NUMERICAL ANALYSIS II

MUZAMMIL TANVEER

mtanveer8689@gmail.com

0316-7017457

Dedicated to

My honorable Teacher

&

My Parents

Lecture # 1

Operator:

	E	Shift Operator
Capital delta	Δ	Forward Difference Operator
Nabla	∇	Backward Difference Operator
Small delta	δ	Central Difference Operator
Mu	μ	Average Operator

Relationship of Operators:

$$\Delta = E - 1$$

$$\nabla = 1 - E^{-1}$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}$$

Role of Shift Operator:

$Ef(x) = f(x+h)$

$E^2f(x) = E(E(f(x)) = Ef(x+h)$

$= f(x+h+h) = f(x+2h)$

$E^3f(x) = f(x+3h)$

. .

. .

. .

$E^nf(x) = f(x+nh)$

And $E^{-n}f(x) = f(x-nh)$

Now $\Delta f(x) = f(x+h) - f(x)$

$= Ef(x) - f(x)$

$\Delta f(x) = (E-1)f(x)$

$\Delta = (E-1)$

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1}f(x)$

$= (1-E^{-1})f(x)$

$\nabla = 1-E^{-1}$

$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$

$= E^{\frac{1}{2}}f(x) - E^{-\frac{1}{2}}f(x)$

$\delta f(x) = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})f(x)$

$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

$\mu f(x) = \frac{f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)}{2}$

$= \frac{E^{\frac{1}{2}}f(x)+E^{-\frac{1}{2}}f(x)}{2}$

$\mu f(x) = \left(\frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}\right)f(x)$

$\mu = \frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}$

$y_k = f(x_k)$

$Ey_k = y_{k+1}$

$\Delta y_k = y_{k+1} - y_k$

$\nabla y_k = y_k - y_{k-1}$

$\delta y_k = y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}}$

$$\mu y_k = \frac{y_{k+\frac{1}{2}} + y_{k-\frac{1}{2}}}{2}$$

Difference Equation:

An equation consists of an independent variable k, dependent variable y_k and one or several difference of the dependent variable y_k as $\Delta y_k, \Delta^2 y_k, \Delta^3 y_k, \ldots \Delta^n y_k$ is called the difference equation.

f(Δ)y_k = F(k)

Example:

1. $\Delta^3 y_k + 3\Delta^2 y_k - \Delta y_k + y_k = 3k+2$
2. $\Delta^2 y_k + 3\Delta y_k - 7y_k = 0$

Lecture # 02

Definition:

The order of difference equation written in form free from Δ's is difference between the highest and lowest subscript of y's. For example

(i) $y_{k+2} - 3y_{k+1} + 2y_k = 2^k + 3$ is a difference equation of order 2

(ii) $y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = 0$ is a homogeneous linear difference equation of order 3.

Definition:

The degree of a difference equation written in a form free Δ's is the highest power of y's. For example

$y_{k+1}^2 y_{k+2}^3 - y_{k+1} y_k - y_k^2 = k$ is a non-homogeneous and non-linear difference equation of order 2 and degree 3.

Difference equation are classified into following two types

(i) Homogeneous Linear difference equation

(ii) Non-Homogeneous Linear difference equation

Definition:

If in the difference equation the function of independent variable is zero (i.e. F(k) = 0 in equation f(Δ)y_k = F(k) then the equation is called Homogeneous Linear difference equation.

Definition:

If in the difference equation the function of independent variable is not zero (i.e. F(k) ≠ 0 in equation f(Δ)y_k = F(k) then the equation is called Non-Homogeneous Linear difference equation.

Solution of Homogeneous Linear Difference Equation:

Question: Solve the difference equation

$$y_{k+3} + 6y_{k+2} + 11y_{k+1} + 6y_k = 0$$

Solution: The given difference equation can be written as

$$E^3y_k + 6E^2y_k + 11Ey_k + 6y_k = 0$$

$$(E^3 + 6E^2 + 11E + 6)y_k = 0$$

The Auxiliary equation is

$$E^3 + 6E^2 + 11E + 6 = 0$$

By Synthetic division

	1	6	11	6
−1	↓	−1	−5	−6
	1	5	6	0

$\Rightarrow$ m = −1 and $m^2 + 5m + 6 = 0$

$$m = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm \sqrt{1}}{2}$$

$$= \frac{-5 \pm 1}{2} \Rightarrow \frac{-5+1}{2}, \quad \frac{-5-1}{2}$$

$\Rightarrow$ m = −1, −2, −3

Thus $y_k = c_1(-1)^k + c_2(-2)^k + c_3(-3)^k$ is the required solution.

Example: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + 4y_k = 0$$

Solution: The given difference equation can be written as

$$E^2y_k - 4Ey_k + 4y_k = 0$$

$$(E^2 - 4E + 4)y_k = 0$$

The Auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

m = 2 and m = 2

$\Rightarrow$ m = 2 , 2

Thus $y_k = (c_1 + c_2k)(2)^k$ is the required solution.

Example: Solve the difference equation

$$y_{k+2} + y_k = 0$$

Solution: The given difference equation can be written as

$$E^2 y_k + y_k = 0$$
$$(E^2 + 1)y_k = 0$$

The Auxiliary equation is

$$m^2 + 1 = 0$$

$$m^2 = -1 = i^2$$

$$m = \pm i$$

$$m = 0 \pm i$$

$$r = |m| = \sqrt{(0)^2 + (1)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$$

Thus $y_k = r^k[c_1 cos\theta k + c_2 sin\theta k]$

$$= (1)^k\left[c_1 cos\frac{\pi}{2}k + c_2 sin\frac{\pi}{2}k\right]$$

$= c_1 cos\frac{k\pi}{2} + c_2 sin\frac{k\pi}{2}$ is the required solution.

Example: Solve the difference equation

$$y_{k+4} + 4y_k = 0$$

Solution: The given difference equation can be written as

$$E^4 y_k + 4y_k = 0$$
$$(E^4 + 4)y_k = 0$$

The Auxiliary equation is

$$m^4 + 4 = 0$$

$$m^4 + 4\,m^2 + 4 - 4m^2 = 0$$

$$(m^2 + 2)^2 - (2m)^2 = 0$$

$$(m^2+2+2m)(m^2+2-2m)=0$$

$(m^2+2+2m)=0$, $(m^2+2-2m)=0$

$m=\frac{-2\pm\sqrt{4-8}}{2}$, $m=\frac{2\pm\sqrt{4-8}}{2}$

$=\frac{-2\pm\sqrt{-4}}{2}$, $=\frac{2\pm\sqrt{-4}}{2}$

$=\frac{-2\pm 2i}{2}$, $=\frac{2\pm 2i}{2}$

$m=-1\pm i$, $m=1\pm i$

$r_1=|m|=\sqrt{(1)^2+(1)^2}=\sqrt{2}$, $r_2=|m|=\sqrt{(1)^2+(1)^2}=\sqrt{2}$

$\theta_1=\tan^{-1}\left(\frac{1}{-1}\right)=-\frac{\pi}{4}$, $\theta_2=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4}$

Thus $y_k=(r_1)^k[c_1 cos\theta_1 k+c_2 sin\theta_1 k]+(r_2)^k[c_3 cos\theta_2 k+c_4 sin\theta_2 k]$

$=\left(\sqrt{2}\right)^k\left[c_1 cos\left(-\frac{\pi}{4}\right)k+c_2 sin\left(-\frac{\pi}{4}\right)k\right]+\left(\sqrt{2}\right)^k\left[c_3 cos\left(\frac{\pi}{4}\right)k+c_4 sin\left(\frac{\pi}{4}\right)k\right]$

$=\left(\sqrt{2}\right)^k\left[c_1 cos\frac{k\pi}{4}-c_2 sin\frac{k\pi}{4}\right]+\left(\sqrt{2}\right)^k\left[c_3 cos\frac{k\pi}{4}+c_4 sin\frac{k\pi}{4}\right]$

$=\left(\sqrt{2}\right)^k\left[c_1 cos\frac{k\pi}{4}-c_2 sin\frac{k\pi}{4}+c_3 cos\frac{k\pi}{4}+c_4 sin\frac{k\pi}{4}\right]$ is the required solution.

Question: Solve the difference equation

$$2y_{k+2}-5y_{k+1}+2y_k=0$$

Solution: The given difference equation can be written as

$2E^2y_k-5Ey_k+2y_k=0$

$(2E^2-5E+2)y_k=0$

The Auxiliary equation is

$2m^2-5m+2=0$ $\Rightarrow$ $m=\frac{5\pm\sqrt{25-16}}{4}$ $=\frac{5\pm\sqrt{9}}{4}$

$=\frac{5\pm 3}{4}$ $=\frac{5+3}{4}$, $\frac{5-3}{4}$ $\Rightarrow$ $m=2,\frac{1}{2}$

Thus $y_k=c_1(2)^k+c_2\left(\frac{1}{2}\right)^k$ is the required solution.

Question: Solve the difference equation

$$y_{k+4} - 6y_{k+3} + 14y_{k+2} - 14y_{k+1} + 5y_k = 0$$

Solution: The given difference equation can be written as

$$E^4 y_k - 6E^3 y_k + 14E^2 y_k - 14E y_k + 5y_k = 0$$

$$(E^4 - 6E^3 + 14E^2 - 14E + 5)y_k = 0$$

The Auxiliary equation is

$$m^4 - 6m^3 + 14m^2 - 14m + 5 = 0$$

By Synthetic Division

1	1	–6	14	–14	5
	↓	1	–5	9	–5
1	1	–5	9	–5	0
	↓	1	–4	5	
	1	–4	5	0	

$\Rightarrow$ m = 1 , 1 and $m^2 - 4m + 5 = 0$

$$m = \frac{4 \pm \sqrt{16-20}}{2} \Rightarrow \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm 2i}{2} = 2 \pm i$$

$$r = |m| = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

Thus $y_k = (c_1 + c_2 k)(1)^k + (\sqrt{5})^k \left[c_3 \cos k \tan^{-1}\left(\frac{1}{2}\right) + c_4 \sin k \tan^{-1}\left(\frac{1}{2}\right)\right]$

$= (c_1 + c_2 k) + (\sqrt{5})^k \left[c_3 \cos k \tan^{-1}\left(\frac{1}{2}\right) + c_4 \sin k \tan^{-1}\left(\frac{1}{2}\right)\right]$ is the required solution.

Question: Solve the difference equation

$$y_{k+1} - 2\cos\beta\, y_k + y_{k-1} = 0$$

Solution: The given difference equation can be written as

$E^2 y_{k-1} - 2cos\beta E y_{k-1} + y_{k-1} = 0$

$(E^2 - 2cos\beta E + 1)\, y_{k-1} = 0$

The Auxiliary equation is

$m^2 - 2cos\beta m + 1 = 0$

$m = \frac{2cos\beta \pm \sqrt{4cos^2\beta - 4}}{2} \Rightarrow = \frac{2cos\beta \pm \sqrt{-4(1-cos^2\beta)}}{2}$

$m = \frac{2cos\beta \pm \sqrt{-4sin^2\beta}}{2} \Rightarrow = \frac{2cos\beta \pm i2sin\beta}{2}$

$m = \cos\beta \pm i \sin\beta$

$r = |m| = \sqrt{(cos\beta)^2 + (sin\beta)^2} = 1$

$\theta = \tan^{-1}\left(\frac{sin\beta}{cos\beta}\right) = \tan^{-1}(tan\beta) = \beta$

Thus $y_k = r^k[c_1 cos\theta k + c_2 sin\theta k]$

$= (1)^k[c_1 cos\beta k + c_2 sin\beta k]$

$= c_1 cos\beta k + c_2 sin\beta k$ is the required solution.

Lecture # 03

Non-Homogenous Linear Difference Equation:

The difference equation of the form

$$f(\Delta)y = F(k) \qquad .....(1)$$

Where $f\left(\Delta\right)=a_0\Delta^n + a_1\Delta^{n-1} + a_2\Delta^{n-2} + a_3\Delta^{n-3} + ... + a_{n-1}\Delta + a_n$

and Δ is a forward difference operator.

The solution of equation (1) consist of two parts

(i) Complementary Solution (Y)
(ii) Particular Solution (Y^*)

The general solution of equation (1) is

$$Y_k = Y + Y^*$$

To find complementary solution of equation (1) we shall find the solution of f(Δ) y = 0

To find particular solution of equation (1) we shall discuss the following type

When the R.H.S of given difference equation is constant (i.e. F(k) = constant). Then in order to find Y^* (particular solution) we shall substitute a trial function y_k = c in the given non-homogeneous difference equation and evaluate the value of c. If the trial function or any term of trial function is present in Y (complementary solution) then the trail function will be multiplied by a suitable k^n.

Question: Solve the difference equation

$$y_{k+3} + 3y_{k+2} - 2y_k = 5$$

Solution: The given difference equation can be written as

$$E^3 y_k + 3E^2 y_k - 2y_k = 5$$

$$\left(E^3 + 3E^2 - 2\right)y_k = 5 \qquad ...(1)$$

For Complementary solution

$$\left(E^3+3E^2-2\right)y_k=0$$

The auxiliary equation

$$m^3+3m^2-2=0$$

	1	3	0	−2
−1	↓	−1	−2	2
	1	2	−2	0

$\Rightarrow$ m = −1 and $m^2+2m-2=0$

$$m=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{12}}{2}$$

$$=\frac{-2\pm2\sqrt{3}}{2}$$

$$m=-1\pm\sqrt{3}$$

$$\Rightarrow m=-1,-1-\sqrt{3},-1+\sqrt{3}$$

$$Thus\ \ Y=c_1(-1)^k+c_2(-1-\sqrt{3})^k+c_3(-1+\sqrt{3})^k$$

For Particular solution

Let y_k = c put in (1)

$$\left(E^3+3E^2-2\right)c=5 \quad ...(1)$$

$$E^3c+3E^2c-2c=5$$

$$c+3c-2c=5$$

$$2c=5 \ \Rightarrow\ c=\frac{5}{2}$$

$$y_k=\frac{5}{2} \ \Rightarrow\ Y^*=\frac{5}{2}$$

$$\Rightarrow Y_k=Y+Y^*$$

$Hence\ Y_k=c_1(-1)^k+c_2(-1-\sqrt{3})^k+c_3(-1+\sqrt{3})^k+\frac{5}{2}$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 2y_{k+1} + 6y_k = 7$$

Solution: The given difference equation can be written as

$$E^2 y_k - 2Ey_k + 6y_k = 7$$

$$\left(E^2 - 2E + 6\right) y_k = 7 \quad ...(1)$$

For Complementary solution

$$\left(E^2 - 2E + 6\right) y_k = 0$$

The auxiliary equation

$$m^2 - 2m + 6 = 0$$

$$m = \frac{2 \pm \sqrt{4-24}}{2} = \frac{2 \pm \sqrt{-20}}{2} \Rightarrow = \frac{2 \pm 2\sqrt{5}i}{2}$$

$$m = 1 \pm \sqrt{5}i \quad \& \quad r = |m| = \sqrt{1+5} = \sqrt{6}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{5}}{1}\right) = \tan^{-1}(\sqrt{5})$$

$$Thus\ Y = \left(\sqrt{6}\right)^k \left[c_1 \cos k\theta + c_2 \sin k\theta\right]$$

For Particular solution

Let y_k = c put in (1)

$$\left(E^2 - 2E + 6\right) c = 7$$

$$E^2 c - 2Ec + 6c = 7$$

$$c - 2c + 6c = 7$$

$$5c = 7 \Rightarrow c = \frac{7}{5} \Rightarrow y_k = \frac{7}{5} \quad \Rightarrow Y^* = \frac{7}{5}$$

$$\Rightarrow Y_k = Y + Y^*$$

$$Hence\ Y_k = \left(\sqrt{6}\right)^k \left[c_1 \cos k\theta + c_2 \sin k\theta\right] + \frac{7}{5} \quad \textit{is the requied general solution.}$$

Question: Solve the difference equation

$$y_{k+2}-3y_{k+1}-4y_k=2$$

Solution: The given difference equation can be written as

$$E^2y_k-3Ey_k-4y_k=2$$

$$\left(E^2-3E-4\right)y_k=2 \quad ...(1)$$

For Complementary solution

$$\left(E^2-3E-4\right)y_k=0$$

The auxiliary equation

$$m^2-3m-4=0$$

$$m=\frac{3\pm\sqrt{9+16}}{2}=\frac{3\pm\sqrt{25}}{2} \Rightarrow =\frac{3\pm 5}{2}$$

$$m=\frac{3+5}{2},\frac{3-5}{2}$$

$$m=4,-1$$

$$Thus \quad Y=c_1(-1)^k+c_2(4)^k$$

For Particular solution

Let y_k = c put in (1)

$$\left(E^2-3E-4\right)c=2$$

$$E^2c-3Ec-4c=2$$

$$c-3c-4c=2$$

$$-6c=2 \Rightarrow c=\frac{-1}{3}$$

$$\Rightarrow y_k=\frac{-1}{3} \Rightarrow Y^*=\frac{-1}{3}$$

$$\Rightarrow Y_k=Y+Y^*$$

$$Hence\ Y_k=c_1(-1)^k+c_2(4)^k-\frac{1}{3} \quad is\ the\ requied\ general\ solution.$$

Question: Solve the difference equation

$$y_{k+3}-3y_{k+2}+3y_{k+1}-y_k=9$$

Solution: The given difference equation can be written as

$$E^3y_k-3E^2y_k+3Ey_k-y_k=9$$

$$\left(E^3-3E^2+3E-1\right)y_k=9 \qquad ...(1)$$

For Complementary solution $\left(E^3-3E^2+3E-1\right)y_k=0$

If we take y_k = c so it is already in complementary solution.

If we take y_k = ck it is also present in complementary solution.

If we take $y_k = ck^2$ it is also present in complementary solution.

So, we take $y_k = ck^3$ which is not in complementary solution.

The auxiliary equation

$$m^3-3m^2+3m-1=0$$

$$\begin{array}{c|cccc} & 1 & -3 & 3 & -1 \\ 1 & \downarrow & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$m=1$, $m^2-2m+1=0$

$(m-1)^2=0 \quad m=1,1 \Rightarrow m=1,1,1$

Thus $Y=\left(c_1+c_2k+c_3k^2\right)(1)^k=c_1+c_2k+c_3k^2$

For Particular solution put $y_k = ck^3$ put in (1)

$$\left(E^3-3E^2+3E-1\right)ck^3=9$$

$$E^3ck^3-3E^2ck^3+3Eck^3-ck^3=9$$

$$c\left(k+3\right)^3-3c\left(k+2\right)^3+3c\left(k+1\right)^3-ck^3=9$$

$$c\left[k^3+9k^2+27k+27\right]-3c\left[k^3+6k^2+12k+8\right]+3c\left[k^3+3k^2+3k+1\right]-ck^3=9$$

$$k^3c+9ck^2+27ck+27c-3ck^3-18ck^2-36ck-24c+3ck^3+9ck^2+9ck+3c-ck^3=9$$

$$6c=9 \Rightarrow c=\frac{3}{2} \quad , \quad y_k=\frac{3}{2}k^3 \Rightarrow Y^*=\frac{3}{2}k^3$$

$$\Rightarrow Y_k=Y+Y^*$$

Hence $Y_k=c_1+c_2k+c_3k^2+\frac{3}{2}k^3$ *is the requied general solution.*

Question: Solve the difference equation

$$y_{k+4} - 6y_{k+3} + 14y_{k+2} - 14y_{k+1} - 5y_k = 1$$

Solution: The given difference equation can be written as

$$E^4 y_k - 6E^3 y_k + 14E^2 y_k - 14E y_k + 5y_k = 1$$

$$\left(E^4 - 6E^3 + 14E^2 - 14E + 5\right) y_k = 1 \quad \ldots(1)$$

For Complementary solution

$$\left(E^4 - 6E^3 + 14E^2 - 14E + 5\right) = 0$$

The auxiliary equation

$$m^4 - 6m^3 + 14m^2 - 14m + 5 = 0$$

	1	−6	14	−14	5
1	↓	1	−5	9	−5
	1	−5	9	−5	0
1	↓	1	−4	5	
	1	−4	5	0	

$$m = 1, 1 \;,\; m^2 - 4m + 5 = 0$$

$$m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$$

$$m = 2 \pm i$$

$$r = |m| = \sqrt{4+1} = \sqrt{5} \;\Rightarrow\; \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

$$Thus\;\; Y = \left(c_1 + c_2 k\right)(1)^k + \left(\sqrt{5}\right)^k \left[c_3 \cos\theta k + c_4 \sin\theta k\right]$$

$$Thus\;\; Y = c_1 + c_2 k + \left(\sqrt{5}\right)^k \left[c_3 \cos\theta k + c_4 \sin\theta k\right]$$

For Particular solution Let $y_k = ck^2$ put in (1)

$$\left(E^4 - 6E^3 + 14E^2 - 14E + 5\right)ck^2 = 1$$

$$E^4ck^2 - 6E^3ck^2 + 14E^2ck^2 - 14Eck^2 + 5ck^2 = 1$$

$$c\left(k+4\right)^2 - 6c\left(k+3\right)^2 + 14c\left(k+2\right)^2 - 14c\left(k+1\right)^2 + 5ck^2 = 1$$

$$c\left(k^2 + 8k + 18\right) - 6c\left(k^2 + 6k + 9\right) + 14c\left(k^2 + 4k + 4\right) - 14c\left(k^2 + 2k + 1\right) + 5ck^2 = 1$$

$$k^2c + 8ck + 16c - 6ck^2 - 36ck - 54c + 14ck^2 + 56ck + 56c - 14ck^2 - 28ck - 14c + 5ck^2 = 1$$

$$4c = 1 \quad \Rightarrow \quad c = \frac{1}{4}$$

$$y_k = \frac{1}{4}k^2$$

$$Y^* = \frac{1}{4}k^2$$

$$\Rightarrow Y_k = Y + Y^*$$

$$Hence\ Y_k = c_1 + c_2k + \left(\sqrt{5}\right)^k \left[c_3 \cos\theta k + c_4 \sin\theta k\right] + \frac{1}{4}k^2$$

$$where \ \ \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

is the required general solution.

Lecture # 04

Non-Homogenous Difference Equation:

Type II:

When the R.H.S of the given non-homogenous difference equation is of the form

$f(k) = \alpha a^k$ where 'α' and 'a' are constant, then in order to find particular solution Y^* we shall substitute $y_k = Aa^k$ (trial function) in the given difference equation and find the value of A. If the trial function is present in complementary solution Y then the trial function will be multiplied by a suitable k^n .

Question: Solve the difference equation

$$y_{k+2} - 6y_{k+1} + 7y_k = 3^k$$

Solution: The given difference equation can be written as

$$E^2 y_k - 6Ey_k + 7y_k = 3^k$$

$$(E^2 - 6E + 7)y_k = 3^k \quad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 6E + 7)y_k = 0$$

The auxiliary equation

$$m^2 - 6m + 7 = 0$$

$$m = \frac{6 \pm \sqrt{36-28}}{2} = \frac{6 \pm 8}{2} = \frac{6 \pm 2\sqrt{2}}{2}$$

$$m = 3 \pm \sqrt{2}$$

$$Y = c_1(3-\sqrt{2})^k + c_2(3+\sqrt{2})^k$$

For particular solution

$$\text{Let } y_k = A3^k \qquad put\ in (1)$$

$$(E^2 - 6E + 7)A3^k = 3^k$$

$$E^2 A3^k - 6E A3^k + 7A3^k = 3^k$$

$$A3^{k+2} - 6A3^{k+1} + 7A3^k = 3^k$$

$$9A3^k - 18A3^k + 7A3^k = 3^k$$

$$9A - 18A + 7A = 1$$

$$-2A = 1$$

$$A = \frac{-1}{2}$$

$$\Rightarrow \quad Y^* = \frac{-1}{2}3^k$$

$$\text{Thus } Y_k = Y + Y^*$$

$Y_k = c_1(3-\sqrt{2})^k + c_2(3+\sqrt{2})^k - \frac{1}{2}3^k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + 4y_k = 2^k$$

Solution: The given difference equation can be written as

$$E^2 y_k - 4Ey_k + 4y_k = 2^k$$

$$(E^2 - 4E + 4)y_k = 2^k \qquad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 4E + 4)y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$(m-2) = 0 \;\; \& \;(m-2) = 0$$

$$m = 2, 2$$

$$Y = (c_1 + c_2 k)(2)^k$$

For particular solution

Let $y_k = A2^k k^2 \qquad put\ in (1)$

$$(E^2 - 4E + 4)\,\mathrm{A}k^2\, 2^k = 2^k$$

$$E^2\,\mathrm{A}k^2\, 2^{k+2} - 4E\,\mathrm{A}k^2\, 2^{k+1} + 4\,\mathrm{A}k^2\, 2^k = 2^k$$

$$\mathrm{A}(k+2)^2\, 2^{k+2} - 4\,\mathrm{A}(k+1)^2\, 2^{k+1} + 4\,\mathrm{A}k^2\, 2^k = 2^k$$

$$4\,\mathrm{A}(k^2 + 4k + 4)2^k - 8\,\mathrm{A}(k^2 + 2k + 1)2^k + 4\,\mathrm{A}k^2\, 2^k = 2^k$$

$$4\,\mathrm{A}k^2 + 16Ak + 16A - 8\,\mathrm{A}k^2 - 16Ak - 8A + 4\,\mathrm{A}k^2 = 1$$

$$8\mathrm{A} = 1$$

$$\mathrm{A} = \frac{1}{8}$$

$$\Rightarrow \quad Y^* = \frac{1}{8} k^2\, 2^k$$

Thus $Y_k = Y + Y^*$

$Y_k = (c_1 + c_2 k)(2)^k + \frac{1}{8} k^2\, 2^k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + 4y_k = 2^{k+2}$$

Solution: The given equation

$$y_{k+2} - 4y_{k+1} + 4y_k = 4.2^k$$

The given difference equation can be written as

$$E^2 y_k - 4Ey_k + 4y_k = 4.2^k$$

$$(E^2 - 4E + 4)y_k = 4.2^k \quad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 4E + 4)y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$(m-2) = 0 \ \& \ (m-2) = 0$$

$$m = 2, 2$$

$$Y = (c_1 + c_2 k)(2)^k$$

For particular solution

Let $y_k = A2^k k^2$ *put in*(1)

$$(E^2 - 4E + 4)Ak^2 2^k = 4.2^k$$

$$E^2 Ak^2 2^{k+2} - 4E Ak^2 2^{k+1} + 4Ak^2 2^k = 4.2^k$$

$$A(k+2)^2 2^{k+2} - 4A(k+1)^2 2^{k+1} + 4Ak^2 2^k = 4.2^k$$

$$4A(k^2+4k+4)2^k - 8A(k^2+2k+1)2^k + 4Ak^2 2^k = 4.2^k$$

$$4Ak^2+16Ak+16A-8Ak^2-16Ak-8A+4Ak^2=4$$

$$8A=4$$

$$A=\frac{1}{2}$$

$$\Rightarrow \quad Y^* = \frac{1}{2}k^2 2^k$$

Thus $Y_k = Y + Y^*$

$Y_k = (c_1 + c_2 k)(2)^k + \frac{1}{2}k^2 2^k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + 3y_k = 3^{k+1}$$

Solution: The given difference equation can be written as

$$E^2 y_k - 4Ey_k + 3y_k = 3^{k+1}$$

$$(E^2 - 4E + 3)y_k = 3^{k+1} \quad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 4E + 3)y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 3 = 0$$

$$m = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2}$$

$$m = \frac{6}{2}, \frac{2}{2}$$

$$m = 1, 3$$

$$Y = c_1(1)^k + c_2(3)^k$$

For particular solution

$$(E^2 - 4E + 3)y_k = 3^k \qquad \_\_\_\_\_\_\_\_(2)$$

$$(E^2 - 4E + 3)y_k = 1 \qquad \_\_\_\_\_\_\_\_(3)$$

For equation (2)

Let $y_k = A3^k k \qquad put\ in(2)$

$$(E^2 - 4E + 3)\,\mathrm{Ak}\,3^k = 3^k$$

$$E^2\,\mathrm{Ak}\,3^k - 4E\,\mathrm{Ak}\,3^k + 3\,\mathrm{A}\,3^k = 3^k$$

$$\mathrm{A(k+2)}3^{k+2} - 4\,\mathrm{A(k+1)}3^{k+1} + 3\,\mathrm{A}\,3^k = 3^k$$

$$9\,\mathrm{A(k+2)}3^k - 12\,\mathrm{A(k+1)}3^k + 3\,\mathrm{A}\,3^k = 3^k$$

$$9\,\mathrm{A(k+2)} - 12\,\mathrm{A(k+1)} + 3\,\mathrm{A}\,k = 1$$

$$9\,\mathrm{Ak} + 18\,\mathrm{A} - 12\,\mathrm{Ak} - 12\,\mathrm{A} + 3\,\mathrm{Ak} = 1$$

$$6\,\mathrm{A} = 1$$

$$\mathrm{A} = \frac{1}{6}$$

$$\Rightarrow \quad Y_1^* = \frac{1}{6}k\,3^k$$

For equation (3)

Let $y_k = Bk$ put in (3)

$$(E^2 - 4E + 3)Bk = 1$$

$$E^2Bk - 4EBk + 3Bk = 1$$

$$B(k+2) - 4B(k+1) + 3Bk = 1$$

$$Bk + 2B - 4Bk - 4B + 3Bk = 1$$

$$-2B = 1$$

$$B = \frac{-1}{2}$$

$$Y_2^* = \frac{-1}{2}k$$

$$\Rightarrow \quad Y^* = Y_1^* + Y_2^*$$

$$Y^* = \frac{1}{6}k\,3^k - \frac{1}{2}k$$

Thus $Y_k = Y + Y^*$

$Y_k = c_1(1)^k + c_2(3)^k + \frac{1}{6}k3^k - \frac{1}{2}k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + 4y_k = 3.2^k + 5.4^k$$

Solution: The given difference equation can be written as

$$E^2 y_k - 4Ey_k + 4y_k = 3.2^k + 5.4^k$$

$$(E^2 - 4E + 4)y_k = 3.2^k + 5.4^k$$

For complementary solution

$$(E^2 - 4E + 4)y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$(m-2) = 0 \;\&\; (m-2) = 0$$

$$m = 2,2$$

$$Y = (c_1 + c_2 k)(2)^k$$

For particular solution

$$(E^2 - 4E + 4)\,y_k = 3.2^k \qquad \_\_\_\_\_\_(i)$$

$$(E^2 - 4E + 4)\,y_k = 5.4^k \qquad \_\_\_\_\_\_(ii)$$

For equation (i)

$$\text{Let } y_k = A2^k k^2 \qquad put\ in\,(i)$$

$$(E^2 - 4E + 4)\,Ak^2\,2^k = 3.2^k$$

$$E^2\,Ak^2\,2^{k+2} - 4E\,Ak^2\,2^{k+1} + 4\,Ak^2\,2^k = 3.2^k$$

$$A(k+2)^2\,2^{k+2} - 4\,A(k+1)^2\,2^{k+1} + 4\,Ak^2\,2^k = 3.2^k$$

$$4\,A(k^2 + 4k + 4)2^k - 8\,A(k^2 + 2k + 1)2^k + 4\,Ak^2\,2^k = 3.2^k$$

$$4\,Ak^2 + 16Ak + 16A - 8\,Ak^2 - 16Ak - 8A + 4\,Ak^2 = 3$$

$$8\,A = 3$$

$$A = \frac{3}{8}$$

$$\Rightarrow \quad Y_1^* = \frac{3}{8}k^2\,2^k$$

For equation (ii)

$$\text{Let } y_k = B4^k \quad \text{put in (3)}$$

$$(E^2 - 4E + 4)B4^k = 5.4^k$$

$$E^2B4^k - 4EB4^k + 4B4^k = 5.4^k$$

$$B4^{k+2} - 4B4^{k+1} + 4B4^k = 5.4^k$$

$$14B4^k - 16B4^k + 4B4^k = 5.4^k$$

$$4B4^k = 5.4^k$$

$$B = \frac{5}{4}$$

$$Y_2^* = \frac{5}{4}4^k$$

$$\Rightarrow \quad Y^* = Y_1^* + Y_2^*$$

$$Y^* = \frac{3}{8}k^2 2^k + \frac{5}{4}4^k$$

Thus $Y_k = Y + Y^*$

$Y_k = (c_1 + c_2 k)(2)^k + \frac{3}{8}k^2 2^k + \frac{5}{4}4^k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + 4y_k = 3.2^k + 5.4^k + 7$$

Solution: The given difference equation can be written as

$$E^2 y_k - 4Ey_k + 4y_k = 3.2^k + 5.4^k + 7$$

$$(E^2 - 4E + 4)y_k = 3.2^k + 5.4^k + 7$$

For complementary solution

$$(E^2 - 4E + 4)y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$(m-2) = 0 \ \& \ (m-2) = 0$$

$m = 2,2$

$Y = (c_1 + c_2 k)(2)^k$

For particular solution

$(E^2 - 4E + 4)\,y_k = 3.2^k$ ______(i)

$(E^2 - 4E + 4)\,y_k = 5.4^k$ ______(ii)

$(E^2 - 4E + 4)\,y_k = 7$ ______(iii)

For equation (i)

Let $y_k = A2^k k^2$ *put in* (i)

$(E^2 - 4E + 4)\,Ak^2 2^k = 3.2^k$

$E^2 Ak^2 2^{k+2} - 4E\,Ak^2 2^{k+1} + 4Ak^2 2^k = 3.2^k$

$A(k+2)^2 2^{k+2} - 4A(k+1)^2 2^{k+1} + 4Ak^2 2^k = 3.2^k$

$4A(k^2 + 4k + 4)2^k - 8A(k^2 + 2k + 1)2^k + 4Ak^2 2^k = 3.2^k$

$4Ak^2 + 16Ak + 16A - 8Ak^2 - 16Ak - 8A + 4Ak^2 = 3$

$8A = 3$

$A = \frac{3}{8}$

$\Rightarrow \quad Y_1^* = \frac{3}{8} k^2 2^k$

For equation (ii)

Let $y_k = B4^k$ put in (3)

$(E^2 - 4E + 4)B4^k = 5.4^k$

$E^2 B4^k - 4EB4^k + 4B4^k = 5.4^k$

$B4^{k+2} - 4B4^{k+1} + 4B4^k = 5.4^k$

$$14B4^k - 16B4^k + 4B4^k = 5.4^k$$

$$4B4^k = 5.4^k$$

$$B = \frac{5}{4}$$

$$Y_2^* = \frac{5}{4}4^k$$

For equation (iii)

Let $y_k = C$ put in (iii)

$$(E^2 - 4E + 4)C = 7$$

$$E^2C - 4EC + 4C = 7$$

$$C - 4C + 4C = 7$$

$$C = 7$$

$$Y_3^* = 7$$

$$\Rightarrow \quad Y^* = Y_1^* + Y_2^* + Y_3^*$$

$$Y^* = \frac{3}{8}k^2\,2^k + \frac{5}{4}4^k + 7$$

Thus $Y_k = Y + Y^*$

$Y_k = (c_1 + c_2k)(2)^k + \frac{3}{8}k^2\,2^k + \frac{5}{4}4^k + 7$ is the required general solution.

Lecture # 05

Non-Homogenous Difference Equation:

Type III:

When the R.H.S of the given difference equation is a polynomial of ‘k’ then in order to find particular solution (Y*) we shall consider a trial function in the form of polynomial of ‘k’ (with the same degree of the given polynomial on the R.H.S of difference equation). And substitute the trial function in the given difference equation and evaluate the values of constants. But if the trial function or any term of trial function is present in complementary solution (Y) then we shall multiply the trial function with a suitable k^n

Question: Solve the difference equation

$$y_{k+2} + y_{k+1} + y_k = k^2 + k + 1$$

Solution: The given difference equation can be written as

$$E^2 y_k + E y_k + y_k = k^2 + k + 1$$

$$(E^2 + E + 1) y_k = k^2 + k + 1 \qquad \text{\_\_\_\_\_}(1)$$

For complementary solution

$$(E^2 + E + 1) y_k = 0$$

The auxiliary equation

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$|m| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{1+3}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}\left(-\sqrt{3}\right)$$

$$Y=(1)^{k}\left[c_1\cos\theta k+c_2\sin\theta k\right]$$

$$Y=c_1\cos\theta k+c_2\sin\theta k$$

For particular solution

Let $y_k=Ak^2+Bk+C \qquad put\ in(1)$

$$(E^2+E+1)\left(Ak^2+Bk+C\right)=k^2+k+1$$

$$E^2\left(Ak^2+Bk+C\right)+E\left(Ak^2+Bk+C\right)+\left(Ak^2+Bk+C\right)=k^2+k+1$$

$$A\left(k+2\right)^2+B(k+2)+\mathrm{C}+\mathrm{A}\left(k+1\right)^2+B(k+1)+\mathrm{C}+Ak^2+Bk+C=k^2+k+1$$

$$A\left(k^2+4k+4\right)+B(k+2)+\mathrm{C}+\mathrm{A}\left(k^2+2k+1\right)+B(k+1)+\mathrm{C}+Ak^2+Bk+C=k^2+k+1$$

By comparing coefficients

$$\mathrm{A}+\mathrm{A}+\mathrm{A}=1$$

$$3\mathrm{A}=1$$

$$\mathrm{A}=\frac{1}{3}$$

$$4\mathrm{A}+\mathrm{B}+2\mathrm{A}+B+B=1$$

$$6\mathrm{A}+3\mathrm{B}=1$$

$$6\left(\frac{1}{3}\right)+3\mathrm{B}=1$$

$$B=-\frac{1}{3}$$

$$4\mathrm{A}+2\mathrm{B}+C+\mathrm{A}+B+C+C=1$$

$$5\mathrm{A}+3\mathrm{B}+3C=1$$

$$5\left(\frac{1}{3}\right)+3\left(\frac{-1}{3}\right)+3C=1$$

$$C=\frac{1}{9}$$

$$\Rightarrow \quad Y^* = \frac{1}{3}k^2 + \left(\frac{-1}{3}\right)k + \frac{1}{9}$$

Thus $Y_k = Y + Y^*$

$Y_k = c_1 \cos\theta k + c_2 \sin\theta k + \frac{1}{3}k^2 - \frac{1}{3}k + \frac{1}{9}$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 4y_k = 9k^2$$

Solution: The given difference equation can be written as

$$E^2 y_k - 4y_k = 9k^2$$

$$(E^2 - 4)y_k = 9k^2 \quad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 4)y_k = 0$$

The auxiliary equation

$$m^2 - 4 = 0$$

$$m^2 = 4$$

$$m = \pm 2$$

$$Y = c_1(-2)^k + c_2(2)^k$$

For particular solution

Let $y_k = Ak^2 + Bk + C \qquad put\ in (1)$

$$(E^2 - 4)\left(Ak^2 + Bk + C\right) = 9k^2$$

$$E^2\left(Ak^2 + Bk + C\right) - 4\left(Ak^2 + Bk + C\right) = 9k^2$$

$$A\left(k+2\right)^2 + B(k+2) + \mathrm{C} - 4\mathrm{A}\mathrm{k}^2 - 4Bk - 4C = 9k^2$$

$$A\left(k^2+4k+4\right)+B(k+2)-4\text{A}k^2-4Bk-3C=9k^2$$

By comparing coefficients

$$\text{A}-4\text{A}=9$$

$$-3\text{A}=9$$

$$\text{A}=-3$$

$$4\text{A}+\text{B}-4B=0$$

$$4\text{A}-3\text{B}=0$$

$$4\left(-3\right)+3\text{B}=0$$

$$B=-4$$

$$4\text{A}+2\text{B}-3C=0$$

$$4\left(-3\right)+2\left(-4\right)=3C$$

$$C=\frac{-3}{20}$$

$$\Rightarrow\ Y^*=-3k^2-4k-\frac{20}{3}$$

Thus $Y_k=Y+Y^*$

$Y_k=c_1(-2)^k+c_2(2)^k-3k^2-4k-\dfrac{20}{3}$ is the required general solution.

Question: Solve the difference equation

$$\Delta^2 y_k+2\Delta y_k+y_k=3k+2$$

Solution: As we know $\Delta=E-1$

$$\left(E-1\right)^2 y_k+2\left(E-1\right)y_k+y_k=3k+2$$

$$\left(E^2-2E+1\right)y_k+2\left(E-1\right)y_k+y_k=3k+2$$

$$E^2y_k - 2Ey_k + y_k + 2Ey_k - 2y_k + y_k = 3k + 2$$

$$E^2y_k = 3k + 2 \qquad \_\_\_\_\_(1)$$

For complementary solution

$$E^2y_k = 0$$

The auxiliary equation

$m^2 = 0$

$m = 0, 0$

$Y = (c_1 + c_2k)(0)^k = 0$

For particular solution

Let $y_k = Ak + B \qquad put\ in(1)$

$E^2(Ak + B) = 3k + 2$

$A(k+2) + B = 3k + 2$

$Ak + 2A + B = 3k + 2$

By comparing coefficients

$A = 3$

$2\text{A} + B = 2$

$2(3) + B = 2$

$B = -4$

$\Rightarrow \quad Y^* = 3k - 4$

Thus $Y_k = Y + Y^*$

$\Rightarrow \quad Y_k = 0 + 3k - 4 = 3k - 4$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 3y_{k+1} + 2y_k = k^2$$

Solution: The given difference equation can be written as

$$E^2 y_k - 3Ey_k + 2y_k = k^2$$

$$(E^2 - 3E + 2)y_k = k^2 \qquad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 3E + 2)y_k = 0$$

The auxiliary equation

$$m^2 - 3m + 2 = 0$$

$$m = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \frac{3-1}{2}, \frac{3+1}{2}$$

$$m = 1, 2$$

$$Y = c_1(1)^k + c_2(2)^k$$

$$Y = c_1 + c_2(2)^k$$

For particular solution

Let $y_k = Ak^3 + Bk^2 + Ck \qquad put\ in\,(1)$

$$(E^2 - 3E + 2)\left(Ak^3 + Bk^2 + Ck\right) = k^2$$

$$E^2\left(Ak^3 + Bk^2 + Ck\right) - 3E\left(Ak^3 + Bk^2 + Ck\right) + 2\left(Ak^3 + Bk^2 + Ck\right) = k^2$$

$$A(k+2)^3 + B(k+2)^2 + C(k+2) - 3A(k+1)^3 - 3B(k+1)^2 - 3C(k+1) + 2Ak^3 + 2Bk^2 + 2Ck = k^2$$

$$A\left(k^3+12k^2+12k+8\right)+B\left(k^2+4k+4\right)+C(k+2)-3\left[A\left(k^3+3k^2+3k+1\right)+B\left(k^2+2k+1\right)+C(k+1)\right]+2Ak^3+2Bk^2+2Ck=k^2$$

By comparing coefficients

$$A - 3A + 2A = 0$$

$$12A + B - 9A - 3B + 2B = 1$$

$$3A = 1$$

$$A = \frac{1}{3}$$

$$12A + 4B + C - 9A - 6B - 3C + 2C = 0$$

$$3A - 2B = 0$$

$$3\left(\frac{1}{3}\right) = 2B$$

$$B = \frac{1}{2}$$

$$8A + 4B + 2C - 3A - 3B - 3C = 0$$

$$5A + B - C = 0$$

$$5\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right) = C$$

$$C = \frac{13}{6}$$

$$\Rightarrow \quad Y^* = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{13}{6}k$$

Thus $Y_k = Y + Y^*$

$Y_k = c_1 + c_2(2)^k + \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{13}{6}k$ is the required general solution.

Question: Solve the difference equation

$$2y_{k+2} + 5y_{k+1} + 2y_k = 2^k + k^2$$

Solution: The given difference equation can be written as

$$2E^2 y_k + 5Ey_k + 2y_k = 2^k + k^2$$

$$(2E^2 + 5E + 2)y_k = 2^k + k^2 \qquad \_\_\_\_\_(1)$$

For complementary solution

$$(2E^2+5E+2)y_k=0$$

The auxiliary equation

$$2m^2+5m+2=0$$

$$m=\frac{-5\pm\sqrt{25-16}}{4}=\frac{-5\pm 9}{4}=\frac{-5+3}{4},\frac{-5-3}{4}$$

$$m=\frac{-1}{2},-2$$

$$Y=c_1\left(\frac{-1}{2}\right)^k+c_2(-2)^k$$

For particular solution

$$(2E^2+5E+2)y_k=2^k \qquad \_\_\_\_\_\_\text{(i)}$$

$$(2E^2+5E+2)y_k=k^2 \qquad \_\_\_\_\_\_\text{(ii)}$$

For equation (i)

Let $y_k = A2^k$ $\quad$ *put in* (i)

$$(2E^2+5E+2)A2^k=2^k$$

$$2E^2A2^k+5EA2^k+2A2^k=2^k$$

$$2A2^{k+2}+5A2^{k+1}+2A2^k=2^k$$

$$8A2^k+10A2^k+2A2^k=2^k$$

$$8A+10A+2A=1$$

$$20A=1$$

$$A=\frac{1}{20}$$

$$\Rightarrow \quad Y_1^* = \frac{1}{20}2^k$$

For equation (ii)

Let $y_k = Bk^2 + Ck + D \qquad put\ in$ (ii)

$$(2E^2 + 5E + 2)\left(Bk^2 + Ck + D\right) = k^2$$

$$2E^2\left(Bk^2 + Ck + D\right) + 5E\left(Bk^2 + Ck + D\right) + 2\left(Bk^2 + Ck + D\right) = k^2$$

$$2\left[B(k+2)^2 + C(k+2) + D\right] + 5\left[B(k+1)^2 + C(k+1) + D\right] + 2\left(Bk^2 + Ck + D\right) = k^2$$

$$2\left[B\left(k^2+4k+4\right) + C(k+2) + D\right] + 5\left[B\left(k^2+2k+1\right) + C(k+1) + D\right] + 2Bk^2 + 2Ck + 2D = k^2$$

By comparing coefficient

$$2B + 5B + 2B = 1$$

$$B = \frac{1}{9}$$

$$8B + 2C + 10B + 5C + 2C = 0$$

$$18B + 9C = 0$$

$$18\left(\frac{1}{9}\right) + 9C = 0$$

$$C = \frac{-2}{9}$$

$$8B + 4C + 2D + 5B + 5C + 5D + 2D = 0$$

$$13B + 9C + 9D = 0$$

$$9D = -13\left(\frac{1}{9}\right) - 9\left(\frac{-2}{9}\right) \quad \Rightarrow \quad D = \frac{5}{81}$$

$$Y_2^* = \frac{1}{9}k^2 - \frac{2}{9}k + \frac{5}{81}$$

$$Y^* = Y_1^* + Y_2^*$$

$$Y^* = \frac{1}{20}2^k + \frac{1}{9}k^2 - \frac{2}{9}k + \frac{5}{81}$$

Thus $Y_k = Y + Y^*$

$$Y_k = c_1\left(\frac{-1}{2}\right)^k + c_2(-2)^k + \frac{1}{20}2^k + \frac{1}{9}k^2 - \frac{2}{9}k + \frac{5}{81}$$ is the required general solution.

Question: Solve the difference equation

$$y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = k^2$$

Solution: The given difference equation can be written as

$$E^4 y_k - 2E^3 y_k + 2E^2 y_k - 2Ey_k + y_k = k^2$$

$$\left(E^4 - 2E^3 + 2E^2 - 2E + 1\right) y_k = k^2 \qquad \text{\_\_\_\_\_(1)}$$

For complementary solution

$$(E^4 - 2E^3 + 2E^2 - 2E + 1) y_k = 0$$

The auxiliary equation

$$m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$$

By synthetic division

	1	−2	2	−2	1
1	↓	1	−1	1	−1
	1	−1	1	−1	0
1	↓	1	0	1	
	1	0	1	0	

$$m = 1, 1 \text{ and } m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

$$m = 0 \pm i$$

$$|m| = \sqrt{(0)^2 + (1)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

$$Y = (c_1 + c_2\,\mathrm{k})(1)^k + (1)^k\left(c_3\cos\theta k + c_4\sin\theta k\right)$$

$$Y = (c_1 + c_2\,\mathrm{k}) + \left(c_3\cos\theta k + c_4\sin\theta k\right)$$

For particular solution

Let $y_k = Ak^4 + Bk^3 + Ck^2$ $\quad$ *put in* (1)

$$(\mathrm{E}^4 - 2E^3 + 2E^2 - 2E + 1)\left(Ak^4 + Bk^3 + Ck^2\right) = k^2$$

$$\mathrm{E}^4\left(Ak^4 + Bk^3 + Ck^2\right) - 2E^3\left(Ak^4 + Bk^3 + Ck^2\right) + 2E^2\left(Ak^4 + Bk^3 + Ck^2\right)$$

$$-2E\left(Ak^4 + Bk^3 + Ck^2\right) + \left(Ak^4 + Bk^3 + Ck^2\right) = k^2$$

$$A(k+4)^4 + B(k+4)^3 + C(k+4)^2 - 2\left[A(k+3)^4 + B(k+3)^3 + C(k+3)^2\right]$$

$$+2\left[A(k+2)^4 + B(k+2)^3 + C(k+2)^2\right] - 2\left[A(k+1)^4 + B(k+1)^3 + C(k+1)^2\right]$$

$$+\left(Ak^4 + Bk^3 + Ck^2\right) = k^2$$

$$\because (a+b)^4 = a^4 + b^4 + 4a^3b + 4ab^3 + 6a^2b^2$$

$$A\left(k^4 + 256 + 16k^3 + 256k + 96k^2\right) + B\left(k^3 + 12k^2 + 48k + 64\right) + C\left(k^2 + 16 + 8k\right)$$

$$-2\left[A\left(k^4 + 81 + 12k^3 + 108k + 54k^2\right) + B\left(k^3 + 9k^2 + 27k + 27\right) + C\left(k^2 + 9 + 6k\right)\right]$$

$$+2\left[A\left(k^4 + 16 + 8k^3 + 32k + 24k^2\right) + B\left(k^3 + 6k^2 + 12k + 8\right) + C\left(k^2 + 4 + 4k\right)\right]$$

$$-2\left[A\left(k^4 + 1 + 4k^3 + 4k + 6k^2\right) + B\left(k^3 + 3k^2 + 3k + 1\right) + C\left(k^2 + 1 + 2k\right)\right]$$

$+\left(Ak^4+Bk^3+Ck^2\right)=k^2$

By comparing coefficients

$$96A+12B+C-108A-18B-2C+48A+12B+2C-12A-6B-2C+C=1$$

$$A=\frac{1}{24}$$

$$256A+48B+8C-216A-54B-12C+64A+24B+8C-8A-6B-4C=0$$

$$96A+12B=0$$

$$69\left(\frac{1}{24}\right)+12B=0$$

$$B=-\frac{1}{3}$$

$$256A+64B+16C-162A-54B-18C+32A+16B+8C-2A-2B-2C=0$$

$$124A+24B+4C=0$$

$$\frac{124}{24}-\frac{24}{3}+4C=0$$

$$C=\frac{17}{24}$$

$$\Rightarrow \quad Y^*=\frac{1}{24}k^4-\frac{1}{3}k^3+\frac{17}{24}k^2$$

Thus $Y_k=Y+Y^*$

$Y_k=\left(c_1+c_2\,\mathrm{k}\right)+\left(c_3\cos\frac{\pi}{2}k+c_4\sin\frac{\pi}{2}k\right)+\frac{1}{24}k^4-\frac{1}{3}k^3+\frac{17}{24}k^2$ is the required general solution.

Lecture # 06

Non-Homogenous Difference Equation:

Type IV:

When the R.H.S of the given non-homogenous difference is of the form

$$f(k) = a^k \phi(k)$$

Where $\phi(k)$ is a polynomial in k, then in order to find particular solution Y* we shall substitute a trial function.

$$y_k = a^k \left(A_0 + A_1 k + A_2 k^2 + ... \right)$$

in the given difference equation and then evaluate the values of $\left(A_0, A_1, A_2, ... \right)$. If the trial function or any term of trial function is present in complementary solution (Y) then the trial function will be multiplied by a suitable k^n

Question: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + 4y_k = 3^k \left(k^2 + 1 \right)$$

Solution: The given difference equation can be written as

$$(E^2 - 4E + 4)y_k = 3^k \left(k^2 + 1 \right) \quad \_\_\_\_\_\_\text{(i)}$$

For complementary solution

$$(E^2 - 4E + 4)y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$(m-2) = 0 \ \& \ (m-2) = 0$$

$$m = 2,2$$

$$Y = (c_1 + c_2 k)(2)^k$$

For particular solution

Let $y_k = 3^k\left(Ak^2 + Bk + C\right)$ *put in* (i)

$$(E^2 - 4E + 4)3^k\left(Ak^2 + Bk + C\right) = 3^k\left(k^2 + 1\right)$$

$$E^2.3^k\left(Ak^2 + Bk + C\right) - 4E.3^k\left(Ak^2 + Bk + C\right) + 4.3^k\left(Ak^2 + Bk + C\right) = 3^k\left(k^2 + 1\right)$$

$$3^{k+2}\left(A(k+2)^2 + B(k+2) + C\right) - 43^{k+1}\left(A(k+1)^2 + B(k+1) + C\right) + 4.3^k\left(Ak^2 + Bk + C\right) = 3^k\left(k^2 + 1\right)$$

$$9.3^k\left[A(k^2 + 4k + 4) + B(k+2) + C\right] - 12.3^k\left(A(k^2 + 2k + 1) + B(k+1) + C\right) + 4.3^k\left(Ak^2 + Bk + C\right) = 3^k\left(k^2 + 1\right)$$

$$9\left[A(k^2 + 4k + 4) + B(k+2) + C\right] - 12\left(A(k^2 + 2k + 1) + B(k+1) + C\right) + 4\left(Ak^2 + Bk + C\right) = \left(k^2 + 1\right)$$

$$\left(9A - 12A + 4A\right)k^2 + \left(36A + 9B - 24A - 12B + 4B\right)k + 36A + 18B + 9C - 12A - 12B - 12C + C = \left(k^2 + 1\right)$$

On comparing coefficients

A = 1

12A+B = 0

12(1) + B = 0

B = −12

24A+6B+C = 1

24(1) + 6(−12)+C = 1

C = 49

$$\Rightarrow \quad Y^* = 3^k\left(k^2 - 12k + 49\right)$$

Thus $Y_k = Y + Y^*$

$Y_k = (c_1 + c_2 k)(2)^k + 3^k\left(k^2 - 12k + 49\right)$ is the required general solution.

Question: Solve the difference equation

$$y_{k+3}+5y_{k+2}-8y_{k+1}-4y_k=2^k.k$$

Solution: The given difference equation can be written as

$$E^3y_k-5E^2y_k+8Ey_k-4y_k=2^k.k$$

$$\left(E^3-5E^2+8E-4\right)y_k=2^k.k \qquad \_\_\_\_\_\text{(i)}$$

For complementary solution

$$(E^3-5E^2+8E-4)y_k=0$$

The auxiliary equation

$$m^3-5m^2+8m-4=0$$

By synthetic division

	1	–5	8	–4
1	↓	1	–4	4
	1	–4	4	0

$$m=1 \;\; and \;\; m^2-4m+4=0$$

$$(m-2)^2=0$$

$$\Rightarrow \quad m=2,2$$

$$Y=c_1(1)^k+\left(c_2+c_3\,\text{k}\right)2^k$$

For particular solution

Let $y_k=2^k\left(Ak^3+Bk^2\right)$ $\qquad put\ in\,\text{(i)}$

$$\left(E^3-5E^2+8E-4\right)2^k\left(Ak^3+Bk^2\right)=2^k.k$$

$$\left(E^32^k\left(Ak^3+Bk^2\right)-5E^22^k\left(Ak^3+Bk^2\right)+8E2^k\left(Ak^3+Bk^2\right)-4.2^k\left(Ak^3+Bk^2\right)\right)=2^k.k$$

$$2^{k+3}\left[A(k+3)^3+B(k+3)^2\right]-5.2^{k+2}\left[A(k+2)^3+B(k+2)^2\right]+8.2^k\left[A(k+1)^3+B(k+1)^2\right]$$
$$+4.2^k\left(Ak^3+Bk^2\right)=2^k.k$$

$$8.2^k\left[A(k^3+9k^2+27k+27)+B(k^2+6k+9)\right]-20.2^k\left[A(k^3+6k^2+12k+8)+B(k^2+4k+4)\right]$$
$$+16.2^k\left[A(k^3+3k^2+3k+1)+B(k^2+2k+1)\right]-4.2^k\left[\left(Ak^3+Bk^2\right)\right]=2^k.k$$

$$8\left[A(k^3+9k^2+27k+27)+B(k^2+6k+9)\right]-20\left[A(k^3+6k^2+12k+8)+B(k^2+4k+4)\right]$$

$$+16\left[A(k^3+3k^2+3k+1)+B(k^2+2k+1)\right]-4\left[\left(Ak^3+Bk^2\right)\right]=k$$

$$\left(8A-20A+16A-4A\right)k^3+\left(72A+8B-120A-20B+48A+16B-4B\right)k^2$$

$$+\left(216A+48B-240A-80B+48A+32B\right)k+\left(216A+72B-160A-80B+16A+16B\right)=k$$

On comparing

24Ak+(72A+8B) = k

24A = 1

$$A=\frac{1}{24}$$

72A+8B = 0

$$72\left(\frac{1}{24}\right)+8B=0$$

$$B=\frac{-3}{8}$$

$$Y^*=2^k\left(\frac{1}{24}k^3-\frac{3}{8}k^2\right)$$

Thus $Y_k=Y+Y^*$

$Y_k=c_1(1)^k+\left(c_2+c_3\,\mathrm{k}\right)2^k+2^k\left(\frac{1}{24}k^3-\frac{3}{8}k^2\right)$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 7y_{k+1} + 8y_k = 2^k\left(k^2 - k\right)$$

Solution: The given difference equation can be written as

$$E^2 y_k - 7Ey_k - 8y_k = 2^k\left(k^2 - k\right)$$

$$(E^2 - 7E - 8)y_k = 2^k\left(k^2 - k\right) \quad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 7E - 8)y_k = 0$$

The auxiliary equation

$$m^2 - 7m - 8 = 0$$

$$m = \frac{7 \pm \sqrt{49+32}}{2} = \frac{7 \pm \sqrt{81}}{2} = \frac{7 \pm 9}{2}$$

$$m = -1, 8$$

$$Y = c_1(-1)^k + c_2(8)^k$$

For particular solution

Let $y_k = 2^k\left(Ak^2 + Bk + C\right) \qquad put\ in\,(\text{i})$

$$(E^2 - 7E - 8)2^k\left(Ak^2 + Bk + C\right) = 2^k\left(k^2 - k\right)$$

$$E^2.2^k\left(Ak^2 + Bk + C\right) - 7E.2^k\left(Ak^2 + Bk + C\right) - 8.2^k\left(Ak^2 + Bk + C\right) = 2^k\left(k^2 - k\right)$$

$$2^{k+2}\left[A(k+2)^2 + B(k+2) + C\right] - 7.2^{k+1}\left[A(k+1)^2 + B(k+1) + C\right] - 8.2^k\left[Ak^2 + Bk + C\right] = 2^k\left(k^2 - k\right)$$

$$4.2^k\left[A\left(k^2+4k+4\right) + B(k+2) + C\right] - 14.2^k\left[A\left(k^2+2k+1\right) + B(k+1) + C\right] - 8.2^k\left[Ak^2 + Bk + C\right] = 2^k\left(k^2 - k\right)$$

$$4\left[A\left(k^2+4k+4\right) + B(k+2) + C\right] - 14\left[A\left(k^2+2k+1\right) + B(k+1) + C\right] - 8\left[Ak^2 + Bk + C\right] = \left(k^2 - k\right)$$

$$(4A - 14A - 8A)k^2 + (16A + 4B - 28A - 14B - 8B)k + (16A + 8B + 4C - 14A - 14B - 14C - 8C) = \left(k^2 - k\right)$$

$$-18Ak^2 + (-12A - 18B)k + (2A - 6B - 18C) = \left(k^2 - k\right)$$

On comparing coefficients

$$-18A = 1$$

$$A = \frac{-1}{18}$$

$$-12A - 18B = -1 \quad \Rightarrow \quad 18B = -12\left(\frac{-1}{18}\right) + 1$$

$$B = \frac{5}{54}$$

$$2A - 6B - 18C = 0 \quad \Rightarrow \quad 18C = 2\left(\frac{-1}{18}\right) - 6\left(\frac{5}{54}\right)$$

$$C = -\frac{1}{27}$$

$$\Rightarrow \quad Y^* = 2^k\left(\frac{-1}{18}k^2 + \frac{5}{54}k - \frac{1}{27}\right)$$

Thus $Y_k = Y + Y^*$

$Y_k = c_1(-1)^k + c_2(8)^k + 2^k\left(\frac{-1}{18}k^2 + \frac{5}{54}k - \frac{1}{27}\right)$ is the required general solution.

Lecture # 07

Non-Homogenous Difference Equation:

Type V:

When the R.H.S of the given non-homogeneous difference equation is of the form f(k) = sinAk or cosAk

Where A is constant, then in order to find particular solution Y* we shall substitute $y_k = c_1 \sin Ak + c_2 \cos Ak$ in the given difference equation and then evaluate the values of c_1 and c_2. If the trial function or any term of trial function is present in complementary solution Y then the trial function will be multiplied by suitable k^n.

Question: Solve the difference equation

$$y_{k+2} - 7y_{k+1} + 12y_k = \sin 3k$$

Solution: The given difference equation can be written as

$$E^2 y_k - 7Ey_k + 12y_k = \sin 3k$$

$$(E^2 - 7E + 12)y_k = \sin 3k \qquad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2 - 7E + 12)y_k = 0$$

The auxiliary equation

$$m^2 - 7m + 12 = 0$$

$$m = \frac{7 \pm \sqrt{49 - 48}}{2} = \frac{7 \pm 1}{2}$$

$$m = 4, 3$$

$$Y = c_1(3)^k + c_2(4)^k$$

For particular solution

$$Let\ \ y_k = A\sin 3k + B\cos 3k \qquad\qquad Put\,in\,(1)$$

$$(E^2 - 7E + 12)\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$E^2\left(A\sin 3k + B\cos 3k\right) - 7E\left(A\sin 3k + B\cos 3k\right) + 12\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$\left(A\sin 3(k+2) + B\cos 3(k+2)\right) - 7\left(A\sin 3(k+1) + B\cos 3(k+1)\right) + 12\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$\left(A\sin(3k+6) + B\cos(3k+6)\right) - 7\left(A\sin(3k+3) + B\cos(3k+3)\right) + 12\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$A\sin(3k+6) + B\cos(3k+6) - 7A\sin(3k+3) - 7B\cos(3k+3) + 12A\sin 3k + 12B\cos 3k = \sin 3k$$

$$A\left[\sin 3k\cos 6 + \cos 3k\sin 6\right] + B\left[\cos 3k\cos 6 - \sin 3k\sin 6\right] - 7A\left[\sin 3k\cos 3 + \sin 3\cos 3k\right]$$
$$-7B\left[\cos 3k\cos 3 - \sin 3k\sin 3\right] + 12A\sin 3k + 12B\cos 3k = \sin 3k$$

$$A\left[\sin 3k\cos 6 + \sin 6\cos 3k - 7\sin 3k\cos 3 - 7\sin 3\cos 3k + 12\sin 3k\right]$$

$$+B\left[\cos 3k\cos 6 - \sin 3k\sin 6 - 7\cos 3k\cos 3 + 7\sin 3k\sin 3 + 12\cos 3k\right] = \sin 3k$$

$$A\left[\sin 3k\left(\cos 6 - 7\cos 3 + 12\right) + \cos 3k\left(\sin 6 - 7\sin 3\right)\right]$$

$$+B\left[\cos 3k\left(\cos 6 - 7\cos 3 + 12\right) - \sin 3k\left(\sin 6 - 7\sin 3\right)\right] = \sin 3k$$

Say $\mathrm{M} = \cos 6 - 7\cos 3 + 12 \qquad \& \qquad N = \sin 6 - 7\sin 3$

$$A\left[\mathrm{M}\sin 3k + N\cos 3k\right] + B\left[M\cos 3k - N\sin 3k\right] = \sin 3k$$

$$\left(AM - BN\right)\sin 3k + \left(AN + BM\right)\cos 3k = \sin 3k$$

By comparing coefficients of sin3k and cos3k

$$\text{AM} - \text{BN} = 1$$

$$\text{AN} + \text{BM} = 0$$

Now by cross multiplication

$$\text{AM} - \text{BN} - 1 = 0$$

$$\text{AN} + \text{BM} + 0 = 0$$

M −N −1 M −N −1

N M 0 N M 0

$$\frac{A}{-0+M}=\frac{B}{-N-0}=\frac{1}{M^2+N^2}$$

$$A=\frac{M}{M^2+N^2}\quad , \quad B=\frac{-N}{M^2+N^2}$$

$$Y^*=A\sin 3k+B\cos 3k$$

$$Y_k=Y+Y^*$$

$Y_k=c_1(3)^k+c_2(4)^k+A\sin 3k+B\cos 3k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2}+3y_{k+1}+y_k=\sin k$$

Solution: The given difference equation can be written as

$$E^2y_k+3Ey_k+y_k=\sin k$$

$$(E^2+3E+1)y_k=\sin k \qquad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2+3E+1)y_k=0$$

The auxiliary equation

$$m^2+3m+1=0$$

$$m=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}$$

$$m=\frac{-3+\sqrt{5}}{2},\frac{-3-\sqrt{5}}{2}$$

$$Y=c_1\left(\frac{-3+\sqrt{5}}{2}\right)^k+c_2\left(\frac{-3-\sqrt{5}}{2}\right)^k$$

For particular solution

$$Let\ \ y_k = A\sin k + B\cos k \qquad\qquad Put\,in\,(1)$$

$$(E^2+3E+1)\left(A\sin k + B\cos k\right) = \sin k$$

$$E^2\left(A\sin k + B\cos k\right) + 3E\left(A\sin k + B\cos k\right) + \left(A\sin k + B\cos k\right) = \sin k$$

$$\left(A\sin(k+2) + B\cos(k+2)\right) + 3\left(A\sin(k+1) + B\cos(k+1)\right) + \left(A\sin k + B\cos k\right) = \sin k$$

$$A\left[\sin k\cos 2 + \sin 2\cos k\right] + B\left[\cos k\cos 2 - \sin k\sin 2\right] + 3A\left[\sin k\cos 1 + \sin 1\cos k\right]$$
$$+3B\left[\cos k\cos 1 - \sin k\sin 1\right] + A\sin k + B\cos k = \sin k$$

$$A\left[\sin k\cos 2 + \sin 2\cos k + 3\sin k\cos 1 + 3\sin 1\cos k + \sin k\right]$$

$$+B\left[\cos k\cos 2 - \sin k\sin 2 + 3\cos k\cos 1 - 3\sin k\sin 1 + \cos k\right] = \sin k$$

$$A\left[\sin k\left(\cos 2 + 3\cos 1 + 1\right) + \cos k\left(\sin 2 + 3\sin 1\right)\right]$$

$$+B\left[\cos k\left(\cos 2 + 3\cos 1 + 2\right) - \sin k\left(\sin 2 + 3\sin 1\right)\right] = \sin k$$

Say $\quad \mathrm{M} = \cos 2 + 3\cos 1 + 1 \qquad \& \qquad N = \sin 2 + 3\sin 1$

$$A\left[\mathrm{M}\sin k + N\cos k\right] + B\left[M\cos k - N\sin k\right] = \sin k$$

$$\left(AM - BN\right)\sin k + \left(AN + BM\right)\cos k = \sin k$$

By comparing coefficients of sink and cosk

$$\mathrm{AM - BN = 1}$$

$$\mathrm{AN + BM = 0}$$

Now by cross multiplication

$$\mathrm{AM - BN - 1 = 0}$$

$$\mathrm{AN + BM + 0 =}$$

M	–N	–1	M	–N	–1
N	M	0	N	M	0

$$\frac{A}{-0+M}=\frac{-B}{0+N}=\frac{1}{M^2+N^2}$$

$$A=\frac{M}{M^2+N^2} \quad , \quad B=\frac{-N}{M^2+N^2}$$

$$Y^*=A\sin k+B\cos k$$

$$Y_k=Y+Y^*$$

$Y_k=c_1\left(\frac{-3+\sqrt{5}}{2}\right)^k+c_2\left(\frac{-3-\sqrt{5}}{2}\right)^k+A\sin k+B\cos k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2}-7y_{k+1}+12y_k=\cos k$$

Solution: The given difference equation can be written as

$$E^2y_k-7Ey_k+12y_k=\cos k$$

$$(E^2-7E+12)y_k=\cos k \qquad \_\_\_\_\_(1)$$

For complementary solution

$$(E^2-7E+12)y_k=0$$

The auxiliary equation

$$m^2-7m+12=0$$

$$m=\frac{7\pm\sqrt{49-48}}{2}=\frac{7\pm1}{2}$$

$$m=4,3$$

$$Y=c_1(3)^k+c_2(4)^k$$

For particular solution

$$Let\ \ y_k=A\sin k+B\cos k \qquad Put\,in\,(1)$$

$$(E^2 - 7E + 12)\left(A\sin k + B\cos k\right) = \cos k$$

$$E^2\left(A\sin k + B\cos k\right) - 7E\left(A\sin k + B\cos k\right) + 12\left(A\sin k + B\cos k\right) = \cos k$$

$$\left(A\sin(k+2) + B\cos(k+2)\right) - 7\left(A\sin(k+1) + B\cos(k+1)\right) + 12\left(A\sin k + B\cos k\right) = \cos k$$

$$A\left[\sin k\cos 2 + \sin 2\cos k\right] + B\left[\cos k\cos 2 - \sin k\sin 2\right] - 7A\left[\sin k\cos 1 + \sin 1\cos k\right]$$
$$-7B\left[\cos k\cos 1 + \sin k\sin 1\right] + 12A\sin k + 12B\cos k = \cos k$$

$$A\left[\sin k\cos 2 + \sin 2\cos k - 7\sin k\cos 1 - 7\sin 1\cos k + 12\sin k\right]$$

$$+B\left[\cos k\cos 2 - \sin k\sin 2 - 7\cos k\cos 1 + 7\sin k\sin 1 + 12\cos k\right] = \cos k$$

$$A\left[\sin k\left(\cos 2 - 7\cos 1 + 12\right) + \cos k\left(\sin 2 - 7\sin 1\right)\right]$$

$$+B\left[\cos k\left(\cos 2 - 7\cos 1 + 12\right) - \sin k\left(\sin 2 - 7\sin 1\right)\right] = \cos k$$

Say $M = \cos 2 - 7\cos 1 + 12$ & $N = \sin 2 - 7\sin 1$

$$A\left[M\sin k + N\cos k\right] + B\left[M\cos k - N\sin k\right] = \cos k$$

$$\left(AM - BN\right)\sin k + \left(AN + BM\right)\cos k = \cos k$$

By comparing coefficients of sink and cosk

$$AM - BN = 1$$

$$AN + BM = 0$$

Now by cross multiplication

$$AM - BN - 1 = 0$$

$$AN + BM - 1 = 0$$

$$\frac{A}{N - 0} = \frac{-B}{-M - 0} = \frac{1}{M^2 + N^2}$$

$$A = \frac{N}{M^2 + N^2} \quad , \quad B = \frac{M}{M^2 + N^2}$$

$$Y^* = A\sin k + B\cos k$$

$$Y_k = Y + Y^*$$

$Y_k = c_1(3)^k + c_2(4)^k + A\sin k + B\cos k$ is the required general solution.

Question: $\Delta^2 y_k + \Delta y_k = \cos k$

Solution: $\left(\Delta^2 + \Delta\right) y_k = \cos k$

$$\left((E-1)^2 + E - 1\right) y_k = \cos k$$

$$\left(E^2 - 2E + 1 + E - 1\right) y_k = \cos k$$

$$\left(E^2 - E\right) y_k = \cos k \qquad \_\_\_\_\_\_\_(1)$$

For complementary solution

$$\left(E^2 - E\right) y_k = 0$$

The auxiliary equation is

$$m^2 - m = 0$$

$$m(m-1) = 0$$

$$m = 0, 1$$

$$Y = c_1(0)^k + c_2(1)^k$$

$$Y = c_2$$

For particular solution

$$Let \;\; y_k = A\sin k + B\cos k \qquad Put\, in\, (1)$$

$$\left(E^2 - E\right)\left(A\sin k + B\cos k\right) = 0$$

$$E^2\left(A\sin k + B\cos k\right) - E\left(A\sin k + B\cos k\right) = \cos k$$

$$\left(A\sin(k+2) + B\cos(k+2)\right) - A\sin(k+1) - B\cos(k+1) = \cos k$$

$$A\left[\sin k\cos 2 + \sin 2\cos k\right] + B\left[\cos k\cos 2 - \sin k\sin 2\right] - A\left[\sin k\cos 1 + \sin 1\cos k\right]$$

$-B\left[\cos k\cos 1-\sin k\sin 1\right]=\cos k$

$A\left[\sin k\cos 2+\sin 2\cos k-\sin k\cos 1-\sin 1\cos k\right]$

$+B\left[\cos k\cos 2+\text{sinksin}\,2-\cos k\cos 1+\text{sinksin}\,1\right]=\text{cosk}$

$A\left[\sin k\left(\cos 2-\cos 1\right)+\cos k\left(\sin 2-\sin 1\right)\right]+B\left[\cos k\left(\cos 2-\cos 1\right)-\sin k\left(\sin 2-\sin 1\right)\right]=\cos k$

Say $\quad \text{M}=\cos 2-\cos 1$ & $N=\sin 2-\sin 1$

$A\left[\text{M}\sin k+N\cos k\right]+B\left[M\cos k-N\sin k\right]=\cos k$

$\left(AM-BN\right)\sin k+\left(AN+BM\right)\cos k=\cos k$

By comparing coefficients of sink and cosk

$$\text{AM} - \text{BN} = 0$$

$$\text{AN} + \text{BM} = 1$$

By cross multiplication

$$\frac{A}{N-0}=\frac{-B}{-M-0}=\frac{1}{M^2+N^2}$$

$$A=\frac{N}{M^2+N^2}\ ,\quad B=\frac{M}{M^2+N^2}$$

$$Y^*=A\sin k+B\cos k$$

$$Y_k=Y+Y^*$$

$Y_k=c_2+A\sin k+B\cos k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2}-7y_{k+1}+12y_k=\sin 3k+2+2^k$$

Solution: The given difference equation can be written as

$$E^2y_k-7Ey_k+12y_k=\sin 3k+2+2^k$$

$$(E^2-7E+12)y_k=\sin 3k+2+2^k$$

For complementary solution

$$(E^2 - 7E + 12)y_k = 0$$

The auxiliary equation

$$m^2 - 7m + 12 = 0$$

$$m = \frac{7 \pm \sqrt{49-48}}{2} = \frac{7 \pm 1}{2}$$

$$m = 4, 3$$

$$Y = c_1(3)^k + c_2(4)^k$$

For particular solution

$$(E^2 - 7E + 12)y_k = \sin 3k \quad \_\_\_(i)$$

$$(E^2 - 7E + 12)y_k = 2 \quad \_\_\_(ii)$$

$$(E^2 - 7E + 12)y_k = 2^k \quad \_\_\_(iii)$$

$$Let \;\; y_k = A\sin 3k + B\cos 3k \qquad Put\, in\, (i)$$

$$(E^2 - 7E + 12)\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$E^2\left(A\sin 3k + B\cos 3k\right) - 7E\left(A\sin 3k + B\cos 3k\right) + 12\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$\left(A\sin 3(k+2) + B\cos 3(k+2)\right) - 7\left(A\sin 3(k+1) + B\cos 3(k+1)\right) + 12\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$\left(A\sin(3k+6) + B\cos(3k+6)\right) - 7\left(A\sin(3k+3) + B\cos(3k+3)\right) + 12\left(A\sin 3k + B\cos 3k\right) = \sin 3k$$

$$A\sin(3k+6) + B\cos(3k+6) - 7A\sin(3k+3) - 7B\cos(3k+3) + 12A\sin 3k + 12B\cos 3k = \sin 3k$$

$$A\left[\sin 3k\cos 6 + \sin 6\cos 3k\right] + B\left[\cos 3k\cos 6 - \sin 3k\sin 6\right] - 7A\left[\sin 3k\cos 3 + \sin 3\cos 3k\right]$$
$$-7B\left[\cos 3k\cos 3 - \sin 3k\sin 3\right] + 12A\sin 3k + 12B\cos 3k = \sin 3k$$

$$A\left[\sin 3k\cos 6 + \sin 6\cos 3k - 7\sin 3k\cos 3 - 7\sin 3\cos 3k + 12\sin 3k\right]$$

$$+B\left[\cos 3k\cos 6 - \sin 3k\sin 6 - 7\cos 3k\cos 3 + 7\sin 3k\sin 3 + 12\cos 3k\right] = \sin 3k$$

$$A\left[\sin 3k\left(\cos 6 - 7\cos 3 + 12\right) + \cos 3k\left(\sin 6 - 7\sin 3\right)\right]$$

$$+B\left[\cos 3k\left(\cos 6-7\cos 3+12\right)-\sin 3k\left(\sin 6-7\sin 3\right)\right]=\sin 3k$$

Say $\quad \mathrm{M}=\cos 6-7\cos 3+12 \quad \& \quad N=\sin 6-7\sin 3$

$$A\left[\mathrm{M}\sin 3k+N\cos 3k\right]+B\left[M\cos 3k-N\sin 3k\right]=\sin 3k$$

$$\left(AM-BN\right)\sin 3k+\left(AN+BM\right)\cos 3k=\sin 3k$$

By comparing coefficients of sin3k and cos3k

$$\mathrm{AM}-\mathrm{BN}=1$$

$$\mathrm{AN}+\mathrm{BM}=0$$

Now by cross multiplication

$$\mathrm{AM}-\mathrm{BN}-1=0$$

$$\mathrm{AN}+\mathrm{BM}+0=0$$

$$\frac{A}{-0+M}=\frac{-B}{0+N}=\frac{1}{M^2+N^2}$$

$$A=\frac{M}{M^2+N^2}, \quad B=\frac{-N}{M^2+N^2}$$

$$Y_1^*=A\sin 3k+B\cos 3k$$

For equation (ii)

$$Let \;\; y_k=C \qquad\qquad Put\,in\,(ii)$$

$$(E^2-7E+12)C=2$$

$$(E^2C-7EC+12C)=2$$

$$C-7C+12C=2$$

$$6C=2 \qquad \Rightarrow \; C=\frac{1}{3}$$

$$Y_2^*=\frac{1}{3}$$

$$Let\ \ y_k = D\,2^k \qquad\qquad Put\,in\,(iii)$$

$$(E^2 - 7E + 12)\,D\,2^k = 2^k$$

$$(E^2\,D\,2^k - 7E\,D\,2^k + 12\,D\,2^k) = 2^k$$

$$D\,2^{k+2} - 7\,D\,2^{k+1} + 12\,D\,2^k = 2^k$$

$$4\,D - 14\,D + 12\,D = 1$$

$$2\,D = 1$$

$$D = \frac{1}{2}$$

$$Y_3^* = \frac{1}{2}2^k$$

$$Y^* = Y_1^* + Y_2^* + Y_3^*$$

$$Y^* = A\sin 3k + B\cos 3k + \frac{1}{3} + \frac{1}{2}2^k$$

$$Y_k = Y + Y^*$$

$Y_k = c_1(3)^k + c_2(4)^k + A\sin 3k + B\cos 3k + \frac{1}{3} + \frac{1}{2}2^k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 4y_{k+1} + y_k = \cos^2 k$$

Solution: The given difference equation can be written as

$$E^2 y_k - 4Ey_k + y_k = \cos^2 k$$

$$(E^2 - 4E + 1)y_k = \cos^2 k \quad \_\_\_\_\_(i)$$

For complementary solution

$$(E^2 - 4E + 1)y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 1 = 0$$

$$m = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2}$$

$$m = 2 \pm \sqrt{3}$$

$$m = 2+\sqrt{3}\ ,\ 2-\sqrt{3}$$

$$Y = c_1(2+\sqrt{3})^k + c_2(2-\sqrt{3})^k$$

For particular solution

$$(E^2 - 4E + 1)\,y_k = \cos^2 k = \frac{1+\cos 2k}{2} = \frac{1}{2} + \frac{1}{2}\cos 2k$$

$$(E^2 - 4E + 1)\,y_k = \frac{1}{2} \qquad \_\_\_\_(i)$$

$$(E^2 - 4E + 1)\,y_k = \frac{1}{2}\cos 2k \qquad \_\_\_\_(ii)$$

For equation (i)

Let $y_k = A$ $\quad$ *put in* (i)

$$(E^2 - 4E + 1)A = \frac{1}{2}$$

$$(E^2 A - 4EA + A) = \frac{1}{2}$$

$$A - 4A + A = \frac{1}{2}$$

$$-2A = \frac{1}{2}$$

$$A = \frac{-1}{4}$$

$$Y_1^* = \frac{-1}{4}$$

For equation (ii)

$$Let \quad y_k = B\sin 2k + C\cos 2k \qquad\qquad Put\,in\,(ii)$$

$$(E^2 - 4E + 1)\left(B\sin 2k + C\cos 2k\ \right) = \frac{1}{2}\cos 2\mathrm{k} \qquad \_\_\_\_(\mathrm{ii})$$

$$E^2\left(\mathrm{B}\sin 2k + C\cos 2k\right) - 4E\left(\mathrm{B}\sin 2k + C\cos 2k\right) + \left(\mathrm{B}\sin 2k + C\cos 2k\right) = \frac{1}{2}\cos 2k$$

$$\left(\mathrm{B}\sin 2(k+2) + C\cos 2(k+2)\right) - 4\left(\mathrm{B}\sin 2(k+1) + C\cos 2(k+1)\right) + \left(\mathrm{B}\sin 2k + C\cos 2k\right) = \frac{1}{2}\cos 2k$$

$$\mathrm{B}\sin(2k+4) + C\cos(2k+4) - 4\mathrm{B}\sin(2k+2) - 4C\cos(2k+2) + \mathrm{B}\sin 2k + C\cos 2k = \frac{1}{2}\cos 2k$$

$$B\left[\sin 2k\cos 4 + \sin 4\cos 2k\right] + C\left[\cos 2k\cos 4 - \sin 2k\sin 4\right] - 4B\left[\sin 2k\cos + \sin 2\cos 2k\right]$$

$$-4C\left[\cos 2k\cos 2 - \sin 2k\sin 2\right] + B\sin 2k + C\cos 2k = \frac{1}{2}\cos 2k$$

$$B\left[\sin 2k\cos 4 + \sin 4\cos 2k - 4\sin 2k\cos 2 - 4\sin 2\cos 2k + \sin 2k\right]$$

$$+C\left[\cos 2k\cos 4 - \sin 2k\sin 4 - 4\cos 2k\cos 2 + 4\sin 2k\sin 2 + \cos 2k\right] = \frac{1}{2}\cos 2k$$

$$B\left[\sin 2k\left(\cos 4 - 4\cos 2 + 1\right) + \cos 2k\left(\sin 4 - 4\sin 2\right)\right]$$

$$+C\left[\cos 2k\left(\cos 4 - 4\cos 2 + 1\right) - \sin 2k\left(\sin 4 - 4\sin 2\right)\right] = \frac{1}{2}\cos 2k$$

Say $\quad \mathrm{M} = \cos 4 - 4\cos 2 + 1 \quad \& \quad N = \sin 4 - 4\sin 2$

$$B\left[\mathrm{M}\sin 2k + N\cos 2k\right] + C\left[M\cos 2k - N\sin 2k\right] = \frac{1}{2}\cos 2k$$

$$\left(BM - CN\right)\sin 2k + \left(BN + BM\right)\cos 2k = \frac{1}{2}\cos 2k$$

By comparing coefficients of sin2k and cos2k

$$\mathrm{BM} - \mathrm{CN} = 0$$

$$\text{BN} + \text{CM} = \frac{1}{2}$$

$$\Rightarrow \qquad 2BN + 2CM = 1$$

Now by cross multiplication

$$\text{BM} - \text{CN} + 0 = 0$$

$$2\text{BN} + 2\text{CM} - 1 = 0$$

$$\frac{B}{N-0} = \frac{C}{-M-0} = \frac{1}{2M^2 + 2N^2}$$

$$B = \frac{N}{2\left(M^2 + N^2\right)} \quad , \quad C = \frac{-M}{2\left(M^2 + N^2\right)}$$

$$Y_2^* = B\sin 2k + C\cos 2k$$

$$Y^* = Y_1^* + Y_2^*$$

$$Y^* = \frac{-1}{4} + B\sin 2k + C\cos 2k$$

$$\Rightarrow \quad Y_k = Y + Y^*$$

$Y_k = c_1(2+\sqrt{3})^k + c_2(2-\sqrt{3})^k - \frac{1}{4} + B\sin 2k + C\cos 2k$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$$

Solution: The given difference equation can be written as

$$E^2 y_k - 2Ey_k + y_k = \sin 5k + \cos 5k + 6$$

$$(E^2 - 2E + 1)y_k = \sin 5k + \cos 5k + 6$$

For complementary solution

$$(E^2 - 2E + 1)y_k = 0$$

The auxiliary equation

$$m^2-2m+1=0$$

$$(m-1)^2=0$$

$$m=1,1$$

$$Y=(c_1+c_2k)(1)^k$$

$$Y=(c_1+c_2k)$$

For particular solution

$$(E^2-2E+1)y_k=\sin 5k+\cos 5k \qquad \_\_\_\_(i)$$

$$(E^2-2E+1)y_k=6 \qquad \_\_\_\_(ii)$$

For equation (i)

$$Let\ \ y_k=A\sin 5k+B\cos 5k \qquad Put\ in\ (i)$$

$$(E^2-2E+1)(A\sin 5k+B\cos 5k)=\sin 5k+\cos 5k$$

$$E^2(A\sin 5k+B\cos 5k)-2E(A\sin 5k+B\cos 5k)+(A\sin 5k+B\cos 5k)=\sin 5k+\cos 5k$$

$$A\sin(5k+10)+B\cos(5k+10)-2A\sin(5k+5)-2B\cos(5k+5)+A\sin 5k+B\cos 5k=\sin 5k+\cos 5k$$

$$A[\sin 5k\cos 10+\sin 10\cos 5k]+B[\cos 5k\cos 10-\sin 5k\sin 10]-2A[\sin 5k\cos 5+\sin 5\cos 5k]$$
$$-2B[\cos 5k\cos 5-\sin 5k\sin 5]+A\sin 5k+B\cos 5k=\sin 5k+\cos 5k$$

$$A[\sin 5k\cos 10+\sin 10\cos 5k-2\sin 5k\cos 5-2\sin 5\cos 5k+\sin 5k]$$

$$+B[\cos 5k\cos 10-\sin 5k\sin 10-2\cos 5k\cos 5+2\sin 5k\sin 5+\cos 5k]=\sin 5k+\cos 5k$$

$$A\left[\sin 5k(\cos 10-2\cos 5+1)+\cos 5k(\sin 10-2\sin 5)\right]$$

$$+B\left[\cos 5k(\cos 10-2\cos 5+1)-\sin 5k(\sin 10-2\sin 5)\right]=\sin 5k+\cos 5k$$

Say $\quad \mathrm{M}=\cos 10-2\cos 5+1 \quad \& \quad N=\sin 10-2\sin 5$

$$A[\mathrm{M}\sin 5k+N\cos 5k]+B[M\cos 5k-N\sin 5k]=\sin 5k+\cos 5k$$

$$\left(AM - BN\right)\sin 5k + \left(AN + BM\right)\cos 5k = \sin 5k + \cos 5k$$

By comparing coefficients of sin5k and cos5k

$$AM - BN = 1$$

$$AN + BM = 1$$

Now by cross multiplication

$$AM - BN - 1 = 0$$

$$AN + BM - 1 = 0$$

$$\frac{A}{N+M} = \frac{-B}{-M+N} = \frac{1}{M^2+N^2}$$

$$\Rightarrow \quad A = \frac{N+M}{M^2+N^2} \quad , \quad B = \frac{M-N}{M^2+N^2}$$

$$Y_1^* = A\sin 5k + B\cos 5k$$

Now for equation (ii)

$$Let \quad y_k = Ck^2 \quad put\ in\ (ii)$$

$$(E^2 - 2E + 1)Ck^2 = 6$$

$$C\left(k+2\right)^2 - 2C\left(k+1\right)^2 + Ck^2 = 6$$

$$C\left(k^2 + 4k + 4\right) - 2C\left(k^2 + 2k + 1\right) + Ck^2 = 6$$

$$Ck^2 + 4Ck + 4C - 2Ck^2 - 4Ck - 2C + Ck^2 = 6$$

$$2C = 6$$

$$\Rightarrow \quad C = 3$$

$$\Rightarrow \quad Y_2^* = 3k^2$$

$$Y^* = Y_1^* + Y_2^*$$

$$Y^* = A\sin 5k + B\cos 5k + 3k^2$$

$$Y_k = Y + Y^*$$

$\Rightarrow \quad Y_k = (c_1 + c_2 k) + A\sin 5k + B\cos 5k + 3k^2$ is the required general solution.

Non-Homogenous Difference Equation:

Type VI:

When the R.H.S of the given non-homogeneous difference equation is of the form $\quad f(k) = a^k \sin Ak \ \ or\, a^k \cos Ak$

Where a and A is constant, then in order to find particular solution Y* we shall substitute $y_k = a^k (c_1 \sin Ak + c_2 \cos Ak)$ in the given difference equation and then evaluate the values of c_1 and c_2. If the trial function or any term of trial function is present in complementary solution Y then the trial function will be multiplied by suitable k^n.

Question: Solve the difference equation

$$y_{k+2} + 13y_{k+1} + 3y_k = 3^k \cos 4k$$

Solution: The given difference equation can be written as

$$E^2 y_k + 13Ey_k + 3y_k = 3^k \cos 4k$$

$$(E^2 + 13E + 3)y_k = 3^k \cos 4k \qquad \_\_\_\_\_(i)$$

For complementary solution

$$(E^2 + 13E + 3)y_k = 0$$

The auxiliary equation

$$m^2 + 13m + 3 = 0$$

$$m = \frac{-13 \pm \sqrt{169 - 12}}{2} = \frac{-13 \pm \sqrt{157}}{2}$$

$$m = \frac{-13 + \sqrt{157}}{2}, \frac{-13 - \sqrt{157}}{2}$$

$$Y = c_1\left(\frac{-13+\sqrt{157}}{2}\right)^k + c_2\left(\frac{-13-\sqrt{157}}{2}\right)^k$$

For particular solution

$$Let\quad y_k = 3^k\left(A\sin 4k + B\cos 4\text{k}\right) \qquad\qquad Put\,in\,(i)$$

$$(E^2+13E+3)3^k\left(A\sin 4k + B\cos 4\text{k}\right) = 3^k\cos 4k$$

$$E^2.3^k\left(A\sin 4k + B\cos 4k\right) + 13E.3^k\left(A\sin 4k + B\cos 4k\right) + 3.3^k\left(A\sin 4k + B\cos 4k\right) = 3^k\cos 4k$$

$$3^{k+2}\left(A\sin(4k+8) + B\cos(4k+8)\right) + 13.3^{k+1}\left(A\sin(4k+4) + B\cos(4k+4)\right) + 3.3^k\left(A\sin 4k + B\cos 4k\right) = 3^k\cos 4k$$

$$9\left(A\sin(4k+8) + B\cos(4k+8)\right) + 39\left(A\sin(4k+4) + B\cos(4k+4)\right) + 3\left(A\sin 4k + B\cos 4k\right) = \cos 4k$$

$$9\left[A\left(\sin 4k\cos 8 + \sin 8\cos 4k\right) + B\left(\cos 4k\cos 8 - \sin 4k\sin 8\right)\right]$$

$$+39\left[A\left(\sin 4k\cos 4 + \sin 4\cos 4k\right) + B\left(\cos 4k\cos 4 - \sin 4k\sin 4\right)\right]$$

$$+3A\sin 4k + 3B\cos 4k = \cos 4k$$

$$A\left[9\sin 4k\cos 8 + 9\sin 8\cos 4k + 39\sin 4k\cos 4 + 39\sin 4\cos 4\text{k} + 3\sin 4\text{k}\right]$$

$$+B\left[9\cos 4k\cos 8 - 9\sin 4k\sin 8 + 39co\text{s}\,4k\cos 4 - 39\sin 4k\sin 4 + 3co\text{s}\,4\right] = \cos 4k$$

$$A\left[\sin 4k\left(9\cos 8 + 39\cos 4 + 3\right) + \cos 4k\left(9\sin 8 + 39\sin 4\right)\right]$$

$$+B\left[\cos 4k\left(9\cos 8 + 39\cos 4 + 3\right) - \sin 4k\left(9\sin 8 + 39\sin 4\right)\right] = \cos 4k$$

Say $\quad \text{M} = 9\cos 8 + 39\cos 4 + 3$ & $\qquad N = 9\sin 8 + 39\sin 4$

$$A\left[\text{M}\sin 4k + N\cos 4k\right] + B\left[M\cos 4k - N\sin 4k\right] = \cos 4k$$

$$\left(AM - BN\right)\sin 4k + \left(AN + BM\right)\cos 4k = \cos 4k$$

By comparing coefficients of sin4k and cos4k

$$\text{AM} - \text{BN} = 0$$

$$\text{AN} + \text{BM} = 1$$

Now by cross multiplication

$$\text{AM} - \text{BN} + 0 = 0$$

$$AN + BM - 1 = 0$$

M	$-N$	0	M	$-N$	0
N	M	-1	N	M	-1

$$\frac{A}{N-0} = \frac{-B}{-M-0} = \frac{1}{M^2+N^2}$$

$$A = \frac{N}{M^2+N^2} \quad , \quad B = \frac{M}{M^2+N^2}$$

$$Y^* = 3^k\left(A\sin 4k + B\cos 4k\right)$$

$$Y_k = Y + Y^*$$

$Y_k = c_1\left(\dfrac{-13+\sqrt{157}}{2}\right)^k + c_2\left(\dfrac{-13-\sqrt{157}}{2}\right)^k + 3^k\left(A\sin 4k + B\cos 4k\right)$ is the required general solution.

Question: Solve the difference equation

$$y_{k+2} + y_{k+1} + y_k = 2^k \sin k \cos 3k$$

Solution: The given difference equation can be written as

$$E^2 y_k + E y_k + y_k = 2^k \sin k \cos 3k$$

$$(E^2 + E + 1) y_k = 2^k \sin k \cos 3k$$

For complementary solution

$$(E^2 + E + 1) y_k = 0$$

The auxiliary equation

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$|m| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}\left(-\sqrt{3}\right)$$

$$Y = (1)^k \left(c_1 \cos\theta k + c_2 \sin\theta k\right)$$

$$Y = \left(c_1 \cos\theta k + c_2 \sin\theta k\right)$$

For particular solution

$$(E^2 + E + 1)y_k = \frac{2^k}{2}\left(2\,\text{sink}\cos 3k\right)$$

$$(E^2 + E + 1)y_k = \frac{2^k}{2}\left(\sin(1+3)k + \sin(1-3)k\right)$$

$$(E^2 + E + 1)y_k = \frac{2^k}{2}\left[\sin 4k - \sin 2k\right]$$

$$(E^2 + E + 1)y_k = \frac{1}{2}2^k \sin 4k - \frac{1}{2}2^k \sin 2k$$

$$(E^2 + E + 1)y_k = \frac{1}{2}2^k \sin 4k \qquad \_\_\_\_(i)$$

$$(E^2 + E + 1)y_k = -\frac{1}{2}2^k \sin 2k \qquad \_\_\_\_(ii)$$

For equation (i)

$$Let \quad y_k = 2^k\left(A\sin 4k + B\cos 4k\right) \qquad Put\,in\,(i)$$

$$(E^2 + E + 1)2^k\left(A\sin 4k + B\cos 4k\right) = \frac{1}{2}2^k \sin 4k$$

$$E^2.2^k\left(A\sin 4k+B\cos 4k\right)+E.2^k\left(A\sin 4k+B\cos 4k\right)+2^k\left(A\sin 4k+B\cos 4k\right)=\frac{1}{2}2^k\sin 4k$$

$$2^{k+2}\left[A\sin(4k+8)+B\cos(4k+8)\right]+2^{k+1}\left[A\sin(4k+4)+B\cos(4k+4)\right]+2^k\left(A\sin 4k+B\cos 4k\right)=\frac{1}{2}2^k\sin 4k$$

$$4A[\sin 4k\cos 8+\sin 8\cos 4k]+4B[\cos 4k\cos 8-\sin 4k\sin 8]+2A[\sin 4k\cos 4+\sin 4\cos 4k]$$

$$+2B[\cos 4k\cos 4-\sin 4k\sin 4]+A\sin 4k+B\cos 4k=\frac{1}{2}\sin 4k$$

$$A\left[4\sin 4k\cos 8+4\sin 8\cos 4k+2\sin 4k\cos 4+2\sin 4\cos 4k+\sin 4k\right]$$

$$+B\left[4\cos 4k\cos 8-4\sin 4k\sin 8+2\cos 4k\cos 4-2\sin 4k\sin 4+\cos 4k\right]=\frac{1}{2}\sin 4k$$

$$A\left[\sin 4k\left(4\cos 8+2\cos 4+1\right)+\cos 4k\left(4\sin 8+2\sin 4\right)\right]$$

$$+B\left[\cos 4k\left(4\cos 8+2\cos 4+1\right)-\sin 4k\left(4\sin 8+2\sin 4\right)\right]=\frac{1}{2}\sin 4k$$

$$Say\ \ M=4\cos 8+2\cos 4+1\ \ \&\ \ N=4\sin 8+2\sin 4$$

$$A\left[M\sin 4k+N\cos 4k\right]+B\left[M\cos 4k-N\sin 4k\right]=\frac{1}{2}\sin 4k$$

$$\left(AM-BN\right)\sin 4k+\left(AN+BM\right)\cos 4k=\frac{1}{2}\sin 4k$$

By comparing coefficients of sin4k and cos4k

$$AM-BN=\frac{1}{2}$$

$$2AM-2BN=1$$

$$AN+BM=0$$

By cross multiplication

$$2AM-2BN-1=0$$

$$AN+BM+0=0$$

$$\frac{A}{0+M}=\frac{-B}{0+N}=\frac{1}{2M^2+2N^2}$$

$$A=\frac{M}{2\left(M^2+N^2\right)} \quad , \quad B=\frac{-N}{2\left(M^2+N^2\right)}$$

$$Y_1^*=2^k\left(A\sin 4k+B\cos 4k\right)$$

Now for equation (ii)

$$Let \;\; y_k=2^k\left(\mathrm{C}\sin 2k+D\cos 2k\right) \qquad\qquad Put\,in\,(ii)$$

$$(E^2+E+1)2^k\left(\mathrm{C}\sin 2k+D\cos 2k\right)=-\frac{1}{2}2^k\sin 2k$$

$$E^2.2^k\left(\mathrm{C}\sin 2k+D\cos 2k\right)+E.2^k\left(\mathrm{C}\sin 2k+D\cos 2k\right)+2^k\left(\mathrm{D}\sin 2k+C\cos 2k\right)=\frac{-1}{2}2^k\sin 2k$$

$$2^{k+2}\left[\mathrm{C}\sin(2k+4)+D\cos(2k+4)\right]+2^{k+1}\left[\mathrm{C}\sin(2k+2)+D\cos(2k+2)\right]+2^k\left(\mathrm{C}\sin 2k+D\cos 2k\right)=\frac{-1}{2}2^k\sin 2k$$

$$4C\left[\sin 2k\cos 4+\sin 4\cos 4k\right]+4D\left[\cos 2k\cos 4-\sin 2k\sin 4\right]+2C\left[\sin 2k\cos 2+\sin 2\cos 2k\right]$$

$$+2D\left[\cos 2k\cos 2-\sin 2k\sin 2\right]+C\sin 2k+D\cos 2k=-\frac{1}{2}\sin 2\mathrm{k}$$

$$C\left[4\sin 2k\cos 4+4\sin 4\cos 2k+2\sin 2k\cos 2+2\sin 2\cos 2k+\sin 2k\right]$$

$$+D\left[4\cos 2k\cos 4-4\sin 2k\sin 4+2\cos 2k\cos 2-2\sin 2k\sin 2+\cos 2k\right]=-\frac{1}{2}\sin 2k$$

$$C\left[\sin 2k\left(4\cos 4+2\cos 2+1\right)+\cos 2k\left(4\sin 4+2\sin 2\right)\right]$$

$$+D\left[\cos 2k\left(4\cos 4+2\cos 2+1\right)-\sin 2\mathrm{k}\left(4\sin 2+2\sin 2\right)\right]=-\frac{1}{2}\sin 2k$$

$$Say \;\; M=4\cos 4+2\cos 2+1 \;\;\&\;\; N=4\sin 4+2\sin 2$$

$$C\left[\mathrm{M}\sin 2k+N\cos 2k\right]+D\left[M\cos 2k-N\sin 2k\right]=-\frac{1}{2}\sin 2k$$

$$\left(CM-DN\right)\sin 2k+\left(CN+DM\right)\cos 2k=-\frac{1}{2}\sin 2k$$

By comparing coefficients of sin2k and cos2k

$$CM - DN = -\frac{1}{2} \Rightarrow 2CM - 2DN = -1$$

$$CN + DM = 0$$

Now by cross multiplication

$$2CM - 2DN + 1 = 0$$

$$CN + DM + 0 = 0$$

$$\frac{C}{-0-M} = \frac{-D}{0-N} = \frac{1}{2M^2 + 2N^2}$$

$$C = \frac{-M}{2\left(M^2 + N^2\right)} \quad , \quad D = \frac{N}{2\left(M^2 + N^2\right)}$$

$$Y_2^* = 2^k\left(C\sin 2k + D\cos 2k\right)$$

$$Y^* = Y_1^* + Y_2^*$$

$$Y^* = 2^k\left(A\sin 4k + B\cos 4k\right) + 2^k\left(C\sin 2k + D\cos 2k\right)$$

$$Y_k = Y + Y^* \Rightarrow Y_k = \left(c_1\cos\theta k + c_2\sin\theta k\right) + 2^k\left(A\sin 4k + B\cos 4k\right) + 2^k\left(C\sin 2k + D\cos 2k\right)$$

is the required general solution.

Lecture # 08

Type VII

Simultaneous Difference Equation:

Question: Solve $2x + 4y = 7$, $x - 8y = 9$

Solution: $2x + 4y = 7$ ______(i)

$x - 8y = 9$ ______(ii)

Multiplying (i) by 2 and add with (ii)

$$4x + 8y = 14$$

$$\underline{x - 8y = 9}$$

$$5x = 23$$

$$x = \frac{23}{5}$$

Put in (ii) $\Rightarrow \frac{23}{5} - 8y = 9$

$$\Rightarrow 8y = \frac{23}{5} - 9 = \frac{23-45}{5}$$

$$\Rightarrow 8y = \frac{-22}{5}$$

$$\Rightarrow y = \frac{-22}{5\times 8} = \frac{-11}{20}$$

$$S.S = \left\{\frac{23}{5}, \frac{-11}{20}\right\}$$

Question: Solve the following system of D.E

$$u_{n+1} - u_n + 3v_n = 7$$

$$3v_{n+1} + v_n + 2u_n = 6$$

Solution:

The given system can be written as

$$Eu_n - u_n + 3v_n = 7$$

$$3Ev_n + v_n + 2u_n = 6$$

$$(E-1)u_n + 3v_n = 7 \qquad \_\_\_\_(i)$$

$$(3E+1)v_n + 2u_n = 6 \qquad \_\_\_\_(ii)$$

From (i) $\quad 3v_n = 7 - (E-1)u_n$

$$v_n = \frac{7}{3} - \frac{1}{3}(E-1)u_n \qquad \_\_\_\_(iii)$$

Put in (ii) $\Rightarrow \quad (3E+1)\left(\frac{7}{3} - \frac{1}{3}(E-1)u_n\right) + 2u_n = 6$

$$(3E+1)\frac{7}{3} - \frac{1}{3}(E-1)(3E+1)u_n + 2u_n = 6$$

$$3E\left(\frac{7}{3}\right) + \frac{7}{3} - \frac{1}{3}(3E^2 - 2E - 1)u_n + 2u_n = 6$$

$$3\left(\frac{7}{3}\right) + \frac{7}{3} - \frac{1}{3}(3E^2 - 2E - 1)u_n + 2u_n = 6$$

Multiplying by 3 $\Rightarrow \quad 21 + 7 - (3E^2 - 2E - 1)u_n + 6u_n = 18$

$$28 - (3E^2 - 2E - 1 - 6)u_n = 18$$

$$(3E^2 - 2E - 7)u_n = 10 \qquad \_\_\_\_(iv)$$

For complementary solution we have

$$\left(3E^2-2E-7\right)u_n=0$$

The auxiliary equation

$$3m^2-2m-7=0$$

$$m=\frac{2\pm\sqrt{4+84}}{6}=\frac{2\pm\sqrt{88}}{6}$$

$$m=\frac{2\pm2\sqrt{22}}{6}=\frac{1\pm\sqrt{22}}{3}$$

$$m=\frac{1}{3}\pm\frac{\sqrt{22}}{3}$$

$$u=c_1\left(\frac{1}{3}-\frac{\sqrt{22}}{3}\right)^n+c_2\left(\frac{1}{3}+\frac{\sqrt{22}}{3}\right)^n$$

For particular solution we have

Let $u_n=A$ put in (iv)

$$\left(3E^2-2E-7\right)A=10$$

$$3E^2A-2EA-7A=10$$

$$3A-2A-7A=10$$

$$A=\frac{-5}{3}$$

$$\Rightarrow\quad u^*=\frac{-5}{3}$$

$$Thus,\quad u_n=u+u^*$$

$$u_n=c_1\left(\frac{1}{3}-\frac{\sqrt{22}}{3}\right)^n+c_2\left(\frac{1}{3}+\frac{\sqrt{22}}{3}\right)^n-\frac{5}{3}\qquad \text{\_\_\_\_(v)}$$

Put in (iii) $\Rightarrow$ $$v_n=\frac{7}{3}-\frac{1}{3}(E-1)\left[c_1\left(\frac{1}{3}-\frac{\sqrt{22}}{3}\right)^n+c_2\left(\frac{1}{3}+\frac{\sqrt{22}}{3}\right)^n-\frac{5}{3}\right]$$

$$v_n=\frac{7}{3}-\frac{1}{3}\left[c_1E\left(\frac{1}{3}-\frac{\sqrt{22}}{3}\right)^n+c_2E\left(\frac{1}{3}+\frac{\sqrt{22}}{3}\right)^n-E\frac{5}{3}-c_1\left(\frac{1}{3}-\frac{\sqrt{22}}{3}\right)^n+c_2\left(\frac{1}{3}+\frac{\sqrt{22}}{3}\right)^n+\frac{5}{3}\right]$$

$$v_n=\frac{7}{3}-\frac{1}{3}\left[c_1\left(\frac{1}{3}-\frac{\sqrt{22}}{3}\right)^{n+1}+c_2\left(\frac{1}{3}+\frac{\sqrt{22}}{3}\right)^{n+1}-c_1\left(\frac{1}{3}-\frac{\sqrt{22}}{3}\right)^n+c_2\left(\frac{1}{3}+\frac{\sqrt{22}}{3}\right)^n\right] \quad \text{\_(iv)}$$

From (v) and (vi)

$S.S=\{u_n,v_n\}$ is the required general solution.

Question: Solve the following system of D.E

$$u_{n+1}+u_n-3v_n=2^n$$

$$3v_{n+1}-v_n+4u_n=n^2$$

Solution:

The given system can be written as

$$(E+1)u_n-3v_n=2^n \quad \text{\_\_\_\_(i)}$$

$$(3E-1)v_n+4u_n=n^2 \quad \text{\_\_\_\_}(ii)$$

From (i) $$3v_n=(E+1)u_n-2^n$$

$$v_n=\frac{1}{3}(E+1)u_n-\frac{1}{3}2^n \quad \text{\_\_\_\_\_\_}(iii)$$

Put in (ii) $\Rightarrow$ $$(3E-1)\left(\frac{1}{3}(E+1)u_n-\frac{1}{3}2^n\right)+4u_n=n^2$$

$$\frac{1}{3}(3E-1)(E+1)u_n-\frac{1}{3}(3E-1)2^n+4u_n=n^2$$

$$\frac{1}{3}\left(3E^2+2E-1\right)u_n-\frac{1}{3}\left(3.E2^n-2^n\right)+4u_n=n^2$$

$$\left(3E^2+2E-1\right)u_n-\left(3.2^{n+1}-2^n\right)+12u_n=3n^2$$

$$\left(3E^2+2E-1+12\right)u_n=3n^2+3.2^{n+1}-2^n$$

$$\left(3E^2+2E+11\right)u_n=3n^2+6.2^n-2^n$$

$$\left(3E^2+2E+11\right)u_n=3n^2+5.2^n \qquad \text{\_\_\_(iv)}$$

For complementary solution we have

$$\left(3E^2+2E+11\right)u_n=0$$

The auxiliary equation

$$3m^2+2m+11=0$$

$$m=\frac{-2\pm\sqrt{4-132}}{6}=\frac{-2\pm\sqrt{-128}}{6}$$

$$m=\frac{-2\pm4\sqrt{8}i}{6}$$

$$m=\frac{-1}{3}\pm\frac{2\sqrt{8}}{3}i$$

$$|m|=\sqrt{\left(\frac{-1}{3}\right)^2+\left(\frac{2\sqrt{8}}{3}\right)}=\sqrt{\frac{1}{9}+\frac{32}{9}}$$

$$|m|=\sqrt{\frac{1+32}{9}}=\sqrt{\frac{33}{9}}=\sqrt{\frac{11}{3}}$$

$$\theta=\tan^{-1}\left(\frac{\frac{2\sqrt{8}}{3}}{\frac{-1}{3}}\right)=\tan^{-1}\left(-2\sqrt{8}\right)$$

$$Thus \quad u = \left(\sqrt{\frac{11}{3}}\right)^n \left[c_1 \cos\theta n + c_2 \sin\theta n\right]$$

For particular solution

$$\left(3E^2 + 2E + 11\right)u_n = 3n^2 \qquad \text{\_\_\_\_(v)}$$

$$\left(3E^2 + 2E + 11\right)u_n = 5.2^n \qquad \text{\_\_\_\_(vi)}$$

Let $u_n = An^2 + Bn + C$ put in (v)

$$\left(3E^2 + 2E + 11\right)\left(An^2 + Bn + C\right) = 3n^2$$

$$3E^2\left(An^2 + Bn + C\right) + 2E\left(An^2 + Bn + C\right) + 11\left(An^2 + Bn + C\right) = 3n^2$$

$$3\left[A(n+2)^2 + B(n+2) + C\right] + 2\left[A(n+1)^2 + B(n+1) + C\right] + 11\left(An^2 + Bn + C\right) = 3n^2$$

$$3\left[A\left(n^2 + 4n + 4\right) + B(n+2) + C\right] + 2\left[A\left(n^2 + 2n + 1\right) + B(n+1) + C\right] + 11\left(An^2 + Bn + C\right) = 3n^2$$

By comparing coefficients

$$3A + 2A + 11A = 3$$

$$A = \frac{3}{16}$$

$$12A + 3B + 4A + 2B + 11B = 0$$

$$16A + 16B = 0$$

$$16B = -16A \Rightarrow 16B = -16\left(\frac{3}{16}\right)$$

$$B = \frac{-3}{16}$$

$$12A + 6B + 3C + 4A + 2B + 2C + 11C = 0$$

$$16A + 8B + 16C = 0$$

$$16C = -16A - 8B$$

$$16C = -16\left(\frac{3}{16}\right) - 8\left(\frac{-3}{16}\right)$$

$$16C = -3 + \frac{3}{2} \quad \Rightarrow 16C = \frac{-6+3}{2}$$

$$C = \frac{-3}{32}$$

$$\Rightarrow \quad u_1^* = \frac{3}{16}n^2 - \frac{3}{16}n - \frac{3}{32}$$

For equation (vi) Let $u_n = D2^n$ put in (vi)

$$\left(3E^2 + 2E + 11\right)D2^n = 5.2^n$$

$$3E^2D2^n + 2ED2^n + 11D2^n = 5.2^n$$

$$3D2^{n+2} + 2D2^{n+1} + 11D2^n = 5.2^n$$

$$12D + 4D + 11D = 5$$

$$27D = 5$$

$$\Rightarrow D = \frac{5}{27}$$

$$\Rightarrow \quad u_2^* = \frac{5}{27}2^n$$

$$\Rightarrow \quad u^* = u_1^* + u_2^* \Rightarrow u^* = \frac{3}{16}n^2 - \frac{3}{16}n - \frac{3}{32} + \frac{5}{27}2^n$$

Thus, $u_n = u + u^*$

$$u_n = \left(\sqrt{\frac{11}{3}}\right)^n \left[c_1 \cos\theta n + c_2 \sin\theta n\right] + \frac{3}{16}n^2 - \frac{3}{16}n - \frac{3}{32} + \frac{5}{27}2^n \quad \text{\_\_\_\_(vii)}$$

Put in (iii)

$$v_n = \frac{1}{3}(E+1)\left[\left(\sqrt{\frac{11}{3}}\right)^n \left[c_1 \cos\theta n + c_2 \sin\theta n\right] + \frac{3}{16}n^2 - \frac{3}{16}n - \frac{3}{32} + \frac{5}{27}2^n\right] - \frac{1}{3}2^n$$

$$v_n=\frac{1}{3}\left[\left(\sqrt{\frac{11}{3}}\right)^{n+1}\{c_1\cos\theta(n+1)+c_2\sin\theta(n+1)\}+\frac{3}{16}(n+1)^2-\frac{3}{16}(n+1)-\frac{3}{32}+\frac{5}{27}2^{n+1}\right]$$

$$+\frac{1}{3}\left[\left(\sqrt{\frac{11}{3}}\right)^{n}\{c_1\cos\theta n+c_2\sin\theta \text{n}\}+\frac{3}{16}n^2-\frac{3}{16}n-\frac{3}{32}+\frac{5}{27}2^{n}\right]-\frac{1}{3}2^n \quad \text{\_\_\_\_\_(viii)}$$

From (vii) and (viii)

$S.S=\{u_n,v_n\}$ is the required general solution.

Question: Solve the following system of D.E

$$u_{n+1}+v_n-u_n=2^n$$

$$3v_{n+1}+2v_n+u_n=7$$

Solution:

The given system can be written as

$$(E-1)u_n+v_n=2^n \quad \text{\_\_\_\_(i)}$$

$$(3E+2)v_n+u_n=7 \quad \text{\_\_\_\_}(ii)$$

From (i) $\quad v_n=2^n-(E-1)u_n \quad$ ____(iii)

Put in (ii) $\Rightarrow \quad (3E+2)\left(2^n-(E-1)u_n\right)+u_n=7$

$$(3E+2)2^n-(3E+2)(E-1)u_n+u_n=7$$

$$3.2^{n+1}+2.2^n-\left(3E^2-E-2\right)u_n+u_n=7$$

$$(6+2)2^n-\left(3E^2-E-3\right)u_n=7$$

$$\left(3E^2-E-3\right)u_n=8.2^n-7 \quad \text{\_\_\_\_\_(iv)}$$

For complementary solution we have

$$\left(3E^2-E-3\right)u_n=0$$

The auxiliary equation

$$3m^2 - m - 3 = 0$$

$$m = \frac{1 \pm \sqrt{1+36}}{6} = \frac{1 \pm \sqrt{37}}{6}$$

$$m = \frac{1}{6} \pm \frac{\sqrt{37}}{6}$$

$$u = c_1\left(\frac{1}{6} - \frac{\sqrt{37}}{6}\right)^n + c_2\left(\frac{1}{6} + \frac{\sqrt{37}}{6}\right)^n$$

For particular solution

$$\left(3E^2 - E - 3\right)u_n = 8.2^n \qquad \text{\_\_\_\_(v)}$$

$$\left(3E^2 - E - 3\right)u_n = -7 \qquad \text{\_\_\_\_(vi)}$$

For equation (v) Let $u_n = A2^n$ put in (v)

$$\left(3E^2 - E - 3\right)A2^n = 8.2^n$$

$$3E^2 A2^n - EA2^n - 3A2^n = 8.2^n$$

$$3A2^{n+2} - A2^{n+1} - 3A2^n = 8.2^n$$

$$12A - 2A - 3A = 8$$

$$7A = 8$$

$$\Rightarrow A = \frac{8}{7}$$

$$\Rightarrow \quad u_1^* = \frac{8}{7}2^n$$

For equation (vi) Let $u_n = B$ put in (vi)

$$\left(3E^2 - E - 3\right)B = -7$$

$$3B - B - 3B = -7$$

$$-B=-7$$

$$\Rightarrow B=7$$

$$\Rightarrow \quad u_2^*=7$$

$$\Rightarrow \quad u^*=u_1^*+u_2^* \Rightarrow u^*=\frac{8}{7}2^n+7$$

$$Thus, \quad u_n=u+u^*$$

$$u_n=c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^n+c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^n+\frac{8}{7}2^n+7 \qquad \text{\_\_\_\_(vii)}$$

Put in (iii)

$$v_n=2^n-(E-1)\left[c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^n+c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^n+\frac{8}{7}2^n+7\right]$$

$$v_n=2^n-\left[c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^{n+1}+c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^{n+1}+\frac{8}{7}2^{n+1}+7-c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^n-c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^n-\frac{8}{7}2^n-7\right]$$

$$v_n=2^n-\left[c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^{n+1}+c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^{n+1}+\left(\frac{16}{7}-\frac{8}{7}\right)2^n-c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^n-c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^n\right]$$

$$v_n=2^n-\left[c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^{n+1}+c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^{n+1}+\frac{8}{7}2^n-c_1\left(\frac{1}{6}-\frac{\sqrt{37}}{6}\right)^n-c_2\left(\frac{1}{6}+\frac{\sqrt{37}}{6}\right)^n\right] \qquad \text{\_(viii)}$$

From (vii) and (viii)

$S.S=\{u_n,v_n\}$ is the required general solution.

Formation of Difference Equation:

Question: Determine the corresponding difference equation

$$y_k=c_1 2^k+c_2 3^k+\frac{1}{2} \qquad \text{\_\_\_\_(i)}$$

Solution:

From equation (i) it is clear that

$$y_{k+1} = c_1 2^{k+1} + c_2 3^{k+1} + \frac{1}{2}$$

$$y_{k+1} = 2c_1 2^k + 3c_2 3^k + \frac{1}{2} \qquad \text{\_\_\_\_\_(ii)}$$

$$y_{k+2} = 4c_1 2^k + 9c_2 3^k + \frac{1}{2} \qquad \text{\_\_\_\_\_(iii)}$$

From eq (ii) and (iii) we have

$$4c_1 2^k + 6c_2 3^k + (1 - 2y_{k+1}) = 0$$

$$8c_1 2^k + 18c_2 3^k + (1 - 2y_{k+2}) = 0$$

By cross multiplication

$$\frac{c_1 2^k}{6(1-2y_{k+2}) - 18(1-2y_{k+1})} = \frac{-c_2 3^k}{4(1-2y_{k+2}) - 8(1-2y_{k+1})} = \frac{1}{72-48}$$

$$c_1 2^k = \frac{1}{24}\left[6(1-2y_{k+2}) - 18(1-2y_{k+1})\right]$$

$$c_2 3^k = \frac{1}{24}\left[8(1-2y_{k+2}) - 4(1-2y_{k+1})\right]$$

Putting these values in eq (i) we have

$$y_k = \frac{1}{24}\left[6(1-2y_{k+2}) - 18(1-2y_{k+1})\right] + \frac{1}{24}\left[8(1-2y_{k+2}) - 4(1-2y_{k+1})\right] + \frac{1}{2}$$

$$24y_k = \left[6(1-2y_{k+2}) - 18(1-2y_{k+1})\right] + \left[8(1-2y_{k+2}) - 4(1-2y_{k+1})\right] + 12$$

$$24y_k = (6-4)(1-2y_{k+2}) + (-18+8)(1-2y_{k+1}) + 12$$

$$24y_k = 2(1-2y_{k+2}) - 10(1-2y_{k+1}) + 12$$

$$24y_k = 2\left[1 - 2y_{k+2} - 5 + 10y_{k+1} + 6\right]$$

$$12y_k = -2y_{k+2} + 10y_{k+1} + 2$$

$$6y_k = -y_{k+2} + 5y_{k+1} + 1$$

$y_{k+2} - 5y_{k+1} + 6y_k = 1$ is the required difference solution.

Question: Solve the difference equation

$y_{k+2} - 4y_{k+1} + 4y_k = \sin k + 2^k$ with condition $Y_0 = 0 = Y_1$

Solution: The given difference equation can be written as

$$\left(E^2 - 4E + 4\right) y_k = \sin k + 2^k$$

For complementary solution

$$\left(E^2 - 4E + 4\right) y_k = 0$$

The auxiliary equation

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$\Rightarrow Y = (c_1 + c_2 k) 2^k$$

For particular solution

$$\left(E^2 - 4E + 4\right) y_k = \sin k \qquad \text{\_\_\_\_(i)}$$

$$\left(E^2 - 4E + 4\right) y_k = 2^k \qquad \text{\_\_\_\_(ii)}$$

For equation (i)

$$Let \;\; y_k = \left(A \sin k + B \cos k\right) \qquad Put\, in\, (i)$$

$$(E^2 - 4E + 4)\left(A \sin k + B \cos k\right) = \sin k$$

$$E^2\left(A \sin k + B \cos k\right) - 4E\left(A \sin k + B \cos k\right) + 4\left(A \sin k + B \cos k\right) = \sin k$$

$$A \sin(k+2) + B \cos(k+2) - 4\left[A \sin(k+1) + B \cos(k+1)\right] + 4A \sin k + 4B \cos k = \sin k$$

$$\left[A\left(\sin k\cos 2+\sin 2\cos k\right)+B\left(\cos k\cos 2-\sin k\sin 2\right)\right]$$

$$-4\left[A\left(\sin k\cos 1+\sin 1\cos k\right)+B\left(\cos k\cos 1-\sin k\sin 1\right)\right]$$

$$+4A\sin k+4B\cos k=\sin k$$

$$A\left[\sin k\cos 2+\sin 2\cos k-4\sin k\cos 1-4\sin 1\cos k+4\sin k\right]$$

$$+B\left[\cos k\cos 2-\sin k\sin 2-4\cos k\cos 1+4\sin k\sin 1+4\cos k\right]=\sin k$$

$$A\left[\sin k\left(\cos 2-4\cos 1+4\right)+\cos k\left(\sin 2-4\sin 1\right)\right]$$

$$+B\left[\cos k\left(\cos 2-4\cos 1+4\right)-\sin k\left(\sin 2-4\sin 1\right)\right]=\sin k$$

Say $\quad \mathrm{M}=\cos 2-4\cos 1+4$ & $N=\sin 2-4\sin 1$

$$A\left[\mathrm{M}\sin k+N\cos k\right]+B\left[M\cos k-N\sin k\right]=\sin k$$

$$\left(AM-BN\right)\sin k+\left(AN+BM\right)\cos k=\sin k$$

By comparing coefficients of sink and cosk

$$\mathrm{AM}-\mathrm{BN}=1$$

$$\mathrm{AN}+\mathrm{BM}=0$$

Now by cross multiplication

$$\mathrm{AM}-\mathrm{BN}-1=0$$

$$\mathrm{AN}+\mathrm{BM}-0=0$$

$$\frac{A}{0+M}=\frac{-B}{0+N}=\frac{1}{M^2+N^2}$$

$$A=\frac{M}{M^2+N^2} \quad , \quad B=\frac{-N}{M^2+N^2}$$

$$Y_1^*=A\sin k+B\cos k$$

For equation (ii)

$$Let \quad y_k=C2^k k^2 \qquad\qquad Put\,in\,(ii)$$

$$(E^2-4E+4)C2^k k^2=2^k$$

$$E^2C2^k k^2-4EC2^k k^2+4C2^k k^2=2^k$$

$$C2^{k+2}(k+2)^2-4C2^{k+1}(k+1)^2+4C2^k k^2=2^k$$

$$4C\left(k^2+4k+4\right)-8C\left(k^2+2k+1\right)+4Ck^2=1$$

$$4Ck^2+16Ck+16C-8Ck^2-16Ck-8C+4Ck^2=1$$

$$8C=1$$

$$\Rightarrow \quad C=\frac{1}{8}$$

$$Y_2^*=\frac{1}{8}k^2 2^k$$

$$Y^*=Y_1^*+Y_2^* \Rightarrow Y^*=A\sin k+B\cos k+\frac{1}{8}k^2 2^k$$

$$Y_k=Y+Y^* \Rightarrow Y_k=\left(c_1+c_2k\right)2^k+A\sin k+B\cos k+\frac{1}{8}k^2 2^k$$

$$Y_0=c_1+B$$

$$c_1+B=0 \qquad \because Y_0=0$$

$$c_1=-B \quad \Rightarrow \quad c_1=\frac{N}{M^2+N^2}$$

$$Y_k'=\left(c_1+c_2k\right)2^k+c_2 2^k+A\cos k-B\sin k+\frac{1}{8}k^2 2^k+\frac{2}{8}k2^k$$

$$Y_0'=c_1+c_2+A$$

$$0=c_1+c_2+A \qquad \because Y_0'=0$$

$$c_2=-c_1-A$$

$$c_2=\frac{-N}{M^2+N^2}-\frac{M}{M^2+N^2}=\frac{-M-N}{M^2+N^2}=-\left(\frac{M+N}{M^2+N^2}\right)$$

By putting values

$$Y_k = \left(\frac{N}{M^2+N^2} - \left(\frac{M+N}{M^2+N^2} \right) k \right) 2^k + \frac{M}{M^2+N^2} \sin k - \frac{N}{M^2+N^2} \cos k + \frac{1}{8} k^2 2^k$$

Question: Determine the corresponding difference equation

$$y_k = c_1 k^2 + c_2 k + c_3 \qquad \text{\_\_\_\_(i)}$$

Solution:

From equation (i) it is clear that

$$y_{k+1} = c_1 (k+1)^2 + c_2 (k+1) + c_3 \qquad \text{\_\_\_\_(ii)}$$

$$y_{k+2} = c_1 (k+2)^2 + c_2 (k+2) + c_3 \qquad \text{\_\_\_\_(iii)}$$

$$y_{k+3} = c_1 (k+3)^2 + c_2 (k+3) + c_3 \qquad \text{\_\_\_\_(iv)}$$

Subtracting (ii) from (iii)

$$y_{k+2} - y_{k+1} = c_1 \left\{ (k+2)^2 - (k+1)^2 \right\} + c_2 \left\{ (k+2) - (k+1) \right\}$$

$$y_{k+2} - y_{k+1} = c_1 \left\{ (k^2+4k+4) - (k^2+2k+1) \right\} + c_2 \left\{ k+2-k-1 \right\}$$

$$y_{k+2} - y_{k+1} = c_1 \left\{ k^2+4k+4-k^2-2k-1 \right\} + c_2 (1)$$

$$y_{k+2} - y_{k+1} = c_1 (2k+3) + c_2 \qquad \text{\_\_\_(v)}$$

Subtracting (iii) from (iv)

$$y_{k+3} - y_{k+2} = c_1 \left\{ (k+3)^2 - (k+2)^2 \right\} + c_2 \left\{ (k+3) - (k+2) \right\}$$

$$y_{k+3} - y_{k+2} = c_1 \left\{ (k^2+6k+9) - (k^2+4k+4) \right\} + c_2 \left\{ k+3-k-2 \right\}$$

$$y_{k+3} - y_{k+2} = c_1 \left\{ k^2+6k+9-k^2-4k-4 \right\} + c_2 (1)$$

$$y_{k+3} - y_{k+2} = c_1 (2k+5) + c_2 \qquad \text{\_\_\_(vi)}$$

Subtracting (v) from (vi)

$$\left(y_{k+3}-y_{k+2}\right)-\left(y_{k+2}-y_{k+1}\right)=c_1\left\{\left(2k+5\right)-\left(2k+3\right)\right\}$$

$$y_{k+3}-y_{k+2}-y_{k+2}+y_{k+1}=c_1\left(2k+5-2k-3\right)$$

$$y_{k+3}-2y_{k+2}+y_{k+1}=2c_1$$

$$c_1=\frac{1}{2}\left[y_{k+3}-2y_{k+2}+y_{k+1}\right]$$

Put the value of c_1 in (vi)

$$y_{k+3}-y_{k+2}=\frac{1}{2}\left[y_{k+3}-2y_{k+2}+y_{k+1}\right]\left(2k+5\right)+c_2$$

$$2y_{k+3}-2y_{k+2}=\left[y_{k+3}-2y_{k+2}+y_{k+1}\right]\left(2k+5\right)+2c_2$$

$$2c_2=2y_{k+3}-2y_{k+2}-\left(2k+5\right)y_{k+3}+2\left(2k+5\right)y_{k+2}-\left(2k+5\right)y_{k+1}$$

$$2c_2=\left(2-2k-5\right)y_{k+3}-\left(2-4k-10\right)y_{k+2}-\left(2k+5\right)y_{k+1}$$

$$2c_2=\left(-2k-3\right)y_{k+3}-\left(-4k-8\right)y_{k+2}-\left(2k+5\right)y_{k+1}$$

$$2c_2=-\left(2k+3\right)y_{k+3}+\left(4k+8\right)y_{k+2}-\left(2k+5\right)y_{k+1}$$

$$c_2=\frac{1}{2}\left[-\left(2k+3\right)y_{k+3}+\left(4k+8\right)y_{k+2}-\left(2k+5\right)y_{k+1}\right]$$

Put the value of c_1 and c_2 in (ii)

$$y_{k+1}=\frac{1}{2}\left[y_{k+3}-2y_{k+2}+y_{k+1}\right]\left(k+1\right)^2+\frac{1}{2}\left[-\left(2k+3\right)y_{k+3}+\left(4k+8\right)y_{k+2}-\left(2k+5\right)y_{k+1}\right]\left(k+1\right)+c_3$$

$$2y_{k+1}=\left(k+1\right)^2y_{k+3}-2\left(k+1\right)^2y_{k+2}+\left(k+1\right)^2y_{k+1}-\left(k+1\right)\left(2k+3\right)y_{k+3}+\left(k+1\right)\left(4k+8\right)y_{k+2}-\left(k+1\right)\left(2k+5\right)y_{k+1}+2c_3$$

$$2c_3=\left(k^2+2k+1\right)y_{k+3}-\left(2k^2+5k+3\right)y_{k+3}-2\left(k^2+2k+1\right)y_{k+2}$$

$$+\left(4k^2+12k+8\right)y_{k+2}-\left(k^2+2k+1\right)y_{k+1}+\left(2k^2+7k+5\right)y_{k+1}+2y_{k+1}$$

$$2c_3=\left(2k^2+5k+3-k^2-2k-1\right)y_{k+3}+\left(2k^2+4k+2-4k^2-12k-8\right)y_{k+2}$$

$$+\left(-k^2-2k-1+2k^2+7k+5+2\right)y_{k+1}$$

$$2c_3=\left(k^2+3k+2\right)y_{k+3}+\left(-2k^2-8k-6\right)y_{k+2}+\left(k^2+5k+6\right)y_{k+1}$$

$$c_3=\frac{1}{2}\left[\left(k^2+3k+2\right)y_{k+3}-\left(2k^2+8k+6\right)y_{k+2}+\left(k^2+5k+6\right)y_{k+1}\right]$$

Put the value of c_1 , c_2 and c_3 in (i)

$$y_k=k^2.\frac{1}{2}\left[y_{k+3}-2y_{k+2}+y_{k+1}\right]+k.\frac{1}{2}\left[-(2k+3)y_{k+3}+(4k+8)y_{k+2}-(2k+5)y_{k+1}\right]$$
$$+\frac{1}{2}\left[\left(k^2+3k+2\right)y_{k+3}-\left(2k^2+8k+6\right)y_{k+2}+\left(k^2+5k+6\right)y_{k+1}\right]$$

$$2y_k=k^2\left[y_{k+3}-2y_{k+2}+y_{k+1}\right]+k\left[-(2k+3)y_{k+3}+(4k+8)y_{k+2}-(2k+5)y_{k+1}\right]$$
$$+\left[\left(k^2+3k+2\right)y_{k+3}-\left(2k^2+8k+6\right)y_{k+2}+\left(k^2+5k+6\right)y_{k+1}\right]$$

$$2y_k=\left(k^2-2k^2-3k+k^2+3k+2\right)y_{k+3}+\left(-2k^2+4k^2+8k-2k^2-8k-6\right)y_{k+2}$$
$$+\left(k^2-2k^2-5k+k^2+5k+6\right)y_{k+1}$$

$$2y_{k+1}=2y_{k+3}-6y_{k+2}+6y_{k+1}$$

$$y_k=y_{k+3}-3y_{k+2}+3y_{k+1}$$

$y_{k+3}-3y_{k+2}+3y_{k+1}-y_k=0$ is the required difference equation.

Question: Determine the corresponding difference equation

$$y_k=(c_1+c_2k)3^k \quad \_\_\_\_\text{(i)}$$

Solution:

From equation (i) It is clear that

$$y_{k+1}=\left(c_1+c_2(k+1)\right)3^{k+1}$$

$$y_{k+1}=3\left(c_1+c_2(k+1)\right)3^k \qquad \_\_\_\_\_\_\text{(ii)}$$

$$y_{k+2}=9\left(c_1+c_2(k+2)\right)3^k \qquad \_\_\_\_\_\_\text{(iii)}$$

From equation (ii) and (iii)

$$3c_1 3^k + 3c_2(k+1)3^k - y_{k+1} = 0$$

$$9c_1 3^k + 9c_2(k+2)3^k - y_{k+2} = 0$$

By cross multiplication

$$\frac{c_1 3^k}{3(k+1)y_{k+2} + 9(k+2)y_{k+1}} = \frac{-c_2 3^k}{3y_{k+2} + 9y_{k+1}} = \frac{1}{27(k+2) - 27(k+1)}$$

$$c_1 3^k = \frac{3(k+1)y_{k+2} + 9(k+2)y_{k+1}}{27}$$

$$c_1 = \frac{1}{9.3^k}\left[(k+1)y_{k+2} + 3(k+2)y_{k+1}\right]$$

$$c_2 3^k = \frac{-3y_{k+2} - 9y_{k+1}}{27}$$

$$c_2 = \frac{1}{9.3^k}\left[-y_{k+2} - 3y_{k+1}\right]$$

Put the value of c_1 and c_2 in (i)

$$y_k = \left(\frac{1}{9.3^k}\left[(k+1)y_{k+2} + 3(k+2)y_{k+1}\right] + \frac{1}{9.3^k}\left[-y_{k+2} - 3y_{k+1}\right]k\right)3^k$$

$$y_k = \frac{3^k}{9.3^k}\left[(k+1)y_{k+2} + 3(k+2)y_{k+1} - ky_{k+2} - 3ky_{k+1}\right]$$

$$y_k = \frac{1}{9}\left[(k+1-k)y_{k+2} + (3k+6-3k)y_{k+1}\right]$$

$$y_k = \frac{1}{9}\left[y_{k+2} + 6y_{k+1}\right]$$

$$9y_k = y_{k+2} + 6y_{k+1}$$

$y_{k+2} + 6y_{k+1} - 9y_k = 0$ is the required difference equation.

Question: Determine the corresponding difference equation

$$y_k = c_1 k^2 + c_2 k + 9 \qquad \_\_\_\_(i)$$

Solution:

From equation (i) It is clear that

$$y_{k+1} = c_1(k+1)^2 + c_2(k+1) + 9 \qquad \text{\_\_\_\_\_\_(ii)}$$

$$y_{k+2} = c_1(k+2)^2 + c_2(k+2) + 9 \qquad \text{\_\_\_\_\_\_(iii)}$$

From equation (ii) and (iii)

$$c_1(k+1)^2 + c_2(k+1) + (9 - y_{k+1}) = 0$$

$$c_1(k+2)^2 + c_2(k+2) + (9 - y_{k+2}) = 0$$

By cross multiplication

$$\frac{c_1}{(k+1)(9-y_{k+2})-(k+2)(9-y_{k+1})} = \frac{-c_2}{(k+1)^2(9-y_{k+2})-(k+2)^2(9-y_{k+1})} = \frac{1}{(k+1)^2(k+2)-(k+2)^2(k+1)}$$

$$c_1 = \frac{(k+1)(9-y_{k+2})+(k+2)(9-y_{k+1})}{(k+1)^2(k+2)-(k+2)^2(k+1)}$$

$$c_1 = \frac{9k+9-(k+1)y_{k+2}-9k-18+(k+2)y_{k+1}}{(k+2)(k+1)\{k+1-k-2\}}$$

$$c_1 = \frac{-(k+1)y_{k+2}+(k+2)y_{k+1}-9}{-(k+2)(k+1)}$$

$$c_1 = \frac{(k+1)y_{k+2}-(k+2)y_{k+1}+9}{(k+2)(k+1)}$$

$$c_2 = -\left[\frac{(k+1)^2(9-y_{k+2})+(k+2)^2(9-y_{k+1})}{(k+1)(k+2)\{k+1-k-2\}}\right]$$

$$c_2 = -\left[\frac{9(k^2+2k+1)-(k+1)^2 y_{k+2}-9(k^2+4k+4)+(k+2)^2 y_{k+1}}{-(k+1)(k+2)}\right]$$

$$c_2=\frac{9k^2+18k+9-(k+1)^2y_{k+2}-9k^2-36k-36+(k+2)^2y_{k+1}}{(k+1)(k+2)}$$

$$c_2=\frac{-(k+1)^2y_{k+2}+(k+2)^2y_{k+1}-18k-27}{(k+1)(k+2)}$$

Put the value of c_1 and c_2 in (i)

$$y_k=\left[\frac{(k+1)y_{k+2}-(k+2)y_{k+1}+9}{(k+2)(k+1)}\right]k^2+\left[\frac{-(k+1)^2y_{k+2}+(k+2)^2y_{k+1}-18k-27}{(k+1)(k+2)}\right]k+9$$

$$y_k=\left[\frac{k^2(k+1)y_{k+2}-k^2(k+2)y_{k+1}+9k^2-k(k^2+2k+1)y_{k+2}+k(k^2+4k+4)y_{k+1}-18k^2-27k+9(k^2+3k+2)}{(k^2+3k+2)}\right]$$

$$y_k=\left[\frac{(k^3+k^2-k^3-2k^2-k)y_{k+2}+(-k^3-2k^2+k^3+4k^2+4k)y_{k+1}+9k^2-18k^2-27k+9k^2+27k+18}{(k^2+3k+2)}\right]$$

$$y_k=\left[\frac{(-k^2-k)y_{k+2}+(2k^2+4k)y_{k+1}+18}{(k^2+3k+2)}\right]$$

$$(k^2+3k+2)y_k=(-k^2-k)y_{k+2}+(2k^2+4k)y_{k+1}+18$$

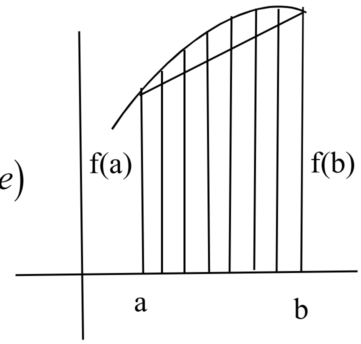
$$(k^2+3k+2)y_k=-(k^2+k)y_{k+2}+(2k^2+4k)y_{k+1}+18$$

$(k^2+k)y_{k+2}-(2k^2+4k)y_{k+1}+(k^2+3k+2)y_k=18$ is the required difference equation.

Lecture # 09

Numerical Integration:

$$\int_a^b f(x)\,dx = Area$$



$$Area\ of\ Trapezium = \frac{1}{2}(Distance\ between\ //\ side)(Sum\ of\ //\ side)$$

$$Area\ of\ Trapezium = \frac{1}{2}(b-a)\big(f(a)+f(b)\big)$$

Consider a definite integral $I = \int_a^b f(x)\,dx$ ______(i)

Let us divide the interval [a,b] into n equal parts such that

$x_0 = 0$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 3h$

. .

. .

. .

$x_n = x_0 + nh$

And y = f(x) take the values of $y_0, y_1, y_2, ..., y_n$ for $x_0, x_1, x_2, ..., x_n$

Then from eq (i) we have

$I = \int_{x_0}^{x_n} f(x)\,dx$ ______(ii)

Now using the Newton's forward difference interpolating formula

$$P(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + ... = f(x) \text{_____(iii)}$$

Where $P=\frac{x-x_0}{h} \Rightarrow x=x_0+Ph \;\; \Rightarrow dx=hdP$

From (ii) and (iii) we have

$$when \quad x=x_0 \Rightarrow P=0$$

$$when \quad x=x_n \;\; \Rightarrow P=\frac{x_n-x_0}{h}=n \quad \because x_n=x_0+nh$$

$$I=\int_0^n\left[y_0+P\Delta y_0+\frac{P(P-1)}{2!}\Delta^2 y_0+\frac{P(P-1)(P-2)}{3!}\Delta^3 y_0+....\right]h\,dP$$

$$I=h\int_0^n\left[y_0+P\Delta y_0+\frac{1}{2}\left(P^2-P\right)\Delta^2 y_0+\frac{1}{6}\left(P^3-3P^2+2P\right)\Delta^3 y_0+....\right]dP$$

$$I=h\left[Py_0+\frac{P^2}{2}\Delta y_0+\frac{1}{2}\left(\frac{P^3}{3}-\frac{P^2}{2}\right)\Delta^2 y_0+\frac{1}{6}\left(\frac{P^4}{4}-P^3+P^2\right)\Delta^3 y_0+....\right]_0^n$$

$$I=h\left[ny_0+\frac{n^2}{2}\Delta y_0+\frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0+\frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0+\frac{1}{24}\left[\frac{n^5}{5}-\frac{3n^4}{2}+\frac{11n^3}{3}-3n^2\right]\Delta^4 y_0\right.$$

$$\left.+\frac{1}{120}\left[\frac{n^6}{6}-2n^5+\frac{35n^4}{4}-\frac{50n^3}{3}+12n^2\right]\Delta^5 y_0+\frac{1}{720}\left[\frac{n^7}{7}-\frac{15n^6}{6}+17n^5-\frac{225n^4}{4}+\frac{273n^3}{3}-60n^2\right]\Delta^6 y_0+...\right]$$

This is called Newton Cotes quadrature formula for equi-spaced data. This is also called general quadrature formula.

Lecture # 10

As we know this relation

$$\int_{x_0}^{x_n} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0 + \frac{1}{24}\left[\frac{n^5}{5}-\frac{3n^4}{2}+\frac{11n^3}{3}-3n^2\right]\Delta^4 y_0\right.$$

$$\left.+\frac{1}{120}\left[\frac{n^6}{6}-2n^5+\frac{35n^4}{4}-\frac{50n^3}{3}+12n^2\right]\Delta^5 y_0 + \frac{1}{720}\left[\frac{n^7}{7}-\frac{15n^6}{6}+17n^5-\frac{225n^4}{4}+\frac{273n^3}{3}-60n^2\right]\Delta^6 y_0 + \dots\right]$$

______(A)

1-Trapezoidal Rule:

Put n = 1 in equation (A) and by neglecting 2nd order and higher order differences we have

$$\int_{x_0}^{x_n} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[2y_0 + \Delta y_0]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[2y_0 + y_1 - y_0] \qquad \because \Delta y_0 = y_1 - y_0$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[y_0 + y_1]$$

Similarly,

$$\int_{x_1}^{x_2} f(x)dx = h\left[y_1 + \frac{1}{2}\Delta y_2\right]$$

$$= \frac{h}{2}[2y_1 + \Delta y_2]$$

$$= \frac{h}{2}[2y_1 + y_2 - y_1] \qquad \because \Delta y_2 = y_2 - y_1$$

$$\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}[y_1 + y_2]$$

Also $$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}[y_2 + y_3]$$

. .

. .

. .

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

By adding above integrals, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\big[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})\big]$$

This is called Trapezoidal rule for evaluating definite integrals. This formula is suitable if n is multiple of 1.

2- Simpson's 1/3 Rule:

Put n = 2 in equation (A) and by neglecting third order and higher order differences, we get

$$\int_{x_0}^{x_2} f(x)\,dx = h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}[6y_0 + 6y_1 - 6y_0 + y_2 - 2y_1 + y_0]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly, $$\int_{x_2}^{x_4} f(x)\,dx = h\left[2y_2 + 2\Delta y_2 + \frac{1}{3}\Delta^2 y_2\right]$$

$$= h\left[2y_2 + 2(y_3 - y_2) + \frac{1}{3}(y_4 - 2y_3 + y_2)\right]$$

$$= \frac{h}{3}[6y_2 + 6y_3 - 6y_2 + y_4 - 2y_3 + y_2]$$

$$= \frac{h}{3}[y_2 + 4y_3 + y_4]$$

Also $$\int_{x_4}^{x_6} f(x)\,dx = \frac{h}{3}[y_4 + 4y_5 + y_6]$$

. .

. .

. .

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

By adding above integrals, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + ... + y_{n-2}) + 4(y_1 + y_3 + y_5 + ... + y_{n-1})\right]$$

This is called Simpson's 1/3 rule for evaluating definite integral. This formula is suitable if n is multiple of 2.

3-Simpson's 3/8 rule:

Put n = 3 in equation (A) and by neglecting fourth order and higher order differences, we get

$$\int_{x_0}^{x_3} f(x)\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{27}{3} - \frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{3}{8}\Delta^3 y_0\right]$$

$$\int_{x_0}^{x_3} f(x)\,dx \;= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + y_3 - 3y_2 + 3y_1 - y_0\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly, $\int_{x_3}^{x_6} f(x)\,dx = h\left[3y_3 + \frac{9}{2}\Delta y_3 + \frac{9}{4}\Delta^2 y_3 + \frac{3}{8}\Delta^3 y_3\right]$

$$\int_{x_0}^{x_3} f(x)\,dx \;= h\left[3y_3 + \frac{9}{2}(y_4 - y_3) + \frac{9}{4}(y_5 - 2y_4 + y_3) + \frac{3}{8}(y_6 - 3y_5 + 3y_4 - y_3)\right]$$

$$= \frac{3h}{8}\left[8y_3 + 12y_4 - 12y_3 + 6y_5 - 12y_4 + 6y_3 + y_6 - 3y_5 + 3y_4 - y_3\right]$$

$$= \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

Also $\int_{x_6}^{x_9} f(x)\,dx = \frac{3h}{8}\left[y_6 + 3y_7 + 3y_8 + y_9\right]$

. .

. .

. .

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

By adding above integrals, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \ldots. + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 + \ldots. + y_{n-2} + y_{n-1})\right]$$

This is called Simpson's 3/8 rule for evaluating definite integral. This formula is suitable if n is multiple of 3.

4-Bool's Rule: Put n = 4 in equation (A) and by neglecting fifth order and higher order differences, we get

$$\int_{x_0}^{x_4} f(x)\,dx = h\left[4y_0 + \frac{16}{2}\Delta y_0 + \frac{1}{2}\left(\frac{64}{3} - \frac{16}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{256}{4} - 64 + 16\right)\Delta^3 y_0\right.$$

$$\left. + \frac{1}{24}\left(\frac{1024}{5} - \frac{768}{2} + \frac{704}{3} - 48\right)\Delta^4 y_0\right]$$

$$= h\left[4y_0 + 8\Delta y_0 + \frac{20}{3}\Delta^2 y_0 + \frac{8}{3}\Delta^3 y_0 + \frac{28}{90}\Delta^4 y_0\right]$$

$$= h\left[4y_0 + 8(y_1 - y_0) + \frac{20}{3}(y_2 - 2y_1 + y_0) + \frac{8}{3}(y_3 - 3y_2 + 3y_1 - y_0) + \frac{14}{45}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\right]$$

$$= \frac{2h}{45}\left[90y_0 + 180(y_1 - y_0) + 150(y_2 - 2y_1 + y_0) + 60(y_3 - 3y_2 + 3y_1 - y_0) + 7(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\right]$$

$$= \frac{2h}{45}[90y_0 + 180y_1 - 180y_0 + 150y_2 - 300y_1 + 150y_0 + 60y_3 - 180y_2 + 180y_1 - 60y_0$$

$$+7y_4 - 28y_3 + 42y_2 - 28y_1 + 7y_0]$$

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

Similarly, $$\int_{x_4}^{x_8} f(x)\,dx = h\left[4y_4 + 8\Delta y_4 + \frac{20}{3}\Delta^2 y_4 + \frac{8}{3}\Delta^3 y_4 + \frac{28}{90}\Delta^4 y_4\right]$$

$$\int_{x_4}^{x_8} f(x)\,dx = \frac{2h}{45}[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

Also

$$\int_{x_8}^{x_{12}} f(x)\,dx = \frac{2h}{45}[7y_8 + 32y_9 + 12y_{10} + 32y_{11} + 7y_{12}]$$

. .

. .

. .

$$\int_{x_{n-4}}^{x_n} f(x)\,dx = \frac{2h}{45}[7y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n]$$

By adding above integrals, we have

$$\int_{x_0}^{x_n} f(x)dx = \frac{2h}{45}\left[7(y_0+y_n)+12(y_2+y_6+y_{10}+....+y_{n-2})+14(y_4+y_8+y_{12}+y_{16}+....+y_{n-4})+32(y_1+y_3+...+y_{n-1})\right]$$

This is called Bool's rule for evaluating definite integral. This formula is suitable if n is multiple of 4.

$$\Delta y_0 = (E-1)y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = (E-1)^2 y_0 = (E^2 - 2E + 1)y_0 = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = (E-1)^3 y_0 = (E^3 - 3E^2 + 3E - 1)y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = (E-1)^4 y_0 = (E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

Lecture # 11

As we know this relation

$$\int_{x_0}^{x_n} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \frac{1}{24}\left[\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right]\Delta^4 y_0\right.$$

$$\left. + \frac{1}{120}\left[\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2\right]\Delta^5 y_0 + \frac{1}{720}\left[\frac{n^7}{7} - \frac{15n^6}{6} + 17n^5 - \frac{225n^4}{4} + \frac{273n^3}{3} - 60n^2\right]\Delta^6 y_0 + \ldots\right.$$

______(A)

Weddle's Rule:

Put n = 6 in equation (A) and by neglecting 7th order and higher order differences we have

$$\int_{x_0}^{x_6} f(x)dx = h\left[6y_0 + \frac{36}{2}\Delta y_0 + \frac{1}{2}\left(\frac{216}{3} - \frac{36}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{1296}{4} - 216 + 36\right)\Delta^3 y_0 + \frac{1}{24}\left[\frac{7776}{5} - \frac{3888}{2} + \frac{2376}{3} - 108\right]\Delta^4 y_0\right.$$

$$\left. + \frac{1}{120}\left[\frac{46656}{6} - 15552 + \frac{45360}{4} - \frac{10800}{3} + 432\right]\Delta^5 y_0 + \frac{1}{720}\left[\frac{299936}{7} - \frac{699840}{6} + 132912 - \frac{291600}{4} + \frac{59187}{3} - 2160\right]\Delta^6 y_0\right.$$

$$\int_{x_0}^{x_6} f(x)dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$

$$\int_{x_0}^{x_6} f(x)dx = \frac{3h}{10}\left[20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 + \frac{41}{42}\Delta^6 y_0\right]$$

$$\int_{x_0}^{x_6} f(x)dx = \frac{3h}{10}\left[20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 + \Delta^6 y_0\right]$$

$\because \frac{41}{42} \approx 1$ (Main Step)

$$\int_{x_0}^{x_6} f(x)dx = \frac{3h}{10}\left[20y_0 + 60(y_1 - y_0) + 90(y_2 - 2y_1 + y_0) + 80(y_3 - 3y_2 + 3y_1 - y_0)\right.$$

$$+41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) + 11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)$$

$$\left. +(y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)\right]$$

$$\int_{x_0}^{x_6} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Similarly,

$$\int_{x_6}^{x_{12}} f(x)\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

$$\int_{x_{12}}^{x_{18}} f(x)\,dx = \frac{3h}{10}\left[y_{12} + 5y_{13} + y_{14} + 6y_{15} + y_{16} + 5y_{17} + y_{18}\right]$$

. .

. .

. .

$$\int_{x_{n-6}}^{x_n} f(x)\,dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

By adding above integrals

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{10}\Big[\left(y_0 + y_2 + y_4 + y_8 + \dots + y_{n-4} + y_{n-2} + y_n\right) + 2\left(y_6 + y_{12} + y_{18} + \dots + y_{n-6}\right)$$
$$+6\left(y_3 + y_9 + y_{15} + \dots + y_{n-3}\right) + 5\left(y_1 + y_5 + y_7 + y_{11} + \dots + y_{n-5} + y_{n-1}\right)$$

This is called Weddle's Rule for evaluating definite integrals. This formula is suitable if n is multiple of 6.

Question: Using Trapezoidal, Simpson's 1/3, Simpsons 3/8 and Weddle's rule to evaluate the definite integral $\int_1^7 f(x)\,dx$ by using the following data

x	1	2	3	4	5	6	7
f(x)	2.105	2.808	3.614	4.604	5.857	7.451	9.467

Solution: From the given data it is clear that n = 6 and h = 1

Using Trapezoidal rule

$$I = \frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$I = \frac{1}{2}\left[\left(2.105 + 9.467\right) + 2\left(2.808 + 3.614 + 4.604 + 5.857 + 7.451\right)\right] = 30.1165$$

Using Simpson's 1/3 rule for n = 6 we have

$$I=\frac{h}{3}\left[\left(y_0+y_6\right)+2\left(y_2+y_4\right)+4\left(y_1+y_3+y_5\right)\right]$$

$$I=\frac{1}{3}\left[\left(2.105+9.467\right)+2\left(3.614+5.857\right)+4\left(2.808+4.604+7.451\right)\right]=29.9886$$

Using Simpson's 3/8 rule for n = 6 we have

$$I=\frac{3h}{8}\left[\left(y_0+y_6\right)+2\left(y_3\right)+3\left(y_1+y_2+y_4+y_5\right)\right]$$

$$I=\frac{3}{8}\left[\left(2.105+9.467\right)+2\left(4.604\right)+3\left(2.808+3.614+5.857+7.451\right)\right]=29.9887$$

Using Weddle's rule for n = 6 we have

$$I=\frac{3h}{10}\left[\left(y_0+y_2+y_4\right)+2y_6+6y_3+5\left(y_1+y_5\right)\right]$$

$$I=\frac{3}{10}\left[\left(2.105+3.614+5.857\right)+2\left(9.467\right)+6\left(4.604\right)+5\left(2.808+7.451\right)\right]=29.751$$

Question: Evaluate the integral $\int_{100}^{200}\frac{dx}{\ln x}$. Taking n = 4 using Trapezoidal rule, Simpson's 1/3 rule and Bool's rule.

Solution: Given that a = 100 , b = 200 and n = 4.

Thus $$h=\frac{b-a}{n}=\frac{200-100}{4}=25$$

We make the table

x	100	125	150	175	200
$f(x)=\frac{1}{\ln x}$	0.2171	0.2071	0.1996	0.1936	0.1887

Using Trapezoidal rule for n = 4 we have

$$I=\frac{h}{2}\left[\left(y_0+y_4\right)+2\left(y_1+y_2+y_3\right)\right]$$

$$I=\frac{25}{2}\left[\left(0.2171+0.1887\right)+2\left(0.2071+0.1996+0.1936\right)\right]=20.08$$

Using Simpson's 1/3 rule for n = 4 we have

$$I=\frac{h}{3}\left[\left(y_0+y_4\right)+2\left(y_2\right)+4\left(y_1+y_3\right)\right]$$

$$I=\frac{25}{3}\left[\left(0.2171+0.1887\right)+2\left(0.1996\right)+4\left(0.2071+0.1936\right)\right]=20.065$$

Using Bool's rule for n = 4 we have

$$I=\frac{2h}{45}\left[7\left(y_0+y_4\right)+12\left(y_2\right)+32\left(y_1+y_3\right)\right]$$

$$I=\frac{2\times 25}{45}\left[7\left(0.2171+0.1887\right)+12\left(0.1996\right)+32\left(0.2071+0.1936\right)\right]=20.06$$

Question: Apply Simpson's 3/8 rule to evaluate the definite integral $\int_0^{\frac{\pi}{8}} e^{\sin x}\,dx$.

Solution: Given that a = 0 , b = $\frac{\pi}{8}$ and n = 3.

Thus $h=\frac{b-a}{n}=\frac{\frac{\pi}{8}-0}{3}=\frac{\pi}{24}$

We arrange the table

x	0	$\frac{\pi}{24}$	$\frac{\pi}{12}$	$\frac{\pi}{8}$
$f(x)=e^{\sin x}$	1	1.00228	1.00458	1.00688

Using Simpson's 3/8 rule for n = 3 we have

$$I=\frac{3h}{8}\left[\left(y_0+y_3\right)+3\left(y_1+y_2\right)\right]$$

$$I=\frac{3.\frac{\pi}{24}}{8}\left[(1+1.00688)+3(1.00228+1.00458)\right]=\frac{\pi}{64}(8.02746)=0.39405$$

Question: Using the combination of Simpson's 1/3 and Simpson' 3/8 rule.

Evaluate the integral $\int_1^8 f(x)\,dx$ By the given table

x	1	2	3	4	5	6	7	8
f(x)	1	4	9	16	25	36	49	64

Solution: Here n = 7 , $h=\frac{b-a}{n}=\frac{8-1}{7}=\frac{7}{7}=1$

We arrange the table for both Simpson's 1/3 and Simpson's 3/8 rule

For Simpson's 3/8 For Simpson's 1/3

n	x_0	x_1	x_2	x_3				
				x_0	x_1	x_2	x_3	x_4
x	1	2	3	4	5	6	7	8
f(x)	1	4	9	16	25	36	49	64

$$I=I_{\frac{3}{8}}+I_{\frac{1}{3}}$$

$$I=\frac{3h}{8}\left[(y_0+y_3)+3(y_1+y_2)\right]+\frac{h}{3}\left[(y_0+y_4)+2y_2+4(y_1+y_3)\right]$$

$$I=\frac{3}{8}\left[(1+16)+3(4+9)\right]+\frac{1}{3}\left[(16+64)+2(36)+4(25+49)\right]=21+149.33=170.33$$

Question: Using the combination of Simpson's 1/3 and Simpson' 3/8 rule.

Evaluate the integral $\int_0^{\frac{\pi}{8}} e^{\sin x}\,dx$

Solution: Here n = 5

[Least value of Simpson's 1/3 is 2 and Simpson's 3/8 is 3 i.e 2+3 = 5]

$$h=\frac{b-a}{n}=\frac{\frac{\pi}{8}-0}{5}=\frac{\pi}{40}$$

We arrange the table for both Simpson's 1/3 and Simpson's 3/8 rule

For Simpson's 3/8 For Simpson's 1/3

n	x_0	x_1	x_2	x_3		
				x_0	x_1	x_2
x	0	$\frac{\pi}{40}$	$\frac{\pi}{20}$	$\frac{3\pi}{40}$	$\frac{\pi}{10}$	$\frac{\pi}{8}$
f(x)	1	1.0014	1.0027	1.0041	1.0055	1.0069

$$I = I_{\frac{3}{8}} + I_{\frac{1}{3}}$$

$$I = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right] + \frac{h}{3}\left[(y_0 + y_2) + 4(y_1)\right]$$

$$I = \frac{3.\frac{\pi}{40}}{8}\left[(1 + 1.0041) + 3(1.0014 + 1.0027)\right] + \frac{\frac{\pi}{40}}{3}\left[(1.0041 + 1.0069) + 4(1.0055)\right]$$

$$I = 0.2361 + 0.1579$$

$$I = 0.39404$$

Lecture # 12

Numerical solution of ODE's (ordinary differential equations) or Initial value problem:

(i) Picard's Method:

Consider an initial value problem

$$\frac{dy}{dx} = F(x,y) \quad ; \quad y_0 = y(x_0) \quad \_\_\_\_(i)$$

Eq (i) can be written as

$$dy = F(x,y)dx$$

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} F(x,y)dx$$

$$[y]_{x_0}^{x} = \int_{x_0}^{x} F(x,y)dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} F(x,y)dx$$

$$y = y_0 + \int_{x_0}^{x} F(x,y_n)dx \qquad \because y(x) = y \, , y(x_0) = y_0$$

This formulation enables us to propose the following iterative scheme.

$$y_{n+1} = y_0 + \int_{x_0}^{x} F(x,y_n)dx \qquad ; n = 0,1,2,.....$$

This is called Picard's method for solving the problem (i).

Question: Use Picard's method to approximate the value of y when x = 0.1 , 0.2, 0.3, 0.4, 0.5. Given that y = 1 at x = 0 and $y' = 1 + xy$ correct to three decimal places.

Solution: Given that $F(x,y) = y' = 1 + xy \quad ; \quad y(0) = 1$

Using Picard's method, we have

$$y_{n+1} = y_0 + \int_{x_0}^{x} F(x, y_n)\,dx$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} (1 + xy_n)\,dx$$

$$y_{n+1} = 1 + \int_{0}^{x} (1 + xy_n)\,dx \quad \therefore\ y_0 = 1, x_0 = 0\ ; n = 0,1,2,\ldots$$

1st Approximation: (n = 0)

$$y_1 = 1 + \int_{0}^{x} (1 + xy_0)\,dx$$

$$y_1 = 1 + \int_{0}^{x} (1 + x)\,dx \quad \because\ y_0 = 1$$

$$y_1 = 1 + \left[x + \frac{x^2}{2}\right]_0^x$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

2nd Approximation: (n = 1)

$$y_2 = 1 + \int_{0}^{x} (1 + xy_1)\,dx$$

$$y_2 = 1 + \int_{0}^{x} \left(1 + x + x^2 + \frac{x^3}{2}\right)dx$$

$$y_2 = 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]_0^x$$

$$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

3rd Approximation: (n = 2)

$$y_3 = 1 + \int_{0}^{x} (1 + xy_2)\,dx$$

$$y_3 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right) dx$$

$$y_3 = 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}\right]_0^x$$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

4th Approximation: (n = 3)

$$y_4 = 1 + \int_0^x \left(1 + xy_3\right) dx$$

$$y_4 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8} + \frac{x^6}{15} + \frac{x^7}{48}\right) dx$$

$$y_4 = 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{105} + \frac{x^8}{384}\right]_0^x$$

$$y_4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{105} + \frac{x^8}{384}$$

Comparison Table:

x	0.1	0.2	0.3	0.4	0.5
y_1	1.105	1.220	1.345	1.480	1.625
y_2	1.105	1.223	1.355	1.505	1.674
y_3	1.105	1.223	1.355	1.505	1.677
y_4	1.105	1.223	1.355	1.505	1.677

Question: Using Picard's method find y(0.2). Given that $y' = x - y \;\; ; \;\; y(0) = 1$

Solution: Given that $F(x, y) = y' = x - y$

Using Picard's method, we have

$$y_{n+1} = y_0 + \int_{x_0}^x F(x, y_n) dx$$

$$y_{n+1} = y_0 + \int_{x_0}^x (x - y_n) dx$$

$$y_{n+1} = 1 + \int_{x_0}^{x} (x - y_n)\, dx \qquad \because x_0 = 0\ ,\ y_0 = 1\ ;\ n = 0,1,2,3,4$$

1st Approximation: (n = 0)

$$y_1 = 1 + \int_0^x (x - y_0)\, dx$$

$$y_1 = 1 + \int_0^x (x - 1)\, dx$$

$$y_1 = 1 + \left[\frac{x^2}{2} - x\right]_0^x$$

$$y_1 = 1 + \frac{x^2}{2} - x$$

$At\ x = 0.2 \quad y_1 = 0.82 \quad \Rightarrow y_1(0.2) = 0.82$

2nd Approximation: (n = 1)

$$y_2 = 1 + \int_0^x (x - y_1)\, dx$$

$$y_2 = 1 + \int_0^x (x - 0.82)\, dx$$

$$y_2 = 1 + \left[\frac{x^2}{2} - 0.82x\right]_0^x$$

$$y_2 = 1 + \frac{x^2}{2} - 0.82x$$

$At\ x = 0.2 \quad y_2 = 0.856 \quad \Rightarrow y_2(0.2) = 0.856$

3rd Approximation: (n = 2)

$$y_3 = 1 + \int_0^x (x - y_2)\, dx$$

$$y_3 = 1 + \int_0^x (x - 0.856)\, dx$$

$$y_3 = 1 + \left[\frac{x^2}{2} - 0.856x \right]_0^x$$

$$y_3 = 1 + \frac{x^2}{2} - 0.856x$$

$$At\ x = 0.2 \quad y_3 = 0.848 \quad \Rightarrow y_3(0.2) = 0.848$$

4th Approximation: (n = 3)

$$y_4 = 1 + \int_0^x (x - y_3)\,dx$$

$$y_4 = 1 + \int_0^x (x - 0.848)\,dx$$

$$y_4 = 1 + \left[\frac{x^2}{2} - 0.848x \right]_0^x$$

$$y_4 = 1 + \frac{x^2}{2} - 0.848x$$

$$At\ x = 0.2 \quad y_4 = 0.850 \quad \Rightarrow y_4(0.2) = 0.850$$

5th Approximation: (n = 4)

$$y_5 = 1 + \int_0^x (x - y_4)\,dx$$

$$y_5 = 1 + \int_0^x (x - 0.850)\,dx$$

$$y_5 = 1 + \left[\frac{x^2}{2} - 0.850x \right]_0^x$$

$$y_5 = 1 + \frac{x^2}{2} - 0.850x$$

$$At\ x = 0.2 \quad y_5 = 0.850 \quad \Rightarrow y_5(0.2) = 0.850$$

Question: Using Picard's method, obtained a solution up to 5th Approximation to the equation $y' = y + x$ such that y(0) = 1. Check your answer by finding the exact solution. Also find y(0.01) and y(0.2).

Solution: Given that $F(x,y) = y' = y + x$

Using Picard's method, we have

$$y_{n+1} = y_0 + \int_{x_0}^{x} F(x, y_n)\,dx$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} (y_n + x)\,dx$$

$$y_{n+1} = 1 + \int_{x_0}^{x} (y_n + x)\,dx \qquad \because x_0 = 0 \text{ , } y_0 = 1 \text{ ; } n = 0,1,2,3,4$$

1st Approximation: (n = 0)

$$y_1 = 1 + \int_0^x (y_0 + x)\,dx$$

$$y_1 = 1 + \int_0^x (1 + x)\,dx \qquad \because y_0 = 1$$

$$y_1 = 1 + \left[x + \frac{x^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2}$$

$$At\ x = 0.1 \qquad y_1 = 1.105$$

$$At\ x = 0.2 \qquad y_1 = 1.22$$

2nd Approximation: (n = 1)

$$y_2 = 1 + \int_0^x (y_1 + x)\,dx$$

$$y_2 = 1 + \int_0^x (1.105 + x)\,dx \quad , \quad y_2 = 1 + \int_0^x (1.22 + x)\,dx$$

$$y_2 = 1 + \left[1.105x + \frac{x^2}{2}\right]_0^x = 1 + 1.105x + \frac{x^2}{2} \text{ , } y_2 = 1 + \left[1.22x + \frac{x^2}{2}\right]_0^x = 1 + 1.22x + \frac{x^2}{2}$$

$$At\ x = 0.1 \qquad y_2 = 1.1155 \quad , \quad At\ x = 0.2 \qquad y_2 = 1.264$$

3rd Approximation: (n = 2)

$$y_3 = 1 + \int_0^x (y_2 + x)\,dx$$

$$y_3 = 1 + \int_0^x (1.1155 + x)\,dx \qquad , \qquad y_3 = 1 + \int_0^x (1.264 + x)\,dx$$

$$y_3 = 1 + \left[1.1155x + \frac{x^2}{2}\right]_0^x = 1 + 1.1155x + \frac{x^2}{2}, \; y_3 = 1 + \left[1.264x + \frac{x^2}{2}\right]_0^x = 1 + 1.264x + \frac{x^2}{2}$$

$At\ x = 0.1 \qquad y_3 = 1.1166 \qquad , \qquad At\ x = 0.2 \qquad y_3 = 1.269$

4th Approximation: (n = 3)

$$y_4 = 1 + \int_0^x (y_3 + x)\,dx$$

$$y_4 = 1 + \int_0^x (1.1166 + x)\,dx \qquad , \qquad y_4 = 1 + \int_0^x (1.269 + x)\,dx$$

$$y_4 = 1 + \left[1.1166x + \frac{x^2}{2}\right]_0^x = 1 + 1.1166x + \frac{x^2}{2}, \; y_4 = 1 + \left[1.269x + \frac{x^2}{2}\right]_0^x = 1 + 1.269x + \frac{x^2}{2}$$

$At\ x = 0.1 \qquad y_4 = 1.1167 \qquad , \qquad At\ x = 0.2 \qquad y_4 = 1.274$

5th Approximation: (n = 4)

$$y_5 = 1 + \int_0^x (y_4 + x)\,dx$$

$$y_5 = 1 + \int_0^x (1.1167 + x)\,dx \qquad , \qquad y_5 = 1 + \int_0^x (1.274 + x)\,dx$$

$$y_5 = 1 + \left[1.1167x + \frac{x^2}{2}\right]_0^x = 1 + 1.1167x + \frac{x^2}{2}, \; y_5 = 1 + \left[1.274x + \frac{x^2}{2}\right]_0^x = 1 + 1.274x + \frac{x^2}{2}$$

$At\ x = 0.1 \qquad y_5 = 1.1167 \qquad , \qquad At\ x = 0.2 \qquad y_5 = 1.274$

$\Rightarrow \qquad y(0.1) = 1.1167$

$\Rightarrow \qquad y(0.2) = 1.274$

Exact Values:

$$y' = \frac{dy}{dx} = y + x$$

Which is linear equation

$$\frac{dy}{dx} - y = x \quad \_\_\_\_(i)$$

$$I.F = e^{-\int 1dx} = e^{-x}$$

Multiplying (i) by I.F

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}x$$

$$d\left(ye^{-x}\right) = xe^{-x}$$

On integration

$$\int d\left(ye^{-x}\right) = \int xe^{-x}$$

$$ye^{-x} = x\frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1}dx$$

$$ye^{-x} = -xe^{-x} + \int e^{-x}dx$$

$$ye^{-x} = -xe^{-x} + \frac{e^{-x}}{-1} + c$$

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

Multiplying by e^{x}

$$y = -x - 1 + ce^{x}$$

Initial condition y(0) = 1

$1 = -0-1+c\ e^{0}$

$1+1 = c \quad \Rightarrow \quad c = 2$

$$y = -x - 1 + 2e^x$$

For check x = 0

$$y = -0 - 1 + 2e^0 \ = -1 + 2 = 1$$

$\Rightarrow$ y(0) = 1

For x = 0.1 $\quad y = -0.1 - 1 + 2e^{0.1} \quad \Rightarrow \quad y = 1.1103$

For x = 0.2 $\quad y = -0.2 - 1 + 2e^{0.2} \quad \Rightarrow \quad y = 1.2428$

Euler's Method:

Consider an initial value problem

$$\frac{dy}{dx} = F(x, y) \ ; y_0 = y(x_0) \ \_\_\_\_(i)$$

Using the definition of derivative, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

If h > 0 is very small, then

$$\frac{dy}{dx} \cong \frac{y(x+h) - y(x)}{h} \quad \_\_\_\_(ii)$$

From (i) and (ii) we have

$$\frac{y(x+h) - y(x)}{h} = F(x, y)$$

$$y(x+h) - y(x) = hF(x, y)$$

$$y(x+h) = y(x) + hF(x, y)$$

This formulation enables us to propose the following iterative scheme.

Algorithm I: $y_{n+1} = y_n + hF(x_n, y_n) \ \_\_\_\_(iii) \quad ; n = 0, 1, 2, ...$

This is called Euler's method for solving the problem (i).

Question: With h = 0.1 Find the numerical solution on 0 < x < 1 by using Euler's method for $y' = y^2 + 2x - x^4 ; y(0) = 0$ Also compare your results with the exact solution $y = x^2$

Solution: Given that $F(x,y) = y' = y^2 + 2x - x^4 ; y(0) = 0$

And h = 0.1 on the interval [0,1]

$$h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} = \frac{1-0}{0.1} = 10$$

$\Rightarrow x_0 = 0 , x_1 = x_0 + h = 0 + 0.1 = 0.1 , x_2 = 0.2 , x_3 = 0.3, x_4 = 0.4, x_5 = 0.5, x_6 = 0.6$

$\Rightarrow x_7 = 0.7 , x_8 = 0.8 , x_9 = 0.9, x_{10} = 1$

Using Euler's method, we have

$$y_{n+1} = y_n + hF(x_n, y_n)$$

$$y_{n+1} = y_n + 0.1\left(y_n^2 + 2x_n - x_n^4\right)$$

1st Approximation: (n = 0)

$$y_1 = y_0 + 0.1\left(y_0^2 + 2x_0 - x_0^4\right)$$

$$y_1 = 0 + 0.1(0 + 0 - 0) = 0$$

2nd Approximation: (n = 1)

$$y_2 = y_1 + 0.1\left(y_1^2 + 2x_1 - x_1^4\right)$$

$$y_2 = 0 + 0.1\left(0 + 2(0.1) - (0.1)^4\right) = 0.0019$$

3rd Approximation: (n = 2)

$$y_3 = y_2 + 0.1\left(y_2^2 + 2x_2 - x_2^4\right)$$

$$y_3 = 0.0019 + 0.1\left((0.0019)^2 + 2(0.2) - (0.2)^4\right) = 0.042$$

4th Approximation: (n = 3)

$$y_4 = y_3 + 0.1\left(y_3^2 + 2x_3 - x_3^4\right)$$

$$y_4 = 0.042 + 0.1\left((0.042)^2 + 2(0.3) - (0.3)^4\right) = 0.101$$

5th Approximation: (n = 4)

$$y_5 = y_4 + 0.1\left(y_4^2 + 2x_4 - x_4^4\right)$$

$$y_5 = 0.101 + 0.1\left((0.101)^2 + 2(0.4) - (0.4)^4\right) = 0.179$$

6th Approximation: (n = 5)

$$y_6 = y_5 + 0.1\left(y_5^2 + 2x_5 - x_5^4\right)$$

$$y_6 = 0.179 + 0.1\left((0.179)^2 + 2(0.5) - (0.5)^4\right) = 0.276$$

7th Approximation: (n = 6)

$$y_7 = y_6 + 0.1\left(y_6^2 + 2x_6 - x_6^4\right)$$

$$y_7 = 0.276 + 0.1\left((0.276)^2 + 2(0.6) - (0.6)^4\right) = 0.391$$

8th Approximation: (n = 7)

$$y_8 = y_7 + 0.1\left(y_7^2 + 2x_7 - x_7^4\right)$$

$$y_8 = 0.391 + 0.1\left((0.391)^2 + 2(0.7) - (0.7)^4\right) = 0.522$$

9th Approximation: (n = 8)

$$y_9 = y_8 + 0.1\left(y_8^2 + 2x_8 - x_8^4\right)$$

$$y_9 = 0.522 + 0.1\left((0.522)^2 + 2(0.8) - (0.8)^4\right) = 0.645$$

10^{th} Approximation: (n = 9)

$$y_{10} = y_9 + 0.1\left(y_9^2 + 2x_9 - x_9^4\right)$$

$$y_{10} = 0.645 + 0.1\left((0.645)^2 + 2(0.9) - (0.9)^4\right) = 0.801$$

11^{th} Approximation: (n = 10)

$$y_{11} = y_{10} + 0.1\left(y_{10}^2 + 2x_{10} - x_{10}^4\right)$$

$$y_{11} = 0.801 + 0.1\left((0.801)^2 + 2(1) - (1)^4\right) = 0.965$$

Comparison with exact solution:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$y' = y^2 + 2x - x^4$	0	0.0019	0.042	0.101	0.179	0.276	0.391	0.522	0.645	0.801	0.965
$y = x^2$	0	0.01	0.04	0.09	0.16	0.25	0.36	0.49	0.64	0.81	1

Question: : With h = 0.1 Find the numerical solution on 0 < x < 2 by using Euler's method for $y' = y^3 - 8x^3 + 2$; $y(0) = 0$ Also compare your results with the exact solution $y = 2x$

Solution: Given that $F(x,y) = y' = y^3 - 8x^3 + 2$; $y(0) = 0$

And h = 0.1 on the interval [0,1]

$$h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} = \frac{2-0}{0.1} = 20$$

$$\Rightarrow x_0 = 0 \text{ , } x_1 = x_0 + h = 0 + 0.1 = 0.1 \text{ , } x_2 = 0.2 \text{ , } x_3 = 0.3, \ldots\ldots x_{20} = 2$$

Using Euler's method, we have

$$y_{n+1} = y_n + hF(x_n, y_n)$$

$$y_{n+1} = y_n + 0.1\left(y_n^3 - 8x_n^3 + 2\right) ; n = 0,1,2,\ldots,20$$

1st Approximation: (n = 0)

$$y_1 = y_0 + 0.1\left(y_0^3 - 8x_0^3 + 2\right)$$

$$y_1 = 0 + 0.1(0 - 0 + 2) = 0.2$$

2nd Approximation: (n = 1)

$$y_2 = y_1 + 0.1\left(y_1^3 - 8x_1^3 + 2\right)$$

$$y_2 = 0.2 + 0.1\left((0.2)^3 - 8(0.1)^3 + 2\right) = 0.4$$

3rd Approximation: (n = 2)

$$y_3 = y_2 + 0.1\left(y_2^3 - 8x_2^3 + 2\right)$$

$$y_3 = 0.4 + 0.1\left((0.4)^3 - 8(0.2)^3 + 2\right) = 0.6$$

4th Approximation: (n = 3)

$$y_4 = y_3 + 0.1\left(y_3^3 - 8x_3^3 + 2\right)$$

$$y_4 = 0.6 + 0.1\left((0.6)^3 - 8(0.3)^3 + 2\right) = 0.8$$

5th Approximation: (n = 4)

$$y_5 = y_4 + 0.1\left(y_4^3 - 8x_4^3 + 2\right)$$

$$y_5 = 0.8 + 0.1\left((0.8)^3 - 8(0.4)^3 + 2\right) = 1$$

6th Approximation: (n = 5)

$$y_6 = y_5 + 0.1\left(y_5^3 - 8x_5^3 + 2\right)$$

$$y_6 = 1 + 0.1\left((1)^3 - 8(0.5)^3 + 2\right) = 1.2$$

7^th Approximation: (n = 6)

$$y_7 = y_6 + 0.1\left(y_6^3 - 8x_6^3 + 2 \right)$$

$$y_7 = 1.2 + 0.1\left(\left(1.2\right)^3 - 8\left(0.6\right)^3 + 2\right) = 1.4$$

8^th Approximation: (n = 7)

$$y_8 = y_7 + 0.1\left(y_7^3 - 8x_7^3 + 2 \right)$$

$$y_8 = 1.4 + 0.1\left(\left(1.4\right)^3 - 8\left(0.7\right)^3 + 2\right) = 1.6$$

9^th Approximation: (n = 8)

$$y_9 = y_8 + 0.1\left(y_8^3 - 8x_8^3 + 2 \right)$$

$$y_9 = 1.6 + 0.1\left(\left(1.6\right)^3 - 8\left(0.8\right)^3 + 2\right) = 1.8$$

10^th Approximation: (n = 9)

$$y_{10} = y_9 + 0.1\left(y_9^3 - 8x_9^3 + 2 \right)$$

$$y_{10} = 1.8 + 0.1\left(\left(1.8\right)^3 - 8\left(0.9\right)^3 + 2\right) = 2$$

11^th Approximation: (n = 10)

$$y_{11} = y_{10} + 0.1\left(y_{10}^3 - 8x_{10}^3 + 2 \right)$$

$$y_{11} = 2 + 0.1\left(\left(2\right)^3 - 8\left(1\right)^3 + 2\right) = 2.2$$

12^th Approximation: (n = 11)

$$y_{12} = y_{11} + 0.1\left(y_{11}^3 - 8x_{11}^3 + 2 \right)$$

$$y_{12} = 2.2 + 0.1\left(\left(2.2\right)^3 - 8\left(1.1\right)^3 + 2\right) = 2.4$$

13th Approximation: (n = 12)

$$y_{13} = y_{12} + 0.1\left(y_{12}^3 - 8x_{12}^3 + 2\right)$$

$$y_{13} = 2.4 + 0.1\left((2.4)^3 - 8(1.2)^3 + 2\right) = 2.6$$

14th Approximation: (n = 13)

$$y_{14} = y_{13} + 0.1\left(y_{13}^3 - 8x_{13}^3 + 2\right)$$

$$y_{14} = 2.6 + 0.1\left((2.6)^3 - 8(1.3)^3 + 2\right) = 2.8$$

15th Approximation: (n = 14)

$$y_{15} = y_{14} + 0.1\left(y_{14}^3 - 8x_{14}^3 + 2\right)$$

$$y_{15} = 2.8 + 0.1\left((2.8)^3 - 8(1.4)^3 + 2\right) = 3$$

16th Approximation: (n = 15)

$$y_{16} = y_{15} + 0.1\left(y_{15}^3 - 8x_{15}^3 + 2\right)$$

$$y_{16} = 3 + 0.1\left((3)^3 - 8(1.5)^3 + 2\right) = 3.2$$

17th Approximation: (n = 16)

$$y_{17} = y_{16} + 0.1\left(y_{16}^3 - 8x_{16}^3 + 2\right)$$

$$y_{17} = 3.2 + 0.1\left((3.2)^3 - 8(1.6)^3 + 2\right) = 3.4$$

18th Approximation: (n = 17)

$$y_{18} = y_{17} + 0.1\left(y_{17}^3 - 8x_{17}^3 + 2\right)$$

$$y_{18} = 3.4 + 0.1\left((3.4)^3 - 8(1.7)^3 + 2\right) = 3.6$$

19th Approximation: (n = 18)

$$y_{19} = y_{18} + 0.1\left(y_{18}^3 - 8x_{18}^3 + 2\right)$$

$$y_{19} = 3.6 + 0.1\left((3.6)^3 - 8(1.8)^3 + 2\right) = 3.8$$

20th Approximation: (n = 19)

$$y_{20} = y_{19} + 0.1\left(y_{19}^3 - 8x_{19}^3 + 2\right)$$

$$y_{20} = 3.8 + 0.1\left((3.8)^3 - 8(1.9)^3 + 2\right) = 4$$

21st Approximation: (n = 20)

$$y_{21} = y_{20} + 0.1\left(y_{20}^3 - 8x_{20}^3 + 2\right)$$

$$y_{24} = 4 + 0.1\left((4)^3 - 8(2)^3 + 2\right) = 4.2$$

Comparison with exact solution:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$y' = y^2 - 8x^3 + 2$	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$y = 2x$	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2

x	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$y' = y^2 - 8x^3 + 2$	2.6	2.8	3	3.2	3.4	3.6	3.8	4
$y = 2x$	2.6	2.8	3	3.2	3.4	3.6	3.8	4

Lecture # 13

$y = F(x)$ is called explicit function

$y = F(x,y)$ is called implicit function.

Euler method is also called implicit Euler function. i.e.

$$y_{n+1} = y_n + h\,F(x_n\,,\,y_n)$$

Semi-Implicit Euler's Method:

Consider an initial value problem

$$\frac{dy}{dx} = F(x,y) \;\; ; \; y_0 = y(x_0) \quad \_\_\_\_(i)$$

Using the definition of derivative, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

If $h > 0$ is very small, then

$$\frac{dy}{dx} \cong \frac{y(x+h) - y(x)}{h} \quad \_\_\_\_(ii)$$

From (i) and (ii) we have

$$\frac{y(x+h) - y(x)}{h} = F(x,y)$$

$$y(x+h) - y(x) = hF(x,y)$$

$$y(x+h) = y(x) + hF(x,y)$$

This formulation enables us to propose the following iterative scheme.

Algorithm **<u>II</u>**:

$$y_{n+1} = y_n + hF(x_n, y_{n+1}) \quad \_\_\_\_(iii) \qquad ; n = 0,1,2,....$$

This is called semi-implicit Euler's method for solving the problem (i)

To implement implicit scheme, we introduce two steps (predictor corrector technique) for solving initial value problems. Thus, from the scheme (iii) we shall introduce the following two step scheme.

Algorithm <u>III</u>: (Two step scheme of semi-implicit Euler's method)

For given $y_0 = y(x_0)$ *find* y_{n+1}

I: $u_n = y_n + hF(x_n, y_n)$

II: $y_{n+1} = y_n + hF(x_n, u_n)$; $n = 0,1,2,....$

Example: With h = 0.1 find the numerical solution on 0 < x < 1 by using semi-implicit Euler's method for $y' = y^2 + 2x - x^4$, $y(0) = 0$. Also compare your results with the exact solution $y = x^2$

Solution: Given that $F(x, y) = y' = y^2 + 2x - x^4$, $y(0) = 0$

And h = 0.1 on the interval (0,1). Thus,

$$h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} = \frac{1-0}{0.1} = 10$$

$$\Rightarrow x_0 = 0, x_1 = 0.1, x_2 = 0.2, ...x_{10} = 1$$

Using two step Semi-Euler method

I: $u_n = y_n + hF(x_n, y_n)$

II: $y_{n+1} = y_n + hF(x_n, u_n)$; $n = 0,1,2,....$

From this we have

I: $u_n = y_n + 0.1(y_n^2 + 2x_n - x_n^4)$

II: $y_{n+1} = y_n + 0.1(u_n^2 + 2x_n - x_n^4)$

1st Approximation: (n = 0, x_0 = 0, y_0 = 0)

I: $u_0 = y_0 + 0.1(y_0^2 + 2x_0 - x_0^4) \quad \Rightarrow \quad u_0 = 0 + 0.1\left(0^2 + 2(0) - 0^4\right) = 0$

II: $y_1 = y_0 + 0.1(u_0^2 + 2x_0 - x_0^4) \quad \Rightarrow \quad y_1 = 0 + 0.1\left(0^2 + 2(0) - 0^4\right) = 0$

2nd Approximation: ($n = 1$, $x_1 = 0.1$, $y_1 = 0$)

I: $u_1 = y_1 + 0.1(y_1^2 + 2x_1 - x_1^4)$

$$u_1 = 0 + 0.1\left(0^2 + 2(0.1) - (0.1)^4\right) = 0.01999$$

II: $y_2 = y_1 + 0.1(u_1^2 + 2x_1 - x_1^4)$

$$y_2 = 0 + 0.1\left((0.01999)^2 + 2(0.1) - (0.1)^4\right) = 0.0200$$

3rd Approximation: ($n = 2$, $x_2 = 0.2$, $y_2 = 0.0200$)

I: $u_2 = y_2 + 0.1(y_2^2 + 2x_2 - x_2^4)$

$$u_2 = 0.02 + 0.1\left((0.02)^2 + 2(0.2) - (0.2)^4\right) = 0.05988$$

II: $y_3 = y_2 + 0.1(u_2^2 + 2x_2 - x_2^4)$

$$y_3 = 0.02 + 0.1\left((0.05988)^2 + 2(0.2) - (0.2)^4\right) = 0.06019$$

4th Approximation: ($n = 3$, $x_3 = 0.3$, $y_3 = 0.06019$)

I: $u_3 = y_3 + 0.1(y_3^2 + 2x_3 - x_3^4)$

$$u_3 = 0.06019 + 0.1\left((0.06019)^2 + 2(0.3) - (0.3)^4\right) = 0.1197$$

II: $y_4 = y_3 + 0.1(u_3^2 + 2x_3 - x_3^4)$

$$y_4 = 0.06019 + 0.1\left((0.1197)^2 + 2(0.3) - (0.3)^4\right) = 0.1208$$

5th Approximation: ($n = 4$, $x_4 = 0.4$, $y_4 = 0.1208$)

I: $u_4 = y_4 + 0.1(y_4^2 + 2x_4 - x_4^4)$

$$u_4 = 0.1208 + 0.1\left((0.1208)^2 + 2(0.4) - (0.4)^4\right) = 0.1997$$

II: $y_5 = y_4 + 0.1(u_4^2 + 2x_4 - x_4^4)$

$$y_5 = 0.1208 + 0.1\left((0.1997)^2 + 2(0.4) - (0.4)^4\right) = 0.2022$$

6th Approximation: (n = 5, $x_5 = 0.5$, $y_5 = 0.2022$)

I: $u_5 = y_5 + 0.1(y_5^2 + 2x_5 - x_5^4)$

$$u_5 = 0.2022 + 0.1\left((0.2022)^2 + 2(0.5) - (0.5)^4\right) = 0.300$$

II: $y_6 = y_5 + 0.1(u_5^2 + 2x_5 - x_5^4)$

$$y_6 = 0.2022 + 0.1\left((0.3)^2 + 2(0.5) - (0.5)^4\right) = 0.3049$$

7th Approximation: (n = 6, $x_6 = 0.6$, $y_6 = 0.3049$)

I: $u_6 = y_6 + 0.1(y_6^2 + 2x_6 - x_6^4)$

$$u_6 = 0.3049 + 0.1\left((0.3049)^2 + 2(0.6) - (0.6)^4\right) = 0.4212$$

II: $y_7 = y_6 + 0.1(u_6^2 + 2x_6 - x_6^4)$

$$y_7 = 0.3049 + 0.1\left((0.4212)^2 + 2(0.6) - (0.6)^4\right) = 0.4296$$

8th Approximation: (n = 7, $x_7 = 0.7$, $y_7 = 0.4296$)

I: $u_7 = y_7 + 0.1(y_7^2 + 2x_7 - x_7^4)$

$$u_7 = 0.4296 + 0.1\left((0.4296)^2 + 2(0.7) - (0.7)^4\right) = 0.5640$$

II: $y_8 = y_7 + 0.1(u_7^2 + 2x_7 - x_7^4)$

$$y_8 = 0.4296 + 0.1\left((0.5640)^2 + 2(0.7) - (0.7)^4\right) = 0.5774$$

9th Approximation: (n = 8, $x_8 = 0.8$, $y_8 = 0.5774$)

I: $u_8 = y_8 + 0.1(y_8^2 + 2x_8 - x_8^4)$

$$u_8 = 0.5774 + 0.1\left((0.5774)^2 + 2(0.8) - (0.8)^4\right) = 0.7298$$

II: $y_9 = y_8 + 0.1(u_8^2 + 2x_8 - x_8^4)$

$$y_9 = 0.5774 + 0.1\left((0.7298)^2 + 2(0.8) - (0.8)^4\right) = 0.7497$$

10th Approximation: (n = 9, $x_9 = 0.9$, $y_9 = 0.7497$)

I: $u_9 = y_9 + 0.1(y_9^2 + 2x_9 - x_9^4)$

$$u_9 = 0.7497 + 0.1\left((0.7497)^2 + 2(0.9) - (0.9)^4\right) = 0.9203$$

II: $y_{10} = y_9 + 0.1(u_9^2 + 2x_9 - x_9^4)$

$$y_{10} = 0.7497 + 0.1\left((0.9202)^2 + 2(0.9) - (0.9)^4\right) = 0.9488$$

11th Approximation: (n = 10, $x_{10} = 1$, $y_{10} = 0.9488$)

I: $u_{10} = y_{10} + 0.1(y_{10}^2 + 2x_{10} - x_{10}^4)$

$$u_{10} = 0.9488 + 0.1\left((0.9488)^2 + 2(1) - (1)^4\right) = 1.1388$$

II: $y_{11} = y_1 + 0.1(u_{10}^2 + 2x_{10} - x_{10}^4)$

$$y_{11} = 0.9488 + 0.1\left((1.1388)^2 + 2(1) - (1)^4\right) = 1.1784$$

Comparison with exact solution:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$y' = y^2 + 2x - x^4$	0	0.02	0.0602	0.1208	0.2022	0.3049	0.4296	0.5774	0.7497	0.9488	1.1784
$y = x^2$	0	0.01	0.04	0.09	0.16	0.25	0.36	0.49	0.64	0.81	1

Implicit Euler's Method:

Consider an initial value problem

$$\frac{dy}{dx} = F(x,y) \;\; ; y_0 = y(x_0) \quad \_\_\_\_(i)$$

Using the definition of derivative, we have

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h)-y(x)}{h}$$

If h > 0 is very small, then

$$\frac{dy}{dx} \cong \frac{y(x+h)-y(x)}{h} \qquad \_\_\_\_(ii)$$

From (i) and (ii) we have

$$\frac{y(x+h)-y(x)}{h} = F(x,y)$$

$$y(x+h)-y(x) = hF(x,y)$$

$$y(x+h) = y(x) + hF(x,y)$$

This formulation enables us to propose the following iterative scheme.

Algorithm IV:

For given $y_0 = y(x_0)$, Find y_{n+1}

$$y_{n+1} = y_n + hF(x_{n+1}, y_{n+1})\ ;\ n = 0,1,2,.....$$

This is called Implicit Euler method for solving problem (i)

To implement Algorithm IV, we shall propose the following two step scheme.

Algorithm V: (Two step scheme of Implicit Euler's method)

For given $y_0 = y(x_0)$ *find* y_{n+1}

I: $u_n = y_n + hF(x_n, y_n)$

II: $y_{n+1} = y_n + hF(x_{n+1}, u_n)$; $n = 0,1,2,....$

Modified Euler's method (Trapezoid method)

Consider an initial value problem

$$\frac{dy}{dx}=F(x,y)\ \ ;\ y_0=y(x_0)\quad \_\_\_\_(i)$$

Using Trapezoidal rule of n = 1 , we have

$$\int_a^x f(x)dx=\frac{x-a}{2}\left[f(a)-f(x)\right]$$

If 'f' is differentiable then

$$\int_a^x f'(x)dx=\frac{x-a}{2}\left[f'(a)-f'(x)\right]$$

By using Fundamental theorem of calculus

$$f(x)-f(a)=\frac{x-a}{2}\left[f'(a)-f'(x)\right]$$

$$f(x)=f(a)+\frac{x-a}{2}\left[f'(a)-f'(x)\right]$$

This formulation enables us to propose the following iterative scheme.

Algorithm <u>VI</u>:

$$f(x_{n+1})=f(x_n)+\frac{x_{n+1}-x_n}{2}\left[f'(x_n)-f'(x_{n+1})\right]$$

$$f(x_{n+1})=f(x_n)+\frac{h}{2}\left[f'(x_n)-f'(x_{n+1})\right]$$

Or $$y_{n+1}=y_n+\frac{h}{2}\left[y_n'-y_{n+1}'\right]$$

Or $$y_{n+1}=y_n+\frac{h}{2}\left[F(x_n,y_n)+F(x_{n+1},y_{n+1})\right];n=0,1,2,....$$

This is called Modified Euler's method for solving problem (i).

Algorithm VI is also an implicit scheme. Therefore, to implement the algorithm VI we shall propose the following two step algorithm.

Algorithm VII: (Two step scheme for Modified Euler's method)

For given $y_0 = y(x_0)$ *find* y_{n+1}

I: $u_n = y_n + hF(x_n, y_n)$

II: $y_{n+1} = y_n + h\left[F(x_n, \mathrm{y}_n) + \mathrm{F}(x_{n+1}, \mathrm{y}_n)\right]$; $\mathrm{n} = 0,1,2,....$