LINEAR AND NON - LINEAR MODELS

CHAPTER 3

Mathematical modeling is a powerful tool for understanding complex relationships between variables. In this pursuit, two fundamental approaches have emerged: linear models and nonlinear models. These models serve as the foundation for statistical analysis, prediction, and decision-making in various fields.

Linear models are characterized by their simplicity and proportionality. They assume a straightforward relationship between variables, where the effect of one variable on another is constant. This linearity enables easy interpretation and prediction, making linear models a popular choice for many applications.

However, real-world phenomena often exhibit complex, nonlinear relationships. Nonlinear models capture these intricacies, allowing for curved, interactive, and dynamic relationships between variables. While more challenging to interpret and analyze, nonlinear models offer a more nuanced understanding of complex systems, making them essential in fields like physics, biology, and economics.

In this chapter we will learn about;

- Deterministic and population growth models
- Introduction to rectangular coordinates system
- Introduction to functions, their graphs and slopes
- Finding linear and non – linear models
- Exponential and logarithmic functions
- Ratio, rate and proportion
- Proportion and the golden ratio
- Exercises about linear and non – linear models

Deterministic Models

Deterministic model is a mathematical model that predicate the outcome of the system based on a set of fixed inputs. That is Deterministic model describes systems where the output is uniquely determined by the input, with no randomness or uncertainty. For example Y = a + bx and $Area = \pi r^2$ are examples of deterministic models. Solved examples are as follows.

Characteristics

1. Predictable outcomes
2. No randomness or uncertainty
3. Unique solution
4. Cause-and-effect relationships

1. Find the output of the system y = 2x + 3 when x = 4.

Solution: y = 11

2. Solve the differential equation dy/dx = 2x, given y(0) = 1.

Solution: $y = x^2 + 1$

3. Evaluate the function $f(x) = 3x^2 - 2x + 1$ at x = 2.

Solution: *f* (2) = 9

4. Find the equilibrium point of the system $x' = 2x - 3$

Solution: x = 3/2

5. Solve the system of linear equations:

2x + 3y = 7 ; x - 2y = -3

Solution: x = 1, y = 2

6. Find the derivative of $f(x) = 4x^3 - 2x^2 + x$

Solution: $f'(x) = 12x^2 - 4x + 1$

7. Find the critical points of $f(x) = x^3 - 2x^2 - 5x + 1$

Solution: $x \approx 2.53\ or\ x \approx -0.8$

Population Growth Models

Population growth models describe the change in population size over time.

Types of Models

- Exponential Growth Model
- Logistic Growth Model
- Malthusian Growth Model
- Verhulst Model

8. Exponential Growth Model: Find the population size after 5 years, given an initial population of 1000, growth rate of 0.05, and exponential growth.

Solution:

The exponential growth model is given by P(t) = P_0 e^{rt} , where P_0 is the initial population, r is the growth rate, and t is time. P(5) = 1000 $e^{(0.05\times5)} \approx 1276.78$

9. Logistic Growth Model: Solve the logistic growth equation $\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)$, with r = 0.2, K = 1000, and P(0) = 500.

Solution:

The logistic growth model accounts for carrying capacity (K). The solution involves separating variables and integrating. $P(t) = \frac{1000}{\left(1+e^{-0.2t}\right)}$

10. Malthusian Growth Model

Evaluate the Malthusian growth model $P(t) = P_0 e^{rt}$, with P_0=500, r=0.03, and t=10.

Solution:

The Malthusian growth model assumes exponential growth without limits.
P(10) ≈ 674.03

11. Verhulst Model: Solve the Verhulst equation $\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)^2$, with r = 0.2, K = 1000, and P(0) = 500.

Solution:

The Verhulst model modifies the logistic growth model with a quadratic term.

$$P(t) = \frac{1000}{\left(1+(1-0.5)e^{-0.2t}\right)}$$

12. Population Doubling Time: Find the population doubling time for an exponential growth model with r = 0.04.

Solution:

The population doubling time is calculated using the formula $T = \frac{\ln(2)}{r}$

$T = \frac{\ln(2)}{r} \approx 17.33$ years

13. Comparative Growth Rates: Compare the growth rates of two populations, one growing exponentially (r = 0.05) and one logistically (r = 0.05, K = 1000).

Solution:

Exponential growth is faster initially, but logistic growth slows due to carrying capacity. Exponential growth initially faster, but logistic growth slows as population approaches carrying capacity.

14. Equilibrium Population Size: Find the equilibrium population size for a logistic growth model with r = 0.1 and K = 500.

Solution:

The equilibrium population size occurs when $\frac{dP}{dt} = 0$

P = K = 500

15. Differential Equation Solution: Solve the differential equation $\frac{dP}{dt} = 0.02P - 0.01P^2$, with P(0) = 100.

Solution: The solution involves separating variables and integrating.

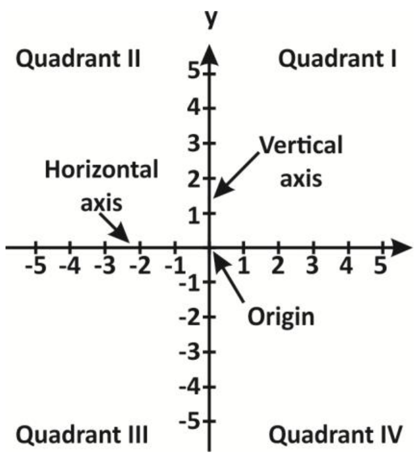
$$P(t) = \frac{1000}{\left(1 + 0.01e^{-0.02t}\right)}$$

16. Model Comparison: Compare the predictions of exponential, logistic, and Malthusian growth models.

Solution:

Exponential growth overestimates, logistic growth accounts for carrying capacity, and Malthusian growth underestimates.

Introduction to Rectangular Coordinate Systems

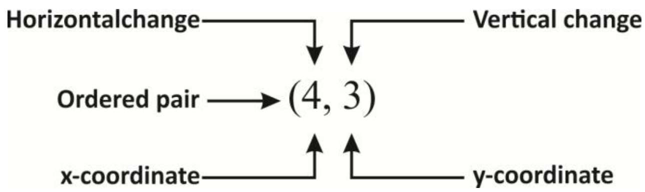


In mathematics we encounter a problem of locating a point in a plane. One way to solve the problem is to use a rectangular coordinate system.

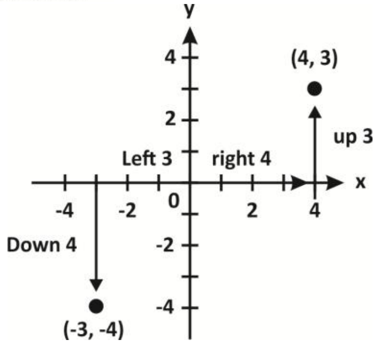
A **rectangular coordinate system** is formed by two number lines, one horizontal and one vertical, that intersect at the zero point of each line. The point of intersection is called the **origin**. The two number lines are called the **coordinate axes**, or simply the **axes**. Frequently, the horizontal axis is labeled the x-axis and the vertical axis is labeled the y-axis. In this case, the axes form what is called the **xy-plane**.

The two axes divide the plane into four regions called **quadrants**, which are numbered counterclockwise, using Roman numerals, from I to IV, starting at the upper right.

Each point in the plane can be identified by a pair of numbers called an ordered pair. An **ordered pair** is a pair of coordinates, and the order in which the coordinates are listed matters. The first number of the ordered pair measures a horizontal change from the y-**axis** and is called the **abscissa**, or **x-coordinate**. The second number of the ordered pair measures a vertical change from the x-axis and is called the **ordinate**, or **y-coordinate**. The ordered pair (x, y) associated with a point is also called the **coordinates** of the point.

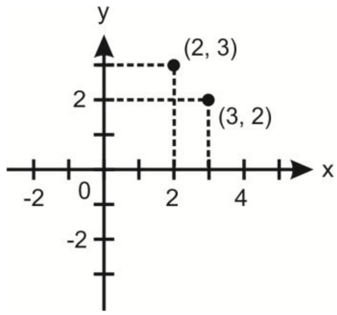


To **graph**, or **plot**, a point means to place a dot at the coordinates of the point. For example, to graph the ordered pair $(4,3)$ start at the origin. Move 4 units to the right and then 3 units up. Draw a dot. To graph $(-3,-4)$ start at the origin. Move 3 units left and then 4 units down. Draw a dot.



The **graph of an ordered pair** is the dot drawn at the coordinates of the point in the plane. The graphs of the ordered pairs $(4,3)$ and $(-3,-4)$ are shown at the upper right.

The graphs of the points whose coordinates Are $(2,3)$ and $(3,2)$ are shown at the right. Note that they are different points. The order in which the numbers in an ordered pair are listed is important. If the axes are labeled with letters other than x and y, then we refer to the ordered pair using the given labels. For instance, if the horizontal axis is labeled t and the vertical axis is labeled d, then the ordered pairs are written as (t, d). We sometimes refer to the first number in an ordered pair as the **first coordinate** of the ordered pair and to the second number as the **second coordinate** of the ordered pair. One purpose of a coordinate system is to draw a picture of the solutions of an equation in two variables. Examples of equations in two variables are shown at the right. A **solution of an equation in two variables** is an ordered pair that makes the equation a true statement. For instance, as shown below, $(2,4)$ is a solution of $y = 3x - 2$ but $(3,-1)$ is not a solution of the equation.



$y = 3x - 2$
$x^2 + y^2 = 25$
$s = t^2 - 4t + 1$

$$\begin{array}{l|l} \multicolumn{2}{l}{\underline{y = 3x-2}} \\ 4 & 3(2)-2 \\ 4 & 6-2 \\ \multicolumn{2}{l}{4 = 4} \end{array}$$

- $x = 2, y = 4$
- *Checks.*

$$\begin{array}{l|l} \multicolumn{2}{l}{\underline{y = 3x-2}} \\ -1 & 3(3)-2 \\ -1 & 9-2 \\ \multicolumn{2}{l}{-1 \neq 7} \end{array}$$

- $x = 3, y = -1$
- *Does not checks.*

17. Is $(-2,1)$ a solution of $y=3x+7$?

Solution:

Yes, because $1=3(-2)+7$

Introduction to Functions

A **function** is a correspondence, or relationship, between two sets called the **domain** and **range** such that for each element of the domain there corresponds exactly one element of the range.

Or Let A and B be two non-empty sets such that:

i) *f* is a relation from A to B that is , *f* is a subset of A × B

ii) Dom *f* = A

iii) First element of no two pairs of f are equal, then f is said to be a function from A to B.

The function *f* is also written as: $f: A \to B$ which is read: *f* is a function from A to B.

If (x, y) in an element of *f* when regarded as a set of ordered pairs, We write y = *f* (x). y is called the **dependent** value of *f* for x that is **independent** value or image of x under *f*.

The process of finding *f* (x) for a given value of x is called **evaluating the function**.

Examples

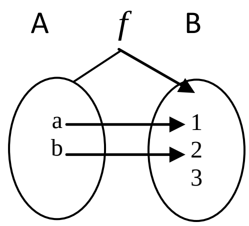
The ordered pairs, the graph, and the equation are all different ways of expressing the correspondence, or relationship, between the two variables. These are called a function. Here are some additional examples of functions, along with a specific example of each correspondence.

To each real number 5	there corresponds $\longrightarrow$	its square 25
To each score on an exam 87	there corresponds $\longrightarrow$	a grade B
To each student Alexander Sterling	there corresponds $\longrightarrow$	a student identification number S18723519

An important fact about each of these correspondences is that each result is unique. For instance, for the real number 5, there is exactly one square, 25.

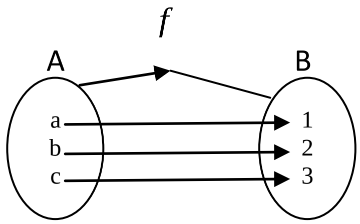
Into function

A function $f: A \to B$ is said to be into function if Range $f \neq$ B or Range $f \subset B$ as shown in figure.



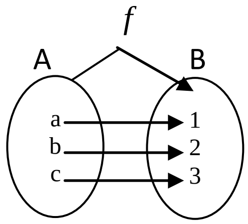
Onto function (Surjective function)

A function $f : A \rightarrow B$ is said to be onto function if Range f = B.



One-one function

A function $f : A \rightarrow B$ is called (1 – 1) function if different elements of A has different images in B as shown in figure.



(1-1) and onto function (Bijective function)

(Range f = B and 1 – 1)

A function f which is both one to one and onto is called Bijective function.

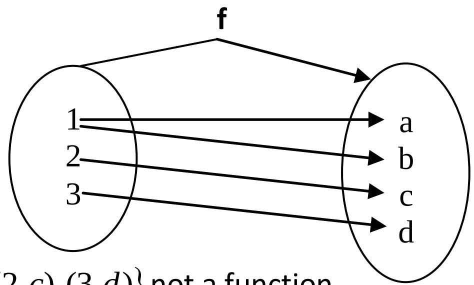
(1-1) and into (Injective function)

A function f which is both one to one and into is called Injective function.

18. Which of the following diagrams represent functions and of which type?

i.

Solution:

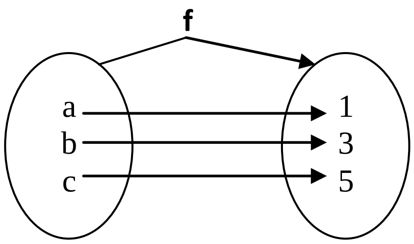


$R = \{(1,a),(1,b),(2,c),(3,d)\}$ not a function.

Since there are two ordered pairs that have same first element

ii.

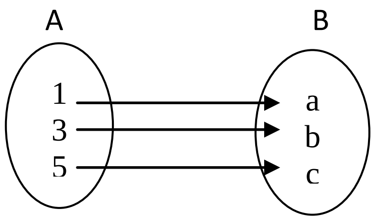
Solution:



$R = \{(a,1),(b,3),(c,5)\}$

Both conditions are satisfied. So is a function. R is one – one and onto so R is bijective function also.

iii.

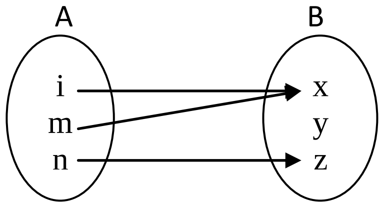


Solution:

$R = \{(1,a),(3,b),(5,c)\}$

Different element has different images so is one-one.
Range R = B so is onto function (1 – 1) & on to i.e. bijective function.

iv.



Solution:

$R = \{(i,x),(m,x),(n,z)\}$

No two ordered pairs of R have same 1st element.
Domain R = A
R is a function from A to B and into function because Range of f $\subset$B

19. Let $f(x) = x^2$. Find the domain and range of *f*.

Solution: $f(x)$ is defined for every real number x.
Further for every real number x, $f(x) = x^2$ is a non-negative real number. So
Domain *f* = Set of all real numbers. Or $(-\infty, \infty)$
Range *f* = Set of all non-negative real numbers. Or $[0, \infty)$

20. Let $f(x) = \dfrac{x}{x^2 - 4}$. Find the domain and range of *f*.

Solution: At $x = 2$ and $x = -2$, $f(x) = \dfrac{x}{x^2 - 4}$ is not defined. So
Domain *f* = Set of all real numbers except –2 and 2
Range *f* = Set of all real numbers.

21. Let $f(x) = \sqrt{x^2 - 9}$. Find the domain and range of *f*.

Solution:

We see that if x is in the interval $-3 < x < 3$, a square root of a negative number is obtained. Hence no real number $y = \sqrt{x^2 - 9}$ exists. So
Domain $f = \{x \in R : |x| \geq 3\} = (-\infty, -3] \cup [3, +\infty)$
Range *f* = set of all positive real numbers = $[0, +\infty)$

22. Given $f(x) = x^3 - 2x^2 + 4x - 1$, find

(i) $f(0)$ (ii) $f(1)$ (iii) $f(-2)$ (iv) $f(1 + x)$ (v) $f\left(1/x\right),\ x \neq 0$

Solution: $f(x) = x^3 - 2x^2 + 4x - 1$

(i) $f(0) = (0)^3 - 2(0)^2 + 4(0) - 1 = 0 - 0 + 0 - 1 = -1$

(ii) $f(1) = (1)^3 - 2(1)^2 + 4(1) - 1 = 1 - 2 + 4 - 1 = 2$

(iii) $f(-2) = (-2)^3 - 2(-2)^2 + 4(-2) - 1 = -8 - 8 - 8 - 1 = -25$

(iv) $f(1+x) = (1+x)^3 - 2(1+x)^2 + 4(1+x) - 1 = \cancel{1} + 3x + 3x^2 + x^3 - 2 - \cancel{4x} - 2x^2 + 4 + \cancel{4x} - \cancel{1}$

$= x^3 + x^2 + 3x + 2$

(v) $f\left(1/x\right) = \left(1/x\right)^3 - 2\left(1/x\right)^2 + 4\left(1/x\right) - 1 = \frac{1}{x^3} - \frac{2}{x^2} + \frac{4}{x} - 1,\ x \neq 0$

23. Evaluate $f(t) = 2t^2 - 3t + 1$ **when** $t = -2$

Solution:

$f(t) = 2t^2 - 3t + 1$

$f(-2) = 2(-2)^2 - 3(-2) + 1 = 15$

24. Evaluate $f(z) = z^2 - z$ **when** $z = -3$

Solution:

$f(z) = z^2 - z$

$f(z) = (-3)^2 - (-3) = 9 + 3 = 12$

25. The surface area of a cube (the sum of the areas of each of the six faces) is given by $SA(s) = 6s^2$ **where** $SA(s)$ **is the surface area of the cube and s is the length of one side of the cube. Find the surface area of a cube that has a side of length 10 centimeters.**

Solution:

$SA(s) = 6s^2$

$SA(10) = 6(10)^2 = 600$

The surface area of the cube is 600 square centimeters.

26. A diagonal of a polygon is a line segment from one vertex to a nonadjacent vertex, as shown. The total number of diagonals of a polygon is given by $N(s)=\frac{s^2-3s}{2}$ where $N(s)$ is the total number of diagonals and s is the number of sides of the polygon. Find the total number of diagonals of a polygon with 12 sides.

Solution:

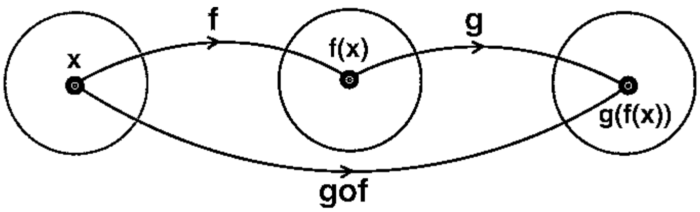
$$N(s)=\frac{s^2-3s}{2}$$

$$N(s)=\frac{(12)^2-3(12)}{2}=54$$

A polygon with 12 sides has 54 diagonals.

Composition of Functions

Let f : X → Y and g : Y → Z be two functions. Then the composition of f and g, denoted by g ∘ f, is defined as the function g ∘ f : X → Z given by g ∘ f (x) = g(f (x)), ∀ x ∈X.



27. Let the real valued functions *f* and *g* be defined by

$f(x) = 2x + 1$ and $g(x) = x^2 - 1$

Obtain the expressions for (i) $fg(x)$ (ii) $gf(x)$ (iii) $f^2(x)$ (iv) $g^2(x)$

Solution:

(i) $fg(x) = f(g(x)) = f(x^2 - 1) = 2(x^2 - 1) + 1 = 2x^2 - 2 + 1 = 2x^2 - 1$

(ii) $gf(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 1 = 4x^2 + 4x + 1 - 1 = 4x^2 + 4x$

(iii) $f^2(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 2 + 1 = 4x + 3$

(iv) $g^2(x) = g(gx) = g(x^2 - 1) = (x^2 - 1)^2 - 1 = x^4 - 2x^2 + 1 - 1 = x^4 - 2x^2$

We observe from (i) and (ii) that $fg(x) \neq gf(x)$

28. The real valued functions f and g are defined below. $f(x) = 2x + 1$, $g(x) = \frac{3}{x-1}$, $x \neq 1$

Find (a) fog(x) (b) gof(x) (c) fof(x) (d) gog(x)

Solution:

(a) $fog(x) = f(g(x)) = f\left(\frac{3}{x-1}\right)$

$$= 2\left(\frac{3}{x-1}\right)+1=\frac{6}{x-1}+1=\frac{6+x-1}{x-1}=\frac{5+x}{x-1}$$

(b) gof(x) $= g(f(x)) = g(2x+1) = \dfrac{3}{2x+1-1} = \dfrac{3}{2x}$

(c) fof (x) $= f(f(x)) = f(2x+1) = 2(2x+1)+1 = 4x+2+1 = 4x+3$

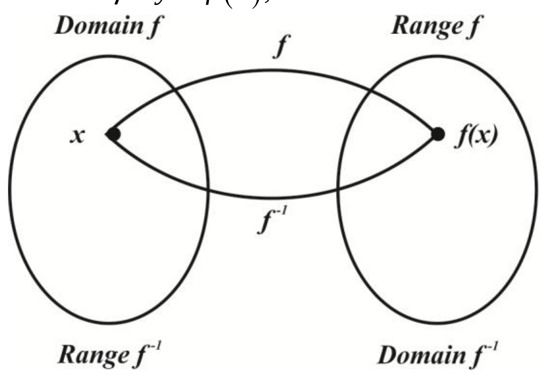
(d) gog(x) $= g(g(x)) = g\left(\dfrac{3}{x-1}\right)$

$$= \frac{3}{\frac{3}{x-1}-1} = \frac{3}{\frac{3-x+1}{x-1}} = \frac{3}{\frac{4-x}{x-1}} = \frac{3(x-1)}{4-x}$$

Inverse of a Function

Let f be a one-one function from X onto Y. The inverse function of f denoted by f^{-1} is a function from Y onto X and is defined by:

$x = f^{-1}(y), \forall y \in Y$ if and only if $y = f(x), \forall x \in X$.



29. Let $f : R \to R$ be the function defined by
$f(x) = 2x + 1$. Find $f^{-1}(x)$

Solution:

We find the inverse of f as follows:

Write $f(x) = 2x + 1 = y$

or $y = 2x + 1 \Rightarrow 2x = y - 1$

$\Rightarrow \quad x = \dfrac{y-1}{2}$ ______ I ; $f(x) = y \Rightarrow x = f^{-1}(y)$ ____ II

Compare I & II

$$\therefore \quad f^{-1}(y) = \frac{1}{2}(y-1)$$

To find $f^{-1}(x)$, replace y by x.

$$f^{-1}(x) = \frac{1}{2}(x-1)$$

30. For the real valued function f defined below, find (a) $f^{-1}(x)$, (b) $f^{-1}(-1)$ and verify $f(f^{-1}(x))= f^{-1}(f(x))=x$

(i) $f(x) = -2x + 8$

Solution:

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ ------------- I

Then $y = -2x + 8 \Rightarrow y - 8 = -2x$

$= -y + 8 = 2x \Rightarrow x = \dfrac{8-y}{2}$ _______ II

Compare I & II

$f^{-1}(x) = \dfrac{8-x}{2}$

put $x = -1$

$f^{-1}(-1) = \dfrac{8-(-1)}{2} = \dfrac{8+1}{2} = \dfrac{9}{2}$

Verification:

$f(f^{-1}(x)) = f\left(\dfrac{8-x}{2}\right)$

$= -\cancel{2}\left(\dfrac{8-x}{\cancel{2}}\right)+8 = -\cancel{8}+x+\cancel{8} = x$ _______ III

Again $f^{-1}(f(x)) = f^{-1}(-2x + 8)$

$= \dfrac{8-(-2x+8)}{2} = \dfrac{\cancel{8}+2x-\cancel{8}}{2} = \dfrac{\cancel{2}x}{\cancel{2}} = x$ _______ IV

From III & IV $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(ii) $f(x) = 3x^3 + 7$

Solution:

Take $f(x) = y \Rightarrow x = f^{-1}(y)$ __________ I

Then $y = 3x^3 + 7 \Rightarrow y - 7 = 3x^3$

$\dfrac{y-7}{3} = x^3 \Rightarrow x = \left(\dfrac{y-7}{3}\right)^{\frac{1}{3}}$ __________ II

Compare I and II

$f^{-1}(y) = \left(\dfrac{y-7}{3}\right)^{1/3}$

Replace y by x

$f^{-1}(x) = \left(\dfrac{x-7}{3}\right)^{1/3}$

Put $x = -1$, $f^{-1}(-1) = \left(\dfrac{-1-7}{3}\right)^{\frac{1}{3}} = \left(\dfrac{-8}{3}\right)^{1/3}$

Verification:

$$f(f^{-1}(x)) = f\left(\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right) = 3\left(\frac{x-7}{3}\right)^{3\times\frac{1}{3}} + 7$$

$$= \cancel{3}\left(\frac{x-7}{\cancel{3}}\right) + 7 = x - \cancel{7} + \cancel{7} = x \quad \text{\_\_\_\_\_\_\_\_\_ III}$$

And $f^{-1}(f(x)) = f^{-1}(3x^3 + 7)$

$$= \left(\frac{3x^3 + \cancel{7} - \cancel{7}}{3}\right)^{1/3} = \left(\frac{\cancel{3}x^3}{\cancel{3}}\right)^{1/3} = x^{3\times\frac{1}{3}} = x \quad \text{\_\_\_\_\_\_\_\_\_ IV}$$

From III & IV $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(iii) $f(x) = (-x + 9)^3$

Solutoin:

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ _________ I

Then $y = (-x + 9)^3 \Rightarrow y^{1/3} = -x + 9$

$\Rightarrow x = 9 - y^{1/3}$ _________ II

Compare I and II

$f^{-1}(y) = 9 - y^{1/3} \Rightarrow f^{-1}(x) = 9 - x^{1/3}$ (Replace y by x)

Put $x = -1$, $f^{-1}(-1) = 9 - (-1)^{1/3}$

Verification:

$f(f^{-1}(x)) = f(9 - x^{1/3}) = (-(9 - x^{1/3}) + 9)^3$

$= (-\cancel{9} + x^{1/3} + \cancel{9})^3 = x^{3\times\frac{1}{3}} = x$ _________ III

Also $f^{-1}(f(x)) = f^{-1}((-x + 9)^3)$

$= 9 - [(-x + 9)^3]^{1/3} = 9 - (-x + 9)$

$= \cancel{9} + x - \cancel{9} = x$ _________ IV

From III & IV $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(iv) $f(x) = \dfrac{2x + 1}{x - 1}$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ _________ I

Then $y = \dfrac{2x+1}{x-1} \Rightarrow y(x-1) = 2x + 1$

$yx - y = 2x + 1 \Rightarrow yx - 2x = y + 1$

$x(y-2) = y + 1 \Rightarrow x = \dfrac{y+1}{y-2}$ _________ II

Compare I and II

$f^{-1}(y) = \frac{y+1}{y-2}$ Replace y by x

$f^{-1}(x) = \frac{x+1}{x-2}$

Put $x = -1$ $\Rightarrow f^{-1}(-1) = \frac{-1+1}{-1-2} = \frac{0}{-3} = 0$

Verification:

$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right)$

$$= \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1} = \frac{\frac{2x+\cancel{2}+x-\cancel{2}}{x-2}}{\frac{\cancel{x}+1-\cancel{x}+2}{x-2}} = \frac{3x}{\cancel{x-2}} \times \frac{\cancel{x-2}}{3} = x \quad \_\_\_\_\_\_\_\_ \text{III}$$

Now $f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right)$

$$= \frac{\left(\frac{2x+1}{x-1}\right)+1}{\frac{2x+1}{x-1}-2} = \frac{\frac{2x+\cancel{1}+x-\cancel{1}}{x-1}}{\frac{\cancel{2x}+1-\cancel{2x}+2}{x-1}} = \frac{\cancel{3}x}{\cancel{x-1}} \times \frac{\cancel{x-1}}{\cancel{3}} = x \quad \_\_\_\_\_\_\_\_ \text{IV}$$

From III and IV $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

31. without finding the inverse, state the domain and range of f^{-1}

(i) $f(x) = \sqrt{x+2}$

Solution:

We see that f(x) is defined for $x+2 \geq 0 \Rightarrow x \geq -2$ so

Domain of $f(x) = [-2, \infty)$

Range of $f(x) = [0, \infty)$

Domain of $f^{-1}(x) = [0, \infty)$

Range of $f^{-1}(x) = [-2, \infty)$

(ii) $f(x) = \frac{x-1}{x-4}$, $x \neq 4$

Solution:

Domain of $f(x) = R - \{4\}$

Range 0f $f(x) = R - \{1\}$

Domain of $f^{-1}(x) = R - \{1\}$

Range of $f^{-1}(x) = R - \{4\}$

(iii) **f(x) = $\frac{1}{x+3}$, $x \neq -3$**

Solution:

Domain of f(x) = R – {–3}
Range of f(x) = R – {0}
Domain of f^{-1} (x) = R – {0}
Range of f^{-1} (x) = R – {–3}

(iv) **f(x) = $(x - 5)^2$, $x \geq 5$**

Solution:

Domain of f(x) = [5, ∞)
Range of f(x) = [0, ∞)
Domain of f^{-1}(x) = [0, ∞)
Range of f^{-1}(x) = [5, ∞)

Linear Functions and Linear Growth Models

A linear function is one that can be written in the form $f(x) = mx + b$, where m is the coefficient of x and b is a constant. Here are some other examples of linear functions.

$f(x) = 2x + 5$ m=2, b=5

$g(t) = \frac{2}{3}t - 1$ m=2/3, b=-1

$v(s) = -2s$ m=-2, b=0

32. Are the given functions linear functions?

a. $f(x) = 2x^2 + 5$ *b.* $g(x) = 1 - 3x$

Solution:

a. Because $f(x) = 2x^2 + 5$ has an x^2 term, *f* is not a linear function.

b. Because $g(x) = 1 - 3x$ can be written in the form $f(x) = mx + b$ as $g(x) = -3x + 1$ $(m = -3 \ \ and \ \ b = 1)$, g is a linear function.

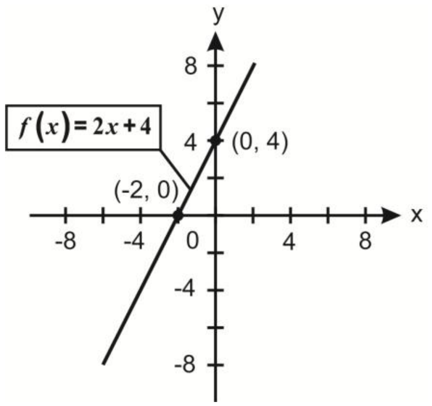
Graph of an Function

The graph of a linear function is a straight line. Observe that when the graph crosses the x-axis, the y-coordinate is 0. When the graph crosses the y-axis, the x-coordinate is 0. The table confirms these observations.

33. Graph $f(x)=2x+4$

Solution:

x	f(x) = 2x + 4	(x,y)
-3	f(-3) = 2(-3) + 4 = -2	(-3,-2)
-2	f(-2) = 2(-2) + 4 = 0	(-2,0)
-1	f(-1) = 2(-1) + 4 = 2	(-1,2)
0	f(0) = 2(0) + 4 = 4	(0,4)
1	f(1) = 2(1) + 4 = 6	(1,6)



34. Find the x- and y-intercepts of the graph $g(x)=-3x+2$

Solution:

For x – intercept: Put $y=g(x)=0$

$$g(x)=-3x+2\Rightarrow -3x+2=0\Rightarrow x=\frac{2}{3}\Rightarrow$$

The x-intercept is $\left(\frac{2}{3},0\right)$

For y – intercept: Put $x=0$

$$g(x)=-3x+2\Rightarrow g(0)=-3(0)+2\Rightarrow y=2\Rightarrow$$

The y-intercept is $(0,2)$

35. Find the x- and y-intercepts of the graph $g(x)=\frac{1}{2}x+3$

Solution:

For x – intercept: Put $\boldsymbol{y=g(x)=0}$

$g(x)=\frac{1}{2}x+3\Rightarrow \frac{1}{2}x+3=0\Rightarrow x=-6\Rightarrow$ The x-intercept is $(-6,0)$

For y – intercept: Put $\boldsymbol{x=0}$

$g(x)=\frac{1}{2}x+3\Rightarrow g(0)=\frac{1}{2}(0)+3\Rightarrow y=3\Rightarrow$ The y-intercept is $(0,3)$

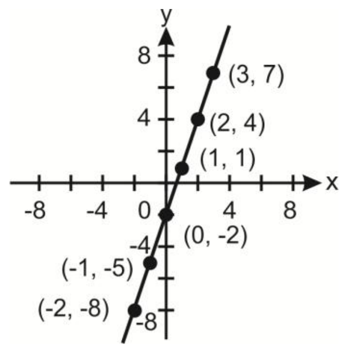
36. Graph $y = 3x - 2$

Solution:

To find ordered-pair solutions, select various values of x and calculate the corresponding values of y. Plot the ordered pairs. After the ordered pairs have been graphed, draw a smooth curve through the points. It is convenient to keep track of the solutions in a table.

When choosing values of x, we often choose integer values because the resulting ordered pairs are easier to graph.

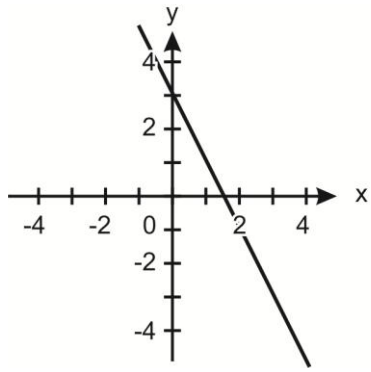
X	3x-2 = y	(x,y)
-2	3(-2)-2 = -8	(-2,-8)
-1	3(-1)-2 = -5	(-1,-5)
0	3(0)-2 = -2	(0,-2)
1	3(1)-2 = 4	(1,1)
2	3(2)-2 = 7	(2,4)
3	3(3)-2 = 7	(3,7)



37. Graph $y = -2x + 3$

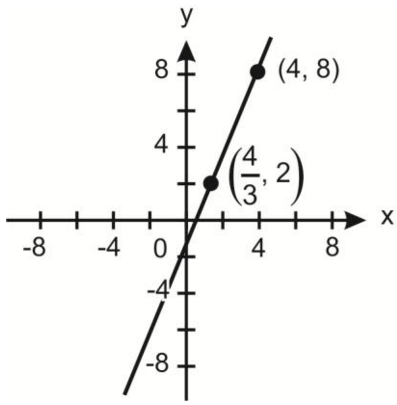
Solution:

X	-2x+3 = y	(x,y)
-2	-2(-2)+3 = 7	(-2,7)
-1	-2(-1)+3 = 5	(-1,5)
0	-2(0)+3 = 3	(0,3)
1	-2(1)+3 = 1	(1,1)
2	-2(2)+3 = -1	(2,-1)
3	-2(3)+3 = -3	(3,-3)



Note

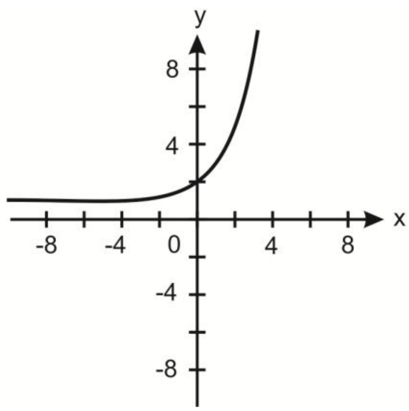
The graph of $y = 3x - 2$ is shown below. Note that the ordered pair $\left(\frac{4}{3}, 2\right)$ is a solution of the equation and is a point on the graph. The ordered pair $(4,8)$ is not a solution of the equation and is not a point on the graph. **Every ordered-pair solution of the equation is a point on the graph, and every point on the graph is an ordered-pair solution of the equation**.



38. Graph $f\left(x\right)=2-\frac{3}{4}x$

Solution:

x	$f\left(x\right)=2-\frac{3}{4}x$	(x,y)
-3	$f\left(-3\right)=2-\frac{3}{4}\left(-3\right)=4\frac{1}{4}$	$\left(-3,4\frac{1}{4}\right)$
-2	$f\left(-2\right)=2-\frac{3}{4}\left(-2\right)=3\frac{1}{2}$	$\left(-2,3\frac{1}{2}\right)$
-1	$f\left(-1\right)=2-\frac{3}{4}\left(-1\right)=2\frac{3}{4}$	$\left(-1,2\frac{3}{4}\right)$
0	$f\left(0\right)=2-\frac{3}{4}\left(0\right)=2$	$\left(0,2\right)$
1	$f\left(1\right)=2-\frac{3}{4}\left(1\right)=1\frac{1}{4}$	$\left(1,1\frac{1}{4}\right)$
2	$f\left(2\right)=2-\frac{3}{4}\left(2\right)=\frac{1}{2}$	$\left(2,\frac{1}{2}\right)$
3	$f\left(3\right)=2-\frac{3}{4}\left(3\right)=-\frac{1}{4}$	$\left(3,-\frac{1}{4}\right)$



39. After a parachute is deployed, a function that models the height of the parachutist above the ground is $f\left(t\right)=-10t+2800$ where $f\left(t\right)$ is the height, in feet, of the parachutist t seconds after the parachute is deployed. Find the intercepts on the vertical and horizontal axes and explain what they mean in the context of the problem.

Solution:

For Horizontal – intercept: Put $y=g\left(t\right)=0$

$f\left(t\right)=-10t+2800\Rightarrow-10t+2800=0\Rightarrow t=280$

The intercept on the horizontal axis is $\left(280,0\right)$. This means that the parachutist reaches the ground 280 seconds after the parachute is deployed.

For Vertical – intercept: Put $t=0$

$f\left(t\right)=-10t+2800\Rightarrow f\left(0\right)=-10\left(0\right)+2800\Rightarrow y=2800$

The intercept on the vertical axis is $\left(0,2800\right)$. This means that the parachutist is 2800 feet above the ground when the parachute is deployed.

Note that the parachutist reaches the ground when $f\left(t\right)=0$.

40. A function that models the descent of a certain small airplane is given by $g(t) = -20t + 8000$ where $g(t)$ is the height, in feet, of the airplane t seconds after it begins its descent. Find the intercepts on the vertical and horizontal axes, and explain what they mean in the context of the problem.

Solution:

For Horizontal – intercept: Put $y = g(t) = 0$

$g(t) = -20t + 8000 \Rightarrow -20t + 8000 = 0 \Rightarrow t = 400$

The intercept on the horizontal axis is $(400, 0)$. This means that the plane reaches the ground 400 seconds after beginning its descent.

For Vertical – intercept: Put $t = 0$

$g(t) = -20t + 8000 \Rightarrow g(0) = -20(0) + 8000 \Rightarrow y = 8000$

The intercept on the vertical axis is $(0, 8000)$. This means that the plane is at an altitude of 8000 feet when it begins its descent.

Slope of Graph of linear Function

For a linear function given by the slope of the graph of the function $f(x) = mx + b$ is m, the coefficient of the variable.

41. What is the slope of each of the following?

a. $y = -2x + 3$ **b.** $f(x)x + 4$ **c.** $g(x) = 3 - 4x$ **d.** $y = \frac{1}{2}x - 5$

Solution:

a. -2 **b.** 1 **c.** -4 **d.** $\frac{1}{2}$

Slope of a Line (when two points are given)

Let (x_1, y_1) and (x_2, y_2) be two points on a nonvertical line. Then the **slope** of the line through the two points is the ratio of the change in the y-coordinates to the change in the x-coordinates.

$$m = \frac{change\,in\,y}{change\,in\,x} = \frac{y_2 - y_1}{x_2 - x_1};\quad x_1 \neq x_2$$

42. Why is the restriction $x_1 \neq x_2$ required in the definition of slope?

Solution:

If $x_1 - x_2$ then the difference $x_2 - x_1 = 0$. This would make the denominator 0, and division by 0 is not defined.

43. Find the slope of the line between the two points.

a. $(-4,-3)$ *and* $(-1,1)$

b. $(-2,3)$ *and* $(1,-3)$

c. $(-1,-3)$ *and* $(4,-3)$

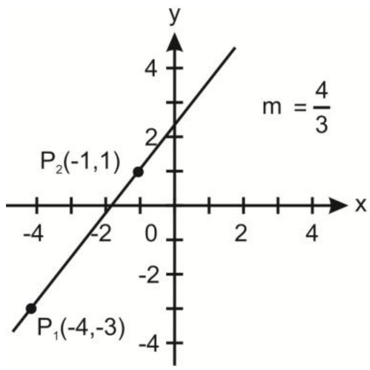
d. $(4,3)$ *and* $(4,-1)$

Solution:

a. $(x_1,y_1)=(-4,-3),\quad (x_2,y_2)=(-1,1)$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-3)}{1-(-4)}=\frac{4}{3}$$

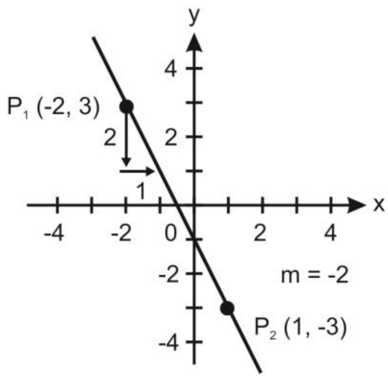
The slope is $\frac{4}{3}$. A **positive** slope indicates that the line slopes upward to the right. For this particular line, the value of y increases by $\frac{4}{3}$ when x increases by 1.



b. $(x_1,y_1)=(-2,3),(x_2,y_2)=(1,-3)$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-3-3}{1-(-2)}=\frac{-6}{3}=-2$$

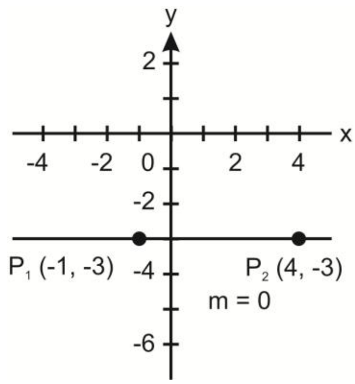
The slope is -2. A **negative** slope indicates that the line slopes downward to right. For this particular line, the value of y decreases by 2 when x increases by 1.

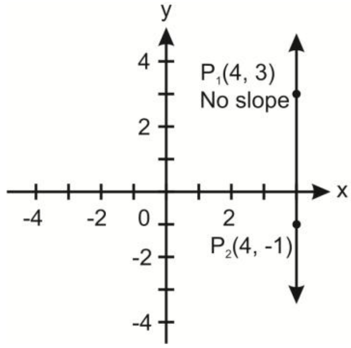


c. $(x_1,y_1)=(-1,-3),(x_2,y_2)=(4,-3)$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-3-(-3)}{4-(-1)}=\frac{0}{5}=0$$

The slope is 0. A zero slope indicates that the line is horizontal. For a horizontal line, the value of y stays the same when x increases by any amount.





d. $(x_1, y_1) = (4,3), (x_2, y_2) = (4,-1)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-3}{4-4} = \frac{-4}{0} = 0$$

If the denominator of the slope formula is zero, the line has no slope. Sometimes we say that the slope of a vertical line is **undefined**.

44. Find the slope of the line between the two points.

a. $(-6,5)$ *and* $(4,-5)$

b. $(-5,0)$ *and* $(-5,7)$

c. $(-7,-2)$ *and* $(8,8)$

d. $(-6,7)$ *and* $(1,7)$

Solution:

a. $(x_1, y_1) = (-6,5), (x_2, y_2) = (4,-5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-5}{4-(-6)} = \frac{-10}{10} = -1$$

b. $(x_1, y_1) = (-5,0), (x_2, y_2) = (-5,7)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-0}{-5-(-5)} = \frac{7}{0} \quad \text{(undefined)}$$

c. $(x_1, y_1) = (-7,-2), (x_2, y_2) = (8,8)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-(-2)}{8-(-7)} = \frac{10}{15} = \frac{2}{3}$$

d. $(x_1, y_1) = (-6,7), (x_2, y_2) = (1,7)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-7}{1-(-6)} = \frac{0}{7} = 0$$

45. Find the slope and inclination of the line joining the points:

(i) $(-2,4),(5,11)$

Solution:

Here $x_1 = -2,\ y_1 = 4,\ x_2 = 5,\ y_2 = 11$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-4}{5-(-2)} = \frac{11-4}{5+2} = \frac{\cancel{7}}{\cancel{7}} = 1$$

$$Tan\alpha = m \quad \Rightarrow Tan\alpha = 1$$

$$\Rightarrow \alpha = Tan^{-1}(1) = \frac{\pi}{4} \quad \Rightarrow \text{So inclination} = \alpha = \frac{\pi}{4}$$

(ii) (3,–2), (2, 7)

Solution:

$x_1 = 3,\ y_1 = -2,\ x_2 = 2,\ y_2 = 7$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-(-2)}{2-3}$$

$$= \frac{7+2}{-1} = \frac{9}{-1} = -9$$

Now $Tan\alpha = m \quad \Rightarrow Tan\alpha = -9$

$$\alpha = \text{Tan}^{-1}(-9) = (96.34)^0$$

(iii) (4,6), (4, 8)

Solution:

$x_1 = 4,\ y_1 = 6,\ x_2 = 4,\ y_2 = 8$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-6}{4-4} = \frac{2}{0} = \infty$$

Now $Tan\alpha = m \quad \Rightarrow Tan\alpha = \infty$

$$\alpha = Tan^{-1}(\infty) = \frac{\pi}{2} = 90^o$$

46. The function $T(x) = -6.5x + 20$ approximates the temperature $T(x)$, in degrees Celsius, at x kilometers above sea level. What is the slope of this function? Write a sentence that explains the meaning of the slope in the context of this application.

Solution:

For the linear function $T(x) = -6.5x + 20$ the slope is the coefficient of x. Therefore, the slope is -6.5. The slope means that the temperature is decreasing (because the slope is negative) 6.5°C for each 1-kilometer increase in height above sea level.

47. The distance that a homing pigeon can fly can be approximated by $d(t) = 50t$ where $d(t)$ is the distance, in miles, flown by the pigeon in t hours. Find the slope of this function. What is the meaning of the slope in the context of the problem?

Solution:

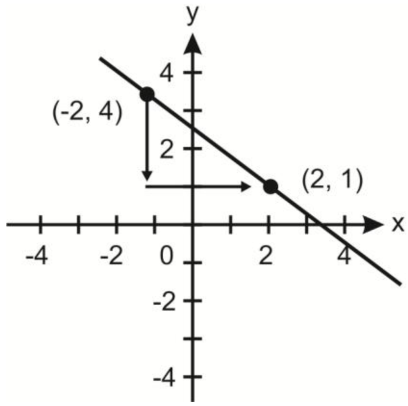
For the linear function $d(t) = 50t$ the slope is the coefficient of t. Therefore, the slope is 50. This means that a homing pigeon can fly 50 miles for each 1 hour of flight time.

Slope Intercept Form of a Straight Line

Let m be the slope and c be the y – intercept of a non – vertical line then the equation of line is $f(x) = mx + b$, this equation is called the slope–intercept form of a straight line.

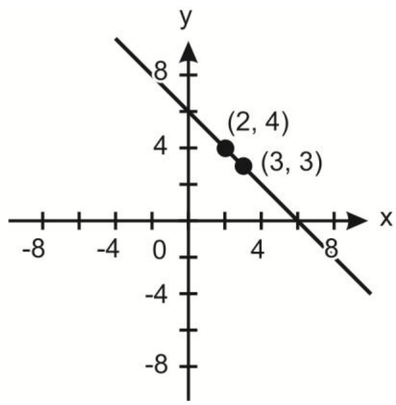
48. Draw the line that passes through $(-2,4)$ and has slope $-\frac{3}{4}$.

Solution:



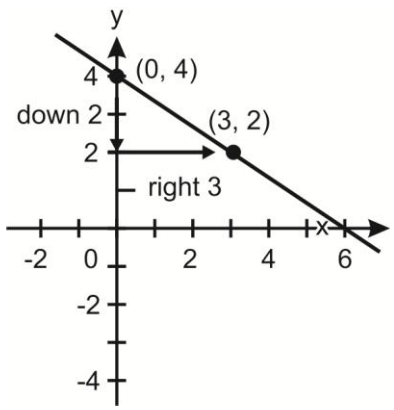
49. Draw the line that passes through $(2,4)$ and has slope -1.

Solution:



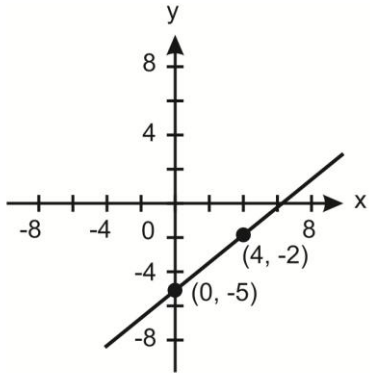
50. Graph $f(x)=-\frac{2}{3}x+4$ by using the slope and y-intercept.

Solution:



51. Graph $f(x)=\frac{3}{4}x-5$ by using the slope and y-intercept.

Solution:



52. Suppose that a car uses 0.04 gallon of gas per mile driven and that the fuel tank, which holds 18 gallons of gas, is full. Using this information, determine a linear model for the amount of fuel remaining in the gas tank after driving x miles.

Solution:

The slope is the rate at which the car is using fuel, 0.04 gallon per mile. Because the car is consuming the fuel, the amount of fuel in the tank is decreasing. Therefore, the slope is negative and we have m = - 0.04.

The amount of fuel in the tank depends on the number of miles x the car has been driven. Before the car starts (that is, when x = 0), there are 18 gallons of gas in the tank. The y-intercept is $(0,18)$

Using this information, we can create the linear function.

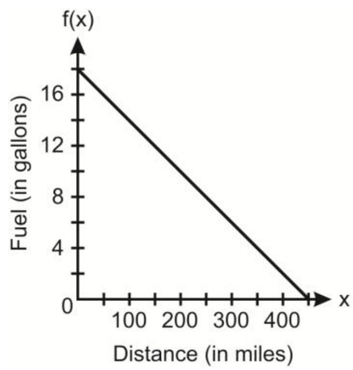
$f(x)=mx+b$

$f(x)=-0.04x+18$

The linear function that models the amount of fuel remaining in the tank is given by $f(x)=-0.04x+18$ where $f(x)$ is the amount of fuel, in gallons, remaining after driving x miles. The graph of the function is shown follows.

The x-intercept of a graph is the point at which $f(x)=0$.

For this application, $f(x)=0$ when there are 0 gallons of fuel remaining in the tank.



For x – intercept: Put $y=f(x)=0$

$f(x)=-0.04x+18$

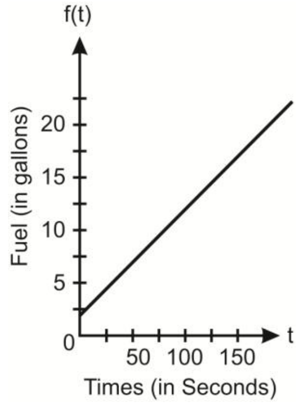
$\Rightarrow -0.04x+18=0 \Rightarrow x=450$

The car can travel 450 miles before running out of gas.

53. Suppose a 20-gallon gas tank contains 2 gallons when a motorist decides to fill up the tank. If the gas pump fills the tank at a rate of 0.1 gallon per second, find a linear function that models the amount of fuel in the tank t seconds after fueling begins.

Solution:

When fueling begins, at t = 0 there are 2 gallons of gas in the tank. Therefore, the y-intercept is $(0,2)$. The slope is the rate at which fuel is being added to the tank. Because the amount of fuel in the tank is increasing, the slope is positive and we have m = 0.1. To find the linear function, replace m and b by their respective values.



$f(t) = mt + b$

$f(t) = 0.1t + 2$

The linear function is $f(t) = 0.1t + 2$ where $f(t)$ is the number of gallons of fuel in the tank t seconds after fueling begins.

54. The boiling point of water at sea level is 100°C.
The boiling point decreases 3.5°C per 1 kilometer increase in altitude. Find a linear function that gives the boiling point of water as a function of altitude.

Solution:

$f(a) = ma + b$

$f(a) = -3.5a + 100$

The linear function is $f(a) = -3.5a + 100$ where $f(a)$ is the boiling point of water in degrees Celsius at an altitude of 'a' kilometers above sea level.

Point–Slope Formula of a Straight Line

Let (x_1, y_1) be a point on a line and let m be the slope of the line. Then the equation of the line can be found using the point–slope formula

$y - y_1 = m(x - x_1)$

55. Find the equation of the line that passes through $(1,-3)$ and has slope -2.

Solution:

$y - y_1 = m(x - x_1)$

$y - (-3) = -2(x - 1)$

$y + 3 = -2x + 2$

$y = -2x - 1$

56. Find an equation of the line through (–4,–6) and perpendicular to a line having slope $\frac{-3}{2}$.

Solution:

pt $(-4,-6)$

Slope of given line $=-\frac{3}{2}$

Slope of required perpendicular line $=\frac{2}{3}$

Equation of required perpendicular line is

$y+6=\frac{2}{3}(x+4) \Rightarrow 3y+18=2x+8$

$\Rightarrow 2x-3y+8-18=0 \Rightarrow 2x-3y-10=0$

57. Find an equation of the line through (11, –5) and parallel to a line with slope –24.

Solution:

pt $(11,-5)$, Slope $=-24$

Slope of required line which is parallel $=-24$

Equation of required line is

$y-(-5)=-24(x-11) \Rightarrow y+5=-24x+264$

$\Rightarrow y+5+24x-264=0 \Rightarrow 24x+y-259=0$

58. Find the equation of the line that passes through $(-2,2)$ and has slope $-\frac{1}{2}$.

Solution:

$$y-y_1=m(x-x_1) \Rightarrow y-2=-\frac{1}{2}(x-(-2)) \Rightarrow y=-\frac{1}{2}x+1$$

59. Find an equation of the straight line if

(a) its slope is 2 and *y*-intercept is 5

(b) it is perpendicular to a line with slope –6 and its *y*-intercept is $\frac{4}{3}$

Solution:

(a) The slope and *y*-intercept of the line are respectively:

$m = 2$ and $c = 5$

y=mx+c $\Rightarrow$ $y = 2x + 5$

(Slope-intercept form: $y = mx + c$) is the required equation.

(b) The slope of the given line is $m_1 = -6$

The slope of the required line is: $m_2 = -\frac{1}{m_1} = \frac{1}{6}$

The slope and y –intercept of the required line are respectively:

$$m = \frac{1}{6} \qquad (\text{slope of } \perp \text{ line is } -6) \text{ and } c = \frac{4}{3}$$

Thus $y = \frac{1}{6}(x) + \frac{4}{3}$ or $6y = x + 8$ is the required equation.

60. Write down an equation of the straight line passing through (5, 1) and parallel to a line passing through the points (0, –1), (7, –15).

Solution: Let m be the slope of the required straight line, then

$$m = \frac{-15-(-1)}{7-0} = -2 \qquad (\because \text{Slopes of parallel lines are equal})$$

As the point (5, 1) lies on the required line having slope –2 so, by point-slope form of equation of the straight line, we have

$y - (1) = -2(x - 5)$

or $y = -2x + 11$

or $2x + y - 11 = 0$ is an equation of the required line

61. Find an equation of line through the points (–2,1) and (6,–4).

Solution:

Using two-points form of the equation of straight line, the required equation is

$$y - 1 = \frac{-4-1}{6-(-2)}\left[x-(-2)\right]$$

or $y - 1 = \frac{-5}{8}(x+2) \Rightarrow 8y - 8 = -5x - 10$ or $5x + 8y + 2 = 0$

62. Based on data from the Kelley Blue Book, the value of a certain car decreases approximately \$250 per month. If the value of the car 2 years after it was purchased was \$14,000, find a linear function that models the value of the car after x months of ownership. Use this function to find the value of the car after 3 years of ownership.

Solution:

Let V represent the value of the car after x months. Then $V = 14000$ when (2 years is 24 months). A solution of the equation is $(24, -14000)$.The car is decreasing in value at a rate of \$250 per month. Therefore, the slope is -250. Now use the point–slope formula to find the linear equation that models the function.

$V - V_1 = m(x - x_1)$

$V - 14000 = -250(x - 24)$

$V - 14000 = -250x + 6000$

$V = -250x + 20000$

A linear function that models the value of the car after x months of ownership is

$V(x) = -250x + 20000$

To find the value of car after 3 years (36 months), evaluate the function when x = 36.

$V(x) = -250x + 20000$

$V(36) = -250(36) + 20000 = 11000$

The value of the car is $11,000 after 3 years of ownership.

63. During a brisk walk, a person burns about 3.8 calories per minute. If a person has burned 191 calories in 50 minutes, determine a linear function that models the number of calories burned after t minutes.

Solution:

$C - C_1 = m(t - t_1)$

$C - 191 = 3.8(t - 50)$

$C - 191 = 3.8t - 190$

$C = 3.8t + 1$

A linear function that models the number of calories burned after minutes is

$C(t) = 3.8t + 1$

64. Find the equation of the line that passes through $P_1(6, -4)$ **and** $P_2(3, 2)$**.**

Solution:

Find the slope of the line between the two points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-4)}{3 - 6} = \frac{6}{-3} = -2$$

Use the point–slope formula to find the equation of the line.

$y - y_1 = m(x - x_1)$

$y - (-4) = -2(x - 6)$

$y + 4 = -2x + 12$

$y = -2x + 8$

65. Find the equation of the line that passes through $P_1(-2,3)$ and $P_2(4,1)$.

Solution:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-3}{4-(-2)}=\frac{-2}{6}=-\frac{1}{3}$$

$$y-y_1=m(x-x_1)$$

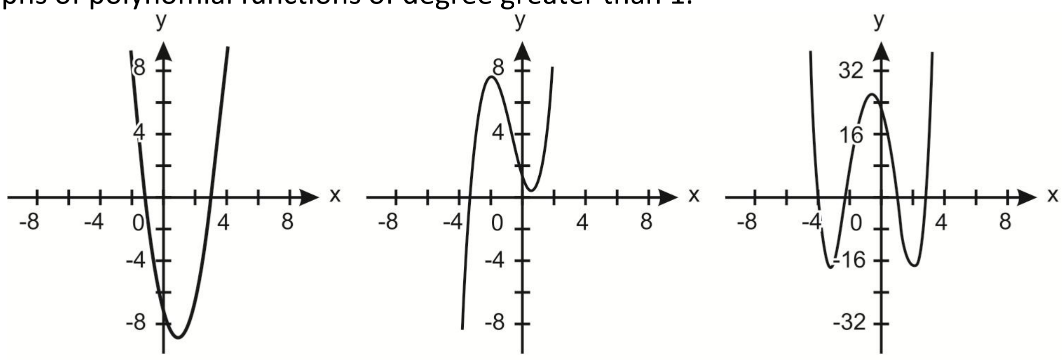
$$y-3=-\frac{1}{3}\left[x-(-2)\right]$$

$$y-3=-\frac{1}{3}x-\frac{2}{3}$$

$$y=-\frac{1}{3}x+\frac{7}{3}$$

Nonlinear functions

Polynomial functions of degree greater than 1 are part of a class of functions that are called **nonlinear functions** because their graphs are not straight lines. Here are some graphs of polynomial functions of degree greater than 1.



g(x) = 2x^2-4x-7 2nd degree   p(x) = 1-2x+3x^2+x^3 3rd degree   z(x) = x^4+2x^3-13x^2-14x+24 4rd degree

Just as a first-degree polynomial function is also called a linear function, some of the other polynomial functions have special names.

Polynomial Function	Name
First-degree	Linear function
Second-degree	Quadratic function
Third-degree	Cubic function
Fourth-degree	Quartic function

Polynomial Function

A function of the form $P(x) = a_n x^n + a_{n-1} xn^{-1} + ... + a_1 x + a_0 ; a_n \neq 0$

Where **n** is a non – negative integer and the coefficients $a_n, a_{n-1}, ..., a_1, a_0$ are real numbers. It can be considered as a polynomial function of x.

Degree of the Polynomial

The highest power of x in a polynomial in x is called the **degree of the polynomial**. So the expression is the polynomial of degree n.

Here are some examples of polynomial functions and their degrees.

Polynomial Function	Degree
$f(x) = 2x - 3$	1
$g(x) = 2x^2 - 4x - 7$ $f(t) = t^2$	2
$\rho(x) = 1 - 2x - 3x^2 - x^3$ $s(t) = 5t^3 - t - 8$	3
$z(x) = x^4 - 2x^3 - 13x^2 - 14x + 24$	4

Things to Remember

- Constant Polynomial/Equation has degree 0.
- Linear Polynomial/Equation has degree 1.
- Quadratic Polynomial has degree 2.
- Cubic Polynomial/Equation has degree 3.
- Degree 4, is a quadratic polynomial.
- 0 is a polynomial of degree 0.
- A polynomial having one term is called monomial.
- A polynomial having two terms is called binomial.
- A polynomial having three terms is called trinomial.

66. Let $f(x) = 3x - 2x^2 - 4x^3 + 2$

- **a. Write the polynomial in standard form.**
- **b. Name the degree of the function.**
- **c. Evaluate the function when x = 3.**

Solution:

a. Write the polynomial in decreasing powers of x.

$$f(x) = -4x^3 - 2x^2 + 3x + 2$$

b. The degree is 3, the largest exponent on the variable.

c. Value of function at x = 3.

$f(x) = -4x^3 - 2x^2 + 3x + 2$

$f(3) = -4(3)^3 - 2(3)^2 + 3(3) + 2$

$f(3) = -4(27) - 2(9) + 9 + 2 = -108 - 18 + 9 + 2 = -115$

67. Let $g(t) = -4t + 3t^2 + 5$

- **a. Write the polynomial in standard form.**
- **b. Name the degree of the function.**
- **c. Evaluate the function when x = 3.**

Solution:

a. $g(t) = 3t^2 - 4t + 5$

b. The degree is 2, the largest exponent on the variable.

c. Value of function at x = 3.

$g(t) = 3t^2 - 4t + 5$

$g(3) = 3(3)^2 - 4(3) + 5$

$= 27 - 12 + 5 = 20$

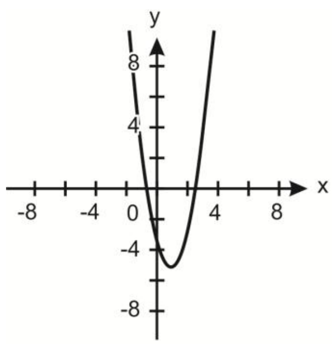
Quadratic Function

A quadratic function in x is a function that can be written in the form $f(x) = ax^2 + bx + c$; where a, b and c are real numbers and $a \neq 0$.

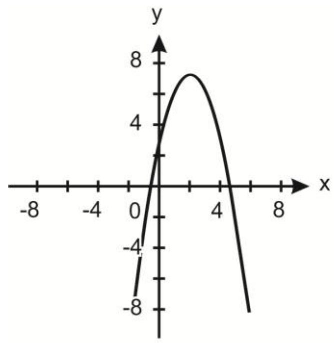
Or A quadratic function in x is **2nd Degree Polynomial** in x.

Examples: $f(x) = x^2 - 3x + 1, g(x) = -2t^2 - 4, h(p) = 4 - 2p - p^2$

The graph of a quadratic function in a single variable x is a parabola. The graphs of two such quadratic functions are shown below.



a > 0



a < 0

The figure on the left is the graph of $f(x) = 2x^2 - 4x - 3$. The value of a is positive $(a = 2)$ and the graph opens up.

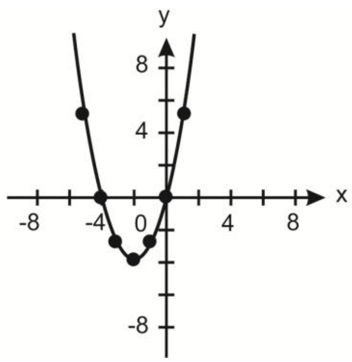
The figure on the right is the graph of $f(x) = -x^2 + 4x + 3$. The value of a is negative $(a = -1)$ and the graph opens down.

68. Graph $y = x^2 + 4x$

Solution:

Select various values of x and calculate the corresponding values of y. Plot the ordered pairs. After the ordered pairs have been graphed, draw a smooth curve through the points. The following table shows some ordered pair solutions.

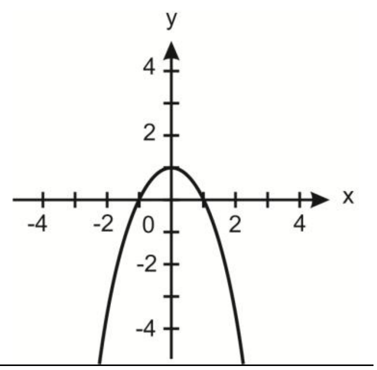
x	x^2+4x = y	(x,y)
-5	$(-5)^2$+4(-5) = 5	(-5,5)
-4	$(-4)^2$+4(-4) = 0	(-4,0)
-3	$(-3)^2$+4(-3) = -3	(-3,-3)
-2	$(-2)^2$+4(-2) = -4	(-2,-4)
-1	$(-1)^2$+4(-1) = -3	(-1,-3)
0	$(0)^2$+4(0) = 0	(0,0)
1	$(1)^2$+4(1) = 5	(1,5)



69. Graph $y = -x^2 + 1$

Solution:

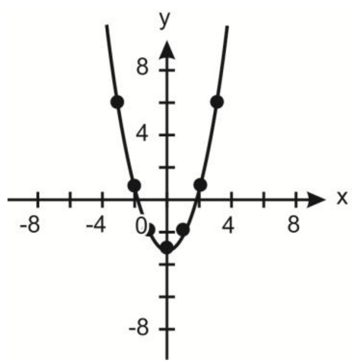
x	$-x^2$+1 = y	(x,y)
-3	$-(-3)^2$+1 = -8	(-5,5)
-2	$-(-2)^2$+1 = -3	(-4,0)
-1	$-(-1)^2$+1 = 0	(-3,-3)
0	$-(0)^2$+1 = 1	(-2,-4)
1	$-(1)^2$+1 = 0	(-1,-3)
2	$-(2)^2$+1 = -3	(0,0)
3	$-(3)^2$+1 = -8	(1,5)



70. Graph $h(x) = x^2 - 3$

Solution:

x	h(x)=x^2-3	(x,y)
-3	h(-3)=$(-3)^2$-3= 6	(-3,6)
-2	h(-2)=$(-2)^2$-3= 1	(-2,1)
-1	h(-1)=$(-1)^2$-3= -2	(-1,-2)
0	h(0)=(0^2-3= -3	(0,-3)
1	h(1)=$(1)^2$-3= -2	(1,-2)
2	h(2)=$(2)^2$-3= 1	(2,1)
3	h(3)=$(3)^2$-3= 6	(3,6)



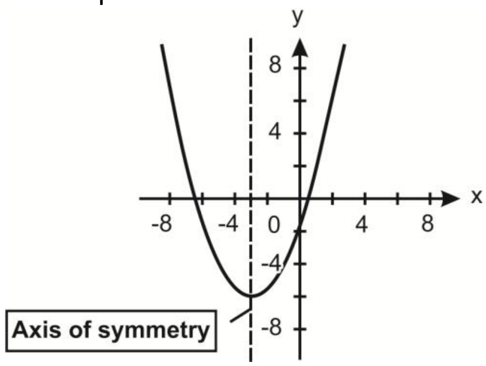
Vertex of the parabola

The point at which the graph of a parabola has a minimum or a maximum is called the vertex of the parabola. The vertex of a parabola is the point with the smallest y-coordinate when a > 0 and the point with the largest y-coordinate when a < 0.

Let $f(x) = ax^2 + bx + c$ be the equation of a parabola. The coordinates of the vertex are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Axis of symmetry

The axis of symmetry of the graph of a quadratic function is a vertical line that passes through the vertex of the parabola.



71. Find the vertex of the parabola whose equation is $y = -3x^2 + 6x + 1$

Solution:

$$x = -\frac{b}{2a} = -\frac{6}{2(-3)} = 1$$

$$y = -3x^2 + 6x + 1$$

$$y = -3(1)^2 + 6(1) + 1$$

$$y = 4$$

The vertex is $(1, 4)$

72. Find the vertex of the parabola whose equation is $y = x^2 - 2$

Solution:

$$x = -\frac{b}{2a} = -\frac{0}{2(1)} = 0$$

$$y = (0)^2 - 2 = -2$$

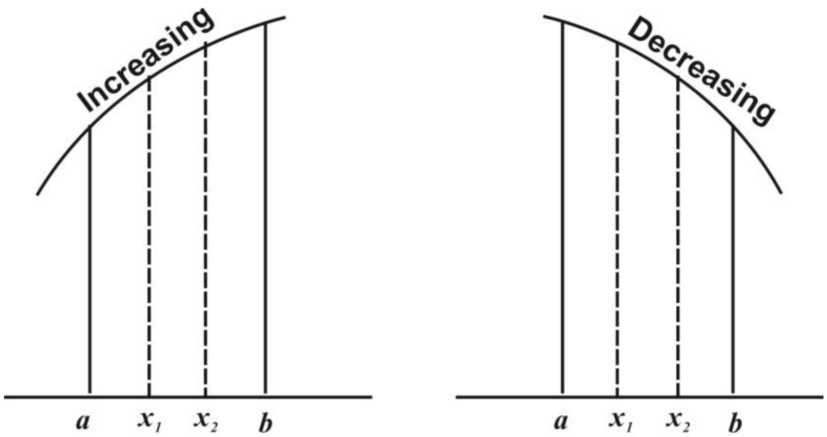
The vertex is $(0, -2)$

Increasing and Decreasing Functions

Let *f* be defined on an interval (a, b) and let $x_1, x_2 \in$ (a, b). Then

(i) *f* is increasing on the interval (a, b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

(ii) *f* is decreasing on the interval (a, b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.



$f(x_2) > f(x_1)$ if $x_2 > x_1$ $f(x_2) < f(x_1)$ if $x_2 > x_1$

Fist Derivative Theorem

Let *f* be a differentiable function on the open interval (a,b). Then

i. *f* is increasing on (a,b) if $f'(x) > 0$ for each $x \in$ (a, b).

ii. *f* is decreasing on (a,b) if $f'(x) < 0$ for each $x \in$ (a, b).

iii. *f* is neither increasing nor decreasing on (a,b) if $f'(x) = 0$ for each $x \in$ (a, b).

Stationary Point

Any point where *f* is neither increasing nor decreasing

Critical value or Critical Point

If $c \in Domf \text{ and } f'(c) = 0 \text{ or } f'(c)$ does not exists then c is called Critical value or Critical Point.

Relative Maxima / Maximum and Relative Minima / Minimum

Function f has relative maxima at c if $f''(c) < 0$ and function f has relative minima at c if $f''(c) > 0$.

Point of Inflection

Point where the function f is increasing before x = 0 and also after x = 0, such point is called the point of inflection.

73. Show that $f(x)=x^2$ is increasing or decreasing function on the interval $(-\infty,\infty)$.

Solution:

Given that $f(x)=x^2$ then $f'(x)=2x$

If $x>0$ then $f'(x)=2x>0$ so $f(x)=x^2$ is increasing in $(0,\infty)$

If $x<0$ then $f'(x)=2x<0$ so $f(x)=x^2$ is decreasing in $(-\infty,0)$

If $x=0$ then $f'(x)=2x=0$ so $f(x)=x^2$ is neither increasing nor decreasing.

Hence x = 0 is stationary point. And function has minimum at x = 0.

74. Find the maximum value of $f(x)=-2x^2+4x+3$.

Solution:

$$x=\frac{b}{2a}=-\frac{4}{2(-2)}=1$$

$$f(x)=-2x^2+4x+3$$

$$f(1)=-2(1)^2+4(1)+3$$

$$f(1)=5$$

The maximum value of the function is 5, the y-coordinate of the vertex.

75. Find the minimum value of $f(x)=2x^2-3x+1$.

Solution:

$$x=-\frac{b}{2a}=-\frac{-3}{2(2)}=\frac{3}{4}$$

$$f(x)=2x^2-3x+1$$

$$f\left(\frac{3}{4}\right)=2\left(\frac{3}{4}\right)^2-3\left(\frac{3}{4}\right)+1$$

$$f\left(\frac{3}{4}\right)=-\frac{1}{8}$$

The vertex is $\left(\frac{3}{4},-\frac{1}{8}\right)$.

The minimum value of the function is $-\frac{1}{8}$, the y-coordinate of the vertex.

76. Find the extreme values for the function $f(x)=1-x^3$.

Solution:

Given that $f(x)=1-x^3$ then

$f'(x)=-3x^2$ and $f''(x)=-6x$

$f'(x)=0 \Rightarrow -3x^2=0 \Rightarrow x=0$

$f''(0)=0$ gives no information.

$f'(0-\in)=-3(0-\in)^2=-3\in^2<0$

$f'(0+\in)=-3(0+\in)^2=-3\in^2<0$

First derivative does not change sign at x = 0.

Hence $(0, f(0)=1)$ is a point of inflection.

77. Find the extreme values for the function $f(x)=x^2-x-2$.

Solution:

Given that $f(x)=x^2-x-2$ then

$f'(x)=2x-1$ and $f''(x)=2$

$f'(x)=2x-1=0 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$

$f''\left(\frac{1}{2}\right)=2>0$. So $f(x)$ is minimum at $x=\frac{1}{2}$

$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)-2=-\frac{9}{4}$ is required minimum value.

78. Find the extreme values for $f(x)=5+3x-x^3$

Solution:

Given that $f(x)=5+3x-x^3$ then

$f'(x)=3-3x^2$ and $f''(x)=-6x$

$f'(x)=3-3x^2=0 \Rightarrow x=\pm1$

$f''(1)=-6<0$. So $f(x)$ is maximum at $x=1$

$f(1)=5+3(1)-(1)^3=7$ is required maximum value.

$f''(-1)=6>0$. So $f(x)$ is minimum at $x=\frac{1}{2}$

$f(-1)=5+3(-1)-(-1)^3=3$ is required minimum value.

79. The vertex of a parabola that opens up is $(-4,7)$. What is the minimum value of the function?

Solution:

The minimum value of the function is 7, the y-coordinate of the vertex.

80. A mining company has determined that the cost c, in dollars per ton, of mining a mineral is given by $C(x)=0.2x^2-2x+12$, where x is the number of tons of the mineral that is mined. Find the number of tons of the mineral that should be mined to minimize the cost. What is the minimum cost?

Solution:

To find the number of tons of the mineral that should be mined to minimize the cost and to find the minimum cost, find the x- and y-coordinates of the vertex of the graph of $C(x)=0.2x^2-2x+12$.

$$x=-\frac{b}{2a}=-\frac{-2}{2(0.2)}=5$$

To minimize the cost, 5 tons of the mineral should be mined.

$$C(x)=0.2x^2-2x+12 \Rightarrow C(5)=0.2(5)^2-2(5)+12=7$$

The minimum cost per ton is $7.

81. The height s, in feet, of a ball thrown straight up is given by $s(t)=-16t^2+64t+4$, where t is the time in seconds after the ball is released. Find the time it takes the ball to reach its maximum height. What is the maximum height?

Solution:

$$x=-\frac{b}{2a}=-\frac{64}{2(-16)}=2$$

The ball reaches its maximum height in 2 seconds.

$$s(t)=-16t^2+64t+4$$

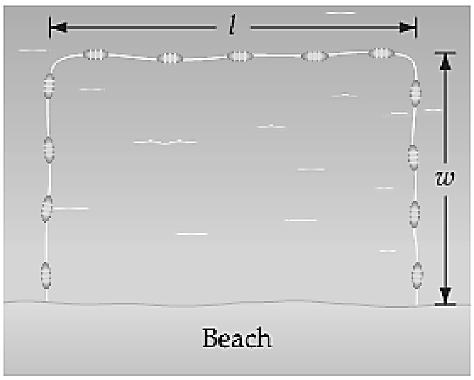
$$s(2)=-16(2)^2+64(2)+4=68$$

The maximum height of the ball is 68 feet.

82. A lifeguard has 600 feet of rope with buoys attached to lay out a rectangular swimming area on a lake. If the beach forms one side of the rectangle, find the dimensions of the rectangle that will enclose the greatest swimming area.

Solution:

Let *l* represent the length of the rectangle, let w represent the width of the rectangle, and let A (which is unknown) represent the area of the rectangle. See the figure. Use these variables to write expressions for the perimeter and area of the rectangle.



Perimeter: $w+l+w=600$

$2w+l=600$

$l=-2w+600$

Area: $A = lw$

$A = (-2w + 600)w$

$A = -2w^2 + 600w$

Find the w-coordinate of the vertex.

$$w = -\frac{b}{2a} = -\frac{600}{2(-2)} = 150$$

The width is 150 feet. To find *l*, replace w by 150 in $l = -2w + 600$ and solve for *l*.

$l = -2w + 600$

$l = -2(150) + 600 = 300$

The dimensions of the rectangle with maximum area are 150 feet by 300 feet.

83. A mason is forming a rectangular floor for a storage shed. The perimeter of the rectangle is 44 feet. What dimensions will give the floor a maximum area?

Solution

Perimeter: $w + l + w + l = 44 \Rightarrow w + l = 22 \Rightarrow l = -w + 22$

Area: $A = lw \Rightarrow A = (-w + 22)w \Rightarrow A = -w^2 + 22w$

Find the w-coordinate of the vertex: $w = -\frac{b}{2a} = -\frac{22}{2(-1)} = 11$

The width is 11 feet. To find *l*, replace w by 11 in $l = -w + 22$ and solve for *l*.

$l = w + 22 \Rightarrow l = -11 + 22 = 11$

The length is 11 feet. The dimensions of the rectangle with maximum area are 11 feet by11 feet.

84. An open box is made from a square piece of cardboard that measures 50 inches on a side. To construct the box, squares inches on a side are cut from each corner of the cardboard. The remaining flaps are folded up to create a box.

a. Express the volume of the box as a polynomial function in x.
b. What is the volume of the box when squares 5 inches on a side are cut out?
c. Is it possible for the value of to be 30? Explain your answer.

Solution:

a. The volume of a box is a product of its length, width, and height.
From the diagram, the length is 50 – 2x, the width is 50 – 2x, and the height is x. Therefore, the volume is given by

$V = lwh$

$V(x) = (50 - 2x)(50 - 2x)x$

$V(x) = 4x^3 - 200x^2 + 2500x$

The volume is given by $V(x) = 4x^3 - 200x^2 + 2500x$.

b. To find the volume when squares 5 inches on a side are cut out, evaluate the volume function when x = 5.

$V(x) = 4x^3 - 200x^2 + 2500x$

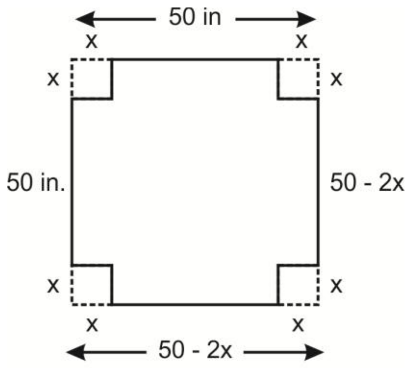
$V(5) = 4(5)^3 - 200(5)^2 + 2500(5)$

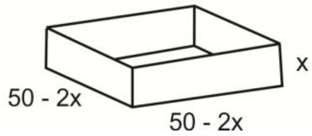
$V(5) = 4(125) - 200(25) + 2500(5) = 8000$

When squares 5 inches on a side are removed, the volume of the box is 8000 cubic inches.

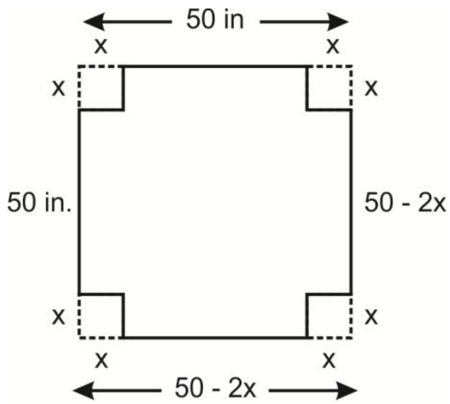
c. If x = 30, then the value of 50 – 2x would be $50 - 2x = 50 - 2(30) = -10$

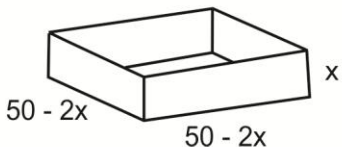
Because a length of -10 inches is not possible, the value of x cannot be 30.





85. An open box is made from a square piece of cardboard that measures 50 inches on a side. To construct the box, squares inches on a side are cut from each corner of the cardboard. The remaining flaps are folded up to create a box. Express the surface area of the box as a function of x. The surface area is the sum of the areas of the four sides of the box and its bottom.





Solution:

The surface area is the sum of the areas of the four sides of the box and the area of its bottom.

$$S(x) = \underbrace{4}_{\text{4 sides}}\overbrace{(50-2x)x}^{\text{Area of each side}} + \overbrace{(50-2x)(50-2x)}^{\text{Area of the base}}$$

$$= 200x - 8x^2 + 2500 - 200x + 4x^2$$

$$= -4x^2 + 2500$$

86. A lighthouse is 3 miles south of a port.

A ship leaves the port and sails east at 15 mph.

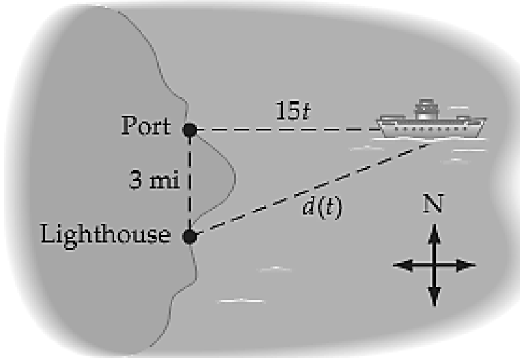
a. Express the distance $d(t)$, in miles, between the ship and the lighthouse in terms of t, the number of hours the ship has been sailing.

b. Find the distance of the ship from the lighthouse after 3 hours. Round to the nearest tenth

Solution:

Because the ship is sailing at 15 mph, after t hours the ship has traveled 15t miles, as shown in the diagram.

Using the Pythagorean Theorem $c^2 = a^2 + b^2$, where c is the length of the hypotenuse of a right triangle and a and b are the lengths of the legs, we have



$$\left[d(t)\right]^2 = (15t)^2 + 3^2$$

$$\left[d(t)\right]^2 = 225t^2 + 9$$

$$d(t) = \sqrt{225t^2 + 9}$$

To find the distance after 3 hours, replace by 3 and simplify.

$$d(3) = \sqrt{225(3)^2 + 9}$$

$$d(3) = \sqrt{2034} \approx 45.1$$

After 3 hours, the ship is approximately 45.1 miles from the lighthouse.

87. A plane flies directly over a radar station at an altitude of 2 miles and a speed of 400 miles per hour.

a. Express the distance $d(t)$ in miles, between the plane and the radar station in terms of t, the number of hours after the plane passes over the radar station.

b. Find the distance of the plane from the radar station after 3 hours. Round to the nearest tenth

Solution:

a. $\left[d(t)\right]^2 = 2^2 + (400t)^2$

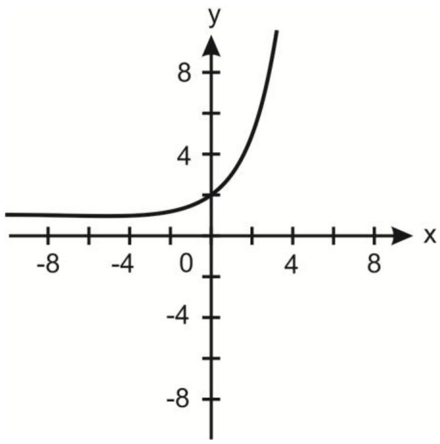
$\left[d(t)\right]^2 = 4 + 160000t^2$

$d(t) = \sqrt{4 + 160000t^2}$

b. $d(t) = \sqrt{4 + 160000t^2}$

$d(t) = \sqrt{4 + 160000(3)^2}$

$d(t) \approx 1200$



88. One of the considerations for a retail company is the cost of maintaining its inventory. The annual inventory cost is the cost of storing the items plus the cost of reordering the items. A lighting store has determined that the annual cost, in dollars, of storing *x* 25-watt halogen bulbs is 0.10x. The annual cost, in dollars, of reordering *x* 25-watt halogen bulbs is $\frac{0.15x+2}{x}$.

a. Express the inventory cost, $C(x)$, for the halogen bulbs in terms of x.

b. Find the inventory cost if the company wants to maintain an inventory of 150, 25-watt halogen bulbs. Round to the nearest cent.

Solution:

a. $C(x)=0.10x+\frac{0.15x+2}{x}$

b. $C(x)=0.10x+\frac{0.15x+2}{x}$

$$C(150)=0.10(150)+\frac{0.15(150)+2}{150}\approx 15.16$$

The inventory cost is $\$15.16$.

89. A manufacturer has determined that the total cost, in dollars, of producing x straight-back wooden chairs is given by $C(x)=35x+500$. The average cost per chair, $A(x)$, is the quotient of the total cost and x.

a. Find the function for the average cost per chair.

b. What is the average cost per chair when the manufacturer produces 40 chairs?

Solution:

a. Let $A(x)$ be the average cost per chair

$$A(x)=\frac{total\ cost}{x}=\frac{C(x)}{x}$$

$$A(x)=\frac{35x+500}{x}$$

b. $A(x)=\frac{35x+500}{x}$

$$A(40)=\frac{35(40)+500}{(40)}=47.50$$

The average cost is $\$47.50$ per chair.

Exponential Function

The exponential function is defined by $f(x)=b^x$ where b is called the base, b > 0, $b\neq 1$and x is any real number.

90. Evaluate $f(x)=3^x$ at $x=2, x=-4$ *and* $x=\pi$ Round approximate results to the nearest hundred thousandth.

Solution:

$$f(2)=3^2=9$$

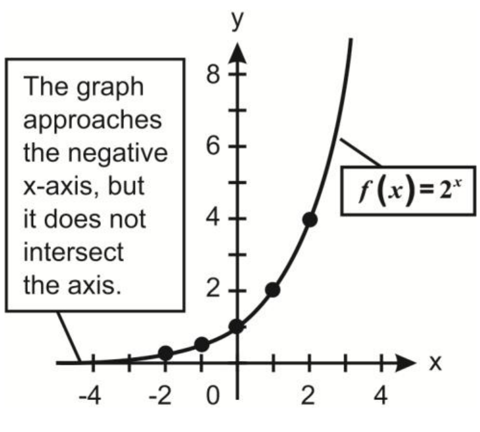
$$f(-4)=3^{-4}=\frac{-1}{3^4}=\frac{1}{81}$$

$$f(\pi)=3^{\pi}\approx 3^{3.1415927}\approx 31.54428$$

Graphs of Exponential Functions/ Exponential Growth Model

The graph of $f(x)=2^x$ is shown in Figure. The coordinates of some of the points on the graph are given in the table.

x	f(x) = 2^x	(x,y)
-2	f(-2) = 2^{-2} = 1/4	(-2,1/4)
-1	f(-1) = 2^{-1} = 1/2	(-1,1/2)
0	f(0) = 2^0 = 1	(0,1)
1	f(1) = 2^1 = 2	(1,2)
2	f(2) = 2^2 = 4	(2,4)
3	f(3) = 2^3 = 8	(3,8)



Observe that the values of y increase as x increases. This is an **exponential growth function**. This is typical of the graphs of all exponential functions for which the base is greater than 1. For the function $f(x)2^x, b=2$ which is greater than 1.

91. Evaluate $f(x)=\left(\frac{1}{2}\right)^x$ $at\ \ x=3, x=-1\ \ and\ \ x=\sqrt{3}$ **Round approximate results to the nearest hundred thousandth. Also graph it.**

Solution:

$$f(x)=\left(\frac{1}{2}\right)^x$$

$$f(3)=\left(\frac{1}{2}\right)^3=\frac{1}{8}$$

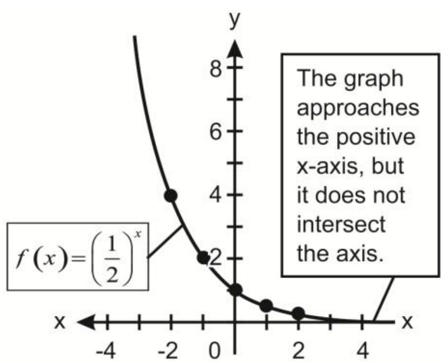
$$f(-1)=\left(\frac{1}{2}\right)^{-1}=\frac{1}{\frac{1}{2}}=2$$

$$f\left(\sqrt{3}\right)=\left(\frac{1}{2}\right)^{\sqrt{3}}\approx\left(\frac{1}{2}\right)^{1.732}\approx 0.301$$

92. Graph the function $f(x)=\left(\frac{1}{2}\right)^x$

Solution:

x	$f(x)=\left(\frac{1}{2}\right)^x$	**(x,y)**
-3	$f(-3)=\left(\frac{1}{2}\right)^{-3}=8$	$(-3,8)$
-2	$f(-2)=\left(\frac{1}{2}\right)^{-2}=4$	$(-2,4)$
-1	$f(-1)=\left(\frac{1}{2}\right)^{-1}=2$	$(-1,2)$
0	$f(0)=\left(\frac{1}{2}\right)^{0}=1$	$(0,1)$
1	$f(1)=\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$	$\left(1,\frac{1}{2}\right)$
2	$f(2)=\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$	$\left(2,\frac{1}{4}\right)$



Observe that the values of y decrease as x increases. This is an **exponential decay function**. This is typical of the graphs of all exponential functions for which the positive base is less than 1. For the function $f(x)=\left(\frac{1}{2}\right)^x, b=\frac{1}{2}$ which is less than 1.

93. Is $f\left(x\right)=0.25^x$ an exponential growth function or an exponential decay function?

Solution:

The base is 0.25, which is less than 1. The function is an exponential decay function.

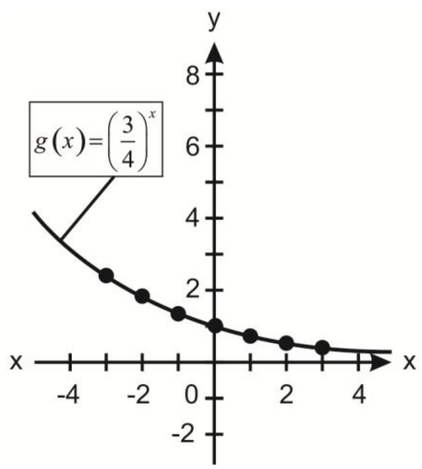
94. State whether $g\left(x\right)=\left(\frac{3}{4}\right)^x$ is an exponential growth function or an exponential decay function. Then graph the function.

Solution:

Because the base $\frac{3}{4}$ is less than 1, g is an exponential decay function. Because it is an exponential decay function, the y-values will decrease as x increases. The y-intercept of the graph is the point $\left(0,1\right)$ and the graph also passes through $\left(\frac{1,3}{4}\right)$.

Plot a few additional points. Then draw a smooth curve through the points, as shown in the figure.

x	$g\left(x\right)=\left(\frac{3}{4}\right)^x$	(x,y)
-3	$g\left(-3\right)=\left(\frac{3}{4}\right)^{-3}=\frac{64}{27}$	$\left(-3,\frac{64}{27}\right)$
-2	$g\left(-2\right)=\left(\frac{3}{4}\right)^{-2}=\frac{16}{9}$	$\left(-2,\frac{16}{9}\right)$
-1	$g\left(-1\right)=\left(\frac{3}{4}\right)^{-1}=\frac{4}{3}$	$\left(-1,\frac{4}{3}\right)$
0	$g\left(0\right)=\left(\frac{3}{4}\right)^{0}=1$	$\left(0,1\right)$
1	$g\left(1\right)=\left(\frac{3}{4}\right)^{1}=\frac{3}{4}$	$\left(1,\frac{3}{4}\right)$
2	$g\left(2\right)=\left(\frac{3}{4}\right)^{2}=\frac{9}{16}$	$\left(2,\frac{9}{16}\right)$
	$g\left(3\right)=\left(\frac{3}{4}\right)^{3}=\frac{27}{64}$	$\left(3,\frac{27}{64}\right)$

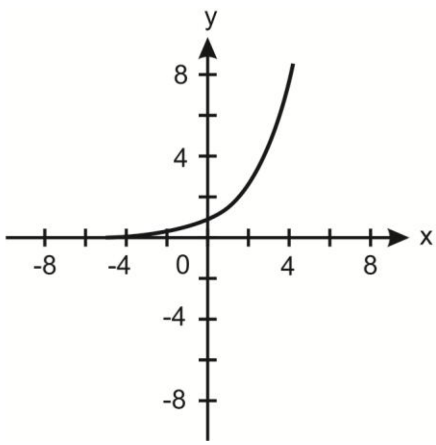


95. State whether $f(x)=\left(\frac{3}{2}\right)^x$ is an exponential growth function or an exponential decay function. Then graph the function.

Solution:

Because the base $\frac{3}{2}$ is greater than 1, *f* is an exponential growth function.

x	$g(x)=\left(\frac{3}{2}\right)^x$	(x,y)
-3	$f(-3)=\left(\frac{3}{2}\right)^{-3}=\frac{8}{27}$	$\left(-3,\frac{8}{27}\right)$
-2	$f(-2)=\left(\frac{3}{2}\right)^{-2}=\frac{4}{9}$	$\left(-2,\frac{4}{9}\right)$
-1	$f(-1)=\left(\frac{3}{2}\right)^{-1}=\frac{2}{3}$	$\left(-1,\frac{2}{3}\right)$
0	$f(0)=\left(\frac{3}{2}\right)^{0}=1$	$(0,1)$
1	$f(1)=\left(\frac{3}{2}\right)^{1}=\frac{3}{2}$	$\left(1,\frac{3}{2}\right)$
2	$f(2)=\left(\frac{3}{2}\right)^{2}=\frac{9}{4}$	$\left(2,\frac{9}{4}\right)$
3	$f(3)=\left(\frac{3}{2}\right)^{3}=\frac{27}{8}$	$\left(3,\frac{27}{8}\right)$



The Number 'e'

The number e is defined as the number that $\left(1+\frac{1}{n}\right)^n$ approaches as n increases without bound. i.e. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

The letter e was chosen in honor of the Swiss mathematician Leonhard Euler. The value of e accurate to eight decimal places is 2.71828183.

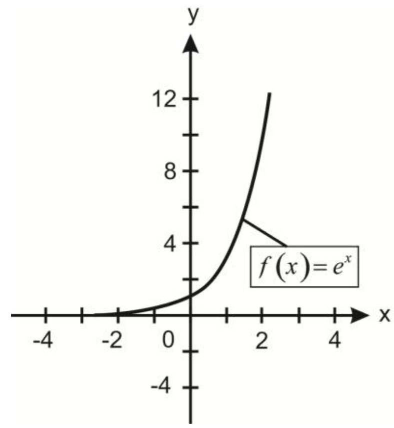
The Natural Exponential Function

For all real numbers x, the function defined by $f(x)=e^x$ is called the natural exponential function.

96. Graph $f(x)=e^x$.

Solution:

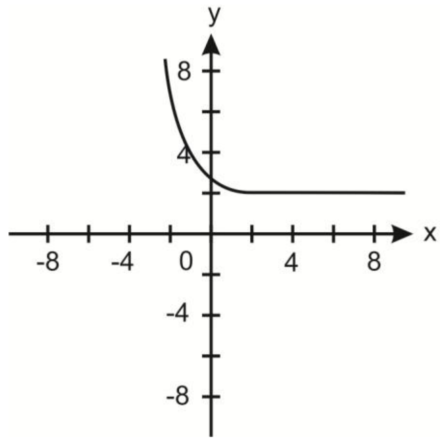
x	-2	-1	0	1	2
f(x) = e^x	0.1	0.4	1.0	2.7	7.4



97. Graph $f(x)=e^{-x}+2$.

Solution

x	-2	-1	0	1	2
f(x) = e^{-x}+2	9.4	4.7	3	2.4	2.1



98. When an amount of money P is placed in an account that earns compound interest, the value A of the money after t years is given by the compound interest formula $A=P\left(1+\frac{r}{n}\right)^{nt}$ where r is the annual interest rate as a decimal and n is the number of compounding periods per year. Suppose $500 is placed in an account that earns 8% interest compounded daily. Find the value of the investment after 5 years.

Solution:

Use the compound interest formula. Because interest is compounded daily, n = 365

$$A=P\left(1+\frac{r}{n}\right)^{nt} \quad =500\left(1+\frac{0.08}{365}\right)^{365(5)} \quad \approx 500(1.491759)\approx 745.88$$

After 5 years, there is $\$745.88$ in the account.

99. The radioactive isotope iodine-131 is used to monitor thyroid activity. The number of grams N of iodine-131 in the body t hours after an injection is given by $N(t)=1.5\left(\frac{1}{2}\right)^{\frac{t}{193.7}}$. Find the number of grams of the isotope in the body 24 hours after an injection. Round to the nearest ten-thousandth.

Solution:

$$N(t)=1.5\left(\frac{1}{2}\right)^{\frac{t}{193.7}}$$

$$N(24)=1.5\left(\frac{1}{2}\right)^{\frac{24}{193.7}}$$

$$\approx 1.5(0.9177)\approx 1.3766$$

After 24 hours, there is approximately 1.3766 grams of the isotope in the body.

100.A cup of coffee is heated to 160°F and placed in a room that maintains a temperature of 70°F. The temperature T of the coffee after t minutes is given by $T(t)=70+90e^{-0.0485t}$. Find the temperature of the coffee 20 minutes after it is placed in the room. Round to the nearest degree

Solution:

$$T(t)=70+90e^{-0.0485t}$$

$$T(20)=70+90e^{-0.0485(20)}\approx 70+34.1\approx 104.1$$

After 20 minutes the temperature of the coffee is about 104°F.

101.The function $A(t)=200e^{-0.014t}$ gives the amount of aspirin, in milligrams, in a patient's bloodstream t minutes after the aspirin has been administered. Find the amount of aspirin in the patient's bloodstream after 45 minutes. Round to the nearest milligram.

Solution:

$$A(t)=200e^{-0.014t}$$

$$A(45)=200e^{-0.014(45)}\approx 107$$

After 45 minutes, there is approximately 107 milligrams of aspirin in the patient's bloodstream.

Logarithm

For b > 0, $b\neq 1, y=\log_b x$ is equivalent to $x=b^y$.

102. Which of the following is the logarithmic form of $4^3 = 64$?

a. $\log_4 3 = 64$ **b.** $\log_3 4 = 64$ **c.** $\log_4 64 = 3$

Solution:

$\log_4 64 = 3$ is equivalent to $4^3 = 64$.

Remark

The equation $y = \log_b x$ is the logarithmic form of $b^y = x$ and is the exponential form of $y = \log_b x$. These two forms state exactly the same relationship between x and y.

103. Write a Logarithmic Equation in Exponential Form and an Exponential Equation in Logarithmic Form

a. **Write $2 = \log_{10}(x+5)$ in exponential form.**

b. **Write $2^{3x} = 64$ in logarithmic form.**

Solution:

a. $2 = \log_{10}(x+5)$ if and only if $10^2 = x+5$

b. $2^{3x} = 64$ if and only if $\log_2 64 = 3x$

104. Write a Logarithmic Equation in Exponential Form and an Exponential Equation in Logarithmic Form

a. **Write $\log_2(4x) = 10$ in exponential form.**

b. **Write $10^3 = 2x$ in logarithmic form.**

Solution:

a. $\log_2(4x) = 10$ if and only if $2^{10} = 4x$

b. $10^3 = 2x$ if and only if $\log_{10}(2x) = 3$

Equality of Exponents Property

If b > 0 and $b^x = b^y$, then $x = y$

105. Evaluate the logarithms. $\log_8 64 \; and \; \log_2\left(\frac{1}{8}\right)$

Solution:

a.

$$\log_8 64 = x$$
$$8^x = 64$$
$$8^x = 8^2$$
$$x = 2$$
$$\log_8 64 = 2$$

b.

$$\log_2\left(\frac{1}{8}\right) = x$$
$$2^x = \frac{1}{8}$$
$$2^x = 2^{-3}$$
$$x = -3$$
$$\log_2\left(\frac{1}{8}\right) = -3$$

106. Evaluate the logarithms. $\log_{10}(0.001)$ *and* $\log_5 125$.

Solution

a.

$\log_{10} 0.001 = x$

$10^x = 0.001$

$10^x = 10^{-3}$

$x = -3$

$\log_{10} 0.001 = -3$

b.

$\log_5 125 = x$

$5^x = 125$

$5^x = 5^3$

$x = 3$

$\log_5 125 = 3$

107. Solve $\log_3 x = 2$

Solution:

$\log_3 x = 2$

$3^2 = x$

$9 = x$

108. Solve $\log_2 x = 6$

Solution:

$\log_2 x = 6$

$2^6 = x$

$64 = x$

Common and Natural Logarithms

The function defined by $f(x) = \log_{10} x$ is called the **common logarithmic function**. It is customarily written without the base as $f(x) = \log x$

The function defined by $f(x) = \log_e x$ is called the **natural logarithmic function**. It is customarily written as $f(x) = \ln x$.

109. Solve each of the following equations. Round to the nearest thousandth.

$\log x = -1.5$; $\ln x = 3$

Solution:

$\log x = -1.5 \Rightarrow 10^{-1.5} = x \Rightarrow 0.032 \approx x$

$\ln x = 3 \Rightarrow e^3 = x \Rightarrow 20.086 \approx x$

110. Solve each of the following equations. Round to the nearest thousandth.

$\log x = -2.1$; $\ln x = 2$

Solution:

$\log x = -2.1 \Rightarrow 10^{-2.1} = x \Rightarrow 0.008 \approx x$

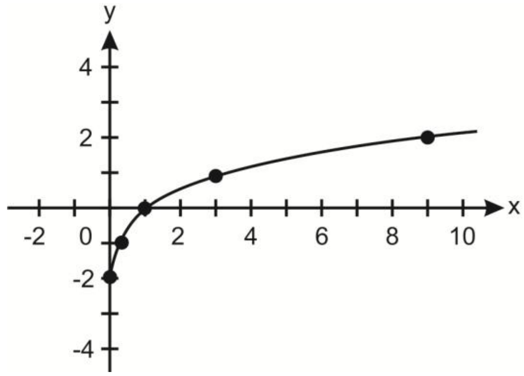
$\ln x = 2 \Rightarrow e^2 = x \Rightarrow 7.389 \approx x$

111. **Graph** $f(x)=\log_3 x$

Solution:

$\log_3 x = y \Rightarrow x = 3^y$

$x = 3^y$	1/9	1/3	1	3	9
y	-2	-1	0	1	2

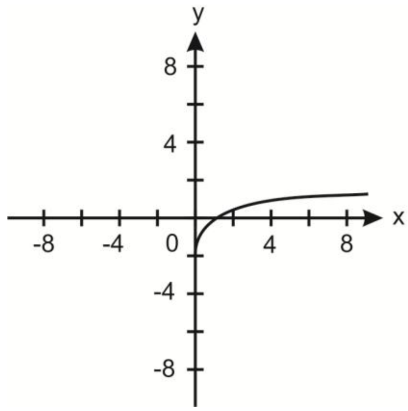


112. **Graph** $f(x)=\log_5 x$

Solution:

$\log_5 x = y \Rightarrow x = 5^y$

$x = 5^y$	1/25	1/5	1	5	25
y	-2	-1	0	1	2



113. During the 1980s and 1990s, the average time T of a major league baseball game tended to increase each year. If the year 1981 is represented by x = 1, then the function $T(x)=149.57+7.63\ln x$ approximates the average time T, in minutes, of a major league baseball game for the years x = 1 to x = 19.

a. Use the function to determine the average time of a major league baseball game during the 1981 season and during the 1999 season. Round to the nearest hundredth of a minute

b. By how much did the average time of a major league baseball game increase from 1981 to 1999?

Solution:

a. The year 1981 is represented by x = 1 and the year 1999 by x = 19.
$T(1)=149.57+7.63\ln(1)=149.57$. In 1981 the average time of a major league baseball game was about 149.57 minutes.
$T(19)=149.57+7.63\ln(19)\approx 172.04$. In 1999 the average time of a major league baseball game was about 172.04 minutes.

b. $T(19)-T(1)\approx 172.04-149.57=22.47$
From 1981 to 1999, the average time of a major league baseball game increased by about 22.47 minutes.

114. The following function models the average typing speed S, in words per minute, of a student who has been typing for t months.

$S(t)=5+29\ln(t+1)\ ;\ 0\le t\le 9$

a. Use the function to determine the student's average typing speed when the student first started to type and the student's average typing speed after 3 months. Round to the nearest whole word per minute

b. By how much did the typing speed increase during the 3 months?

Solution:

a. $S(0)=5+29\ln(0+1)=5$

The average typing speed when the student first started to type was 5 words per minute.

$S(3)=5+29\ln(3+1)\approx 45$

The average typing speed after 3 months was about 45 words per minute.

b. $S(3)-S(0)=45-5=40$

The typing speed increased by 40 words per minute during the 3 months.

The Richter Scale Magnitude of an Earthquake

An earthquake with an intensity of I has a Richter scale magnitude of $M=\log\left(\frac{I}{I_0}\right)$

Where I_0 is the measure of the intensity of a zero-level earthquake

115. Find the Richter scale magnitude of the 2003 Amazonas, Brazil earthquake, which had an intensity of $I=12589254I_0$. Round to the nearest tenth.

Solution:

$$M=\log\left(\frac{I}{I_0}\right)=\log\left(\frac{12589254I_0}{I_0}\right)=\log(12589254)\approx 7.1$$

The 2003 Amazonas, Brazil earthquake had a Richter scale magnitude of 7.1.

116. Previous example: What is the Richter scale magnitude of an earthquake whose intensity is twice that of the Amazonas, Brazil earthquake.

That is Find the Richter scale magnitude of the 2003 Amazonas, Brazil earthquake, which had an intensity of $I=2(12589254I_0)$. Round to the nearest tenth

Solution:

$I=2(12589254I_0)=25178508I_0$

$$M=\log\left(\frac{I}{I_0}\right)=\log\frac{25178508I_0}{I_0}=\log(25178508)\approx 7.4$$

The Richter scale magnitude of an earthquake whose intensity is twice that of the Amazonas, Brazil earthquake is 7.4.

117. Find the intensity of the 2003 Colina, Mexico earthquake, which measured 7.6 on the Richter scale. Round to the nearest thousand.

Solution:

$$M = \log\left(\frac{I}{I_0}\right) = 7.6 \Rightarrow \frac{1}{I_0} = 10^{7.6} \Rightarrow I = 10^{7.6} I_0 \Rightarrow I \approx 398107171 I_0$$

The 2003 Colina, Mexico earthquake had an intensity that was approximately 39,811,000 times the intensity of a zero-level earthquake.

118. On April 29, 2003, an earthquake measuring 4.6 on the Richter scale struck Fort Payne, Alabama. Find the intensity of the quake. Round to the nearest thousand.

Solution:

$$M = \log\left(\frac{I}{I_0}\right) = 4.6 \Rightarrow \frac{1}{I_0} = 10^{4.6} \Rightarrow I = 10^{4.6} I_0 \Rightarrow I \approx 39811 I_0$$

The April 29, 2003 earthquake had an intensity that was approximately 40,000 times the intensity of a zero-level earthquake.

The pH of a Solution

The pH of a solution with a hydronium-ion concentration of H^+ moles per liter is given by $pH = -\log\left[H^+\right]$

119. Find the pH of each liquid. Round to the nearest tenth.

a. **Orange juice containing an H^+ concentration of 2.8×10^{-4} mole per liter.**

b. **Milk containing an H^+ concentration of 3.97×10^{-7} mole per liter.**

c. **A baking soda solution containing an H^+ concentration of mole 3.98×10^{-9} per liter.**

Solution:

a. $pH = -\log\left[H^-\right]$

$pH = -\log\left[2.8 \times 10^{-4}\right] \approx 3.6$

b. $pH = -\log\left[H^+\right]$

$pH = -\log\left[3.97 \times 10^{-7}\right] \approx 6.4$

c. $pH = -\log\left[H^+\right]$

$pH = -\log\left[3.98 \times 10^{-9}\right] \approx 8.4$

120. Find the pH of each liquid. Round to the nearest tenth.

a. **A cleaning solution containing an H^+ concentration of 2.41×10^{-13} mole per liter.**

b. **A cola soft drink containing an H^+ concentration of 5.07×10^{-4} mole per liter.**

c. **Rainwater containing an H^+ concentration of 6.31×10^{-5} mole per liter .**

Solution:

a. $pH = -\log\left[H^{+}\right]$

$pH = -\log\left[2.41\times10^{-13}\right] \approx 12.6$

b. $pH = -\log\left[H^{+}\right]$

$pH = -\log\left[5.07\times10^{-4}\right] \approx 3.3$

c. $pH = -\log\left[H^{+}\right]$

$pH = -\log\left[5.07\times10^{-4}\right] \approx 3.3$

121. A sample of blood has a pH of 7.3. Find the hydronium-ion concentration of the blood.

Solution:

$pH = -\log\left[H^{+}\right]$

$7.3 = -\log\left[H^{+}\right]$

$-7.3 = \log\left[H^{+}\right]$

$10^{-7.3} = H^{+}$

$5.0\times10^{-8} \approx H^{+}$

The hydronium-ion concentration of the blood is about 5.01×10^{-8} mole per liter.

122. The water in the Great Salt Lake in Utah has a pH of 10.0. Find the hydronium-ion concentration of the water.

Solution:

$pH = -\log\left[H^{+}\right]$

$10.0 = -\log\left[H^{+}\right] \Rightarrow -10.0 = \log\left[H^{+}\right]$

$10^{-10.0} = H^{+} \Rightarrow 1.0\times10^{-10} \approx H^{+}$

The hydronium-ion concentration of the water in the Great Salt Lake in Utah is 1.0×10^{-10} mole per liter.

Ratio, Rate and Proportion

When a fraction is represented in the form of a:b, then it is a ratio whereas a proportion states that two ratios are equal. Here, a and b are any two integers. The ratio and proportion are the two important concepts, and it is the foundation to understand the various concepts in mathematics as well as in science.

Ratio

A comparison of two numbers or quantities in the same units. Ratios can be written as a fraction $\frac{2}{3}$, with a colon $2:3$, or as two numbers separated by a the word to **2 to 3**." For example, the ratio of children to students in a music class could be written as 20:30, which simplifies to 2:3. Also the fractional form of the ratio 12 to 5 is $\frac{12}{5}$.

Rate

A comparison of two number or quantities in different units and a special type of ratio where the two quantities are measured in different units. For example, miles per hour are a rate that compares a number of miles in one hour. Rates can be expressed as a number or a percentage, and can be equal to any value, including negative numbers.

Unit Rate

A unit rate is a rate in which the number in the denominator is 1. To find a unit rate, divide the number in the numerator of the rate by the number in the denominator of the rate. For example, for the rate $\frac{135 miles}{6 gal} = 22.5$ the unit rate is $\frac{22.5 miles}{gal}$

123.Calculate a Unit Rate: A dental hygienist earns $\$780$ for working a 40-hour week. What is the hygienist's hourly rate of pay?

Solution:

The hygienist's rate of pay is $\frac{\$780}{40h}$

To find the hourly rate of pay, divide 780 by 40. i.e. 780 $\div$ 40 = 19.5

$$\frac{\$780}{40h} = \frac{\$19.5}{h}$$

The hygienist's hourly rate of pay is $\$19.50$ per hour.

124. Solve an Application of Unit Rates: A teacher earns a salary of $\$34{,}200$ per year. Currently the school year is 180 days. If the school year were extended to 220 days, as is proposed in some states, what annual salary should the teacher be paid if the salary is based on the number of days worked per year?

Solution:

Find the current salary per day.

$$\frac{\$34200}{180\ days} = \frac{\$190}{day}$$

Multiply the salary per day by the number of days in the proposed school year.

$\frac{\$190}{day} \times 220\ days = \$41800$

The teacher's annual salary should be $41,800.

125. Determine the more economical purchase: Which is the more economical purchase? An 18 ounce jar of peanut butter priced at $3.49 or a 12 ounce jar of peanut butter priced $2.59?

Solution:

Finding the unit price we have $\frac{\$3.49}{18\ oz} = \frac{\$0.194}{oz}$ and

$\frac{\$2.59}{12\ oz} = \frac{\$0.216}{oz}$ that is $\$0.194 < \$0.1216$

The item with the lower unit price is the more economical purchase. The more economical purchase is the 18 ounce jar of peanut butter priced at $3.49.

126. Determine a ratio in simplest form: A survey revealed that, on average, eighth-graders watch approximately 21 hours of television each week. Find the ratio, as a fraction in simplest form, of the number of hours spent watching television to the total number of hours in a week.

Solution:

1 week to hours $= \frac{24\,h}{day} \times 7\,days = 168\,hours$

Required ratio $= \frac{21h}{week} = \frac{21h}{168\,hours} = \frac{1}{8}$

127. Calculate the student–faculty ratio at Oregon State University. Round to the nearest whole number. Write the ratio using the word to.

University	Men	Women	Faculty
Oregon State University	7509	6478	1352
University of Oregon	6742	7710	798

Solution:

Total number of students = 7509 + 6478 = 13,987

Required ratio $\frac{13987}{1352} \approx \frac{10.3454}{1} \approx \frac{10}{1}$

The ratio is approximately 10 to 1.

Ratio among three or more quantities

Ratio can be set up between more than two quantities. For example, the marks obtained by the students who stood up first, second and third in semester 1, are 90, 75 and 60 respectively, the ratio among the marks will be written as

90 : 75 : 60

6 : 5 : 4 (in lowest form)

128. Salaries of A, B, C and D are Rs. 5500, Rs. 3500 and Rs. 4000 respectively. Find the ratio among the salaries.

Solution:

Salary of A	**:**	**Salary B**	**:**	**Salary of C**	**:**	**Salary of D**
4500	:	5500	:	3500	:	4000
9	:	11	:	7	:	8

Thus the required ratio is A: B : C : D = 9 : 11 : 7 : 8

129. Amina, zainab and HIra got eidi in the ratio 2 : 3 : 5 if Zainab got Rs. 30 as eidi, find the eidi of the others two.

Solution:

Amina's eidi	**:**	**Zainab eidi**	**:**	**Hira's eidi**
2	:	3	:	5

Since Zainab got Rs. 30, therefore, to make Zainab's share equivalent to 30, we multiply elements by 10. Thus, The required eidi is

Amina's edi = 2 x 10 = 20

Zainab's edi = 3 x 10 = 30

Hira's edi = 5 x 10 = 50

Distributing a quantity in a given Ratio

In daily life, we need to distribute the quantities in a given ratio. Working rule for distribution is discussed with the help of following examples.

130.Distribute Rs. 2500 among two Friends nm the ratio of 3 : 2

Solution:

Total amount = 2500

Given ratio = $3:2$

Sum of ratio = $3+2=5$

Part of first friend = $\frac{3}{5}\times 2500=3\times 500=1500$

Part of second friend $=\frac{2}{5}\times 2500=2\times 500=1000$

131. In three shelves 450 books are kept in the ratio of 3 : 5 : 7. Find the number of books in each shelf.

Solution:

Total books to be distributed = 450

Ratio of books kept in shelves = $3:5:7$

Sum of the ratios of elements = $3+5+7=15$

Number of books in 1^{st} shelf = $\frac{3}{15}\times 450 = 3\times 30 = 90$

Number of books in 2^{nd} shelf $= \frac{5}{15}\times 450 = 5\times 30 = 150$

Number of books in 3^{rd} shelf $= \frac{7}{15}\times 450 = 210$

132. Distribute 68 camels among three brothers in the ratio of $\frac{1}{9}:\frac{1}{3}:\frac{1}{2}$

Solution:

Total camels to be distributed $=68$

Ratio of the shares $=\frac{1}{9}:\frac{1}{3}:\frac{1}{2}$

Multiply by 18, (L.C.M) of 9,3,2) $=\frac{1}{9}\times 18:\frac{1}{3}\times 18:\frac{1}{2}\times 18$

Sum of elements of the ratio $=2:6:6$

Shares of first brother $=\frac{2}{17}\times 68 = 2\times 4 = 8$ camels

Shares of second brother $=\frac{6}{17}\times 68 = 6\times 4 = 24$ camels

Shares of third brother $=\frac{9}{17}\times 68 = 9\times 4 = 36$ camels

133. Three men invested Rs. 18000, Rs. 12000 and Rs. 6000 respectively in a business. After some time they have a profit of Rs. 36000 distribute this profit among three men.

Solution:

Ratio of the investment $=18000:12000:6000$

Ratio of the investment $=3:2:1$

Sum of the elements of the ratio $=3+2+1=6$

Shares of first man in profit $=\frac{3}{6}\times 36000 = 3\times 6000 = 18000$

Shares of second man in profit $=\frac{2}{6}\times 36000 = 2\times 6000 = 12000$

Shares of third man in profit $=\frac{1}{6}\times 36000 = 1\times 6000 = 6000$

134.Nazir started a business with a capital of Rs. 1000 Six months later Qadir invest Rs. 15000. After eight months, Bashir joined by investing RS. 2000. They got Rs. 2900 at the end of the year. Find the share of the each in the profit.

Solution:

Nazir's duration of investment $=12$ months

Qadir's duration of investment $=6$ months

Bashir's duration of investment $=4$ months

Multiplying their capitl by their duration of investment, we have

Ratio of the capital $=12\times 10000:6\times 15000:4\times 20000$

Ratio of the capital $=1,20,000:90,000:80,000$

In simplified from $=12$

Sum of the elements of the ratio $=12+9+8=29$

Total profit $=2900$

Nazir's share in profit $=\dfrac{12}{29}\times 2900=12\times 100=1200$

Qadir's share in profit $=\dfrac{9}{29}\times 2900=9\times 100=900$

Nazir's share in profit $=\dfrac{8}{29}\times 2900=8\times 100=800$

135.Distribute an amount of Rs. 6200 in three parts such that double times the second part and 5 times the 3rd part are mutually equal.

Solution: Let the parts be A, B and C here

Double the 1st part $=2A$

3 times the 2nd part $=3B$

5 times the 3rd part $=5C$

As they are mutually equal , so $2A=3B=5C$

Dividing the L.C.M of 2,3 and 5 5 i.e, 30, we get

$$\Rightarrow \frac{2A}{30}=\frac{3B}{30}=\frac{5C}{30}\Rightarrow \frac{A}{15}=\frac{B}{10}=\frac{C}{6}$$

$$\Rightarrow A:B:C=15:10:6$$

Sum of ratios $=15+10+6=31$

Total amount of the be distributed $=6200$

Thus First part $=\dfrac{15}{31}\times 6200=15\times 200=3000$

second part $=\dfrac{10}{31}\times 6200=10\times 200=2000$

third part $=\dfrac{6}{31}\times 6200=6\times 200=1200$

136. A profit of Rs. 60,000 is to be distributed in four parterres in the ratio of

$$\frac{1}{2}:\frac{7}{12}:\frac{2}{3}:\frac{3}{4}$$

Solution:

Ratio of the partners = $\frac{1}{2}:\frac{7}{12}:\frac{2}{3}:\frac{3}{4}$

Multiplying by 12 (the L.C.M) $=6:7:8:9$

Sum of ratios $=6+7+8+9=30$

Total profit $=60,000$

Shares of first partner $=\frac{6}{30}\times 60,000=6\times 2000=12000$

Shares of first partner $=\frac{7}{30}\times 60,000=7\times 2000=14000$

Shares of first partner $=\frac{8}{30}\times 60,000=8\times 2000=16000$

Shares of first partner $=\frac{9}{30}\times 60,000=9\times 2000=18000$

Note: We can check the solution by adding the distribution as

$12,000+14000+16,000+18,000 \ = \ 60,000$

Continued Ratio

In the previous section, we have learnt about the concept of ratio its elements and have solved related problems. Now let us learn the concept of continued ratio. If the two ratios a : b and b:c are given for three quantities a,b and c, then the ratio a:b:c is called continued ratio. It can be written as

$$\begin{array}{l} a:b \\ \quad b:c \\ \hline a:b:c \end{array}$$

Note: a:b:c is called the continued ratio and the element present both the ratios is called their common element.

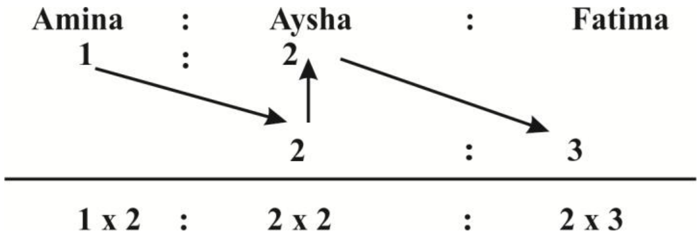
137. Amna and Aysha's ages are in the ratio 1: 2, while Ayesha and Fatima's age are in ratio of 2:3. Find the continued ratio between the ages of Amna and Fatima.

Solution:

The ratio of Amina and Aysha 's age = 1 : 2

The ratio of Aysha and Fatima's ages = 2 : 3

As in the two ratios Aysha's age is common, so keeping Aysha's age in between, we have

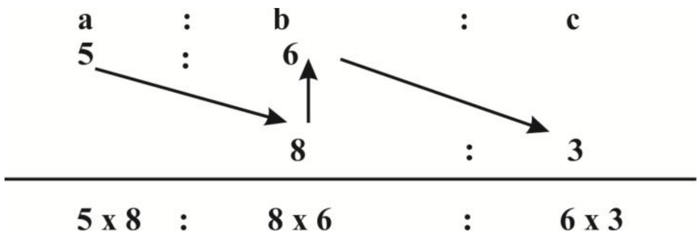


Continued ratio 2 : 4 : 6

In simplest form 1 : 2 : 3

The ratio of Amina and Fatima is 1:3

138.Find a :b : c , when a:b = 5 : 6 and b : c = 8 : 3



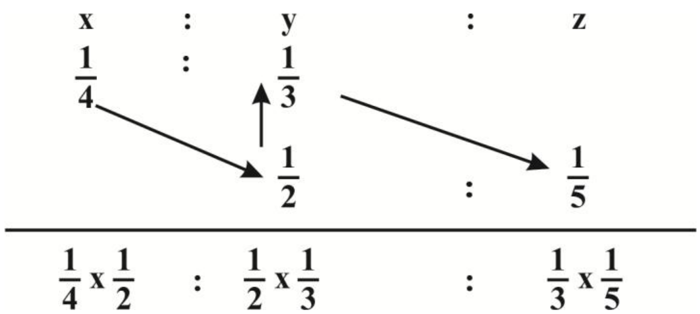
Continued ratio 40 : 48 : 18

In simplest form 20 : 24 : 9

Therefore , a: b : c = 20 : 24 : 9

139.Find $x:y:z$ **, when** $x:y=\frac{1}{4}:\frac{1}{3}$ **and** $y:z=\frac{1}{2}:\frac{1}{5}$

Solution:



Continued ratio

Multiplying by 120 (L.C.M of 8,6 and 15), we have

$\frac{1}{8}\times 120:\frac{1}{6}\times 120:\frac{1}{15}\times 120$

15 : 20 : 8

Thus, $x:y:z=15:20:8$

Dividing a Quantity into a given ratio by applying continued ratio

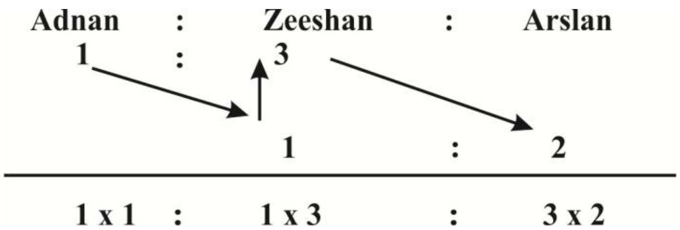
The method of dividing a quantity into a given ratio by applying continued ratio is illustrated through the examples below.

140.The sum of Rs. 5000 is to be divided among Adnan, Zeeshan and Arslan in such a way that Arslan gets twice as that of Zeeshan and Zeeshan get 3 times as that of Adnan. How much will be the share of each?

Solution:

If Adnan gets 1, then Zeeshan gets 3 thus Adnan : Zeeshan = 1 : 3

If Zeeshan gets 1, then Arslan gets 2, thus Zeeshan : Arslan = 1: 2



Continued ratio = 1 : 3 : 6

Sum of ratios = 1 + 3 + 6 = 10

Total amount to be distributed = 5000

Adnan's share $= \frac{1}{10} \times 5000 = 1 \times 500 = 500$

Zeeshan share $= \frac{3}{10} \times 5000 = 3 \times 500 = 1500$

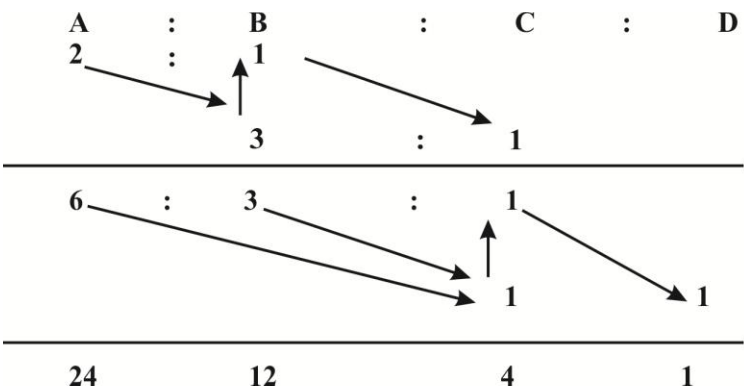
Arslan's share $= \frac{6}{10} \times 5000 = 6 \times 500 = 3000$

Dividing a quantity into more than three ratios

The method of dividing a quantity into more than two ratios, by applying continued ratio is illustrated through the examples below

141.If A : B = 2 : 1 , B : C = 3 : 1 and C : D = 4 : 1 , distribute an amount of Rs. 8200 in the four partners

Solution:



Continued ratio $=24:12:4:1$

Sum of ratios $=24+12+4+1=41$

A's share $=\frac{21}{41}\times 8200=21\times 200=1800$

B's share $=\frac{12}{41}\times 8200=12\times 200=2400$

C's share $=\frac{4}{41}\times 8200=4\times 200=800$

D's share $=\frac{1}{11}\times 8200=1\times 200=200$

Proportions

A proportion is an equation that states that two ratios or rates are equal to each other or Equality of two ratios or rates. Proportions are often denoted using the symbol "::" or "=". The definition of a proportion can be stated as follows: If $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios or rates, then $\frac{a}{b}=\frac{c}{d}$ is a proportion. Each of the four members in a proportion is called a **term**. The second and third terms of the proportion are called the **means** and the first and fourth terms are called the **extremes**. For example, if a train travels 100 kilometers per hour, it would take 5 hours to travel 500 kilometers. The proportion would be written as 100km/hr = 500km/5hrs. Also the proportion of 3 is to 5 as 12 is to 20 will be 3:5::12:20 or $\frac{3}{5}=\frac{12}{20}$

142.For the proportion $\frac{5}{8}=\frac{10}{16}$ **.**

- **a.** Name the first and third term.
- **b.** Write the product of the means.
- **c.** Write the product of the extremes.

Solution:

a. The first term is 5. The third term is 10.

b. The product of the means is 8(10) = 80

c. The product of the extremes is 5(16) = 80.

Cross-Products Method of Solving a Proportion

If $\frac{a}{b}=\frac{c}{d}$ then $ad=bc$

143. Solve $\frac{8}{5}=\frac{n}{6}$

Solution:

$$\frac{8}{5}=\frac{n}{6}\Rightarrow 5\times n=8\times 6\Rightarrow 5n=48\Rightarrow n=\frac{48}{5}\Rightarrow n=9.6$$

144. Solve $\frac{42}{x}=\frac{5}{8}$

Solution:

$$\frac{42}{x}=\frac{5}{8}\Rightarrow 5\times x=8\times 42\Rightarrow 42$$

$$\Rightarrow 5x=336\Rightarrow x=\frac{336}{5}$$

$$\Rightarrow x=67.5$$

145. If you travel 290 miles in your car on 15 gallons of gasoline, how far can you travel in your car on 12 gallons of gasoline under similar driving conditions?

Solution:

$$\frac{290\,miles}{15\,gallons}=\frac{x\,miles}{12\,gallons}$$

$$\Rightarrow\frac{290}{15}=\frac{x}{12}\Rightarrow 15\times x=290\times 12\Rightarrow 15x=3480\Rightarrow x=\frac{3480}{15}\Rightarrow x=232$$

You can travel 232 miles on 12 gallons of gasoline.

146. On a map, a distance of 2 centimeters represents 15 kilometers. What is the distance between two cities that are 7 centimeters apart on the map?

Solution:

$$\frac{15\,kilometers}{2\,centimeters}=\frac{x\,kilometers}{7\,centimeters}$$

$$\frac{15}{2}=\frac{x}{7}\Rightarrow 2x=105\Rightarrow x=\frac{105}{2}\Rightarrow x=52.5$$

The distance between two cities 52.5 kilometers

147. The table below shows three of the universities in the Big Ten Conference and their student–faculty ratios. There are approximately 31,100 full-time undergraduate students at Michigan State University. Approximate the number of faculty at Michigan State University.

University	Student – Faculty Ratio
Michigan State University	13 to 1
University of Illinois	15 to 1
University of Iowa	11 to 1

Solution:

Let F = the number of faculty members.

Write a proportion and then solve the proportion for F.

$$\frac{13\,students}{1\,Faculty}=\frac{31,100\ \ students}{F\ Faculty}$$

$$\Rightarrow\frac{13}{1}=\frac{31,100}{F}\Rightarrow 13F=31,100\Rightarrow F\Rightarrow\frac{31,100}{13}\Rightarrow F\approx 2392$$

There are approximately 2392 faculty members at Michigan State University.

148.In the United States, the average annual number of deaths per million people aged 5 to 34 from asthma is 3.5. Approximately how many people aged 5 to 34 die from asthma each year in this country? Use a figure of 150,000,000 for the number of U.S. residents who are 5 to 34 years old.

Solution:

Let D=the number of people aged 5 to 34 who die each year from asthma in the United States. Write and solve proportion. One rate is 3.5 deaths per million people

$$\frac{3.5\,deatgs}{1000,000\ \ people}=\frac{D\ \ deaths}{150,000,000\ \ people}\Rightarrow\frac{3.5}{1000,000}=\frac{D}{150,000,000}$$

$$\Rightarrow 1000,000D=3.5\times150,000,000\Rightarrow 1000,000D=525,000,000$$

$$\Rightarrow D=\frac{525,000,000}{1000,000}\Rightarrow D=525$$

In the United States, approximately 525 people aged 5 to 34 die each year from asthma.

149.Find the value of the x in the following proportion.

i. $x:21:25:35$ **ii.** $5:x:2:10$

Solution:

i. $x:21::25:35$

we know that

Product of extremes = Product of Means

$\Rightarrow x\times35=21\times35$

Dividing both the sides by 35

$$\frac{x\times35}{35}=\frac{21\times25}{35}\Rightarrow x=15$$

ii. $5:x::2:10$

We know that

Product of Extremes = Product of means

$\Rightarrow\ 5\times10=x\times2$

Dividing both the sides by 2

$$\Rightarrow\frac{5\times10}{2}=\frac{x+2}{2}\Rightarrow x=25$$

150.Find the fourth term in the proportion 10 , 11 , 30

Solution:

Let the fourth term be x, then the proportion is

$10 : 14 :: 30 : x$

Now, Product of extremes = Product of means

$\Rightarrow 10\times x = 14\times 30$

$\Rightarrow \frac{10\times x}{10} = \frac{14\times 30}{10}$

$x = 42$

151.Find the third term in the proportion 3,6,20

Solution:

Let the third term be x then proportion is

$3:6::x:20$

Now Product of extremes = product means

$\Rightarrow 3\times 20 = 6\times x$

$\Rightarrow \frac{3\times 20}{6} = \frac{6\times x}{6}$

$10 = x$

Thus the third term is 10

Third proportion

If x is the third proportion to the numbers a and b then, $a:b::b:x$

152.Find the third proportion to the two numbers 3 , 9

Solution:

Let the third proportion be x, then the proportion is

$3:9::9:x$

Now, Product of extremes = Product of means

$\Rightarrow 3\times x = 9\times 9$

$\Rightarrow x = \frac{9\times 9}{3} = 27$

Note: The difference of third term and third proportion in examples 3 & 4

153.Find the mean proportion to the two numbers 3 , 9

Solution:

Let the mean proportion be x, then the proportion is

$3:x::x:9$

Now Product of extremes = product means

$\Rightarrow 3\times 9 = x\times x$

$\Rightarrow x^2 = 27$

$\Rightarrow x = \pm\sqrt{27}$ *or* $x = \pm\sqrt{9\times 3} = \pm 3\sqrt{3}$

Types of the Proportions : There are three types of the proportions, namely, Direct Proportion, Inverse Proportion and compound proportion.

Direct proportion: The relation of two ratios in which increase in one quantity causes the increase in the other quantity in the same ratio or decrease in one quantity causes the decrease in the other quantity in the same ratio is called the direct proportion. Proportion is a very important topic of Mathematics used in daily life. We explain it with the help of following examples.

154.A person digs a tunnel of 5 km in 10 days. In how many days he will dig a 15 km long?

Solution:

Length of tunnel in km	Days
5	10
15	x

Product of extremes = product means

$\Rightarrow 5\times x = 10\times 10$

$\Rightarrow x = \frac{5\times 10}{5}$

$\Rightarrow x = 30$ days

155.A man earns Rs. 1500 in 2 weeks. What will he earn in 2 days working 6 days in solution:

Solution:

Days	Rupees
12	1500
2	x

Note: In less days he will earn less amount so, it is a direct proportion

Now $12:2::1500:x$

Product of extremes = product means

$\Rightarrow 12\times x = 2\times 1500$

$\Rightarrow x = \frac{2\times 1500}{12} = \frac{1500}{6} = 250$

or x = Rs. 250

156.A car runs 81 km in 4.5 liters of petrol. How far will it run by 20 liters?

Solution:

Petrol (liters)	Distance (Kms)
4.5	81
20	x

Now 4.5 : 20 :: 81 : x

Product of extremes = product means

$\Rightarrow 4.5\times x = 20\times 81$

$\Rightarrow x = \frac{20\times 81}{45} = \frac{20\times 81}{45}\times 10 = 350\,kms$

Thus in 20 liters car will run 360 kms

Inverse Proportion: The relation between two ratios, in which increase in one quantity causes the decrease in quantity and vice versa, is called the inverse proportion. See the examples below.

157.10 men do a work in 4 days. In how many days 20 men do the same job?

Solution:

Men	Days
10	4
20	x

Note: That if there are more persons, then it will take less days to finish the same work. So it is an inverse proportion.

Now, $20:10::4:x$

Product of extremes = product means

$\Rightarrow 20\times x = 10\times 4$

$\Rightarrow x = \dfrac{10\times 4}{20} = 2$ days

158.60 days food is available for 210 students in a hostel after 14 days 20 more students came. For how many days, the remaining food will be enough?

Solution:

Number of Students	Days
210	46
230	x

Note: That it is an inverse proportion. Also 14 days have passed and if students remained 210, then food will be enough up to remaining 46 days.

Now $230:210::46:x$

Product of extremes = product means

$\Rightarrow 230\times x = 210\times 46$

$\Rightarrow x = \dfrac{210\times 46}{230} = 42$ days

Thus food will be enough for 42 days

159.A road can be constructed by 120 men in 180 days. If the road is to be completed in 150 days. How many more men are required?

Solution:

Men	Days
180	120
150	x

It is an inverse proportion, so $150:180::120:x$

Product of extremes = product means

$\Rightarrow 150\times x = 180\times 120$

$\Rightarrow x = \dfrac{180\times 120}{150} = 144$ men

As required men to finish the job = 144

Men already working = 120

More men required = 144 – 120 = 21 men

160.In a factory, 20 machines are used to complete a piece of work in 30 days. If 8 machines break down how long will it take now to complete the job?

Solution:

Machines	Days
20	30
12	x

It is an inverse proportion, so, $12:20::30:x$

Product of extremes = product means

$$\Rightarrow 12\times x = 20\times 30$$

$$\Rightarrow x = \frac{20\times 30}{12}=\frac{600}{12}=50$$

Thus number of days required = 50 days.

Compound Proportion

The relationship between two or more proportions is known as the Compound Proportion. The method of solving questions relating to the compound proportion is illustrated with the help of following examples.

161.If 35 laborers dig 805 cubic centimeters of the earth in 5 hours. How much earth will 30 labors dig in 6 hour?

Solution:

Labors	hours	Earth (cm^3)
35	5	805
30	6	x

Note: That when labors are increased, earth will be increased, so it is direct proportion. Also when time is increased earth is increased, so it is also direct proportion.

$$\left.\begin{matrix}35:30\\ 5:\ 6\end{matrix}\right\}::805:x$$

Product of extremes = product means

$$\Rightarrow 35\times 5\times x = 30\times 6\times 805$$

$$\Rightarrow x = \frac{30\times 6\times 805}{35\times 5}=828$$

Thus the earth will be dug = 828 cm^3.

162.Rs. 8000 rupees are sufficient for a family of 4 members for 40 days. For how many days Rs. 15000 will be sufficient for a family of 5 members?

Solution:

Rupees	Members	Days
8000	4	40
15000	5	x

Note: That when rupees are increase, days are also increased so it is direct proportion. Again when family members are increased, days are increased so it is an inverse proportion

$$\left.\begin{matrix} 8000:15000 \\ 5:\ 4 \end{matrix}\right\} :: 40 : x$$

Product of extremes = product means

$$\Rightarrow 800 \times 5 \times x = 15000 \times 4 \times 40$$

$$\Rightarrow \quad x = \frac{15000 \times 4 \times 40}{8000 \times 5} = 60 \text{ days}$$

163.195 men working 10 hours daily can finish a job I 20 days. How many employed to finish the job in 15 days if they work 13 hours daily?

Solution:

Hours	Days	Men
10	20	195
13	15	x

Note: That when hours are increased, the numbers of men are decreased, so it is inverse proportion. Also when days are decreased, the number of men be increased, it is inverse proportion.

$$\left.\begin{matrix} 13:10 \\ 15:20 \end{matrix}\right\} :: 195 : x$$

Product of extremes = product means

$$\Rightarrow 13 \times 15 \times x = 20 \times 10 \times 195$$

$$\Rightarrow \quad x = \frac{20 \times 10 \times 195}{13 \times 15} = 200 \text{ men}$$

As 195 men are already working, so more men = 200 – 195 = 5

Note: If question is “how many men” are required, then number of men = 200

164. A contractor got a contract to build a road in 30 days and employed 20 men to do the work Only 1/4 work was completed in 10 days. How many more men be employed, so the work will be completed in time?

Solution:

Days	Work	Men
10	1/4	20
20	3/4	x

Note: That work is 1, when completed 1/4 , the remaining work = 1 – 1/4 = 3/4 . Also when days are increased, less men can do the job in specified time, so it is inverse. When work is increased, more men are required so it is direct proportion.

$$\left.\begin{matrix} 20 : 10 \\ 1/4 : 3/4 \end{matrix}\right\} :: 20 : x$$

Product of extremes = product means

$$\Rightarrow 20 \times \frac{1}{4} \times x = 10 \times \frac{3}{4} \times 20$$

$$\Rightarrow x = \frac{10}{20} \times \frac{3}{4} \times \frac{4}{1} = 30 \text{ men}$$

Men required to finish job in time = 30

Men are already working = 20

More men required = 30 – 20 = 10

165. 15 tailors work for 8 hour a day can produce 360 shirts in 6 days. If 3 tailors leave the job, how many shirts can be produced of remaining tailors work 10 hours a day for 8 days?

Solution:

Tailors	Hours	Days	Shirts
15	8	6	360
12	10	8	x

Note: That when tailors are decreased, shirts are decreased, so it is direct proportion. When hours are increased, shirts are increased, so it is direct proportion. When days are increased, shirts are increased so it is direct proportion.

$$\left.\begin{matrix} 15 : 12 \\ 8 : 10 \\ 6 : 8 \end{matrix}\right\} : 360 : x$$

Product of extremes = product means

$$\Rightarrow 15 \times 8 \times 6 \times x = 12 \times 10 \times 8 \times 360$$

$$\Rightarrow x = \frac{12 \times 10 \times 8 \times 360}{15 \times 8 \times 6} = 480 \text{ shirts}$$

Thus numbers of shirts = 480

Proportion and the Golden Ratio

In Mathematics, two quantities are said to be in golden ratio, if their ratio is equal to the ratio of their sum to the larger of the two quantities. The golden ratio, also known as the golden sections, the golden number, golden proportion, golden mean or the divine proportion, is a ratio between two numbers that equals approximately 1.618. Usually written as the Greek letter phi, it is strongly associated with the Fibonacci sequence, a series of numbers wherein each number is added to the last.

Mathematically it is written in the form $\phi = \frac{1+\sqrt{5}}{2} = 1.61803$

The Greeks claimed that the most visually pleasing division of the line had the following property for the ratios of its lengths: ratio of the long piece to the short piece = ratio of the entire line segment to the long piece. That is, $\frac{L}{1} = \frac{L+1}{L}$.

This statement of proportion can be solved to find that L has a special value, denoted by the Greek letter ϕ (phi, pronounced "fie" or "fee"), which is $\phi = \frac{1+\sqrt{5}}{2} = 1.61803....$

Remark

- The Greek letter ϕ is the first letter in the Greek spelling of Phydias, the name of a Greek sculptor who may have used the golden ratio in his work.
- The Golden Ratio is also equal to **2 × sin(54°)**, get your calculator and check!

How to Calculate It?

You can use that formula to try and calculate ϕ yourself.

First **guess** its value, then do this calculation again and again:

A) divide 1 by your value (=1/value) B) add 1

C) now use *that* value and start again at A

With a calculator, just keep pressing "1/x", "+", "1", "=", around and around.

Start with 2 and got this:

value	1/value	1/value + 1
2	1/2 = 0.5	0.5 + 1 = **1.5**
1.5	1/1.5 = 0.666...	0.666... + 1 = **1.666...**
1.666...	1/1.666... = 0.6	0.6 + 1 = **1.6**
1.6	1/1.6 = 0.625	0.625 + 1 = **1.625**
1.625	1/1.625 = 0.6153...	0.6154... + 1 = **1.6153...**

It gets closer and closer to ϕ the more we go.

But there are better ways to calculate it to thousands of decimal places quite quickly.

166.Suppose the line segment in Figure is divided according to the golden ratio. If the length of the longer piece labeled x is 5 centimeters, how long is the entire line segment?

Solution:

Because the line segment is divided in the golden ratio, we know that $\frac{x}{y} = \phi$. We solve for y by multiplying both sides by y and dividing both sides by ϕ:

$$\frac{x}{y} = \phi \rightarrow y = \frac{x}{\phi}$$

Substituting x=5 cm and the approximate value 1.6 for ϕ, we find $y = \frac{5cm}{\phi} \approx 3.1cm$

The entire segment has a length of x + y,
so its total length is approximately 5 cm + 3.1 cm = 8.1 cm.

167.Household Golden ratios: Consider the following household items with the given dimensions. Which item comes closest to having the proportions of golden ratio?

- Standard sheet of paper: 8.5 in x 11 in
- 8 x 10 picture frame: 8 in x 10 in
- HDTV (high-definition television), which comes in many sizes but always with a 16:9 ratio of width to height

Solution:

The ratio of the sides of a standard sheet of paper is 11/8.5 ≈ 1.29, which is 20% less than the golden ratio. The ratio of the sides of a standard picture frame is 10/8 = 1.25, which is 23% less than the golden ratio. The HDTV ratio is 16/9 ≈ 1.78 which is about 10% more than the golden ratio. Of the three objects, the high-definition television is closest to being a golden rectangle.

168. Calculate the value of the golden ratio φ using quadratic equations.

Solution:

We know,

$\varphi = 1 + 1/\varphi$

Multiplying both sides by φ,

$\varphi^2 = \varphi + 1$

On rearranging, we get,

$\varphi^2 - \varphi - 1 = 0$

The above equation is a quadratic equation and can be solved using quadratic formula:

$$\varphi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values of a = 1, b = -1 and c = -1, we get,

$$\varphi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

The solution can be simplified to a positive value giving:

$$\varphi = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

Note that we are not considering the negative value, as φ is the ratio of lengths and it cannot be negative.

Therefore, $\varphi = \frac{1}{2} + \frac{\sqrt{5}}{2}$

Fibonacci Sequence and Golden Ratio

There is a special relationship between the Golden Ratio and the Fibonacci Sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

(The next number is found by adding up the two numbers before it.)

And here is a surprise: when we take any two successive *(one after the other)* Fibonacci Numbers, **their ratio is very close to the Golden Ratio**.

In fact, the bigger the pair of Fibonacci Numbers, the closer the approximation.

Let us try a few:

A	B	B/A
2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
⋮	⋮	⋮
144	233	1.618055556...
233	377	1.618025751...
⋮	⋮	⋮

We don't have to start with 2 and 3, here I randomly chose 192 and 16 (and got the sequence 192, 16,208,224,432,656,1088,1744,2832,4576,7408,11984,19392,31376,...):

A	B	B / A
192	**16**	0.08333333...
16	208	13
208	224	1.07692308...
224	432	1.92857143...
⋮	⋮	⋮
7408	11984	1.61771058...
11984	19392	1.61815754...
⋮	⋮	⋮

The connection between the Fibonacci numbers and the golden ratio becomes clear when we compute the ratios of successive Fibonacci numbers, as shown in Table. Note that, as we go further out in the sequence, the ratios of successive Fibonacci numbers get closer and closer to the golden ratio ϕ = 1.61803...

Ratios of Successive Fibonacci Numbers

$F_3/F_2 = 2/1 = 2.0$	$F_{11}/F_{10} = 89/55 \approx 1.618182$
$F_4/F_3 = 3/2 = 1.5$	$F_{12}/F_{11} = 144/89 \approx 1.617978$
$F_5/F_4 = 5/3 \approx 1.667$	$F_{13}/F_{12} = 233/144 \approx 1.618056$
$F_6/F_5 = 8/5 = 1.600$	$F_{14}/F_{13} = 377/233 \approx 1.618026$
$F_7/F_6 = 13/8 = 1.625$	$F_{15}/F_{14} = 610/377 \approx 1.618037$
$F_8/F_7 = 21/13 \approx 1.6154$	$F_{16}/F_{15} = 987/610 \approx 1.618033$
$F_9/F_8 = 34/21 \approx 1.61905$	$F_{17}/F_{16} = 1597/987 \approx 1.618034$
$F_{10}/F_9 = 55/34 \approx 1.61765$	$F_{18}/F_{17} = 2584/1597 \approx 1.618034$

169. The 14th term in the sequence is 377. Find the next term.
Solution:

We know that 15^{th} term = 14^{th} term × the golden ratio.

F_{15} = 377 × 1.618034 ≈ 609.99 = 610

Therefore, the 15^{th} term in the Fibonacci sequence is 610.

The Most Irrational

Golden Ratio is the **most** irrational number. Here is why ...

We saw before that the Golden Ratio can be defined in terms of itself, like this:

$$\phi = 1 + 1/\phi$$

(In numbers: 1.61803... = 1 + 1/1.61803...)

That can be expanded into this fraction that goes on forever
(called a *"continued fraction"*):

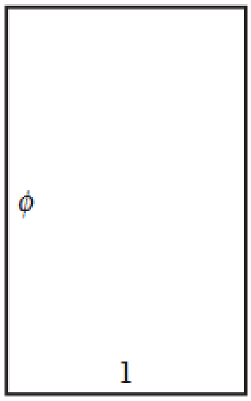
$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

So, it neatly slips in between simple fractions.

Note: many other irrational numbers are close to rational numbers, such as Pi = 3.14159265... is pretty close to 22/7 = 3.1428571...)

What is Golden Rectangle?

In geometry, a golden rectangle is defined as a rectangle whose side lengths are in the golden ratio. The golden rectangle exhibits a very special form of self-similarity. All rectangles that are created by adding or removing a square are golden rectangles as well. its short side. A golden rectangle can be of any size, but its sides must have a ratio of $\phi \approx \frac{8}{5}$. Figure shows a golden rectangle.



The golden rectangle was considered by the Greeks to be of the most pleasing proportions, and its shape figures in ancient architecture. The same motif is used in modern architecture such as the buildings of Le Corbusier (whose only work in North America is the Carpenter Center at Harvard).

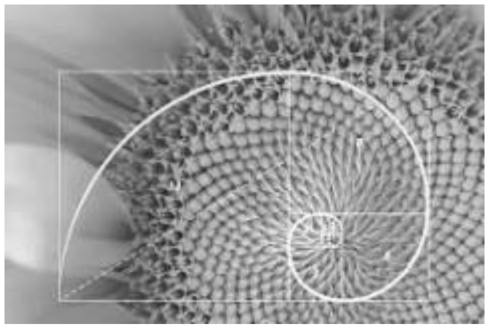
Drawing It

Here is one way to draw a rectangle with the Golden Ratio:

- Draw a square of size "1"
- Place a dot half way along one side
- Draw a line from that point to an opposite corner
- Now turn that line so that it runs along the square's side
- Then you can extend the square to be a rectangle with the Golden Ratio!

Use of golden ratio in nature

One of the most remarkable occurrences of the Golden Ratio in nature is seen in the formation of spirals. Examples include the patterns found in sunflowers, pinecones, and seashells. These spirals exhibit a consistent growth rate, adhering closely to the Golden Ratio.



Exercise

1) In Exercises i–vi, graph each equation.

i. $y=2x-1$ ii. $y=-3x+2$ iii. $y=\frac{2}{3}x+1$

iv. $y=-\frac{x}{2}-3$ v. $y=\frac{1}{2}x^2$ vi. $y=|x-41|$

2) In Exercises i–iv, evaluate the function for the given value.

i. $f(x)=2x+7$; $x=-2$ ii. $f(x)=1-3x$; $x=-4$

iii. $f(t)=t^2-t-3$; $t=3$ iv. $T(p)=\frac{p^2}{p-2}$; $p=0$

3) In Exercises i–xiv, find the x- and y-intercepts of the graph of the equation.

i. $f(x)=3x-6$ ii. $f(x)=2x+8$ iii. $y=\frac{2}{3}x-4$

iv. $y=-\frac{3}{4}x+6$ v. $y=-x-4$ vi. $y=-\frac{x}{2}+1$

vii. $3x+4y=12$ viii. $5x-2y=10$ ix. $2x-3y=9$

x. $4x+3y=8$ xi. $\frac{x}{2}+\frac{y}{3}=1$ xii. $\frac{x}{3}-\frac{y}{2}=1$

xiii. $x-\frac{y}{2}=1$ xiv. $-\frac{x}{4}-\frac{y}{3}=1$

4) In Exercises i–vi, find the slope of the line containing the two points.

i. $(1,3),(3,1)$ ii. $(2,3),(5,1)$

iii. $(-1,4),(2,5)$ iv. $(3,-2),(1,4)$

v. $(-1,3),(-4,5)$ vi. $(-13,-2),(-3,2)$

5) In Exercises i–viii, find the equation of the line that passes through the given point and has the given slope.

i. $(0,5), m=2$ ii. $(2,3), m=\frac{1}{2}$ iii. $(-1,7), m=-3$

iv. $(0,0), m=\frac{1}{2}$ v. $(3,5), m=-\frac{2}{3}$ vi. $(0,-3), m=-1$

vii. $(-2,-3), m=0$ viii. $(4,-5), m=-2$

6) In Exercises i–viii, find the equation of the line that passes through the given points.

i. $(0,2),(3,5)$ ii. $(0,-3),(-4,5)$ iii. $(0,3),(2,0)$

iv. $(-2,-3),(-1,-2)$ v. $(2,0),(0,-1)$ vi. $(3,-4),(-2,-4)$

vii. $(-2,5),(2,-5)$ viii. $(2,1),(-2,-3)$

7) In Exercises i-vi, a. write (if necessary) the polynomial function in standard form, b. give the degree of the polynomial function, and c. evaluate the function for the given values of the variable.

i. $f(x)=2x^2+4x-10, f(2)$ ii. $f(x)=1+2x^2, f(-2)$

iii. $g(x)=x^2+2x^3-3x-1, g(-2)$ iv. $s(t)=1-t^2-t^4, s(3)$

v. $y(z)=2z^3-3z^2+4z-z^5+6, y(-2)$ vi. $p(x)=-2x^3, p(-3)$

8) The real valued functions f and g are defined below.

Find (a) fog(x) (b) gof(x) (c) fof(x) (d) gog(x)

(i) $f(x) = \sqrt{x+1}, g(x)=\frac{1}{x^2}$

(ii) $f(x)= \frac{1}{\sqrt{x-1}}, x \neq 1, g(x)=(x^2+1)^2$

(iii) $f(x) = 3x^4-2x^2$, $g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

9) Find domain and range and sketch the graph.

(i) $g(x) = 2x - 5$

(ii) $g(x) = \sqrt{x^2 - 4}$

(iii) $g(x) = \sqrt{x+1}$

(iv) $g(x) = |x - 3|$

(v) $g(x) = \begin{cases} 6x+7, & x \leq -2 \\ 4x-3, & -2 < x \end{cases}$

(vi) $g(x) = \begin{cases} x-1, & x < 3 \\ 2x+1, & 3 \leq x \end{cases}$

(vii) $g(x) = \frac{x^2+3x+2}{x+1}, x \neq -1$

(viii) $g(x) = \frac{x^2-16}{x-4}, x \neq 4$

10) In Exercises i-vi, find the minimum or maximum value of each quadratic function. State whether the value is a minimum or a maximum.

i. $f(x)=x^2-2x+3$ ii. $f(x)=-2x^2+4x-5$

iii. $f(x)=x^2+3x-1$ iv. $f(x)=2x^2+4x$

v. $f(x)=3x^2+3x-2$ vi. $f(x)=x^2-5x+3$

11) Find the extreme values for the following functions defined as;

i. $f(x)=5x^2-6x+2$ ii. $f(x)=3x^2$

iii. $f(x)=3x^2+4x+5$ iv. $f(x)=2x^3-2x^2-36x+3$

v. $f(x)=x^4-4x^2$ vi. $f(x)=(x-2)^2(x-1)$

12) Show that $y=\frac{\ln x}{x}$ has maximum value at x = e.

13) Show that $y=x^x$ has maximum value at $x=\frac{1}{e}$

14) Given $f(x)=3^x$, evaluate

a. $f(2)$ b. $f(0)$ c. $f(-2)$

15) Given $H(x)=2^x$, evaluate

a. $H(-3)$ b. $H(0)$ c. $H(2)$

16) Given $G(r)=\left(\frac{1}{2}\right)^{2r}$, evaluate

a. $G(0)$ b. $G\left(\frac{3}{2}\right)$ c. $G(-2)$

17) In Exercises i –x, graph the equation.

i. $f(x)=2^x+1$ ii. $f(x)=3^x-2$ iii. $g(x)=3^{x/2}$

iv. $h(x)=3^{-x/2}$ v. $f(x)=2^{x+3}$ vi. $g(x)=4^{-x}+1$

vii. $H(x)=2^{2x}$ viii. $F(x)=2^{-x}$ ix. $f(x)=e^{-x}$

x. $y(x)=e^{2x}$

18) In Exercises i–viii, write the exponential equation in logarithmic form.

i. $7^2=49$ ii. $10^3=1000$ iii. $5^4=625$

iv. $2^{-3}=\frac{1}{8}$ v. $10^{-4}=0.0001$ vi. $3^5=243$

vii. $10^y=x$ viii. $e^y=x$

19) In Exercises i–viii, write the logarithmic equation in exponential form.

i. $\log_3 81=4$ ii. $\log_2 16=4$ iii. $\log_5 125=3$

iv. $\log_4 64=3$ v. $\log_4 \frac{1}{16}=-2$ vi. $\log_2 \frac{1}{16}=-4$

vii. $\ln x=y$ viii. $\log x=y$

20) In Exercises i–viii, evaluate the logarithm.

i. $\log_3 81$ ii. $\log_2 49$ iii. $\log_5 100$ iv. $\log 0.001$

v. $\log_3 \frac{1}{9}$ vi. $\log_7 \frac{1}{7}$ vii. $\log_2 64$ viii. $\log 0.01$

21) In Exercises i–viii, solve the equation for x.

i. $\log_3 x = 2$ ii. $\log_5 x = 1$ iii. $\log_7 x = -1$

iv. $\log_8 x = -2$ v. $\log_3 x = -2$ vi. $\log_5 x = 3$

vii. $\log_4 x = 0$ viii. $\log 4x = -1$

22) In Exercises i–vi, graph the function.

i. $g(x) = \log_2 x$ ii. $g(x) = \log_4 x$

iii. $f(x) = \log_3(2x-1)$ iv. $f(x) = -\log_2 x$

v. $f(x) = \log_2(x-1)$ vi. $f(x) = \log_3(x-2)$

23) In July 2004, the federal minimum wage was $5.15 per hour, and the minimum wage in California was $6.75. How much greater is an employee's pay for working 35 hours and earning the California minimum wage rather than the federal minimum wage?

24) Which is the more economical purchase, 32 ounces of detergent for $2.99 or 48 ounces of detergent for $3.99?

25) Solve the following for the ratio;

a. According to the National Low Income Housing Coalition, a minimum-wage worker ($5.15 per hour) living in New Jersey would have to work 120 hours per week to afford the rent on an average two-bedroom apartment and be within the federal standard of 30% of income for housing. Find the ratio, as a fraction in simplest form, of the number of hours a minimum-wage worker would spend working per week to the total number of hours in a week.

b. Although a minimum-wage worker in New Jersey would have to work 120 hours per week to afford a two-bedroom rental, the national average is 60 hours of work per week. For this "average" worker, find the ratio, written using the word to, of the number of hours per week spent working to the number of hours spent not working.

26) The length of the road is 25 kilometers. $\frac{3}{5}$ of it has been completed. Find the length incomplete part.

27) Allowance Rs. 15000 in the ratio 2:4:6

3. A profit of Rs 1025 is distributed among three partners in the ratio of $\frac{4}{3}:\frac{7}{6}:\frac{11}{12}$.

Find the profit of each partner.

28) Basit , Tariq and Anwar start business contributing Rs. 9000 for a year, Rs. 6000 for 9 months and Rs. 3000 for 7 months respectively. Find the share of each from the total profit of Rs. 3375.

29) Distribute stock of 600 electric fans to the three dealers in the ratio of 3:5:4.

30) The length of Pakistan flag is 1.6 m. The ratio between green and white the length of green part.

31) Zubair and Bilal entered in a contract of construction of a road. They spent Rs. 150000 and 18000 respectively. Find the share of each in the profit of Rs. 55000.

32) The profits of a firm are shared by its two partners in the ratio 7 : 5. If the partner receiving the larger amount of this year's profits receives $28,000, what amount does the other partner receive?

33) Find a : b: c when

i. a : b = 5 : 2 , b : c = 3 : 5 ii. a : b = 3 : 4 , c : b = 5 : 6

iii. a : b = 3 : 2 , c : b = 1 : 4

34) Find $x:y:z$ when

i. $x:y=\frac{1}{3}:\frac{1}{2}$: $y:z=\frac{1}{2}:\frac{1}{5}$ ii. $z:x=\frac{1}{4}:\frac{1}{3}$: $y:x=\frac{1}{6}:\frac{1}{7}$

35) The ratio of A and B's share is 5 to 11 and the ratio of B and C's share is 9 to 10 Find the continued ratio

36) Distribute Rs. 420 among Akram. Aslam and Anwar in such a way that Aslam : Anwar = 4 : 3 and Aslam : Akram = 2 : 3

37) The ratio between the shares of A and is 1 : 2 and the ratio between the shares of B and C is 3 : 4 if the profit of Rs. 20,400 is to be divided, how much will each get?

38) Find the missing number x in the followings

i. $x:3::2:5$ ii. $6:x::3:9$ iii. $10:20::x:4$ iv. $12:2::1500:x$

39) Find the fourth proportional 4 , 6, 20

40) Find the mean proportional to 24 and 8

41) Find the third proportional to $\sqrt{2},\sqrt{8}$

42) If price of one dozen of eggs is Rs. 280. How many eggs can be purchased in Rs. 420.

43) If 20 pens cost Rs. 300. Find the number of pens that can be bought for Rs. 480.

44) A tube-well produce 6400 gallons of water in 160 minutes. How long will it take to produce gallons of water?

45) At the time shadow of 36 dm. electric pole is measured to be 45 dm. The shadow of MInar of Badshahi Mosque is 610 dm. what is the height of MInar?

46) Ahmad earns Rs. 6300 in 3 weeks. Find his 5 days income when he works 6 day a week?

47) Akram travels 144 km distance in 2 hours. What distance will it travel in 50 minutes with same speed?

48) The cost of construction of 1.5 miles of a road is Rs. 4,20,000. What will be the cost of 4.5 miles of such road?

49) 6 men can plough a field in 14 hours. How long will it take 4 men do the same work?

50) An air plane travels a certain distance in 170 minutes at a speed of 450 km/h. how much time will it take to cover the same distance if speed is decreased to 340 km/h.

51) 7 days food is available for 72 soldiers in a camp, 12 more soldieries came. For how many days will the food be enough?

52) 35 men dig a trench in 16 days . In how many days 28 men will dig this trench?

53) 10 men have a ration of 21 days in a house. If 3 men leave the house. For how many days will ration be sufficient?

54) 20 men can finish a job in 13 days. How many more men are required to do the same job in 4 days.

55) 45 men take 6 hours to dig a garden. How many men will be required to dig the garden in $7\frac{1}{2}$ hours?

56) If a work is finished in 10 days by 30 workers working 8 hours daily. In how the same work will be fished by 20 workers working 10 hours a day?

57) If 6 pumps raise 108 liters of water in 12 minutes, how long will 4 pumps take of water.

58) 15 machines prepare 360 sweaters in 6 days, 3 machines get out of order. How many sweaters can be prepared in 10 days by the remixing machines?

59) 30 men repair a road in 56 days by working 6 hours daily. In how many days 45 men will repair the same road by working 7 hours daily.

60) If cost 8 persons Rs. 9600 to stay at a hotel for 5 days. What it cost 9 persons to stay at same hotel for 7 days.

61) 66 men working 9 hours a day can complete a job in 36 days. How many hours a day 81 men work to complete the job in 33 days.

62) A group of 42 workers can construct a house in 60 days working 8 hours a day. How many days are required to construct the same house by 60 workers, if they work 7 hours daily.

63) In a factory, 60 men work 8 hours daily to prepare 600 machines 12 days. If 800 machine are required in 16 days. How many men are needed, if they work 6 hours daily.

64) Explain the golden ratio in terms of proportions of line segments.

65) How is a golden rectangle formed?

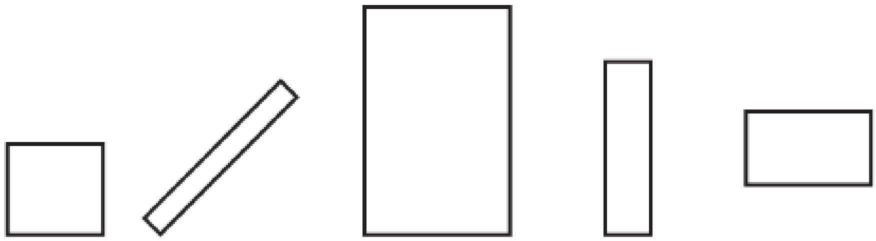
66) What evidence suggests that the golden ratio and golden rectangle hold particular beauty?

67) What is a logarithmic spiral? How is it formed from a golden rectangle?

68) What is the Fibonacci sequence?

69) What is the connection between the Fibonacci sequence and the golden ratio? Give some examples of the Fibonacci sequence in nature.

70) Measure the sides of each rectangle in Figure, and compute the ratio of the long side to the short side for each rectangle. Which ones are golden rectangles?



71) Dimensions of Golden Rectangles. Consider the following lengths of one side of a golden rectangle. Find the length of the other side. Notice that the other side could be either longer or shorter than the given side. Use the approximation $\phi \approx 1.62$ for your work.

i. 2.7 inches ii. 5.8 meters iii. 12.6 kilometers iv. 0.66 centimeter

72) Find at least three everyday objects with rectangular shapes (for example, cereal boxes, windows). In each case, measure the side lengths and calculate the ratio. Are any of these objects golden rectangles? Explain.

73) Draw a line segment 6 inches long. Now sub-divide it according to the golden ratio. Verify your work by computing the ratio of the whole segment length to the long segment length and the ratio of the long segment length to the short segment length.

74) A line is subdivided according to the golden ratio, with the smaller piece having a length of 5 meters. What is the length of the entire line?