2}$

A is set

B is uper bound and

choose $a \in \mathbb{R}$ such that A is not upper bound of A.

Consider the interval $[a, b]$

if $\frac{a+b}{2}$ is upper bound $\Rightarrow [a, \frac{a+b}{2}]$

if $\frac{a+b}{2}$ is not upper bound $\Rightarrow [\frac{a+b}{2}, b]$

consider $I_1 = [a_1, b_1]$  $l(I_1) = \frac{b-a}{2}$

$I_2 = [a_2, b_2]$  $l(I_2) = \frac{b-a}{2^2}$

$\downarrow$

$I_n = [a_n, b_n]$ ; $l(I_n) = \frac{b-a}{2^n}$

then $l(I_\infty) = (0)$ if $\bigcap_{n=1}^{n} I_n \Rightarrow p \in I_n$ th

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(i) we show that $\rho$ is upper bound
(ii) if $v < \rho$ Then $v$ is not upper bound of A.

(i) prof: let $\rho$ is not upper bound
$\Rightarrow s \in A$ such that $\rho < s$
$\Rightarrow s - \rho > 0 \Rightarrow \delta = s - \rho$


Choose $[a_k, b_k]$ such that
$b_k - a_k < \delta$
$b_k - a_k < s - \rho \longrightarrow$ (i)
$a_k \leq \rho \leq b_k \longrightarrow$ (ii)
$a_k \leq \rho \longrightarrow$ (iii)
$b_k < s$, which is contradouction to The fact that $b_k$ is upper bound and Hence $\rho$ is upper

(ii) $v < \rho \Rightarrow \rho - v > 0$
let $\delta = \rho - v$


choose Interval $[a_m, b_m]$
such that $b_m - a_m < \delta$
$\Rightarrow b_m - a_m \leq \rho - v \longrightarrow$ (i)
$a_m \leq \rho \leq b_m \rightarrow$ (ii)
$\rho \leq b_m$
$-\rho \geq -b_m \longrightarrow$ (iii)
(i) + (iii)
$-a_m < -v$
$\boxed{a_m > v}$

# (Archimedian property:-)

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Let $x, y \in \mathbb{R}$ and $x > 0$, Then there exist a positive integer $n$ such that $nx > y$.

Proof:- we discuss the following cases.
Case (I): If $y \leq 0$ Then $nx > 0 \geq y$
$\Rightarrow nx > y$ for positive integer $n$.
Case(II) If $y > 0$. we prove that there exists positive integer $n$ such that $nx > y$ let us suppose that $nx \leq y$
$\forall$ +ve integer $n$.
Let $A = \{nx : n \in N\}$. Then since
$nx \leq y \quad \forall\ n \in N$
$\Rightarrow A$ is bounded above. So let $\alpha$ is lub of $A$. Then $nx \leq \alpha \quad \forall\ n \in N$
i.e $mx \leq \alpha \quad \forall\ m \in N$
$m = m+1$
Then $(n+1)x \leq \alpha$
$\Rightarrow nx + x \leq \alpha$
$\Rightarrow nx \leq \alpha - x \quad \forall\ n \in N$
But $\alpha - x < \alpha$ and $\alpha$ is lub of $A$.

**Density Theorem:-** Between any two (2019) distinct real number there is a rational number.

Proof:- Let $x, y \in \mathbb{R}$ such that
$x < y \Rightarrow y - x > 0$

we need to find a number $\frac{m}{n}$
Such that $x < \frac{m}{n} \le y$, $m, n \in \mathbb{Z}, n \neq 0$
Since $y - x > 0$, $1 \in \mathbb{R}$ So by
Archimedian property there exists a
+ve integer $n$ such that $n(y-x) > 1$

$$\Rightarrow nx < m < nx + n(y-x)$$
$$\Rightarrow nx < m < \cancel{nx} + ny - \cancel{nx}$$
$$\Rightarrow nx < m < ny$$
$$\Rightarrow x < \frac{m}{n} < y$$

So we can find between any two real numbers a rational number.

**Corollary:-** If $x$ and $y$ are any two distinct real numbers then there exist irrational number between them.

**Proof:-** Let $x, y \in \mathbb{R}$ Such that

$$x < y \Rightarrow \frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$$

So density Theorem there exists a rational $r$ such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$
$$\Rightarrow x < \sqrt{2}\, r < y$$

But $r\sqrt{2}$ is an irrational number
So between any two real numbers there is irrational number.

## Nighbarhood of real Number.

Let $p \in \mathbb{R}$ Then the nbhd of $p$ is denoted $N_p$. Then A subset $A$ of $\mathbb{R}$ is said to be nbhd of $p$ if there exists an open interval $I$ such that

$$p \in I \subseteq A$$

$(p-\epsilon, p+\epsilon)$

we can write $I$ in the form $(p-\epsilon_1, p+\epsilon)$

A set $A$ is nbhd of $p$ if there exists $\epsilon > 0$ such that

$$p \in (p-\epsilon, p+\epsilon) \subseteq A$$

If $N_p$ is the nbhd of $p$ then $N_p \setminus \{p\}$ is called deleted nbhd of $p$. i.e. $\boxed{2 \in (0,3) \in (-1,5) \text{ So } (-1,5) \text{ is nbhd of } 2}$

$(-1,5) \setminus 2 = (-1,2) \cup (2,5)$ is the deleted nbhd of $2$.

If $(a,b)$ is an open interval and $x \in (a,b)$ Then

$$x \in (a,b) \subseteq (a,b)$$

$[1, 4]$

$1 \in (1-\epsilon, 1+\epsilon) \not\subseteq [1,4]$

$4 \in (4-\epsilon, 4+\epsilon) \not\subseteq [1,4]$

$x \in (1,4)$

$x \in (x-\epsilon, x+\epsilon) \subseteq (1,4)$.

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## Limit point of a Set:-

Let $\mathbb{R}$ be the set of real numbers and $A \subseteq \mathbb{R}$ Then a point $x_0 \in \mathbb{R}$ is limit point of $A$ if for any open interval $I$ such that $p \in I$ we have $(I \setminus \{p\}) \cap A \neq \phi$.

Let $A = [1, 3]$, let $x_0 \in \mathbb{R}$.

$x_0 \in [1, 3]$.

Then let $I$ be any interval such $x_0 \in I$, Then $(I \setminus \{x_0\}) \cap A \neq \phi$

$A^d = [1, 3]$

$B = (3, 5]$, 3 is a limit point of $B$.

So $B^d = [3, 5]$.

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Theorem :- Let $S$ be a non-empty subset of real number and $p \in \mathbb{R}$. Then the point $p$ is a limit point of $S$ iff every nbhd of $p$ contains infinitely many points of $S$.

Proof :- Let $p$ is limit point of $S$, i.e $p \in S^d (S')$

we show that every nbhd $N_p(\delta)$ of $p$ contains infinitly many points

Assume $N_p(\delta) \cap S = \{x_1, x_2, x_3 --- x_n\}$

Let $\epsilon = \text{Min}\{(p-x_1), (p-x_2), p-x_3 --- (p-x_n)\}$

and consider $N_p(\epsilon)$ and $(N_p(\epsilon) \setminus \{p\}) \cap S = \phi$

$\Rightarrow p \notin S^d$, contradaction.

Therefore accept that every nbhd of contains infinitly many point of S.

Conversly :- Let every nbhd $N_p(\delta)$ of $p$ contain infinitly many points of $S$

i.e $N_p(\delta) \cap S =$ infinit set

$\Rightarrow (N_p(\delta) \setminus \{p\}) \cap S =$ infinit.

$\Rightarrow (N_p(\delta) \setminus \{p\}) \cap S \neq \phi$

$\Rightarrow p \in S^d$

$\Rightarrow p$ is the limit point of $S$.

Theorem :- If $S$ is infinite and bound above subset of $\mathbb{R}$ and $M = \sup(S)$ such that $M \notin S$ Then $M \in S^d$ (i.e $M$ is limit of $S$). If A bounded above set does not contain it supremum $M$. Then $M$ is the limit of $S$.

Proof: Let $M = \sup(S)$ we show that $M$ is the limit point of $S$.
let $\epsilon > 0$ and $(M-\epsilon, M+\epsilon)$
since $M$ is supremum
$\Rightarrow$ $M-\epsilon$ is not upper bound.
$\Rightarrow$ There exits $x \in S$. Such that
$M-\epsilon < x$ But $M > x$
So $M-\epsilon < x < M < M+\epsilon$
$\Rightarrow$ $x \in (M-\epsilon, M+\epsilon)$, $x \neq M$, $x \in S$.
$\Rightarrow$ $((M-\epsilon, M+\epsilon) \setminus \{M\}) \cap S \neq \phi$

$\Rightarrow$ $M \in S^d$ $\Rightarrow$ $M$ is limit point of $S$.

Theorem: If $S$ is infinite and bound below subset of $\mathbb{R}$ and $m = \inf(S)$ such that $m \notin S$. Then $m \in S^d$ (i.e $m$ is limit of $S$). If A bounded below set does not contain it Infimum Then $m$ is the limit point of $S$.

Proof:- Given that $m = \inf(S)$.
Consider $(m-\epsilon, m+\epsilon)$
Then $m+\epsilon$ is not a lower bound of $S$.
$\Rightarrow$ There exists some $x \in S$ Such that
$m+\epsilon > x$ and $m = \inf(S)$
$\Rightarrow$ $m < x$ $\Rightarrow$ $x > m$
$m-\epsilon < m < x < m+\epsilon$
$\Rightarrow$ $x \in (m-\epsilon, m+\epsilon)$, $x \neq m$, $x \in S$

$\Rightarrow ((m-\epsilon, m+\epsilon)\setminus\{m\}) \cap S \neq \phi$

$\Rightarrow m \in S^d \Rightarrow m$ is a limit point of $S$

Theorem :- (**Bolzano weistrass Theorem**).

Statment :- Every bounded and infinite set of real numbers has at least one limit point.

Proof :- Let $S$ be infinite and bounded subset of $\mathbb{R}$.

we show that $S$ has at least one limit point.

since $S$ is bounded so there exists $a, b \in \mathbb{R}$ ~~such~~ such that $S \subseteq [a, b]$

Let $A = \{y \in \mathbb{R} : (-\infty, y) \cap S = \text{finite set}\}$.

$(-\infty, a) \cap S = \phi \Rightarrow a \in A$

[illegible]

$(-\infty, b) \cap S = \text{infinite set} \Rightarrow b \notin A$

$\Rightarrow A$ is bounded above.

$\Rightarrow$ let $K = \sup(A)$ we show that $K$ is the limit point of $S$.

$((K-\delta, K+\delta)\setminus\{K\}) \neq \phi$

$(K-\delta, K+\delta) = (-\infty, K+\delta) \setminus (-\infty, K-\delta)$

$(K-\delta, K+\delta) \cap S = ((-\infty, K+\delta) \setminus (-\infty, K-\delta)) \cap S$

$= (-\infty, K+\delta) \cap S \setminus (-\infty, K-\delta) \cap S$

since $K$ is sup $\Rightarrow K-\delta$ is not $\sup A \Rightarrow \exists p \in A$

such that $K-\delta < p$

$\Rightarrow (-\infty, p) \cap S$ = finite set

$(-\infty, K-\delta) \cap S$ = finite set $K-\delta < p$

$K = \sup\{A\}$

$K+\delta \notin A$

$\Rightarrow (-\infty, K+\delta) \cap S$ = infinite set

$(K-\delta, K+\delta)$ = infinite

$(K-\delta, K+\delta) \setminus \{K\} \cap S$ = infinite

$K \in S'$ . K is a limit point of S.

H.W

Theorem - The derived set of a set is closed

Proof:- let $S \neq \phi \subseteq \mathbb{R}$ and

$S'$ is denoted the derived set of S.

we need prove that $S'$ is closed.

Equivalently, we have to show that $(S')^c$ is open.

For this, let $x \in (S')^c$

$\Rightarrow x \notin S'$

i.e $x$ is not a limit point of S,

Thus, a particular $\delta > 0$, there must exist an open interval / open nbhd say

$N_\delta(x) = (x-\delta, x+\delta)$ such that

$\Rightarrow (x-\delta, x+\delta) \cap S$ = finite points $\rightarrow$ (1)

let $y \in N_\delta(x)$

As $N_\delta(x)$ is open set / open nbhd

To $y$ is the interior point of $N_\delta(x)$.
$\Rightarrow$ for $\delta > 0$ $\exists$ an $(y-\delta, y+\delta)$ is a open nbhd such that
$y \in (y-\delta, y+\delta) \subset N_\delta(x)$ [$\subset$ = contains
$\Rightarrow (y-\delta, y+\delta) \subset N_\delta(x)$ and from Eq(1)
$(y-\epsilon, y+\epsilon) \cap S$ = finite points.

$\Rightarrow \{(y-\epsilon, y+\epsilon) - \{y\}\} \cap S$ = finite points

$y$ is not a limit point of $S$.
i.e $y \notin S'$
$\Rightarrow y \in (S')^c$
As $y \in N_\delta(x)$
$\Rightarrow y \in (S')^c$
$\Rightarrow N_\delta(x) \subset (S')^c$
As $x \in N_\delta(x) \subset (S')^c$
$\Rightarrow x \in \text{int}\,(S')^c$
As
$x$ is arbitrary point of $(S')^c$
Thus each point of $(S')^c$ is its interior points.
$\Rightarrow (S')^c$ is open

So $S'$ is closed.

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Theorem:- A Subset $S \subseteq R$ has a limit point iff $S$ contains a subset which bounded and Infinite.

Proof:- Let us suppose $S$ has a limit point. we show that $S$ contain a subset which is infinite and bounded.

Let $p \in R$ be limit point of $S$, so by def for every open set $(x-\delta, x+\delta)$ we have

$(x-\delta, x+\delta) \setminus \{x\} \cap S$ = infinite set

$(x-\delta, x+\delta) \cap S$ = infinite = $B$ (say)

$B \subseteq S$ and $B \subseteq (x-\delta, x+\delta)$

Since $(x-\delta, x+\delta)$ is bounded

So $B$ is bounded

$B$ is the subset of $S$ which is bounded and infinite.

Conversly:-

Let $S$ has subset $B$ which is bounded and infinite.

we show that $S$ has limit point

By previous Theorem $B$ has limit point Say $p \in R$.

So for every open set $(p-\epsilon, p+\epsilon)$ we have $(p-\epsilon, p+\epsilon) \setminus \{p\} \cap B \neq \phi$

But $B \subseteq S$

$\Rightarrow (p-\epsilon, p+\epsilon) \setminus \{p\} \cap S \neq \phi$

$\Rightarrow p \in S'$

So $S$ has limit point

**Theorem:-** If $A$ is infinite subset of $\mathbb{R}$ and $m$ is glb and $M$ is lub of $A$ Then all the limit Point of $A$ contained in $[m, M]$ i.e $A' \subseteq [m, M]$ (OR) The derived set of infinite and bounded set is bounded.

**Proof:-** we show that for any $p > M$ is not limit point of $A$

let $p > M \Rightarrow (p - M) > 0$ , b.s ÷ 2

$$\Rightarrow \frac{p-M}{2} > 0$$

Let $\epsilon = \frac{p-M}{2}$

$$p - \epsilon = p - \frac{p-M}{2}$$

$$= \frac{p+M}{2} > \frac{M+M}{2} = M$$

$p - \epsilon > M$ But $M \geq x \ \forall x \in A$

As $M = lub\{A\}$

$p - \epsilon > x \quad \forall x \in A$

$p + \epsilon > p - \epsilon > x$

$\Rightarrow x \notin (p-\epsilon, p+\epsilon) \quad \forall x \in A$

$(p-\epsilon, p+\epsilon) \setminus \{p\} \cap A = \phi$

So $p \notin A'$

Let $q < m$ we show that $q \notin A'$

since $m > q \Rightarrow m - q > 0 \Rightarrow \frac{m-q}{2} > 0$

let $\epsilon = \frac{m-q}{2}$

$q + \epsilon < m \leq x \quad \forall x \in A$

$x \notin (q-\epsilon, q+\epsilon) \quad \forall x \in A$

$(q-\epsilon, q+\epsilon) \setminus \{q\} \cap A = \phi$ So $q \notin A'$

$\Rightarrow A' \subseteq [m, M]$

H.W

**Theorem:-** Show that $\log_2 5$ is an irrational number.

**Proof:-** suppose $\log_2 5$ is rational number.

Rational number is the form $\frac{p}{q}, q \neq 0$ where $p, q$ is a natural number.

$\log_2 5 = \frac{p}{q}$, for natural numbers $p$ and $q$.

By definition of logarithmic function

$$2^{p/q} = 5 \implies 2^p = 5^q$$

The right hand side $2^p$ is even and left hand side $5^q$ is odd.

No two natural numbers satisfy above equation.

Therefore, this is a contradiction to the assumption (1)

Thus,

$\log_2 5$ is an irrational number

Hence the result

CHP # 1 : END