METHODS OF MATHEMATICAL PHYSICS (MMP)

MUHAMMAD USMAN HAMID

The course provides a foundation to solve PDE's, ODE's, IE's with special emphasis on wave, heat and Laplace equations, formulation and some theory of these equations are also intended.

RECOMMENDED BOOK:

Problems and Methods in Physics and Applied Mathematics by Khalid Latif Mir

MMP by Lal Din Baig

This work is dedicated

To

Sir Anwar Khan

by

“a relatively insignificant student”

MUHAMMAD USMAN HAMID

INTRODUCTION TO DIFFERENTIAL EQUATION

DIFFERENCE EQUATION:

An equation involving differences (derivatives) is called difference equation.

DIFFERENTIAL EQUATION

An equation that relate a function to its derivative in such a way that the function itself can be determined.

OR an equation containing the derivatives of one dependent variable with respect to one or more independent variables is said to be a differential equation. It has two types:

i. **Ordinary differential equation (ODE)**
ii. **Partial differential equation (PDE)**

ORDINARY DIFFERENTIAL EQUATION

A differential equation that contains only one independent variable is called ODE.

EXAMPLES:

$y_x + xy = x^2$, $y''(x) - y'(x) + 6y = 0$ **And in general** $y = f(x)$

PARTIAL DIFFERENTIAL EQUATION

A differential equation that contains, in addition to the dependent variable and the independent variables, one or more partial derivatives of the dependent variable is called a partial differential equation.

In general, it may be written in the form

$$f\ (x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{xy}, \dots)\ =\ 0$$

involving several independent variables x,y, an unknown function 'u' of these variables, and the partial derivatives $u_x, u_y, \dots, u_{xx}, u_{xy}, \dots$, of the function. Subscripts on dependent variables denote differentiations, e.g.,

$u_x = \frac{\partial u}{\partial x}$ *and* $u_y = \frac{\partial u}{\partial y}$. $u_{xx} - u_{yy} = 0$, **are partial differential equations**

In general $u = u\ (x, y)$

HOMOGENEOUS DIFFERENTIAL EQUATION

An equation which always posseses that trivial solution i.e. $u = 0$ is called Homogeneous DE.

Or DE for which $u = 0$ is a solution is called a Homogeneous DE.

EXAMPLES: $uu_{xy} + u_{xy} = 0$, $u_{xx} + u_{yy} = 0$,

$uu_{xx} + 2yu_{xy} + 3xu_{yy} = 0$, $(u_x)^2 + (u_y)^2 = 0$

NON - HOMOGENEOUS DIFFERENTIAL EQUATION

An equation which always posseses that non - trivial solution i.e. $u \neq 0$ is called non -Homogeneous DE.

EXAMPLES: $u_{xx} + u_{yy} = f(x, y)$, $uu_{xx} + 2yu_{xy} + 3xu_{yy} = 4sinx$,

$(u_x)^2 + (u_y)^2 = 1$,

THE ORDER OF A PARTIAL DIFFERENTIAL EQUATION

The order of a partial differential equation is the order of the highest ordered partial derivative appearing in the equation.

For example $u_{xx} + 2xu_{xy} + u_{yy} = e^y$ is a second-order partial differential equation,

And $u_{xxy} + xu_{yy} + 8u = 7y$ is a third-order partial differential equation.

THE DEGREE OF A PARTIAL DIFFERENTIAL EQUATION

The degree of PDE is the highest power of variable appear in PDE. For example $u_x + u_y = u + xy$ is of degree one. And $(u_{xx})^2 = (1+u_y)^{1/2}$ is of degree two.

LINEAR PARTIAL DIFFERENTIAL EQUATION

A differential equation is said to be linear if it is linear in the unknown function (dependent variable) and all its derivatives with coefficients depending only on the independent variables.

For example, the equation

$yu_{xx} + 2xyu_{yy} + u = 1$ and $u_{xx} + u_{yy} = u$ are linear differential equations

NON LINEAR PARTIAL DIFFERENTIAL EQUATION

A differential equation is said to be nonlinear if the unknown function (dependent variable) and all its derivatives with coefficients depending only on the independent variables do not occur linearly.

For example, $u_{xx} + uu_y = 1$, $\frac{dy}{dx} + \sqrt{y} = x$

INITIAL CONDITIONS:

If all conditions are given at the same value of the independent variable, then they are called initial conditions.

For example for a differential equation of order one

$a(x,y)u_x = f(x,y) \Rightarrow u_x = g(x,y)$

Then $u_x = g(x,y)$ with $u(a) = u_0, u'(a) = u_1$ then x = a is an initial condition.

INITIAL VALUE PROBLEM (IVP):

A DE along with initial conditions defines an IVP.

For example, the partial differential equation (PDE)

$u_t - u_{xx} = 0$, $0 < x < l$, $t > 0$, with. $u(x, 0) = \sin x$, $0 \le x \le l$, $t > 0$, is IVP

BOUNDRY CONDITIONS:

If the conditions are given at the end points of the intervals of definition (i.e. for different value of the independent variables) are at the boundary of the domain of definition then they are called boundary conditions.

For example $u'' + 2u' + 3u = 0$ with $u(0) = 0, u(2) = 1$ is a BVP

BOUNDRY VALUE PROBLEM (BVP):

A DE along with boundary conditions defines an IVP.

For example, the partial differential equation (PDE)

$u_t - u_{xx} = 0$, $0 < x < l$, $t > 0$,

with B.C. $u(0, t) = 0$, $t \ge 0$, and B.C. $u(l, t) = 0$, $t \ge 0$,

PRINCIPLE OF SUPERPOSITION:

According to this principle, if we know 'n' solutions "$u_1, u_2, u_3, \dots \dots \dots u_n$" we can construct other as linear combination.

<u>Statement</u>: if $u_1, u_2, u_3, \dots \dots \dots u_n$ are solutions of a linear, homogeneous PDE then $W = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ where $c_1, c_2, \dots \dots, c_n$ are constant is also a solution of the equation.

EXISTENCE THEOREM:

An ODE of order 'n' i.e.

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y^1 + a_0(x)y = f(x)$$

Has exactly 'n' liear indedpedently solutions.

THEOREM:

Let $a_0(x), a_1(x), \dots\dots, a_n(x)$ and $g(x)$ be continuous on an interval $I: [a, b]$ and let $a_n(x) \neq 0$ for every 'x' in this interval. If $x = x_0$ is any point in this interval then a unique solution $y(x)$ of IVP

$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y^1 + a_0 y = g(x)$ with initial conditions exists on this interval.

THEOREM:

Let y_1, y_2,…….., y_n be 'n' solutions of the linear (Homogeneous or Non Homogeneous) ODE on an interval "I" then the set of solutions is linearly independent on "I" if and only if $W(y_1, y_2, \dots\dots.., y_n)(x) \neq 0$ for every 'x' in the interval. Where W is called Wronskian function.

NOTE:

i. **$W(y_1, y_2, \dots\dots.., y_n)(x) \neq 0$ is a Wronskian of 'n' functions of independent varaiable 'x'**

ii. **Wronskian of two functions $y_1(x) = y_1$ and $y_2(x) = y_2$ can be defined as $W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$**

iii. **Wronskian of three functions $y_1(x) = y_1$, $y_2(x) = y_2$ and $y_3(x) = y_3$ can be defined as $W(y_1, y_2, y_3)(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}$**

iv. **Wronskian of 'n' functions $y_1(x) = y_1$, $y_2(x) = y_2$, $y_3(x) = y_3$ ……………$y_n(x) = y_n$ can be defined as**

$$W(y_1, y_2, \dots\dots.., y_n)(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ y''_1 & y''_2 & \cdots & y''_n \\ \vdots & \vdots & \vdots & \vdots \\ {y_1}^{n-1} & {y_2}^{n-1} & \cdots & {y_n}^{n-1} \end{vmatrix}$$

SOLUTION OF DIFFERENTIAL EQUATION

(DIFFERENTIAL FUNCTION):

A function in ‘x’ (for an ODE) or functions of more than one variables (for PDE), such that when it is substituted in the given DE, it is satisfied i.e. the DE takes the form 0 = 0 for all values of the independent variable(s) in the specified domain. It is the relation between the variables not involving differential coefficents.

FUNDAMENTAL SET OF SOLUTIONS:

The set of linearly independent solutions y_i ; $i = 1, 2, 3 \dots \dots$ of the homogeneous or non - homogeneous DE’s is called Fundamental set of solutiosn on the interval ‘I’

EIGENVALUE PROBLEMS:

If an IVP or BVP contains a parameter λ in the DE and non – trivial solution(s) corresponding to certain values of λ can be found then the problem is called Eigenvalue Problem, and the corresponding values of λ are called Eigenvalues of the problem.

EXAMPLE: (UoS,2018 –I, II)

Solve the problem $\frac{d^2u}{dx^2}+\lambda u=0$ **where** λ **is parameter and the boundry conditions are** $u(0)=0$ **and** $u(a)=0$

Solution: **if** $u=0$ **then** λ **has trivial solutions.**

Now for non – trivial solutions we will discuss three cases;

CASE I: when $\lambda=0$

$\frac{d^2u}{dx^2}+\lambda u=0 \Rightarrow \frac{d^2u}{dx^2}+(0)u=0 \Rightarrow \frac{d^2u}{dx^2}=0 \Rightarrow u(x)=c_1x+c_2$ **……….(i)**

Now using BC's $u(0)=0 \Rightarrow c_2=0$ **and** $u(a)=0 \Rightarrow c_1=0$

$(i) \Rightarrow u(x)=0$ **which is trivial solution. So** $\lambda=0$ **is not an eigenvalue.**

CASE II: when $\lambda<0$ **then we may take** $\lambda=-m^2$

Then $\frac{d^2u}{dx^2}+\lambda u=0 \Rightarrow \frac{d^2u}{dx^2}+(-m^2)u=0 \Rightarrow (D^2-m^2)u=0 \Rightarrow D=\pm m$

Then general solution becomes $u(x)=c_1e^{mx}+c_2e^{-mx}$ **……….(ii)**

Now using BC's $u(0)=0$ **and** $u(a)=0 \Rightarrow c_1=0, c_2=0$

$(ii) \Rightarrow u(x)=0$ **which is trivial solution. So** $\lambda<0$ **is not an eigenvalue.**

CASE III: when $\lambda>0$ **then we may take** $\lambda=m^2$

Then $\frac{d^2u}{dx^2}+\lambda u=0 \Rightarrow \frac{d^2u}{dx^2}+(m^2)u=0 \Rightarrow (D^2+m^2)u=0 \Rightarrow D=\pm im$

Then general solution becomes $u(x)=c_1Cosmx+c_2Sinmx$ **……….(iii)**

Now using BC's $u(0)=0 \Rightarrow c_1=0$

and $u(a)=0 \Rightarrow c_2\neq 0$ **then** $Sinm(a)=0 \Rightarrow m=\frac{n\pi}{a}\equiv m_n$

$then\ \lambda_n={m_n}^2=\frac{n^2\pi^2}{a^2}$ **are the eigenvalues for the non – trivial solution**

$u_n(x)=c_nSin\sqrt{\lambda_n}x$ **corresponding to** λ_n

GENERAL FORMS OF FIRST-ORDER LINEAR EQUATIONS IN TWO VARIABLES

For the general first-order linear partial differential equation

$a(x, y)\,u_x + b(x, y)\,u_y + c(x, y)\,u = d(x, y)$

SECOND-ORDER EQUATIONS IN ONE INDEPENDENT VARIABLE

The general linear second-order partial differential equation in one dependent variable u may be written as

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G$$

in which we assume $A_{ij} = A_{ji}$ and A_{ij} , B_i, F, and G are real-valued functions defined in some region of the space $(x_1, x_2, \ldots, x_n)$

SECOND-ORDER EQUATIONS IN TWO INDEPENDENT VARIABLES

Second-order equations in the dependent variable u and the independent variables x, y can be put in the form $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ where the coefficients are functions of 'x' and 'y' and do not vanish simultaneously. We shall assume that the function 'u' and the coefficients are twice continuously differentiable in some domain in R^2.

CLASSIFICATION OF SECOND-ORDER LINEAR EQUATIONS

The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry.

The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, represents

hyperbola if $B^2 - 4AC$ is positive i.e. $B^2 - 4AC > 0$

parabola if $B^2 - 4AC$ is zero i.e. $B^2 - 4AC = 0$

or ellipse if $B^2 - 4AC$ is negative i.e. $B^2 - 4AC < 0$

for example:

(i) **The heat equation** $\frac{1}{K}u_t = u_{xx}$ **is parabolic.**

(ii) **The wave equation** $\frac{1}{c^2}u_{tt} = u_{xx}$ **is hyperbolic.**

(iii) **The potential (Laplace) equation** $\nabla^2 u = u_{xx} + u_{yy} = 0$ **is elliptic.**

METHOD OF SEPARATION OF VARIABLES

During the last two centuries several methods have been developed for solving partial differential equations. Among these, a technique known as the method of separation of variables is perhaps the oldest systematic method for solving partial differential equations.

- **Its essential feature is to transform the partial differential equations by a set of ordinary differential equations.**
- **The required solution of the partial differential equations is then exposed as a product u (x, y) = X (x) Y (y) ≠ 0**
  or as a sum u (x, y) = X (x)+Y (y)

where X (x) and Y (y) are functions of x and y, respectively.

IMPORTANCE: Many significant problems in partial differential equations can be solved by the method of separation of variables. This method has been considerably refined and generalized over the last two centuries and is one of the classical techniques of applied mathematics, mathematical physics and engineering science. Usually, the first-order partial differential equation can be solved by separation of variables without the need for Fourier series.This method is used to convert PDE into ODE.

Example:

Solve by method of separation of variables

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \Rightarrow u_{xx} - u_t = 0,$ **where 'x' is real and** $a \le x \le b,\ \ t \ge 0$

Solution: given that $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

let $u \equiv u(x, t) = X(x)\, T(t) \equiv XT$

$\Rightarrow \quad \frac{\partial^2}{\partial x^2}(XT) - \frac{\partial}{\partial t}(XT) = 0 \Rightarrow X''(x)\, T(t) - X(x)\, T'(t) = 0$

Dividing XT on both sides $\Rightarrow \frac{X''\, T}{XT} - \frac{XT'}{XT} = 0$

$\Rightarrow \frac{X''}{X} - \frac{T'}{T} = 0 \Rightarrow \quad \frac{X''}{X} = \frac{T'}{T}$

Since L.H.S of this equation is a function of x only and the R.H.S is a function of 't' only

$\Rightarrow \frac{X''}{X} = \frac{T'}{T} = \lambda \Rightarrow \frac{X''}{X} = \lambda$ **and** $\frac{T'}{T} = \lambda$ **where** λ **is separation constant.**

Consequently, gives two ordinary differential equations

$X''(x) - \lambda X(x) = 0$ **and** $\frac{T'}{T} = \lambda \Rightarrow \int \frac{T'}{T}\, dt = \lambda \int dt \Rightarrow T = ae^{\lambda t}$

These equations have solutions given, respectively, by

$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ **and** $T(t) = ae^{\lambda t}$

where A and B are arbitrary integrating constants.

Consequently, the general solution is given by

$u(x,t) = X(x)T(t) = \left[Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}\right]\left[ae^{\lambda t}\right] = \left[aAe^{\sqrt{\lambda}x} + aBe^{-\sqrt{\lambda}x}\right]e^{\lambda t}$

$u(x,t) = \left[Ce^{\sqrt{\lambda}x} + De^{-\sqrt{\lambda}x}\right]e^{\lambda t}$ **where** $C = aA, D = aB$

Example: (UoS,2015 – II, 2018 – I)

Solve by method of separation of variables

$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \Rightarrow u_{xx} - \frac{1}{c^2} u_{tt} = 0,$ **where 'x' is real and** $a \le x \le b,\ \ t \ge 0$

Solution: given that $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

let $u \equiv u(x, t) = X(x)\, T(t) \equiv XT$

$\Rightarrow \frac{\partial^2}{\partial x^2}(XT) - \frac{1}{c^2}\frac{\partial}{\partial t^2}(XT) = 0 \Rightarrow \ X''(x)\, T(t) - \frac{1}{c^2} X(x)\, T''(t) = 0$

Dividing XT on both sides $\Rightarrow \frac{X''\,T}{XT} - \frac{1}{c^2}\frac{XT''}{XT} = 0$

$\Rightarrow \frac{X''}{X} - \frac{1}{c^2}\frac{T''}{T} = 0 \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$

Since L.H.S of this equation is a function of x only and the R.H.S is a function of 't' only

$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \Rightarrow \frac{X''}{X} = \lambda$ **and** $\frac{1}{c^2}\frac{T''}{T} = \lambda$ **where** λ **is separation constant.**

Consequently, gives two ordinary differential equations

$X''(x) - \lambda X(x) = 0$ **and** $T''(t) - \lambda c^2 T(t) = 0$

These equations have solutions given, respectively, by

$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ **and** $T(t) = Ce^{\sqrt{\lambda}ct} + De^{-\sqrt{\lambda}ct}$

where A , B,C and D are arbitrary integrating constants.

Consequently, the general solution is given by

$u(x,t) = X(x)T(t) = \left[Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}\right]\left[Ce^{\sqrt{\lambda}ct} + De^{-\sqrt{\lambda}ct}\right]$

Example: (UoS,2017 – II, 2018 – II, 2019 – I) Solve by method of separation of variables $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0 \Rightarrow u_{xx}+u_{yy}-\frac{1}{c^2}u_{tt}=0$, **where** $x\in[a,b], y\in[c,d], t\geq 0$

Solution: given that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0$

let $u\equiv u(x,y,t)=X(x)\,Y(y)\,T(t)\equiv XYT$

$\Rightarrow \frac{\partial^2}{\partial x^2}(XYT)+\frac{\partial}{\partial y^2}(XYT)-\frac{1}{c^2}\frac{\partial}{\partial t^2}(XYT)=0 \Rightarrow X''YT+XY''T-\frac{1}{c^2}XYT''=0$

Dividing XYT on both sides $\Rightarrow \frac{X''YT}{XYT}+\frac{XY''T}{XYT}-\frac{1}{c^2}\frac{XYT''}{XYT}=0$

$\Rightarrow \frac{X''}{X}+\frac{Y''}{Y}-\frac{1}{c^2}\frac{T''}{T}=0 \Rightarrow \frac{X''}{X}+\frac{Y''}{Y}=\frac{1}{c^2}\frac{T''}{T}$

$\Rightarrow \frac{X''}{X}+\frac{Y''}{Y}=\frac{1}{c^2}\frac{T''}{T}=\lambda \Rightarrow \frac{X''}{X}+\frac{Y''}{Y}=\lambda$ **and** $\frac{1}{c^2}\frac{T''}{T}=\lambda$

Now solving for $\frac{1}{c^2}\frac{T''}{T}=\lambda$ **we get** $T(t)=c_1e^{\sqrt{\lambda}ct}+c_2e^{-\sqrt{\lambda}ct}$

And solving $\frac{X''}{X}+\frac{Y''}{Y}=\lambda \Rightarrow \frac{X''}{X}=\lambda-\frac{Y''}{Y}$

$\Rightarrow \frac{X''}{X}=p$ **and** $\Rightarrow \lambda-\frac{Y''}{Y}=p$ **wher p is another constant.**

Consequently, gives two ordinary differential equations

$X''(x)-pX(x)=0$ **and** $Y''(y)+(\pm q^2)Y(y)=0$ **where** $\pm q^2=\lambda-p$

These equations have solutions given, respectively, by

$X(x)=c_3e^{\sqrt{p}x}+c_4e^{-\sqrt{p}x}$

and $Y(y)=c_5e^{+qy}+c_6e^{-qy}$ **for** $(+q)$

or $Y(y)=c_5Cosqy+c_6Sinqy$ **for** $(-q)$

Consequently, the general solution is given by

$u(x,y,t)=X(x)Y(y)T(t)=$

$$\left[c_3e^{\sqrt{p}x}+c_4e^{-\sqrt{p}x}\right]\left[c_5e^{\pm qy}+c_6e^{\mp qy}\right]\left[c_1e^{\sqrt{\lambda}ct}+c_2e^{-\sqrt{\lambda}ct}\right]$$

Example:

Solve by method of separation of variables $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\Rightarrow$ **$u_{xx} + u_{yy} = 0$, where** $x \in [a, b], y \in [c, d]$

Solution: given that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

let $u \equiv u(x, y) = X(x)Y(y) \equiv XY$

$\Rightarrow \frac{\partial^2}{\partial x^2}(XY) + \frac{\partial}{\partial y^2}(XY) = 0 \Rightarrow X''(x)\, Y(y) + X(x)\, Y''(y) = 0$

Dividing XY on both sides

$$\Rightarrow \frac{X''\,Y}{XY} + \frac{XY''}{XY} = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$

Since the L.H.S of this equation is a function of x only and the R.H.S is a function of t only

$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \Rightarrow \frac{X''}{X} = \lambda$ **and** $-\frac{Y''}{Y} = \lambda$ **where λ is separation constant.**

Consequently, gives two ordinary differential equations

$X''(x) - \lambda X(x) = 0$ **and** $Y''(y) + \lambda Y(y) = 0$

These equations have solutions given, respectively, by

$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ **and** $Y(y) = CSin\sqrt{\lambda}y + DCos\sqrt{\lambda}y$

where A , B,C and D are arbitrary integrating constants.

Consequently, the general solution is given by

$$u(x, y) = X(x)Y(y) = \left[Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}\right]\left[CSin\sqrt{\lambda}y + DCos\sqrt{\lambda}y\right]$$

STURM LIOUVILLE SYSTEM (SL – SYSTEM)

The functions discussed in this chapter arise as solution of second order DE's which appear in special, rather than in general physical problems. So, these functions are usually known as "The Special Functions of Mathematical Physics"

SELF ADJOINT OPERATOR:

An operatior 'A' defined over a linear space of functions is called Self Adjoint if $\langle u, Av\rangle = \langle v, Au\rangle$ which is equivalent to

$\int_a^b u(x)[Av(x)]dx = \int_a^b [Au(x)]v(x)dx$ where the functions 'u' and 'v' are supposed to be real. In case of complex functions a slight modification is necessary.

EXAMPLES (JUST READ):

- **Sturm liouville Differential operator is Self Adjoint.**
- **The Hormonic oscillator equation is Self Adjoint.**
- **Legendre's equation is Self Adjoint.**
- **Laguerre's equation and Hermite equation are not Self adjoint but could be made using few conditions.**

STURM LIOUVILLE EQUATION (SL - EQUATION):

The SL equation is named after the German Mathematician John Sturm (1803 – 1855) and the French Mathematician Joseph Liouville (1809 - 1887), who did pioneering work on this DE and related problems.

Defination: The Second order Ordinary Linear Homogeneous DE of the form

$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$

$\Rightarrow p(x)u'' + p'(x)u' + q(x)u + \lambda\, r(x)u = 0$ **is called a Sturm Liouville equation OR briefly an SL equation. If $p(x), p'(x), q(x), r(x)$ are real and continuous over an interval [a,b].**

STURM LIOUVILLE (SL) DIFFERENTIAL OPERATOR:

A self adjoint operator of the form $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$ is called SL differential operator. This operator is a second order linear differential operator because it operate on everything to the right, not just by ordinary multiplication but also by the operation of differentiation.

REMARK (JUST READ):

- **SL differential operator $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$ is called <u>normal operator</u> if $p(x) \neq 0$ in the range $x \in (-\infty, \infty)$**
- **In terms of SL differential operator SL equation can also be written as**

  $L(u) + \lambda\, r(x)u = 0$

LINEAR SECOND ORDER DE's AND SL EQUATION

There are may second order linear ODE's which appear in physical and engineering problems. Some of these are as follows;

EQUATION OF SIMPLE HARMONIC MOTION:

An equation of the form $\frac{d^2x}{dt^2} + \mu^2 x = 0 \Rightarrow x_{tt} + \mu^2 x = 0$ **with condition on x(t) of the form** $x(t = 0) = x_0$, $\quad x'(t = 0) = 0$

In this equation $p(x) = 1, q(x) = 0,\ r(x) = 1$

LEGENDRE'S DIFFERENTIAL EQUATION:

An equation of the form

$$\frac{d}{dx}\left\{(1 - x^2)\frac{du}{dx}\right\} + n(n + 1)u = 0 \Rightarrow (1 - x^2)u'' - 2xu' + n(n + 1)u = 0$$

with condition on u(x) of the form $u(\pm 1)$ **are finite. In this equation '**$x = \pm 1$**' are singular points.**

In this equation $p(x) = (1 - x^2), q(x) = 0,\ r(x) = 1$

BESSELE'S DIFFERENTIAL EQUATION:

An equation of the form

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (k^2x^2 - v^2)u = 0 \Rightarrow x^2u'' + xu' + k^2x^2u - v^2u = 0$$ **where the solution of this DE are called Bessele's Functions.**

In this equation $p(x) = x^2\ , q(x) = k^2x^2\ ,\ r(x) = -1$

THE HERMITE EQUATION:

An equation of the form

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \lambda u = 0 \Rightarrow u'' - 2xu' + \lambda u = 0\ ;\ 0 \le x < \infty$$

In this equation $p(x) = 1\ , q(x) = 0\ ,\ r(x) = 1$

THE LAGUERRE EQUATION:

An equation of the form

$$x\frac{d^2u}{dx^2}+(1-x)\frac{du}{dx}+\lambda u=0 \Rightarrow xu''+(1-x)u'+\lambda u=0\ ;\ x>0$$

with $x=0$ a singular point. In this equation $p(x)=x\ ,q(x)=0\ ,\ r(x)=1$

THE CHEBYSHEV EQUATION:

An equation of the form

$$(1-x^2)\frac{d^2u}{dx^2}-x\frac{du}{dx}+\alpha^2u=0 \Rightarrow (1-x^2)u''-xu'+\alpha^2u=0\ ;\ x>0$$ **with**

$-1<x<1$ i.e $x\in(-1,1)$ with $p(x)=(1-x^2)\ ,q(x)=0\ ,\ r(x)=1$

AIRY'S EQUATION:

An equation of the form $\frac{d^2u}{dx^2}+xu=0 \Rightarrow u''+xu=0\ ;\ x\in\mathbb{R}$

In this equation $p(x)=1\ ,q(x)=x\ ,\ r(x)=0$

<u>POSSIBLE QUESTION:</u> ***<u>Describe about introduction to SL System of equations.</u>***

SINGULAR POINTS: For SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\}+q(x)u+\lambda\, r(x)u=0$ **the points at which $p(x)$ *and* $r(x)$ vanishes (i.e. become zero) over any interval [a,b] are called singular points.**

REGULAR POINTS: For SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\}+q(x)u+\lambda\, r(x)u=0$ **the points at which $p(x)$ *and* $r(x)$ do not vanishes (i.e. become zero) over any interval [a,b] are called regular points.**

WEIGHT FUNCTION: In SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ **the continuous, non negative, real function** $r(x)$ **on any interval 'I' is called Weight Function.**

SINGULAR SL EQUATION:

The SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ **is called Singular SL equation in the interval [a,b] if the points** $p(x)$ *and* $r(x)$ **vanishes (i.e. become zero) at any point the interval [a,b].**

EXAMPLES:

- **Legendre's DE** $\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$ **with** $p(x) = (1-x^2), q(x) = 0,\ r(x) = 1$ **is singular at** $x = \pm 1$**.**
- **Bessele's DE is singular after few arrangements. At** $x = 0$

SINGULAR SL SYSTEM: A singular SL equation together with suitable linear homogeneous conditions on $u(x)$ **leads to a singular SL system.**

REGULAR SL EQUATION:

The SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ **is called regular SL equation in the interval [a,b] if the points** $p(x)$ *and* $r(x)$ **do not vanishes (i.e. become zero) at any point the interval [a,b].**

EXAMPLES:

- $u''(x) + \lambda u(x) = 0$ **with** $p(x) = 1 > 0, q(x) = 0, r(x) = 1 > 0$ **is regular SL equation in every interval.**
- **Legendre's DE** $\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$ **with** $p(x) = (1-x^2), q(x) = 0,\ r(x) = 1$ **is singular or is not regular at** $x = \pm 1$**.**
- **Bessele's DE is singular or is not regular. At** $x = 0$

REGULAR SL SYSTEM: A Regular SL equation together with suitable end ponint conditions leads to a regular SL system.

Conditions are $\alpha u(a) + \alpha' u'(a) = 0$ **and** $\beta u(b) + \beta' u'(b) = 0$

PERIODIC SL EQUATION:

The SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ **is called periodic SL equation in the interval [a,b] if the points** $p(x), q(x)$ *and* $r(x)$ **are periodic funcitons of period** $b - a$ **.**

EXAMPLES:

- $u'' + \lambda u = 0$ **with** $u(-\pi) = u(+\pi)$ *and* $u'(-\pi) = u'(+\pi)$ **is periodic SL equation.**
- **With period** $2\,\pi$**, The Mathieu DE** $u'' + \lambda u + 16dCos2xu = 0$ **with** $u(-\pi) = u(+\pi)$ *and* $u'(-\pi) = u'(+\pi)$ **is periodic SL equation.**

PERIODIC SL SYSTEM: A Periodic SL equation together with suitable end ponint conditions leads to a Periodic SL system.

Conditions are $u(a) = u(b)$ *and* $u'(a) = u'(b)$

BOUNDRY CONDITIONS ASSOCIATED WITH SL SYSTEM

- **The boundry conditions** $\alpha u(a) + \alpha' u'(a) = 0$ **and** $\beta u(b) + \beta' u'(b) = 0$ **are called <u>Separated</u> Boundry Conditions are <u>Unmixed</u> Boundry Conditions.**
- **If the Separated Boundry conditions are of the form** $u(a) = c_1$ *and* $u(b) = c_2$ **then they are called <u>Drichlet</u> BC's**
- **If the Separated Boundry conditions are of the form** $u'(a) = c'_1$ *and* $u'(b) = c'_2$ **then they are called <u>Neumann</u> BC's**
- **If the Separated Boundry conditions are of the form** $u(a) = u(b)$ *and* $u'(a) = u'(b)$ **then they are called <u>Periodic</u> BC's**

EIGENVALUE PROBLEMS:

A non – zero solution of an SL sytem (Regular or Periodic) is said to be an eigensolution or eigenfunction corresponding to a value of the parameter λ in SL equation. The value of λ then called sn eigenvalue of the DE.

OR If an IVP or BVP contains a parameter λ in the DE and non – trivial solution(s) corresponding to certain values of λ can be found then the problem is called Eigenvalue Problem, and the corresponding values of λ are called Eigenvalues of the problem.

EXAMPLE:

Find the eigenvalue and eigenfunctions (solutions) fo the regular SL system $u'' + \lambda u = 0$ where λ is parameter and the boundry conditions are $u(0) = 0$ and $u(\pi) = 0$

Solution: (the end point conditions shows that the system is regular but not periodic) with $a = 0, b = \pi, p(x) = 1, q(x) = 0, r(x) = 1$

Now $u'' + \lambda u = 0 \Rightarrow D = \pm i\sqrt{\lambda}$

Then general solution becomes $u(x) = ACos\sqrt{\lambda}x + BSin\sqrt{\lambda}x \qquad 0 \le x \le \pi$

Now using BC's $u(0) = 0 \Rightarrow A = 0 \qquad \Rightarrow u(x) = BSin\sqrt{\lambda}x$

$u(\pi) = 0 \Rightarrow BSin\sqrt{\lambda}\pi = 0 \Rightarrow B \neq 0(gives\ trivial\ solution\ )\ Sin\sqrt{\lambda}\pi = 0$

$\Rightarrow \sqrt{\lambda}\pi = n\pi \Rightarrow \sqrt{\lambda} = n$ **; $n = \pm 1, \pm 2, \ldots$….ommiting ,0, because gives trivial solution.**

Hence $\lambda = \lambda_n = n^2$ **; $n = \pm 1, \pm 2, \ldots$…. are the eigenvalues for the non – trivial solution where the eigenfunctions are**

$u_n(x) = b_n Sinnx$ **; $n = \pm 1, \pm 2, \ldots$… whrer the constants b_n are in general different for each solution.**

EXAMPLE: (UoS,2015 – I, 2018 – I) **Show that if $u(x)$ and $v(x)$ are periodic solutions of the Mathieu equation with period π having the distinct eigenvalues then $\int_0^{\pi} u(x)v(x)dx = 0$**

Solution:

we know that the Mathieu DE with period π is $u'' + \lambda u + 16dCos2xu = 0$ with end point conditions $u(0) = u(\pi)$ $and\ u'(0) = u'(\pi)$

now if 'u' and 'v' are solutions of given equation corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively then

$u'' + \lambda_1 u + 16dCos2xu = 0$ ....(i) with end point conditions $u(0) = u(\pi)$ $and\ u'(0) = u'(\pi)$

Similarly for 'v' we have

$v'' + \lambda_2 v + 16dCos2xv = 0$ ....(ii) with end point conditions $v(0) = v(\pi)$ $and\ v'(0) = v'(\pi)$

Multiplying (i) with 'v' and (ii) with 'u' and subtracting we obtain

$(\lambda_1 - \lambda_2)uv = v''u - u''v$ **.............(iii)**

$\Rightarrow \int_0^{\pi} (\lambda_1 - \lambda_2)uvdx = \int_0^{\pi} v''u - u''vdx$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^{\pi} u(x)v(x)dx = \int_0^{\pi} \frac{d}{dx}(v'u - u'v)dx$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^{\pi} u(x)v(x)dx = |v'u - u'v|_0^{\pi}$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^{\pi} u(x)v(x)dx = u(\pi)v'(\pi) - u'(\pi)v(\pi) - u(0)v'(0) + u'(0)v(0)$

Now using end conditions.

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^{\pi} u(x)v(x)dx = u(\pi)v'(\pi) - u'(\pi)v(\pi) - u(\pi)v'(\pi) + u'(\pi)v(\pi)$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^{\pi} u(x)v(x)dx = 0 \Rightarrow (\lambda_1 - \lambda_2) \neq 0 \Rightarrow \int_0^{\pi} u(x)v(x)dx = 0$

as required.

EXAMPLE: (UoS,2014 – II)

Determine eigenvalue of the system $u'' + \lambda u = 0$ with boundry conditions are $u(0) = u(\pi)$ and $u'(0) = 2u'(\pi)$

Solution: Since we have $u'' + \lambda u = 0$

Then $u'' + \lambda u = 0 \Rightarrow D = \pm i\sqrt{\lambda}$

Then general solution becomes $u(x) = ACos\sqrt{\lambda}x + BSin\sqrt{\lambda}x$

Then $u'(x) = -\sqrt{\lambda}ASin\sqrt{\lambda}x + \sqrt{\lambda}BCos\sqrt{\lambda}x$

Now using BC's

$u(0) = u(\pi) \Rightarrow ACos\sqrt{\lambda}(0) + BSin\sqrt{\lambda}(0) = ACos\sqrt{\lambda}(\pi) + BSin\sqrt{\lambda}(\pi)$

$\Rightarrow \left(Cos\sqrt{\lambda}(\pi) - 1\right)A + BSin\sqrt{\lambda}(\pi) = 0$ ………….(i)

Similarly using BC's $u'(0) = 2u'(\pi)$

$\Rightarrow -\sqrt{\lambda}ASin\sqrt{\lambda}(0) + \sqrt{\lambda}BCos(0) = -2\sqrt{\lambda}ASin\sqrt{\lambda}(\pi) + 2\sqrt{\lambda}BCos\sqrt{\lambda}(\pi)$

$\Rightarrow 2ASin\sqrt{\lambda} + \left(1 - 2Cos\sqrt{\lambda}(\pi)\right)B = 0$ ………….(ii)

Being homogeneous both equations have trivial solution. i.e. A = 0, B = 0.

For non – trivial solution we must have

$$\begin{vmatrix} Cos\sqrt{\lambda}(\pi) - 1 & Sin\sqrt{\lambda}(\pi) \\ 2ASin\sqrt{\lambda} & 1 - 2Cos\sqrt{\lambda}(\pi) \end{vmatrix} = 0 \Rightarrow Cos\sqrt{\lambda}(\pi) = 1 \Rightarrow \lambda = \lambda_n = 4n^2$$

Corresponding eigne functions are $u_n = A_nCos2nx + B_nSin2nx$

ORTHOGONAL FUNCTIONS: Functions $u(x)$ and $v(x)$ defined over [a,b] are said to be orthogonal w.r.to a weight function $w(x)$ if

$$\int_a^b w(x)u(x)v(x)dx = 0$$

SQUARE INTEGRABLE FUNCTION: A function $f(x)$ is said to be square integrable with respect to a weight function $w(x) > 0$ over an interval [a,b] if $\int_a^b w(x)|f(x)|^2 dx < \infty$

If $w(x) = 1$ then $\int_a^b |f(x)|^2 dx < \infty$ in this case $f(x)$ is simply called square integrable.

EXAMPLES:

- Legendre's DE $\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$ is square integrable.
- Bessele's DE is square integrable.

LAGRANGE'S IDENTITY: (UoS,2013 – II, 2017 –I , II)

Suppose $u(x)$ and $v(x)$ are two solutions of an SL equation, then the following identity must hold

$$uL(v) - vL(u) = \frac{d}{dx}\{p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\}$$

Which is called Differential form of Langrange's idenetity. While the integral form is given as follows

$$\int_a^b [uL(v) - vL(u)]dx = \left|p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\right|_a^b$$

PROOF: Since $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$ therefore

$$uL(v) - vL(u) = u\frac{d}{dx}\left\{p(x)\frac{dv}{dx}\right\} + q(x) - v\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} - q(x)$$

$$uL(v) - vL(u) = up(x)v'' + up'(x)v' - vp(x)u'' - vp'(x)u'$$

$$uL(v) - vL(u) = p(x)(uv'' - u''v) + p'(x)(uv' - u'v)$$

$$uL(v) - vL(u) = \frac{d}{dx}\{p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\}$$

$\Rightarrow uL(v) - vL(u) = \frac{d}{dx}\{p(x)W(u,v)(x)\}$, W is called Wronskian of 'u' ,'v'

Taking integral from 'a' to 'b'

$$\int_a^b [uL(v) - vL(u)]dx = \int_a^b \frac{d}{dx}\{p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\}dx$$

$$\int_a^b [uL(v) - vL(u)]dx = \left|p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\right|_a^b$$

IMPORTANCE: By using Lagrange's identity, we may prove reality, orthogonality and simplicity of eigenvalues of an SL system (regular or periodic)

REALITY OF EIGENVALUES

THEOREM – I : (UoS,2015 – II, 2017 –II) یہ یا اس سے اگلا رزلٹ، ان میں سے کوئی ایک یا دونوں کی اکٹھی سٹیٹمینٹ پوچھ لی جاتی ہے۔

The eigenvalues of an SL system (regular) are real.

PROOF: Let 'u' be eigenfunction corresponding to eigenvalue 'λ' then

$u(x) \neq 0 \ ; \forall x \epsilon (a, b)$

Now as $L(u) + \lambda\, r(x)u = 0 \Rightarrow L(u) = -\lambda\, r(x)u$ **……………..(i)**

If possible let 'λ' be complex then $\bar{L}(u) = -\bar{\lambda}\, r(x)\bar{u}$

Now 'p','q','r' are real, therefore L is real hence

$\Rightarrow L(u) = -\bar{\lambda}\, r(x)\bar{u}$ **……………..(ii)**

Now from Lagrange's identity

$\int_a^b [uL(v) - vL(u)]dx = \left| p(x)\big(u(x)v'(x) - u'(x)v(x)\big) \right|_a^b$

Taking $v = \bar{u}$ **then**

$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = \left| p(x)\big(u(x)\bar{u}'(x) - u'(x)\bar{u}(x)\big) \right|_a^b$ **……………..(A)**

Now for a regular SL system $\alpha u(a) + \alpha' u'(a) = 0$ **and** $\beta u(b) + \beta' u'(b) = 0$

Similarly $\alpha \bar{u}(a) + \alpha' \bar{u}'(a) = 0$ **and** $\beta \bar{u}(b) + \beta' \bar{u}'(b) = 0$

If we substitute the values of $u(a), u(b), \bar{u}'(a), \bar{u}'(b)$ **in R.H.S of (A) we find it will be zero. Hence** $\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = 0$

Using (i) and (ii) $\int_a^b \big[u\big(-\bar{\lambda}\, r(x)\bar{u}\big) - \bar{u}\big(-\lambda\, r(x)u\big)\big]dx = 0$

$\Rightarrow \int_a^b [-\bar{\lambda}\, ru\bar{u} + \lambda\, ru\bar{u}]dx = 0 \Rightarrow \int_a^b (\lambda - \bar{\lambda}) ru\bar{u}dx = 0 \Rightarrow \int_a^b (\lambda - \bar{\lambda}) r|u|^2 dx = 0$

Now as $r(x) > 0$ **also** $|u|^2 > 0$ **therefore** $\int_a^b r|u|^2 dx > 0$

Then $(\lambda - \bar{\lambda}) = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ **is real as required.**

THEOREM – II : (UoS,2015 – II, 2017 –II) یہ یااس سے پچھلا رزلٹ، یادونوں کی اکٹھی سٹیٹمینٹ پوچھ لی جاتی ہے۔

The eigenvalues of an SL system (periodic) are real.

PROOF: Let ‘u’ be eigenfunction corresponding to eigenvalue ‘λ’ then $u(x) \neq 0$; $\forall x \epsilon (a, b)$

Now as $L(u) + \lambda\, r(x)u = 0 \Rightarrow L(u) = -\lambda\, r(x)u$ ……………..(i)

If possible let ‘λ’ be complex then $\bar{L}(u) = -\bar{\lambda}\, r(x)\bar{u}$

Now ‘p’,’q’,’r’ are real, therefore L is real hence

$\Rightarrow L(u) = -\bar{\lambda}\, r(x)\bar{u}$ ……………..(ii)

Now from Lagrange’s identity

$$\int_a^b [uL(v) - vL(u)]dx = \left| p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\right|_a^b$$

Taking $v = \bar{u}$ then

$$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = \left| p(x)\big(u(x)\bar{u}'(x) - u'(x)\bar{u}(x)\big)\right|_a^b \quad \text{……….(A)}$$

Now for a periodic SL system $u(a) = u(b), u'(a) = u'(b), p(a) = p(b)$ and if B.C’s are singular then $p(a) = p(b) = 0$ and R.H.S of (A) will be zero.

Hence $\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = 0$

Using (i) and (ii) $\int_a^b \big[u\big(-\bar{\lambda}\, r(x)\bar{u}\big) - \bar{u}\big(-\lambda\, r(x)u\big)\big]dx = 0$

$\Rightarrow \int_a^b \big[-\bar{\lambda}\, ru\bar{u} + \lambda\, ru\bar{u}\big]dx = 0 \Rightarrow \int_a^b (\lambda - \bar{\lambda}) ru\bar{u}dx = 0$

$\Rightarrow \int_a^b (\lambda - \bar{\lambda}) r|u|^2 dx = 0$

Now as $r(x) > 0$ also $|u|^2 > 0$ therefore $\int_a^b r|u|^2 dx > 0$

Then $(\lambda - \bar{\lambda}) = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real as required.

ORTHOGONALITY OF EIGENVALUES

THEOREM: (UoS,2011, 2013)

Eigenfunctions of a regular or periodic SL system corresponding to distinct eigenvalues are orthogonal w.r.to weight function $r(x)$.

PROOF:

Let 'λ_m' and 'λ_n' be eigenvalues of an SL system with eigenfunctions $u_m(x)$ and $u_n(x)$ respectively then using Lagrange's identity

$L(u_m) = -\lambda_m\, r(x)u_m$ and $L(u_n) = -\lambda_n\, r(x)u_n$ with boundry conditions of the regular or periodic type.

Again using Lagrange's identity for $u_m(x)$ and $u_n(x)$

$$\int_a^b [u_m L(u_n) - u_n L(u_m)]dx = \left|p(x)\big(u_m(x)u_n{}'(x) - u_m{}'(x)u_n(x)\big)\right|_a^b$$

For a regular or periodic SL system, R.H.S = 0

Hence $\int_a^b [u_m L(u_n) - u_n L(u_m)]dx = 0$

Using (i) and (ii) $\int_a^b [u_m(-\lambda_n\, r(x)u_n) - u_n(-\lambda_m\, r(x)u_m)]dx = 0$

$\Rightarrow \int_a^b [-\lambda_n\, r u_m u_n + \lambda_m\, r u_m u_n]dx = 0 \Rightarrow \int_a^b (\lambda_m - \lambda_n) r u_m u_n dx = 0$

$\Rightarrow (\lambda_m - \lambda_n) \neq 0 \Rightarrow \int_a^b u_m u_n r dx = 0$

This shows that eigenvalues are orthogonal w.r.to weight function $r(x)$.

EXAMPLE: Determine eigenvalues and eigenfuctions of the problem $u'' + \lambda^2 u = 0$; $0 < x < \pi$ with the boundry conditions $u'(0) + 2u'(\pi) = 0$ and $u(\pi) = 0$

Solution: **Given** $u'' + \lambda^2 u = 0 \Rightarrow D = \pm i\lambda$

Then general solution becomes $u(x) = c_1 Cos\lambda x + c_2 Sin\lambda x$

$\Rightarrow u'(x) = -\lambda c_1 Sin\lambda x + \lambda c_2 Cos\lambda x$

Now using BC's $u(\pi) = 0 \Rightarrow c_1 Cos\lambda\pi + c_2 Sin\lambda\pi = 0$ **…………(i)**

Now using BC's $u'(0) + 2u'(\pi) = 0$

$-\lambda c_1 Sin\lambda 0 + \lambda c_2 Cos\lambda 0 + 2[-\lambda c_1 Sin\lambda\pi + \lambda c_2 Cos\lambda\pi] = 0$

$\Rightarrow \lambda c_2 - 2\lambda c_1 Sin\lambda\pi + 2\lambda c_2 Cos\lambda\pi = 0$

$\Rightarrow \lambda c_2 Cos\lambda\pi - \lambda c_1 2Sin\lambda\pi Cos\lambda\pi + 2\lambda c_2 Cos^2\lambda\pi = 0$

$\Rightarrow -\lambda c_1 Sin2\lambda\pi + \lambda c_2 Cos\lambda\pi(1 + 2Cos\lambda\pi) = 0$

$\Rightarrow c_1 Sin2\lambda\pi - c_2 Cos\lambda\pi(1 + 2Cos\lambda\pi) = 0$ …………(ii)

Substituiting c_2 **from (i) into (ii) we get**

$c_1(2 + Cos\lambda\pi) = 0 \Rightarrow c_1 \neq 0 \Rightarrow 2 + Cos\lambda\pi = 0 \Rightarrow Cos\lambda\pi = -2$ **which cannot be satisfied for any real value of** λ**. Therefore the problem has only complex eigenvalues and complex eigenfunctions.**

EXAMPLE:

Solve BVP defined by $u'' + \lambda^2 u = 0$ **with** $u(0) = 0, u(1) = 1; 0 < x < 1$

Solution: **Given** $u'' + \lambda^2 u = 0 \Rightarrow D = \pm i\lambda$

Then general solution becomes $u(x) = c_1 Cos\lambda x + c_2 Sin\lambda x$

Now using BC's $u(0) = 0 \Rightarrow c_1 = 0$

then given solution reduces to $u(x) = c_2 Sin\lambda x$

Now using BC's $u(1) = 1 \Rightarrow c_2 = \frac{1}{Sin\lambda}$

Hence the solution can be written as $u_\lambda(x) = \frac{Sin\lambda x}{Sin\lambda}$

EXAMPLE: (UoS,2019 – I)

Express the function $f(x)=\begin{cases}1 & ;0\le x\le\frac{1}{2}\\ 0 & ;\frac{1}{2}\le x\le 1\end{cases}$ **defined in the interval [0,1] in terms of eigenfunctions of the SL problem** $y''+\lambda^2 y=0$ **with the BC's** $5y(1)+y'(1)=0$ **and** $y(0)=0$ **where** $0<x<1$

OR

Determine eigenvalues and eigenfuctions of the problem

$u''+\lambda^2 u=0\,;0<x<1$ **with the boundry conditions** $5u(1)+u'(1)=0$ **and** $u(0)=0$

Solution: **Given** $u''+\lambda^2 u=0\Rightarrow D=\pm i\lambda$

Then general solution becomes $u(x)=c_1 Cos\lambda x+c_2 Sin\lambda x$

Now using BC's $u(0)=0\Rightarrow c_1=0$

Then $u(x)=c_2 Sin\lambda x$ **and** $u'(x)=\lambda c_2 Cos\lambda x$

Now using BC's $5u(1)+u'(1)=0$

$5[c_2 Sin\lambda]+\lambda c_2 Cos\lambda=0\Rightarrow c_2[5Sin\lambda+\lambda Cos\lambda]=0$

$\Rightarrow c_2\neq 0\Rightarrow [5Sin\lambda+\lambda Cos\lambda]=0$

Given problem has infinite numbers of eigenvalues which satisfy the equation $tan\lambda_n=-\frac{\lambda_n}{5}$ **where corresponding eigenfunctions are**

$u_n=c_n Sin\lambda_n x$ **n = 1,2,3,………**

SIMPLICITY OF EIGENVALUES

"The eigenvalues of a regular SL system are simple". i.e. to each eigenvalue there corresponds only one linearly independent eigenfunction.

In other words, if $u(x)$ and $v(x)$ are eigenfunctions corresponding to the same eigenvalue, then they must differ by a multiplicative constant.

PROOF: If possible let $u(x)$ and $v(x)$ be two linearly independent solutions corresponding to the same eigenvalue λ then using Lagrange's identity

$L(u) = -\lambda\, r(x)u$ **and** $L(v) = -\lambda\, r(x)v$

Then $uL(v) - vL(u) = -\lambda\, r(x)uv + \lambda\, r(x)uv = 0$ **…………..(i)**

But from Lagrange's identity, we have

$uL(v) - vL(u) = \frac{d}{dx}\{p(x)W(u,v)(x)\}$

Thus using (i) $\frac{d}{dx}\{p(x)W(u,v)(x)\} = 0 \quad ; \forall x \epsilon [a,b]$

Since in $[a,b]$ **,** $\Rightarrow p(x) \neq 0 \Rightarrow W(u,v)(x) = 0$

It follows that $u(x)$ and $v(x)$ be linearly dependent solutions. i.e. to each eigenvalue there corresponds only one linearly independent eigenfunction.

ABEL'S FORMULA

If $u(x)$ and $v(x)$ are any two solutions of a regular or periodic SL equation, then $p(x)W(u,v)(x) = constant \quad ; \forall x \epsilon [a,b]$

PROOF:

Since for a regular or periodic SL equation $uL(v) - vL(u) = 0$ for any pair of solutions. Hence from Lagrange's identity

$\frac{d}{dx}\{p(x)W(u,v)(x)\} = 0 \quad ; \forall x \epsilon [a,b]$

$\Rightarrow p(x)W(u,v)(x) = constant \quad ; \forall x \epsilon [a,b]$

THEOREM: **Any eigenvalue 'λ' can be related to its eigenfunction $u(x)$ by Rayleigh quotient** $\lambda = \frac{|-pu(x)u\prime(x)|_a^b + \int_a^b pu\prime^2(x)dx}{\int_a^b r(x)u^2 dx}$

This result cannot be used to determine eigenvalues, however, interesting and important results can be obtained from it.

EXAMPLE: **Using Rayleigh quotient, discuss the signe of eigenvalue(s) of the SL system**

$u'' + \lambda u = 0$ **with** $u(0) = 0, u(l) = 0, p(x) = 1, q(x) = 0, r(x) = 1$

Solution: **here** $a = 0, b = l, p(x) = 1, q(x) = 0, r(x) = 1$

Therefore using formula $\lambda = \frac{|-pu(x)u\prime(x)|_a^b + \int_a^b pu\prime^2(x)dx}{\int_a^b r(x)u^2 dx} = \frac{\int_0^l u\prime^2(x)dx}{\int_0^l u^2 dx}$

This result cannot be used to determine eigenvalues, however, interesting and important results about the eigenvalues can be obtained from it.

COMPLETENESS OF EIGENVALUES (just read)

"The eigenvalues of an SL system are complete"

OR" the set of eigenfunctions of an SL system are complete"

OR " Every function $u(x)$; $x \epsilon [a, b]$ can be represented in terms of these eigenfunctions as $u(x) = \sum_{n=1}^{\infty} c_n u_n(x)$; $x \epsilon [a, b]$

OR " A set of functions is said to be complete, if any function can be written as a linear combination of the function in the set, with constant coefficients."

This is the generalization of the concept of the Fourier Series.

REMARKS:

- **Legendre's polynomials are a complete set on $I = [-1, 1]$**
- **Laguere polynomials are a complete set on $I = [0, \infty)$**
- **Hermite polynomials are a complete set on $I = (-\infty, \infty)$**
- **The eigenvalues of a Regular SL system are simple.i.e. Regular SL system have multiplicity 1.**
- **The eigenvalues of a Periodic SL system have multiplicity 2.**

SL OPERATOR IS SELF ADJOINT

For self adjointness $\langle u, Lv\rangle = \langle v, Lu\rangle \Rightarrow \langle u, Lv\rangle - \langle v, Lu\rangle = 0$

Or $\int_a^b [uL(v) - vL(u)]dx = 0$

From Lagrange's identity we have; $\int_a^b [uL(v) - vL(u)]dx = |p(uv' - u'v)|_a^b$

But for periodic and regular SL system R.H.S = 0

Thus $\int_a^b [uL(v) - vL(u)]dx = 0$

This means that SL operator 'L' is self adjoint for regular or periodic SL system.

EXAMPLE: **For the SL eigenvalue problem $u'' + \lambda u = 0$ with $u'(0) = 0, u'(l) = 0$ verify the following general results**

i. **There are an infinite number of eigenvalues with a smallest but no largest.**
ii. **The nth eigenfunction has exactly 'n-1' zeros.**
iii. **The eigenfunctions are orthogonal and form a complete set.**

Solution:

Then general solution becomes $u(x) = c_1 Cos\sqrt{\lambda}x + c_2 Sin\sqrt{\lambda}x$

$\Rightarrow u'(x) = -\sqrt{\lambda}c_1 Sin\sqrt{\lambda}x + \sqrt{\lambda}c_2 Cos\sqrt{\lambda}x$

Now using BC's $u'(0) = 0 \Rightarrow c_2 = 0$ then given solution reduces to

$u'(x) = -\sqrt{\lambda}c_1 Sin\sqrt{\lambda}x$

Now using BC's $u'(l) = 0 \Rightarrow -\sqrt{\lambda}c_1 Sin\sqrt{\lambda}l = 0 \Rightarrow -\sqrt{\lambda} \neq 0,\ c_1 Sin\sqrt{\lambda}l = 0$

$\Rightarrow \sqrt{\lambda}l = n\pi \quad ; n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots$

$\Rightarrow \lambda = \frac{n^2\pi^2}{l^2} \quad ; n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots$

Hence for the eigenvalues $\Rightarrow \lambda \equiv \lambda_n = \frac{n^2\pi^2}{l^2} \quad ; n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots$

Thus the eigenfunctions are given by $\Rightarrow u \equiv u_n = c_n Cos\sqrt{\lambda_n}x = c_n Cos\frac{n\pi x}{l}$

i. **It is clear that** $\lambda_0 = 0$ **is the smallest eigenvalue. The others eigenvalues are** λ_n **for** $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots..$ **obviously there are no other largest eigenvalue.**

   The eigenfunction corresponding to the nth eigenvalues is

   $\Rightarrow u_{n-1} = c_{n-1} Cos\frac{(n-1)\pi x}{l}$; $n \geq 1$

   Or $u_n = c_n Cos\frac{n\pi x}{l}$; $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots..$

ii. **Now we will prove that** $u_n(x)$ **has exactly 'n-1' zeros. When 'n' takes the largest values, i.e.** $n = 0, 1, 2, \ldots\ldots\ldots\ldots..$ **n-1**

   When n = 0 then $u_0 = c_0$ **has no zero as expected.**

   When n = 1 then $u_1 = c_1 Cos\frac{\pi x}{l}$ **then a zero of this function occur when** $\frac{\pi x}{l} = \frac{\pi}{2}$ **i.e. at** $x = \frac{l}{2}$ **which lies in the interval** $(0, l)$ **and there is no other zero in this interval. Therefore , the eigenfunction has exactly 'one' zero in the interval** $[0, l]$ **the next zero occur at** $x = \frac{3l}{2} \notin (0, l)$

   Similarly, When n = 2 then $u_2 = c_2 Cos\frac{2\pi x}{l}$; $0 < x < l$ **then a zero of this function occur when** $\frac{2\pi x}{l} = \frac{\pi}{2}$ **i.e. at** $x_1 = \frac{l}{4}$ **and the second zero is given by** $\frac{2\pi x}{l} = \frac{3\pi}{2}$ **i.e. at** $x_2 = \frac{3l}{4}$ **also the third zero is given by** $\frac{2\pi x}{l} = \frac{5\pi}{2}$ **i.e. at** $x_3 = \frac{5l}{4} \notin (0, l)$

iii. **The eigenfunctions '**u_n**' are said to be orthogonal. From the theory of Fourier series we know that they form a complete set for the half interval** $[0, l]$**. Every function defined in this interval and satisfying some conditions can be written as**

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x)$$

EXAMPLE: **Show that the following boundry conditions yield self adjoint problems.**

i. $u(0) = 0, u(l) = 0$ ii. $u'(0) = 0, u(l) = 0$

iii. $u(a) = u(b)$ and $p(a)u'(a) = p(b)u'(b)$

Solution: Given $u'' + \lambda u = 0 \Rightarrow D = \pm i\sqrt{\lambda}$

For self adjointness we have $\int_a^b [uL(v) - vL(u)]dx = |p(uv' - u'v)|_a^b = 0$

i. Here we have $a = 0, b = l, u(0) = 0, u(l) = 0, v(0) = 0, v(l) = 0$

therefore $\int_0^l [uL(v) - vL(u)]dx = |p(uv' - u'v)|_0^l = 0$

$\int_0^l [uL(v) - vL(u)]dx = p(l)\big(u(l)v'(l) - u'(l)v(l)\big) - p(0)\big(u(0)v'(0) - u'(0)v(0)\big) = 0$

$\int_0^l [uL(v) - vL(u)]dx = 0$ i.e. condition satisfied for self adjointness.

ii. Here we have $a = 0, b = l, u'(0) = 0, u(l) = 0, v'(0) = 0, v(l) = 0$

therefore $\int_a^b [uL(v) - vL(u)]dx = |p(uv' - u'v)|_a^b = 0$

$\int_0^l [uL(v) - vL(u)]dx = p(l)\big(u(l)v'(l) - u'(l)v(l)\big) - p(0)\big(u(0)v'(0) - u'(0)v(0)\big) = 0$

$\int_0^l [uL(v) - vL(u)]dx = 0$ i.e. condition satisfied for self adjointness.

iii. Here we have $u(a) = u(b)$ and $p(a)u'(a) = p(b)u'(b)$

also $v(a) = v(b)$ and $p(a)v'(a) = p(b)v'(b)$

therefore $\int_a^b [uL(v) - vL(u)]dx = |p(uv' - u'v)|_a^b = 0$

$\int_a^b [uL(v) - vL(u)]dx = p(b)\big(u(b)v'(b) - u'(b)v(b)\big) - p(a)\big(u(a)v'(a) - u'(a)v(a)\big) = 0$

$\int_a^b [uL(v) - vL(u)]dx = 0$ i.e. condition satisfied for self adjointness.

BOUNDRY CONDITIONS OF 1- D HEAT EQUATION

i. **THE DRICHLET BC's or BC's OF 1st KIND:**

Booundry conditions of the form $u(0,t) = u_0(t)$ and $u(l,t) = u_1(t)$; $t > 0$ are called Drichlet boundry conditions.

Physical Meaning: This condition tells that the temperature at the boundry of a body may be controlled in some way without being held constant.

ii. **THE NEUMANN BC's or BC's OF 2nd KIND:**

Booundry conditions of the form $u_x(0,t) = \gamma(t)$ and $u_x(l,t) = \delta(t)$ are called Neumann boundry conditions. Where γ and δ are functions of time. And in particular, γ and δ may be zero. If $\gamma = 0$ then there is no flow at x = 0

Physical Meaning: This condition tells that the rate of flow of heat is specified at one or more boundry.

iii. **THE ROBBIN BC's or BC's OF 3rd KIND:**

Booundry conditions of the form $\propto_1 u(0,t) + \propto_2 u_x(0,t) = constant$ and $\beta_1 u(l,t) + \beta_2 u_x(l,t) = constant$ are called Robbin boundry conditions.

Physical Meaning: This condition tells about the proportionality between the rate of transfer of heat to the difference of temperature between the two bodies. i.e. both will be Proportional.

iv. **MIXED BC's or BC's OF 4th KIND:**

If more than one boundry points involved the BC's are called Mixed BC's. these are of the form $u(x_0,t) = u(x_1,t)$ and $u_x(x_0,t) = u_x(x_1,t)$

MATHEMATICAL MODELS

Usually, in almost all physical phenomena (or physical processes), the dependent variable $u = u(x, y, z, t)$ is a function of three space variables, x, y, z and time variable t.

The three basic types of second-order partial differential equations are:

(a) The wave equation $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0 \Rightarrow u_{tt} - c^2\nabla^2 u = 0$

(b) The heat equation $u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0 \Rightarrow u_t - k\nabla^2 u = 0$

(c) The Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0 \Rightarrow \nabla^2 u = 0$

WAVE: A wave is a disturbance that carries energy from one place to another. For example, wave produced on the string.

There are two types of waves.

MECHANICAL WAVE: Waves which required any medium for their propogation.

e.g. (i) Sound waves (ii) water waves.

ELECTROMEGNATIC WAVE: Waves which do not required any medium for their propogation.

e.g. (i) Radio waves (ii) X - Rays.

Mechanical waves have two types

TRANSVERSE WAVES:

In the case of transverse waves, the motion of particles of the medium is perpendicular to the motion of waves.

e.g. Waves produced on water surface

LONGITUDINAL WAVES:

In the case of longitudinal waves, the particles of the medium move back and forth along the direction of propogation of wave.

e.g. Waves produced in an elastic spring.

GENERAL FORM OF WAVE EQUATION

In general, the wave equation may be written as $u_{tt} = c^2\nabla^2 u$

where the Laplace operator may be one, two, or three dimensional.

The importance of the wave equation stems from the facts that this type of equation arises in many physical problems; for example, sound waves in space, electrical vibration in a conductor, torsional oscillation of a rod, shallow water waves, linearized supersonic flow in a gas, waves in an electric transmission line, waves in magnetohydrodynamics, and longitudinal vibrations of a bar.

ONE DIMENSIONAL WAVE EQUATION (UoS; 2017 – I , II)

An equation of the form $u_{tt} = c^2 u_{xx}$ where $c^2 = \frac{T}{\rho}$ is called the one-dimensional wave equation. Where u (x,t) is a function of displacement at position x in time 't' and 'c' denotes the velocity of wave equation.

PROOF : Let us consider a stretched string of length *l* fixed at the end points. The

problem here is to determine the equation of motion which characterizes the position u (x,t) of the string at time t after an initial disturbance is given.

In order to obtain a simple equation, we make the following assumptions:

1. The string is flexible and elastic, that is the string cannot resist bending moment and thus the tension in the string is always in the direction of the tangent to the existing profile of the string.

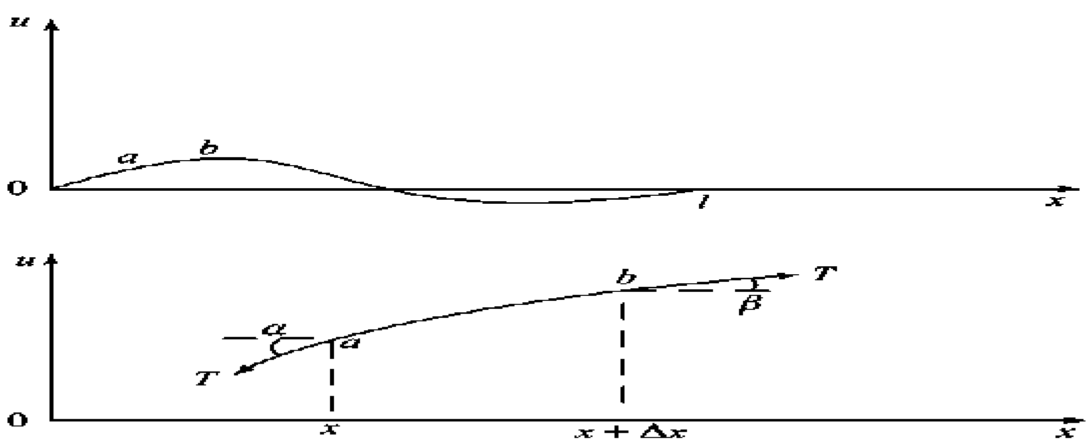
2. There is no elongation of a single segment of the string and hence, by Hooke's law, the tension is constant.

3. The weight of the string is small compared with the tension in the string.

4. The deflection is small compared with the length of the string.

5. The slope of the displaced string at any point is small compared with unity.

6. There is only pure transverse vibration.



We consider a differential element of the string. Let T be the tension at the end points as shown in Figure. The forces acting on the element of the string in the vertical direction are $T\sin\beta - T\sin\alpha$

By Newton's second law of motion, the resultant force is equal to the mass times the acceleration. Hence,

$T\sin\beta - T\sin\alpha = \rho\delta s\, u_{tt}$ ……………(i) $\quad\therefore \rho = m/\delta s$

where ρ is the line density and δs is the smaller arc length of the string.

Since the slope of the displaced string is small, we have $\delta s \simeq \delta x$

Since the angles α and β are small $\sin\alpha \simeq \tan\alpha,\ \sin\beta \simeq \tan\beta$

Thus, equation (i) becomes $\tan\beta - \tan\alpha = \frac{\rho}{T}\delta x\, u_{tt}$ ……………(ii)

But, from calculus we know that $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and x + δx:

$\tan\alpha = u_x(x,t)$ **and** $\tan\beta = u_x(x+\delta x, t)$ **at time t.**

Then Equation (ii) may thus be written as

$\frac{1}{\delta x}[(u_x)_{x+\delta x} - (u_x)_x] = \frac{\rho}{T}u_{tt}$

$\frac{1}{\delta x}[u_x(x+\delta x, t) - u_x(x,t)] = \frac{\rho}{T}u_{tt}$

$\lim_{\delta x\to 0}\frac{1}{\delta x}[u_x(x+\delta x, t) - u_x(x,t)] = \frac{\rho}{T}u_{tt}$ $\quad$ **∴limit have no effect on R.H.S**

$u_{tt} = c^2 u_{xx}$ ……………(iii)

where $c^2 = \frac{T}{\rho}$. This is called the one-dimensional wave equation.

D'ALEMBERT'S SOLUTION OF WAVE EQUATION

This is the general method for the solution of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Let $w = x - ct$ and $z = x + ct$, so that the function $u(x,t)$ is now a function of the new variables 'w' and 'z'

Using the rules for partial differentiation we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\frac{\partial}{\partial x}(x - ct) + \frac{\partial u}{\partial z}\frac{\partial}{\partial x}(x + ct) \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial z} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z} \text{ ....(i)}$$

Similarly $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial w}\frac{\partial}{\partial t}(x - ct) + \frac{\partial u}{\partial z}\frac{\partial}{\partial t}(x + ct) \Rightarrow \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial w} + c\frac{\partial u}{\partial z} \Rightarrow \frac{\partial}{\partial t} = -c\frac{\partial}{\partial w} + c\frac{\partial}{\partial z} \text{ ......(ii)}$$

Hence $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial w} + \frac{\partial u}{\partial z}\right) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2}$ **.........(iii)**

Similarly Hence $\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial w^2} - 2c^2\frac{\partial^2 u}{\partial w \partial z} + c^2\frac{\partial^2 u}{\partial z^2}$ **.........(iv)**

Thus putting all values in $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ we get the form

$$\Rightarrow \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\left[c^2\frac{\partial^2 u}{\partial w^2} - 2c^2\frac{\partial^2 u}{\partial w \partial z} + c^2\frac{\partial^2 u}{\partial z^2}\right] \Rightarrow \frac{\partial^2 u}{\partial w \partial z} = 0$$

$$\Rightarrow \frac{\partial}{\partial w}\frac{\partial u}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} = s(z) \Rightarrow u = \int s(z)dz + a\ function\ of\ w$$

$$\Rightarrow u(x,t) = \varphi(z) + \Psi(w)$$

$$\Rightarrow u(x,t) = \varphi(x - ct) + \Psi(x + ct)$$

This solution is called D'Alembert's Solution of the wave equation.

EXAMPLE:

Sove the problem $u_{tt} = u_{xx} + h$; $0 \leq x \leq 1$ …………..(i)

with conditions $u(x,0) = x(1-x)$ and $u_t(x,0) = 0$ …………..(ii)

$u(0,t) = 0 = u(1,t)$ …………..(iii) where 'h' is constant.

Solution:

We suppose that $u(x,t) = v(x,t) + w(x)$ is the solution of PDE (i) then on substituiting in (i), (ii), (iii) we get

$v_{tt} - (v_{xx} + w'') = h$; $0 \leq x \leq 1$ …………..(iv)

$v(x,0) + w(x) = x(1-x)$ …………..(v) $v_t(x,0) = 0$ ………..(vi)

$v(0,t) + w(0) = 0$ ……(vii) $v(1,t) + w(1) = 0$…….(viii)

The DE (iv) together with the BC's and IC's (vi) to (viii) are equivalent to the following two IVP/BVP

$-w'' = h$; $w(0) = 0$; $w(1) = 0$ …………..(ix)

and $v_{tt} - v_{xx} = 0$ …………..(x) $v(x,0) = x(1-x) - w(x)$ …………..(xi)

$v_t(x,0) = 0$ ………..(xii)

$v(0,t) = 0$……..(xiii)

$v(1,t) = 0$ ………..(xiv)

General solution of ODE in (ix) is given by

$$w(x) = -\frac{1}{2}hx^2 + c_1x + c_2$$

On applying the BC's we get $c_1 = \frac{h}{2}$; $c_2 = 0$ we get

$$w(x) = -\frac{1}{2}hx^2 + \frac{1}{2}hx = \frac{h}{2}x(1-x)$$

TRANSIENT TEMPERATURE DISTRIBUTION: for wave equation we consider $u(x,t) = v(x,t) + w(x)$, in this phenomenon $u(x,t)$ is called transient solution, $v(x,t)$ is non steady state solution and $w(x)$ is steady state solution.

EXAMPLE: (UoS; 2017)

Sove the problem $u_{tt} = c^2 u_{xx} + x^2 \ ; 0 \le x \le 1$ **…………..(i)**

with conditions $u(x,0) = u_t(x,0) = 0$ **…………..(ii);**

$u(0,t) = 0 = u(1,t)$ **…………..(iii)**

Solution: We suppose that $u(x,t) = v(x,t) + w(x)$ **is the solution of PDE (i) then on substituiting in (i), (ii), (iii) we get**

$v_{tt} - c^2(v_{xx} + w'') = x^2 \ ; 0 \le x \le 1$ **…………..(iv)**

$v(x,0) + w(x) = 0$ **…………..(v)** $v_t(x,0) = 0$ **………..(vi)**

$v(0,t) + w(0) = 0$ **……(vii)** $v(1,t) + w(1) = 0$ **…....(viii)**

The DE (iv) together with the BC's and IC's (vi) to (viii) are equivalent to the following two IVP/BVP

$-c^2 w'' = x^2 \ ; \ w(0) = 0 \ ; w(1) = 0$**………..(ix) and** $v_{tt} - c^2 v_{xx} = 0$ **……..(x)**

$v(x,0) = -w(x) \ , \ v_t(x,0) = 0 \ , \ v(0,t) = 0 \ , v(1,t) = 0$ **……....(xi)**

General solution of ODE in (ix) is given by $w(x) = -\frac{1}{12c^2}x^4 + c_1 x + c_2$

On applying the BC's we get $c_1 = \frac{1}{12c^2}$; $c_2 = 0$ **we get**

$w(x) = -\frac{1}{12c^2}x^4 + \frac{1}{12c^2}x = \frac{1}{12c^2}(x - x^4)$

Solving (x) by separating variables

$v(x,t) = \sum_n Sinn\pi x(a_n Cosn\pi ct + b_n Sinn\pi ct)$ **…………….(xii)**

Now initial conditions (xi) give respectively

$\sum_n a_n Sinn\pi x = \frac{1}{12c^2}(x^4 - x)$ **and** $\sum_n b_n Sinn\pi x = 0$ **; n = 1,2,3,…………**

From these equations we obtained; $b_n = 0, a_n = 2\int_0^1 \frac{1}{12c^2}(x^4 - x)Sinn\pi x dx \ ; \forall n$

(xii) $\Rightarrow v(x,t) = \sum_n a_n Sinn\pi x Cosn\pi ct$

Then the complete solution is

$u(x,t) = \sum_n a_n Sinn\pi x Cosn\pi ct + \frac{1}{12c^2}(x - x^4)$

HEAT: **Heat is a form of energy that transferred from hot body to the cold body, by means of thermal contact. It is denoted by 'q'**

CONDUCTION OF HEAT: **In this mode heat is transmitted through actual contact between particles (molecules) of the medium.**

CONVECTION OF HEAT: **In this mode heat is transmitted through gases or liquids by actual motion of particles (molecules) of the medium.**

RADIATION OF HEAT: **In this mode heat is transmitted through electromagnetic waves. Or by means of heat waves or thermal radiations. Medium is not essential for it. i.e. heat can take places in vaccume also.**

SPECIFIC HEAT OF SUBSTANCE (MATERIAL) : **The quantity of heat required to raise the temperature of 1g of material by $1C^\circ$ and it is denoted by C and mathematically could be written as $\Delta q = Cm\Delta u$**

HEAT FLUX (THERMAL FLUX) : **Is the rate of heat energy transfer through a given surface per unit surface area. Its unit is watt or Js^{-1}**

THERMAL CONDUCTIVITY: **The quantity of heat flowing per second across a plate (of the material) of unit area and unit thickness, when the temperature difference between opposite sides is $1C^\circ$**

It determines how good a conductor the material is . It is large for good conductors and small for bad conductors.

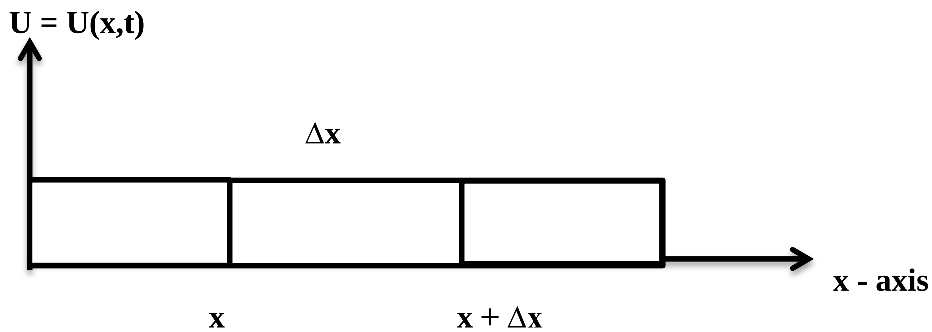
SOME FACTORS ON WHICH RATE OF FLOW OF HEAT DEPENDS

- **Area as** $q(x,t) \propto A$
- **Length as** $q(x,t) \propto \frac{1}{L}$
- **Change in temperature as** $q(x,t) \propto \Delta u$

ONE DIMENSIONAL HEAT EQUATION

An equation of the form $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$ is called heat equation. Where U= U(x,t) is a temperature of a body at 'x' position in time 't' and 'K' is called diffusivity or thermal conductivity of the material.

PROOF:



Let us consider the flow of heat through a uniform rod of length 'l' and cross sectional area 'A' then

Density of rod = $\rho = mass/volume = m/A\Delta x$ i.e $m = \rho A\Delta x$

We choose the x – axis along the length of the rod with origin at one end of the rod. Then temperature at point 'x' from origin at time 't' will be U = U(x,t)

Let flow of heat = q (x,t)

(quantity of heat entering per second through unit area perpendicular to the direction of flow)

Also Heat generation = γ and heat stored per second = cm $\frac{\partial u}{\partial t}$ = cρAΔx $\frac{\partial u}{\partial t}$

Now using law of conservation of heat energy

(Quantity of heat which entered) + (heat generated inside the rod)

= (Quantity of heat which leave) + (quantity of heat stored)

$$q(x,t)A + \gamma A\Delta x = q(x+\Delta x,t)A + c\rho A\Delta x\frac{\partial u}{\partial t}$$

dividing both sides by 'A' we get $q(x,t) + \gamma\Delta x = q(x+\Delta x,t) + c\rho\Delta x\frac{\partial u}{\partial t}$

dividing both sides by Δx we get $\frac{1}{\Delta x}[q(x,t) - q(x+\Delta x,t)] + \gamma = c\rho\frac{\partial u}{\partial t}$

Applying Δx → 0 $-\frac{\partial q}{\partial x} + \gamma = c\rho\frac{\partial u}{\partial t}$

Now by using Fourier law of heat conductivity which is q = - KΔu

Then (i) becomes $-\frac{\partial}{\partial x}(-\kappa\Delta u) + \gamma = c\rho\frac{\partial u}{\partial t}$ **then we get** $\kappa\frac{\partial^2 u}{\partial x^2} + \gamma = c\rho\frac{\partial u}{\partial t}$

For standard form we suppose $\gamma = 0$ and cρ = 1 (i.e. no heat generation)

Then $\kappa\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

Or $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$ **which is required heat equation in one dimension**

In general $u_t = \kappa\nabla^2 u$ **or** $\nabla^2 u = \frac{1}{\kappa}u_t$

EXAMPLE:

Show that solution of heat flowing problem is unique.

Solution:

Consider $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$ **............(i)** $x \in (x_1, x_2)$; $t > t_0$
With BC's $u(x_1, t) = 0$ **and** $u(x_2, t) = 0$
with original temperature distribution $u(x, t_0) = 0$
for unique solution of (i) we suppose on contrary that equation (i) has two solutions 'u' and 'v' then by principle of superposition $w = u - v$ **is also a solution of (i) and satirsfy equation (i)**
$\Rightarrow \frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}$ **With BC's** $w(x_1, t) = 0 = w(x_2, t)$ **and** $w(x, t_0) = 0$
$\Rightarrow kw_{xx} = w_t$
Now we will prove $w(x, t) = 0$
Define $I(t) = \int_{x_1}^{x_2} w^2(x, t)dx$ **............(ii) clearly** $I(t) \geq 0$ **and** $I(t_0) = 0$
Diff. (ii) w.r.to 't' $\Rightarrow I'(t) = \int_{x_1}^{x_2} 2ww_t dx$
$\Rightarrow I'(t) = \int_{x_1}^{x_2} 2w(kw_{xx})dx = 2k\int_{x_1}^{x_2} ww_{xx}dx = 2k\left[|ww_x|_{x_1}^{x_2} - \int_{x_1}^{x_2} w_x w_x dx\right]$
$\Rightarrow I'(t) = 0 - 2k\int_{x_1}^{x_2} {w_x}^2 dx$ **since** $w(x_1, t) = 0 = w(x_2, t)$
$\Rightarrow I'(t) = 2k\int_{x_1}^{x_2} {w_x}^2 dx \leq 0 \Rightarrow I'(t) \leq 0 \Rightarrow I'(t) < 0$ **or** $I'(t) = 0$
If $I'(t) < 0$ **then** $I(t)$ **is decreasing function.**
If $I'(t) = 0$ **then** $I(t)$ **is constant function.**
This result together with the fact that $I(t) \geq 0$; $\forall t \geq t_0$ **and** $I(t_0) = 0$
$\Rightarrow I(t) \geq 0$; $\forall t \geq t_0$
$(ii) \Rightarrow w(x, t) = 0 \Rightarrow u - v = 0 \Rightarrow u = v$ **(contradiction)**
Hence solution is unique.

MAXIMUM PRINCIPLE FOR THE HEAT EQUATION:
Let $u(x, t)$ **be a continuous differentiable function that satisfy the heat equation** $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$; $x \in (0, l)$; $t > 0$ **with BC's** $u(0, t) = 0 = u(l, t)$ **then** $u(x, t)$ **attains its maximum value at** $t = 0$ **for some** $x \in [0, l]$
In other words $\underset{\substack{t \geq 0 \\ 0 \leq x \leq l}}{Max}\ u(x, t) = \underset{\substack{t = 0 \\ 0 \leq x \leq l}}{max}\ u(x, 0)$
This principle is called Maximum Principle.

LAPLACE TRANSFORMATION WITH APPLICATIONS

Because of their simplicity, Laplace transforms are frequently used to solve a wide class of partial differential equations. Like other transforms, Laplace transforms are used to determine particular solutions. In solving partial differential equations, the general solutions are difficult, if not impossible, to obtain. The transform technique sometimes offers a useful tool for finding particular solutions. The Laplace transform is closely related to the complex Fourier transform, so the Fourier integral formula can be used to define the Laplace transform and its inverse.

INTEGRAL TRANSFORMATION

Consider a set $K(x,y) = \{f(x); f \text{ is function of } x \text{ over } [a,b]\}$ then integral transformation is defined as

$T\{f(x)\} = F(y) = \int_a^b f(x)K(x,y)dx$ **where $K(x,y)$ is kernel of T.**

LAPLACE TRANSFORMATION

If $f(t)$ is defined for all values of $t > 0$, then the Laplace transform of $f(t)$ is denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$ and is defined by the integral

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt$$

If $F(s)$ is laplace transform of $f(t)$ then $f(t)$ is called the <u>INVERSE LAPLACE TRANSFORM</u> of $F(s)$ i.e. $\mathcal{L}^{-1}\{F(s)\} = f(t)$

QUESTION: Show that $\mathcal{L}\{c\} = \frac{c}{s}$ where 'c' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{c\} = \int_0^\infty e^{-st}\, cdt = c\int_0^\infty e^{-st}\, dt = c\left|-\frac{e^{-st}}{s}\right|_0^\infty = \frac{c}{s}$

QUESTION: Show that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ where 'a' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left|-\frac{e^{-(s-a)t}}{(s-a)}\right|_0^\infty = \frac{1}{s-a}$

QUESTION: Show that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ where '$n > 0$'

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then for n = 1;

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, tdt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}dt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s}$$

In above $te^{-st} \to 0$ as $t \to \infty$

for n = 2;

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\, t^2\, dt = \left|-\frac{t^2\, e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} 2t\, dt = \left|-\frac{t^2\, e^{-st}}{s}\right|_0^\infty +$$

$\frac{2}{s}\int_0^\infty e^{-st}\, dt = \frac{2}{s^3}$ In this part $t^2\, e^{-st}, te^{-st} \to 0$ as $t \to \infty$

And in general

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt = \left|-\frac{t^n\, e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} nt^{n-1}\, dt$$

$$\mathcal{L}\{t^n\} = \left|-\frac{t^n\, e^{-st}}{s}\right|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\} =$$

$$\frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\ldots\ldots\ldots..\frac{2}{s}\cdot\frac{1}{s}\mathcal{L}\{t^0\}$$

$$\mathcal{L}\{t^n\} = \frac{(n-1)(n-1)(n-1)\ldots\ldots\ldots.3.2.1}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^n}\cdot\frac{1}{s}$$

Hence $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ where '$n > 0$'

QUESTION: Show that $\mathcal{L}\{Sinat\} = \frac{a}{s^2+a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Sinat\} = \int_0^\infty e^{-st}\, Sinatdt$

$\therefore \int_0^\infty e^{at}\, Sinbtdt = \frac{e^{at}}{a^2+b^2}[aSinbt - bCosbt]$ therefore

$\mathcal{L}\{Sinat\} = \left|\frac{e^{-st}}{s^2+a^2}[-sSinat - aCosat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-a)\right] = \frac{a}{s^2+a^2}$

QUESTION: Show that $\mathcal{L}\{Cosat\} = \frac{s}{s^2+a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Cosat\} = \int_0^\infty e^{-st}\, Cosatdt$

$\therefore \int_0^\infty e^{at}\, Cosbtdt = \frac{e^{at}}{a^2+b^2}[aCosbt + bSinbt]$ therefore

$\mathcal{L}\{Cosat\} = \left|\frac{e^{-st}}{s^2+a^2}[-sCosat + aSinat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-s)\right] = \frac{s}{s^2+a^2}$

QUESTION: Show that $\mathcal{L}\{Sinhat\} = \frac{a}{s^2-a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Sinhat\} = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\left[\int_0^\infty e^{-st}\, e^{at}dt - \int_0^\infty e^{-st}\, e^{-at}dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{a}{s^2-a^2}$

QUESTION: Show that $\mathcal{L}\{Coshat\} = \frac{s}{s^2-a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then

$$\mathcal{L}\{Coshat\} = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{1}{2}\left[\int_0^\infty e^{-st}\, e^{at}dt + \int_0^\infty e^{-st}\, e^{-at}dt\right]$$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt + \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{s}{s^2-a^2}$$

FUNCTION OF EXPONENTIAL ORDER: A function f (t) is said to be <u>of exponential order</u> as $t \to \infty$ if there exist real constants M and c such that $|f\,(t)| \leq Me^{ct}$ for $0 \leq t < \infty$.

FUNCTION OF CLASS 'A': A function f (t) which is peicewise continuous and is of exponential order is said to be function of class A.

EXISTENCE THEOREM OF LAPLACE TRANSFORMATION:

(UoS; 2013,2015)

Let f be piecewise continuous in the interval $[0, T]$ for every positive T, and let f be of exponential order, that is, $f\,(t) = O\,(e^{at})$ as $t \to \infty$ for some $a > 0$. Then, the Laplace transform of $f\,(t)$ exists for $Res > a$.

OR sufficient condition for the existence of Laplace transformation is that it should be a function of class A.

Proof: Since f is piecewise continuous and of exponential order, we have

$$|\mathcal{L}\{f(t)\}| = \left|\int_0^\infty e^{-st}\, f(t)dt\right| \leq \int_0^\infty e^{-st}\, |f(t)|dt \leq \int_0^\infty e^{-st}\, Me^{at}\, dt = M\int_0^\infty e^{-(s-a)t}\, dt$$

$|\mathcal{L}\{f(t)\}| \leq \frac{M}{s-a}$ **Thus the Laplace transform of $f\,(t)$ exists for $Res > a$.**

Remark: $F(s) = s^2$ **is not L.T. of any piecewise continuous function of exponential order, because** s^2 **does not approaches to zero as** $s \to \infty$ **i.e.** $\mathcal{L}^{-1}\{s^2\}$ **does not exists.**

QUESTION: Show that $\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ **where** α **is any real.**

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$

$$\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-st}\, t^{\alpha}\, dt = \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{\alpha} \frac{1}{s} du = \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-u}\, u^{\alpha}\, du \;\ldots\ldots\ldots(i)$$

Since by definition of Gamma function we have

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u}\, u^{\alpha-1}\, du \Rightarrow \Gamma(\alpha+1) = \int_0^{\infty} e^{-u}\, u^{\alpha}\, du \quad (i) \Rightarrow \mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

USEFUL RESULTS:

- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ **then** $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha\Gamma(\alpha)}{s^{\alpha+1}}$
- $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\}$

QUESTION: Find $\mathcal{L}\left\{t^{1/2}\right\}$ **and** $\mathcal{L}\left\{t^{-1/2}\right\}$

SOLUTION: Since $\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

Put $\alpha = \frac{1}{2}$ **Then** $\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ **now using** $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha\Gamma(\alpha)}{s^{\alpha+1}}$

we have $\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s.s^{\frac{1}{2}}}$

Then $\mathcal{L}\left\{t^{1/2}\right\} = \frac{1}{2s}\frac{\sqrt{\pi}}{\sqrt{s}}$ **as** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **thus** $\mathcal{L}\left\{t^{1/2}\right\} = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$

Put $\alpha = -\frac{1}{2}$**Then** $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$ **now we have** $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}$

Then $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}}$ **as** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **thus** $\mathcal{L}\left\{t^{-1/2}\right\} = \sqrt{\frac{\pi}{s}}$

QUESTION: Find $\mathcal{L}\{t^{k/2}\}$ where 'k' is an odd positive integer. $\mathcal{L}\{t^{5/2}\}$ =?

SOLUTION: Suppose k = m + 1 where 'm' is any positive integer.

Then using $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\}$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\} = \mathcal{L}\left\{t^{\frac{2m+1}{2}}\right\} = \mathcal{L}\left\{t^{m+\frac{1}{2}}\right\} = \frac{m+\frac{1}{2}}{s}\mathcal{L}\left\{t^{m+\frac{1}{2}-1}\right\} = \frac{m+\frac{1}{2}}{s}.\frac{m-\frac{1}{2}}{s}.\mathcal{L}\left\{t^{m-\frac{1}{2}-1}\right\}$$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\} = \frac{2m+1}{2s}.\frac{2m-1}{2s}.\frac{2m-3}{2s}\ldots\ldots..\frac{3}{2s}.\frac{1}{2s}.\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} = \frac{(2m+1).(2m+1).(2m+1)\ldots\ldots..3.1}{(2s)^{m+1}}\sqrt{\frac{\pi}{s}}$$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\} = \frac{(2m+1).(2m+1-2).(2m+1-4)\ldots\ldots..3.1}{(2s)^{m+1}}\sqrt{\frac{\pi}{s}} = \frac{(k).(k-2).(k-4)\ldots\ldots..3.1}{(2)^{\frac{k+1}{2}}}\sqrt{\frac{\pi}{s^{k+2}}}$$

Where we use $2m + 1 = k \Rightarrow m = (k-1)/2$

If k = 5 then $\mathcal{L}\left\{t^{\frac{5}{2}}\right\} = \frac{5.3.1}{(2)^{\frac{5+1}{2}}}\sqrt{\frac{\pi}{s^{5+2}}} = \frac{15}{(2)^3}\sqrt{\frac{\pi}{s^7}}$

PROPERTIES OF LAPLACE TRANSFORMS

LINEARITY PROPERTY: THE LAPLACE TRANSFORMATION $\mathcal{L}$ IS LINEAR.

Proof. Let $u(t) = af(t) + bg(t)$ where a and b are constants.

We have, by definition

$$\mathcal{L}\{u(t)\} = \int_0^{\infty} e^{-st} f(t)dt = \int_0^{\infty} e^{-st}[af(t) + bg(t)]dt$$

$$\mathcal{L}\{u(t)\} = a\int_0^{\infty} e^{-st} f(t)dt + b\int_0^{\infty} e^{-st} g(t)dt = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ **hence proved.**

1st SHIFTING PROPERTY (1st TRANSLATION THEOREM):

If $F(s)$ is the laplace transformation of $f(t)$ Then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Proof. By definition, we have

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^{\infty} e^{-st} e^{at} f(t)dt = \int_0^{\infty} e^{-(s-a)t} f(t)dt = F(s-a)$$

This result also known as 1st shifting theorem or 1st translation theorem.

EXAMPLES:

i. If $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ then $\mathcal{L}\{t^2\, e^{t}\} = \frac{2}{(s-1)^3}$

ii. If $\mathcal{L}\{Sinwt\} = \frac{w}{s^2+w^2}$ then $\mathcal{L}\{e^{at}\, Sinwt\} = \frac{w}{(s-a)^2+w^2}$

iii. If $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2}$ then $\mathcal{L}\{e^{at}\, Coswt\} = \frac{s-a}{(s-a)^2+w^2}$

iv. If $\mathcal{L}\{t^{n}\} = \frac{n!}{s^{n+1}}$ then $\mathcal{L}\{e^{at}\, t^{n}\} = \frac{n!}{(s-a)^{n+1}}$

Question: Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\}$

Answer: in this question we will use the first shifting theorem according to which $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2-1^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2-1^2}\right\} = e^{-t}\, Cosht$

Question: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\}$

Answer: in this question we will use the first shifting theorem according to which $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-1^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-1^2}\right\} = e^{-t}\, Sinht$

Question: Find $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\}$

Answer: in this question we will use the first shifting theorem according to which $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{2}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\}\left\{\frac{2}{s}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} - e^{-2t}\mathcal{L}^{-1}\left\{\frac{2}{s}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = 3e^{-t} - 2e^{-2t}$ since $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

SCALING PROPERTY: If $F(s)$ is the laplace transformation of (t) , then

$\mathcal{L}\,[f\,(at)] = \frac{1}{a}\,F\,(\frac{s}{a})$ **with a > 0**

Proof. By definition we have

$$\mathcal{L}\,\{f(at)\} = \int_0^\infty e^{-st}\,f(at)dt = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t'}\,f(t')dt' = \frac{1}{a}\,F\,(\frac{s}{a})$$

putting $at = t'$ This result also known as Rule of Scale.

EXAMPLES:

i. **If $\mathcal{L}\{Cost\} = \frac{s}{s^2+1}$ then $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2} = \frac{1}{w}\left[\frac{s/w}{(s/w)^2+1}\right]$**

ii. **If $\mathcal{L}\{e^{t}\} = \frac{1}{s-1}$ then $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} = \frac{1}{a}\left[\frac{1}{\left(\frac{s}{a}-1\right)}\right]$**

DIFFERENTIATION PROPERTY: (UoS; 2011,2014)

Let f be continuous and f' piecewise continuous, in $0 \le t \le T$ for all $T > 0$. Let f also be of exponential order as $t \to \infty$ Then, the Laplace transform of $f'\,(t)$ exists and is given by

$$\mathcal{L}[f'\,(t)] = s\mathcal{L}[f(t)] - f\,(0) = sF(s) - f\,(0)$$

Proof. If $f(t)$ is continuous and $f'(t)$ is sectionally continuous on the interval $[0,\infty)$ and both are of exponential order then

$$\mathcal{L}\,\{f'(t)\} = \int_0^\infty e^{-st}\,f'(t)dt = |e^{-st}f(t)|_0^\infty - (-s)\int_0^\infty e^{-st}\,f(t)dt$$

$$\mathcal{L}\,\{f'(t)\} = [0 - f(0)] + s\mathcal{L}\,\{f(t)\}$$

$$\mathcal{L}[f'\,(t)] = s\mathcal{L}[f(t)] - f\,(0) = sF(s) - f\,(0)$$

If f' and f'' satisfy the same conditions imposed on f and f' respectively, then, the Laplace transform of $f''\,(t)$ can be obtained immediately by applying the preceding theorem; that is

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f\,'(0) = s^2F(s) - sf(0) - f'\,(0)$$

Proof. If $f(t), f'(t)$ are continuous and $f''(t)$ is sectionally continuous on the interval $[0, \infty)$ and all are of exponential order then

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)dt = |e^{-st} f'(t)|_0^\infty - (-s)\int_0^\infty e^{-st} f'(t)dt$$

$$\mathcal{L}\{f''(t)\} = [0 - f'(0)] + s\mathcal{L}\{f'(t)\} = -f'(0) + s[sF(s) - f(0)]$$

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f'(0) = s^2F(s) - sf(0) - f'(0)$$

Clearly, the Laplace transform of $f^n(t)$ can be obtained in a similar manner by successive application.The result may be written as

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - \cdots \ldots \ldots - sf^{n-2}(0) - f^{n-1}(0)$$

INTEGRATION PROPERTY :

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

PROOF:

Consider $g(\tau) = \int_0^t f(\tau)\, d\tau \Rightarrow g'(\tau) = f(t) \Rightarrow \mathcal{L}[g'(\tau)] = \mathcal{L}[f(t)]$

$\Rightarrow sG(s) - g(0) = \mathcal{L}[f(t)] \Rightarrow s\mathcal{L}[g(\tau)] - 0 = \mathcal{L}[f(t)]$

$\Rightarrow \mathcal{L}[g(\tau)] = \frac{F(s)}{s} \quad \Rightarrow \mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$

Question: Solve the initial value problem $u' - 2u = 0$ with $u(0) = 1$

Answer: Given $u' - 2u = 0$

$\Rightarrow \mathcal{L}\{u'\} - 2\mathcal{L}\{u\} = 0 \Rightarrow sU(s) - u(0) - 2U(s) = 0$

Using $u(0) = 1 \quad \Rightarrow sU(s) - 1 - 2U(s) = 0 \Rightarrow U(s) = \frac{1}{s-2}$

$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \qquad \Rightarrow u(t) = e^{2t}$ required answer.

Question:

Solve the initial value problem $u'' + 4u' + 3u = 0$ **with** $u(0) = 1, u'(0) = 0$

Answer: **Given** $u'' + 4u' + 3u = 0$

$\Rightarrow \mathcal{L}\{u''\} + 4\,\mathcal{L}\{u'\} + 3\mathcal{L}\{u\} = 0$

$\Rightarrow s^2U(s) - su(0) - u'(0) + 4sU(s) - 4u(0) + 3U(s) = 0$

$\Rightarrow s^2U(s) - s + 4sU(s) - 4 + 3U(s) = 0$ **since** $u(0) = 1, u'(0) = 0$

$\Rightarrow U(s) = \frac{s+4}{s^2+4s+2} \Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+2}\right\}$

$\Rightarrow u(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s+1} - \frac{1/2}{s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1/2}{s+3}\right\}$

$\Rightarrow u(t) = e^{-\mathrm{t}}\,\mathcal{L}^{-1}\left\{\frac{3/2}{s}\right\} - e^{-3\mathrm{t}}\,\mathcal{L}^{-1}\left\{\frac{1/2}{s}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \frac{3}{2}e^{-\mathrm{t}} - \frac{1}{2}e^{-3\mathrm{t}}$ **since** $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

UNIT STEP FUNCTION: A real valued function $H: R \to R$ **is defined as**

$H\,(t-\xi) = \begin{cases} 1 & ; \quad t \geq \xi \\ 0 & ; \quad t < \xi \end{cases}$ **When** $\xi = 0$; $H\,(t) = \begin{cases} 1 & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases}$

CONVOLUTION FUNCTION / FAULTUNG FUNCTION OF LAPLACE TRANSFORMATION.

The function $(f * g)(t) = \int_0^t f\,(t-\xi)\,g\,(\xi)\,d\xi$ **is called the convolution of the functions** f **and** g **regarding laplace transformation.**

THE CONVOLUTION SATISFIES THE FOLLOWING PROPERTIES:

1. $f * g = g * f$ **(commutative).**

2. $f * (g * h) = (f * g) * h$ **(associative).**

3. $f * (\alpha g + \beta h) = \alpha\,(f * g) + \beta\,(f * h)$ **(distributive),**

where α **and** β **are constants.**

USEFUL RESULT:

$$(f * g)(t) = \int_0^t f\,(t-\xi)g\,(\xi)d\xi = \int_0^\infty H\,(t-\xi)f\,(t-\xi)\,g\,(\xi)\,d\xi$$

CONVOLUTION / FAULTUNG THEOREM OF LAPLACE TRANSFORMATION (UoS; 2015)

If $F(s)$ and $G(s)$are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then the Laplace transform of the convolution $(f * g)(t)$ is the product $F(s)G(s)$

OR $\mathcal{L}^{-1}\{F(s)G(s)\} = f * g \Rightarrow \mathcal{L}\{f * g\} = F(s)G(s)$

PROOF: By definition, we have

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}\,(f * g)\,dt$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st} \int_0^t f(t-\xi)\,g(\xi)\,d\xi\,dt$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st} \int_0^t f(\xi) g(t-\xi)\,d\xi\,dt \qquad \text{since } f * g = g * f$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}\left[\int_0^\infty H(t-\xi) f(\xi) g(t-\xi)\,d\xi\right] dt$$

By reversing the order of integration, we have

$$\mathcal{L}\{f * g\} = \int_0^\infty \left[\int_0^\infty e^{-st} H(t-\xi) g(t-\xi)\,dt\right] f(\xi)\,d\xi$$

If we introduce the new variable $\eta = (t-\xi)$ in the inner integral, we obtain

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\,d\xi\left[\int_{-\xi}^\infty e^{-s(\xi+\eta)} H(\eta) g(\eta)\,d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\,d\xi\left[\int_{-\xi}^0 e^{-s(\xi+\eta)} H(\eta) g(\eta)\,d\eta + \int_0^\infty e^{-s(\xi+\eta)} H(\eta) g(\eta)\,d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\,d\xi\left[\int_{-\xi}^0 e^{-s(\xi+\eta)} 0.g(\eta)\,d\eta + \int_0^\infty e^{-s(\xi+\eta)}.1.g(\eta)\,d\eta\right] \quad \text{by step function}$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\,d\xi\left[\int_0^\infty e^{-s(\xi+\eta)} g(\eta)\,d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi \int_0^\infty e^{-s\eta} g(\eta)\,d\eta$$

$$\mathcal{L}\{f * g\} = F(s)G(s)$$

PROBLEM: Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s^2+9)}\right\}$

Solution: **Here we have** $H(s) = F(s)G(s)$

then taking $F(s) = \frac{1}{s^2} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \Rightarrow f(t) = t$

$G(s) = \frac{3}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)}\right\} \Rightarrow g(t) = Sin3t$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(t-\xi)g(\xi)d\xi = \int_0^t (t-\xi)\,Sin3(\xi)\,d\xi$$

$$h(t) = \int_0^t t\,Sin3(\xi)\,d\xi - \int_0^t \xi\,Sin3(\xi)d\xi = \left|-\frac{tCos3\xi}{3} + \frac{\xi Cos3\xi}{3} - \frac{Sin3\xi}{9}\right|_0^t$$

$$h(t) = -\frac{Sin3t}{9} + \frac{t}{3} = \frac{1}{9}(3t - Sin3t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2(s^2+9)}\right\}$$

PROBLEM: Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$

Solution: **Here we have** $H(s) = F(s)G(s)$

then taking $F(s) = \frac{s}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)}\right\} \Rightarrow f(t) = Cos3t$

$G(s) = \frac{1}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)}\right\} \Rightarrow g(t) = \frac{1}{3}Sin3t$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(t-\xi)g(\xi)d\xi = \frac{1}{3}\int_0^t Cos3(t-\xi)\,Sin3(\xi)\,d\xi$$

$$h(t) = \frac{1}{3}\int_0^t (Cos3tCos3\,\xi + Sin3tSin3\,\xi)\,Sin3(\xi)\,d\xi$$

$$h(t) = \frac{1}{3}\int_0^t Cos3tCos3\xi Sin3\xi + Sin3tSin^2 3\xi\,d\xi$$

$$h(t) = \frac{1}{6}Cos3t\int_0^t 2Cos3\xi Sin3\xi d\xi + \frac{1}{3}Sin3t\int_0^t Sin^2 3\xi\,d\xi$$

$$h(t) = \frac{1}{6}Cos3t\int_0^t Sin6\xi d\xi + \frac{1}{6}Sin3t\int_0^t \left(\frac{1-Cos6\xi}{2}\right)\,d\xi$$

$$h(t) = \frac{1}{6}Cos3t\left|-\frac{Cos6\xi}{6}\right|_0^t + \frac{1}{6}Sin3t\left|\xi - \frac{Sin6\xi}{6}\right|_0^t$$

$$h(t) = \frac{1}{36}Cos3t(1 - Cos6t) + \frac{1}{6}Sin3t\left(t - \frac{Sin6t}{6}\right)$$

$$h(t)=\frac{1}{36}Cos3t-\frac{1}{36}Cos3tCos6t+\frac{1}{6}tSin3t-\frac{1}{36}Sin3tSin6t$$

$$h(t)=-\frac{1}{36}[Cos3tCos6t+Sin3tSin6t]+\frac{1}{6}tSin3t+\frac{1}{36}Cos3t$$

$$h(t)=-\frac{1}{36}[Cos(6t-3t)]+\frac{1}{6}tSin3t+\frac{1}{36}Cos3t$$

$$h(t)=-\frac{1}{36}Cos3t+\frac{1}{6}tSin3t+\frac{1}{36}Cos3t=-\frac{1}{6}tSin3t=\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$$

PROBLEM: (UoS; Past Paper)

Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$

Solution: **Here we have** $H(s)=F(s)G(s)=\frac{1}{s^2+6s+13}=\frac{1}{(s+3+2i)(s+3-2i)}$

$$F(s)=\frac{1}{s+3+2i}\Rightarrow\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+3+2i}\right\}\Rightarrow f(t)=e^{-(3+2i)t}$$

$$G(s)=\frac{1}{s+3-2i}\Rightarrow\mathcal{L}^{-1}\{G(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+3-2i}\right\}\Rightarrow g(t)=e^{-(3-2i)t}$$

Now using Convolution theorem

$$h(t)=f*g=\int_0^t f(\xi)g(t-\xi)d\xi=\int_0^t e^{-(3+2i)\xi}\,e^{-(3-2i)(t-\xi)}d\xi$$

$$h(t)=e^{-(3-2i)t}\int_0^t e^{-(3+2i)\xi}\,e^{(3-2i)\xi}d\xi$$

$$h(t)=e^{-(3-2i)t}\int_0^t e^{-4i\xi}\,d\xi$$

$$h(t)=e^{-(3-2i)t}\left|\frac{e^{-4i\xi}}{-4i}\right|_0^t=\frac{e^{-(3-2i)t}}{-4i}\left|e^{-4it}-e^0\right|=\frac{e^{-(3-2i)t}}{-4i}\left|e^{-4it}-1\right|$$

$$h(t)=\frac{e^{-3t}}{2}\left[\frac{e^{-2it}-e^{2it}}{-2i}\right]=\frac{e^{-3t}}{2}\left[\frac{e^{2it}-e^{-2it}}{2i}\right]$$

$$h(t)=\frac{e^{-3t}}{2}Sin2t$$

PROBLEM:

Use covolution theorem to calculate laplace transform of

$$f(t) = \int_0^t (t - \beta)^3 e^{\beta} Sin\beta d\beta$$

Solution:

Let $f(t) = g * h = \int_0^t (t - \beta)^3 e^{\beta} Sin\beta d\beta$ **…………..(i)**

Comparing with $g * h = \int_0^t g(t - \beta) h(\beta) d\beta$ **…………..(ii) we get**

$g(t - \beta) = (t - \beta)^3 \Rightarrow g(t) = t^3$ **and** $h(\beta) = e^{\beta} Sin\beta \Rightarrow h(t) = e^t Sint$

Now $\mathcal{L}\{f(t)\} = \mathcal{L}\{g * h\} = F(s)G(s) = \mathcal{L}\{g(t)\}.\mathcal{L}\{h(t)\} = \mathcal{L}\{t^3\}.\mathcal{L}\{e^t Sint\}$

$$\mathcal{L}\{f(t)\} = \frac{3!}{s^{3+1}} \cdot \frac{1}{(s-1)^2 + 1^2} = \frac{6}{s^4(s^2 - 2s + 1)}$$

THE GAUSSIAN INTEGRAL (UoS; 2015 – I)

Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ **or** $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Solution: consider $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ **and** $I = \int_{-\infty}^{\infty} e^{-y^2} dy$

then multiplying both $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} dxdy$

Now using polar coordinates

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} rdrd\theta = \int_0^{2\pi} d\theta \left(-\frac{1}{2}\right) \int_0^{\infty} e^{-r^2} (-2r)dr = \pi \Rightarrow I = \sqrt{\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \Rightarrow 2\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

LAPLACE TRANSFORM OF STEP FUNCTION:

The <u>Heaviside unit step</u> function is defined by

$$H\,(t\,-\,a)\;=\begin{cases}0 & t < a\\ 1 & t \geq a\end{cases}\quad \text{where } a \geq 0$$

Now, we will find its Laplace transform.

$$\mathcal{L}\{H\,(t\,-\,a)\} = \int_0^\infty e^{-st}\,H\,(t\,-\,a)\,dt$$

$$\mathcal{L}\{H\,(t\,-\,a)\} = \int_0^a e^{-st}\,H\,(t\,-\,a)\,dt + \int_a^\infty e^{-st}\,H\,(t\,-\,a)\,dt$$

$$\mathcal{L}\{H\,(t\,-\,a)\} = \int_0^a e^{-st}\,.\,0\,dt + \int_a^\infty e^{-st}\,.\,1\,dt$$

$$\mathcal{L}\{H\,(t\,-\,a)\} = \int_a^\infty e^{-st}\,dt = \left|\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s}\quad ;\; s > 0$$

THEOREM: (UoS; 2014 , 2015)

If $f(t)$ is a function of exponential order 'c' then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)\;\; ; s > a$$

PROOF: Consider $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Differentiating w.r.to 's'

$$\Rightarrow \frac{d}{ds} F(s) = (-1)\int_0^\infty e^{-st}\; tf(t)dt = (-1)\mathcal{L}\{tf(t)\} \Rightarrow (-1)\frac{d^1}{ds^1}F(s) = \mathcal{L}\{t^1 f(t)\}$$

Again differentiating w.r.to 's'

$$\Rightarrow \frac{d^2}{ds^2} F(s) = (-1)(-1)\int_0^\infty e^{-st}\,(-t)\,tf(t)dt = (-1)^2\int_0^\infty e^{-st}\; t^2 f(t)dt = (-1)^2\mathcal{L}\{t^2 f(t)\}$$

$$\Rightarrow (-1)^2\frac{d^2}{ds^2}F(s) = (-1)^2\mathcal{L}\{t^2 f(t)\}$$

Continuing this process, we get the required

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)\;\; ; s > a \qquad \therefore (-1)^n = (-1)^{-n}$$

REMARK: $\mathcal{L}\{t^{-n} f(t)\} = \frac{d^n}{ds^n} F(s)$

LAPLACE TRANSFORMATION OF LOGRITHMIC FUNCTION:

(UoS; 2015 – I)

Show that $\mathcal{L}\{lnt\} = \frac{1}{s}(\Gamma'(1) - lns)$

SOLUTION: by using definition

$\mathcal{L}\{lnt\} = \int_0^\infty e^{-st}\ lntdt = \int_0^\infty e^{-u}\ ln\left(\frac{u}{s}\right)\frac{du}{s}$ **by putting st = u**

$\mathcal{L}\{lnt\} = \frac{1}{s}\int_0^\infty e^{-u}\ lnudu - \frac{1}{s}\int_0^\infty e^{-u}\ lnsdu = \frac{1}{s}(I) - \frac{1}{s}lns\int_0^\infty e^{-u}\ du$

$\mathcal{L}\{lnt\} = \frac{1}{s}(I) - \frac{1}{s}lns(1) = \frac{1}{s}(I) - \frac{1}{s}lns$ **………….(i)**

Now consider $I = \int_0^\infty e^{-u}\ lnudu$

Since $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du \Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-u}u^{\alpha}\ du \Rightarrow \Gamma'(1) = \int_0^\infty e^{-u}u^{\alpha}\, lnu\, du$

Put $\alpha = 0 \Rightarrow \Gamma'(1) = \int_0^\infty e^{-u}\, lnu\, du = I$

Thus $\mathcal{L}\{lnt\} = \frac{1}{s}(\Gamma'(1) - lns)$

where $\Gamma'(1) \approx 0.57721$ **is called Euler's constant.**

THE GAMMA FUNCTION:

Gamma function can be defined as follows $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du$

USEFUL RESULTS:

- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

Proof: since $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du \Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-u}u^{\alpha}\ du$

$\Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-u}u^{\alpha}\ du = \left|u^{\alpha}\frac{e^{-u}}{-1}\right|_0^\infty - \int_0^\infty \left|\frac{e^{-u}}{-1}\right| \alpha u^{\alpha-1}\ du$

$\Rightarrow \Gamma(\alpha+1) = 0 + \alpha\int_0^\infty e^{-u}u^{\alpha-1}\ du = \alpha\Gamma(\alpha)$

- $\Gamma(1) = 1$ **we can prove it using** $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du$ **with** $\alpha = 1$

- $\Gamma(\alpha+1)=\alpha!$

Proof: since $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$

$put\ \ \alpha=1 \Rightarrow \Gamma(2)=1.\Gamma(1)=1.1=1!$

$put\ \ \alpha=2 \Rightarrow \Gamma(3)=2.\Gamma(2)=2.1=2!$

$put\ \ \alpha=3 \Rightarrow \Gamma(4)=3.\Gamma(3)=3.2.1=3!$

: :

Then $\Gamma(\alpha)=\alpha-1! \Rightarrow \Gamma(\alpha+1)=\alpha!$

SECOND SHIFTING (TRANSLATION) THEOREM:

If $F(s)$ **and** $G(s)$ **are the Laplace transforms of** $f(t)$ **and** $g(t)$ **respectively, then**

$$\mathcal{L}[H\,(t-a)\,f\,(t-a)] = e^{-as}\,F(s) = e^{-as}\,\mathcal{L}\{f\,(t)\}$$

Or $\mathcal{L}^{-1}\{e^{-as}\,F(s)\}=H\,(t-a)\,f\,(t-a)$

Proof: By definition

$$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_0^\infty e^{-st}\,H\,(t-a)\,f\,(t-a)\,dt$$

$$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_0^a e^{-st}\,H\,(t-a)f\,(t-a)\,dt+\int_a^\infty e^{-st}\,H\,(t-a)f\,(t-a)\,dt$$

$$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_a^\infty e^{-st}\,f\,(t-a)\,dt$$

Introducing the new variable $\xi=t-a$**, we obtain**

$$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_0^\infty e^{-(\xi+a)s}\,f\,(\xi)\,d\xi = e^{-as}\int_0^\infty e^{-\xi s}\,f\,(\xi)\,d\xi$$

$$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = e^{-as}\mathcal{L}\{\,f\,(t\,)\} = e^{-as}F(s)$$

REMARK:

1st Shifting theorem enables us to calculate Laplace transform of the function of the form $e^{kt}\,f(t)$ **where the 2nd Shifting theorem in similar way enables us to calculate inverse Laplace transform of the function of the form** $e^{-as}F(s)$

COROLLARY: Prove that $\mathcal{L}\{p(t)f(t)\} = P(-D)F(s)$ where $p(t)$ is a polynomial in 't'.

SOLUTION:

Since $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots\ldots\ldots\ldots + a_n t^n = \sum_{i=1}^{n} a_i t^i$ **Then**

$$\mathcal{L}\{p(t)f(t)\} = \mathcal{L}\{\sum_{i=1}^{n} a_i t^i f(t)\} = \sum_{i=1}^{n} a_i \mathcal{L}\{t^i f(t)\} = \sum_{i=1}^{n} a_i (-1)^i \frac{d^i}{ds^i} F(s)$$

$$\mathcal{L}\{p(t)f(t)\} = \sum_{i=1}^{n} a_i (-1)^i D^i F(s) = \sum_{i=1}^{n} a_i (-D)^i F(s) = P(-D)F(s)$$

LAPLACE TRANSFORMATION OF BESSEL'S FUNCTION

EXAMPLE: (UoS; 2014,2019 – I)

Find Laplace Tranformation of $J_0(t) = \frac{1}{\pi}\int_0^{\pi} Cos(tSin\theta)d\theta$ **also find** $\mathcal{L}\{J_0(at)\}$

Solution: By definition

$$\mathcal{L}\{J_0(t)\} = \int_0^{\infty} e^{-st} J_0(t)\, dt = \int_0^{\infty} e^{-st} \left[\frac{1}{\pi}\int_0^{\pi} Cos(tSin\theta)d\theta\right] dt$$

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\pi}\int_0^{\pi}\left[\int_0^{\infty} e^{-st} Cos(tSin\theta)\, dt\right] d\theta = \frac{1}{\pi}\int_0^{\pi} I\, d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now $I = \int_0^{\infty} e^{-st} Cos(Sin\theta)t\, dt = \mathcal{L}\{Cos(Sin\theta)t\} = \frac{s}{s^2+(Sin\theta)^2}$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{1}{\pi}\int_0^{\pi} \frac{s}{s^2+Sin^2\theta} d\theta = \frac{2}{\pi}\int_0^{\pi/2} \frac{s}{s^2+Sin^2\theta} d\theta \qquad \because \int_0^{a} f(x)dx = 2\int_0^{a/2} f(x)dx$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2} \frac{s}{s^2+Sin^2\left(\frac{\pi}{2}-\theta\right)} d\theta \quad \because \int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2} \frac{s}{s^2+Cos^2\theta} d\theta = \frac{2}{\pi}\int_0^{\pi/2} \frac{sSec^2\theta}{Sec^2\theta(s^2+Cos^2\theta)} d\theta = \frac{2}{\pi}\int_0^{\pi/2} \frac{sSec^2\theta}{s^2Sec^2\theta+1} d\theta$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2} \frac{sSec^2\theta}{s^2(1+Tan^2\theta)+1} d\theta = \frac{2}{\pi}\int_0^{\pi/2} \frac{sSec^2\theta}{(s^2+1)+(s^2Tan^2\theta)} d\theta$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2} \frac{sSec^2\theta}{s^2\left[\frac{s^2+1}{s^2}+Tan^2\theta\right]} d\theta = \frac{2}{\pi s}\int_0^{\pi/2} \frac{Sec^2\theta}{\left[\frac{s^2+1}{s^2}+Tan^2\theta\right]} d\theta$$

$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi s}\int_0^{\infty} \frac{dx}{a^2+x^2}$ **by putting** $x = Tan\theta \Rightarrow dx = Sec^2\theta d\theta$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi s}\left|\frac{1}{a}Tan^{-1}\frac{x}{a}\right|_0^{\infty} = \frac{2}{\pi s}\left(\frac{\pi}{2a}\right) = \frac{1}{as} = \frac{1}{s}\cdot\frac{s}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}} \qquad \because a^2 = \frac{s^2+1}{s^2}$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

- **To find $\mathcal{L}\{J_0(at)\}$ see last portion of next example.**

EXAMPLE: Given the Bessel's functions of the first kind and positive integral order satisfy the recurrence relations $J_1 = -J'_0, \quad J_{n+1} = J_{n-1} - 2J'_n \ ; n \geq 1$

with $J_0(0) = 1\ , J_n(0) = 0\ ; n > 0$ **then show that** $\mathcal{L}\{J_n(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}$

also find $\mathcal{L}\{J_n(at)\} \quad ; a > 0$

Solution: We will prove the result by mathematical induction.

Using first recurrence relation:

$$\mathcal{L}\{J_1(t)\} = \mathcal{L}\{-J'_0(t)\} = -\mathcal{L}\{J'_0(t)\} = -[s\mathcal{L}\{J_0(t)\} - J_0(0)] = -\frac{1}{\sqrt{s^2+1}} + 1$$

$$\mathcal{L}\{J_1(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^1}{\sqrt{s^2+1}} \qquad \text{result is true for n = 0}$$

For n = 1:

$$J_2 = J_0 - 2J'_1 \Rightarrow \mathcal{L}\{J_2(t)\} = \mathcal{L}\{J_0(t)\} - 2\mathcal{L}\{J'_1(t)\} = \frac{1}{\sqrt{s^2+1}} - 2[s\mathcal{L}\{J_1(t)\} - J_1(0)]$$

$$\Rightarrow \mathcal{L}\{J_2(t)\} = \frac{1}{\sqrt{s^2+1}} - 2\left[s.\frac{\left(\sqrt{s^2+1}-s\right)^1}{\sqrt{s^2+1}} - 0\right] = \frac{1}{\sqrt{s^2+1}} - \frac{2s\left(\sqrt{s^2+1}-s\right)}{\sqrt{s^2+1}} = \frac{\left(\sqrt{s^2+1}-s\right)^2}{\sqrt{s^2+1}}$$

$$\Rightarrow \mathcal{L}\{J_2(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^2}{\sqrt{s^2+1}} \quad \text{result is true for n = 1}$$

Suppose that result is true for n = k. $\qquad \Rightarrow \mathcal{L}\{J_k(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^k}{\sqrt{s^2+1}}$

Now we will check the result For n = k+1:

$$J_{k+1} = J_{k-1} - 2J'_k \Rightarrow \mathcal{L}\{J_{k+1}\} = \mathcal{L}\{J_{k-1}\} - 2\mathcal{L}\{J'_k\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}} - 2[s\mathcal{L}\{J_k(t)\} - J_k(0)]$$

$$\Rightarrow \mathcal{L}\{J_{k+1}\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}} - 2\left[s.\frac{\left(\sqrt{s^2+1}-s\right)^k}{\sqrt{s^2+1}} - 0\right] = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}} - \frac{2s\left(\sqrt{s^2+1}-s\right)^k}{\sqrt{s^2+1}}$$

$$\Rightarrow \mathcal{L}\{J_{k+1}\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}}\left[1 - 2s\left(\sqrt{s^2+1} - s\right)\right] = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}}\left(\sqrt{s^2+1} - s\right)^2$$

$$\Rightarrow \mathcal{L}\{J_{k+1}\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k+1}}{\sqrt{s^2+1}} \quad \text{result is true for n = k+1}$$

So induction complete and result is proved.i.e. $\mathcal{L}\{J_n(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}$

Now to find $\mathcal{L}\{J_0(at)\} \quad ; a > 0$ **we will use rule of scale. i.e** $\mathcal{L}\,[f\,(at)] = \frac{1}{a}\,F\left(\frac{s}{a}\right)$

Then $\mathcal{L}\,[J_n(at)] = \frac{1}{a}\,F_n\left(\frac{s}{a}\right) = \frac{1}{a}.\frac{\left(\sqrt{\left(\frac{s}{a}\right)^2+1}-\left(\frac{s}{a}\right)\right)^n}{\sqrt{\left(\frac{s}{a}\right)^2+1}} = \frac{\left(\sqrt{s^2+a^2}-s\right)^n}{a^n\sqrt{s^2+a^2}}$

Then for n = 0 $\mathcal{L}\,[J_0(at)] = \frac{\left(\sqrt{s^2+a^2}-s\right)^0}{a^0\sqrt{s^2+a^2}} = \frac{1}{\sqrt{s^2+a^2}}$

EXAMPLE: (UoS; 2018 – I)

Show that $\mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \frac{1}{2}log_e\left\{1+\frac{a^2}{s^2}\right\}$

Solution: We will use the result $\mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_s^\infty F(s')ds'$ **…………(i)**

provided $\lim_{x\to 0}\left\{\frac{f(x)}{x}\right\}$ **exists**

now $\lim_{x\to 0}\left\{\frac{f(x)}{x}\right\} = \lim_{x\to 0}\left\{\frac{1-Cosax}{ax}\right\} = \lim_{x\to 0}\left\{\frac{aSinax}{1}\right\} = 0$

$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{1 - Cosax\} = \mathcal{L}\{1\} - \mathcal{L}\{Cosax\} = \frac{1}{s} - \frac{s}{s^2+a^2} \quad ; s > a$

Hence

$(i) \Rightarrow \mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_s^\infty F(s')ds' \; (i) \Rightarrow \mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \int_s^\infty \left(\frac{1}{s\prime} - \frac{s\prime}{s\prime^2+a^2}\right)ds'$

$\Rightarrow \mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \left|lns' - \frac{1}{2}ln(s'^2+a^2)\right|_s^\infty = \left|ln\frac{s'}{\sqrt{s\prime^2+a^2}}\right|_s^\infty = 0 - ln\sqrt{\frac{s^2}{s^2+a^2}} = ln\sqrt{\frac{s^2+a^2}{s^2}}$

Thus $\mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \frac{1}{2}ln\left\{1+\frac{a^2}{s^2}\right\} = \frac{1}{2}log_e\left\{1+\frac{a^2}{s^2}\right\}$

EXAMPLE: Find $\mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\}$ **and deduce** $\mathcal{L}\left\{\frac{Sin^2t}{t}\right\} = \frac{1}{2}ln\left(\frac{s^2+4}{s^2}\right) \quad ; s > 1$

Solution: We will use the result $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$ **…………(i)**

provided $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ **exists**

now $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\} = \lim_{t\to 0}\left\{\frac{e^{at}-Cosbt}{t}\right\} = \lim_{t\to 0}\left\{\frac{ae^{at}+bSinbt}{1}\right\} = a$

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at} - Cosbt\} = \mathcal{L}\{e^{at}\} - \mathcal{L}\{Cosbt\} = \frac{1}{s-a} - \frac{s}{s^2+b^2}$

$F(u) = \frac{1}{u-a} - \frac{u}{u^2+b^2}$

Hence $(i) \Rightarrow \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du \Rightarrow \mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = \int_s^\infty \left(\frac{1}{u-a} - \frac{u}{u^2+b^2}\right)du$

$\Rightarrow \mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = \left|ln(u-a) - \frac{1}{2}ln(u^2+b^2)\right|_s^\infty = \left|ln\frac{u-a}{\sqrt{u^2+b^2}}\right|_s^\infty = \left|ln\frac{1-\frac{a}{u}}{u\sqrt{1+\left(\frac{b}{u}\right)^2}}\right|_s^\infty$

$\Rightarrow \mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = 0 - ln\frac{s-a}{\sqrt{s^2+b^2}} = ln\frac{\sqrt{s^2+b^2}}{s-a}$

Thus $\mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = ln\frac{\sqrt{s^2+b^2}}{s-a}$

Now putting a = 0, b = 2 we get $\mathcal{L}\left\{\frac{e^{0}-Cos2t}{t}\right\} = ln\frac{\sqrt{s^2+2^2}}{s-0} \Rightarrow \mathcal{L}\left\{\frac{1-Cos2t}{t}\right\} = ln\frac{\sqrt{s^2+4}}{s}$

Hence $\mathcal{L}\left\{\frac{Sin^2 t}{t}\right\} = \frac{1}{2}ln\left(\frac{s^2+4}{s^2}\right)\quad ; s > 1$

NULL FUNCTION: A function $N(x)$ is called Null Function if $\int_0^\infty N(x)dx = 0$

HEAVISIDE EXPANSION THEOREMS

THEOREM – I :

If $M(s)$ and $N(s)$ are polynomials of degree 'm' and 'n' respectively with m < n and $N(s)$ has 'n' distinct zeros a_i ; $i = 1,2,3\ldots\ldots$ none of which is zero of $M(s)$ then

$$\mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n}\frac{M(a_i)}{N'(a_i)}e^{a_i t}$$

Proof: Given $M(s)$ and $N(s)$ are polynomials of degree 'm' and 'n' respectively

Let $N(s) = a_0 + a_1 s + a_2 s^2 + \ldots\ldots\ldots + a_n s^n = (s-a_1)(s-a_2)\ldots\ldots\ldots(s-a_n)$

Then consider $\frac{M(s)}{N(s)} = \frac{c_1}{s-a_1} + \frac{c_2}{s-a_2} + \cdots\ldots\ldots + \frac{c_n}{s-a_n} = \sum_{i=1}^{n}\frac{c_i}{s-a_i}$ ....(i)

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \mathcal{L}^{-1}\left\{\sum_{i=1}^{n}\frac{c_i}{s-a_i}\right\} = \sum_{i=1}^{n} c_i\,\mathcal{L}^{-1}\left\{\frac{1}{s-a_i}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n} c_i\, e^{a_i t}$ ....(ii)

$(i) \Rightarrow c_i = \lim_{s\to a_i}\left[(s-a_i)\frac{M(s)}{N(s)}\right] =$

$\lim_{s\to a_i}[M(s)]\,.\lim_{s\to a_i}\left[\frac{(s-a_i)}{N(s)}\right] = M(a_i).\lim_{s\to a_i}\left[\frac{1}{N'(s)}\right]$

$(i) \Rightarrow c_i = \frac{M(a_i)}{N'(a_i)}$

$(ii) \Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n}\frac{M(a_i)}{N'(a_i)}e^{a_i t}$

THEOREM – II :

If $M(s)$ and $N(s)$ are polynomials of degree 'm' and 'n' respectively with m < n and if $N(s)$ has a repeated root a_1 of multiplicity 'r' while othere roots $\sum_{i=2}^{n} a_i$ are not repeated then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=2}^{n}\frac{M(a_i)}{N'(a_i)}e^{a_i t} + \sum_{j=1}^{r}\frac{1}{(j\text{-}1!)}\left\{\frac{d^{j-1}}{ds^{j-1}}(s-a_i)^j\frac{M(s)}{N(s)}\right\}e^{a_1 t}\Bigg|_{s=a_1}$$

Proof: Since $N(s)$ has a repeated root a_1 of multiplicity 'r' while othere roots $a_2, a_3, \dots. a_n$ are not repeated it means

$$N(s) = (s-a_1)^r(s-a_2)\dots\dots..(s-a_n)$$

$$\Rightarrow \frac{M(s)}{N(s)} = \frac{M(s)}{(s-a_1)^r(s-a_2)\dots\dots(s-a_n)}$$

Then in terms of Partial fraction we will be as follows

$$\frac{M(s)}{N(s)} = \frac{d_r}{(s-a_1)^r} + \frac{d_{r-1}}{(s-a_1)^{r-1}} + \frac{d_{r-2}}{(s-a_1)^{r-2}} + \cdots\dots\dots\dots + \frac{d_{r-l}}{(s-a_1)^{r-l}} + \cdots\dots. + \frac{1}{(s-a_1)^1} +$$

$$\frac{d_1}{(s-a_1)^1} + \frac{c_2}{(s-a_2)} + \frac{c_3}{(s-a_3)} + \cdots\dots\dots\dots + \frac{c_n}{(s-a_n)} \qquad \dots\dots\dots\dots\dots\dots(A)$$

Multiplying $(s-a_1)^r$ on both sides

$$(s-a_1)^r\frac{M(s)}{N(s)} = d_r + d_{r-1}(s-a_1) + \cdots\dots\dots\dots. + d_1(s-a_1)^{r-1} + \sum_{i=2}^{\infty} c_i\frac{(s-a_1)^r}{(s-a_i)} \quad \dots\dots\dots..(B)$$

Now taking $\lim\limits_{s\to a_1}$ on both sides we get

$$d_r = \lim_{s\to a_1}\left[(s-a_1)^r\frac{M(s)}{N(s)}\right]$$

$$d_{r-1} = \lim_{s\to a_1}\frac{d}{ds}\left[(s-a_1)^r\frac{M(s)}{N(s)}\right] \qquad \textbf{diff.w.to 's'}$$

$$d_{r-2} = \frac{1}{2!}\lim_{s\to a_1}\frac{d^2}{ds^2}\left[(s-a_1)^r\frac{M(s)}{N(s)}\right] \qquad \textbf{again diff.w.to 's'}$$

$$d_{r-l} = \frac{1}{l!}\lim_{s\to a_1}\left(\frac{d}{ds}\right)^l\left[(s-a_1)^r\frac{M(s)}{N(s)}\right] \qquad \textbf{again diff.w.to 's' } l - time$$

Now by second translation theorem

$\mathcal{L}^{-1}\{e^{-as}\, F(s)\} = H\,(t - a)\, f\,(t - a)$

or $\mathcal{L}[H\,(t - a)\, f\,(t - a)] = e^{-as}\, F(s) = e^{-as}\, \mathcal{L}\{f\,(t)\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-a_1)^r}\right\} = e^{a_1 t}\mathcal{L}^{-1}\left\{\frac{1}{s^r}\right\} = e^{a_1 t}\frac{t^{r-1}}{(r-1)!}$

Now by 'A' we have $\frac{M(s)}{N(s)} = \sum_{l=1}^{r}\frac{d_l}{(s-a_1)^l} + \sum_{i=2}^{n}\frac{c_i}{(s-a_i)}$

Then taking laplace inverse on both sides

$\mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \mathcal{L}^{-1}\left\{\sum_{l=1}^{r}\frac{d_l}{(s-a_1)^l}\right\} + \mathcal{L}^{-1}\left\{\sum_{i=2}^{n}\frac{c_i}{(s-a_i)}\right\} = \sum_{l=1}^{r}\mathcal{L}^{-1}\left\{\frac{d_l}{(s-a_1)^l}\right\} + \sum_{i=2}^{n}\mathcal{L}^{-1}\left\{\frac{c_i}{(s-a_i)}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{l=1}^{r}\frac{1}{(l-1)!}\lim_{s\to a_1}\left(\frac{d}{ds}\right)^{l-1}\left[(s-a_1)^l\frac{M(s)}{N(s)}\right]e^{a_1 t} + \sum_{i=2}^{n}\lim_{s\to a_i}(s-a_i)\frac{M(s)}{N(s)}$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=2}^{n}\frac{M(a_i)}{N'(a_i)}e^{a_i t} + \sum_{j=1}^{r}\frac{1}{(j-1)!}\left\{\frac{d^{j-1}}{ds^{j-1}}(s-a_i)^j\frac{M(s)}{N(s)}\right\}e^{a_1 t}\Big|_{s=a_1}$

EXAMPLE: (UoS; 2018 – I)

Using Heaviside Expansion theorem evaluate $\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2 s^3}\right\}$

Solution: **Given that** $F(s) = \frac{s+2}{(s-1)^2 s^3}$ **has a pole at s = 1 of order'2' and at s = 0 of order '3'**

Then in terms of Partial fraction we will be as follows

$F(s) = \frac{d_1}{(s-1)^2} + \frac{d_2}{(s-1)} + \frac{c_1}{s^3} + \frac{c_2}{s^2} + \frac{c_3}{s}$

Now using Heaviside formula

$d_1 = \lim_{s\to 1}[(s-1)^2 F(s)] = \lim_{s\to 1}\left[(s-1)^2\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 1}\left[\frac{s+2}{s^3}\right] = 3$

$d_2 = \lim_{s\to 1}\frac{d}{ds}[(s-1)^2 F(s)] = \lim_{s\to 1}\frac{d}{ds}\left[(s-1)^2\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 1}\frac{d}{ds}\left[\frac{s+2}{s^3}\right] = -8$

$c_1 = \lim_{s\to 0}[s^3 F(s)] = \lim_{s\to 0}\left[s^3\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 0}\left[\frac{s+2}{(s-1)^2}\right] = 2$

$c_2 = \lim_{s\to 0}\frac{d}{ds}[s^3 F(s)] = \lim_{s\to 0}\frac{d}{ds}\left[s^3\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 0}\frac{d}{ds}\left[\frac{s+2}{(s-1)^2}\right] = 5$

$c_3 = \frac{1}{2!}\lim_{s\to 0}\frac{d^2}{ds^2}[s^3 F(s)] = \frac{1}{2}\lim_{s\to 0}\frac{d^2}{ds^2}\left[s^3\frac{s+2}{(s-1)^2 s^3}\right] = \frac{1}{2}\lim_{s\to 0}\frac{d^2}{ds^2}\left[\frac{s+2}{(s-1)^2}\right] = 8$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = d_1\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + d_2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + c_1\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + c_3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = 3te^t - 8e^t + 2\frac{t^2}{2} + 5t + 8 = (3t-8)e^t + (t^2 + 5t + 8)$

EXAMPLE: Using Heaviside Expansion theorem evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+2\alpha s+b^2)}\right\}$

Solution: Given that $F(s)=\frac{1}{s^2(s^2+2\alpha s+b^2)}=\frac{1}{s^2(s-s_1)(s-s_2)}$ **has simple poles at** $s=s_1, s=s_2$ **and a pole of order '2' at s = 0**

Then in terms of Partial fraction we will be as follows

$$F(s)=\frac{1}{s^2(s^2+2\alpha s+b^2)}=\frac{1}{s^2(s-s_1)(s-s_2)}=\frac{d_1}{s^2}+\frac{d_2}{s}+\frac{c_1}{(s-s_1)}+\frac{c_2}{(s-s_2)}$$

Where we take $s_1=-\alpha-i\beta$ **,** $s_2=-\alpha+i\beta$ **then** $s_1s_2=\alpha^2+\beta^2=b^2$

Now using Heaviside formula

$$d_1=\lim_{s\to 0}[s^2F(s)]=\lim_{s\to 0}\left[s^2\frac{1}{s^2(s-s_1)(s-s_2)}\right]=\lim_{s\to 0}\left[\frac{1}{(s-s_1)(s-s_2)}\right]=\frac{1}{b^2}$$

$$d_2=\lim_{s\to 0}\frac{d}{ds}[s^2F(s)]=\lim_{s\to 0}\frac{d}{ds}\left[s^2\frac{1}{s^2(s-s_1)(s-s_2)}\right]=\lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{(s-s_1)(s-s_2)}\right]$$

$$d_2=\lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{s^2+2\alpha s+b^2}\right]=\lim_{s\to 0}\frac{-(2s+2\alpha)}{(s^2+2\alpha s+b^2)^2}=\frac{-2\alpha}{b^4}$$

$$c_1=\lim_{s\to s_1}[(s-s_1)F(s)]=\lim_{s\to s_1}\left[(s-s_1)\frac{1}{s^2(s-s_1)(s-s_2)}\right]=\lim_{s\to s_1}\left[\frac{1}{s^2(s-s_2)}\right]$$

$$c_1=\frac{1}{{s_1}^2(s_1-s_2)}$$

$$c_2=\lim_{s\to s_2}[(s-s_2)F(s)]=\lim_{s\to s_2}\left[(s-s_2)\frac{1}{s^2(s-s_1)(s-s_2)}\right]=\lim_{s\to s_2}\left[\frac{1}{s^2(s-s_1)}\right]$$

$$c_2=\frac{1}{{s_2}^2(s_2-s_1)}$$

Now as $s_1=-\alpha-i\beta=\sqrt{\alpha^2+\beta^2}e^{-i\theta}\Rightarrow {s_1}^2=(\alpha^2+\beta^2)e^{-2i\theta}=b^2e^{-2i\theta}$

$s_2=-\alpha+i\beta=\sqrt{\alpha^2+\beta^2}e^{i\theta}\Rightarrow {s_2}^2=(\alpha^2+\beta^2)e^{2i\theta}=b^2e^{2i\theta}$ **then**

$$s_1-s_2=-2i\beta$$

Then $c_1=\frac{1}{{s_1}^2(s_1-s_2)}=-\frac{e^{2i\theta}}{2ib^2\beta}$ **and** $c_2=\frac{1}{{s_2}^2(s_2-s_1)}=\frac{e^{-2i\theta}}{2ib^2\beta}$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\}=f(t)=d_1\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+d_2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+c_1\mathcal{L}^{-1}\left\{\frac{1}{(s-s_1)}\right\}+c_2\mathcal{L}^{-1}\left\{\frac{1}{(s-s_2)}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\}=f(t)=\frac{t}{b^2}-\frac{2\alpha}{b^4}+c_1e^{s_1t}+c_2e^{s_2t}=\frac{t}{b^2}-\frac{2\alpha}{b^4}-\frac{e^{2i\theta}}{2ib^2\beta}.e^{s_1t}+\frac{e^{-2i\theta}}{2ib^2\beta}e^{s_2t}$$

EXAMPLE: (UoS; 2015 – II)

Find the general solution of the differential equation evaluate

$y''(t) + k^2 y(t) = f(t)$

Solution: Given that $y''(t) + k^2 y(t) = f(t)$

$\Rightarrow \mathcal{L}\{y''(t)\} + k^2 \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$

$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + k^2 Y(s) = F(s) \Rightarrow s^2 Y(s) + k^2 Y(s) = F(s) + sy(0) + y'(0)$

$\Rightarrow Y(s) = \frac{c_1 + c_2 s + F(s)}{s^2 + k^2}$ **where we use** $y'(0) = c_1, y(0) = c_2$

Now $\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{c_1}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{c_2 s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{c_1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} + c_2 \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{c_1}{k} Sinkt + c_2 Coskt + \frac{1}{k} Sinkt * f(t)$

$\Rightarrow y(t) = \frac{c_1}{k} Sinkt + c_2 Coskt + \frac{1}{k}\int_0^t e^{-st}\, Sink(t - \xi) f(\xi)\, d\xi$

EXAMPLE: (UoS; 2017 – I , II)

Slove the IVP $y''(t) + ty'(t) - y(t) = 0$ **with** $y(0) = 0, y'(0) = 1$

Solution: Given that $y''(t) + ty'(t) - y(t) = 0$

$\Rightarrow \mathcal{L}\{y''(t)\} + \mathcal{L}\{ty'(t)\} - \mathcal{L}\{y(t)\} = 0$

$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + \left(-\frac{d}{ds}\right)\mathcal{L}\{y'(t)\} - Y(s) = 0$

$\Rightarrow s^2 Y(s) - 1 - \left(\frac{d}{ds}\right)\{sY(s) - y(0)\} - Y(s) = 0$

where we use $y(0) = 0, y'(0) = 1$

$\Rightarrow s^2 Y(s) - 1 - sY'(s) - Y(s) - Y(s) = 0$ **where we use** $y(0) = 0, y'(0) = 1$

$\Rightarrow Y'(s) + \frac{2-s^2}{s} Y(s) = -\frac{1}{s}$ **this will have in** $I.F = s^2 e^{-\frac{s^2}{2}}$

Thus $\Rightarrow Y(s) = \frac{1}{s^2} + ce^{\frac{s^2}{2}} \Rightarrow Y(s) = \frac{1}{s^2}$ **when** $s \to \infty$ $then\ c = 0$

Now $\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \Rightarrow y(t) = t$

EXAMPLE:

Slove the IVP $u'' - au = f(t)$ **with** $u(0) = u_0, u'(0) = u_1$

Solution: Given that $u'' - au = f(t)$

$\Rightarrow \mathcal{L}\{u''\} - a\mathcal{L}\{u\} = \mathcal{L}\{f(t)\}$

$\Rightarrow s^2U(s) - su(0) - u'(0) - aU(s) = F(s)$

$\Rightarrow s^2U(s) - su_0 - u_1 - aU(s) = F(s)$ **where we use** $u(0) = u_0, u'(0) = u_1$

$\Rightarrow (s^2 - a)U(s) = F(s) + su_0 + u_1$

$\Rightarrow U(s) = \frac{F(s)}{s^2-a} + u_0.\frac{s}{s^2-a} + u_1.\frac{1}{s^2-a}$

Now $\Rightarrow \mathcal{L}^{-1}\{U(s)\} = u(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2-a}\right\} + u_0\mathcal{L}^{-1}\left\{\frac{s}{s^2-a}\right\} + u_1\mathcal{L}^{-1}\left\{\frac{1}{s^2-a}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = u(t) = \frac{1}{\sqrt{a}}\mathcal{L}^{-1}\left\{F(s).\frac{\sqrt{a}}{s^2-(\sqrt{a})^2}\right\} + u_0\mathcal{L}^{-1}\left\{\frac{s}{s^2-(\sqrt{a})^2}\right\} + \frac{u_1}{\sqrt{a}}\mathcal{L}^{-1}\left\{\frac{\sqrt{a}}{s^2-(\sqrt{a})^2}\right\}$

$\Rightarrow u(t) = \frac{1}{\sqrt{a}}\mathcal{L}^{-1}\{f(t) * Sinh\sqrt{a}t\} + u_0Cosh\sqrt{a}t + \frac{u_1}{\sqrt{a}}Sinh\sqrt{a}t$

$\Rightarrow u(t) = \frac{1}{\sqrt{a}}\int_0^t e^{-st}\, Sinh\sqrt{a}(t\, - \xi)f\,(\xi)\, d\xi + u_0Cosh\sqrt{a}t + \frac{u_1}{\sqrt{a}}Sinh\sqrt{a}t$

MELLIN INTEGRAL TRANSFORMATION:

For a well behaved function 'f' Mellin Integral Transformation is defined as

$M\{f(t): s\} = f^*(s) = \int_0^\infty f(t)t^{s-1}dt$

INVERSE MELLIN INTEGRAL TRANSFORMATION:

For a well behaved function 'f' Inverse Mellin Integral Transformation is defined as

$M^{-1}\{f^*(s): t\} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} f^*(s)t^{-s}ds \;\; ; t > 0; r = R(s)$

THE LAPLACE INVERSION INTEGRAL
or THE FOURIER MELLIN INTEGRAL
or DERIVATION OF INVERSION INTEGRAL

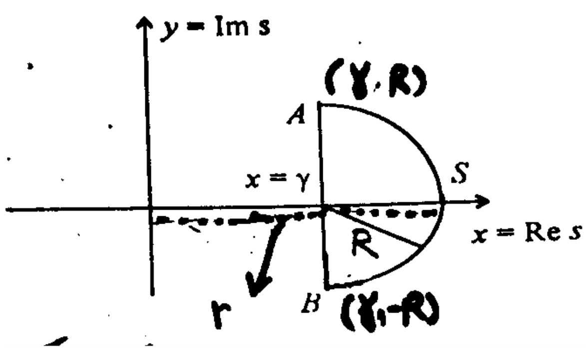
STATEMENT :

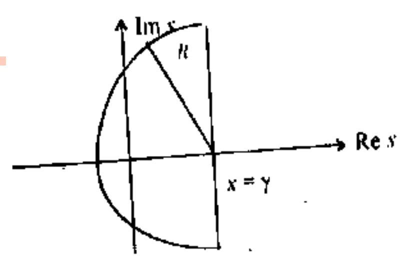
If $f(t)$ is inverse Laplace Transformation of $F(s)$ and all singularties of $F(s)$ in the complex plane 'S' lie to the left of the line $x=\gamma$ then

$$f(t)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR}e^{st}F(s)ds$$

Proof:

Draw the line $x=\gamma$ in the 'S' plane and mark the points $A=(\gamma,R)$ and $B=(\gamma,-R)$ on this line and draw a semicircle S of radius R to the right of the line $x=\gamma$. Let $C=\overline{AB}\cup S$ be the closed contour consisting of the line segment $\overline{AB}$ and S.





Let the function $F(z)=\int_0^\infty e^{-zt}f(t)dt$ is an analytic function on and within the contour C. if 's' is any point inside C then by Cauchy Integral Theorem

$$F(s)=\frac{1}{2\pi i}\oint\frac{F(z)}{z-s}dz\Rightarrow F(s)=\frac{1}{2\pi i}\oint\frac{1}{z-s}\int_0^\infty e^{-zt}f(t)dt\,dz$$

$\Rightarrow F(s)=\frac{1}{2\pi i}\int_0^\infty f(t)\left[\oint\frac{1}{z-s}e^{-zt}dz\right]dt$ **interchanging the order of integration.**

$\Rightarrow F(s)=\frac{1}{2\pi i}\int_0^\infty f(t)\left[\int_{-s}\frac{e^{-zt}}{z-s}dz+\int_A^B\frac{e^{-zt}}{z-s}dz\right]dt=\frac{1}{2\pi i}\int_0^\infty f(t)\int_A^B\frac{e^{-zt}}{z-s}dz\,dt$ **by Jordan's**

Also $\int_A^B \frac{e^{-zt}}{z-s}dz = \lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR}\frac{e^{-zt}}{z-s}dz = -\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{-zt}}{z-s}dz$

$\Rightarrow F(s) = \frac{-1}{2\pi i}\int_0^\infty f(t)\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{-zt}}{z-s}dz\,dt = \frac{1}{2\pi i}\int_0^\infty f(t)\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{-zt}}{s-z}dz\,dt$

$\Rightarrow F(s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[\int_0^\infty e^{-zt}f(t)dt\right]\frac{1}{s-z}dz$ again changing the order of integration.

$\Rightarrow F(s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{F(z)}{s-z}dz$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(z)\mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\}dz = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{zt}F(z)dz$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}F(s)ds = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR}e^{st}F(s)ds$

SPECIAL CASE:

Now suppose $F(s)$ has poles only to the left of the line $x = ReS = \gamma$ then we can enclose all those poles in a contour C on the left of $x = \gamma$ then

$\Rightarrow f(t) = \frac{1}{2\pi i}\oint_c e^{st}F(s)ds = \frac{1}{2\pi i}\sum_j(2\pi i R_j) = \sum_j R_j$

where $R_j = residue\ of\, e^{st}F(s)$ at the poles $s = s_j$

EXAMPLE:

Use Laplace Inversion Intgral (or Rasidue method) evaluate $\mathcal{L}^{-1}\left\{\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$

Solution:

Given $F(s) = \frac{s^3+2s^2+1}{s^2(s^2+1)} = \frac{s^3+2s^2+1}{s^2(s+i)(s-i)}$ has simple poles at $s = \pm i$ and a pole of order '2' at $s = 0$

Now using $R(f,\alpha) = \lim_{s\to\alpha}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$R(f,0) = R_0 = \lim_{s\to 0}\frac{d}{ds}[s^2e^{st}F(s)] = \lim_{s\to 0}\frac{d}{ds}\left[s^2e^{st}.\frac{s^3+2s^2+1}{s^2(s^2+1)}\right] = t$

$R(f,i) = R_1 = \lim_{s\to i}[(s-i)e^{st}F(s)] = \lim_{s\to i}\left[(s-i)e^{st}\frac{s^3+2s^2+1}{s^2(s+i)(s-i)}\right]$

$R(f,i) = R_1 = \frac{1-i}{2i}e^{it}$

$R(f,-i) = R_2 = \lim_{s\to -i}[(s+i)e^{st}F(s)] = \lim_{s\to -i}\left[(s+i)e^{st}\frac{s^3+2s^2+1}{s^2(s+i)(s-i)}\right]$

$R(f,-i) = R_2 = \frac{1+i}{2i}e^{-it}$

Now $\Rightarrow f(t) = \sum_j R_j = R_0 + R_1 + R_2$

$\Rightarrow f(t) = \sum_j R_j = t + \frac{1-i}{2i}e^{it} + \frac{1+i}{2i}e^{-it}$

$\Rightarrow f(t) = \sum_j R_j = t + \frac{1-i}{2i}(Cost + iSint) + \frac{1+i}{2i}(Cost - iSint)$

$\Rightarrow f(t) = \sum_j R_j = t + Cost + Sint$ after solving.

EXAMPLE:

Use Laplace Inversion Intgral (or Rasidue method) evaluate $\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s^2+1)}\right\}$

Solution: Given $F(s) = \frac{2s+1}{s(s^2+1)} = \frac{2s+1}{s(s+i)(s-i)}$ has simple poles at $s = 0, \pm i$

Now using $R(f,\alpha) = \lim_{s\to\alpha}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$R(f,0) = R_0 = \lim_{s\to 0}[se^{st}F(s)] = \lim_{s\to 0}\left[se^{st}.\frac{2s+1}{s(s^2+1)}\right] = 1$

$R(f,i) = R_1 = \lim_{s\to i}[(s-i)e^{st}F(s)] = \lim_{s\to i}\left[(s-i)e^{st}\frac{2s+1}{s(s+i)(s-i)}\right]$

$R(f,i) = R_1 = \frac{1+2i}{-2i}e^{it}$

$R(f,-i) = R_2 = \lim_{s\to -i}[(s+i)e^{st}F(s)] = \lim_{s\to -i}\left[(s+i)e^{st}\frac{2s+1}{s(s+i)(s-i)}\right]$

$R(f,-i) = R_2 = \frac{1-2i}{-2i}e^{-it}$

Now $\Rightarrow f(t) = \sum_j R_j = R_0 + R_1 + R_2$

$\Rightarrow f(t) = \sum_j R_j = 1 - \frac{1+2i}{2i}e^{it} - \frac{1-2i}{2i}e^{-it}$

$\Rightarrow f(t) = \sum_j R_j = t - \frac{1+2i}{2i}(Cost + iSint) - \frac{1-2i}{2i}(Cost - iSint)$

$\Rightarrow f(t) = \sum_j R_j = 1 + 2Sint - Cost$ after solving.

EXAMPLE:

Use Laplace Inversion Intgral (or Rasidue method) evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$

Solution: Given $F(s) = \frac{1}{s^2(s+1)}$ **has simple pole at** $s = -1$ **and a pole of order '2' at** $s = 0$

Now using $R(f,\alpha) = \lim_{s\to\alpha} \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$$R(f,0) = R_0 = \lim_{s\to 0}\frac{d}{ds}[s^2 e^{st}F(s)] = \lim_{s\to 0}\frac{d}{ds}\left[s^2 e^{st}.\frac{1}{s^2(s+1)}\right] = t-1$$

$$R(f,-1) = R_1 = \lim_{s\to -1}[(s-i)e^{st}F(s)] = \lim_{s\to i}\left[(s+1).e^{st}.\frac{1}{s^2(s+1)}\right] = e^{-t}$$

Now $\Rightarrow f(t) = \sum_j R_j = R_0 + R_1$

$\Rightarrow f(t) = \sum_j R_j = t - 1 + e^{-t}$

In order to find a solution of linear partial differential equations, the following formulas and results are useful.

If $\mathcal{L}[u(x,t)] = U(x,s)$ **then**

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\,U(x,s) - u(x,0)$$

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\,U(x,s) - su(x,0) - u_t(x,0)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mathcal{L}\left\{\frac{\partial^n u}{\partial t^n}\right\} = s^n\,U(x,s) - s^{n-1}u(x,0) - \cdots \ldots\ldots\ldots. -su_{t_{n-2}}(x,0) - u_{t_{n-1}}(x,0)$$

Similarly, it is easy to show that

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial}{\partial x}\,U(x,s)\ ,\ \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2}\,U(x,s)\ ,\ \ldots\ldots\ldots\ldots\ldots,\ \mathcal{L}\left\{\frac{\partial^n u}{\partial x^n}\right\} = \frac{\partial^n}{\partial x^n}U(x,s)$$

EXAMPLE:

Use Laplace Transformation method to solve BVP

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; 0 < x < a; \quad 0 \le t < \infty$$

$$u(0,t) = 1, \ u(1,t) = 1 \ ; t > 0 \ , u(x,0) = 1 + Sin\pi x$$

Solution:

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} \Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s\,U(x,s) - u(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s\,U(x,s) - (1 + Sin\pi x)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - s\,U(x,s) = -1 - Sin\pi x$ …………(i)

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ …………(ii)

For Chractristic (auxiliary) solution

$(i) \Rightarrow (D^2 - s)U(x,s) = -1 - Sin\pi x \Rightarrow D^2 - s = 0 \Rightarrow D = \pm\sqrt{s}$

Then $U_c(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$

For Particular solution

Consider $U_p(x,s) = \frac{-1 - Sin\pi x}{D^2 - s} = \frac{-e^{0x}}{D^2 - s} - img\frac{e^{i\pi x}}{D^2 - s} = \frac{-1}{0^2 - s} - \frac{Sin\pi x}{(i\pi)^2 - s} = \frac{1}{s} - \frac{Sin\pi x}{-\pi^2 - s}$

Then $U_p(x,s) = \frac{1}{s} + \frac{Sin\pi x}{\pi^2 + s}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{Sin\pi x}{\pi^2 + s}$

$\Rightarrow U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{Sin\pi x}{\pi^2 + s}$ …………(iii)

Now using BC's

$u(0,t) = 1 \Rightarrow \mathcal{L}\{u(0,t)\} = \mathcal{L}\{1 = t^0\} \Rightarrow U(0,s) = \frac{1}{s}$

$u(1,t) = 1 \Rightarrow \mathcal{L}\{u(1,t)\} = \mathcal{L}\{1 = t^0\} \Rightarrow U(1,s) = \frac{1}{s}$

$(iii) \Rightarrow U(0,s) = \frac{1}{s} = c_1 e^0 + c_2 e^0 + \frac{1}{s} + \frac{Sin(0)}{\pi^2 + s} \Rightarrow c_1 + c_2 + \frac{1}{s} = \frac{1}{s} \Rightarrow c_1 = -c_2$

$(iii) \Rightarrow U(1,s) = \frac{1}{s} = c_1 e^{\sqrt{s}(1)} + c_2 e^{-\sqrt{s}(1)} + \frac{1}{s} + \frac{Sin\pi}{\pi^2+s} \Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s} - \frac{1}{s} = 0$

$\Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \Rightarrow -c_2 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \qquad \therefore c_1 = -c_2$

$\Rightarrow c_2\left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right] = 0 \Rightarrow c_2 = 0 \ , \ \left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right] \neq 0$

$\Rightarrow c_2 = 0 \Rightarrow c_1 = 0 \qquad \therefore c_1 = -c_2$

$(iii) \Rightarrow U(x,s) = \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s} \qquad \therefore c_1 = c_2 = 0$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{Sin\pi x}{\pi^2+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + Sin\pi x \mathcal{L}^{-1}\left\{\frac{1}{s-(-\pi^2)}\right\}$

$\Rightarrow u(x,t) = 1 + Sin\pi x e^{-\pi^2 t}$ **required solution.**

EXAMPLE: (UoS; 2017 – II)

Use Laplace Transformation method to solve BVP

$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t); t > 0, x > 0$

$u(x,0) = u_t(x,0) = 0,\ u(0,t) = f(t)$, $\lim_{x\to\infty} u(x,t) = 0$

Solution:

Given $u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\}$

$\Rightarrow s^2 U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow s^2 U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) \Rightarrow s^2 U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - \frac{s^2}{\alpha^2} U(x,s) = 0$

This is Homogeneous DE of 2nd order therefore

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm \frac{s}{a}$

Then $U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$ **…………(i)**

Now using BC's

$u(0,t) = f(t) \Rightarrow \mathcal{L}\{u(0,t)\} = \mathcal{L}\{f(t)\} \Rightarrow U(0,s) = F(s)$

$\lim_{x\to\infty} u(x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} U(x,s) = 0$

$(i) \Rightarrow U(0,s) = F(s) = c_1 e^{\frac{s}{a}(0)} + c_2 e^{-\frac{s}{a}(0)} \Rightarrow c_1 + c_2 = F(s)$

$(i) \Rightarrow \lim_{x\to\infty} U(x,s) = 0 = \lim_{x\to\infty}\left[c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}\right] = c_1 e^{\infty} + c_2 e^{-\infty}$

$\Rightarrow c_1 = 0 \quad then \quad c_2 = F(s) \qquad \therefore c_1 + c_2 = F(s)$

Thus $(i) \Rightarrow U(x,s) = F(s) e^{-\frac{s}{a}x}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{F(s) e^{-\frac{s}{a}x}\right\}$

$\Rightarrow u(x,t) = H\left(t - \frac{x}{a}\right) f\left(t - \frac{x}{a}\right)$ **where** $H\left(t - \frac{x}{a}\right) f\left(t - \frac{x}{a}\right) = \begin{cases} 0 & t < \frac{x}{a} \\ f(t) & t \geq \frac{x}{a} \end{cases}$

EXAMPLE:

Use Laplace Transformation method to solve BVP

$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) - g$

$u(x,0) = u_t(x,0) = 0,\ u(0,t) = 0\ ,\lim_{x\to\infty} u_x(x,t) = 0$

Solution: Given $u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) - g \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\} - g\mathcal{L}\{1\}$

$\Rightarrow s^2 U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow s^2 U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow s^2 U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - \frac{s^2}{\alpha^2} U(x,s) = \frac{g}{\alpha^2 s}$ **…………(i)**

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ **…………(ii)**

For Chractristic (auxiliary) solution

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U_c(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$

For Particular solution

Consider $U_p(x,s) = \frac{\frac{g}{\alpha^2 s}}{D^2-\frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}e^{0x}}{0^2-\frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}}{-\frac{s^2}{\alpha^2}} = -\frac{g}{s^3}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$ …………(iii)

Now using BC's

$u\,(0,t) = 0 \Rightarrow \mathcal{L}\{u\,(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$\lim_{x\to\infty} u_x\,(x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u_x\,(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} \frac{\partial}{\partial x} U\,(x,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1 e^{0} + c_2 e^{-0} - \frac{g}{s^3} \Rightarrow c_1 + c_2 = \frac{g}{s^3}$

$(iii) \Rightarrow \lim_{x\to\infty} \frac{\partial}{\partial x} U\,(x,s) = 0 = \lim_{x\to\infty}\left[c_1 \frac{s}{a} e^{\frac{s}{a}x} - \frac{s}{a} c_2 e^{-\frac{s}{a}x}\right] = c_1 \frac{s}{a} e^{\infty} + c_2 \frac{s}{a} e^{-\infty}$

$\Rightarrow c_1 \frac{s}{a} e^{\infty} = 0 \Rightarrow c_1 = 0 \;\; since \; \frac{s}{a} e^{\infty} \neq 0, \;\; then \;\; c_2 = \frac{g}{s^3} \quad \therefore c_1 + c_2 = \frac{g}{s^3}$

Thus $(iii) \Rightarrow U(x,s) = \frac{g}{s^3} e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \frac{g}{2!}\mathcal{L}^{-1}\left\{e^{-\frac{x}{a}s}.\frac{2!}{s^{2+1}}\right\} - \frac{g}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}$

$\Rightarrow u\,(x,t) = \frac{g}{2} H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 - \frac{g}{2}(t^2)$

$\Rightarrow u\,(x,t) = \frac{g}{2}\left[H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 - (t^2)\right]$

where $H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 = \begin{cases} 0 & t < \frac{x}{a} \\ t^2 & t \geq \frac{x}{a} \end{cases}$

EXAMPLE: (UoS; 2017)

Use Laplace Transformation method to solve BVP

$u_{xx}(x,t) = u_{tt}(x,t); t > 0;\ 0 < x < 1$

$u(0,t) = 0 = u(1,t),\ u(x,0) = Sin\pi x\ \ , u_t(x,0) = -Sin\pi x$

Solution:

Given $u_{xx}(x,t) = u_{tt}(x,t) \Rightarrow \mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = s^2\, U(x,s) - su(x,0) - u_t(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = s^2\, U(x,s) - sSin\pi x + Sin\pi x$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - s^2\, U(x,s) = -sSin\pi x + Sin\pi x$ …………(i)

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ …………(ii)

For Chractristic (auxiliary) solution

$\Rightarrow (D^2 - s^2)U(x,s) = 0 \Rightarrow D^2 - s^2 = 0 \Rightarrow D = \pm s$

Then $U_c(x,s) = c_1 e^{sx} + c_2 e^{-sx}$

For Particular solution

Consider

$U_p(x,s) = \frac{(1-s)Sin\pi x}{D^2-s^2} = (1-s)img\frac{e^{i\pi x}}{D^2-s^2} = (1-s)\frac{Sin\pi x}{(i\pi)^2-s} = (1-s)\frac{Sin\pi x}{-\pi^2-s}$

$U_p(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$ …………(iii)

Now using BC's $u(0,t) = 0 \Rightarrow \mathcal{L}\{u(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$u(1,t) = 0 \Rightarrow \mathcal{L}\{u(1,t)\} = 0 \Rightarrow U(1,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1 e^0 + c_2 e^{-0} + \frac{(s-1)Sin\pi(0)}{\pi^2+s} \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$(iii) \Rightarrow U(1,s) = 0 = c_1e^s + c_2e^{-s} + \frac{(s-1)Sin\pi}{\pi^2+s} \Rightarrow c_1e^s + c_2e^{-s} = 0 \Rightarrow c_1e^s - c_1e^{-s} = 0$

$\Rightarrow c_1(e^s - e^{-s}) = 0 \Rightarrow c_1 = 0 \;\; as\; (e^s - e^{-s}) \neq 0 \Rightarrow c_2 = 0$

Thus $(iii) \Rightarrow U(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = Sin\pi x\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} - \frac{Sin\pi x}{\pi}\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$

$\Rightarrow u\,(x,t) = Sin\pi xCos\pi t - \frac{Sin\pi x}{\pi}Sin\pi t = Sin\pi x\left[Cos\pi t - \frac{Sin\pi x}{\pi}\right]$

EXAMPLE: (UoS; 2019 – I)

A uniform bar of length 'l' is fixed at one end. Let the force

$f(\,t) = \begin{cases} 0 & t < 0 \\ f_0 & t > 0 \end{cases}$ **be suddenly applied at the end $= \; l$, if the bar is initially at rest, find the longitudinal displacement for $t > 0$ using Laplace Transformation the motion of bar is govern by the differential system**

$u_{tt} \; = \alpha^2 u_{xx}; t > 0, 0 < x < 1$ **and α is constant.**

$u(x,0) = u(0,t) = u_t(x,0) = 0, \quad u_x(l,t) = \frac{f_0}{E}$ **where E is constant.**

Solution:

Given $u_{tt}\,(x,t) = \alpha^2 u_{xx}\,(x,t) \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2\mathcal{L}\{u_{xx}\}$

$\Rightarrow s^2\,U(x,s) - su(x,0) - u_t(x,0) = \alpha^2\frac{\partial^2}{\partial x^2}U(x,s)$

$\Rightarrow s^2\,U(x,s) - (0) - (0) = \alpha^2\frac{\partial^2}{\partial x^2}U(x,s) \Rightarrow s^2\,U(x,s) = \alpha^2\frac{\partial^2}{\partial x^2}U(x,s)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - \frac{s^2}{\alpha^2}\,U(x,s) = 0$

This is Homogeneous DE of 2nd order therefore

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right)U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U(x,s) = c_1e^{\frac{s}{a}x} + c_2e^{-\frac{s}{a}x}$ …………(i)

Now using BC's

$$u\,(0,t)=0\Rightarrow \mathcal{L}\{u\,(0,t)\}=0\Rightarrow U(0,s)=0$$

$$u_x(l,t)=\frac{f_0}{E}\Rightarrow \mathcal{L}\{u_x(l,t)\}=\mathcal{L}\left\{\frac{f_0}{E}\right\}\Rightarrow \frac{\partial}{\partial x}U\,(l,s)=\frac{F_0}{E}$$

$$(i)\Rightarrow U(0,s)=F(\,s)=c_1e^{0}+c_2e^{-0}\Rightarrow c_1+c_2=0\Rightarrow c_2=-c_1$$

Then $U(x,s)=c_1e^{\frac{s}{a}x}-c_1e^{-\frac{s}{a}x}$ **...........(ii)**

$$\Rightarrow \frac{\partial}{\partial x}U(x,s)=c_1\frac{s}{a}e^{\frac{s}{a}x}+c_1\frac{s}{a}e^{-\frac{s}{a}x}$$

Then using $\frac{\partial}{\partial x}U\,(l,s)=\frac{F_0}{E}$ **we get**

$$\Rightarrow \frac{\partial}{\partial x}U(x,s)=\frac{F_0}{E}=c_1\frac{s}{a}e^{\frac{s}{a}x}+c_1\frac{s}{a}e^{-\frac{s}{a}x}\Rightarrow c_1=\frac{F_0}{E\left(\frac{s}{a}e^{\frac{s}{a}x}+\frac{s}{a}e^{-\frac{s}{a}x}\right)}$$

Hence $(ii)\Rightarrow U(x,s)=\frac{F_0}{E\left(\frac{s}{a}e^{\frac{s}{a}x}+\frac{s}{a}e^{-\frac{s}{a}x}\right)}.\left(e^{\frac{s}{a}x}-e^{-\frac{s}{a}x}\right)=\frac{F_0}{E}\frac{\left(e^{\frac{s}{a}x}-e^{-\frac{s}{a}x}\right)}{\frac{s}{a}\left(e^{\frac{s}{a}x}+e^{-\frac{s}{a}x}\right)}$

Taking Laplace inverse on both sides

$$u(x,t)=\mathcal{L}^{-1}\left\{\frac{F_0}{E}\frac{\left(e^{\frac{s}{a}x}-e^{-\frac{s}{a}x}\right)}{\frac{s}{a}\left(e^{\frac{s}{a}x}+e^{-\frac{s}{a}x}\right)}\right\}$$

which is required longitudinal displacement for t > 0

THEOREM: Let $f(t)$ be a piecewise continuous function for $t \geq 0$ and of exponential order. If $f(t)$ is periodic with period T then show that

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

PROOF: By definition, we have

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt$$

In the 2nd integral on the right put $t = u + T \Rightarrow dt = du$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + \int_0^\infty e^{-s(u+T)} f(u+T)\, du$$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + e^{-sT} \int_0^\infty e^{-su} f(u+T)\, du$$

Since given functionis periodic with period T therefore $f(u+T) = f(u)$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + e^{-sT} \int_0^\infty e^{-su} f(u)\, du$$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + e^{-sT} \mathcal{L}\{f(u)\}$$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + e^{-sT} \mathcal{L}\{f(t)\}$$

$$(1 - e^{-sT})\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$ **As required the result.**

THEOREM: If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$

PROOF: By definition, we have

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\int_s^\infty F(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds \qquad \text{integrating.}$$

$$\int_s^\infty F(s)\, ds = \int_0^\infty f(t) \left[\int_s^\infty e^{-st}\, ds\right] dt \qquad \text{changing the order of integration.}$$

$$\int_s^\infty F(s)\, ds = \int_0^\infty f(t) \left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Hence $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$

FOURIER TRANSFORMATION AND INTEGRALS WITH APPLICATIONS

FOURIER TRANSFORMATION: If $f\ (x)$ is a continuous, piecewise smooth, and absolutely integrable function, then the Fourier transform of $f\ (x)$with respect to $x \in R$ is denoted by $F\ (k)$and is defined by

$$\mathcal{F}\{f\ (x)\} = F\ (k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f\ (x)\ dx$$

where k is called the Fourier transform variable and $exp\ (-ikx)$ is called the kernel of the transform.

Then, for all $x \in R$, the <u>INVERSE FOURIER TRANSFORM</u> of $F\ (k)$ is defined by

$$\mathcal{F}^{-1}\{F\ (k)\} = f\ (x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F\ (k)\ dk$$

CONDITION FOR EXISTENCE OF FOURIER TRANSFORMATION

Fourier Transforamtion and Inverse Fourier Transformation exist if

(i) The function $f(x)$ or $F(k)$ is continuous or piecewise continuous over $(-\infty, \infty)$ and bounded.

(ii) The function $f(x)$ or $F(k)$ are absolutely integrable i.e. $\int_{-\infty}^{\infty}|f(x)|dx$ or $\int_{-\infty}^{\infty}|F(k)|dk$ this condition is sufficient for existence of Fourier Transforamtion and Inverse Fourier Transformation.

Example: (UoS; 2014 – II, 2015 – I)

Show that for a Guassian Function $\mathcal{F}\left\{Ne^{-ax^2}\right\} = \frac{N}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$; $a > 0$, N **is constant.**

Solution. We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} . e^{-ax^2} dx$$

$$\mathcal{F}\{f(x)\} = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-ax^2} dx = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(x-\frac{ik}{2a}\right)^2+\frac{k}{4a^2}^2\right]} dx$$

Consider $ikx - ax^2$

$$= -a\left(x^2 - \frac{ikx}{a}\right)$$

$$= -a\left[\left(x^2 - \frac{2ikx}{2a}\right) + \left(\frac{ik}{2a}\right)^2 - \left(\frac{ik}{2a}\right)^2\right]$$

$$= -a\left[\left(x - \frac{ik}{2a}\right)^2 + \frac{k}{4a^2}^2\right]$$

$$\mathcal{F}\{f(x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2} dx$$

Put $a\left(x-\frac{ik}{2a}\right)^2 = P^2 \Rightarrow \sqrt{a}\left(x-\frac{ik}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2} dx = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-P^2} . \frac{dP}{\sqrt{a}}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}}.\sqrt{\pi} \qquad \therefore \int_{-\infty}^{\infty} e^{-P^2} dP = \sqrt{\pi}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \mathcal{F}\left\{Ne^{-ax^2}\right\} = \frac{N}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$$

Example: Find the Fourier transform of a box function

$$f(x) = \begin{cases} 1 , & |x| < a \text{ or } -a < x < a \\ 0 & , |x| > a \end{cases}$$

Solution. Let we have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} f(x)dx + \int_{-a}^{a} e^{ikx} f(x)dx + \int_{a}^{\infty} e^{ikx} f(x)dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} .0dx + \int_{-a}^{a} e^{ikx} .1dx + \int_{a}^{\infty} e^{ikx} .0dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ikx} dx = \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ika}-e^{-ika}}{2i}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$$

Example: (UoS; 2013, 2014)

Find the Fourier transform of $g(\mathrm{x})=\frac{\mathrm{a}}{x^2+a^2}$

Solution. Let we have, by definition

$$\mathcal{F}\{g(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\mathrm{ikx}}\,g(x)\mathrm{dx}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\mathrm{ikx}}\,\frac{\mathrm{a}}{x^2+a^2}\mathrm{dx}$$

$$\mathcal{F}\{g(z)\}=\frac{a}{\sqrt{2\pi}}\oint_c\frac{e^{\mathrm{ikz}}}{z^2+a^2}\mathrm{dz}\qquad\text{replacing 'x' with 'z'}$$

$$\therefore e^{\mathrm{ikz}}=e^{\mathrm{ik}(x+iy)}=e^{\mathrm{ikx}}.e^{\mathrm{i^2ky}}=e^{\mathrm{ikx}}.e^{-\mathrm{ky}}\to 0\quad as\quad y\to\infty\Rightarrow e^{\mathrm{ikz}}\to 0\ ;k>0$$

Similarly $e^{\mathrm{ikz}}\to 0\ ;k<0$ **when** $y\to-\infty$

Let $g(z)=\frac{e^{\mathrm{ikz}}}{z^2+a^2}\Rightarrow z=\pm ai$ **are the simple poles of** $g(z)$

Now using $R(f,\alpha)=\lim_{s\to\alpha}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$$R(g,\alpha i)=R_1=\lim_{z\to\alpha i}(z-\alpha i)\frac{e^{\mathrm{ikz}}}{(z-\alpha i)(z+\alpha i)}=\lim_{z\to\alpha i}\frac{e^{\mathrm{ikz}}}{(z+\alpha i)}=\frac{e^{\mathrm{ik}(\alpha i)}}{2\alpha i}=\frac{e^{-\alpha\mathrm{k}}}{2\alpha i}$$

Similarly

$$R(g,-\alpha i)=R_2=\lim_{z\to-\alpha i}(z+\alpha i)\frac{e^{\mathrm{ikz}}}{(z-\alpha i)(z+\alpha i)}=\lim_{z\to\alpha i}\frac{e^{\mathrm{ikz}}}{(z-\alpha i)}=\frac{e^{\mathrm{ik}(-\alpha i)}}{-2\alpha i}=\frac{e^{\alpha\mathrm{k}}}{-2\alpha i}$$

Now $\Rightarrow\mathcal{F}\{g(x)\}=\frac{a}{\sqrt{2\pi}}\oint_c\frac{e^{\mathrm{ikz}}}{z^2+a^2}\mathrm{dz}=\frac{a}{\sqrt{2\pi}}.2\pi i\sum_j R_j=\frac{a}{\sqrt{2\pi}}.2\pi i[R_1+R_2]$

Now we use $2\pi i$ **for the contour as a semi circle in upper half plane and** $-2\pi i$ **for the contour as a semi circle in lower half plane**

$$\Rightarrow\mathcal{F}\{g(x)\}=\frac{a}{\sqrt{2\pi}}.[(2\pi i)R_1+(2\pi i)R_2]=\frac{a}{\sqrt{2\pi}}.2\pi i[R_1-R_2]$$

$$\Rightarrow\mathcal{F}\{g(x)\}=\frac{a}{\sqrt{2\pi}}.2\pi i\left[\frac{e^{-\alpha\mathrm{k}}}{2\alpha i}+\frac{e^{\alpha\mathrm{k}}}{2\alpha i}\right]=\sqrt{\frac{\pi}{2}}\left[e^{-\alpha\mathrm{k}}+e^{\alpha\mathrm{k}}\right]$$

$$\Rightarrow\mathcal{F}\{g(x)\}=\sqrt{\frac{\pi}{2}}\left[e^{\alpha|\mathrm{k}|}+e^{\alpha|\mathrm{k}|}\right]\qquad\therefore k>0,k<0\Rightarrow|\mathrm{k}|=\pm k$$

$$\Rightarrow\mathcal{F}\{g(x)\}=\sqrt{\frac{\pi}{2}}.2e^{\alpha|\mathrm{k}|}=\sqrt{2\pi}e^{\alpha|\mathrm{k}|}$$

PROPERTIES OF FOURIER TRANSFORMS

LINEARITY PROPERTY: THE FOURIER TRANSFORMATION $\mathcal{F}$ IS LINEAR.

Proof. Let $u(x) = af(x) + bg(x)$ where a and b are constants.

We have, by definition

$$\mathcal{F}\{u(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, u(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, [af(x) + bg(x)]dx$$

$$\mathcal{F}\{u(x)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f(x)dx + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, g(x)dx$$

$$\mathcal{F}\{u(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

$$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$ hence proved.

LINEARITY PROPERTY: THE INVERSE FOURIER TRANSFORMATION $\mathcal{F}^{-1}$ IS LINEAR.

Proof. Let $U(k) = aF(k) + bG(k)$ where a and b are constants.

We have, by definition

$$\mathcal{F}^{-1}\{U(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, U(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, [aF(k) + bG(k)]dk$$

$$\mathcal{F}^{-1}\{U(k)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, F(k)dk + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, G(k)dk$$

$$\mathcal{F}^{-1}\{aF(k) + bG(k)\} = a\mathcal{F}^{-1}\{F(k)\} + b\mathcal{F}^{-1}\{G(k)\}$$ hence proved.

SHIFTING PROPERTY: Let $\mathcal{F}\{f(x)\}$ be a Fourier transform of $f(x)$. Then

(i) $\mathcal{F}[f(x - a)] = e^{ika}F(k)$ where 'a' is a real constant.

Proof. From the definition, we have, for a > 0,

$$\mathcal{F}[f(x - a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f(x - a)dx$$

Put $x - a = x' \Rightarrow dx = dx'$ also as $x \to \pm\infty$ then $x' \to \pm\infty$

$$\mathcal{F}[f(x - a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(x'+a)}\, f(x')dx' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}.e^{ika}\, f(x')dx'$$

$$\mathcal{F}[f(x - a)] = e^{ika}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}\, f(x')dx' = e^{ika}\mathcal{F}\{f(x)\} = e^{ika}F(k)$$

(ii) $\mathcal{F}\,[e^{\mathrm{iax}} f\,(x)] = F(k+a)$ **where 'a' is a real constant.**

Proof. From the definition, we have, for a > 0,

$$\mathcal{F}\,[e^{\mathrm{iax}} f\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, e^{\mathrm{iax}} f\,(x)\,\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k+a)\mathrm{x}} f\,(x)\,\mathrm{dx} = F(k+a)$$

SCALING PROPERTY: If $\mathcal{F}$ is the Fourier transform of f, then

$\mathcal{F}\,[f\,(cx)] = \left(\frac{1}{|c|}\right) F\left(\frac{k}{c}\right)$ **where c is a real nonzero constant.**

Proof. For $c \neq 0$ we have $\mathcal{F}\,[f\,(cx)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}} f\,(cx)\mathrm{dx}$

$$\mathcal{F}\,[f\,(cx)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}\left(\frac{x'}{c}\right)} f\,(x')\frac{dx'}{c} = \frac{1}{c}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}\left(\frac{x'}{c}\right)} f\,(x')dx' = \frac{1}{c}F\left(\frac{k}{c}\right)$$

Since $c \neq 0$ then either $c < 0$ or $c > 0$

If $c > 0$ then $\mathcal{F}\,[f\,(cx)] = \frac{1}{+c}F\left(\frac{k}{c}\right)$ If $c < 0$ then $\mathcal{F}\,[f\,(cx)] = \frac{1}{-c}F\left(\frac{k}{c}\right)$

Hence $\mathcal{F}\,[f\,(cx)] = \left(\frac{1}{|c|}\right) F\left(\frac{k}{c}\right)$

CONJUGATION PROPERTY: Let f is real then $F\,(-k) = \overline{F\,(k)}$

Proof. Since f is real therefore $f\,(x) = \overline{f\,(x)}$ then by defination

$$F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}} f\,(x)\mathrm{dx}$$

$$\overline{F\,(k)} = \mathcal{F}\,[\overline{f\,(x)}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, \overline{f\,(x)}\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(-k)\mathrm{x}} f\,(x)\mathrm{dx} = F\,(-k)$$

Hence $F\,(-k) = \overline{F\,(k)}$

ATTENUATION PROPERTY: (UoS; 2015 – II, 2018 – I)

For a function $f\,(x)$ the result will be , $\mathcal{F}\,[e^{ax} f\,(x)] = F\,(k - ai)$

Proof. By definition $F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}} f\,(x)\mathrm{dx}$

Then $\mathcal{F}\,[e^{ax} f\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, e^{ax} f\,(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, e^{-i^2 ax} f\,(x)\mathrm{dx}$

$$\mathcal{F}\,[e^{ax} f\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k-ai)\mathrm{x}} f\,(x)\mathrm{dx} \quad \text{..........(i)}$$

Also $F\,(k - ai) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k-ai)\mathrm{x}} f\,(x)\mathrm{dx}$ **..........(ii)**

Thus from (i) and (ii) $\mathcal{F}\,[e^{ax} f\,(x)] = F\,(k - ai)$

MODULATION PROPERTY(i): $\mathcal{F}\,[Cosax f\,(x)] \;= \frac{1}{2}[F\,(k+a)+F\,(k-a)]$

Proof. By definition $\mathcal{F}\,[Cosax f\,(x)] \;=\; \mathcal{F}\,[\left(\frac{e^{iax}+e^{-iax}}{2}\right) f\,(x)]$

$\mathcal{F}\,[Cosax f\,(x)] = \frac{1}{2}[\,\mathcal{F}\,\{e^{iax} f\,(x)\} + \mathcal{F}\,\{e^{-iax} f\,(x)\}] = \frac{1}{2}[F\,(k+a)+F\,(k-a)]$

MODULATION PROPERTY (ii): $\mathcal{F}\,[Sinax f\,(x)] \;= \frac{1}{2i}[F\,(k+a)-F\,(k-a)]$

Proof. By definition $\mathcal{F}\,[Sinax f\,(x)] \;=\; \mathcal{F}\,[\left(\frac{e^{iax}-e^{-iax}}{2i}\right) f\,(x)]$

$\mathcal{F}\,[Cosax f\,(x)] = \frac{1}{2i}[\,\mathcal{F}\,\{e^{iax} f\,(x)\} - \mathcal{F}\,\{e^{-iax} f\,(x)\}] = \frac{1}{2i}[F\,(k+a)-F\,(k-a)]$

ROPERTY: if $f\,(x)$ is real and even then $F\,(k)$ is real.

Proof. Since f is real therefore $f\,(x)=\overline{f\,(x)}$ ....(i) and $f\,(-x)=f\,(x)$ ……(ii) then by defination

$$F\,(k)=\mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f\,(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f\,(-x)dx$$

$$F\,(k)=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-ikx'}\, f\,(x')(-dx') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'}\, \overline{f\,(x')}dx'$$

Hence $F\,(k)=\overline{F\,(k)}$ then $F\,(k)$ is real.

ROPERTY: if $f\,(x)$ is real and odd then $F\,(k)$ is pure imaginary.

Proof. Since f is real therefore $f\,(x)=\overline{f\,(x)}$ ………..(i)

and is odd $f\,(-x)=-f\,(x)$ ………(ii) then by defination

$$F\,(k)=\mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f\,(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, (-f\,(-x))dx$$

$$F\,(k)=\mathcal{F}\,[f(x)] = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f\,(-x)dx$$

$$F\,(k)=\frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-ikx'}\, f\,(x')(-dx') = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'}\, f\,(x')dx'$$

Hence $F\,(k)=-\overline{F\,(k)}$ or $\overline{F\,(k)}=-F\,(k)$ then $F\,(k)$ is pure imaginary.

ROPERTY: if $f\ (x)$ is complex then $\mathcal{F}\left[\overline{f(-x)}\right] = \overline{F\ (k)}$

Proof. by definition

$$\mathcal{F}\left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ \overline{f(-x)}\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{\mathrm{ik}(-x')}\ \overline{f(x')}(-dx')$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\mathrm{ik}x'}\ \overline{f(x')}dx' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik}x'}\ \overline{f(x')}dx'$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik}x}\ \overline{f(x)}dx \qquad \text{replacing } x' \text{ with x}$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\ f(x)dx} = \overline{F\ (k)}$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \overline{F\ (k)} \qquad \text{as required.}$$

DIFFERENTIATION PROPERTY (higher derivative theorem):

Let f be continuous and piecewise smooth in $(-\infty, \infty)$. Let $f(x)$ approach zero as $|x| \to \infty$. If f and f' are absolutely integrable, then

$$\mathcal{F}\left[f'\ (x)\right] = (-ik)\mathcal{F}\left[f\ (x)\right] = (-ik)F\ (k)$$

Proof.

$$\mathcal{F}\left[f'\ (x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f'\ (x)\mathrm{dx}$$

$$\mathcal{F}\left[f'(x)\right] = \frac{1}{\sqrt{2\pi}}\left[\left|e^{\mathrm{ikx}} f\ (x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\mathrm{ikx}}(ik) f\ (x)\mathrm{dx}\right]$$

$$\mathcal{F}\left[f'(x)\right] = \frac{1}{\sqrt{2\pi}}\left[0 + (-ik)\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}\right]$$

$$\mathcal{F}\left[f'\ (x)\right] = (-ik)\mathcal{F}\left[f\ (x)\right] = (-ik)F\ (k)$$

For n = 2

$$\mathcal{F}\left[f''\ (x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f''\ (x)\mathrm{dx}$$

$$\mathcal{F}\left[f''(x)\right] = \frac{1}{\sqrt{2\pi}}\left[\left|e^{\mathrm{ikx}} f'(x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\mathrm{ikx}}(ik) f'(x)\mathrm{dx}\right]$$

$$\mathcal{F}\left[f''(x)\right] = \frac{1}{\sqrt{2\pi}}\left[0 + (-ik)\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f'(x)\mathrm{dx}\right]$$

$$\mathcal{F}\left[f''(x)\right] = (-ik)\mathcal{F}\left[f'(x)\right] = (-ik)(-ik)F\ (k) = (-ik)^2\ F\ (k)$$

This result can be easily extended. If f and its first (n − 1) derivatives are continuous, and if its nth derivative is piecewise continuous, then

$$\mathcal{F}\,[f^n(x)] = (-ik)^n \mathcal{F}\,[f\,(x)] = (-ik)^n\, F\,(k) \quad n = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots$$

provided f and its derivatives are absolutely integrable. In addition, we assume that f and its first (n − 1) derivatives tend to zero as |x| tends to infinity.

CONVOLUTION FUNCTION / FAULTUNG FUNCTION

The function $(f * g)\,(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\,(x - \xi)\, g\,(\xi)\, d\xi$

is called the convolution of the functions f and g over the interval (−∞,∞)

NOTE: The convolution satisfies the following properties:

1. $f * g = g * f$ **(commutative)**

2. $f * (g * h) = (f * g) * h$ **(associative)**

3. $* \,(ag + bh) = a\,(f * g) + b\,(f * h)$ **, (distributive)**

where a and b are constants.

PROPERTY: $f * g = g * f$

PROOF: since by definition $(f * g)\,(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\,(x - \xi)\, g\,(\xi)\, d\xi$

Put $x - \xi = \propto \;\Rightarrow d\xi = -d \propto$ $also\ \xi = x - \propto$ **and if** $\xi \to \pm\infty$ **then** $\propto \to \mp\infty$ **then**

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f\,(\propto) g\,(x - \propto)\,(-d \propto) = g * f$$

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g\,(x - \propto) f\,(\propto)\;\;(d \propto) = g * f$$

Hence $f * g = g * f$

CONVOLUTION / FAULTUNG THEOREM (UoS; 2013 – I)

If $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution $(f * g)$ is the product $F(k)G(k)$. That is,

$$\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$$

Or, equivalently, $\quad \mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$

Or

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk = (f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$$

PROOF: By definition, we have

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk$$

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}\, g(x')\,dx'\right\}dk$$

By changing the order of integration

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x')} F(k)\,dk\right] g(x')\,dx'$$

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x')\,g(x')\,dx'$$

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi = (f * g)(x)$$

Where we replace ξ with x'

Hence $\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$

Or $\quad \mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$

PARSEVAL'S FORMULA OF 1ST AND 2ND KIND (UoS; 2019 – I, 2018 – I)

Theorem given by Marc Anotoine des Chenes Parseval (1755 – 1836)

1ST KIND: According to this formula $\int_{-\infty}^{\infty}|f(\mathrm{x})|^2\,dx = \int_{-\infty}^{\infty}|F(\mathrm{k})|^2\,dk$

PROOF: The convolution formula gives

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik}x}\,F(k)G(k)\mathrm{dk} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi$$

$$\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi = \int_{-\infty}^{\infty} e^{-\mathrm{ik}x}\,F(k)G(k)\mathrm{dk}$$

which is, by putting $x = 0$ $\int_{-\infty}^{\infty} f(\xi)g(-\xi)\,d\xi = \int_{-\infty}^{\infty} F(k)G(k)\mathrm{dk}$

$$\int_{-\infty}^{\infty} f(x)g(-x)\,dx = \int_{-\infty}^{\infty} F(k)G(k)\mathrm{dk}$$

Putting $g(-x) = \overline{f(x)}$ **then** $g(x) = \overline{f(-x)} \Rightarrow \mathcal{F}\{g(x)\} = \mathcal{F}\left\{\overline{f(-x)}\right\}$

$\Rightarrow G(k) = \overline{F(k)}$ $\quad\therefore \mathcal{F}\left\{\overline{f(-x)}\right\} = \overline{F(k)}$ **for complex** f.

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}\,dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}\mathrm{dk}$$

where the bar denotes the complex conjugate.

$$\Rightarrow \int_{-\infty}^{\infty}|f(\mathrm{x})|^2\,dx = \int_{-\infty}^{\infty}|F(\mathrm{k})|^2\,dk$$

In terms of the notation of the norm, this is $\|f\| = \|F\|$

2ND KIND: According to this formula

$$\int_{-\infty}^{\infty} F(k)G(k)\,dk = \int_{-\infty}^{\infty} f(u)g(-u)\,du$$

PROOF: The convolution formula gives

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik}x}\,F(k)G(k)\mathrm{dk} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)g(x-u)du$$

by putting $x = 0$ **we get** $\int_{-\infty}^{\infty} F(k)G(k)\,\mathrm{dk} = \int_{-\infty}^{\infty} f(u)g(-u)du$

BOUNDEDNESS AND CONTINUITY OF FOURIER TRANSFORMATION

If $f(x)$ is piecewise smooth and absolutely integrable function on the interval $(-\infty,\infty)$ then its fourier transformation $F(k)$ is bounded and continuous.

PROOF: given that $f(x)$ is piecewise smooth and absolutely integrable function i.e. $J = \int_{-\infty}^{\infty}|f(x)|dx$

now by definition $F(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$

For boundedness taking mod on both sides

$\Rightarrow |F(k)| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx\right| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{ikx}||f(x)|dx$

$\Rightarrow |F(k)| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(x)|dx$ since $|e^{ikx}| = 1$

$\Rightarrow |F(k)| \leq \frac{1}{\sqrt{2\pi}}.J$ since $J = \int_{-\infty}^{\infty}|f(x)|dx$

$\Rightarrow |F(k)| \leq \lambda$ where $\lambda = \frac{1}{\sqrt{2\pi}}.J \in R$

$\Rightarrow F(k)$ is bounded.

Now for continuity of $F(k)$ we have

$F(k+h) - F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k+h)x} f(x)dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$

$F(k+h) - F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\left(e^{ihx} - 1\right) f(x)dx = I(k,h)$ say

$\lim_{h\to 0}[F(k+h) - F(k)] = \lim_{h\to 0} I(k,h)$ ……….(i)

Now $\lim_{h\to 0} I(k,h)$ exists if $I(k,h)$ is uniformly convergent.

For this consider

$\Rightarrow |I(k,h)| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\left(e^{ihx} - 1\right) f(x)dx\right|$

$\Rightarrow |I(k,h)| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{ikx}||e^{ihx} - 1||f(x)|dx$

$\Rightarrow |I(k,h)| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (1)\,|Coshx + iSinhx - 1||f(x)|dx$

$\Rightarrow |I(k,h)| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|(Coshx - 1) + iSinhx||f(x)|dx$

$$\Rightarrow |\mathrm{I}(k,h)| \leq \frac{1}{\sqrt{2\pi}}.\sqrt{2}\int_{-\infty}^{\infty}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx}$$

$$\Rightarrow |\mathrm{I}(k,h)| \leq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx}$$

$$\Rightarrow \lim_{h\to 0}|\mathrm{I}(k,h)| \leq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{h\to 0}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx} \quad \to 0$$

$$\Rightarrow \lim_{h\to 0}|\mathrm{I}(k,h)| \leq 0 \Rightarrow \lim_{h\to 0}\mathrm{I}(k,h) = 0$$

$$(i) \Rightarrow \lim_{h\to 0}[F\ (k+h) - F\ (k)] = 0$$

$\Rightarrow \lim_{h\to 0} F\ (k+h) = F\ (k) \Rightarrow F\ (k)$ **is continuous.**

Hence If $f(x)$ is piecewise smooth and absolutely integrable function on the interval $(-\infty,\infty)$ then its fourier transformation $F(k)$ is bounded and continuous.

RIEMANN LEBESQUE THEOREM

If $f(x)$ is piecewise smooth and absolutely integrable function then

$$\lim_{|\mathrm{k}|\to\infty} F\ (k) = 0$$

PROOF: given that $f(x)$ is piecewise smooth and absolutely integrable function i.e. $J = \int_{-\infty}^{\infty}|f(x)|dx$

now by definition $F\ (k) = \mathcal{F}\ [f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$

$$F\ (k) = \frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]$$

$$\Rightarrow |F\ (k)| = \left|\frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]\right|$$

$$\Rightarrow |F\ (k)| = \left|\frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]\right|$$

$$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left|\,|f(x)|\frac{|e^{\mathrm{ikx}}|}{|ik|}\right|_{-\infty}^{\infty} + \left|-\int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right|$$

$$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left[\lim_{x\to\infty}\frac{|f(x)|}{|k|} - \lim_{x\to-\infty}\frac{|f(x)|}{|k|}\right] + \int_{-\infty}^{\infty}\frac{|e^{\mathrm{ikx}}|}{|ik|}|f'(x)|\mathrm{dx}$$

$$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left[\lim_{x\to\infty}\frac{|f(x)|}{|k|} - \lim_{x\to-\infty}\frac{|f(x)|}{|k|}\right] + \int_{-\infty}^{\infty}\frac{1}{|k|}|f'(x)|\mathrm{dx} \quad \text{……..(i)}$$

Since $f(x)$ is absolutely integrable function then $\lim_{x\to\pm\infty}|f(x)| = 0$

$(i) \Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}.\frac{1}{|k|}\int_{-\infty}^{\infty}|f'(x)|\,\mathrm{dx}$ ……..(ii)

Since $f(x)$ is piecewise smooth then $f'(x)$ will be piecewise continuous and therefore $\int_{-\infty}^{\infty}|f'(x)|\,dx = I$

$(ii) \Rightarrow \lim_{|k|\to\infty}|F\ (k)| \leq \lim_{|k|\to\infty}\frac{1}{\sqrt{2\pi}}.\frac{1}{|k|}.I = 0 \Rightarrow \lim_{|k|\to\infty}|F\ (k)| = 0$

FOURIER TRANSFORM OF THE FUNCTION OF THE FORM $[x^n f(x)]$

Let f be piecewise continuous on the interval $[-l, l]$ for every positive 'l' and $\int_{-\infty}^{\infty}|x^n f(x)|$ converges then

$\mathcal{F}\ [x^n f(x)] = \frac{1}{i^n}F^n(k) = i^{-n}F^n(k)$; $n = 0, 1, 2, \ldots$...............

Proof. By definition $\mathcal{F}\ [f\ (x)] = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f(x)\mathrm{dx}$

$\Rightarrow F'(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ (ix)f(x)\mathrm{dx}$ diff. w.r.to 'k'

$\Rightarrow i^{-1}F'(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ (x)f(x)\mathrm{dx} = \mathcal{F}\ [xf(x)] = i^{-1}F^1(k)$

$\Rightarrow F''(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ (ix)^2 f(x)\mathrm{dx}$ again diff. w.r.to 'k'

$\Rightarrow i^{-2}F''(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ (x^2)f(x)\mathrm{dx} = \mathcal{F}\ [x^2 f(x)] = i^{-2}F^2(k)$

Continuing in this mannar we can get the required result as follows;

$\mathcal{F}\ [x^n f(x)] = i^{-n}F^n(k) = \frac{1}{i^n}F^n(k)$; $n = 0, 1, 2, \ldots$...............

$\mathcal{F}\ [x^n f(x)] = (-i)^n\frac{d^n}{dk^n}F(k)$; $n = 0, 1, 2, \ldots$...............

Where we use the result $i^{-n} = \left(\frac{1}{i}\right)^n = \left(\frac{1}{i}\times\frac{i}{i}\right)^n = \left(\frac{i}{i^2}\right)^n = (-i)^n$

FOURIER TRANSFORM OF AN INTEGRAL

Let f be piecewise continuous on the interval $(-\infty,\infty)$ and that

$\int_{-\infty}^{\infty}|f(x)| < \infty$ **also $F(0) = 0$ with $\mathcal{F}\,[f\,(x)] = F(k)$ then**

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,\mathrm{d}x'\right\} = \frac{1}{-ik}F(k) = \frac{i}{k}F(k)$$

Proof. Let $g(x) = \int_{-\infty}^{x} f(x')\,\mathrm{d}x'$ ……………..(i)

Given that $\mathcal{F}\,[f\,(x)] = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, f(x)\mathrm{dx}$

$\Rightarrow F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ **putting k = 0 also $e^{0} = 1$**

$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{dx} = 0$ **since $F(0) = 0$**

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\mathrm{dx} = 0 \Rightarrow \lim_{x\to\infty}\int_{-\infty}^{x} f(x')\,\mathrm{dx}' = 0 \Rightarrow \lim_{x\to\infty} g(x) = 0$$

Now from (i) we get by using Leibniz Rule

$$g'(x) = f(x') \Rightarrow \mathcal{F}\,\{g'(x)\} = \mathcal{F}\,\{f(x')\} \Rightarrow (-ik)\mathcal{F}\,\{g(x)\} = F(k)$$

$$\Rightarrow \mathcal{F}\,\{g(x)\} = \frac{1}{-i\mathrm{k}}F(k)$$

$$\Rightarrow \mathcal{F}\,\{g(x)\} = \mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,\mathrm{dx}'\right\} = \frac{1}{-ik}F(k) = \frac{i}{\mathrm{k}}F(k)$$

FOURIER INTEGRAL THEOREM

(UoS; 2019 – I, 2014 – II, 2015 – I)

If $f\,(x)$ is real valued function over $(-\infty,+\infty)$ and the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ is absolutely convergent then $f\,(x) = \frac{1}{\pi}\int_{0}^{\infty} dk \int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

PROOF: Since $\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ is absolutely convergent then F.T and I.F.T of function exists.

$f\,(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk}$ **since $\mathcal{F}^{-1}\,\{F\,(k)\} = f\,(x)$**

$$f\,(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk} + \int_{0}^{\infty} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk}\right] \text{…………(i)}$$

Put in 1st term $-k = k' \Rightarrow dk = -dk'$ **also if** $k \to -\infty, 0$ **then** $k' \to \infty, 0$

$$(i) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{ik'x} F(-k')(-dk') + \int_{0}^{\infty} e^{-ikx} F(k)dk\right]$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{ik'x} F(-k')dk' + \int_{0}^{\infty} e^{-ikx} F(k)dk\right]$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{ikx} F(-k)dk + \int_{0}^{\infty} e^{-ikx} F(k)dk\right] \quad \text{replacing } k' \text{ with k}$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\left[e^{ikx}\overline{F(k)} + e^{-ikx}F(k)\right]dk \;\ldots\ldots\ldots\ldots(ii) \qquad \therefore F(-k) = \overline{F(k)}$$

Consider $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')dx'$

$$\Rightarrow \overline{F(k)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'}\, \overline{f(x')}dx' \qquad \text{taking conjugate}$$

Then $e^{-ikx}F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x')} f(x')dx'$

Also $e^{ikx}\overline{F(k)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(x-x')}\, \overline{f(x')}dx'$

Since $f(x)$ **is real therefore** $\overline{f(x')} = f(x')$

Now $e^{ikx}\overline{F(k)} + e^{-ikx}F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{ik(x-x')} + e^{-ik(x-x')}\right] f(x')dx'$

$$e^{ikx}\overline{F(k)} + e^{-ikx}F(k) = \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}\left[e^{ik(x-x')} + e^{-ik(x-x')}\right] f(x')dx'$$

$$e^{ikx}\overline{F(k)} + e^{-ikx}F(k) = \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Cos k(x-x')\, f(x')dx'$$

$$(ii) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Cos k(x-x')\, f(x')dx'\, dk$$

$$f(x) = \frac{2}{2\pi}\int_{0}^{\infty} dk \int_{-\infty}^{\infty} Cos k(x-x') f(x')dx'$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} dk \int_{-\infty}^{\infty} Cos k(x-x') f(x')dx' \qquad \text{as required.}$$

THE FOURIER TRANSFORMS OF STEP AND IMPULSE FUNCTIONS

The Heaviside unit step function is defined by

$$H(x-a) = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases} \qquad \text{where } a \geq 0$$

The Fourier transform of the Heaviside unit step function can be easily determined. We consider first

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} H(x-a)dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{ikx} H(x-a)dx + \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx} H(x-a)dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{ikx}\,.0dx + \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\,.1dx = \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\,dx$$

This integral does not exist. However, we can prove the existence of this integral by defining a new function

$$H(x-a)e^{-\alpha x} = \begin{cases} 0 & x < a \\ e^{-\alpha x} & x \geq a \end{cases}$$

This is evidently the unit step function as $\alpha \to 0$. Thus, we find the Fourier transform of the unit step function as

$$\mathcal{F}[H(x-a)] = \lim_{\alpha\to 0}\mathcal{F}[H(x-a)e^{-\alpha x}]$$

$$\mathcal{F}[H(x-a)] = \lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} H(x-a)e^{-\alpha x}dx$$

$$\mathcal{F}[H(x-a)] = \lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\,e^{-\alpha x}dx = \lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{i(k-\alpha)x}\,dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\,dx = \frac{e^{ika}}{\sqrt{2\pi}ik} \qquad \text{For } a = 0 \Rightarrow \mathcal{F}[H(x)] = \frac{1}{\sqrt{2\pi}ik}$$

An impulse function is defined by

$$p(x) = \begin{cases} h & a-\varepsilon < x < a+\varepsilon \\ 0 & x \leq a-\varepsilon \text{ or } x \geq a+\varepsilon \end{cases}$$

where h is large and positive, $a > 0$, and ε is a small positive constant, This type of function appears in practical applications; for instance, a force of large magnitude may act over a very short period of time.

The Fourier transform of the impulse function is

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, p\,(x\,)dx$$

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a-\varepsilon} e^{ikx}\, p\,(x\,)dx + \frac{1}{\sqrt{2\pi}}\int_{a-\varepsilon}^{a+\varepsilon} e^{ikx}\, p\,(x\,)dx + \frac{1}{\sqrt{2\pi}}\int_{a+\varepsilon}^{\infty} e^{ikx}\, p\,(x\,)dx$$

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{1}{\sqrt{2\pi}}\int_{a-\varepsilon}^{a+\varepsilon} he^{ikx}\, dx = \frac{h}{\sqrt{2\pi}}\left|\frac{e^{ikx}}{ik}\right|_{a-\varepsilon}^{a-\varepsilon}$$

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{h}{\sqrt{2\pi}}\cdot\frac{1}{ik}\left(e^{ik(a+\varepsilon)} - e^{ik(a-\varepsilon)}\right)$$

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{h}{\sqrt{2\pi}}\cdot\frac{e^{ika}}{ik}\left(e^{ik\varepsilon} - e^{-ik\varepsilon}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{ika}\left(\frac{e^{ik\varepsilon}-e^{-ik\varepsilon}}{2ik\varepsilon}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{ika}\left(\frac{Sink\varepsilon}{k\varepsilon}\right)$$

Now if we choose the value of $h = (\frac{1}{2\varepsilon})$ then the impulse defined by

$$I\,(\varepsilon) = \int_{-\infty}^{\infty} p\,(x\,)dx = \int_{a-\varepsilon}^{a+\varepsilon}\frac{1}{2\varepsilon}dx = 1$$

which is a constant independent of ε. In the limit as $\varepsilon \to 0$, this particular function $p_\varepsilon\,(x\,)$ with h = (1/2ε) satisfies $\lim_{\varepsilon\to 0} p_\varepsilon\,(x\,) = 0\ ; x \neq 0$

and $\lim_{\varepsilon\to 0} I\,(\varepsilon) = 1$

Thus, we arrive at the result $\delta\,(x - a) = 0,\ x \neq a$, and $\int_{-\infty}^{\infty}\delta(x-a)dx = 1$

This is the Dirac delta function

We now define the Fourier transform of $\delta(x)$ as the limit of the transform of $p_\varepsilon\,(x\,)$. We then consider

$$\mathcal{F}\,[\delta\,(x - a)\,] = \lim_{\varepsilon\to 0}\mathcal{F}\,[p_\varepsilon\,(x\,)\,] = \lim_{\varepsilon\to 0}\frac{e^{ika}}{\sqrt{2\pi}}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) = \frac{e^{ika}}{\sqrt{2\pi}}$$

in which we note that, by L'Hospital's rule, $\lim_{\varepsilon\to 0}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) = 1$

When a = 0, we obtain $\mathcal{F}\,[\delta\,(x)\,] = \frac{1}{\sqrt{2\pi}}$

FOURIER COSINE TRANSFORMATION AND INVERSE

Let $f\ (x)$ be defined for $0\ \le\ x\ <\ \infty$, and extended as an even function in $(-\infty, \infty)$ satisfying the conditions of Fourier Integral formula. Then, at the points of continuity, the <u>Fourier cosine transform</u> of $f(x)$ and its <u>inverse transform</u> are defined by

$$\mathcal{F}_C\,\{f\ (x)\} = \ F_c\ (k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f\ (x)\,Coskxdx$$

$$\mathcal{F}^{-1}{}_C\,\{F_c\ (k)\} = f\ (x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c\ (k)\,Coskxdk$$

FOURIER SINE TRANSFORMATION AND INVERSE

Let $f\ (x)$ be defined for $0\ \le\ x\ <\ \infty$, and extended as an odd function in $(-\infty, \infty)$ satisfying the conditions of Fourier Integral formula. Then, at the points of continuity, the <u>Fourier sine transform</u> of $f(x)$ and its <u>inverse transform</u> are defined by

$$\mathcal{F}_s\,\{f\ (x)\} = \ F_s\ (k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f\ (x)\,Sinkxdx$$

$$\mathcal{F}^{-1}{}_s\,\{F_s\ (k)\} = f\ (x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s\ (k)\,Sinkxdk$$

Example: (Just read)

Show that $\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right)$; $a > 0$

Solution: We have, by definition

$$\mathcal{F}_C\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, Coskxdx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-(a-ik)x} + e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} + \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right) \quad ; a > 0$$

Example: (Just read)

Show that $\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right)$; $a > 0$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, Sinkxdx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)dx = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-(a-ik)x} - e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} - \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right) \quad ; a > 0$$

Example: (Just read)

Show that $\mathcal{F}_s^{-1}\left\{\frac{1}{k}e^{-sk}\right\} = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right)$

Solution: To prove this we use the standard definite integral

$$\sqrt{\frac{\pi}{2}}\mathcal{F}_s^{-1}\{e^{-sk}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-sk}\, Sinkxdk = \frac{x}{s^2+x^2}$$

Integrating both sides w.r.to 's' from 's' to '∞'

$$\int_0^{\infty}\frac{e^{-sk}}{k}Sinkxdk = \int_s^{\infty}\frac{xds}{s^2+x^2} = \left|tan^{-1}\left(\frac{x}{s}\right)\right|_s^{\infty} = \frac{\pi}{2} - tan^{-1}\left(\frac{x}{s}\right)$$

Consequently

$$\mathcal{F}_s{}^{-1}\left\{\frac{1}{k}e^{-sk}\right\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-sk}}{k}Sinkxdk = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{x}{s}\right)$$

Example: (UoS; Past papers)

Show that $\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(a^2+k^2)^2}$; $a > 0$

Solution: We have, by definition

$$\mathcal{F}_C\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\,Coskxdx$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}xe^{-ax}\,Coskxdx$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\left|x\left(\int e^{-ax}Coskxdx\right)\right|_0^{\infty} - \int_0^{\infty}\left(\int e^{-ax}Coskxdx\right)dx\right] \text{........(i)}$$

Now using formula $\int e^{ax}Cosbxdx = \frac{e^{ax}}{a^2+b^2}[aCosbx + bSinbx]$ **one becomes**

$\mathcal{F}_C\{xe^{-ax}\} =$

$$\sqrt{\frac{2}{\pi}}\left[\left|x.\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^{\infty} - \int_0^{\infty}\left(\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right)dx\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[(0-0) + \frac{a}{a^2+k^2}\int_0^{\infty}e^{-ax}Coskxdx - \frac{k}{a^2+k^2}\int_0^{\infty}e^{-ax}Sinkxdx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^{\infty} - \frac{k}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^{\infty}\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left\{0 - \left(\frac{-a}{a^2+k^2}\right)\right\} - \frac{k}{a^2+k^2}\left\{0 - \left(\frac{-k}{a^2+k^2}\right)\right\}\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a^2}{(a^2+k^2)^2} + \frac{k^2}{(a^2+k^2)^2}\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-k^2}{(a^2+k^2)^2}\right]$$ **as required.**

Example: (UoS; Past papers)

Show that $\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2ak}{(a^2+k^2)^2} \quad ; \ a > 0$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, Sinkxdx$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\, Sinkxdx$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\left|x\left(\int e^{-ax}Sinkxdx\right)\right|_0^\infty - \int_0^\infty\left(\int e^{-ax}Sinkxdx\right)dx\right] \text{........(i)}$$

Now using formula $\int e^{ax}Sinbxdx = \frac{e^{ax}}{a^2+b^2}[aSinbx - bCosbx]$ **one becomes**

$$\mathcal{F}_s\{xe^{-ax}\} =$$

$$\sqrt{\frac{2}{\pi}}\left[\left|x.\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^\infty - \int_0^\infty\left(\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right)dx\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[(0-0) + \frac{a}{a^2+k^2}\int_0^\infty e^{-ax}Sinkxdx + \frac{k}{a^2+k^2}\int_0^\infty e^{-ax}Coskxdx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^\infty + \frac{k}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^\infty\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left\{0 - \left(\frac{-k}{a^2+k^2}\right)\right\} + \frac{k}{a^2+k^2}\left\{0 - \left(\frac{-a}{a^2+k^2}\right)\right\}\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{ak}{(a^2+k^2)^2} + \frac{ak}{(a^2+k^2)^2}\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2ak}{(a^2+k^2)^2} \quad ; \ a > 0$$ **as required.**

Example: (UoS; 2013 – I)

Calculate Fourier Sine Transform of the function $f\,(x) = e^{-x}Cosx$

Solution: We have, by definition

$$\mathcal{F}_s\,\{f\,(x)\} = F_s\,(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f\,(x)\,Sinkxdx$$

$$\mathcal{F}_s\,\{e^{-x}Cosx\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}Cosx\,Sinkxdx = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\,(2SinkxCosx)dx$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\,[Sin(kx+x) + Sin(kx-x)]dx$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\,Sin(k+1)xdx + \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\,Sin(k-1)xdx$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}I_1 + \frac{1}{\sqrt{2\pi}}I_2 \qquad \text{.................(i)}$$

Now using formula $\int e^{ax}Sinbxdx = \frac{e^{ax}}{a^2+b^2}[aSinbx - bCosbx]$

$$I_1 = \int_0^\infty e^{-x}\,Sin(k+1)xdx = \left|\frac{e^{-x}}{(-1)^2+(k+1)^2}[(-1)Sin(k+1)x - (k+1)Cos(k+1)x]\right|_0^\infty$$

$$I_1 = \left[0 - \frac{e^0}{1+(k+1)^2}\{-0-(k+1)(1)\}\right] = \frac{1}{1+k^2+2k+1}(k+1) = \frac{(k+1)}{k^2+2k+2}$$

Similarly

$$I_2 = \int_0^\infty e^{-x}\,Sin(k-1)xdx = \left|\frac{e^{-x}}{(-1)^2+(k-1)^2}[(-1)Sin(k-1)x - (k-1)Cos(k-1)x]\right|_0^\infty$$

$$I_2 = \left[0 - \frac{e^0}{1+(k-1)^2}\{-0-(k-1)(1)\}\right] = \frac{1}{1+k^2-2k+1}(k-1) = \frac{(k-1)}{k^2-2k+2}$$

$$(i) \Rightarrow \mathcal{F}_s\,\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\left[\frac{(k+1)}{k^2+2k+2} + \frac{(k-1)}{k^2-2k+2}\right]$$

$$\Rightarrow \mathcal{F}_s\,\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\left[\frac{2k^3}{k^4+4}\right]$$

$$\Rightarrow \mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{k^3}{k^4+4}\right]$$

Example: (UoS; 2015 – I)

Calculate Fourier Sine Transform of the function $f(x) = \begin{cases} Sinx & 0 \leq x < \pi \\ 0 & x > \pi \end{cases}$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,Sinkxdx$$

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\pi Sinx\,Sinkxdx + \sqrt{\frac{2}{\pi}}\int_\pi^\infty 0.\,Sinkxdx$$

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\pi Sinx\,Sinkxdx = \sqrt{\frac{2}{\pi}}\left(-\frac{1}{2}\right)\int_0^\pi (-2SinxSinkx)\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\int_0^\pi [Cos(kx+x) - Cos(kx-x)]\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\int_0^\pi Cos(k+1)x\,dx + \frac{1}{\sqrt{2\pi}}\int_0^\pi Cos(k-1)x\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\left|\frac{Sin(k+1)x}{k+1}\right|_0^\pi + \frac{1}{\sqrt{2\pi}}\left|\frac{Sin(k-1)x}{k-1}\right|_0^\pi$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left|\frac{Sin(kx-x)}{k-1} - \frac{Sin(kx+x)}{k+1}\right|_0^\pi = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{Sin(k\pi-\pi)}{k-1} - \frac{Sin(k\pi+\pi)}{k+1}\right) - 0\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{Sink\pi Cos\pi - Cosk\pi Sin\pi}{k-1} - \frac{Sink\pi Cos\pi + Cosk\pi Sin\pi}{k+1}\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{-Sink\pi}{k-1} + \frac{Sink\pi}{k+1}\right] \quad \text{since } Cos\pi = -1$$

$$\mathcal{F}_s\{f(x)\} = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{1}{k+1} - \frac{1}{k-1}\right]$$

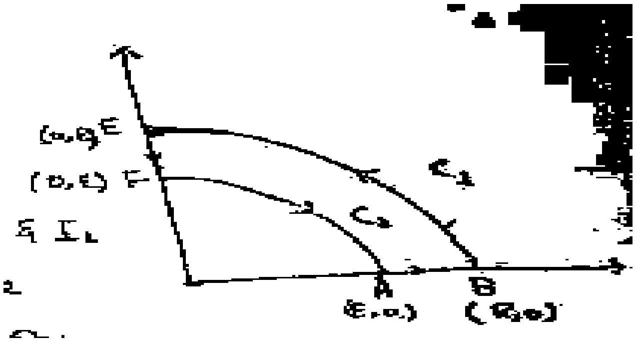
$$\mathcal{F}_s\{f(x)\} = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{k-1-k-1}{(k+1)(k-1)}\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{-2}{k^2-1}\right]$$

$$\mathcal{F}_s\{f(x)\} = -\sqrt{\frac{2}{\pi}}\left[\frac{Sink\pi}{k^2-1}\right]$$

Example: **Evaluate** $\mathcal{F}_c\{x^{\propto-1}\}$ *and* $\mathcal{F}_s\{x^{\propto-1}\}$

Solution:



We have by definition

$$\mathcal{F}_c\{x^{\propto-1}\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1}\,Coskxdx \quad \ldots\ldots\ldots\ldots\ldots.(I_1)$$

$$\mathcal{F}_s\{x^{\propto-1}\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1}\,Sinkxdx \quad \ldots\ldots\ldots\ldots\ldots.(I_2)$$

Firstly we calculate I_1, I_2 **for this we consider the complex valued function**

$f(z) = z^{\propto-1}e^{-kz} \quad ; \; 0 < \propto < 1$

Which is analytic in the closed contour I_1 **then by Cauchy Theorem**

$$\oint_c f(z)dz = 0$$

$$\int_A^B f(z)dz + \int_{c_1} f(z)dz + \int_E^F f(z)dz + \int_{c_2} f(z)dz = 0$$

If $\epsilon \to 0, R \to 0$ **then by Jordan theorem** $\int_{c_1} f(z)dz = 0 \;,\; \int_{c_2} f(z)dz = 0$

$$\int_A^B f(z)dz + \int_E^F f(z)dz = 0$$

$$\int_\epsilon^R x^{\propto-1}e^{-kx}dx + \int_R^\epsilon (iy)^{\propto-1}e^{-k(iy)}(idy) = 0$$

$$\int_0^\infty x^{\propto-1}e^{-kx}dx = -\int_\infty^0 (i)^{\propto-1}(y)^{\propto-1}e^{-k(iy)}(idy)$$

$$\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (i)^{\propto}(y)^{\propto-1}e^{-k(iy)}dy$$

$$(i)^{-\propto}\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}e^{-k(iy)}dy$$

$$\therefore (i)^{-\propto} = \left(Cos\frac{\pi}{2} + iSin\frac{\pi}{2}\right)^{-\propto} = \left(e^{i\frac{\pi}{2}}\right)^{-\propto} = e^{-i\frac{\pi}{2}\propto}$$

$$e^{-i\frac{\pi}{2}\propto}\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}e^{-k(iy)}dy$$

$$\left(Cos\frac{\pi}{2}\propto - iSin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}(Cosky - iSinky)dy$$

Comparing real and imaginary parts

$$\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}(Cosky)dy \qquad \text{............(i)}$$

$$\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}(Sinky)dy \qquad \text{............(ii)}$$

Put x = y in both above

$$(i) \Rightarrow \int_0^\infty x^{\propto-1}(Coskx)dx = \left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx \qquad \text{............(iii)}$$

$$(ii) \Rightarrow \int_0^\infty x^{\propto-1}(Sinkx)dx = \left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx \qquad \text{............(iv)}$$

Multiplying $\sqrt{\frac{2}{\pi}}$ on both sides of (iii)

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1}(Coskx)dx = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty \left(\frac{t}{k}\right)^{\propto-1} e^{-t}\frac{dt}{k} \quad \therefore kx = t, x = \frac{t}{k}$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \frac{1}{k^\propto}\sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty (t)^{\propto-1}e^{-t}dt$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\frac{\Gamma(\propto)}{k^\propto}$$

Multiplying $\sqrt{\frac{2}{\pi}}$ on both sides of (iv)

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1}(Sinkx)dx = \sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \frac{1}{k}\sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty \left(\frac{t}{k}\right)^{\propto-1} e^{-t}dt \quad \therefore kx = t, x = \frac{t}{k}$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \frac{1}{k^\propto}\sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty (t)^{\propto-1}e^{-t}dt$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\frac{\Gamma(\propto)}{k^\propto}$$

Theorem : Let f (x) and its first derivative vanish as x → ∞. If $F_c\,(k)$ is the Fourier cosine transform, then $\mathcal{F}_C\,\{f''\,(x)\} = -k^2 F_c\,(k) - \sqrt{\frac{2}{\pi}} f'\,(0)$

PROOF: Consider $f\,(x)$ is real and $\lim_{x\to\infty}|f\,(x)| = 0$ then

$$\mathcal{F}_C\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f''\,(x)\,Coskxdx$$

$$\mathcal{F}_C\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\left[|Coskxf'\,(x)|_0^{\infty} - \int_0^{\infty} f'\,(x)\,(-kSinkx)dx\right]$$

$$\mathcal{F}_C\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\left[\lim_{x\to\infty}|Coskxf'\,(x)| - \lim_{x\to 0}|Coskxf'\,(x)| + k\int_0^{\infty} f'\,(x)\,Sinkxdx\right]$$

$$\mathcal{F}_C\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\left[0 - f'\,(0) + k\int_0^{\infty} f'\,(x)\,Sinkxdx\right]$$

$$\mathcal{F}_C\,\{f''\,(x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\,(0) + k\left\{\sqrt{\frac{2}{\pi}}|Sinkxf\,(x)|_0^{\infty} - \sqrt{\frac{2}{\pi}}\int_0^{\infty} f\,(x)\,(kCoskx)dx\right\}\right]$$

$$\mathcal{F}_C\,\{f''\,(x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\,(0) + k\left\{\sqrt{\frac{2}{\pi}}|Sinkxf\,(x)|_0^{\infty} - k\sqrt{\frac{2}{\pi}}\int_0^{\infty} f\,(x)\,(Coskx)dx\right\}\right]$$

$$\mathcal{F}_C\,\{f''\,(x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\,(0) + k\left\{\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Sinkxf\,(x)| - \lim_{x\to 0}|Sinkxf\,(x)|) - kF_c\,(k)\right\}\right]$$

$$\mathcal{F}_C\,\{f''\,(x)\} = -k^2 F_c\,(k) - \sqrt{\frac{2}{\pi}} f'\,(0)$$

In a similar manner, the Fourier cosine transforms of higher-order derivatives of f (x) can be obtained.

Theorem : Let f (x) and its first derivative vanish as x → ∞. If $F_s(k)$ is the Fourier cosine transform, then $\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}kf(0) - k^2F_s(k)$

PROOF: Consider $f(x)$ is real and $\lim_{x\to\infty}|f(x)| = 0$ then

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\, Sinkxdx$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[|Sinkxf'(x)|_0^\infty - \int_0^\infty f'(x)\,(kCosx)dx\right]$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[\lim_{x\to\infty}|Sinkxf'(x)| - \lim_{x\to 0}|Sinkxf'(x)| - k\int_0^\infty f'(x)\, Coskxdx\right]$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[0 - 0 - k\int_0^\infty f'(x)\, Coskxdx\right]$$

$$\mathcal{F}_s\{f''(x)\} = -k\left[\sqrt{\frac{2}{\pi}}|Coskxf(x)|_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,(-kSinkx)dx\right]$$

$$\mathcal{F}_s\{f''(x)\} =$$
$$-k\left[\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Coskxf(x)| - \lim_{x\to 0}|Coskxf(x)|) + k\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,(Sinkx)dx\right]$$

$$\mathcal{F}_s\{f''(x)\} = -k\left[\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Coskxf(x)| - \lim_{x\to 0}|Coskxf(x)|) + kF_s(k)\right]$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}kf(0) - k^2F_s(k)$$

In a similar manner, the Fourier sine transforms of higher-order derivatives of $f(x)$ can be obtained.

REMARK:

- $\mathcal{F}[f^n(x)] = (-ik)^n\mathcal{F}[f(x)] = (-ik)^n F(k)\ \ n = 0, 1, 2, \ldots\ldots\ldots$
- **If** $\mathcal{F}\{u_t\} = \mathcal{F}\{u_x\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}\{u(x,t)\} = (-ik)\,\mathcal{F}\{u(x,t)\}$ **when 'x' varies not 't'**
- **When range of spatial variable is infinite then Fourier transform is used rather than the sine or cosine.**
- **If boundry conditions are of the form $u(0,t) = value$ then use Sine transform, while conditions are of the form $u_x(0,t) = value$ then use Cosine transform,**

EXAMPLE: Solve the potential equation for the potential $u(x,y)$ in the semi infinite strip $0 < x < c;\ y > 0$ that satisfies the following conditions;

$u(0,y) = 0\ ; \quad u_y(x,0) = 0\ ;\ u_x(c,y) = f(y)$

Solution: the potential equation is given as $u_{xx} + u_{yy} = 0\ ;\ 0 < x < c;\ y > 0$

Since the BC's are in the form $u_y(x,0) = constant$ **therefor we use fourier cosine transform w.r.to 'y'**

$$\mathcal{F}_C\{u_{xx}\} + \mathcal{F}_C\{u_{yy}\} = 0 \Rightarrow \frac{d^2}{dx^2}\mathcal{F}_C\{u(x,y)\} + \mathcal{F}_C\{u_{yy}\} = 0$$

$$\Rightarrow \frac{d^2}{dx^2}U_C(x,k) + \left[-k^2U_c(x,k) - \sqrt{\frac{2}{\pi}}u_y(x,0)\right] = 0$$

$$\Rightarrow \frac{d^2}{dx^2}U_C(x,k) - k^2U_c(x,k) = 0$$

Then general solution will be $U_c(x,k) = c_1e^{kx} + c_2e^{-kx}$ **………….(i)**

Now using BC's $u(0,y) = 0 \Rightarrow \mathcal{F}_C\{u(0,y)\} = 0 \Rightarrow U_c(0,k) = 0$

$$(i) \Rightarrow U_c(0,k) = 0 = c_1e^0 + c_2e^0 \Rightarrow c_1 = -c_2$$

Now $\frac{d}{dx}U_C(x,k) = c_1ke^{kx} - c_2ke^{-kx}$ **………….(ii)**

using BC's $u_x(c,y) = f(y) \Rightarrow \mathcal{F}_C\{u_x(c,y)\} = f(y) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k)$

$$(ii) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = c_1ke^{kc} - c_2ke^{-kc}$$

$$\Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = -c_2ke^{kc} - c_2ke^{-kc} \qquad \text{since } c_1 = -c_2$$

$$\Rightarrow F_c(k) = -c_2k\left(e^{kc} + e^{-kc}\right) \Rightarrow c_2 = -\frac{F_c(k)}{2k\left(\frac{e^{kc}+e^{-kc}}{2}\right)} = -\frac{F_c(k)}{2kCoshkc}$$

$$\Rightarrow c_2 = -\frac{F_c(k)}{2kCoshkc} \Rightarrow c_1 = \frac{F_c(k)}{2kCoshkc} \qquad \text{since } c_1 = -c_2$$

Then $(i) \Rightarrow U_c(x,k) = \frac{F_c(k)}{2kCoshkc}e^{kx} - \frac{F_c(k)}{2kCoshkc}e^{-kx}$

$$U_c(x,k) = \frac{F_c(k)}{kCoshkc}\left(\frac{e^{kx}-e^{-kx}}{2}\right) = \frac{F_c(k)}{kCoshkc}Sinhkx$$

$$\Rightarrow \mathcal{F}_C^{-1}\{U_c(x,k)\} = \mathcal{F}_C^{-1}\left\{\frac{F_c(k)}{kCoshkc}Sinhkx\right\}$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{F_c(k)}{kCoshkc} Sinhkx\,Coskxdk = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc} F_c(k)dk$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f\,(y')\,Cosky'dy'\right]dk$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^\infty \int_0^\infty \frac{SinhkxCoskxCosky\prime}{kCoshkc} f\,(y')dy'dk$$

EXAMPLE: Solve the problem using Fourier Transformation method
$u_t = u_{xx}$ **with** $u(0,t) = u_0$; $u(x,0) = 0$; $x > 0, t > 0, u_0 > 0$
Solution: BC's suggest that we should use fourier sine transform w.r.to 'x'

$$\mathcal{F}_s\,\{u_t\} = \mathcal{F}_s\,\{u_{xx}\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}_s\,\{u(x,t)\} = \mathcal{F}_s\,\{u_{xx}\}$$

$$\Rightarrow \frac{d}{dt}U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku(0,t) - k^2U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku_0 - k^2U_s\,(k,t)$$

$$\Rightarrow \frac{d}{dt}U_s\,(k,t) + k^2U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku_0 \quad \text{................(i)}$$

This is 1st order, linear, non – homogeneous ODE
Therefore I.F. = $e^{\int k^2dt} = e^{k^2t}$

$$(i) \Rightarrow e^{k^2t}\frac{\partial}{\partial t}U_s\,(k,t) + k^2U_s\,(k,t)e^{k^2t} = \sqrt{\frac{2}{\pi}}ku_0e^{k^2t}$$

$$\Rightarrow \int \frac{d}{dt}e^{k^2t}\,U_sdt = \int \sqrt{\frac{2}{\pi}}ku_0e^{k^2t}dt \;\; + \text{Cosntant}$$

$$\Rightarrow e^{k^2t}U_s = \sqrt{\frac{2}{\pi}}ku_0\frac{e^{k^2t}}{k^2} + c \Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} + ce^{-k^2t} \quad \text{................(ii)}$$

Now using IC's $u(x,0) = 0 \Rightarrow \mathcal{F}_s\{u\,(x,0)\} = 0 \Rightarrow U_s(k,0) = 0$

$$(ii) \Rightarrow U_s(k,0) = 0 = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} + ce^0 \Rightarrow c = -\sqrt{\frac{2}{\pi}}\frac{u_0}{k}$$

Thus $(ii) \Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} - \sqrt{\frac{2}{\pi}}\frac{u_0}{k}e^{-k^2t} = \sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)$

$$\Rightarrow \mathcal{F}_s{}^{-1}\{U_s(k,t)\} = \mathcal{F}_s{}^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)\right\}$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)Sinkxdk = \frac{u_0}{k}\frac{2}{\pi}\int_0^\infty\left(1 - e^{-k^2t}\right)Sinkxdk$$

EXAMPLE: Solve the problem using Fourier Transformation method

$u_t = u_{xx}$ **with** $u_x(0,t) = 0$, $u(x,0) = f(x)$; $0 < x < \infty$, $t > 0$

Solution: BC's suggest that we should use fourier cosine transform w.r.to 'x'

$\mathcal{F}_C\{u_t\} = \mathcal{F}_C\{u_{xx}\} \Rightarrow \frac{d}{dt}\mathcal{F}_C\{u(x,y)\} = \mathcal{F}_C\{u_{xx}\}$

$\Rightarrow \frac{d}{dt}U_C(k,t) = \left[-k^2U_c(k,t) - \sqrt{\frac{2}{\pi}}u_x(0,t)\right] = -k^2U_c(k,t) - 0$

$\Rightarrow \frac{d}{dt}U_C(k,t) + k^2U_c(k,t) = 0$ **…………….(i)**

This is 1st order, linear, homogeneous ODE

Then general solution will be $U_c(k,t) = Ae^{-k^2t}$ **………….(ii)**

Now using IC's

$u(x,0) = f(x) \Rightarrow \mathcal{F}_c\{u(x,0)\} = \mathcal{F}_c\{f(x)\} \Rightarrow U_c(k,0) = F_c(k)$

Thus $(i) \Rightarrow U_c(k,0) = F_c(k) = Ae^0 \Rightarrow A = F_c(k)$

$(i) \Rightarrow U_c(k,t) = F_c(k)e^{-k^2t}$

$\Rightarrow \mathcal{F}_C^{-1}\{U_c(k,t)\} = \mathcal{F}_C^{-1}\{F_c(k)e^{-k^2t}\}$

$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)e^{-k^2t}\,Coskxdk$

$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x')\,Coskx'dx'\right]e^{-k^2t}\,Coskxdk$

$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(x')\,Coskx'dx'\right]e^{-k^2t}\,Coskxdk$

Example: (UoS; 2017) : **Solve the problem using Fourier Transformation method** $u_{xx} = u_t$; $0 < x < \infty$, $t \geq 0$

with $u(x,0) = e^{-ax^2}$; $u(x), u'(x) \to 0$ as $x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$$\mathcal{F}\{u_{xx}\} = \mathcal{F}\{u_t\}$$

$$\Rightarrow (-ik)^2 \mathcal{F}\{u(x,t)\} = \frac{d}{dt}\mathcal{F}\{u(x,t)\} \Rightarrow -k^2 U(k,t) = \frac{d}{dt}U(k,t)$$

$$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -k^2 \Rightarrow \int \frac{dU}{U} = -k^2 \int dt \Rightarrow lnU = -k^2 t + A$$

$$\Rightarrow U(k,t) = e^{-k^2 t + A} \Rightarrow U(k,t) = ce^{-k^2} \quad \text{…………….(i)} \quad \text{where } e^A = c$$

Now using IC's

$$u(x,0) = e^{-ax^2} \Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{e^{-ax^2}\}$$

$$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}.e^{-ax^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-ax^2}dx$$

$$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(x-\frac{ik}{2a}\right)^2 + \frac{k}{4a^2}^2\right]} \, dx$$

$$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2} \, dx$$

Put $a\left(x - \frac{ik}{2a}\right)^2 = P^2 \Rightarrow \sqrt{a}\left(x - \frac{ik}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2} \, dx = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-P^2} \, .\frac{dP}{\sqrt{a}}$$

$$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}}.\sqrt{\pi} \quad \therefore \int_{-\infty}^{\infty} e^{-P^2} \, dP = \sqrt{\pi}$$

$$\Rightarrow U(k,0) = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)} \quad \text{…………….(ii)}$$

$$(i) \Rightarrow U(k,0) = ce^0 \Rightarrow c = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$$

$$\text{Thus} \Rightarrow U(k,t) = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)} e^{-k^2} = \frac{1}{\sqrt{2a}}e^{-k^2\left(t+\frac{1}{4a}\right)}$$

Consider $ikx - ax^2$

$$= -a\left(x^2 - \frac{ikx}{a}\right)$$

$$= -a\left[\left(x^2 - \frac{2ikx}{2a}\right) + \left(\frac{ik}{2a}\right)^2 - \left(\frac{ik}{2a}\right)^2\right]$$

$$= -a\left[\left(x - \frac{ik}{2a}\right)^2 + \frac{k}{4a^2}^2\right]$$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2a}} e^{-k^2\left(t+\frac{1}{4a}\right)}\right\}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2a}}.\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-ikx}.e^{-k^2\left(t+\frac{1}{4a}\right)}\,dk$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left\{k^2 + \frac{ikx}{\left(t+\frac{1}{4a}\right)}\right\}\right] dk \text{ ....(iii)}$$

Since $k^2 + \frac{ikx}{\left(t+\frac{1}{4a}\right)} = k^2 + 2(k)\left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right) + \left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2 - \left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2$

$$k^2 - \frac{ikx}{\left(t+\frac{1}{4a}\right)} = \left(k + \frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2 + \frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}$$

$$(iii) \Rightarrow u(x,t) = \frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left(k + \frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2\right] e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)} dk$$

$$\Rightarrow u(x,t) = \frac{e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left(k + \frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2\right] dk \text{ ....(iv)}$$

Now put $\left(t+\frac{1}{4a}\right)\left(k + \frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2 = m^2 \Rightarrow \sqrt{\left(t+\frac{1}{4a}\right)}\left(k + \frac{ix}{2\left(t+\frac{1}{4a}\right)}\right) = m$

$$\Rightarrow \sqrt{\left(t+\frac{1}{4a}\right)}\,dk = dm \Rightarrow dk = \frac{1}{\sqrt{\left(t+\frac{1}{4a}\right)}}\,dm$$

$$(iv) \Rightarrow u(x,t) = \frac{e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}.\sqrt{\left(t+\frac{1}{4a}\right)}}\int_{-\infty}^{\infty} e^{-m^2}\,dm = \frac{e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}.\frac{1}{\sqrt{4a}}\sqrt{4at+1}}.\sqrt{\pi}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4at+1}} e^{\left(\frac{ax^2}{4at+1}\right)}$$

Example: (UoS; 2017 – II) : **Solve the problem using Fourier Transformation method** $u_t(x,t) = \propto^2 u_{xx}(x,t)\ ;\ -\infty < x < \infty\ , t > 0$ **with** $u_x(x,0) = f(x);\ |u(x,0)| < \infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$$\mathcal{F}\{u_t\} = \propto^2 \mathcal{F}\{u_{xx}\}$$

$$\Rightarrow \frac{d}{dt}\mathcal{F}\{u(x,t)\} = \propto^2 (-ik)^2\mathcal{F}\{u(x,t)\} \Rightarrow \frac{d}{dt}U(k,t) = -\propto^2 k^2 U(k,t)$$

$$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -\propto^2 k^2 \Rightarrow \int \frac{dU}{U} = -\propto^2 k^2 \int dt \Rightarrow lnU = -\propto^2 k^2 t + A$$

$$\Rightarrow U(k,t) = e^{-\propto^2 k^2 t + A} \Rightarrow U(k,t) = ce^{-\propto^2 k^2} \quad \ldots\ldots\ldots\ldots\ldots(\text{i}) \ \text{ where } e^A = c$$

Now using IC's

$$u_x(x,0) = f(x)\ and\ |u(x,0)| < \infty \Rightarrow u(x,0) = f(x)$$

$$\Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} \quad \Rightarrow U(k,0) = F(k)$$

$$(i) \Rightarrow U(k,0) = ce^0 \Rightarrow c = F(k)$$

Thus $(i) \Rightarrow U(k,t) = F(k)e^{-\propto^2 k^2 t}$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\left\{F(k)e^{-\propto^2 k^2}\right\}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}.F(k)e^{-\propto^2 k^2 t}dk$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}f(x')dx'\right]e^{-\propto^2 k^2 t}dk$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t}dk\right]f(x')dx' \ \ldots\ldots\ldots\ldots\ldots(\text{iii})$$

Now consider $I = \int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t}dk$

$I = \int_{-\infty}^{\infty} e^{-iku-\beta k^2}dk$ **put** $x - x' = u$ **and** $\propto^2 t = \beta$

$$I = \int_{-\infty}^{\infty} e^{-\beta\left(k^2+\frac{ik}{\beta}\right)}dk$$

$$I = \int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{iu}{2\beta}\right)^2}.e^{-\frac{u^2}{4\beta}}dk$$

$$I = e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{iu}{2\beta}\right)^2}dk \quad \ldots\ldots\ldots.(\text{iv})$$

Consider $k^2 + \frac{ik}{\beta}u$

$= k^2 + 2k\left(\frac{iu}{2\beta}\right) + \left(\frac{iu}{2\beta}\right)^2 - \left(\frac{iu}{2\beta}\right)^2$

$= \left(k + \frac{iu}{2\beta}\right)^2 - \left(\frac{iu}{2\beta}\right)^2$

$= \left(k + \frac{iu}{2\beta}\right)^2 + \frac{u^2}{4\beta^2}$

Put $\beta\left(k+\frac{\mathrm{i}u}{2\beta}\right)^2=\mathrm{P}^2 \Rightarrow \sqrt{\beta}\left(k+\frac{\mathrm{i}u}{2\beta}\right)=P \Rightarrow \sqrt{\beta}dk=dP \Rightarrow dk=\frac{dP}{\sqrt{\beta}}$

$(iv) \Rightarrow I=e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-\mathrm{P}^2}\,.\frac{dP}{\sqrt{\beta}}=\frac{dP}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-\mathrm{P}^2}\,dp=\frac{1}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}.\sqrt{\pi}$

$(iii) \Rightarrow u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}f(x')dx'$

$\Rightarrow u(x,t)=\frac{1}{2\sqrt{\pi}.\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\propto^2 t}}e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$

$\Rightarrow u(x,t)=\frac{1}{2\sqrt{\pi\propto^2 t}.}\int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$

Example:

Solve the problem using Fourier Transformation method $u_{xxxx}=\frac{1}{a^2}u_{tt}$

with $u(x,0)=f(x); u_t(x,0)=ag'(x)$ **and** $g,u,u_x,u_{xx},u_{xxx}\to 0\ as\ x\to\pm\infty$

Solution: since $x\to\pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\{u_{xxxx}\}=\frac{1}{a^2}\mathcal{F}\{u_{tt}\}$

$\Rightarrow (-ik)^4\mathcal{F}\{u(x,t)\}=\frac{1}{a^2}\frac{d^2}{dt^2}\mathcal{F}\{u(x,t)\}\Rightarrow a^2k^4U(k,t)=\frac{d^2}{dt^2}U(k,t)$

$\Rightarrow \frac{d^2}{dt^2}U-a^2k^4U=0$

$\Rightarrow U(k,t)=Ae^{ak^2t}+Be^{-ak^2t}$ **……………(i)**

$\Rightarrow \frac{d}{dt}U(k,t)=Aak^2e^{ak^2t}-Bak^2e^{-ak^2t}$ **……………(ii)**

Now using IC's $u(x,0)=f(x)\Rightarrow \mathcal{F}\{u\,(x,0)\}=\mathcal{F}\{f(x)\}\Rightarrow U(k,0)=F(k)$
Then $(i)\Rightarrow U(k,0)=Ae^0+Be^0\Rightarrow A+B=F(k)$ **……………(iii)**
Also $u_t(x,0)=ag'(x)\Rightarrow \mathcal{F}\{u_t\,(x,0)\}=\mathcal{F}\{ag'(x)\}$
$\Rightarrow \frac{d}{dt}U(k,0)=a(-ik)^1\mathcal{F}\{g'(x)\}\Rightarrow \frac{d}{dt}U(k,0)=-iakG(k)$
Then $(ii)\Rightarrow \frac{d}{dt}U(k,0)=Aak^2e^0-Bak^2e^0\Rightarrow -iakG(k)=Aak^2-Bak^2$

$\Rightarrow -iG(k)=(A-B)k\Rightarrow A-B=-\frac{i}{k}G(k)$ **……………(iv)**

Adding (iii) and (iv) $A = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]$

Subtracting (iii) and (iv) $B = \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]$

Then (i) becomes

$$\Rightarrow U(k,t) = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]e^{ak^2t} + \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]e^{-ak^2t}$$

$$\Rightarrow U(k,t) = F(k)\left[\frac{e^{ak^2t}+e^{-ak^2t}}{2}\right] - \frac{i}{k}G(k)\left[\frac{e^{ak^2t}-e^{-ak^2t}}{2}\right]$$

$$\Rightarrow U(k,t) = F(k)Coshak^2t - \frac{i}{k}G(k)Sinhak^2t$$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\{F(k)Coshak^2t\} - \mathcal{F}^{-1}\left\{\frac{i}{k}G(k)Sinhak^2t\right\}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-ikx}\,F(k)Coshak^2tdk - \int_{-\infty}^{\infty} e^{-ikx}\,\frac{i}{k}G(k)Sinhak^2tdk\right]$$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,U(k,t)dk$ **is our required solution.**

Example: (UoS; 2017 – I, II) : **Solve the problem using Fourier Transformation method** $u_{xx} = \frac{1}{c^2}u_{tt}$

with $u(x,0) = p(x); u_t(x,0) = q(x)$ **and** $u, u_x \to 0\ as\ x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$$\mathcal{F}\{u_{xx}\} = \frac{1}{c^2}\mathcal{F}\{u_{tt}\}$$

$$\Rightarrow (-ik)^2\mathcal{F}\{u(x,t)\} = \frac{1}{c^2}\frac{d^2}{dt^2}\mathcal{F}\{u(x,t)\} \Rightarrow -c^2k^2U(k,t) = \frac{d^2}{dt^2}U(k,t)$$

$$\Rightarrow \frac{d^2}{dt^2}U + c^2k^2U = 0 \Rightarrow U(k,t) = c_1Cosxkt + c_2Sinckt$$

$$\Rightarrow U(k,t) = c_1\left(\frac{e^{ickt}+e^{-ickt}}{2}\right) + c_2\left(\frac{e^{ickt}-e^{-ickt}}{2}\right)$$

$$\Rightarrow U(k,t) = \left(\frac{c_1+c_1}{2}\right)e^{ickt} + \left(\frac{c_1-c_1}{2}\right)e^{-ickt}$$

$$\Rightarrow U(k,t) = Ae^{ickt} + Be^{-ickt} \quad \ldots\ldots\ldots\ldots\ldots(i)$$

$$\Rightarrow \frac{d}{dt}U(k,t) = Aice^{ickt} - Bice^{-ickt} \quad \ldots\ldots\ldots\ldots\ldots(ii)$$

Now using IC's $u(x,0) = p(x) \Rightarrow \mathcal{F}\{u\,(x,0)\} = \mathcal{F}\{p(x)\} \Rightarrow U(k,0) = P(k)$

Then $(i) \Rightarrow U(k,0) = Ae^{0} + Be^{0} \Rightarrow A + B = P(k)$ **...............(iii)**

Also $u_t(x,0) = q(x) \Rightarrow \mathcal{F}\{u_t\,(x,0)\} = \mathcal{F}\{q(x)\} \Rightarrow \frac{d}{dt}U(k,0) = Q(k)$

Then $(ii) \Rightarrow \frac{d}{dt}U(k,0) = Aicke^{0} - Bicke^{0}$

$\Rightarrow Q(k) = ick(A-B)k \Rightarrow A - B = \frac{1}{ick}Q(k)$ **...............(iv)**

Adding (iii) and (iv) $\quad A = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]$

Subtracting (iii) and (iv) $\quad B = \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]$

Then (i) becomes

$$\Rightarrow U(k,t) = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]e^{ickt} + \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]e^{-ickt}$$

$$\Rightarrow U(k,t) = P(k)\left[\frac{e^{ickt}+e^{-ickt}}{2}\right] + \frac{1}{ick}Q(k)\left[\frac{e^{ickt}-e^{-ickt}}{2}\right]$$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} =$$

$$\frac{1}{2}\left[\mathcal{F}^{-1}\{P(k)e^{ickt}\} + \mathcal{F}^{-1}\{P(k)e^{-ickt}\}\right] + \frac{1}{2ick}\mathcal{F}^{-1}\{Q(k)\left(e^{ickt} - e^{-ickt}\right)\} \quad \textbf{....(A)}$$

$$\mathcal{F}^{-1}\{P(k)e^{ickt}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,P(k)e^{ickt}dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(x-ct)k}\,P(k)dk$$

$$\mathcal{F}^{-1}\{P(k)e^{ickt}\} = P(x-ct)$$

Similarly $\quad \mathcal{F}^{-1}\{P(k)e^{-ickt}\} = P(x+ct)$

And consider $\quad q(x) = \mathcal{F}^{-1}\{Q(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,Q(k)dk$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{x-ct}^{x+ct}\int_{-\infty}^{\infty} e^{-ikx}\,Q(k)dkdx$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{x-ct}^{x+ct} e^{-ikx'}\,dx'Q(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{e^{-ikx'}}{-ik}\right|_{x-ct}^{x+ct} Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{-ik}\left[e^{-ik(x+ct)} - e^{-ik(x-ct)}\right]Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{ik}\left[e^{-ik(x-ct)} - e^{-ik(x+ct)}\right]Q(k)dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[e^{ickt} - e^{-ickt}\right]\frac{Q(k)}{k}dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\mathcal{F}^{-1}\left\{\left(e^{ickt} - e^{-ickt}\right)\frac{Q(k)}{k}dk\right\}$$

$$(A) \Rightarrow u(x,t) = \frac{1}{2}\left[P(x+ct) + P(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} q(x')dx'$$

THE DOUBLE FOURIER TRANSFORM AND ITS INVERSE

Let $f(x_1, x_2)$ be a function defined over the whole plane i.e. $-\infty < x_1, x_2 < \infty$ then its fourier transform and inverse are defined as follows;

$$\mathcal{F}\{f(x_1, x_2)\} = F(k_1, k_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)e^{i(k_1x_1+k_2x_2)}\, dx_1 dx_2$$

$$\mathcal{F}^{-1}\{F(k_1, k_2)\} = f(x_1, x_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1, k_2)e^{-i(k_1x_1+k_2x_2)}\, dk_1 dk_2$$

THREE DIMENSIONAL FOURIER TRANSFORM AND ITS INVERSE

Let $f(x_1, x_2, x_3)$ be a function defined over the whole plane i.e. $-\infty < x_1, x_2, x_3 < \infty$ then its fourier transform and inverse are defined as follows;

$$\mathcal{F}\{f(x_1, x_2, x_3)\} = F(k_1, k_2, k_3) =$$
$$\frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)e^{i(k_1x_1+k_2x_2+k_3x_3)}\, dx_1 dx_2 dx_3$$

$$\mathcal{F}^{-1}\{F(k_1, k_2, k_3)\} = f(x_1, x_2, x_3) =$$
$$\frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1, k_2, k_3)e^{-i(k_1x_1+k_2x_2+k_3x_3)}\, dk_1 dk_2 dk_3$$

n - DIMENSIONAL FOURIER TRANSFORM AND ITS INVERSE

$$\mathcal{F}\{f(\textstyle\sum_{i=1}^{n} x_i)\} = F(\sum_{i=1}^{n} k_i) = \frac{1}{(\sqrt{2\pi})^n}\int_{all\ space} f(\sum_{i=1}^{n} x_i)e^{i(\sum_{i=1}^{n} k_i x_i)}\, d\sum_{i=1}^{n} x_i$$

$$\mathcal{F}^{-1}\{F(\textstyle\sum_{i=1}^{n} k_i)\} = f(\sum_{i=1}^{n} x_i) =$$
$$\frac{1}{(\sqrt{2\pi})^n}\int_{all\ space} F(\textstyle\sum_{i=1}^{n} k_i)e^{-i(\sum_{i=1}^{n} k_i x_i)}\, d\sum_{i=1}^{n} k_i$$

FOURIER SERIES

A trigonometric series with any piecewise continuous periodic function $f\ (x)$ of period 2π and of the form $f\ (x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx\ + b_k \sin kx)$ is called the Fourier Series of a real valued function $f(x)$ where the symbol $\sim$ indicates an association of a_0, a_k, and b_k to f in some unique manner.

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\ \ ,\ \ a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx\ \ ,\ \ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx$$

And are called Fourier Coefficiets.

We may also write $f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx\ + b_k \sin kx)$

COMPLEX FORM OF FOURIER SERIES

Fourier Series expansion for in complex form is given as follows

$f\ (x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}\quad ;\ -\pi < x < \pi$ **Where**

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

OR $f\ (x) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi x}{l}}$ **Where** $c_k = \frac{1}{2l}\int_{-l}^{l} f(y)e^{-i\frac{\pi y}{l}}dy$

Example (just read) :Find the Fourier series expansion for the function f(x) = x+ x^2, $-\pi < x < \pi$

Solultion: Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2\pi^2}{3}$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{4}{k^2}Cosk\pi = \frac{4}{k^2}(-1)^k\ ;\ k = 1,2,3,\dots\dots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = -\frac{2}{k}Cosk\pi = -\frac{2}{k}(-1)^k\ ;\ k = 1,2,3,\dots\dots$$

Therefore, the Fourier series expansion for f is

$$f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx\ + b_k \sin kx)$$

$$f\ (x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(\frac{4}{k^2}(-1)^k \cos kx\ - \frac{2}{k}(-1)^k \sin kx)$$

$$f\ (x) = \frac{\pi^2}{3} -\ 4\cos x + 2sinx\ +\ \cos 2x\ -\ \sin 2x - \dots.$$

Example (just read): Find the Fourier series expansion for the function

$$f\ (x) = \begin{cases} -\pi & ; -\pi\ <\ x\ <\ 0 \\ x & ; 0\ <\ x\ <\ \pi \end{cases}$$

Solultion: Here

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] = -\frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Coskxdx + \int_{0}^{\pi} f(x)Coskxdx\right]$$

$$a_k = \frac{1}{k^2\pi}(Cosk\pi - 1) = \frac{1}{k^2\pi}\left[(-1)^k\ - 1\right]\ ;\ \ k = 1, 2, 3, \dots \dots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Sinkxdx + \int_{0}^{\pi} f(x)Sinkxdx\right]$$

$$b_k = \frac{1}{k}(1 - 2Cosk\pi) = \frac{1}{k}\left[1 - 2(-1)^k\right];\ \ k = 1, 2, 3, \dots \dots$$

Therefore, the Fourier series expansion for f is

$$f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx\ + b_k \sin kx)$$

$$f\ (x) = -\frac{\pi}{4}\ + \sum_{k=1}^{\infty}\left[\frac{1}{k^2\pi}\left[(-1)^k\ - 1\right] \cos kx + \frac{1}{k}\left[1 - 2(-1)^k\right] \sin kx)\right]$$

FOURIER INVERSION FORMULA:

The proper inversion formula is given as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k)\, dk$$

The formula nearly states that f is the fourier transform of $F(k)$

where $F(k) = \mathcal{F}\{f(x)\}$

PROOF:

by Fourier integral theorem $f(x) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

$\Rightarrow f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')dx' \int_0^\infty Cosk(x-x')dk$ **changing the order**

$\Rightarrow f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')dx' . \lim_{m\to\infty}\int_0^m Cosk(x-x')dk$ **.........(i)**

Since $\int_{-m}^{m} Cosk(x'-x)dk = 2\int_0^m Cosk(x-x')dk$ **.........(ii)**

Also $\int_{-m}^{m} Sink(x'-x)dk = 0 \Rightarrow i\int_{-m}^{m} Sink(x-x')dk = 0$ **.........(iii)**

On subtraction from (ii) and (iii) we have

$\int_{-m}^{m}[Cosk(x-x') - iSink(x-x')]dk = 2\int_0^m Cosk(x-x')dk$

$\Rightarrow \int_{-m}^{m} e^{-ik(x-x')}dk = 2\int_0^m Cosk(x-x')dk$

$\Rightarrow \int_0^m Cosk(x-x')dk = \frac{1}{2}\int_{-m}^{m} e^{-ik(x-x')}dk$ **.........(iv)**

Hence from (i) and (iv)

$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')dx' . \lim_{m\to\infty}\int_{-m}^{m} e^{-ik(x-x')}dk$

$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')dx' \int_{-\infty}^{\infty} e^{-ik(x-x')}dk$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, dk . \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')dx'$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\, dk$ **as required.**

GREEN'S FUNCTION AND ASSOCIATED BVP's

THE KRONECKER DELTA FUNCTION:

It is denoted by δ_{ij} and can be defined as follows;

$$\delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

DIRAC DELTA FUNCTION

The dirac delta function is defined as follows;

$$\delta(x) = \lim_{\epsilon\to 0} \delta_\epsilon(x) = \begin{cases} \infty & if\ x = 0 \\ 0 & if\ x \neq 0 \end{cases} \quad \text{Or} \quad \delta(x-t) = \begin{cases} \infty & if\ x = t \\ 0 & if\ x \neq t \end{cases}$$

PROPERTIES:

i. $\int_{-\infty}^{\infty} \delta(x)dx = 1$

ii. **For any continuous function $f(x)$;** $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

iii. $\delta(x) = \delta(-x)$

iv. $\delta(ax) = \frac{1}{a}\delta(x) \quad ; a > 0$

v. **SHIFTING PROPERTY: For any continuous function $f(x)$;**

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

vi. **If $\delta(x)$ is continuous differentiable. Then** $\int_{-\infty}^{\infty} f(x)\delta'(x)dx = -f'(0)$

REMARK:

i. **Dirac delta function can be regarded as the generalization of Kronecker delta function. It strictly speaking a "generalized function" or "distribution function" or " a unit impulse function"**

ii. **In kronecker delta function δ_{ij} the indecis i,j, are integral variables, whereas in passing to direc delta function they become real continuous variables.**

1st SHIFTING PROPERTY OF DIRAC DELTA FUNCTION:

For any continuous function $f(x)$; $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

Where $f(x)$ is analytic (regualar or continuous function) at x = 0

Proof: Since $\delta(x)$ has singularity at x = 0, the limits $-\infty$ and ∞ of the integration may be changed to (or replace by) $0-\epsilon$ and $0+\epsilon$ where ϵ is a small positive number.

Since $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} f(x)\delta(x)dx$

Moreover, since $f(x)$ is continuous at x = 0. We obtain in $\lim\limits_{\epsilon\to 0}$ follow;

$f(0-\epsilon) = f(0+\epsilon) = f(0)$

Therefore $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)\lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} f(x)\delta(x)dx$

since $\delta(x)$ has singularity at x = 0. Therefore

$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0).1 = f(0)$

2nd SHIFTING PROPERTY OF DIRAC DELTA FUNCTION:

(UoS, Past Paper)

For any continuous function $f(x)$; $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

Where $f(x)$ is analytic (regualar or continuous function) at x = a

Proof: Consider $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx$

Set $x-a=t$ and write $f(t+a) = g(t) \Rightarrow f(a) = g(0)$

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(t+a)\delta(t)dt = \int_{-\infty}^{\infty} g(t)\delta(t)dt$

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = g(0)$ **by 1st shifting property**

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$ **by hypothesis**

GREEN's FUNCTION

Green's Function is the impulse response of non – homogeneous differential equation with specified initial and boundry conditions.

IMPORTANCE: it provides an important tool in the study of BVP's. it also have an intrinsic value for mathematicians. Such function is the response corresponding to the source unit.

PROPERTIES OF GREEN's FUNCTION: (UoS; S.Q)

i. **Green's Function is denoted by $G(x,x')$**

ii. **$G(x,x')$ is symmetric i.e. $G(x,x') = G(x',x)$**

iii. **$G(x,x')$ as a function of 'x' satisfies the D Equation $\frac{d^2}{dx^2}G(x,x') = 0$ in each of the interval $0 \le x \le x'$ and $x' < x \le l$**

iv. **$G(0,x') = 0$ and $G(l,x') = 0$ which are the same BC's as those satisfied by u**

v. **$G(x,x')$ is continuous function of 'x' in the interval $[0\,,l\,]$**

(in constructing the Green's function, we will make use of its continuity at $x \;=\; x'$ and this can be seen from the following

$$\lim_{x\to x'_+} G(x,x') = \lim_{x\to x'_-} G(x,x')$$

$$\lim_{x\to x'_+} \frac{x'}{l}(x-l) = \lim_{x\to x'_-} \frac{x}{l}(x'-l)$$

$$\frac{x'}{l}(x'-l) = \frac{x'}{l}(x'-l)$$

vi. **If we calculate $G'(x,x') = \frac{d}{dx}G(x,x')$ we find that**

$$G'(x,x') = \begin{cases} \frac{x'-l}{l} & ;0 \le x \le x' \\ \frac{x'}{l} & ;x' < x \le l \end{cases}$$ **and $G'(x,x')$ will be discontinuous at $x = x'$**

(UoS – 2013) define Green's function and write its properties.

AN IMPORTANT RESULT: $\int_0^x \int_0^{x_2} \varphi(x_1)\, dx_1 dx_2 = \int_0^x \left[\int_{x_1}^x dx_2\right] \varphi(x_1) dx_1$

EXAMPLE: Solve the problem $\frac{d^2u}{dx^2} = f(x)$ **with** $u(0) = 0 = u(l)\ ; 0 \le x \le l$

SOLUTION: This a Singular SL system with $p(x) = 1$

$\frac{d^2u}{dx^2} = f(x) \Rightarrow u''(x) = f(x) \Rightarrow \int_0^x u''(x)dx = \int_0^x f(x)dx$

$\Rightarrow |u'(x)|_0^x = \int_0^x f(x')dx' \Rightarrow u'(x) - u'(0) = \int_0^x f(x')dx'$

$\Rightarrow \int_0^x [u'(x) - u'(0)]\, dx = \int_0^x \left[\int_0^x f(x')dx'\right] dx$

$\Rightarrow \int_0^x u'(x)\, dx - \int_0^x u'(0)\, dx = \int_0^x \int_0^{x''} f(x')dx'\, dx''$

$\Rightarrow |u(x)|_0^x - u'(0)|x|_0^x = \int_0^x \left[\int_{x'}^x dx''\right] f(x')dx'$

$\Rightarrow [u(x) - u(0)] - u'(0)[x - 0] = \int_0^x |x''|_{x'}^x\, f(x')dx'$

$\Rightarrow u(x) - xA = \int_0^x (x - x')\, f(x')dx'$ **where** $u'(0) = A$ **(say) and** $u(0) = 0$

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' + xA$ **………….(i)**

Put $x = l \Rightarrow u(l) = \int_0^l (l - x')\, f(x')dx' + lA$

$\Rightarrow \int_0^l (l - x')\, f(x')dx' + lA = 0$ **since** $u(l) = 0$

$\Rightarrow A = -\frac{1}{l}\int_0^l (l - x')\, f(x')dx'$ **………….(ii) use it in (i)**

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' - \frac{x}{l}\int_0^l (l - x')\, f(x')dx'$

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' + \frac{x}{l}\int_0^l (x' - l)\, f(x')dx'$ **………….(iii)**

This is the solution of given problem.

Now we can costruct a Green's Function by solving (iii)

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' + \frac{x}{l}\left[\int_0^x (x' - l)\, f(x')dx' + \int_x^l (x' - l)\, f(x')dx'\right]$

$\Rightarrow u(x) = \int_0^x \left[x - x' + \frac{x}{l}(x' - l)\right] f(x')dx' + \frac{x}{l}\int_x^l (x' - l)\, f(x')dx'$

$\Rightarrow u(x) = \int_0^x \left[x - x' + \frac{xx'}{l} - x\right] f(x')dx' + \frac{x}{l}\int_x^l (x' - l)\, f(x')dx'$

$\Rightarrow u(x) = \frac{x'}{l}\int_0^x (x-l)\,f(x')dx' + \frac{x}{l}\int_x^l (x'-l)\,f(x')dx'$

$\Rightarrow u(x) = \int_0^x G(x,x')\,f(x')dx'$

Where $G'(x,x') = \begin{cases} \frac{x'}{l}(x-l) & ; 0 \le x' < x \\ \frac{x}{l}(x'-l) & ; x < x' \le l \end{cases}$

is called Green's function of given problem.

EXAMPLE: Solve and obtained the associated Green's Function

$\frac{d^2y}{dx^2} + k^2y = f(x)$ **with** $y(0) = 0 = y(l)\ ; 0 \le x \le l$

SOLUTION: This a linear non – homogeneous DE of order 2 with constant coefficients. Its general solution is as follows;

$y = y_c + y_p$

For Charactristic Solution:

$\frac{d^2y}{dx^2} + k^2y = 0 \Rightarrow D^2 + k^2 = 0 \Rightarrow D = \pm ik \quad \Rightarrow y_c = c_1Coskx + c_2Sinkx$

For Charactristic Solution:

For this we will use Wronskian method (Variation of Parameters)

Let $\Rightarrow y_p = u_1Coskx + u_2Sinkx$

Where $u_1 = -\int_{x_0}^{x} \frac{Sinkxf(x)}{W}dx$ **and** $u_2 = \int_{x_0}^{x} \frac{Coskxf(x)}{W}dx$

$\Rightarrow Wronskian = W = \begin{vmatrix} Coskx & Sinkx \\ -kSinkx & kCoskx \end{vmatrix} = k$

Then $u_1 = -\int_{x_0}^{x} \frac{Sinkx'f(x')}{k}dx'$ **and** $u_2 = \int_{x_0}^{x} \frac{Coskx'f(x')}{W}dx'$

$\Rightarrow y_p = -\int_{x_0}^{x} \frac{Sinkx'f(x')}{k}dx'Coskx + \int_{x_0}^{x} \frac{Coskx'f(x')}{W}dx'Sinkx$

$\Rightarrow y_p = \frac{1}{k}\int_{x_0}^{x} [SinkxCoskx' - CoskxSinkx']f(x')dx'$

$\Rightarrow y_p = \frac{1}{k}\int_{x_0}^{x} Sin(kx - kx')f(x')dx' \Rightarrow y_p = \frac{1}{k}\int_{x_0}^{x} Sink(x-x')f(x')dx'$

$\Rightarrow y_p = \frac{1}{k}\int_{0}^{x} Sink(x-x')f(x')dx'$ **with** $x_0 = 0$ **a fixed point**

Thus for $y = y_c + y_p$ we have

$$y(x) = c_1 Coskx + c_2 Sinkx + \frac{1}{k}\int_0^x Sink(x - x')f(x')dx' \quad \text{.............(i)}$$

$$\Rightarrow y(0) = c_1 Cos(0) + c_2 Sin(0) + \frac{1}{k}\int_0^0 Sink(0 - x')f(x')dx'$$

$$\Rightarrow c_1 = 0 \quad \text{since } y(0) = 0$$

$$\Rightarrow y(l) = c_1 Coskl + c_2 Sinkl + \frac{1}{k}\int_0^l Sink(l - x')f(x')dx' \quad \text{put } x = l$$

$$\Rightarrow 0 = 0 + c_2 Sinkl + \frac{1}{k}\int_0^l Sink(l - x')f(x')dx'$$

$$\Rightarrow c_2 = -\frac{1}{kSinkl}\int_0^l Sink(l - x')f(x')dx'$$

Using c_1, c_2 in (i)

$$y(x) = -\frac{Sinkx}{kSinkl}\int_0^l Sink(l - x')f(x')dx' + \frac{1}{k}\int_0^x Sink(x - x')f(x')dx'$$

$$y(x) = -\frac{Sinkx}{kSinkl}\left[\int_0^x Sink(l - x')f(x')dx' + \int_x^l Sink(l - x')f(x')dx' + \right.$$
$$\left.\frac{1}{k}\int_0^x Sink(x - x')f(x')dx'\right]$$

$$y(x) = \frac{1}{k}\int_0^x \left[Sink(x - x') - \frac{Sinkx}{Sinkl}Sink(l - x')\right] f(x')dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$= \frac{1}{k}\int_0^x \left[Sink(x - x') - \frac{Sinkx}{Sinkl}Sink(l - x')\right]\frac{Sinkl}{Sinkl}.f(x')dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y(x) =$$
$$\frac{1}{k}\int_0^x [(SinkxCoskx' - CoskxSinkx')Sinkl -$$
$$Sinkx(SinklCoskx' - CosklSinkx')]\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y(x) = \frac{1}{k}\int_0^x [SinklSinkxCoskx' - SinklCoskxSinkx' - SinklSinkxCoskx' +$$
$$SinkxCosklSinkx']\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y(x) = \frac{1}{k}\int_0^x [-SinklCoskxSinkx' + SinkxCosklSinkx']\frac{f(x')}{Sinkl}dx' -$$
$$\frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y = \frac{1}{k}\int_0^x [-Sinkx'(SinklCoskx + SinkxCoskl)]\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y=\frac{1}{k}\int_0^x[Sinkx'(SinkxCoskl-SinklCoskx)]\frac{f(x')}{Sinkl}dx'-\frac{Sinkx}{kSinkl}\int_x^l Sink(l-x')f(x')dx'$$

$$y=\frac{1}{k}\int_0^x[Sinkx'Sink(x-l)]\frac{f(x')}{Sinkl}dx'-\frac{Sinkx}{kSinkl}\int_x^l Sink(l-x')f(x')dx'$$

$$y(x)=\int_0^x\frac{Sinkx'Sink(x-l)}{kSinkl}f(x')dx'+\int_x^l\frac{SinkxSink(x'-l)}{kSinkl}f(x')dx'$$

$$y(x)=\int_0^l G(x,x')f(x')dx'$$

Where $G(x,x')=\begin{cases}\dfrac{Sinkx'Sink(x-l)}{kSinkl} & ;0\le x'<x\\[2mm] \dfrac{SinkxSink(x'-l)}{kSinkl} & ;x<x'\le l\end{cases}$

is called Green's function of given problem.

Note: $Sinkl\neq 0$ **i.e. 'k' is not eigenvalue of associated homogeneous problem.**

PROPERTIES OF PREVIOUS GREEN's FUNCTION

i. $G(x,x')$ **is symmetric i.e.** $G(x,x')=G(x',x)$

ii. $G(x,x')$ **as a function of 'x' satisfies the D Equation** $\frac{d^2}{dx^2}G(x,x')=0$ **in each of the interval** $0\le x<x'$ **and** $x'<x\le l$

iii. $G(0,x')=0$ **and** $G(l,x')=0$ **are the same BC's as those satisfied by the given Green's function.**

iv. $G(x,x')$ **is continuous function of 'x' in the interval** $[0,l]$ **and particularly at** $x=x'$

v. $G'(x,x')=\frac{d}{dx}G(x,x')$ **exists as**

$G'(x,x')=\begin{cases}\dfrac{Sinkx'Cosk(x-l)}{kSinkl} & ;0\le x'<x\\[2mm] \dfrac{CoskxSink(x'-l)}{kSinkl} & ;x<x'\le l\end{cases}$ **and** $G'(x,x')$ **will be discontinuous at** $x=x'$

REMEMBER: The Greenn's Function technique is used to solve DE of the form $(L_xu)(x)=f(x)+BC's$ **where** L_x **is a linear operator with specified BC's.**

EXITENCE OF GREEN's FUNCTION:

If the homogeneous problem associated with SL system

$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ **with usual BC's has trivial solution then Green's Function exists.**

In other words, if $\lambda = 0$ is not an eigenvalue for $L(u) + \lambda\, r(x)u = 0$ with usual BC's then Green's Function exists.

GREEN's FUNCTION ASSOCIATED WITH REGULAR SL SYSTEM:

Let $L(u) + \lambda\, r(x)u = 0$ be the SL equation with the endpoint conditions $\propto_1 u(a) + \propto_2 u'(a) = 0$ and $\beta_1 u(b) + \beta_2 u'(b) = 0$ which may also be written as $B_1(u) = \propto_1 + \propto_2 \frac{\partial}{\partial x} = 0$ and $B_2(u) = \beta_1 + \beta_2 \frac{\partial}{\partial x} = 0$ where B is a BC's operatior define regular SL system and gives a trivial solution. Then the Green's Function associated with regular SL system has the following properties;

i. **$G(x,t)$ considered as the function of 'x' satisfies the DE $L\{G(x,t)\} = 0$ in each of the interval $a \le x < t$ and $t < x \le b$**

ii. **$B_1(G) = 0$ and $B_2(G) = 0$ are the same BC's as those satisfied by the given Green's function.**

iii. **$G(x,t)$ is continuous function of 'x' in the interval $[a\,, b\,]$**

iv. **$G'(x,t) = \frac{d}{dx}G(x,t)$ will be discontinuous as $x \to t$ and moreover**

$$\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$$

but $\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

EXAMPLE: Solve the problem associated with non – homogeneous DE

$L(u) + \lambda\, r(x)u = f(x)$ where $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$

SOLUTION: The solution of this non – homogeneous DE subject to BC's is closely related to the existence of Green's function associated with homogeneous equation $L(u) + \lambda\, r(x)u = 0$

If the function $G(x,t,\lambda)$ which does not depends on the source function $f(x)$ exists, then solution of given equation can be written as

$u(x) = \int_a^b G(x,t,\lambda)f(t)dt$ where $G(x,t,\lambda)$ is called Green's function and satisfies the equation $L(G) + \lambda\, r(x)G = \delta(x-t)$

EXAMPLE: (UoS,2013, 2014, 2015, 2019 – I)

Construct Green's function associated with the problem $u'' + \lambda u = 0$ with the boundry conditions $u(0) = 0$ and $u(1) = 0$

Solution: here $p(x) = 1 = p(t)$

i. Put $\lambda = 0$ in given equation

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$ ……….(i)

Now using BC's $u(0) = 0$ and $u(1) = 0$ we have $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ which is trivial solution. So $\lambda = 0$ is not an eigenvalue.

ii. $G(x,t)$ as a function of 'x' satisfies the D Equation $\frac{d^2}{dx^2}G(x,t) = 0$ in each of the interval $0 \le x < t$ and $t < x \le 1$ therefore we have

$$G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + B' & ; t < x \le 1 \end{cases}$$

iii. $G(x,t)$ is continuous function of 'x' in the interval $[0\,,1]$ and particularly at $x = t$ therefore

$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$

$\lim_{x\to t^-}(Ax+B) = \lim_{x\to t^+}(A'x+B')$

$At+B = A't+B' \Rightarrow B' = (A-A')t+B$

Hence $G(x,t) = \begin{cases} Ax+B & ;0\le x<t \\ A'x+(A-A')t+B & ;t<x\le 1 \end{cases}$

iv. $G(0,t)=0$ and $G(1,t)=0$ are the same BC's as those satisfied by the given Green's function.i.e.

$G(0,t)=0 \Rightarrow A(0)+B=0 \Rightarrow B=0$

$G(1,t)=0 \Rightarrow A'(1)+(A-A')t+B=0 \Rightarrow A = \frac{A'(t-1)}{t}$ with $B=0$

Then $G(x,t) = \begin{cases} \frac{A'(t-1)}{t}x+0 & ;0\le x<t \\ A'x+\left(\frac{A'(t-1)}{t}-A'\right)t+0 & ;t<x\le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} \frac{A'(t-1)}{t}x & ;0\le x<t \\ A'x-A' & ;t<x\le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ exists and will be discontinuous as $x\to t$ i.e.

$\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+}\frac{d}{dx}(A'x-A') - \lim_{x\to t^-}\frac{d}{dx}\left(\frac{A'(t-1)}{t}x\right) = \frac{1}{1}$

$\lim_{x\to t^+}(A') - \lim_{x\to t^-}\left(\frac{A'(t-1)}{t}\right) = 1$

$A' - \frac{A'(t-1)}{t} = 1$

$A'\left(\frac{1}{t}\right) = 1 \Rightarrow A' = t$

Then $G(x,t) = \begin{cases} \frac{t(t-1)}{t}x & ;0\le x<t \\ tx-t & ;t<x\le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} (t-1)x & ;0\le x<t \\ (x-1)t & ;t<x\le 1 \end{cases}$

This is our required Green's Function.

EXAMPLE: (UoS,2013, 2014, 2015)

Construct Green's function associated with the problem $xu'' + u' + \lambda ru = 0$ with the boundry conditions $u(0)$ is finite and $u(1) = 0$

Solution: here $p(x) = x$ then $p(t) = t$

i. Put $\lambda = 0$ in given equation

$xu'' + u' + \lambda ru = 0 \Rightarrow xu'' + u' = 0 \Rightarrow \frac{d}{dx}(xu')$

$\Rightarrow u(x) = Alnx + B$ ……….(i)

Now using BC's $u(0) = finite$ and $u(1) = 0$ we have $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ which is trivial solution. So $\lambda = 0$ is not an eigenvalue.

ii. $G(x,t)$ as a function of 'x' satisfies the D Equation $xG'' + G' = 0$ in each of the interval $0 \le x < t$ and $t < x \le 1$ therefore we have

$$G(x,t) = \begin{cases} Alnx + B & ; 0 \le x < t \\ A'lnx + B' & ; t < x \le 1 \end{cases}$$

iii. $G(x,t)$ is continuous function of 'x' in the interval $[0\,,1]$ and particularly at $x = t$ therefore

$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$

$\lim_{x\to t^-}(Alnx + B) = \lim_{x\to t^+}(A'lnx + B')$

$Alnt + B = A'lnt + B' \Rightarrow B' = (A - A')lnt + B$

Hence $G(x,t) = \begin{cases} Alnx + B & ; 0 \le x < t \\ A'lnx + (A - A')lnt + B & ; t < x \le 1 \end{cases}$

iv. $G(0,t) = finite$ and $G(1,t) = 0$ are the same BC's as those satisfied by the given Green's function.i.e.

$G(0,t) = finite \Rightarrow Aln(0) + B = finite \Rightarrow A = 0$

$G(1,t) = 0 \Rightarrow A'ln(1) + (A - A')lnt + B = 0 \Rightarrow B = A'lnt$

$\Rightarrow A' = \frac{B}{lnt}$ with $A = 0, ln(1) = 0$

Then $G(x,t) = \begin{cases} A'lnt & ; 0 \le x < t \\ A'lnx & ; t < x \le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ exists and will be discontinuous as $x \to t$ i.e.

$\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+} A'\left(\frac{1}{x}\right) - \lim_{x\to t^-}(0) = \frac{1}{t}$

$A'\left(\frac{1}{t}\right) = \frac{1}{t} \Rightarrow A' = 1$

Then $G(x,t) = \begin{cases} lnt & ; 0 \le x < t \\ lnx & ; t < x \le 1 \end{cases}$ is our required Green's Function.

EXAMPLE: (UoS,2018 – II) **Construct Green's function associated with the problem $xu'' + u' - \frac{n^2}{x}u + \lambda ru = 0$ with the boundry conditions $u(0)$ is finite and $u(1) = 0$**

Solution: here $p(x) = x$ then $p(t) = t$ this is regular system with $q(x) = -\frac{n^2}{x}$

i. Put $\lambda = 0$ in given equation

$xu'' + u' - \frac{n^2}{x}u + (0)ru = 0 \Rightarrow xu'' + u' - \frac{n^2}{x}u = 0$

$\Rightarrow \left(xD^2 + D - \frac{n^2}{x}\right)u = 0$

$\Rightarrow (x^2D^2 + xD - n^2)u = 0$ ……….(i) this is Cauchy Euler equation

Put $x = e^t \Rightarrow lnx = t \Rightarrow xD = \Delta$ and $x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$(i) \Rightarrow (\Delta^2 - \Delta + \Delta - n^2)u = 0 \Rightarrow (\Delta^2 - n^2)u = 0 \Rightarrow \Delta = \pm n$

$\Rightarrow u(x) = Ae^{nt} + Be^{-nt} = A(e^t)^n + B(e^t)^{-n}$

$\Rightarrow u(x) = Ax^n + Bx^{-n}$ ……….(ii)

Now using BC's $u(0) = finite$ and $u(1) = 0$ we have $A = 0, B = 0$

$(ii) \Rightarrow u(x) = 0$ which is trivial solution. So $\lambda = 0$ is not an eigenvalue.

ii. $G(x,t)$ as a function of 'x' satisfies the Differential Equation $x^2G'' + xG' - n^2G = 0$ in each of the interval $0 \le x < t$ and $t < x \le 1$ therefore we have

$$G(x,t) = \begin{cases} Ax^n + Bx^{-n} & ; 0 \le x < t \\ A'x^n + B'x^{-n} & ; t < x \le 1 \end{cases}$$

iii. $G(x,t)$ is continuous function of 'x' in the interval $[0\,,1]$ and particularly at $x = t$ therefore

$$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$$

$$\lim_{x\to t^-}(Ax^n + Bx^{-n}) = \lim_{x\to t^+}(A'x^n + B'x^{-n})$$

$$At^n + Bt^{-n} = A't^n + B't^{-n} \Rightarrow B' = (A - A')t^{2n} + B$$

Hence $G(x,t) = \begin{cases} Ax^n + Bx^{-n} & ; 0 \le x < t \\ A'x^n + (A - A')t^{2n}x^{-n} + Bx^{-n} & ; t < x \le 1 \end{cases}$

iv. $G(0,t) = 0$ and $G(1,t) = 0$ are the same BC's as those satisfied by the given Green's function.i.e.

$$G(0,t) = finite \Rightarrow A(0)^n + B(0)^{-n} = finite \Rightarrow B = 0$$

$$G(1,t) = 0 \Rightarrow A'(1)^n + (A - A')t^{2n}(1)^{-n} + (0)(1)^{-n}$$

$$\Rightarrow A = A'(1 - t^{-2n}) \text{ with } B = 0$$

Then

$$G(x,t) = \begin{cases} A'(1 - t^{-2n})x^n + (0)x^{-n} & ; 0 \le x < t \\ A'x^n + \left(\left(A'(1 - t^{-2n})\right) - A'\right)t^{2n}x^{-n} + (0)x^{-n} & ; t < x \le 1 \end{cases}$$

$$G(x,t) = \begin{cases} A'(1 - t^{-2n})x^n & ; 0 \le x < t \\ A'x^n + A'x^{-n} & ; t < x \le 1 \end{cases}$$

Hence $G(x,t) = \begin{cases} A'(1 - t^{-2n})x^n & ; 0 \le x < t \\ A'(x^n - x^{-n}) & ; t < x \le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ exists and will be discontinuous as $x \to t$ i.e.

$$\lim_{x\to t^+} G'(x,t) \ne \lim_{x\to t^-} G'(x,t)$$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$$\lim_{x\to t^+} A'(nx^{n-1} - nx^{-n-1}) - \lim_{x\to t^-} nA'(0 - t^{-2n})x^{n-1} = \frac{1}{t}$$

$A'(nt^{n-1} - nt^{-n-1}) - nA'(-t^{-2n})t^{n-1} = \frac{1}{t} \quad \Rightarrow A' = \frac{t^n}{n(2+t^{2n})}$ **after solving**

Then $G(x,t) = \begin{cases} \frac{t^n}{n(2+t^{2n})}(1 - t^{-2n})x^n & ; 0 \le x < t \\ \frac{t^n}{n(2+t^{2n})}(x^n - x^{-n}) & ; t < x \le 1 \end{cases}$

is our required Green's Function.

EXAMPLE: (UoS, 2009, 2011) **Construct Green's function associated with the problem** $\frac{d}{dx}\{(1 - x^2)u'\} - \frac{h^2}{1-x^2}u + \lambda ru = 0$ **with the boundry conditions** $u(\pm 1)$ **are finite**

Solution:here $p(x) = 1 - x^2$ **then** $p(t) = 1 - t^2$ **this is singular system**

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d}{dx}\{(1 - x^2)u'\} - \frac{h^2}{1-x^2}u = 0 \Rightarrow (1 - x^2)u'' - 2xu' - \frac{h^2}{1-x^2}u = 0$

$\Rightarrow (1 - x^2)^2u'' - 2x(1 - x^2)u' - h^2u = 0$ **……….(i)**

Put $t = ln\left(\frac{1+x}{1-x}\right) = ln(1 + x) - ln(1 - x) \Rightarrow \frac{dt}{dx} = \frac{2}{1-x^2}$

$u' = \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \frac{2}{1-x^2}\frac{du}{dt} \Rightarrow u'' = \frac{4}{(1-x^2)^2}\left[\frac{d^2u}{dt^2} + x\frac{du}{dt}\right]$ **after solving**

$(i) \Rightarrow (1 - x^2)^2\frac{4}{(1-x^2)^2}\left[\frac{d^2u}{dt^2} + x\frac{du}{dt}\right] - 2x(1 - x^2)\frac{2}{1-x^2}\frac{du}{dt} - h^2u = 0$

$\Rightarrow 4\frac{d^2u}{dt^2} + 4x\frac{du}{dt} - 4x\frac{du}{dt} - h^2u = 0 \Rightarrow \frac{d^2u}{dt^2} - \frac{h^2}{4}u = 0 \Rightarrow D = \pm\frac{h}{2}$

$\Rightarrow u(x) = Ae^{\frac{h}{2}t} + Be^{-\frac{h}{2}t} = A(e^t)^{h/2} + B(e^t)^{-h/2}$

$\Rightarrow u(x) = A\left(\frac{1+x}{1-x}\right)^{h/2} + B\left(\frac{1+x}{1-x}\right)^{-h/2}$ **……….(ii)**

Now using BC's $u(\pm 1) = finite$ **we have** $A = 0, B = 0$

$(ii) \Rightarrow u(x) = 0$ **which is trivial solution. So** $\lambda = 0$ **is not an eigenvalue.**

ii. $G(x,t)$ **as a function of 'x' satisfies the Differential Equation** $\frac{d}{dx}\{(1-x^2)G'\} - \frac{h^2}{1-x^2}G = 0$ **in each of the interval** $0 \le x < t$ **and** $t < x \le 1$ **therefore we have**

$$G(x,t) = \begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2} + B\left(\frac{1+x}{1-x}\right)^{-h/2} & ; -1 \le x < t \\ A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}} + B'\left(\frac{1+x}{1-x}\right)^{-h/2} & ; t < x \le 1 \end{cases}$$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\,,1]$ **and particularly at** $x = t$ **therefore**

$$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$$

$$\lim_{x\to t^-}\left(A\left(\frac{1+x}{1-x}\right)^{h/2} + B\left(\frac{1+x}{1-x}\right)^{-h/2}\right) = \lim_{x\to t^+}\left(A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}} + B'\left(\frac{1+x}{1-x}\right)^{-h/2}\right)$$

$$A\left(\frac{1+t}{1-t}\right)^{h/2} + B\left(\frac{1+t}{1-t}\right)^{-h/2} = A'\left(\frac{1+t}{1-t}\right)^{\frac{h}{2}} + B'\left(\frac{1+t}{1-t}\right)^{-h/2}$$

$$\Rightarrow B' = (A - A')\left(\frac{1+t}{1-t}\right)^{h} + B$$

then

$$G(x,t) = \begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2} + B\left(\frac{1+x}{1-x}\right)^{-h/2} & ; -1 \le x < t \\ A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}} + \left[(A - A')\left(\frac{1+t}{1-t}\right)^{h} + B\right]\left(\frac{1+x}{1-x}\right)^{-h/2} & ; t < x \le 1 \end{cases}$$

$$G(x,t) = \begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2} + B\left(\frac{1+x}{1-x}\right)^{-h/2} & ; -1 \le x < t \\ A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}} + \left[(A - A')\left(\frac{1+t}{1-t}\right)^{h} + B\right]\left(\frac{1+x}{1-x}\right)^{-h/2} & ; t < x \le 1 \end{cases}$$

iv. $G(\pm,t) = finite$ **are the BC's satisfied by the Green's function.**

$$G(-1,t) = finite \Rightarrow A\left(\frac{1+(-1)}{1-(-1)}\right)^{h/2} + B\left(\frac{1+(-1)}{1-(-1)}\right)^{-h/2} = finite \Rightarrow B = 0$$

$$G(1,t) = finite \Rightarrow A'\left(\frac{1+(1)}{1-(1)}\right)^{\frac{h}{2}} + \left[(A - A')\left(\frac{1+t}{1-t}\right)^{h} + B\right]\left(\frac{1+(1)}{1-(1)}\right)^{-\frac{h}{2}} = finite$$

$$\Rightarrow A' = 0$$

Then $\quad G(x,t) = \begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2} & ; -1 \le x < t \\ A\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{-h/2} & ; t < x \le 1 \end{cases}$

v. $\quad G'(x,t) = \frac{d}{dx}G(x,t)$ exists and will be discontinuous as $x \to t$ i.e.

$\lim_{x\to t^+} G'(x,t) \ne \lim_{x\to t^-} G'(x,t)$

But $\quad \lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$$\lim_{x\to t^+}\left(-A\frac{h}{2}\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{-\frac{h}{2}-1}\left[\frac{2}{(1-x)^2}\right]\right) - \lim_{x\to t^-}\left(A\frac{h}{2}\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}-1}\left[\frac{2}{(1-x)^2}\right]\right) = \frac{1}{1-t^2}$$

$$-A\frac{h}{2}\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+t}{1-t}\right)^{-\frac{h}{2}-1}\left[\frac{2}{(1-t)^2}\right] - A\frac{h}{2}\left(\frac{1+t}{1-t}\right)^{\frac{h}{2}-1}\left[\frac{2}{(1-t)^2}\right] = \frac{1}{1-t^2}$$

$\Rightarrow A = -\frac{1}{2h}\left(\frac{1-t}{1+t}\right)^{h/2}$ $\qquad$ after solving

Then $\; G(x,t) = \begin{cases} -\frac{1}{2h}\left(\frac{1-t}{1+t}\right)^{h/2}\left(\frac{1+x}{1-x}\right)^{h/2} & ; -1 \le x < t \\ -\frac{1}{2h}\left(\frac{1-t}{1+t}\right)^{h/2}\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{-h/2} & ; t < x \le 1 \end{cases}$

is our required Green's Function.

EXAMPLE: (UoS, 2017, 2018 – I)

Construct Green's function associated with the problem $u'' + \lambda u = 0$ with the boundry conditions $u(0) + u'(1) = 0$ and $u(1) + 2u'(0) = 0$

Solution: $\quad$ here $p(x) = 1 = p(t)$

i. $\quad$ Put $\lambda = 0$ in given equation

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$ ……….(i)

Now using BC's

$u(0) + u'(1) = 0$ and $u(1) + 2u'(0) = 0$ we have $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ which is trivial solution. So $\lambda = 0$ is not an eigenvalue.

ii. $G(x,t)$ as a function of 'x' satisfies the D Equation $\frac{d^2}{dx^2}G(x,t) = 0$ in each of the interval $0 \le x < t$ and $t < x \le 1$ therefore we have

$$G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + B' & ; t < x \le 1 \end{cases}$$

iii. $G(x,t)$ is continuous function of 'x' in the interval $[0\,,1]$ and particularly at $x = t$ therefore

$$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$$

$$\lim_{x\to t^-}(Ax + B) = \lim_{x\to t^+}(A'x + B')$$

$$At + B = A't + B' \Rightarrow B' = (A - A')t + B$$

Hence $G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + (A - A')t + B & ; t < x \le 1 \end{cases}$

iv. $G(x,t)$ satisfies the BC's

$$G(0,t) + G'(1,t) = 0 \Rightarrow A(0) + B + A' = 0 \Rightarrow B = -A'$$

$$G(1,t) + 2G'(0,t) = 0 \Rightarrow A'(1) + (A - A')t + B + 2A = 0$$

$$\Rightarrow A = A'\left(\frac{t}{t+2}\right) \qquad \text{with } B = -A'$$

Then $G(x,t) = \begin{cases} A'\left(\frac{t}{t+2}\right)x - A' & ; 0 \le x < t \\ A'x + \left(A'\left(\frac{t}{t+2}\right) - A'\right)t - A' & ; t < x \le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} A'\left(\frac{t}{t+2}x - 1\right) & ; 0 \le x < t \\ A'\left[x + \left(\frac{t}{t+2} - 1\right)t - 1\right] & ; t < x \le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ exists and will be discontinuous as $x \to t$ i.e.

$$\lim_{x\to t^+} G'(x,t) \ne \lim_{x\to t^-} G'(x,t)$$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$$\lim_{x\to t^+}(A') - \lim_{x\to t^-} A'\left(\frac{t}{t+2}\right) = \frac{1}{1}$$

$$A' - A'\left(\frac{t}{t+2}\right) = 1 \qquad \Rightarrow A' = \frac{t+2}{2}$$

Then $G(x,t) = \begin{cases} \frac{t+2}{2}\left(\frac{t}{t+2}x - 1\right) & ; 0 \le x < t \\ \frac{t+2}{2}\left[x + \left(\frac{t}{t+2} - 1\right)t - 1\right] & ; t < x \le 1 \end{cases}$

$\Rightarrow G(x,t) = \begin{cases} \frac{t}{2}x - \frac{t+2}{2} & ; 0 \le x < t \\ \frac{t+2}{2}x + \frac{t^2}{2} - \frac{t(t+2)}{2} - \frac{t+2}{2} & ; t < x \le 1 \end{cases}$

Hence $\Rightarrow G(x,t) = \begin{cases} \frac{tx-t-2}{2} & ; 0 \le x < t \\ \frac{(t+2)x-3t-2}{2} & ; t < x \le 1 \end{cases}$ required Green's Function.

EXAMPLE: (UoS, 2017 – II)

Construct Green's function associated with the problem $u'' + \lambda u = 0$ with the boundry conditions $u'(0) = 0$ and $u(1) = 0$

Solution: here $p(x) = 1 = p(t)$

i. Put $\lambda = 0$ in given equation

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$ ……….(i)

Now using BC's $u'(0) = 0$ and $u(1) = 0$ we have $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ which is trivial solution. So $\lambda = 0$ is not an eigenvalue.

ii. $G(x,t)$ as a function of 'x' satisfies the D Equation $\frac{d^2}{dx^2}G(x,t) = 0$ in each of the interval $0 \le x < t$ and $t < x \le 1$ therefore we have

$G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + B' & ; t < x \le 1 \end{cases}$

iii. $G(x,t)$ is continuous function of 'x' in the interval $[0\,,1]$ and particularly at $x = t$ therefore

$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$

$\lim_{x\to t^-}(Ax + B) = \lim_{x\to t^+}(A'x + B')$

$At + B = A't + B' \Rightarrow B' = (A - A')t + B$

Hence $G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + (A - A')t + B & ; t < x \le 1 \end{cases}$

iv. $G(0,t)=0$ and $G(1,t)=0$ are the same BC's as those satisfied by the given Green's function.i.e.

$G'(0,t)=0 \Rightarrow A=0$

$G(1,t)=0 \Rightarrow B=A'(t-1) \qquad$ with $A=0$

Then $G(x,t)=\begin{cases} A'(t-1) & ;0\le x<t \\ A'x+(0-A')t+A'(t-1) & ;t<x\le 1 \end{cases}$

Hence $G(x,t)=\begin{cases} A'(t-1) & ;0\le x<t \\ A'(x-1) & ;t<x\le 1 \end{cases}$

v. $G'(x,t)=\frac{d}{dx}G(x,t)$ exists and will be discontinuous as $x\to t$ i.e.

$\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

But $\quad \lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+}(A') - \lim_{x\to t^-}(0) = \frac{1}{1} \Rightarrow A'=1$

Hence $G(x,t)=\begin{cases} (t-1) & ;0\le x<t \\ (x-1) & ;t<x\le 1 \end{cases}$ required Green's Function.

EXAMPLE: (UoS, 2015 – I)

Construct Green's function associated with the problem $u''+\lambda u=0$ with the boundry conditions $u'(0)=0$ and $u(2)=0$

Solution: here $p(x)=1=p(t)$

i. Put $\lambda=0$ in given equation

$\frac{d^2u}{dx^2}+\lambda u=0 \Rightarrow \frac{d^2u}{dx^2}+(0)u=0 \Rightarrow \frac{d^2u}{dx^2}=0 \Rightarrow u(x)=Ax+B$ ……….(i)

Now using BC's $u'(0)=0$ and $u(2)=0$ we have $A=0, B=0$

$(i) \Rightarrow u(x)=0$ which is trivial solution. So $\lambda=0$ is not an eigenvalue.

ii. $G(x,t)$ as a function of 'x' satisfies the D Equation $\frac{d^2}{dx^2}G(x,t)=0$ in each of the interval $0\le x<t$ and $t<x\le 1$ therefore we have

$G(x,t)=\begin{cases} Ax+B & ;0\le x<t \\ A'x+B' & ;t<x\le 2 \end{cases}$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\,,2]$ **and particularly at** $x = t$ **therefore**

$$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$$

$$\lim_{x\to t^-}(Ax+B) = \lim_{x\to t^+}(A'x+B')$$

$$At+B = A't+B' \Rightarrow B' = (A-A')t+B$$

Hence $G(x,t) = \begin{cases} Ax+B & ;0\le x<t \\ A'x+(A-A')t+B & ;t<x\le 2 \end{cases}$

iv. $G(0,t)=0$ **and** $G(2,t)=0$ **are the same BC's as those satisfied by the given Green's function.i.e.**

$$G'(0,t)=0 \Rightarrow A=0$$

$$G(2,t)=0 \Rightarrow A'(2)+(0-A')t+B=0 \Rightarrow A'(2-t)+B=0$$

$$\Rightarrow B = A'(t-2) \qquad \text{with } A=0$$

Then $G(x,t) = \begin{cases} A'(t-2) & ;0\le x<t \\ A'x+(0-A')t+A'(t-2) & ;t<x\le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} A'(t-2) & ;0\le x<t \\ A'(x-2) & ;t<x\le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x\to t$ **i.e.**

$$\lim_{x\to t^+} G'(x,t) \ne \lim_{x\to t^-} G'(x,t)$$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$$\lim_{x\to t^+}(A') - \lim_{x\to t^-}(0) = \frac{1}{1} \Rightarrow A' = 1$$

Hence $G(x,t) = \begin{cases} (t-2) & ;0\le x<t \\ (x-2) & ;t<x\le 1 \end{cases}$ **required Green's Function.**

MODIFIED GREEN's FUNCTION

When $\lambda = 0$ is an eigenvalue of the SL system defined by $L(u) + \lambda r u = 0$ with $\beta_1(u) = 0, \beta_2(u) = 0$ then the associated Green's function is called modified green's function. And is denoted by $G_M(x,t)$

PROPERTIES OF MODIFIED GREEN's FUNCTION: (UoS; S.Q)

Let $u_0(x)$ be the normalized eigenfunction corresponding to $\lambda = 0$ this means that $\langle u_0(x), u_0(x)\rangle = \int_a^b u_0(x).u_0(x)dx = 1$ then $G_M(x,t)$ will have the following properties;

i. **$G_M(x,t)$ satisfies the D Equation $L[G_M(x,t)] = u_0(t).u_0(t)$ in each of the interval $a \leq x \leq t$ and $t < x \leq b$**
ii. **$\beta_1[G_M(x,t)] = 0$ and $\beta_2[G_M(x,t)] = 0$ which are the same BC's as those satisfied by $G_M(x,t)$**
iii. **$G_M(x,t)$ is continuous function of 'x' in the interval $[a, b\,]$ and particularly at x = t**
iv. **$G'_M(x,t) = \frac{d}{dx}G_M(x,t)$ exists and will be discontinuous as $x \to t$ i.e.**
 $\lim_{x\to t^+} G'_M(x,t) \neq \lim_{x\to t^-} G'_M(x,t)$
 But $\lim_{x\to t^+} G'_M(x,t) - \lim_{x\to t^-} G'_M(x,t) = \frac{1}{p(t)}$
v. **The modified Green's function $G_M(x,t)$ satisfies the orthogonality condition $\int_a^b G_M(x,t).u_0(x)dx = 0$**

EXAMPLE:Construct Green's function associated with the problem
$u'' + \lambda r u = 0$ **with the boundry conditions** $u'(0) = 0$ **and** $u'(1) = 0$

Solution: **here** $p(x) = 1 = p(t)$

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$

$\Rightarrow u'(x) = A$ **……….(i)**

Now using BC's $u'(0) = 0$ **and** $u'(1) = 0$ **we have** $A = 0, B \neq 0$

$(i) \Rightarrow u(x) = B$ **which is non - trivial solution. So** $\lambda = 0$ **is an eigenvalue.**

Therefore we take $u_0(x) = 1$ **as a normalized function.i.e.**

$\langle u_0(x), u_0(x)\rangle = \int_a^b u_0(x).u_0(x)dx = \int_0^1 1dx = 1$

ii. $G_M(x,t)$ **as a function of 'x' satisfies the D Equation**

$\frac{d^2}{dx^2} G_M(x,t) = u_0(x)u_0(t) = 1$ **in each of the interval** $0 \leq x < t$ **and** $t < x \leq 1$ **therefore we have** $G''_M(x,t) = 1 \Rightarrow G'_M(x,t) = x + A$

$\Rightarrow G_M(x,t) = \frac{x^2}{2} + Ax + B$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & ; 0 \leq x < t \\ \frac{x^2}{2} + A'x + B' & ; t < x \leq 1 \end{cases}$$

iii. $G_M(x,t)$ **satisfies the BC's i.e.** $\Rightarrow G'_M(0,t) = 0 \Rightarrow A = 0$ **and**

$\Rightarrow G'_M(1,t) = 0 \Rightarrow A' = -1$

$$\text{thus } \Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + B & ; 0 \leq x < t \\ \frac{x^2}{2} - x + B' & ; t < x \leq 1 \end{cases}$$

iv. $G_M(x,t)$ **is continuous function of 'x' in the interval** $[0,1]$ **and particularly at x = t i.e.**

$\lim_{x \to t^+} G_M(x,t) = \lim_{x \to t^-} G_M(x,t)$

$$\lim_{x\to t^+}\left(\frac{x^2}{2}-x+B'\right)=\lim_{x\to t^-}G'_M\left(\frac{x^2}{2}+B\right)$$

$$\frac{t^2}{2}-t+B'=\frac{t^2}{2}+B\Rightarrow B'=B+t$$

$$\text{thus }\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+B & ;0\le x<t\\ \frac{x^2}{2}-x+B+t & ;t<x\le 1\end{cases}$$

v. $G'_M(x,t)=\frac{d}{dx}G_M(x,t)$ **exists and will be discontinuous as** $x\to t$ **i.e.**

$$\lim_{x\to t^+}G'_M(x,t)\neq\lim_{x\to t^-}G'_M(x,t)$$

But $\lim_{x\to t^+}G'_M(x,t)-\lim_{x\to t^-}G'_M(x,t)=\frac{1}{p(t)}$

$$\lim_{x\to t^+}\left(\frac{2x}{2}-1\right)-\lim_{x\to t^-}\left(\frac{2x}{2}\right)=\frac{1}{1}$$

$$t-1-t=1\Rightarrow -1\neq 1$$

Thus the discontinuity condition does not help to determining the unknown constant B. so we will use orthogonality condition.

vi. **Using orthogonality condition** $\int_0^1 G_M(x,t).u_0(x)dx=0$

$$\int_0^t G_M(x,t).u_0(x)dx+\int_t^1 G_M(x,t).u_0(x)dx=0$$

$$\int_0^t\left(\frac{x^2}{2}+B\right)dx+\int_t^1\left(\frac{x^2}{2}-x+B+t\right)dx=0=0\quad\text{with } u_0(x)=1$$

$B=\frac{t^2}{2}-t+\frac{1}{3}$ **after solving**

Hence our required Green's function is as follows;

$$\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+\frac{t^2}{2}-t+\frac{1}{3} & ;0\le x<t\\ \frac{x^2}{2}-x+\frac{t^2}{2}-t+\frac{1}{3}+t & ;t<x\le 1\end{cases}$$

$$\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+\frac{t^2}{2}-t+\frac{1}{3} & ;0\le x<t\\ \frac{x^2}{2}-x+\frac{t^2}{2}+\frac{1}{3} & ;t<x\le 1\end{cases}$$

EXAMPLE:Construct Green's function associated with the problem

$u'' + \lambda u = 0$ **with the boundry conditions** $u(0) = u(1)$ **and** $u'(0) = u'(1)$

Solution: **here** $p(x) = 1 = p(t)$

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$

$\Rightarrow u'(x) = A$ **……….(i)**

Now using BC's $u(0) = u(1)$ **and** $u'(0) = u'(1)$ **we have** $A = 0, B \neq 0$

$(i) \Rightarrow u(x) = B$ **which is non - trivial solution. So** $\lambda = 0$ **is an eigenvalue.**

Therefore we take $u_0(x) = 1$ **as a normalized function.i.e.**

$\langle u_0(x), u_0(x)\rangle = \int_a^b u_0(x).u_0(x)dx = \int_0^1 1dx = 1$

ii. $G_M(x,t)$ **as a function of 'x' satisfies the D Equation**

$\frac{d^2}{dx^2}G_M(x,t) = u_0(x)u_0(t) = 1$ **in each of the interval** $0 \leq x < t$ **and** $t < x \leq 1$ **therefore we have** $G''_M(x,t) = 1 \Rightarrow G'_M(x,t) = x + A$

$\Rightarrow G_M(x,t) = \frac{x^2}{2} + Ax + B$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & ; 0 \leq x < t \\ \frac{x^2}{2} + A'x + B' & ; t < x \leq 1 \end{cases}$$

iii. $G_M(x,t)$ **satisfies the BC's i.e.** $\Rightarrow G_M(0,t) = G_M(1,t) \Rightarrow A' = A - 1$

and $\Rightarrow G'_M(0,t) = G'_M(1,t) \Rightarrow B' = B - A + \frac{1}{2}$

$$\text{thus } \Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & ; 0 \leq x < t \\ \frac{x^2}{2} + (A-1)x + B - A + \frac{1}{2} & ; t < x \leq 1 \end{cases}$$

iv. $G_M(x,t)$ **is continuous function of 'x' in the interval** $[0, 1]$ **and particularly at x = t i.e.**

$\lim_{x\to t^+} G_M(x,t) = \lim_{x\to t^-} G_M(x,t)$

$$\lim_{x\to t^+}\left(\frac{x^2}{2}+(A-1)x+B-A+\frac{1}{2}\right)=\lim_{x\to t^-}G'_M\left(\frac{x^2}{2}+Ax+B\right)$$

$$\frac{t^2}{2}+(A-1)t+B-A+\frac{1}{2}=\frac{t^2}{2}+At+B\Rightarrow A=\frac{1}{2}-t$$

thus

$$\Rightarrow G_M(x,t)=$$

$$\begin{cases}\frac{x^2}{2}+\left(\frac{1}{2}-t\right)x+B & ;0\le x<t\\ \frac{x^2}{2}+\left(\frac{1}{2}-t-1\right)x+B-\left(\frac{1}{2}-t\right)+\frac{1}{2} & ;t<x\le 1\end{cases}$$

$$\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+\left(\frac{1}{2}-t\right)x+B & ;0\le x<t\\ \frac{x^2}{2}-\left(\frac{1}{2}+t\right)x+t+B & ;t<x\le 1\end{cases}$$

v. $G'_M(x,t)=\frac{d}{dx}G_M(x,t)$ **exists and will be discontinuous as** $x\to t$ **i.e.**

$$\lim_{x\to t^+}G'_M(x,t)\neq\lim_{x\to t^-}G'_M(x,t)$$

But $\lim_{x\to t^+}G'_M(x,t)-\lim_{x\to t^-}G'_M(x,t)=\frac{1}{p(t)}$

$$\lim_{x\to t^+}\left(\frac{2x}{2}+\left(\frac{1}{2}-t\right)\right)-\lim_{x\to t^-}\left(\frac{2x}{2}-\left(\frac{1}{2}+t\right)\right)=\frac{1}{1}$$

$$t+\frac{1}{2}-t-t+\frac{1}{2}+t=1\Rightarrow 1=1$$

Thus the discontinuity condition does not help to determining the unknown constant B. so we will use orthogonality condition.

vi. **Using orthogonality condition** $\int_0^1 G_M(x,t).u_0(x)dx=0$

$$\int_0^t G_M(x,t).u_0(x)dx+\int_t^1 G_M(x,t).u_0(x)dx=0=0$$

$$\int_0^t\left(\frac{x^2}{2}+\left(\frac{1}{2}-t\right)x+B\right)dx+\int_t^1\left(\frac{x^2}{2}-\left(\frac{1}{2}+t\right)x+t+B\right)dx=0$$ **with** $u_0(x)=1$

$$B=\frac{t^2}{2}-\frac{t}{2}+\frac{1}{12}$$ **after solving**

Hence our required Green's function is as follows;

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + \left(\frac{1}{2} - t\right)x + \frac{t^2}{2} - \frac{t}{2} + \frac{1}{12} & ; 0 \le x < t \\ \frac{x^2}{2} - \left(\frac{1}{2} + t\right)x + t + \frac{t^2}{2} - \frac{t}{2} + \frac{1}{12} & ; t < x \le 1 \end{cases}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + \left(\frac{1}{2} - t\right)x + \frac{t^2}{2} - \frac{t}{2} + \frac{1}{12} & ; 0 \le x < t \\ \frac{x^2}{2} - \left(\frac{1}{2} + t\right)x + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12} & ; t < x \le 1 \end{cases}$$

EXAMPLE: (UoS, 2018 –I, II)

Construct Green's function associated with the problem

$u'' + \lambda u = 0$ **with the boundry conditions** $u(-1) = u(1)$ **and** $u'(-1) = u'(1)$

Solution: **here** $p(x) = 1 = p(t)$

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$

$\Rightarrow u'(x) = A$ **..........(i)**

Now using BC's $u(-1) = u(1)$ **and** $u'(-1) = u'(1)$ **we have** $A = 0, B \neq 0$

$(i) \Rightarrow u(x) = B$ **which is non - trivial solution. So** $\lambda = 0$ **is an eigenvalue.**

Therefore we take $u_0(x) = \frac{1}{\sqrt{2}}$ **as a normalized function.i.e.**

$\langle u_0(x), u_0(x)\rangle = \int_a^b u_0(x).u_0(x)dx = \int_{-1}^{1} \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}dx = 1$

ii. $G_M(x,t)$ **as a function of 'x' satisfies the D Equation**

$\frac{d^2}{dx^2}G_M(x,t) = u_0(x)u_0(t) = \frac{1}{2}$ **in each of the interval** $-1 \le x < t$

and $t < x \le 1$ **therefore we have** $G''_M(x,t) = \frac{1}{2} \Rightarrow G'_M(x,t) = \frac{1}{2}x + A$

$$\Rightarrow G_M(x,t) = \frac{x^2}{4} + Ax + B$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B & ; -1 \le x < t \\ \frac{x^2}{4} + A'x + B' & ; t < x \le 1 \end{cases}$$

iii. $G_M(x,t)$ **satisfies the BC's i.e.**

$$\Rightarrow G_M(-1,t) = G_M(1,t) \Rightarrow B' = 1 + B - 2A$$

and $\Rightarrow G'_M(-1,t) = G'_M(1,t) \Rightarrow A' = A - 1$

$$\text{thus } \Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B & ; -1 \le x < t \\ \frac{x^2}{4} + (A-1)x + 1 + B - 2A & ; t < x \le 1 \end{cases}$$

iv. $G_M(x,t)$ **is continuous function of 'x' in the interval** $[-1, 1]$ **and particularly at x = t i.e.**

$$\lim_{x\to t^+} G_M(x,t) = \lim_{x\to t^-} G_M(x,t)$$

$$\lim_{x\to t^+}\left(\frac{x^2}{4} + (A-1)x + 1 + B - 2A\right) = \lim_{x\to t^-} G'_M\left(\frac{x^2}{4} + Ax + B\right)$$

$$\frac{t^2}{4} + (A-1)t + 1 + B - 2A = \frac{t^2}{4} + At + B \Rightarrow A = \frac{1-t}{2}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + B & ; -1 \le x < t \\ \frac{x^2}{4} + \left(\frac{1-t}{2} - 1\right)x + 1 + B - 2\left(\frac{1-t}{2}\right) & ; t < x \le 1 \end{cases}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + B & ; -1 \le x < t \\ \frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + t + B & ; t < x \le 1 \end{cases}$$

v. $G'_M(x,t) = \frac{d}{dx}G_M(x,t)$ **exists and will be discontinuous as** $x \to t$ **and gives no information about unknown. So we will use orthogonality condition.**

vi. **Using orthogonality condition** $\int_{-1}^{1} G_M(x,t).u_0(x)dx = 0$

$$\int_{-1}^{t} G_M(x,t).u_0(x)dx + \int_{t}^{1} G_M(x,t).u_0(x)dx = 0 = 0$$

$$\int_{-1}^{t}\left(\frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + B\right)dx + \int_{t}^{1}\left(\frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + t + B\right)dx = 0 \quad \text{with}$$

$$u_0(x) = \frac{1}{\sqrt{2}} \ne 0$$

$$B = \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} \quad \text{after solving}$$

Hence our required Green's function is as follows;

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} & ; -1 \le x < t \\ \frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + t + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} & ; t < x \le 1 \end{cases}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} & ; -1 \le x < t \\ \frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + \frac{t^2}{4} + \frac{t}{2} + \frac{1}{6} & ; t < x \le 1 \end{cases}$$

EXAMPLE: (UoS, 2013,2014,2015,2017 – I, II)

Solve the problem $\frac{d^2u}{dx^2} = f(x)$ **with** $u(0) = \propto, u(l) = \beta$

SOLUTION: Let $G(x,x')$ **be a Green's function for the associated homogeneous equation or BVP. Then it satisfies the equation**

$\frac{d^2G}{dx^2} = \delta(x - x')$ **…………(i) with** $G(0,x') = 0 = G(l,x')$ **therefore**

$$\Rightarrow G(x,x') = \begin{cases} -\frac{x}{l}(l - x') & ; 0 \le x < x' \\ -\frac{x'}{l}(l - x) & ; x' < x \le l \end{cases}$$

Since from Lagrange's identity

$\int_a^b [uL(v) - vL(u)]dx = \left|p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\right|_a^b$ **…………(ii)**

By comparing given equation with SL equation w get

$p(x) = 1, q(x) = 0$ **and from BC's** $a = 0, b = l$

And $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x) = \frac{d}{dx}\left\{1.\frac{d}{dx}\right\} + 0 = \frac{d^2}{dx^2}$

Then $(ii) \Rightarrow \int_0^l \left[u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right]dx = \left|1\big(u(x)v'(x) - u'(x)v(x)\big)\right|_0^l$

Take $v(x) = G(x,x')$

$\Rightarrow \int_0^l \left[u\frac{d^2G}{dx^2} - G\frac{d^2u}{dx^2}\right]dx = \left|\big(u(x)G'(x,x') - u'(x)G'(x,x')\big)\right|_0^l$

Since from (i) $\frac{d^2G}{dx^2} = \delta(x - x')$ **and also given** $\frac{d^2u}{dx^2} = f(x)$ **therefore**

$$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx = \left|\left(u(x)G'(x,x') - u'(x)G'(x,x')\right)\right|_0^l$$

$= \left(u(l)G'(l,x') - u'(l)G'(l,x')\right) - \left(u(0)G'(0,x') - u'(0)G'(0,x')\right)$ **………(iii)**

Now using $G(x,x') = \begin{cases} -\frac{x}{l}(l-x') & ; 0 \le x < x' \\ -\frac{x'}{l}(l-x) & ; x' < x \le l \end{cases}$ **and** $u(0) = \propto, u(l) = \beta$

$$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx$$

$$= \left(\beta\left(\frac{x'}{l}\right) - \beta'\left(-\frac{x'}{l}\right)(l-l)\right) - \left(\propto\left(-\frac{1}{l}\right)(l-x') - \propto'\left(-\frac{0}{l}\right)(l-x')\right)$$

$$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx = \beta\left(\frac{x'}{l}\right) + \frac{\propto}{l}(l-x')$$

$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx = (\beta - \propto)\frac{x'}{l} + \propto$ **………(iv)**

Now using property of dirac delta

$$\int \delta(x-x')f(x)dx = f(x') \Rightarrow \int \delta(x-x')u(x)dx = u(x')$$

$$(iv) \Rightarrow \int_0^l u(x)\delta(x-x')dx - \int_0^l G(x,x')f(x)dx = (\beta - \propto)\frac{x'}{l} + \propto$$

$$\Rightarrow u(x') - \int_0^l G(x,x')f(x)dx = (\beta - \propto)\frac{x'}{l} + \propto$$

$$\Rightarrow u(x') = \int_0^l G(x,x')f(x)dx + (\beta - \propto)\frac{x'}{l} + \propto$$

$$\Rightarrow u(x) = \int_0^l G(x'',x)f(x'')dx'' + (\beta - \propto)\frac{x}{l} + \propto$$

Where we replace x', with x and x with x''

EXAMPLE: (UoS, 2019 – I) **Determines the Green's function for the exterior dirichlet problem for a unit circle $\nabla^2 u = 0, r > 1; u = f, r = 1$**

Solution: Consider Green's function assume the form

$G(\xi,\eta;x,y) = f(\xi,\eta;x,y) + g(\xi,\eta;x,y)$

where $f(\xi,\eta;x,y)$ known as free space Green's function satisfies

$\nabla^2 f = \delta(\xi - x, \eta - y)$ in domain D and $g(\xi,\eta;x,y)$ satisfies $\nabla^2 g = 0$

so that by superposition $G = f + g$ satisfies the equation

$\nabla^2 G = \delta(\xi - x, \eta - y)$ in domain D

Also $G = 0$ on boundries requires that $g = -f$ on boundries.

Now for Laplace operator f must satisfies $\nabla^2 f = \delta(\xi - x, \eta - y)$ in domain D then for $r = 1$ we have

$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) = 0$ and solution will be $f = c_1 + c_2 logr$

Now applying the condition $\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\partial G}{\partial n}ds = 1$ where n is outward normal to the circle and $C_\epsilon = (\xi - x)^2 + (\eta - y)^2 = \epsilon^2$ We get $f = \frac{1}{2\pi}logr$

Now if we introduce the polar coordinates $\rho, \theta, \sigma, \beta$ by means of the equations

$x = \rho Cos\theta, y = \rho Sin\theta, \xi = \sigma Cos\beta, x = \sigma Sin\beta$

We get $g(\sigma,\beta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\sigma^n(a_n Cosn\beta + b_n Sinn\beta)$

Where $g = \frac{1}{4\pi}log\left(1 + \rho^2 - 2\rho Cos(\beta - \theta)\right)$ on boundry

Now by using the relation $log\left(1 + \rho^2 - 2\rho Cos(\beta - \theta)\right) = 2\sum_{n=1}^{\infty}\frac{\rho^n Cosn(\beta-\theta)}{n}$ and equating the coefficients of $Cosn\beta, Sinn\beta$ to determine a_n, b_n we find $a_n = \frac{\rho^n Cosn\theta}{2\pi n}, b_n = \frac{\rho^n Sinn\theta}{2\pi n}$

It therefor follows that $g(\rho,\theta,\sigma,\beta) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(\rho\sigma)^n Cosn(\beta-\theta)}{n}$

$g(\rho,\theta,\sigma,\beta) = \frac{1}{4\pi}log\left(1 + (\rho\sigma)^2 - 2\rho\sigma Cos(\beta - \theta)\right)$

Hence the required Green's function is as follows;

$$G(\rho,\theta;\sigma,\beta) = \frac{1}{4\pi}log\left(\rho^2 + \sigma^2 - 2\rho\sigma Cos(\beta - \theta)\right) - \frac{1}{4\pi}log\left(1 + (\rho\sigma)^2 - 2\rho\sigma Cos(\beta - \theta)\right)$$

VARIATIONAL METHODS

The subject of calculus of variation or variational method is similar to but more general than the subject of maxima and minima in Calculus.

FUNCTIONAL:

Let M be the set of functions defined over the interval [a,b]

i.e. $M = \{f \mid f: [a, b] \to \mathbb{R}\}$ such that each function is integrable then a rule of function $I: M \to \mathbb{R}$ defined by $I[f(x)] = J \epsilon \mathbb{R}$ is called functional.

STATIONARY VALUE:

The maximum or minimum value of the function or functional is called stationay value OR the point at which the 1st derivative of a function or functional become zero is called Stationary value.

EXTERMAL:

The curve $y = f(x)$ along which the functional 'I' takes the stationary values is called extermal. i.e. if $\delta I[f(x)] = 0$ then $y = f(x)$ is extermal curve.

SOME EXAMPLES OF VARIATIONAL PROBLEMS:

Here we discuss some important problems whose attempted solutions have led to the development of the subject of Calculus of Variation.

Historically there are three such problems;

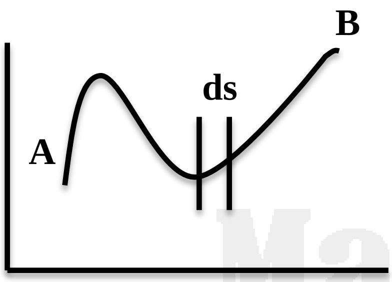
i. **The problems of geodesics: i.e. to find the cuve of minimum length joining two points on given surface.**
ii. **The brachistochrone problems: i.e. to find the path of quickest descent, joining two points in spacew, for a particle moving under gravity.**
iii. **Dido's problems: i.e. the problem of findind curve of given length which encloses maximum area by itself or with a given straight line.**

GEOSDESICS PROBLEM:

Find the curve whose distance between two points is minimum.

EXPLANATION: Let $y = y(x)$ be a curve C on the surface S which is represented by $z = z(x, y)$. Then suppose that A and B be the two points on the curve C. then distance (length) between two points A and B is given by

$l = \int_A^B ds$ ....(i)



In the case of any surface $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

Since curve lies in xy – plane therefore z = 0 then we get

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (y')^2}\,dx$$

$(i) \Rightarrow l = \int_A^B ds = \int_A^B \sqrt{1 + (y')^2}\,dx$ this is our required length.

BRACHISTOCHRONE PROBLEM: (UoS, S.Q)

A particle falls under gravity from A to B. determine the curve along which the time taken by the particle will be minimum.

EXPLANATION: Consider a particle falls under gravity from A to B . then instantaneous velocity is given by $V = \frac{ds}{dt}$ and time taken between A and B is given by $total\ time = \int_A^B dt = \int_A^B \frac{dt}{ds} ds = \int_A^B \frac{1}{\frac{ds}{dt}} ds = \int_A^B \frac{1}{V} ds$ ....(i)

Now using 3rd equation of motion under gravity we get $V = \sqrt{2gy}$

$(i) \Rightarrow total\ time = \frac{1}{\sqrt{2g}} \int_A^B \frac{1}{\sqrt{y}} \sqrt{1 + (y')^2}\,dx = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1+(y')^2}{y}}\,dx$ required.

DIDO's PROBLEM: (UoS, S.Q)

Find the closed curve of given length which enclosed maximum area.

EXPLANATION:

Suppose that $y = y(x)$ is the curve which meet the x – axis at points x_1 and x_2 and enclosed maximum area $A = \int_{x_1}^{x_2} y dx$ and and the length of the same curve given as $l = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$ then the problem reduces to that of maximizing the area in equation $A = \int_{x_1}^{x_2} y dx$ subject to the condition given in $l = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$

$(PU\ 1997, 2000, 2001)$

Discuss 3 well known problmes, viz., geodesic, brachistochrone and dido' and formulate them as variational problems.

FUNDAMENTAL THEOREM ON VARIATIONAL CALCULUS:

(PU, 2002, 2010, 2011)

If $f(x)$ is continuous function in the interval (x_1, x_2) and the integral $\int_{x_1}^{x_2} f(x) g(x) dx$ is identically zero. i.e. $\int_{x_1}^{x_2} f(x) g(x) dx \equiv 0$ where $g(x)$ satisfies the following conditions;

i. **It is an arbitrary function with continuous derivatives in the interval (x_1, x_2)**
ii. **$g(x_1) = g(x_2) = 0$**

Then $f(x) \equiv 0$ for all $x \epsilon [x_1, x_2]$

PROOF: We prove by contradiction. If possible let $f(x) \neq 0$ in (x_1, x_2). Then there is at least one point x_0 in (x_1, x_2) such that $f(x_0) \neq 0$. Then because of continuity of $f(x)$ in (x_1, x_2) there must exists an interval $(x_0 - \delta, x_0 + \delta)$ where $\delta > 0$ surrounding x_0 such that $f(x) > 0$ for all $x \epsilon [x_0 - \delta, x_0 + \delta]$

Since $g(x)$ is arbitrary, it can be taken as

$$g(x) = \begin{cases} (x - x_0 + \delta)^2 (x - x_0 - \delta)^2 & if\ x \epsilon [x_0 - \delta, x_0 + \delta] \\ 0 & otherwise \end{cases}$$

It is clear that $g(x) = 0$ at the endpoints of the interval $(x_0 - \delta, x_0 + \delta)$ and has continuous derivative inside the interval. Then integral $\int_{x_1}^{x_2} f(x)g(x)dx$ becomes $\int_{x_0-\delta}^{x_0-\delta} f(x)(x - x_0 + \delta)^2 (x - x_0 - \delta)^2 dx > 0$

This is contradiction, as $\int_{x_1}^{x_2} f(x)g(x)dx = 0$

Hence $f(x) \equiv 0$ for all $x \epsilon [x_1, x_2]$

EULER LAGRANGE's EQUATION: (UoS, 2013, 2014, 2015)

Let $I = \int_{x_1}^{x_2} F(x, y, y')dx$ where $y = y(x)$ is a continuous function having continuous 1st and 2nd order derivatives satisfying the following endpoint condtions $y_1 = y(x_1)$ and $y_2 = y(x_2)$, also if F is supposed to be have continuous 1st and 2nd order derivatives w.r.to its arguments, then the function $y = y(x)$ will extremise the given integral if it satisfies the following DE

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

PROOF: given that $I = \int_{x_1}^{x_2} F(x, y, y')dx$

$$\delta I = \int_{x_1}^{x_2} \delta F(x, y, y')dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta\left(\frac{dy}{dx}\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y dx + \left[\left|\frac{\partial F}{\partial y'}(\delta y)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx\right]$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y dx - \int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx \qquad \text{since } \delta y(x_1) = 0 = \delta y(x_2)$$

For extermal curve $\delta I = 0$ then $\int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx - \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx = 0$

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y dx = 0$$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ $\quad\quad$ **$\delta y \neq 0, dx \neq 0$ being orbitrary values.**

SPECIAL CASES: (UoS, 2019 – I)

i. **When F is independent of 'y''**

Then $\frac{\partial F}{\partial y'} = 0$ then EL equation becomes as follows;

$\frac{\partial F}{\partial y} = 0$ this is an algebraic equation in 'x' and 'y'. the solution may not satisfy the given boundry conditions.

ii. **When F is independent of 'y'**

Then $\frac{\partial F}{\partial y} = 0$ then EL equation becomes as follows;

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = Constant$$

iii. **When F is independent of 'x'**

Then $\frac{\partial F}{\partial x} = 0$ then EL equation becomes as follows;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} \Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right) y'$$

$$\Rightarrow \left(\frac{\partial F}{\partial y}\right) dy = d\left(\frac{\partial F}{\partial y'}\right) y' \quad\quad \text{............(i)}$$

Since $F = F(y, y') \Rightarrow dF = \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy'$

$\Rightarrow dF = d\left(\frac{\partial F}{\partial y'}\right) y' + \frac{\partial F}{\partial y'} dy'$ $\quad\quad$ **by (i)**

$$\Rightarrow dF = d\left(y' \frac{\partial F}{\partial y'}\right) \Rightarrow d\left(F - y' \frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow F - y' \frac{\partial F}{\partial y'} = constant$$

iv. Suppose 'F' is linear function in y'

i.e. $F(x,y,y') = M(x,y) + N(x,y)y'$ **.........(i)**

$(i) \Rightarrow \frac{\partial F}{\partial y} = \left(\frac{\partial M}{\partial x}\frac{dx}{dy} + \frac{\partial M}{\partial y}\frac{dy}{dy}\right) + \left(\frac{\partial N}{\partial x}\frac{dx}{dy} + \frac{\partial N}{\partial y}\frac{dy}{dy}\right) y'$

$\Rightarrow \frac{\partial F}{\partial y} = \left(\frac{\partial M}{\partial y}\right) + \left(\frac{\partial N}{\partial y}\right) y'$ **.........(ii)**

Again $(i) \Rightarrow \frac{\partial F}{\partial y'} = N(x,y)$

$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\big(N(x,y)\big) = \frac{\partial N}{\partial x}\frac{dx}{dx} + \frac{\partial N}{\partial y}\frac{dy}{dx} = \frac{\partial N}{\partial x} + \frac{\partial N}{\partial y} y'$ **.........(iii)**

now as $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \left(\frac{\partial M}{\partial y}\right) + \left(\frac{\partial N}{\partial y}\right) y' - \frac{\partial N}{\partial x} - \frac{\partial N}{\partial y} y' = 0$

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow M_y(x,y) = N_x(x,y)$ **this is not a DE which may not satisfy the given boundry conditions.**

EULER's LAGRANGE EQUATION IS SECOND ORDER DE

As we know that $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ **..........(i)**

Since $F = F(x,y,y')$ **then** $\frac{\partial F}{\partial y}$ *and* $\frac{\partial F}{\partial y'}$ **are also functions of** x, y *and* y'

Then by using chain rule

$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\frac{dx}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx}$

$\Rightarrow \frac{\partial F}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right) y' + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right) y''$

$\Rightarrow F_y = F_{xy'} + F_{yy'} y' + F_{y'y'} y''$ **which is 2nd oder Differential equation.**

EXTENSION OF EULER LAGRANGE's EQUATION WITH ONE INDEPENDENT VARIABLE AND MANY DEPENDENT VARIABLES:

Let $I = \int_{x_1}^{x_2} F(x, y_k, y_k')dx$; $k = 1, 2, 3, \dots \dots n$ **with the stationary condtions** $y_k(x_1) = constant$ **and** $y_k(x_2) = constant$**, then Euler's Lagrange's equation can be written as** $\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right) = 0$

PROOF: given that $I = \int_{x_1}^{x_2} F(x, y_k, y_k')dx$

$$\delta I = \int_{x_1}^{x_2} \delta F(x, y_k, y_k')dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k}\delta y_k + \frac{\partial F}{\partial y_k'}\delta y_k'\right)dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k'}\delta y_k' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k'}\delta\left(\frac{dy_k}{dx}\right)dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k'}\frac{d}{dx}(\delta y_k)dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx + \left[\left|\frac{\partial F}{\partial y_k'}(\delta y_k)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}(\delta y_k)\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx\right]$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y dx - \int_{x_1}^{x_2}(\delta y_k)\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx \qquad \text{since } \delta y_k(x_1) = 0 = \delta y_k(x_2)$$

For extermal curve $\delta I = 0$ **then** $\int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx - \int_{x_1}^{x_2}(\delta y_k)\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx = 0$

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\right]\delta y_k dx = 0$$

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right) = 0 \ ; \ k = 1, 2, 3, \dots \dots n$$

$\delta y_k \neq 0, dx \neq 0$ **being orbitrary values.**

EXAMPLE: Let $I = \int_{x_1}^{x_2} F(x,\varphi,\psi,\varphi',\psi')dx$ with the stationary condtions $\delta\varphi(x_1) = \delta\varphi(x_2) = 0$ and $\delta\psi(x_1) = \delta\psi(x_2) = 0$ then

$$\frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial\psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\psi'}\right) = 0$$

PROOF: given that $I = \int_{x_1}^{x_2} F(x,\varphi,\psi,\varphi',\psi')dx$

$$\delta I = \int_{x_1}^{x_2} \delta F(x,\varphi,\psi,\varphi',\psi')dx = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial\varphi}\delta\varphi + \frac{\partial F}{\partial\psi}\delta\psi + \frac{\partial F}{\partial\varphi'}\delta\varphi' + \frac{\partial F}{\partial\psi'}\delta\psi'\right)dx$$

$$\delta I = \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi}\delta\varphi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi'}\delta\varphi' dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi}\delta\psi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi'}\delta\psi' dx$$

$$\delta I = \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi}\delta\varphi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi'}\delta\left(\frac{d\varphi}{dx}\right)dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi}\delta\psi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi'}\delta\left(\frac{d\psi}{dx}\right)dx$$

$$\delta I = \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi}\delta\varphi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi}\delta\psi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi'}\frac{d}{dx}(\delta\varphi)dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi'}\frac{d}{dx}(\delta\psi)dx$$

$$\delta I = \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi}\delta\varphi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi}\delta\psi dx + \left[\left|\frac{\partial F}{\partial\varphi'}(\delta\varphi)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}(\delta\varphi)\frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right)dx\right] + \left[\left|\frac{\partial F}{\partial\psi'}(\delta\psi)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}(\delta\psi)\frac{d}{dx}\left(\frac{\partial F}{\partial\psi'}\right)dx\right]$$

$$\delta I = \int_{x_1}^{x_2}\frac{\partial F}{\partial\varphi}\delta\varphi dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial\psi}\delta\psi dx - \int_{x_1}^{x_2}(\delta\varphi)\frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right)dx - \int_{x_1}^{x_2}(\delta\psi)\frac{d}{dx}\left(\frac{\partial F}{\partial\psi'}\right)dx$$

since $\delta\varphi(x_1) = \delta\varphi(x_2) = 0$ and $\delta\psi(x_1) = \delta\psi(x_2) = 0$

$$\delta I = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right)\right)\delta\varphi dx + \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial\psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\psi'}\right)\right)\delta\psi dx$$

For extermal curve $\delta I = 0$ then

$$\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right)\right)\delta\varphi dx + \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial\psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\psi'}\right)\right)\delta\psi dx = 0$$

$$\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right)\right)\delta\varphi dx = 0 \;\; and \;\; \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial\psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\psi'}\right)\right)\delta\psi dx = 0$$

$$\frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial\psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\psi'}\right) = 0$$

$\delta\varphi \neq 0, dx \neq 0, \delta\psi \neq 0$ being orbitrary values.

EXTENSION OF EULER LAGRANGE's EQUATION WITH ONE INDEPENDENT VARIABLE AND ONE DEPENDENT VARIABLE WITH ITS HIGHER ORDER DERIVATIVES:

(UoS, 2017, 2018 – I)

Let $I = \int_{x_1}^{x_2} F\left(x, y, y', y'', y''', \dots\dots y^{(n)}\right) dx$ **with the stationary condtions**

$y(x_1) = y'(x_1) = y''(x_1) = \cdots \dots\dots\dots \dots = y^{(n)}(x_1) = constant$ **and**

$y(x_2) = y'(x_2) = y''(x_2) = \cdots \dots\dots \dots = y^{(n)}(x_2) = constant$, **then Euler's Lagrange's equation can be written as**

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \cdots \dots\dots\dots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

PROOF: given that $I = \int_{x_1}^{x_2} F\left(x, y, y', y'', y''', \dots\dots y^{(n)}\right) dx$

$$\delta I = \int_{x_1}^{x_2} \delta F\left(x, y, y', y'', y''', \dots\dots y^{(n)}\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' + \cdots \dots\dots\dots . + \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)}\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx + \cdots \dots\dots + \int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)} dx \quad \dots\dots\dots\dots\dots(i)$$

Consider $\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta\left(\frac{dy}{dx}\right) dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y) dx$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = \left|\frac{\partial F}{\partial y'}(\delta y)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = -\int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx \qquad \text{since } \delta y(x_1) = 0 = \delta y(x_2)$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = (-1)^1 \int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

Also $\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta\left(\frac{dy'}{dx}\right) dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\frac{d}{dx}(\delta y') dx$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx = \left|\frac{\partial F}{\partial y''}(\delta y')\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} (\delta y')\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx = -\int_{x_1}^{x_2} (\delta y')\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) dx \qquad \text{since } \delta y(x_1) = 0 = \delta y(x_2)$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' dx = \left| -\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)(\delta y) \right|_{x_1}^{x_2} + (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' dx = (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx$$

Similarly $\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(n)}} \delta y^{(n)} dx = (-1)^n \int_{x_1}^{x_2} (\delta y) \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) dx$

Then equation (i) becomes

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + (-1)^1 \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx + (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx + \cdots \ldots \ldots + (-1)^n \int_{x_1}^{x_2} (\delta y) \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) dx$$

For extermal curve $\delta I = 0$ then

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + (-1)^1 \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx + (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx + \cdots \ldots \ldots + (-1)^n \int_{x_1}^{x_2} (\delta y) \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) dx = 0$$

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} + (-1)^1 \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)\right] \delta y dx = 0$$

$$\frac{\partial F}{\partial y} + (-1) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \cdots \ldots \ldots \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

$\delta y \neq 0, dx \neq 0$ being orbitrary values.

EULER LAGRANGE's EQUATION WITH TWO INDEPENDENT VARIABLES: (UoS, 2011)

Let $I = \iint_R F(x, y, u, u_x, u_y)dxdy$ then Euler's Lagrange's equation can be written as $\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$

PROOF: given that $I = \iint_R F(x, y, u, u_x, u_y)dxdy$

$\delta I = \iint_R \delta F(x, y, u, u_x, u_y)dxdy$

$\delta I = \iint_R \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y\right)dxdy$ **………..(i)**

Consider $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u)$

$\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$

$\frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$

Similarly $\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$

$(i) \Rightarrow \delta I = \iint_R \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u\right)dxdy$

$\Rightarrow \delta I = \iint_R \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)dudxdy + \iint_R \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right)dxdy$

$\Rightarrow \delta I = I_1 + I_2$ **………..(ii)**

Consider $I_2 = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right)dxdy$

$I_2 = \oint_c - \left(\frac{\partial F}{\partial u_y}\delta u dx\right) + \left(\frac{\partial F}{\partial u_x}\delta u dy\right) = \oint_c \left(\frac{\partial F}{\partial u_x}dy - \frac{\partial F}{\partial u_y}dx\right)\delta u$ **by Green's theorem**

Since u is prescribed on the boundry therefore due to the closed curve δu must be zero. i.e. $I_2 = 0$

$(ii) \Rightarrow \delta I = \iint_R \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)dudxdy$

$\Rightarrow \iint_R \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)dudxdy = 0$ **for extermal curve $\delta I = 0$**

$\Rightarrow \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$ **since $du \neq 0, dx \neq 0, dy \neq 0$**

Hence required.

EULER LAGRANGE's EQUATION WITH THREE INDEPENDENT VARIABLES:

Let $I = \iiint_V F(x,y,z,u,u_x,u_y,u_z)dxdydz$ then Euler's Lagrange's equation can be written as $\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$

PROOF:

given that $I = \iiint_V F(x,y,z,u,u_x,u_y,u_z)dxdydz$

$\delta I = \iiint_V \delta F(x,y,z,u,u_x,u_y,u_z)dxdydz$

$\delta I = \iiint_V \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z\right)dxdydz$ ………..(i)

Consider $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u)$

$\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$

$\frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$

Similarly $\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$

And $\frac{\partial F}{\partial u_z}\delta u_z = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u$

$(i) \Rightarrow \delta I = \iiint_V \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u\right)dxdydz$

$\Rightarrow \delta I = \iiint_V \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right)dudxdy + \iiint_V \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right)dxdydz$

$\Rightarrow \delta I = I_1 + I_2$ ………..(ii)

Consider $I_2 = \iiint_V \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right)dxdydz$

$I_2 = \iiint_V \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial F}{\partial u_x}\delta u\hat{\imath} + \frac{\partial F}{\partial u_y}\delta u\hat{\jmath} + \frac{\partial F}{\partial u_z}\delta u\hat{k}\right)dv$

$I_2 = \iiint_V \nabla.\vec{G}dv$

$\Rightarrow I_2 = \oiint_s \vec{G}.\vec{n}ds$ **by divergence theorem.**

Since u is prescribed on the boundry therefore due to the closed curve δu must be zero. i.e. $I_2 = 0$

$(ii) \Rightarrow \delta I = \iiint_V \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right)dudxdy$

$\Rightarrow \iiint_V \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right)dudxdy = 0$ **for extermal curve $\delta I = 0$**

$\Rightarrow \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$ **since $du \neq 0, dx \neq 0, dy \neq 0, dz \neq 0$**

Hence required.

PLATEAU'S PROBLEM: (Problem of minimal surface)

In this problem we will find the surface of minimal area which is bounded by a given closed curve.

EXPLANATION:

Consider a surface $z = z(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$ then 1st fundamental form of given surface is

$$(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2$$

Where $= \vec{r}_u.\vec{r}_u = |\vec{r}_u|^2$, $F = \vec{r}_u.\vec{r}_v$, $G = \vec{r}_v.\vec{r}_v = |\vec{r}_v|^2$ are fundamental quantities of the surface. If we take parameters (x,y) and put $u = x, v = y$ then

$$E = |\vec{r}_x|^2 = \left|\frac{\partial x}{\partial x}\hat{\imath} + \frac{\partial y}{\partial x}\hat{\jmath} + \frac{\partial z}{\partial x}\hat{k}\right|^2 = \left|1\hat{\imath} + 0\hat{\jmath} + z_x\hat{k}\right|^2 = \left(\sqrt{|1 + {z_x}^2|}\right)^2$$

$$E = 1 + {z_x}^2$$

$$G = |\vec{r}_y|^2 = \left|\frac{\partial x}{\partial y}\hat{\imath} + \frac{\partial y}{\partial y}\hat{\jmath} + \frac{\partial z}{\partial y}\hat{k}\right|^2 = \left|0\hat{\imath} + 1\hat{\jmath} + z_y\hat{k}\right|^2 = \left(\sqrt{|1 + {z_y}^2|}\right)^2$$

$$G = 1 + {z_y}^2$$

$$F = \vec{r}_x.\vec{r}_y = \left(\frac{\partial x}{\partial x}\hat{\imath} + \frac{\partial y}{\partial x}\hat{\jmath} + \frac{\partial z}{\partial x}\hat{k}\right)\left(\frac{\partial x}{\partial y}\hat{\imath} + \frac{\partial y}{\partial y}\hat{\jmath} + \frac{\partial z}{\partial y}\hat{k}\right) = \left(1\hat{\imath} + z_x\hat{k}\right)\left(1\hat{\jmath} + z_y\hat{k}\right)$$

$$F = z_x z_y$$

Put v = constant then $(ds_1)^2 = E(du)^2 \Rightarrow ds = \sqrt{E}du$

Put u = constant then $(ds_2)^2 = G(dv)^2 \Rightarrow ds = \sqrt{G}dv$

Then $ds = |ds_1 \times ds_2| = |ds_1||ds_2|Sin\theta$

$ds = \sqrt{E}du\sqrt{G}dvSin\theta \Rightarrow ds = \sqrt{EG}dudvSin\theta$ **.........(i)**

if $Cos\theta = \frac{F}{\sqrt{EG}}$ and $Sin\theta = \sqrt{1 - Cos^2\theta} = \frac{\sqrt{EG-F^2}}{\sqrt{EG}}$

$$(i) \Rightarrow ds = \sqrt{EG - F^2}dxdy$$

$$\Rightarrow s = \iiint \sqrt{EG - F^2}dxdy$$

$$\Rightarrow s = \iiint \sqrt{(1+z_x^2)(1+z_y^2)-(z_x z_y)^2}\,dxdy$$

$$\Rightarrow s = \iiint \sqrt{1+z_x^2+z_y^2}\,dxdy$$

Now let $F = F(x,y,z,z_x,z_y) = \sqrt{1+z_x^2+z_y^2}$

Then by using EL equation for two independent variables

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial z}\sqrt{1+z_x^2+z_y^2} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z_x}\sqrt{1+z_x^2+z_y^2}\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_y}\sqrt{1+z_x^2+z_y^2}\right) = 0$$

$$\Rightarrow 0 - \frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0$$

$$\Rightarrow \left(\frac{\left(\sqrt{1+z_x^2+z_y^2}\right)z_{xx} - \frac{z_x z_{xx}}{\sqrt{1+z_x^2+z_y^2}}z_x}{\left(\sqrt{1+z_x^2+z_y^2}\right)^2}\right) + \left(\frac{\left(\sqrt{1+z_x^2+z_y^2}\right)z_{yy} - \frac{z_y z_{yy}}{\sqrt{1+z_x^2+z_y^2}}z_y}{\left(\sqrt{1+z_x^2+z_y^2}\right)^2}\right) = 0$$

$$\Rightarrow \left(\frac{(1+z_x^2+z_y^2)z_{xx} - z_x^2 z_{xx}}{(1+z_x^2+z_y^2)^{3/2}}\right) + \left(\frac{(1+z_x^2+z_y^2)z_{yy} - z_y^2 z_{yy}}{(1+z_x^2+z_y^2)^{3/2}}\right) = 0$$

$$\Rightarrow (1+z_x^2+z_y^2)z_{xx} - z_x^2 z_{xx} + (1+z_x^2+z_y^2)z_{yy} - z_y^2 z_{yy} = 0$$

$\Rightarrow (1+z_y^2)z_{xx} + (1+z_x^2)z_{yy} = 0$ **this is our required.**

CONSTRAIN EXTREMA OR PROBLEMS WITH CONSTRAINTS OR VARIATIONAL PROBLEMS WITH SIDE CONDITIONS OR ISOPERIMETRIC PROBLEMS:

To find the stationary value of a functional $I = \int_{x_1}^{x_2} F(x, y_k, y'_k)dx$ where the argument of F are subjected to constraints or additional conditions such as

i. $G(x, y_k) = constant$
ii. $G(x, y_k, y'_k) = constant$
iii. $\int_{x_1}^{x_2} G(x, y_k, y'_k)dx = constant$

Then we construct a new function involving parameter λ i.e. $H = F + \lambda G$

EULER LAGRANGE EQUATION FOR CONSTRAIN EXTREMA

The extermal curves $y_k = y_k(x)$; $k = 1, 2, 3, \dots \dots \dots .., n$ of the functional $I = \int_{x_1}^{x_2} F(x, y_k, y'_k)dx$ with constraints

$G_j(x, y_k) = constant$; $j = 1, 2, .., n$ **………….(i)**

Then $J = \int_{x_1}^{x_2} F(x, y_k, y'_k)dx + \sum_{i=1}^{m} \lambda_i \int_{x_1}^{x_2} G_i(x, y_k)dx$

$J = \int_{x_1}^{x_2} \left(F(x, y_k, y'_k) + \sum_{i=1}^{m} \lambda_i G_i(x, y_k)\right) dx = \int_{x_1}^{x_2} H\, dx$

With $F(x, y_k, y'_k) + \sum_{i=1}^{m} \lambda_i G_i(x, y_k) = H$ where $\lambda_i = \lambda_i(x)$ are suitably choosen multiplier. It is clear that the Euler Lagrange's equation in this case will be $\frac{\partial H}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'_k}\right) = 0$; $k = 1, 2, 3, \dots \dots \dots .., n$ **………….(ii)**

Then the curves $y_k = y_k(x)$; $k = 1, 2, 3, \dots \dots \dots .., n$ can be obtained from both equations.i.e. (i) and (ii)

GEODESIC:

A geodesic is the curve of shortest length joining two points in space.

EXAMPLE: (UoS, 2017)

Prove that a straight line is the shortest distance between two points in the plane.

PROOF: Since this is the geodesic problem therefore we use the functional

$I = \int_a^b \sqrt{1+(y')^2}dx$ **with** $F = F(x,y,y') = \sqrt{1+(y')^2}$

Since F is not depend on 'y' therefore we use following EL equation;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = 0 \; and \; \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y'} = Constant = C$$

$$\Rightarrow \frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2}\right) = C$$

$$\Rightarrow \frac{y'}{\sqrt{1+(y')^2}} = C \Rightarrow y' = C\sqrt{1+(y')^2}$$

$$\Rightarrow (y')^2 = C^2(1+(y')^2) = C^2 + C^2(y')^2$$

$$\Rightarrow (y')^2 - C^2(y')^2 = C^2 \Rightarrow (1-C^2)(y')^2 = C^2$$

$$\Rightarrow (y')^2 = \frac{C^2}{1-C^2} \Rightarrow y' = \sqrt{\frac{C^2}{1-C^2}}$$

$\Rightarrow y' = \frac{dy}{dx} = a \quad (say)$ **where** $a = \sqrt{\frac{C^2}{1-C^2}}$

$$\Rightarrow \int \frac{dy}{dx}dx = \int a\,dx$$

$\Rightarrow y = ax + c$ **which is straight line.**

The applications of the Calculus of Variations in Mechanics are based on employing Principle of Least Action and Hamilton's Principle; stated as below;

PRINCIPLE OF LEAST ACTION

According to this principle:

Let a particle move in an external field of force which is conservative. If the motion takes place in the interval of the time from t_1 to t_2 where $t_2 > t_1$ then the actual path traced by the particle is the one along which $I = \int_{t_1}^{t_2} Ldt$ is minimum. Where L is the Lagrangian and for a conservative system

$$L = T - V = kinetic\ energy - potential\ energy$$

HAMILTON's PRINCIPLE: (UoS, S.Q)

According to this principle:

The path of motion of a rigid body in the time interval $t_2 - t_1$ is such that the integral $A = \int_{t_1}^{t_2} Ldt$ has a stationary value, where L is the Lagrangian.

EXAMPLE: (UoS, 2015 – I)

Find the equation of the path in space down which a particle will fall from one point to another in shortest possible time.

Solution:

This is the Brachistochrone problem, therefore we use the following functional

$$I = \int_a^b dt \Rightarrow I = \frac{1}{\sqrt{2g}}\int_a^b \sqrt{\frac{1+(y\prime)^2}{y}}\,dt \quad \text{with } F = F(x, y, y') = \frac{1}{\sqrt{2g}}\sqrt{\frac{1+(y\prime)^2}{y}}$$

Since F is not depend on 'x' therefore we use following EL equation;

$$F - y'\left(\frac{\partial F}{\partial y\prime}\right) = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\sqrt{\frac{1+(y\prime)^2}{y}} - y'\frac{1}{\sqrt{2g}}\frac{1}{\sqrt{y}}\frac{\partial}{\partial y\prime}\left(\sqrt{1+(y')^2}\right) = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\left[\sqrt{\frac{1+(y')^2}{y}} - y'\frac{1}{\sqrt{y}}\frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2}\right)\right] = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\left[\sqrt{\frac{1+(y')^2}{y}} - y'\frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+(y')^2}}\right] = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\left[\sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{y}\sqrt{1+(y')^2}}\right] = constant$$

$$\Rightarrow \sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{y(1+(y')^2)}} = \sqrt{2g}(constant) = a(say)$$

$$\Rightarrow \left(\sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{y(1+(y')^2)}}\right)^2 = a^2$$

$$\Rightarrow \frac{1+(y')^2}{y} + \frac{(y')^4}{y(1+(y')^2)} - 2\left(\sqrt{\frac{1+(y')^2}{y}}.\frac{(y')^2}{\sqrt{y(1+(y')^2)}}\right) = a^2$$

$$\Rightarrow \frac{1+(y')^2}{y} + \frac{(y')^4}{y(1+(y')^2)} - \frac{2(y')^2}{y} = a^2$$

$$\Rightarrow \frac{\left(1+(y')^2\right)^2+(y')^4-2(y')^2\left(1+(y')^2\right)}{y(1+(y')^2)} = a^2 \Rightarrow \frac{1}{y(1+(y')^2)} = a^2 \quad \text{after solving}$$

$$\Rightarrow 1 = a^2y(1+(y')^2) \Rightarrow \frac{1}{a^2y} = (1+(y')^2) \Rightarrow (y')^2 = \frac{1}{a^2y} - 1 = \frac{1-a^2y}{a^2y}$$

$$\Rightarrow y' = \frac{dy}{dx} = \sqrt{\frac{1-a^2y}{a^2y}} \Rightarrow \int\frac{\sqrt{a^2y}}{\sqrt{1-a^2y}}dy = \int dx = x + c$$

Put $a^2y = Sin^2\theta \Rightarrow a^2dy = 2Sin\theta Cos\theta d\theta \Rightarrow dy = \frac{2}{a^2}Sin\theta Cos\theta d\theta$

$$\Rightarrow \int\frac{\sqrt{Sin^2\theta}}{\sqrt{1-Sin^2\theta}}.\frac{2}{a^2}Sin\theta Cos\theta d\theta = x + c \Rightarrow \frac{2}{a^2}\int\frac{Sin\theta}{Cos\theta}.Sin\theta Cos\theta d\theta = x + c$$

$$\Rightarrow \frac{2}{a^2}\int Sin^2\theta\, d\theta = x + c \Rightarrow \frac{2}{a^2}\int\frac{1-Cos2\theta}{2}d\theta = \frac{1}{a^2}\int 1 - Cos2\theta\, d\theta = x + c$$

$$\Rightarrow x = \frac{1}{2a^2}(2\theta - Sin2\theta) + b \quad \text{.........(i)}$$

and $y = \frac{1}{2a^2}(2Sin^2\theta) \Rightarrow y = \frac{1}{2a^2}(1 - Cos2\theta)$ **.........(ii)**

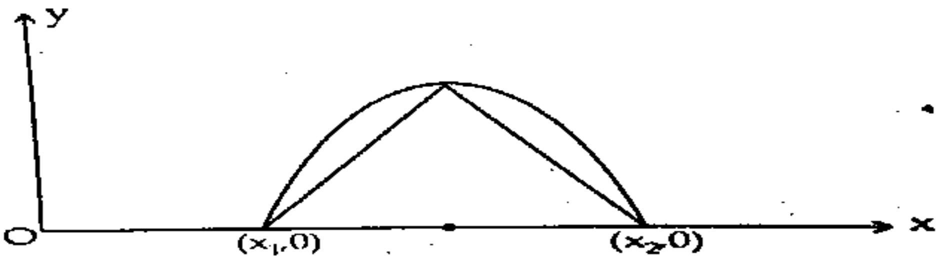
(i) and (ii) are parametric equations of cycloid, where ‘a’ ,’b’ are constants.

Thus the curve downwhich the particle takes the minimum time is cycloid.

DIDO's PROBLEM: (UoS, 2018 – I)

Find the closed curve of given length which enclosed maximum area.

EXPLANATION:



Suppose that $y = y(x)$ is the curve which meet the x – axis at points $A(x_1, 0)$ and $B(x_2, 0)$ and encloses maximum area. Since area enclosed $A = \int_{x_1}^{x_2} y dx$ therefore we have to extremized the functional $l = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+(y')^2}dx$

Here $F = y, G = \sqrt{1+(y')^2}$ and therefore we construct a new function

$$H = F + \lambda G = y + \lambda\sqrt{1+(y')^2}$$

Since there is no explicit dependence on 'x' so we use the special case of EL equation. i.e. $H - y'\frac{\partial H}{\partial y'} = cosntant$

$$\Rightarrow y + \lambda\sqrt{1+(y')^2} - y'\frac{\partial}{\partial y'}\left(y + \lambda\sqrt{1+(y')^2}\right) = c_1$$

$$\Rightarrow y + \lambda\sqrt{1+(y')^2} - \frac{\lambda(y')^2}{\sqrt{1+(y')^2}} = c_1$$

$$\Rightarrow \lambda\left(\frac{1+(y')^2-(y')^2}{\sqrt{1+(y')^2}}\right) = c_1 - y \Rightarrow \lambda\left(\frac{1}{\sqrt{1+(y')^2}}\right) = c_1 - y$$

$$\Rightarrow \frac{c_1-y}{\lambda} = \frac{1}{\sqrt{1+(y')^2}} \Rightarrow \frac{(c_1-y)^2}{\lambda^2} = \frac{1}{1+(y')^2} \Rightarrow 1+(y')^2 = \frac{\lambda^2}{(c_1-y)^2}$$

$$\Rightarrow (y')^2 = \frac{\lambda^2}{(c_1-y)^2} - 1 \Rightarrow (y')^2 = \frac{\lambda^2-(c_1-y)^2}{(c_1-y)^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\sqrt{\lambda^2-(c_1-y)^2}}{c_1-y}$$

$$\Rightarrow \int\frac{c_1-y}{\sqrt{\lambda^2-(c_1-y)^2}}dy = \int dx \Rightarrow -\int\frac{z}{\sqrt{\lambda^2-z^2}}dz = \int dx \qquad \text{put } c_1 - y = z$$

$$\Rightarrow \frac{1}{2}\int(\lambda^2 - z^2)^{-1/2}(-2z)dz = x + c_2$$

$\Rightarrow \frac{1}{2}\frac{(\lambda^2-z^2)^{1/2}}{\frac{1}{2}} = x + c_2 \Rightarrow (\lambda^2 - z^2)^{1/2} = x + c_2 \Rightarrow \lambda^2 - z^2 = (x + c_2)^2$

$\Rightarrow \lambda^2 = (x + c_2)^2 + z^2 \Rightarrow \lambda^2 = (x + c_2)^2 + (y - c_1)^2$

This is an equation of circular arc where the constants c_1, c_2 can be determined by using the given conditions $y(x_1) = 0 = y(x_2)$

INVERSE OF DIDO's PROBLEM:

It can be stated as;

The extermal curves of the functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$ with the endpoint conditions $y(x_1) = y_1, y(x_2) = y_2$ and subject to the constraint $J[y] = \int_{x_1}^{x_2} G(x, y, y')dx = constant$ are the same as the externals of funtioan J with the same endpoint conditions and subject to the constraint $J[y] = constant$

PROOF:

Consider $F \equiv F(t, x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$ and $G \equiv G(t, x, y, \dot{x}, \dot{y}) = \frac{1}{2}(x\dot{y} - \dot{x}y)$

Therefore $H = F + \lambda G = \sqrt{\dot{x}^2 + \dot{y}^2} + \frac{\lambda}{2}(x\dot{y} - \dot{x}y)$

As the EL equations are $\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0$ and $\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0$

In this problem these equations reduce to

$\lambda\dot{y} - \frac{d}{dt}\left(\frac{x}{\sqrt{\dot{x}^2+\dot{y}^2}} - \lambda y\right) = 0$ and $\lambda\dot{x} - \frac{d}{dt}\left(\frac{x}{\sqrt{\dot{x}^2+\dot{y}^2}} - \lambda x\right) = 0$

Which on simplification and integration yield

$2\lambda y - \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c_2$ and $2\lambda x - \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c_1$

On eliminating $\dot{x}, \dot{y}$ we obtain $(x - c_1')^2 + (y - c_2')^2 = \left(\frac{1}{2\lambda}\right)^2$

Where $c_1' = \frac{c_1}{2\lambda}$ **and** $c_2' = \frac{c_2}{2\lambda}$

EXAMPLE:

Find the curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ which give the minimum area of the surface of revolution around y – axis.

Solution:

This is a Dido Problem in xy – plane. We want to find a curve which gives the minimum area of surface of revolution generated around y – axis.

Since curve revolve around y – axis therefore

$Area = \int_A^B 2\pi x ds = 2\pi \int_A^B x\sqrt{1+(y')^2}dx$ **with** $F(x,y,y') = x\sqrt{1+(y')^2}$

Since F is not depend on 'y' therefore we use following EL equation;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = 0 \ and \ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = Constant$$

$\Rightarrow \frac{\partial}{\partial y'}\left(x\sqrt{1+(y')^2}\right) = a\ (say) \Rightarrow \frac{xy'}{\sqrt{1+(y')^2}} = a$ **after solving**

$$\Rightarrow xy' = a\sqrt{1+(y')^2} \Rightarrow x^2(y')^2 = a^2(1+(y')^2) \Rightarrow (x^2-a^2)(y')^2 = a^2$$

$$\Rightarrow (y')^2 = \frac{a^2}{x^2-a^2} \Rightarrow y' = \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}} \Rightarrow \int dy = \int \frac{a}{\sqrt{x^2-a^2}}dx$$

$\Rightarrow y = aCosh^{-1}\left(\frac{x}{a}\right) + c$ **required.**

EXAMPLE:

Find the curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ which give the minimum area of the surface of revolution around x – axis.

Solution:

This is a Dido Problem in xy – plane. We want to find a curve which gives the minimum area of surface of revolution generated around x – axis.

Since curve revolve around x – axis therefore

$Area = \int_A^B 2\pi y ds = 2\pi \int_A^B y\sqrt{1+(y')^2}dx$ **with** $F(x,y,y') = y\sqrt{1+(y')^2}$

Since F is not depend on 'x' therefore we use following EL equation;

$$F - y'\left(\frac{\partial F}{\partial y'}\right) = Constant$$

$$y\sqrt{1+(y')^2} - y'\left(\frac{\partial}{\partial y'}\left(y\sqrt{1+(y')^2}\right)\right) = Constant$$

$$\Rightarrow y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} = a\ (say)$$

$$\Rightarrow \frac{y\left(1+(y')^2\right) - y(y')^2}{\sqrt{1+(y')^2}} = a \Rightarrow y(1+(y')^2 - (y')^2) = a\sqrt{1+(y')^2}$$

$$\Rightarrow y = a\sqrt{1+(y')^2} \Rightarrow y^2 = a^2(1+(y')^2) = a^2 + a^2(y')^2$$

$$\Rightarrow y^2 - a^2 = a^2(y')^2$$

$$\Rightarrow (y')^2 = \frac{y^2-a^2}{a^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\sqrt{y^2-a^2}}{a} \Rightarrow \int \frac{a}{\sqrt{y^2-a^2}}dy = \int dx$$

$\Rightarrow x = aCosh^{-1}\left(\frac{y}{a}\right) + c$ **required.**

EXAMPLE:

On what curves can the functional $I = \int_0^{\frac{\pi}{2}}((y')^2 - y^2)dx$ **with condition** $y(0) = 0, y\left(\frac{\pi}{2}\right) = 1$ **be extremized.**

Solution:

$I = \int_0^{\frac{\pi}{2}}\sqrt{(y')^2 - y^2}dx$ **with** $F(x,y,y') = (y')^2 - y^2$

Since F is not depend on 'y' therefore we use following EL equation;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial y}((y')^2 - y^2) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}((y')^2 - y^2)\right) = 0$$

$$\Rightarrow -2y - \frac{d}{dx}(2y') = 0 \Rightarrow -2(y + y'') = 0 \Rightarrow y'' + y = 0$$

Then general solution will be $y = Acosx + Bsinx$

$\Rightarrow y(0) = 0 \Rightarrow A = 0$ **and** $\Rightarrow y\left(\frac{\pi}{2}\right) = 1 \Rightarrow B = 1$

Hence The general solution will be $y = sinx$

EXAMPLE: (UoS, 2013 – I, 2015 – II)

Find the extermal for $I = \int_0^{\frac{\pi}{2}}((y')^2 + (z')^2 + 2yz)dx$ **with condition** $y(0) = 0, y\left(\frac{\pi}{2}\right) = 1; z(0) = 0, z\left(\frac{\pi}{2}\right) = -1$ **be extremized.**

Solution:

We have $I = \int_0^{\frac{\pi}{2}}((y')^2 + (z')^2 + 2yz)dx$ **with** $F = (y')^2 + (z')^2 + 2yz$

since there are two unknown functions 'y', 'z' (extermal curves) there will be a pair of EL equations;

Since F is not depend on 'y' therefore we use following EL equation;

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$ **……….(i)** **and** $\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z\prime}\right) = 0$ **……….(ii)**

$(i) \Rightarrow \frac{\partial}{\partial y}((y')^2 + (z')^2 + 2yz) - \frac{d}{dx}\left(\frac{\partial}{\partial y\prime}((y')^2 + (z')^2 + 2yz)\right) = 0$

$\Rightarrow 2z - \frac{d}{dx}(2y') = 0 \Rightarrow 2(z - y'') = 0 \Rightarrow y'' = z$ **……….(iii)**

$(ii) \Rightarrow \frac{\partial}{\partial z}((y')^2 + (z')^2 + 2yz) - \frac{d}{dx}\left(\frac{\partial}{\partial z\prime}((y')^2 + (z')^2 + 2yz)\right) = 0$

$\Rightarrow 2y - \frac{d}{dx}(2z') = 0 \Rightarrow 2(y - z'') = 0 \Rightarrow z'' = y$ **……….(iv)**

Using (iii) in (iv) we get $\Rightarrow y^{iv} - y = 0$ **……….(v)**

Then general solution of (v) will be $y = Ae^x + Be^{-x} + Ccosx + Esinx$

And $y'' = z = Ae^x + Be^{-x} - Ccosx - Esinx$

$\Rightarrow y(0) = 0 \Rightarrow A + B + E = 0$ **……….(vi)**

And $\Rightarrow y\left(\frac{\pi}{2}\right) = 1 \Rightarrow Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} + E = 1$ **……….(vii)**

Similarly $\Rightarrow z(0) = 0 \Rightarrow A + B - C = 0$ **……….(viii)**

And $\Rightarrow z\left(\frac{\pi}{2}\right) = -1 \Rightarrow Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} - E = -1$ **……….(ix)**

Adding (v) and (vii) $B = -A$ **also subtraction from (v) and (vii)** $\underline{C = 0}$

Adding (vi) and (viii) $Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} = 0$ **also subtraction from (vi) and (viii)** $\underline{E = 1}$ **then using the relation** $B = -A$ **we get** $\underline{A = 0, B = 0}$

Putting all values we get $y = sinx, z = -sinx$

EXAMPLE:

Find the extermal for $I = \int_0^{\frac{\pi}{2}}((y'')^2 - y^2 + x^2)dx$ **with condition** $y(0) = 1, y\left(\frac{\pi}{2}\right) = 0; y'(0) = 0, y'\left(\frac{\pi}{2}\right) = 1$ **be extremized.**

Solution:

We have $I = \int_0^{\frac{\pi}{2}}((y'')^2 - y^2 + x^2)dx$

with $F = F(x, y, y', y'') = (y'')^2 - y^2 + x^2$

therefore the extermal curve $y = y(x)$ **is obtained by the solving EL equation**

$\frac{\partial F}{\partial y} + (-1)^1 \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$ **……….(i)**

$(i) \Rightarrow -2y + 0 + \frac{d^2}{dx^2}(2y'') = 0 \Rightarrow -2y + 2y^{iv} = 0$

$\Rightarrow y^{iv} - y = 0$ **……….(ii)**

Then general solution of (v) will be $y = Ae^x + Be^{-x} + Ccosx + Esinx$

And $y' = Ae^x - Be^{-x} + Ccosx - Esinx$

$\Rightarrow y(0) = 1 \Rightarrow A + B + C = 1$ **……….(iii)**

And $\Rightarrow y\left(\frac{\pi}{2}\right) = 0 \Rightarrow Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} + E = 0$ **……….(iv)**

Similarly $\Rightarrow y'(0) = 0 \Rightarrow A - B + E = 0$ **……….(v)**

And $\Rightarrow y'\left(\frac{\pi}{2}\right) = 1 \Rightarrow Ae^{\frac{\pi}{2}} - Be^{-\frac{\pi}{2}} - C = -1$ **……….(vi)**

Subtracting and simlifying (iv) and (v) $\left(e^{\frac{\pi}{2}} - 1\right)A + \left(e^{-\frac{\pi}{2}} + 1\right)B = 0$

Adding and simlifying (iii) and (vi) $\left(e^{\frac{\pi}{2}} + 1\right)A - \left(e^{-\frac{\pi}{2}} - 1\right)B - 2 = 0$

$$\Rightarrow \frac{A}{2\left(e^{-\frac{\pi}{2}}+1\right)} = \frac{B}{-2\left(e^{\frac{\pi}{2}}-1\right)} = \frac{1}{-\left(e^{\frac{\pi}{2}}-1\right)\left(e^{-\frac{\pi}{2}}-1\right)-\left(e^{\frac{\pi}{2}}+1\right)\left(e^{-\frac{\pi}{2}}+1\right)}$$

$$\Rightarrow \frac{A}{2\left(e^{-\frac{\pi}{2}}+1\right)} = \frac{B}{-2\left(e^{\frac{\pi}{2}}-1\right)} = -\frac{1}{4}$$

$\Rightarrow A = \frac{1}{2}\left(e^{-\frac{\pi}{2}} + 1\right)$ **and** $\Rightarrow B = -\frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)$

$(iii) \Rightarrow \frac{1}{2}\left(e^{-\frac{\pi}{2}} + 1\right) - \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + C = 1 \Rightarrow C = \frac{1}{2}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)$

$(v) \Rightarrow \frac{1}{2}\left(e^{-\frac{\pi}{2}} + 1\right) + \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + E = 1 \Rightarrow E = -\frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right)$

Hence

$$y = \frac{1}{2}\left(e^{-\frac{\pi}{2}} + 1\right)e^x - \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)e^{-x} + \frac{1}{2}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)cosx - \frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right)sinx$$

EXAMPLE:

Show that the EL equation for the functional $I = \int_a^b F(x, y, z, y', z')dx = 0$ **admit the following 1st integrals;**

i. $\frac{\partial F}{\partial y'} = C$ **if F does not contains 'y'**

ii. $F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = constant$ **if F does not contains 'x'**

Solution: The corresponding EL equations are

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ **……..(i)** **and** $\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$ **……..(ii)**

i. **When F is independent of 'y'**

Then $\frac{\partial F}{\partial y} = 0$ **then EL equation becomes as follows;**

$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = Constant$

ii. **When F is independent of 'x'**

Since $F = F(x, y, z, y', z')$

$\Rightarrow dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y}dz + \frac{\partial F}{\partial y'}dy' + \frac{\partial F}{\partial z'}dz'$ **……..(iii)**

From (i) and (ii) $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ **and** $\frac{\partial F}{\partial z} = \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)$

$\frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$ **and** $\frac{\partial F}{\partial z} = \frac{d}{dz}\left(\frac{\partial F}{\partial z'}\right)\frac{dz}{dx}$

$\frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)y'$ **and** $\frac{\partial F}{\partial z} = \frac{d}{dz}\left(\frac{\partial F}{\partial z'}\right)z'$

$\frac{\partial F}{\partial y}dy = d\left(\frac{\partial F}{\partial y'}\right)y'$ **and** $\frac{\partial F}{\partial z}dz = d\left(\frac{\partial F}{\partial z'}\right)z'$

$(iii) \Rightarrow dF = d\left(\frac{\partial F}{\partial y'}\right)y' + d\left(\frac{\partial F}{\partial z'}\right)z' + \frac{\partial F}{\partial y'}dy' + \frac{\partial F}{\partial z'}dz'$

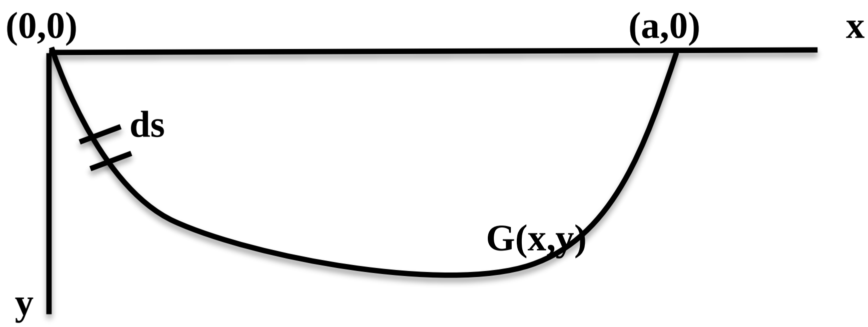
$\Rightarrow dF = d\left(\frac{\partial F}{\partial y'}y'\right) + d\left(\frac{\partial F}{\partial z'}z'\right) \Rightarrow d\left(F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'}\right) = 0$

$\Rightarrow F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = constant$

EXAMPLE: (*BRACHISTOCHRONE PROBLEM*):

A uniform cable is fixed at its ends at the same level in space and is allowed to hang under gravity. Find the final shape of the cable.

SOLUTION:



The final shape of the cable wil correspond to the state of a stable equilibrium or minimum P.E. we choose the coordinate axis as shown in the figure. Let (0,0) and (a,0) be the position of the end points of the cable. The P.E. of the cable is given by $V = mg\bar{y}$ where $\bar{y}$ is the y – coordinate of centroid of the cable. The minimum value of V corresponds to the minimum value of $\bar{y}$

Now y – coordinate of centroid of the curve $y = y(x)$ is given by

$$\bar{y} = \frac{V}{mg} = \frac{mgy}{mg} = \frac{my}{m} = \frac{\int_0^a \rho y ds}{\int_0^a \rho ds} = \frac{\int_0^a y ds}{\int_0^a ds} = \frac{1}{l}\int_0^a y\sqrt{1+(y')^2}dx$$

Where 'l ' is the length of the curve.i.e $l = \int_0^a ds = \int_0^a \sqrt{1+(y')^2}dx$

And we use $\rho = \frac{m}{l} \Rightarrow m = \rho l = \rho \int_0^a ds$

Here $F = y\sqrt{1+(y')^2}, G = \sqrt{1+(y')^2}$ and therefore we construct a new function $H = F + \lambda G = y\sqrt{1+(y')^2} + \lambda\sqrt{1+(y')^2}$

$$\Rightarrow H = (y+\lambda)\sqrt{1+(y')^2}$$

Since there is no explicit dependence on 'x' so we use the special case of EL equation. i.e. $H - y'\frac{\partial H}{\partial y'} = cosntant$

$$\Rightarrow (y+\lambda)\sqrt{1+(y')^2} - y'\frac{\partial}{\partial y'}\left((y+\lambda)\sqrt{1+(y')^2}\right) = c_1$$

$$\Rightarrow (y+\lambda)\sqrt{1+(y')^2} - \frac{\lambda(y')^2}{\sqrt{1+(y')^2}} = c_1$$

$$\Rightarrow (y+\lambda)\left(\frac{1+(y')^2-(y')^2}{\sqrt{1+(y')^2}}\right) = c_1 \Rightarrow (y+\lambda)\left(\frac{1}{\sqrt{1+(y')^2}}\right) = c_1$$

$$\Rightarrow \frac{1}{\sqrt{1+(y')^2}} = \frac{c_1}{y+\lambda} \Rightarrow \frac{(c_1)^2}{(y+\lambda)^2} = \frac{1}{1+(y')^2} \Rightarrow 1+(y')^2 = \frac{(y+\lambda)^2}{(c_1)^2}$$

$$\Rightarrow (y')^2 = \frac{(y+\lambda)^2}{(c_1)^2} - 1 \Rightarrow (y')^2 = \frac{(y+\lambda)^2-(c_1)^2}{(c_1)^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\sqrt{(y+\lambda)^2-(c_1)^2}}{c_1}$$

$$\Rightarrow \int \frac{c_1}{\sqrt{(y+\lambda)^2-(c_1)^2}}dy = \int dx \Rightarrow -\int \frac{c_1}{\sqrt{z^2-(c_1)^2}}dz = \int dx \qquad \text{put } y+\lambda = z$$

$$\Rightarrow c_1 Cosh^{-1}\left(\frac{z}{c_1}\right) = x + c_2 \Rightarrow Cosh^{-1}\left(\frac{z}{c_1}\right) = \frac{x+c_2}{c_1} \Rightarrow \frac{z}{c_1} = Cosh\left(\frac{x+c_2}{c_1}\right)$$

$$\Rightarrow \frac{y+\lambda}{c_1} = Cosh\left(\frac{x+c_2}{c_1}\right) \Rightarrow y = c_1 Cosh\left(\frac{x+c_2}{c_1}\right) - \lambda \qquad \text{.............(i)}$$

$$\Rightarrow y(0) = 0 \;\; and \;\; y(a) = 0 \Rightarrow \frac{\lambda}{c_1} = c_1 Cosh\left(\frac{c_2}{c_1}\right) \;\; and \;\; \frac{\lambda}{c_1} = c_1 Cosh\left(\frac{a+c_2}{c_1}\right)$$

$$\Rightarrow c_1 Cosh\left(\frac{c_2}{c_1}\right) = c_1 Cosh\left(-\frac{a+c_2}{c_1}\right) \Rightarrow \frac{c_2}{c_1} = -\frac{a+c_2}{c_1} \Rightarrow c_2 = -\frac{a}{2}$$

Using $c_2 = -\frac{a}{2}$ and $\lambda = c_1 Cosh\left(\frac{-a}{2c_1}\right)$ then using in (i) we get

$$\Rightarrow y = c_1 Cosh\left(\frac{x-\frac{a}{2}}{c_1}\right) - c_1 Cosh\left(\frac{-a}{2c_1}\right)$$

$$\Rightarrow y = c_1 C\left[osh\left(\frac{x-\frac{a}{2}}{c_1}\right) - Cosh\left(\frac{-a}{2c_1}\right)\right]$$ **This curve is called Catenary.**

EXAMPLE:

Show that a solid of revolution which for a given surface area has maximum volume is a sphere.
OR find the curve which generates a surface of revolution of a given area which enclosed the maximum volume.

SOLUTION:

Let a curve $y = y(x)$ with $y(0) = 0 = y(a)$ be rotated about x – axis so as to generate a surface of revolution. An element of the surface is . therefore total area will be $A = 2\pi\int_0^a y\,ds = 2\pi\int_0^a y\sqrt{1+(y')^2}dx$ **and the volume**

element or solid of revolution is $\pi y^2 dx$ therefore total volume will be
$V = \pi \int_0^a y^2 dx$

Here $F = y^2, G = y\sqrt{1+(y')^2}$ and therefore we construct a new function

$H = F + \lambda G = y^2 + \lambda y\sqrt{1+(y')^2}$

Since there is no explicit dependence on 'x' so we use the special case of EL equation. i.e. $H - y'\frac{\partial H}{\partial y'} = cosntant$

$\Rightarrow y^2 + \lambda y\sqrt{1+(y')^2} - y'\frac{\partial}{\partial y'}\left(y^2 + \lambda y\sqrt{1+(y')^2}\right) = c$

$\Rightarrow y^2 + \lambda y\sqrt{1+(y')^2} - \frac{\lambda y(y')^2}{\sqrt{1+(y')^2}} = c \Rightarrow y^2 + \lambda y\left[\sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}}\right] = c$

$\Rightarrow \lambda y\left[\frac{1+(y')^2-(y')^2}{\sqrt{1+(y')^2}}\right] = c - y^2 \Rightarrow \lambda y = (c - y^2)\sqrt{1+(y')^2}$ ………..(i)

Using $y(0) = 0 \Rightarrow c = 0, \sqrt{1+(y')^2} \neq 0$

$(i) \Rightarrow \lambda y = -y^2\sqrt{1+(y')^2} \Rightarrow \lambda = -y\sqrt{1+(y')^2} \Rightarrow \lambda^2 = y^2[1+(y')^2]$

$\Rightarrow \lambda^2 - y^2 = y^2(y')^2 \Rightarrow (y')^2 = \frac{\lambda^2 - y^2}{y^2} \Rightarrow y' = \frac{\sqrt{\lambda^2 - y^2}}{y} = \frac{dy}{dx}$

$\Rightarrow \int \frac{y}{\sqrt{\lambda^2 - y^2}} dy = \int dx \Rightarrow -\sqrt{\lambda^2 - y^2} = x + a \Rightarrow \lambda^2 - y^2 = (x+a)^2$

$\Rightarrow (x+a)^2 + (y-0)^2 = \lambda^2$ this is an equation of circle centered at $(a, 0)$ having radius λ and hence the surface of revolution is sphere.

EXAMPLE: Find eigenvalue and eigen function of the functional

$I = \int_0^3 [(2x+3)^2(y')^2 - y^2]dx$ subjected to the endpoin conditions

$y(0) = 0 = y(3)$ and side condition $\int_0^3 y^2 dx$

SOLUTION:

Here $F = (2x+3)^2(y')^2 - y^2, G = y^2$ and therefore we construct a new function $H = F + \lambda G = (2x+3)^2(y')^2 - y^2 + \lambda y^2$

$H = (2x+3)^2(y')^2 + (\lambda - 1)y^2$

Using EL equation $\frac{\partial H}{\partial y}-\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right)=0$

$$\frac{\partial}{\partial y}((2x+3)^2(y')^2+(\lambda-1)y^2)-\frac{d}{dx}\left(\frac{\partial}{\partial y'}((2x+3)^2(y')^2+(\lambda-1)y^2)\right)=0$$

$$\Rightarrow (\lambda-1)2y-\frac{d}{dx}((2x+3)^2 2y')=0$$

$$\Rightarrow -2\left[\frac{d}{dx}((2x+3)^2y')-(\lambda-1)y\right]=0 \Rightarrow \frac{d}{dx}((2x+3)^2y')-(\lambda-1)y=0$$

$$\Rightarrow (2x+3)^2y''+2(2x+3)2y'-(\lambda-1)y=0$$

$$\Rightarrow 4\left(x+\frac{3}{2}\right)^2y''+8\left(x+\frac{3}{2}\right)y'+(1-\lambda)y=0$$

$$\Rightarrow \left[4\left(x+\frac{3}{2}\right)^2D^2+8\left(x+\frac{3}{2}\right)D+(1-\lambda)\right]y=0 \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Put $2x+3=e^t \Rightarrow ln(2x+3)=t$

And $\left(x+\frac{3}{2}\right)D=\Delta \Rightarrow \left(x+\frac{3}{2}\right)^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$

$$(i) \Rightarrow [4\Delta^2-4\Delta+8\Delta+(1-\lambda)]y=0$$

$\Rightarrow 4\Delta^2-4\Delta+8\Delta+(1-\lambda)=0$ **since** $y\neq 0$

$$\Rightarrow 4\Delta^2+4\Delta+(1-\lambda)=0 \Rightarrow \Delta=-\frac{1}{2}\pm\frac{1}{2}\sqrt{\lambda}$$

if $\lambda=0$ and $\lambda>0$ **We obtain trivial solution for the given problem**

if $\lambda<0$ **We obtain non – trivial solution for the given problem**

if $\lambda=-\mu^2$ **then** $\Delta=-\frac{1}{2}+\frac{1}{2}\mu i$

and general solution will be $y(x)=e^{-\frac{1}{2}t}\left[c_1Cos\frac{1}{2}\mu t+c_2Sin\frac{1}{2}\mu t\right]$

$$y(x)=(e^t)^{-\frac{1}{2}}\left[c_1Cos\frac{\mu}{2}ln(2x+3)+c_2Sin\frac{\mu}{2}ln(2x+3)\right]$$

$$y(x)=(2x+3)^{-\frac{1}{2}}\left[c_1Cos\frac{\mu}{2}ln(2x+3)+c_2Sin\frac{\mu}{2}ln(2x+3)\right] \quad \ldots\ldots\ldots\text{(ii)}$$

Using $y(0)=0$

$$c_1Cos\frac{\mu}{2}ln(3)+c_2Sin\frac{\mu}{2}ln(3)=0 \quad \ldots\ldots\ldots\text{(iii)}$$

Also Using $y(3) = 0$

$$c_1 Cos\frac{\mu}{2}ln(9) + c_2 Sin\frac{\mu}{2}ln(9) = 0$$

$$\Rightarrow c_1 Cos\mu ln(3) + c_2 Sin\mu ln(3) = 0 \qquad \text{………..(iv)}$$

For non – trivial solution

$$\begin{vmatrix} Cos\frac{\mu}{2}ln(3) & Sin\frac{\mu}{2}ln(3) \\ Cos\mu ln(3) & Sin\mu ln(3) \end{vmatrix} = 0$$

$$\Rightarrow \left(Cos\frac{\mu}{2}ln(3)\right)\left(Sin\mu ln(3)\right) - \left(Cos\mu ln(3)\right)\left(Sin\frac{\mu}{2}ln(3)\right) = 0$$

$$\Rightarrow Sin\left(\mu ln(3) - \frac{\mu}{2}ln(3)\right) = 0 \Rightarrow \mu ln(3) - \frac{\mu}{2}ln(3) = Sin^{-1}(0)$$

$$\Rightarrow \frac{\mu}{2}ln(3) = n\pi \qquad n = 1,2,3,\ldots\ldots\ldots\ldots$$

$$\Rightarrow \mu = \frac{2n\pi}{ln(3)} \Rightarrow \mu_n = \frac{2n\pi}{ln(3)} \qquad n = 1,2,3,\ldots\ldots\ldots\ldots$$

$$(iv) \Rightarrow c_1 Cos\frac{2n\pi}{ln(3)}ln(3) + c_2 Sin\frac{2n\pi}{ln(3)}ln(3) = 0$$

$$\Rightarrow c_1 Cos2n\pi + c_2 Sin2n\pi = 0 \Rightarrow c_1(1) + c_2(0) = 0 \Rightarrow c_1 = 0$$

But $c_2 \neq 0$ we take $c_2 = c_n$ then eigen solution will be as follows;

$$y_n(x) = \frac{c_n}{\sqrt{2x+3}} Sin\frac{n\pi}{ln(3)} ln(2x+3)$$

GEODESIC:

A geodesic is the curve of shortest length joining two points in space.

EXAMPLE:

Find the curve of shortest length between the given points in a plane using polar coordinates.

Solution:

Since we know that $l = \int_A^B ds$ …………..(i)

Also $ds = \sqrt{(dx)^2 + (dy)^2}$ …………..(ii)

Now usig $x = rCos\theta, y = rSin\theta$

$(dx)^2 = (dr)^2 Cos^2\theta + r^2 Sin^2\theta (d\theta)^2 - 2rdrCos\theta Sin\theta$

$(dy)^2 = (dr)^2 Sin^2\theta + r^2 Cos^2\theta (d\theta)^2 + 2rdrCos\theta Sin\theta$

$(ii) \Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dr)^2 + r^2 (d\theta)^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$\Rightarrow ds = \sqrt{r^2 + (r')^2} d\theta$

$(i) \Rightarrow l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2} d\theta$ **subjected to** $r(\theta_1) = c_1$ **and** $r(\theta_2) = c_2$

Here $F = \sqrt{r^2 + (r')^2}$ **Since there is no explicit dependence on 'θ' so we use the special case of EL equation. i.e.** $F - r'\frac{\partial F}{\partial r'} = cosntant$

$\Rightarrow \sqrt{r^2 + (r')^2} - r'\frac{\partial}{\partial r'}\left[\sqrt{r^2 + (r')^2}\right] = c_1$

$\Rightarrow \sqrt{r^2 + (r')^2} - r'\frac{r'}{\sqrt{r^2 + (r')^2}} = c_1 \Rightarrow \frac{r^2 + (r')^2 - (r')^2}{\sqrt{r^2 + (r')^2}} = c \Rightarrow \frac{r^2}{\sqrt{r^2 + (r')^2}} = c_1$

$\Rightarrow \frac{\sqrt{r^2 + (r')^2}}{r^2} = \frac{1}{c_1} \Rightarrow \sqrt{r^2 + (r')^2} = \frac{r^2}{c_1} \Rightarrow r^2 + (r')^2 = \frac{r^4}{c_1^2} \Rightarrow (r')^2 = \frac{r^4}{c_1^2} - r^2$

$\Rightarrow (r')^2 = \frac{r^4 - c_1^2 r^2}{c_1^2} \Rightarrow (r')^2 = \frac{r^2(r^2 - c_1^2)}{c_1^2} \Rightarrow r' = \frac{dr}{d\theta} = \frac{r\sqrt{r^2 - c_1^2}}{c_1}$

$\Rightarrow c_1 \int \frac{1}{r\sqrt{r^2 - c_1^2}} dr = \int d\theta \Rightarrow c_1 \frac{1}{c_1} Sec^{-1}\left(\frac{r}{c_1}\right) = \theta + c_2 \Rightarrow Sec^{-1}\left(\frac{r}{c_1}\right) = \theta + c_2$

$\Rightarrow \frac{r}{c_1} = Sec(\theta + c_2) \Rightarrow \frac{r}{Sec(\theta + c_2)} = c_1 \Rightarrow c_1 = rCos(\theta + c_2)$

$\Rightarrow c_1 = r(Cos\theta Cosc_2 - Sin\theta Sinc_2)$

$\Rightarrow c_1 = (rCos\theta . Cosc_2 - rSin\theta . Sinc_2)$

$\Rightarrow c_1 = (xCosc_2 - ySinc_2)$

$\Rightarrow -xCosc_2 + ySinc_2 + c_1 = 0$

$\Rightarrow -\propto x + \beta y + \gamma = 0$

Which represent the straight line.

EXAMPLE: (UoS, 2015 – II)

Find the curve of shortest length on the surface of sphere.

Solution:

Let A and Bbe the two points on the sphere S. here the problem is to minimize

Since we know that $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ **………(i)**

Now usig $x = rSin\theta Cos\varphi, y = rSin\theta Sin\varphi, z = rCos\theta$

$dx = r[Cos\theta d\theta Cos\varphi - Sin\theta Sin\varphi d\varphi]$

$dy = r[Cos\theta d\theta Sin\varphi + Sin\theta Cos\varphi d\varphi]$

$dz = -rSin\theta d\theta$

$\Rightarrow ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{r^2\left[1 + Sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right]}d\theta$

$\Rightarrow ds = r\sqrt{1 + Sin^2\theta(\varphi')^2}d\theta$

$(i) \Rightarrow l = r\int_{\theta_1}^{\theta_2}\sqrt{1 + Sin^2\theta(\varphi')^2}d\theta$ **subjected to** $r(\theta_1) = c_1$ **and** $r(\theta_2) = c_2$

Here $F = \sqrt{1 + Sin^2\theta(\varphi')^2}$ **then corresponding EL equation will be**

$\frac{\partial F}{\partial \varphi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \varphi'}\right) = 0$

$\Rightarrow 0 - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \varphi'}\right) = 0 \Rightarrow \frac{\partial F}{\partial \varphi'} = Constant \Rightarrow \frac{\partial}{\partial \varphi'}\left(\sqrt{1 + Sin^2\theta(\varphi')^2}\right) = C$

$\Rightarrow \frac{Sin^2\theta\varphi'}{\sqrt{1+Sin^2\theta(\varphi')^2}} = C \Rightarrow Sin^2\theta\varphi' = C\sqrt{1 + Sin^2\theta(\varphi')^2}$

$\Rightarrow Sin^4\theta(\varphi')^2 = C^2(1 + Sin^2\theta(\varphi')^2) \Rightarrow Sin^4\theta(\varphi')^2 = C^2 + C^2Sin^2\theta(\varphi')^2$

$\Rightarrow Sin^4\theta(\varphi')^2 - C^2Sin^2\theta(\varphi')^2 = C^2 \Rightarrow Sin^2\theta(Sin^2\theta - C^2)(\varphi')^2 = C^2$

$\Rightarrow (\varphi')^2 = \frac{C^2}{Sin^2\theta(Sin^2\theta - C^2)} \Rightarrow \varphi' = \frac{d\varphi}{d\theta} = \frac{C}{Sin\theta\sqrt{Sin^2\theta - C^2}}$

$\Rightarrow \varphi' = \frac{d\varphi}{d\theta} = \frac{C}{Sin^2\theta\sqrt{1 - \frac{C^2}{Sin^2\theta}}} = \frac{C.Cosec^2\theta}{\sqrt{1 - C^2Cosec^2\theta}} = \frac{C.Cosec^2\theta}{\sqrt{1 - C^2(1 + Cot^2\theta)}} = \frac{C.Cosec^2\theta}{\sqrt{1 - C^2 - C^2Cot^2\theta}}$

$$\Rightarrow \int d\varphi = \int \frac{C.Cosec^2\theta}{\sqrt{1-C^2-C^2Cot^2\theta}}d\theta \Rightarrow \varphi = \int \frac{C.Cosec^2\theta}{C\sqrt{\left(\frac{\sqrt{1-C^2}}{C}\right)^2-Cot^2\theta}}d\theta$$

$$\Rightarrow \varphi = \int \frac{Cosec^2\theta}{\sqrt{\left(\frac{\sqrt{1-C^2}}{C}\right)^2-Cot^2\theta}}d\theta$$

$\Rightarrow \varphi = \int \frac{-1}{\sqrt{a^2-t^2}}dt$ with $\frac{\sqrt{1-C^2}}{C} = a$; $Cot\theta = t$; $-Cosec^2\theta d\theta = dt$

$$\Rightarrow \varphi = Cos^{-1}\left(\frac{t}{a}\right) + \propto \Rightarrow \varphi = Cos^{-1}\left(\frac{Cot\theta}{a}\right) + \propto \Rightarrow \varphi - \propto = Cos^{-1}\left(\frac{Cot\theta}{a}\right)$$

$$\Rightarrow Cos(\varphi - \propto) = \frac{Cot\theta}{a} \Rightarrow Cos\varphi Cos \propto + Sin\varphi Sin \propto = \frac{1}{a}.\frac{Cos\theta}{Sin\theta}$$

$$\Rightarrow raSin\theta Cos\varphi Cos \propto + raSin\theta Sin\varphi Sin \propto = rCos\theta$$

$$\Rightarrow a(rSin\theta Cos\varphi)Cos \propto + a(rSin\theta Sin\varphi)Sin \propto = rCos\theta$$

$$\Rightarrow aCos \propto x + aSin \propto y = z \Rightarrow Ax + By = z$$

This is an equation of the plane through center of sphere. Hence the curve of shortest length joining A and B is the arc of great circle through A and B.

EXAMPLE: (UoS, 2017 – II, 2019 – I)

Find the geodesic curve for the cylinder $x^2 + y^2 = a^2$

Solution:

We have to minimize $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ **.........(i)**

Now usig $x = rCos\theta, y = rSin\theta, z = z$ **for cylindrical coordinates**

$$dx = -rSin\theta d\theta, dy = rCos\theta d\theta, dz = dz$$

$$\Rightarrow ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{r^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$(i) \Rightarrow l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (z')^2}d\theta$ **subjected to** $r(\theta_1) = c_1$ **and** $r(\theta_2) = c_2$

Here $F = \sqrt{r^2 + (z')^2}$ **then corresponding EL equation will be**

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0$$

$$\Rightarrow 0 - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0 \Rightarrow \frac{\partial F}{\partial z'} = Constant \Rightarrow \frac{\partial}{\partial z'}\left(\sqrt{r^2 + (z')^2}\right) = C$$

$$\Rightarrow \frac{z'}{\sqrt{r^2+(z')^2}} = C \Rightarrow z' = C\sqrt{r^2 + (z')^2} \Rightarrow (z')^2 = C^2(r^2 + (z')^2)$$

$$\Rightarrow (z')^2 - C^2(z')^2 = C^2r^2 \Rightarrow (1 - C^2)(z')^2 = C^2r^2 \Rightarrow (z')^2 = \frac{C^2r^2}{(1-C^2)}$$

$$\Rightarrow z' = \frac{Cr}{\sqrt{1-C^2}} \Rightarrow \frac{dz}{d\theta} = \propto (say) \Rightarrow z = \propto \theta + C' \Rightarrow z - C' = \propto Tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow Tan\left(\frac{z-C'}{\propto}\right) = \frac{y}{x}$$

The intersection of this surface with given cylinder gives required extreme curve.

EXAMPLE:

Find the shortest distance between the points $A(1, -1, 0)$ and $B(2, 1, -1)$ in the plane $15x - 7y + z - 22 = 0$

Solution:

We have to minimize $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$l = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx = \int_A^B \sqrt{1 + (y')^2 + (z')^2}\, dx$$

$$\Rightarrow l = \int_{x_1}^{x_2} \sqrt{1 + (y')^2 + (z')^2}\, dx$$

subjected to constraint $15x - 7y + z - 22$

Here $F = \sqrt{1 + (y')^2 + (z')^2}, G = 15x - 7y + z - 22$

and therefore we construct a new function

$$H = F + \lambda G = \sqrt{1 + (y')^2 + (z')^2} + \lambda(15x - 7y + z - 22)$$

Using EL equation $\quad \frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$$\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)-$$

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)\right)=0$$

$$\Rightarrow -7\lambda-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \quad \text{..............(i)}$$

Also Using EL equation $\frac{\partial H}{\partial z}-\frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right)=0$

$$\frac{\partial}{\partial z}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)-$$

$$\frac{d}{dx}\left(\frac{\partial}{\partial z'}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)\right)=0$$

$$\Rightarrow \lambda-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \quad \text{..............(ii)}$$

Multiplying (ii) with 7 then adding in (i)

$$-\frac{d}{dx}\left(\frac{y'+7z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \Rightarrow \frac{y'+7z'}{\sqrt{1+(y')^2+(z')^2}}=C \quad \text{..............(iii)}$$

Since $15x-7y+z-22=0$

The endpoint conditions satisfied by the functions $y=y(x)$ **and** $z=z(x)$ **are**

$y(1)=-1, y(2)=1, z(1)=0, z(2)=-1$

$\Rightarrow 15-7y'+z'=0 \Rightarrow z'=7y'-15$ **diff. w.r.to 'x'**

$$(iii) \Rightarrow \frac{y'+7(7y'-15)}{\sqrt{1+(y')^2+(7y'-15)^2}}=C$$

$$\Rightarrow y'+49y'-105=C\sqrt{1+(y')^2+49(y')^2+225-210y'}$$

$$\Rightarrow 50y'-105=C\sqrt{50(y')^2-210y'+226}$$

$$\Rightarrow [5(10y'-21)]^2=\left[C\sqrt{50(y')^2-210y'+226}\right]^2$$

$$\Rightarrow 25(10y'-21)^2=C^2(50(y')^2-210y'+226)$$

$$\Rightarrow 25(100(y')^2-420y'+441)=C^2(50(y')^2-210y'+226)$$

$$\Rightarrow (2500-50C^2)(y')^2+(210C^2-11000)y'+(11025-226C^2)=0$$

This is the quadratic equation in y'

Since C was arbitray, we can always choose it , so that the equation has real roots. Let $\propto$ be one such root then $y' = \propto = dy/dx$

$\Rightarrow y = \propto x + \beta$

Now using $y(1) = -1, y(2) = 1, z(1) = 0, z(2) = -1$

$\propto + \beta = -1, 2 \propto + \beta = 1$ **then** $\propto = 2,, \beta = -3$

Then we get $\Rightarrow y = 2x - 3 \Rightarrow y' = 2$

Also for z' we have $\Rightarrow z' = 7y' - 15 = -1$

Then required least distance is $\Rightarrow l = \int_1^2 \sqrt{1 + (y')^2 + (z')^2} dx$

$\Rightarrow l = \int_1^2 \sqrt{1 + 4 + 1} dx = \sqrt{6} |x|_1^2 = \sqrt{6}$

$\Rightarrow l = \sqrt{6}$ **is required least distance.**

EXAMPLE:

Find the shortest distance between the points $A(1, 0, -1)$ and $B(0, -1, 1)$ in the plane $x + y + z = 0$

Solution:

We have to minimize $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$l = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx = \int_A^B \sqrt{1 + (y')^2 + (z')^2} dx$$

$$\Rightarrow l = \int_{x_1}^{x_2} \sqrt{1 + (y')^2 + (z')^2} dx$$

subjected to constraint $x + y + z$

Here $F = \sqrt{1 + (y')^2 + (z')^2}, G = x + y + z$

and therefore we construct a new function

$$H = F + \lambda G = \sqrt{1 + (y')^2 + (z')^2} + \lambda(x + y + z)$$

Using EL equation $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$$\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(x+y+z=0)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2+(z')^2}+\right.\right.$$

$$\left.\left.\lambda(x+y+z=0)\right)\right)=0$$

$$\Rightarrow \lambda-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Also Using EL equation $\frac{\partial H}{\partial z}-\frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right)=0$

$$\frac{\partial}{\partial z}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(x+y+z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial z'}\left(\sqrt{1+(y')^2+(z')^2}+\right.\right.$$

$$\left.\left.\lambda(x+y+z)\right)\right)=0$$

$$\Rightarrow \lambda-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

Subtracting (i) and (ii) we get

$$\frac{d}{dx}\left(\frac{y'-z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \Rightarrow \frac{y'-z'}{\sqrt{1+(y')^2+(z')^2}}=C \qquad \ldots\ldots\ldots\ldots\text{(iii)}$$

Since $x+y+z=0$

The endpoint condition satisfied by the functions $y=y(x)$ **is**

$y(1)=0, y(0)=-1$

$\Rightarrow 1+y'+z'=0 \Rightarrow z'=-1-y'$ **diff. w.r.to 'x'**

$$(iii) \Rightarrow \frac{y'+1+y'}{\sqrt{1+(y')^2+(-1-y')^2}}=C$$

$$\Rightarrow 2y'+1=C\sqrt{1+(y')^2+(y')^2+1+2y'}$$

$$\Rightarrow [2y'+1]^2=\left[C\sqrt{2+2(y')^2+2y'}\right]^2$$

$$\Rightarrow 1+4(y')^2+4y'=C^2(2+2(y')^2+2y')$$

$$\Rightarrow (4-2C^2)(y')^2+(4-2C^2)y'+(1-2C^2)=0$$

This is the quadratic equation in y'

Since C was arbitray, we can always choose it , so that the equation has real roots. Let $\propto$ be one such root then $y' = \propto = dy/dx$

$\Rightarrow y = \propto x + \beta$

Now using $y(1) = 0, y(0) = -1$

$\propto + \beta = -1, \propto (0) + \beta = -1$ **then** $\propto = 1,, \beta = -1$

Then we get $\Rightarrow y = x - 1 \Rightarrow y' = 1$

Also for z' **we have** $\Rightarrow z' = -1 - y' = -2$

Then required least distance is $\Rightarrow l = \int_0^1 \sqrt{1 + (y')^2 + (z')^2}\,dx$

$\Rightarrow l = \int_0^1 \sqrt{1 + 1 + 4}\,dx = \sqrt{6}|x|_0^1 = \sqrt{6}$

$\Rightarrow l = \sqrt{6}$ **is required least distance.**

INTEGRAL EQUATIONS (IE's)

Integral equations are an important tool in solving problems of Applied mathematics and mathematical physics.

A special advantage of using integral equations in dealing with IVP's or BVP's is that the IC's or BC's are automatically incorporated in the resulting integral equation.

INTEGRAL EQUATION: An equation which involves the unknown variable under the integral sigh is called Integral Equation.
EXAMPLES:

- $\int_a^b xu(x)dx = 1$, $\int_a^b K(x,x')u(x')dx' = f(x)$ **are non – homogeneous IE's**
- $\int_a^b K(x,x')u(x')dx' - u^2(x) = 0$ **is homogeneous IE where** $u(x')$ **is the unknown function and** $K(x,x')$ **is called Kernal.**

LINEAR INTEGRAL EQUATION: An equation in which unknown function appears linearly is called Linear Integral Equation.
EXAMPLES:

$\int_a^b xu(x)dx = 1$ **and** $\int_a^b K(x,y)u(y)dy = f(x)$

NON – LINEAR INTEGRAL EQUATION: An equation in which unknown function does not appears linearly is called non – Linear Integral Equation.
EXAMPLE: $\int_a^b K(x,y)u(y)dy = u^2(x)$

HOMOGENEOUS INTEGRAL EQUATION: An Integral Equation in which unknown function vanishes i.e. $u(y) = 0$ **then both sides of equation are equal, such type of equation is called H.I.EQ.**
EXAMPLE: $\int_a^b K(x,y)u(y)dy - u^2(x) = 0$

Non – HOMOGENEOUS INTEGRAL EQUATION: An Integral Equation in which unknown function does not vanishes i.e. $u(y) \neq 0$**, such type of equation is called non – H.I.EQ.**
EXAMPLE: $\int_a^b K(x,y)u(y)dy = f(x)$

FREDHOLM INTEGRAL EQUATION OF 1ST KIND:
An Integral Equation of the form $f(x) = \int_a^b K(x,y)u(y)dy$ where $f(x)$ and $K(x,y)$ are known functions is called F.I.Eq. of 1st kind. And $K(x,y)$ is called Kernal of the IE.

FREDHOLM INTEGRAL EQUATION OF 2nd KIND:
The non – Homogeneous linear Integral Equation of the form
$u(x) = f(x) + \lambda\int_a^b K(x,y)u(y)dy$ where $f(x)$ and $K(x,y)$ are known functions is called F.I.Eq. of 2nd kind. And $K(x,y)$ is called Kernal of the IE.

HOMOGENEOUS AND NON – HOMOGENEOUS F.I.EQUATIONs.
The linear IE $u(x) = \lambda\int_a^b K(x,y)u(y)dy$ is called homogeneous FI Eq. it can be written as an operator equation in the form $u = \lambda Ku \Rightarrow Ku = \frac{1}{\lambda}u$
A F.IE which is not homogeneus is called non – homogeneous or inhomogeneous.

VOLTERA INTEGRAL EQUATION OF 1ST KIND:
If $K(x,y) = 0$ when y>x then Fredholm integral equation of 1st kind assume the form $f(x) = \int_a^x K(x,y)u(y)dy$ and is called Voltera IE of the 1st kind.

VOLTERA INTEGRAL EQUATION OF 2nd KIND:
If $K(x,y) = 0$ when y > x then Fredholm integral equation of 2nd kind assume the form $u(x) = f(x) + \lambda\int_a^x K(x,y)u(y)dy$ and is called Voltera IE of the 2nd kind.

REMEMBER: Voltera IE of 1st kind $f(x) = \int_a^x K(x,y)u(y)dy$ can be convertd into Voltera IE of 2nd kind by differentiating 1st kind w.r.to 'x' and using Leibniz rule i.e. $f'(x) = \int_a^x \frac{\partial}{\partial x}K(x,y)u(y)dy + K(x,x)u(x)$

SINGULAR INTEGRAL EQUATION:
An IE in which either one or both limits of integration are infinite or the integrand become infinite anywhere in the range of integration is called Singular IE.
EXAMPLE: $\int_{-1}^{2}\frac{1}{x}dx$ **in this integrand $\frac{1}{x}$ becomes infinite in $0\epsilon[-1,2]$**

TYPES OF KERNELS

HERMITIAN KERNELS:

If $K(x,y) = \overline{K(y,x)}$ wher bar denots the complex conjugate, then $K(x,y)$ is called Hermitian kernel.

SYMMETRIC KERNELS:

If $K(x,y) = K(y,x)$, then $K(x,y)$ is called "real symmetric kernel" or "merely symmetric kernel".

It is clear that every symmetric kernel is also Hermitian.

CONVOLUTION TYPE KERNELS:

If $K(x,y) = K(x-y)$, then $K(x,y)$ is called convolution type kernel and the corresponding IE is called Convoluton type IE.

SQUARE INTEGRABLE KERNEL:

A kernel $K(x,y)$ defined over $a \le x, y \le b$ i.e. over the square $[a,b] \times [a,b]$ is called square integrable if $\int_a^b \int_a^b |K(x,y)|^2 dxdy < \infty$

i.e. $\int_a^b \int_a^b |K(x,y)|^2 dxdy$ is finite.

SEPARABLE OR DEGENERATE KERNEL:

If a kernel is of the form $K(x,y) = \sum_i g_i(x) h_i(y)$ i.e. it can be expressed as sum of products of functions of x only and y only, then it is called Separable or degenerate kernel.

EXAMPLE:

- **$Sin(x+y), e^{x+y}, xy^2 + x^2y$ are separable.**
- **$Sin(xy), e^{xy}, ln(x+y)$ are not separable.**

STEPS TO FIND SOLUTION OF AN INTEGRAL EQUATION FOR A SEPARABLE KERNEL

(UoS, 2013 – I , 2014 – II , 2019 – I)

- **Start with Fredholm integral equation of 2nd kind in the notation**

  $u(x) = f(x) + \lambda \int_a^b K(x,y)u(y)dy$

with the separable kernel $K(x,y) = \sum_i A_i(x)B_i(y)$ where $A_i(x)$, $B_i(y)$ are linearly independent sets of functions.

- **Susbstitute value of $K(x,y)$**

  $\Rightarrow u(x) = f(x) + \lambda \sum_i A_i(x) \int_a^b B_i(y)u(y)dy$

  $\Rightarrow u(x) = f(x) + \lambda \sum_i C_i A_i(x)$ **with** $C_i = \int_a^b B_i(y)u(y)dy$

  $\Rightarrow u(x) = f(x) + \lambda \sum_k C_k A_k(x)$ **rewriting**

- **Now for** $C_i = \int_a^b B_i(y)u(y)dy$

  $\Rightarrow C_i = \int_a^b B_i(y)[f(y) + \lambda \sum_k C_k A_k(y)]dy$

  $\Rightarrow C_i = \int_a^b B_i(y)f(y)dy + \lambda \sum_k C_k \int_a^b B_i(y)A_k(y)dy$

  $\Rightarrow C_i = f_i + \lambda \sum_k C_k\, a_{ik} \Rightarrow C_i - \lambda \sum_k C_k\, a_{ik} = f_i$

  $\Rightarrow \sum_k C_k \delta_{ik} - \lambda \sum_k C_k\, a_{ik} = f_i$ **since** $\sum_k \delta_{ik} C_k = C_i$

  $\Rightarrow \sum_k (\delta_{ik} - \lambda a_{ik})\, C_k = f_i$ **…………(A)**

- **Now since** $\delta_{ik} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$ **then previous equation is equivalent to a set of n – algebraic equations in 'n' unknown constants $C_1, C_2, C_3, \ldots .. C_n$ and written in full form as follows;**

  $(1 - \lambda a_{11})C_1 - \lambda a_{12}C_2 - \cdots \ldots \ldots . -\lambda a_{1n}C_n = f_1$

  $-\lambda a_{21}C_1 + (1 - \lambda a_{22})C_2 - \cdots \ldots \ldots . -\lambda a_{2n}C_n = f_2$ **…………(B) for all**

  $\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..$

  $-\lambda a_{n1}C_1 - \lambda a_{n2}C_2 - \cdots \ldots \ldots . +(1 - \lambda a_{nn})C_n = f_n$

Then system of equations (A) and (B) will have a unique solution if the matrix of coefficients is non – singular

i.e. if $D(\lambda) = |\delta_{ik} - \lambda a_{ik}| \neq 0$

- **Let** $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_n$ **be the roots of the equation** $D(\lambda) = 0$ **then the system will have a unique solution if** $\lambda \neq \lambda_i$ **(an eigenvalue)**
- **After solving the system** $\sum_k (\delta_{ik} - \lambda a_{ik})\, C_k = f_i$ **we substitute for** C_i **in** $u(x) = f(x) + \lambda \sum_i C_i A_i(x)$ **and obtaing the solution of given IE.**

RESOLVENT KERNEL:

If the solution of the IE $u(x) = f(x) + \lambda \int_a^b K(x,y)u(y)dy$ **is written as** $u(x) = f(x) + \lambda \int_a^b \Gamma(x,y;\lambda)u(y)dy$ **then** $\Gamma(x,y;\lambda)$ **is called Resolvent Kernel**

EXAMPLE: (UoS, 2014 – II)

Solve Fredholm IE of 2nd kind given by $u(x) = x + \lambda \int_0^1 (xy^2 + x^2y)u(y)dy$

Solution: **here** $K(x,y) = xy^2 + x^2y$ **is Separable Kernel.**

- **Given** $u(x) = x + \lambda \int_0^1 (xy^2 + x^2y)u(y)dy$ **………..(i)**

  $\Rightarrow u(x) = x + \lambda \int_0^1 xy^2u(y)dy + \lambda \int_0^1 x^2yu(y)dy$

  $\Rightarrow u(x) = x + \lambda x \int_0^1 y^2u(y)dy + \lambda x^2 \int_0^1 yu(y)dy$

  $\Rightarrow u(x) = x + \lambda xC_1 + \lambda x^2C_2$ **………..(ii)**

  with $C_1 = \int_0^1 y^2u(y)dy$ **and** $C_2 = \int_0^1 yu(y)dy$

  $\Rightarrow u(y) = y + \lambda yC_1 + \lambda y^2C_2$ **rewriting**

- **Now for** $C_i = \int_a^b B_i(y)u(y)dy$

  $\Rightarrow C_i = \int_a^b B_i(y)[f(y) + \lambda \sum_k C_kA_k(y)]dy$

$$\Rightarrow C_1 = \int_0^1 y^2[y + \lambda y C_1 + \lambda y^2 C_2]dy$$

$$\Rightarrow C_1 = \int_0^1 [y^3 + \lambda y^2 C_1 + \lambda y^4 C_2]dy = \frac{1}{4} + \frac{\lambda C_1}{4} + \frac{\lambda C_2}{5}$$

$$\Rightarrow \left(\frac{\lambda}{4} - 1\right) C_1 + \frac{\lambda C_2}{5} = -\frac{1}{4} \qquad \text{..........(iii)}$$

Also $\Rightarrow C_2 = \int_0^1 y[y + \lambda y C_1 + \lambda y^2 C_2]dy$

$$\Rightarrow C_2 = \int_0^1 [y^2 + \lambda y^2 C_1 + \lambda y^3 C_2]dy = \frac{1}{3} + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{4}$$

$$\Rightarrow \frac{\lambda}{3} C_1 + \left(\frac{\lambda}{4} - 1\right) C_2 = -\frac{1}{3} \qquad \text{..........(iv)}$$

Now $(iii) \Rightarrow (5\lambda - 20)C_1 + 4\lambda C_2 = -5 \qquad \times ng\ with\ 20$

And $(iv) \Rightarrow 4\lambda C_1 + (3\lambda - 12)C_2 = -4 \qquad \times ng\ with\ 12$

As above system of two equations is in two unknowns. It will have a unique solution if

$$\begin{vmatrix} 5\lambda - 20 & 4\lambda \\ 4\lambda & 3\lambda - 12 \end{vmatrix} \neq 0$$

$$\Rightarrow (5\lambda - 20)(3\lambda - 12) - 16\lambda^2 \neq 0$$

$$\Rightarrow \lambda^2 + 120\lambda - 240 \neq 0$$

Now from above system of two equations is in two unknowns.

$$\frac{C_1}{16\lambda - 15\lambda + 60} = \frac{C_2}{20\lambda - 20\lambda + 80} = \frac{1}{\lambda^2 + 120\lambda - 240}$$

$$\Rightarrow C_1 = \frac{\lambda + 60}{\lambda^2 + 120\lambda - 240},\ ,\ C_2 = \frac{80}{\lambda^2 + 120\lambda - 240}$$

Thus our required system is from (ii)

$$\Rightarrow u(x) = x + \lambda x\left(\frac{\lambda + 60}{\lambda^2 + 120\lambda - 240}\right) + \lambda x^2\left(\frac{80}{\lambda^2 + 120\lambda - 240}\right)$$

EXAMPLE: (UoS, 2013 – I)

Solve Fredholm IE of 2nd kind given by $u(x) = f(x) + \lambda \int_0^1 (x+y)u(y)dy$

also obtain its Resolven Kernel.

Solution: here $K(x,y) = x + y$ **is Separable Kernel.**

- **Given** $u(x) = f(x) + \lambda \int_0^1 (x+y)u(y)dy$ **………..(i)**

  $\Rightarrow u(x) = f(x) + \lambda \int_0^1 xu(y)dy + \lambda \int_0^1 yu(y)dy$

  $\Rightarrow u(x) = f(x) + \lambda x \int_0^1 u(y)dy + \lambda \int_0^1 yu(y)dy$

  $\Rightarrow u(x) = f(x) + \lambda x C_1 + \lambda C_2$ **………..(ii)**

  with $C_1 = \int_0^1 u(y)dy$ **and** $C_2 = \int_0^1 yu(y)dy$

  $\Rightarrow u(y) = f(y) + \lambda y C_1 + \lambda C_2$ **rewriting**

- **Now for** $C_i = \int_a^b B_i(y)u(y)dy$

  $\Rightarrow C_i = \int_a^b B_i(y)[f(y) + \lambda \sum_k C_k A_k(y)]dy$

  $\Rightarrow C_1 = \int_0^1 u(y)dy = \int_0^1 [f(y) + \lambda y C_1 + \lambda C_2]dy$

  $\Rightarrow C_1 = \int_0^1 f(y)dy + \lambda C_1 \int_0^1 ydy + \lambda C_2 \int_0^1 dy$

  $\Rightarrow C_1 = f_1 + \frac{\lambda C_1}{2} + \lambda C_2$

  $\Rightarrow \left(1 - \frac{\lambda}{2}\right) C_1 - \lambda C_2 = f_1$ **………..(iii)**

  Also $\Rightarrow C_2 = \int_0^1 yu(y)dy = \int_0^1 y[f(y) + \lambda y C_1 + \lambda C_2]dy$

  $\Rightarrow C_2 = \int_0^1 yf(y)dy + \lambda C_1 \int_0^1 y^2 dy + \lambda C_2 \int_0^1 ydy$

  $\Rightarrow C_2 = f_2 + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{2}$

  $\Rightarrow -\frac{\lambda}{3} C_1 + \left(1 - \frac{\lambda}{2}\right) C_2 = f_2$ **………..(iv)**

Now $\Rightarrow \left(1-\frac{\lambda}{2}\right)C_1 - \lambda C_2 = f_1$

And $\Rightarrow -\frac{\lambda}{3}C_1 + \left(1-\frac{\lambda}{2}\right)C_2 = f_2$

As above system of two equations is in two unknowns. It will have a unique solution if

$$\begin{vmatrix} 1-\frac{\lambda}{2} & -\lambda \\ -\frac{\lambda}{3} & 1-\frac{\lambda}{2} \end{vmatrix} \neq 0$$

$$\Rightarrow \left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3} \neq 0$$

Now from above system of two equations is in two unknowns.

$$\frac{C_1}{\lambda f_2 + \left(1-\frac{\lambda}{2}\right)f_1} = \frac{C_2}{\frac{\lambda}{3}f_1 + \left(1-\frac{\lambda}{2}\right)f_2} = \frac{1}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}$$

$$\Rightarrow C_1 = \frac{\lambda f_2 + \left(1-\frac{\lambda}{2}\right)f_1}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}},, C_2 = \frac{\frac{\lambda}{3}f_1 + \left(1-\frac{\lambda}{2}\right)f_2}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}$$

Thus our required system is from (ii)

$$\Rightarrow u(x) = f(x) + \lambda x\left(\frac{\lambda f_2 + \left(1-\frac{\lambda}{2}\right)f_1}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}\right) + \lambda\left(\frac{\frac{\lambda}{3}f_1 + \left(1-\frac{\lambda}{2}\right)f_2}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}\right) \ldots\ldots\ldots\ldots(v)$$

FOR RESOLVENT KERNEL:

Rearranging (v)

$$\Rightarrow u(x) = f(x) + \left(\frac{\lambda}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}\right)\left[\lambda x f_2 + x\left(1-\frac{\lambda}{2}\right)f_1 + \frac{\lambda}{3}f_1 + \left(1-\frac{\lambda}{2}\right)f_2\right]$$

Now substituting $f_1 = \int_0^1 f(y)dy$ **and** $f_2 = \int_0^1 yf(y)dy$

$$\Rightarrow u(x) = f(x) + \left(\frac{\lambda}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}\right)\left[\lambda x\int_0^1 yf(y)dy + x\left(1-\frac{\lambda}{2}\right)\int_0^1 f(y)dy + \frac{\lambda}{3}\int_0^1 f(y)dy + \left(1-\frac{\lambda}{2}\right)\int_0^1 yf(y)dy\right]$$

$$\Rightarrow u(x) = f(x) + \left(\frac{\lambda}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}\right)\int_0^1 \left[\lambda xy + x\left(1-\frac{\lambda}{2}\right) + \frac{\lambda}{3} + \left(1-\frac{\lambda}{2}\right)y\right] f(y)dy$$

$$\Rightarrow u(x) = f(x) + \lambda\int_0^1 \left[\frac{\lambda xy + x\left(1-\frac{\lambda}{2}\right) + \frac{\lambda}{3} + \left(1-\frac{\lambda}{2}\right)y}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}\right] f(y)dy$$

$$\Rightarrow u(x) = f(x) + \lambda\int_0^1 \Gamma(x,y;\lambda)\, f(y)dy$$

Where $\Gamma(x,y;\lambda) = \left[\frac{\lambda xy + x\left(1-\frac{\lambda}{2}\right) + \frac{\lambda}{3} + \left(1-\frac{\lambda}{2}\right)y}{\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}}\right]$ **is the Resolvent Kernel.**

EXAMPLE:

Solve Fredholm IE of 2nd kind given by

$u(x) = f(x) + \lambda\int_0^{2\pi}(SinxSiny)u(y)dy$ **also obtain its Resolven Kernel.**

Solution: **here** $K(x,y) = SinxSiny$ **is Separable Kernel.**

- **Given** $u(x) = f(x) + \lambda\int_0^{2\pi}(SinxSiny)u(y)dy$ **………..(i)**

  $\Rightarrow u(x) = f(x) + \lambda Sinx\int_0^{2\pi}(Siny)u(y)dy$

  $\Rightarrow u(x) = f(x) + \lambda SinxC$ **………..(ii)**

  with $C = \int_0^{2\pi}(Siny)u(y)dy$

  $\Rightarrow u(y) = f(y) + \lambda SinyC$ **rewriting**

- **Now for** $C_i = \int_a^b B_i(y)u(y)dy$

  $\Rightarrow C_i = \int_a^b B_i(y)[f(y) + \lambda\sum_k C_k A_k(y)]dy$

  $\Rightarrow C = \int_0^{2\pi}(Siny)u(y)dy = \int_0^{2\pi}(Siny)[f(y) + \lambda SinyC]dy$

  $\Rightarrow C = \int_0^{2\pi}(Siny)f(y)dy + \lambda\int_0^{2\pi} Sin^2 ydy$

  $\Rightarrow C = f_0 + \lambda C\pi$

  $\Rightarrow (1-\lambda\pi)C = f_0 \Rightarrow C = \frac{f_0}{1-\lambda\pi}$

Thus our required system is from (ii)

$\Rightarrow u(x) = f(x) + \lambda Sinx\left(\frac{f_0}{1-\lambda\pi}\right)$ **………..(iii)**

FOR RESOLVENT KERNEL:

Rearranging (iii)

$\Rightarrow u(x) = f(x) + \frac{\lambda}{1-\lambda\pi} Sinx(f_0)$

Now substituting $f_0 = \int_0^{2\pi}(Siny)f(y)dy$

$\Rightarrow u(x) = f(x) + \frac{\lambda}{1-\lambda\pi}\int_0^{2\pi} SinxSinyf(y)dy$

$\Rightarrow u(x) = f(x) + \lambda\int_0^{2\pi}\left[\frac{SinxSiny}{1-\lambda\pi}\right]f(y)dy$

$\Rightarrow u(x) = f(x) + \lambda\int_0^{2\pi}\Gamma(x,y;\lambda)f(y)dy$

Where $\Gamma(x,y;\lambda) = \left[\frac{SinxSiny}{1-\lambda\pi}\right]$ **is the Resolvent Kernel.**

EXAMPLE: (UoS, 2018 – I)

Solve Fredholm IE of 2nd kind given by

$g(x) = f(x) + \lambda\int_{-1}^{1}(xt + x^2t^2)g(t)dt$ **also obtain its Resolven Kernel.**

Solution: **here** $K(x,t) = xt + x^2t^2$ **is Separable Kernel.**

- **Given** $g(x) = f(x) + \lambda\int_{-1}^{1}(xt + x^2t^2)g(t)dt$ **………..(i)**

  $\Rightarrow g(x) = f(x) + \lambda x\int_{-1}^{1} tg(t)dt + +\lambda x^2\int_{-1}^{1} t^2g(t)dt$

  $\Rightarrow g(x) = f(x) + \lambda xC_1 + \lambda x^2C_2$ **………..(ii)**

  with $C_1 = \int_{-1}^{1} tg(t)dt$ **and** $C_2 = \int_{-1}^{1} t^2g(t)dt$

  $\Rightarrow g(t) = f(t) + \lambda tC_1 + \lambda t^2C_2$ **rewriting**

- **Now for** $C_i = \int_a^b B_i(t)u(t)dt$

  $\Rightarrow C_i = \int_a^b B_i(t)[f(t) + \lambda\sum_k C_kA_k(t)]dt$

$$\Rightarrow C_1 = \int_{-1}^{1} tg(t)dt = \int_{-1}^{1} t[f(t) + \lambda t C_1 + \lambda t^2 C_2]dt$$

$$\Rightarrow C_1 = \int_{-1}^{1} tf(t)dt + \lambda C_1 \int_{-1}^{1} t^2 dt + \lambda C_2 \int_{-1}^{1} t^3 dt$$

$$\Rightarrow C_1 = f_1 + \frac{2}{3}\lambda C_1$$

$$\Rightarrow \left(1 - \frac{2}{3}\lambda\right) C_1 = f_1 \Rightarrow C_1 = \frac{f_1}{\left(1-\frac{2}{3}\lambda\right)} \;\; ; \; \lambda \neq \frac{3}{2} \qquad \text{………..(iii)}$$

Also $$\Rightarrow C_2 = \int_{-1}^{1} t^2 g(t)dt = \int_{-1}^{1} t^2[f(t) + \lambda t C_1 + \lambda t^2 C_2]dt$$

$$\Rightarrow C_2 = \int_{-1}^{1} t^2 f(t)dt + \lambda C_1 \int_{-1}^{1} t^3 dt + \lambda C_2 \int_{-1}^{1} t^4 dt$$

$$\Rightarrow C_2 = f_2 + \frac{2}{5}\lambda C_2$$

$$\Rightarrow \left(1 - \frac{2}{5}\lambda\right) C_2 = f_2 \Rightarrow C_2 = \frac{f_2}{\left(1-\frac{2}{5}\lambda\right)} \;\; ; \; \lambda \neq \frac{5}{2} \qquad \text{………..(iv)}$$

Thus our required system is from (ii)

$$\Rightarrow g(x) = f(x) + \lambda x \left(\frac{f_1}{\left(1-\frac{2}{3}\lambda\right)}\right) + \lambda x^2 \left(\frac{f_2}{\left(1-\frac{2}{5}\lambda\right)}\right) \text{………..(v)}$$

FOR RESOLVENT KERNEL:

As $$\Rightarrow g(x) = f(x) + \lambda x C_1 + \lambda x^2 C_2 \qquad \text{………..(ii)}$$

Substituting $f_1 = \int_{-1}^{1} tf(t)dt$ **and** $f_2 = \int_{-1}^{1} t^2 f(t)dt$ **in (ii)**

$$\Rightarrow g(x) = f(x) + \left(\frac{\lambda x}{\left(1-\frac{2}{3}\lambda\right)}\right)\int_{-1}^{1} tf(t)dt + \left(\frac{\lambda x^2}{\left(1-\frac{2}{5}\lambda\right)}\right)\int_{-1}^{1} t^2 f(t)dt$$

$$\Rightarrow g(x) = f(x) + \lambda \int_{-1}^{1} \left[\frac{xt}{\left(1-\frac{2}{3}\lambda\right)} + \frac{x^2 t^2}{\left(1-\frac{2}{5}\lambda\right)}\right] f(t)dt$$

$$\Rightarrow g(x) = f(x) + \lambda \int_{-1}^{1} [\Gamma(x,t;\lambda)]\, f(t)dt$$

Where $\Gamma(x,t;\lambda) = \left[\frac{xt}{\left(1-\frac{2}{3}\lambda\right)} + \frac{x^2 t^2}{\left(1-\frac{2}{5}\lambda\right)}\right]$ **is the Resolvent Kernel.**

EXAMPLE:

Find the eigenvalue and eigensolution of the IE

$$u(x) = \lambda \int_0^{\pi} (Cos^2xCos2y + Cos^3yCos3x)u(y)dy$$

Solution: Given $(x) = \lambda \int_0^{\pi} (Cos^2xCos2y + Cos^3yCos3x)u(y)dy$ **…..(i)**

$\Rightarrow u(x) = \lambda \int_0^{\pi} Cos^2xCos2yu(y)dy + \lambda \int_0^{\pi} Cos^3yCos3xu(y)dy$

$\Rightarrow u(x) = \lambda Cos^2x \int_0^{\pi} Cos2yu(y)dy + \lambda Cos3x \int_0^{\pi} Cos^3yu(y)dy$

$\Rightarrow u(x) = \lambda Cos^2xC_1 + \lambda Cos3xC_2$ **………..(ii)**

with $C_1 = \int_0^{\pi} Cos2yu(y)dy$ **and** $C_2 = \int_0^{\pi} Cos^3yu(y)dy$

$\Rightarrow u(y) = \lambda Cos^2yC_1 + \lambda Cos3yC_2$ **rewriting**

Now

$\Rightarrow C_1 = \int_0^{\pi} Cos2yu(y)dy = \int_0^{\pi} Cos2y[\lambda Cos^2yC_1 + \lambda Cos3yC_2]dy$

$\Rightarrow C_1 = \lambda \int_0^{\pi} Cos2yCos^2yC_1dy + \lambda \int_0^{\pi} Cos2yCos3yC_2dy$

$\Rightarrow C_1 = C_1\lambda I_1 + C_2\lambda I_2$ **………..(iii)**

Now $I_1 = \int_0^{\pi} Cos2yCos^2ydy = \int_0^{\pi} Cos2y\left(\frac{1+Cos2y}{2}\right)dy$

$I_1 = \frac{1}{2}\int_0^{\pi}[Cos2y + Cos^22y]dy = \frac{1}{2}\left|\frac{Sin2y}{2}\right|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\left(\frac{1+Cos4y}{2}\right)dy$

$I_1 = 0 + \frac{1}{4}\left|y + \frac{Sin4y}{4}\right|_0^{\pi} \Rightarrow I_1 = \frac{\pi}{4}$

Also $I_2 = \int_0^{\pi} Cos2yCos3ydy = \frac{1}{2}\int_0^{\pi} 2Cos2yCos3ydy$

$I_2 = \frac{1}{2}\int_0^{\pi}(Cos5y + Cosy)dy = \frac{1}{2}\left|\frac{Sin5y}{5} + Siny\right|_0^{\pi} \Rightarrow I_2 = 0$

$(iii) \Rightarrow C_1 = C_1\lambda\frac{\pi}{4} \Rightarrow \left(1 - \lambda\frac{\pi}{4}\right)C_1 = 0 \Rightarrow C_1 = 0$

Or $\left(1 - \lambda\frac{\pi}{4}\right)C_1 + 0C_2 = 0$ **………..(iv)**

Now as $C_2 = \int_0^{\pi} Cos^3yu(y)dy$

$\Rightarrow C_2 = \int_0^{\pi} Cos^3y[\lambda Cos^2yC_1 + \lambda Cos3yC_2]dy$

$\Rightarrow C_2 = \lambda C_1 \int_0^{\pi} Cos^3 y Cos^2 y dy + \lambda C_2 \int_0^{\pi} Cos^3 y Cos3y dy$

$\Rightarrow C_2 = \lambda C_1 I_3 + \lambda C_2 I_4$

Now $I_3 = \int_0^{\pi} Cos^3 y Cos^2 y dy = \int_0^{\pi} Cos^5 y dy$

Put $z = y - \frac{\pi}{2} \Rightarrow dz = dy$ also as $y \to 0, \pi$ then $z \to -\frac{\pi}{2}, +\frac{\pi}{2}$ respectively

$I_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Cos^5\left(z + \frac{\pi}{2}\right) dz = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Cos^5 z dz = 0$ because integral is odd.

Now $I_4 = \int_0^{\pi} Cos^3 y Cos3y dy = \int_0^{\pi} Cos^3 y(-3Cosy + 4Cos^3 y) dy$

$I_4 = \int_0^{\pi} (-3Cos^4 y + 4Cos^6 y) dy$

Put $z = y - \frac{\pi}{2} \Rightarrow dz = dy$ also as $y \to 0, \pi$ then $z \to -\frac{\pi}{2}, +\frac{\pi}{2}$ respectively

$I_4 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-3Cos^4\left(z + \frac{\pi}{2}\right) + 4Cos^6\left(z + \frac{\pi}{2}\right)\right) dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-3Sin^4 z + 4Sin^6 z) dz$

$I_4 = \int_0^{\frac{\pi}{2}} (-6Sin^4 z + 8Sin^6 z) dz$

Now using the recursion formulae

$\int_0^{\frac{\pi}{2}} Sin^n z dz = \frac{1.3.5 \ldots (n-1)}{2.4.6 \ldots n} \times \frac{\pi}{2}$ when 'n' is even

$I_4 = -6 \times \frac{1.3}{2.4} \times \frac{\pi}{2} + 8 \times \frac{1.3.5}{2.4.6} \times \frac{\pi}{2} = \frac{\pi}{8}$

$\Rightarrow C_2 = \lambda C_1(0) + \lambda C_2\left(\frac{\pi}{8}\right) \Rightarrow 0C_1 + \left(1 - \lambda\frac{\pi}{8}\right) C_2 = 0$ ………..(v)

From (iv) and (v) we can write

$\left(1 - \lambda\frac{\pi}{4}\right) C_1 + 0C_2 = 0 \Rightarrow 0C_1 + \left(1 - \lambda\frac{\pi}{8}\right) C_2 = 0$

For eigenvalues $\begin{vmatrix} 1 - \lambda\frac{\pi}{4} & 0 \\ 0 & 1 - \lambda\frac{\pi}{8} \end{vmatrix} = 0$

$\Rightarrow \left(1 - \lambda\frac{\pi}{4}\right)\left(1 - \lambda\frac{\pi}{8}\right) = 0 \Rightarrow \left(1 - \lambda\frac{\pi}{4}\right) = 0, \left(1 - \lambda\frac{\pi}{8}\right) = 0$

$\Rightarrow \lambda = \lambda_1 = \frac{4}{\pi}$, $\lambda = \lambda_2 = \frac{8}{\pi}$

For $\lambda = \lambda_1 = \frac{4}{\pi}$ equation (iv) becomes $\left(1 - \lambda\frac{\pi}{4}\right)C_1 + 0C_2 = 0$

$\Rightarrow \left(1 - \frac{4}{\pi}.\frac{\pi}{4}\right)C_1 + 0C_2 = 0 \Rightarrow 0 = 0$

And For $\lambda = \lambda_1 = \frac{4}{\pi}$ equation (v) becomes $0C_1 + \left(1 - \lambda\frac{\pi}{8}\right)C_2 = 0$

$\Rightarrow 0C_1 + \left(1 - \frac{4}{\pi}.\frac{\pi}{8}\right)C_2 = 0 \Rightarrow C_2 = 0$ and C_1 arbitrary

$(ii) \Rightarrow u(x) = \lambda Cos^2 x C_1$

Therefore eigenfunction for $\lambda = \lambda_1 = \frac{4}{\pi}$ is $u^1(x) = \frac{4}{\pi}C_1 Cos^2 x$

$\Rightarrow u^1(x) = Cosntant \times Cos^2 x$

For $\lambda = \lambda_2 = \frac{8}{\pi}$ equation (iv) becomes $\left(1 - \lambda\frac{\pi}{4}\right)C_1 + 0C_2 = 0$

$\Rightarrow \left(1 - \frac{8}{\pi}.\frac{\pi}{4}\right)C_1 + 0C_2 = 0 \Rightarrow C_1 = 0$ and C_2 arbitrary

$(ii) \Rightarrow u(x) = \lambda C_2 Cos^2 3x$

Therefore eigenfunction for $\lambda = \lambda_2 = \frac{8}{\pi}$ is $u^2(x) = \frac{8}{\pi}C_2 Cos^2 3x$

$\Rightarrow u^2(x) = Cosntant \times Cos^2 3x$

THE NEUMANN SERIES FOR THE SOLUTION OF THE FREDHOLM NON – HOMOGENEOUS I.E. OF THE SECOND KIND WITH SEPARABLE KERNEL OR MAY NOT BE A SEPARABLE.

OR METHOD OF SUCCESSIVE APPROXIMATION (UoS, 2018 – I)

- **Start with Fredholm integral equation of 2nd kind in the notation**

  $$u(x) = f(x) + \lambda \int_a^b K(x,y)u(y)dy$$

  Where $f(x)$ and $K(x,y)$ are square integrable functions.

  i.e. $\int_a^b |f(x)|^2\, dx < \infty$ **and** $\int_a^b \int_a^b |K(x,y)|^2\, dxdy < \infty$

  and the kernel may not be separable.

  We want to find condition under which solution converges.

- **First consider the zeroth order approximation and take $u_0(x) = f(x)$**

  $\Rightarrow u_0(y) = f(y)$ **for the solution of IE of 2nd kind.**

- **To obtain first order approximation,**

  Since $u(x) = f(x) + \lambda \int_a^b K(x,y)u(y)dy$

  $$\Rightarrow u_1(x) = f(x) + \lambda \int_a^b K(x,y)u_0(y)dy$$

  $$\Rightarrow u_1(x) = f(x) + \lambda \int_a^b K(x,y')f(y')dy'$$

  Similarly for second approximation;

  $$\Rightarrow u_2(x) = f(x) + \lambda \int_a^b K(x,y)u_1(y)dy$$

  … … … … … … … … … … … … … … … … … … ..

  … … … … … … … … … … … … … … … … … … ..

  And in general we have;

  $$\Rightarrow u_{n+1}(x) = f(x) + \lambda \int_a^b K(x,y)u_n(y)dy$$

  Here $u_n(x) \to u(x)$ as $n \to \infty$ then the approximation is said to converge to $u(x)$.

Now we find convergence of solutio;

As $u_1(x) = f(x) + \lambda \int_a^b K(x, y')f(y')dy'$

- **Then** $u_2(x) = f(x) + \lambda \int_a^b K(x, y)u_1(y)dy$

  $\Rightarrow u_2(x) = f(x) + \lambda \int_a^b K(x, y)\left[f(y) + \lambda \int_a^b K(x, y')f(y')dy'\right]dy$

  $\Rightarrow u_2(x) =$

  $f(x) + \lambda \int_a^b K(x, y)f(y)dy + \lambda^2 \int_a^b \left[\int_a^b K(x, y)K(x, y')dy\right]f(y')dy'$

  $\Rightarrow u_2(x) = f(x) + \lambda \int_a^b K_1(x, y)f(y)dy + \lambda^2 \int_a^b K_2(x, y)\, f(y)dy$

  Where $K_1(x, y) = K(x, y)$ **and** $K_2(x, y) = \int_a^b K(x, y)K(x, y')dy$

- **Similarly**

  $u_3(x) = f(x) + \lambda \int_a^b K_1(x, y)f(y)dy + \lambda^2 \int_a^b K_2(x, y)\, f(y)dy +$

  $\lambda^3 \int_a^b K_3(x, y)\, f(y)dy$

- **And in general;**

  $u_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x, y)\, f(y)dy$

  And $u_n(x) \to u(x)$ **as** $n \to \infty$ **then the approximation is said to converge to** $u(x)$**. And the general series is called <u>Neumann Series</u> with**

  $K_m(x, y) = \int_a^b K_1(x, t)K_{m-1}(t, y)dt = \int_a^b K_{m-1}(x, t)K_1(t, y)dt$

  Where $K_1(x, y) = K(x, y)$

CONDITION FOR CONVERGENCE OF NEUMANN SERIES

Consider a general term of Neumann series and apply the schwarz inequality to it.

$$\left|\int_a^b K_m(x,y)\, f(y)dy\right|^2 \leq \int_a^b |f(y)|^2\, dy \int_a^b |K_m(x,y)|^2\, dy$$

Now let $\int_a^b |f(y)|^2\, dy = D^2$ **and sup** $\left\{\int_a^b |K_m(x,y)|^2\, dy\right\} = C_m^2$

$$\Rightarrow \left|\int_a^b K_m(x,y)\, f(y)dy\right|^2 \leq D^2 C_m^2 \qquad \text{……..(i)}$$

To obtain the formula for C_m^2 **apply Schwarz inequality to it.**

$$|K_m(x,y)|^2 = \left|\int_a^b K_{m-1}(x,t)\, K_1(t,y)dt\right|^2$$

$$|K_m(x,y)|^2 \leq \int_a^b |K_{m-1}(x,t)|^2\, dt \int_a^b |K_1(t,y)|^2\, dt$$

$$\Rightarrow \int_a^b |K_m(x,y)|^2\, dy \leq \int_a^b |K_{m-1}(x,t)|^2\, dt \int_a^b \int_a^b |K_1(t,y)|^2\, dt\, dy$$

$$\Rightarrow C_m^2 \leq C_{m-1}^2 B^2 \qquad \text{with } B^2 = \int_a^b \int_a^b |K_1(t,y)|^2\, dt\, dy$$

Repeating the procedure we obtain

$$C_m^2 = C_{m-1}^2 B^2 \Rightarrow C_{m-1}^2 = C_{m-2}^2 B^2 . B^2 = C_{m-2}^2 (B^2)^2$$

$$\Rightarrow C_m^2 \leq C_1^2 (B^2)^{m-1} \qquad \text{general}$$

$$(i) \Rightarrow \left|\int_a^b K_m(x,y)\, f(y)dy\right|^2 \leq D^2 C_1^2 (B^2)^{m-1}$$

And $\Rightarrow C_m^2 \leq C_{m-1}^2 B^2 \leq C_{m-2}^2 (B^2)^2 \leq \cdots \leq C_1^2 (B^2)^{m-1}$

$$\Rightarrow \left|\int_a^b K_m(x,y)\, f(y)dy\right| \leq D C_1 B^{m-1}$$

$$\Rightarrow \left|\lambda^m \int_a^b K_m(x,y)\, f(y)dy\right| \leq |\lambda^m| D C_1 B^{m-1} \qquad \therefore \times\, |\lambda^m| \text{ on B.Sides}$$

This show that the absolute value of general term in Neumann series is less than or equal to the general term of geometric series $DC_1 \sum_m |\lambda|\, B^{m-1}$ **with common ratio** $B|\lambda|$**. This geometric series will be convergent if** $B|\lambda| < 1$ **or if** $|\lambda| < B^{-1}$**. This is the condition for convergence of geometric series.**

UNIQUENESS SOLUTION OF THE FREDHOLM I.E. OF THE 2ND KIND

Since $u(x) = f(x) + \lambda \int_a^b K(x,y)u(y)dy$

To prove its uniqueness consider if possible, there exists two solutions, then

$$u^1(x) = f(x) + \lambda \int_a^b K(x,y)u^1(y)dy$$

$$u^2(x) = f(x) + \lambda \int_a^b K(x,y)u^2(y)dy$$

Subtracting both

$$u^1(x) - u^2(x) = f(x) + \lambda \int_a^b K(x,y)u^1(y)dy - f(x) - \lambda \int_a^b K(x,y)u^2(y)dy$$

$$\varphi(x) = \lambda \int_a^b K(x,y)[u^1(y) - u^2(y)]dy$$

$$\varphi(x) = \lambda \int_a^b K(x,y)\varphi(y)dy \text{ ....(i)}$$

$$\int_a^b |\varphi(x)|^2\, dx = \left|\lambda \int_a^b \int_a^b K(x,y)\varphi(y)dy\right|^2$$

$$\int_a^b |\varphi(x)|^2\, dx \leq |\lambda|^2 \int_a^b \int_a^b |K(x,y)|^2\, dy \int_a^b |\varphi(y)|^2\, dy$$

$$\int_a^b |\varphi(x)|^2\, dx \leq |\lambda|^2 B^2 \int_a^b |\varphi(y)|^2\, dy$$

$\int_a^b |\varphi(x)|^2\, dx \leq |\lambda|^2 B^2 \int_a^b |\varphi(x)|^2\, dx$ **replacing 'x' with 'y'**

$$\int_a^b [|\varphi(x)|^2 - |\lambda|^2 B^2 |\varphi(x)|^2]\, dx \leq 0$$

$$\int_a^b |\varphi(x)|^2 [1 - |\lambda|^2 B^2]\, dx \leq 0$$

For a convergent Neumann Series $|\lambda| B < 1$ **therefore** $\int_a^b |\varphi(x)|^2\, dx = 0$

$\Rightarrow \varphi(x) = 0$ **hence the solution must be unieque.**

$$(i) \Rightarrow \varphi(x) = 0 = \lambda \int_a^b K(x,y)\varphi(y)dy$$

$$\Rightarrow \lambda \int_a^b K(x,y)u^1(y)dy - \lambda \int_a^b K(x,y)u^2(y)dy = 0$$

$$\Rightarrow f(x) + \lambda \int_a^b K(x,y)u^1(y)dy = f(x) + \lambda \int_a^b K(x,y)u^2(y)dy$$

$$\Rightarrow u^1(x) = u^2(x) = u(x)$$

RESOLVENT KERNEL FOR NEUMANN SERIES

Since Neumann series solution is as

$$u_n(x) = u(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,y)\, f(y)dy$$

$$\Rightarrow u(x) = f(x) + \lambda \int_a^b \left(\sum_{m=1}^{n} \lambda^{m-1} K_m(x,y)\right) f(y)dy \quad \therefore \times\, ing, \div\, ing\ \lambda$$

$$\Rightarrow u(x) = f(x) + \lambda \int_a^b \Gamma(x,y;\lambda)\, f(y)dy$$

With $\Gamma(x,y;\lambda) = \sum_{m=1}^{n} \lambda^{m-1} K_m(x,y)$ **called Resolvent Kernel**

Also $\Gamma(x,y;\lambda) = K_1(x,y) + \lambda K_2(x,y) + \lambda^2 K_3(x,y) + \cdots$ **will be convergent if** $|\lambda|B < 1$ **where** $K_m(x,y)$ **is mth iterated kernel.**

ERROR:

$|R_n| \le \frac{DC_1|\lambda|^{m-1}}{1-|\lambda|B}$ **is the error when term after nth term in Neumann's left out.**

EXAMPLE:

Solve the IE by the method of successive approximation

$$u(x) = f(x) + \lambda \int_0^1 e^{x-y} u(y)dy$$

Solution: By the method of successive approximation solution is

$$\Rightarrow u(x) = f(x) + \lambda \int_0^1 \Gamma(x,y;\lambda)\, f(y)dy \quad \text{..........(i)}$$

With $\Gamma(x,y;\lambda) = \sum_{m=1}^{n} \lambda^{m-1} K_m(x,y)$ **called Resolvent Kernel**

Also $\Gamma(x,y;\lambda) = K_1(x,y) + \lambda K_2(x,y) + \lambda^2 K_3(x,y) + \cdots$

Here $K_1(x,y) = e^{x-y}$

Now since $K_m(x,y) = \int_a^b K_1(x,t) K_{m-1}(t,y)dt$

$$\Rightarrow K_2(x,y) = \int_0^1 K_1(x,t)K_1(t,y)dt = \int_0^1 e^{x-t}e^{t-y}dt = \int_0^1 e^{x-y}dt$$

$$\Rightarrow K_2(x,y) = e^{x-y} \qquad \text{after solving}$$

Similarly $\Rightarrow K_3(x,y) = \int_0^1 K_1(x,t)K_2(t,y)dt = \int_0^1 e^{x-t}e^{t-y}dt$

$$\Rightarrow K_3(x,y) = \int_0^1 e^{x-y}dt \Rightarrow K_3(x,y) = e^{x-y}$$

Similary $K_4 = K_5 = K_6 = \cdots = K_n = K_1 = K$

Hence $\Gamma(x, y; \lambda) = K + \lambda K + \lambda^2 K + \lambda^3 K + \cdots = K(1 + \lambda + \lambda^2 + \lambda^3 + \cdots)$

$\Gamma(x, y; \lambda) = e^{x-y}(1-\lambda)^{-1} = \frac{e^{x-y}}{1-\lambda}$

$(i) \Rightarrow u(x) = f(x) + \frac{\lambda}{1-\lambda}\int_0^1 e^{x-y} f(y)dy$

For convergence $|\lambda|B < 1$ where $B^2 = \int_0^1 \int_0^1 |K_1(x, y)|^2 \, dxdy$

EXAMPLE:

Solve the IE also find the resolvent kernel by the method of successive approximation $u(x) = 1 + \lambda \int_0^1 (1 - 3xy)u(y)dy$

Solution:

$\Gamma(x, y; \lambda) = \sum_{m=1}^{n} \lambda^{m-1} K_m(x, y) = K_1(x, y) + \lambda K_2(x, y) + \lambda^2 K_3(x, y) + \cdots$

Here $K_1(x, y) = 1 - 3xy$

Now since $K_m(x, y) = \int_a^b K_1(x, t)K_{m-1}(t, y)dt$

$\Rightarrow K_2(x, y) = \int_0^1 K_1(x, t)K_1(t, y)dt = \int_0^1 (1 - 3xt)(1 - 3ty)dt$

$\Rightarrow K_2(x, y) = \int_0^1 (1 - 3ty - 3xt + 9xt^2 y)dt = 1 - \frac{3y}{2} - \frac{3x}{2} + 3xy$

$\Rightarrow K_2(x, y) = -\frac{3}{2}(y + x) + (1 + 3xy)$

Similarly $\Rightarrow K_3(x, y) = \int_0^1 K_1(x, t)K_2(t, y)dt$

$\Rightarrow K_3(x, y) = \int_0^1 (1 - 3xt)\left(-\frac{3}{2}(y + t) + (1 + 3ty)\right)dt$

$\Rightarrow K_3(x, y) = \frac{1}{4}(1 + 3xy)$ after solving

Similarly $\Rightarrow K_4(x, y) = \int_0^1 K_1(x, t)K_3(t, y)dt$

$\Rightarrow K_4(x, y) = \frac{1}{4}\int_0^1 (1 - 3xt)(1 + 3ty)dt = \frac{1}{4}\left(1 - \frac{3y}{2} - \frac{3x}{2} + 3xy\right)$

$\Rightarrow K_4(x, y) = \frac{1}{4}K_2(x, y)$

Similarly $K_5(x,y) = \frac{1}{16}K_1(x,y)$ **and** $K_6(x,y) = \frac{1}{16}K_2(x,y)$

Since

$$\Gamma(x,y;\lambda) = \sum_{m=1}^{n} \lambda^{m-1} K_m(x,y) = K_1(x,y) + \lambda K_2(x,y) + \lambda^2 K_3(x,y) + \cdots$$

$$\Gamma(x,y;\lambda) = (K_1 + \lambda^2 K_3 + \lambda^4 K_5 + \cdots) + \left(\lambda K_2 + \lambda^3 K_4 + \lambda^5 K_6 + \cdots\right)$$

$$\Gamma(x,y;\lambda) = \left(K_1 + \frac{\lambda^2}{4}K_1 + \frac{\lambda^4}{16}K_1 + \cdots\right) + \left(\lambda K_2 + \frac{\lambda^3}{4}K_2 + \frac{\lambda^5}{16}K_2 + \cdots\right)$$

$$\Gamma(x,y;\lambda) = K_1\left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right) + \lambda K_2\left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right)$$

$$\Gamma(x,y;\lambda) = \left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right)(K_1 + \lambda K_2)$$

$$\Gamma(x,y;\lambda) = \left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right)\left[(1 - 3xy) + \lambda\left(-\frac{3}{2}(y + x) + (1 + 3xy)\right)\right]$$

$$\Gamma(x,y;\lambda) = \left(1 + \frac{\lambda^2}{4}\right)^{-1}\left[(1 - 3xy) + \lambda\left(-\frac{3}{2}(y + x) + (1 + 3xy)\right)\right]$$

$$\Gamma(x,y;\lambda) = \frac{1}{1+\frac{\lambda^2}{4}}\left[(1 - 3xy) + \lambda\left(-\frac{3}{2}(y + x) + (1 + 3xy)\right)\right]$$

So the solution is

$$u(x) = 1 + \frac{\lambda}{1+\frac{\lambda^2}{4}}\int_0^1 \left[(1 - 3xy) + \lambda\left(-\frac{3}{2}(y + x) + (1 + 3xy)\right)\right](1)dy$$

$$u(x) = 1 + \frac{\lambda}{1+\frac{\lambda^2}{4}}\left(1 - \frac{3}{2}x + \frac{\lambda}{4}\right) \quad \textbf{after solving}$$

EXAMPLE:

Solve the IE also find the resolvent kernel and condition of convergence by the method of successive approximation $u(x) = 1 + \lambda\int_0^{\pi} Sin(x + y)u(y)dy$

Solution: By the method of successive approximation the solution is

$u(x) = f(x) + \lambda\int_0^{\pi}\Gamma(x,y;\lambda)f(y)dy$ **where** $f(x) = 1$ **and**

$$\Gamma(x,y;\lambda) = \sum_{m=1}^{n} \lambda^{m-1} K_m(x,y) = K_1(x,y) + \lambda K_2(x,y) + \lambda^2 K_3(x,y) + \cdots$$

Here $K_1(x,y) = K(x,y) = Sin(x + y)$

$K_1(x,y) = SinxCosy + CosxSiny$

Now since $K_m(x,y) = \int_a^b K_1(x,t)K_{m-1}(t,y)dt$

$\Rightarrow K_2(x,y) = \int_0^\pi K_1(x,t)K_1(t,y)dt = \int_0^\pi Sin(x+t)Sin(t+y)dt$

$\Rightarrow K_2(x,y) = \int_0^\pi (SinxCost + CosxSint)(SintCosy + CostSiny)dt$

$\Rightarrow K_2(x,y) =$

$\int_0^\pi (SinxCosyCostSint + SinxSinyCos^2t + CosxCosySin^2t +$

$CosxSinyCostSint)dt$

$\Rightarrow K_2(x,y) =$

$(SinxCosy + CosxSiny)\int_0^\pi CostSintdt + SinxSiny\int_0^\pi Cos^2tdt +$

$CosxCosy\int_0^\pi Sin^2tdt$

$\Rightarrow K_2(x,y) =$

$Sin(x+y)\left|\frac{Cos^2t}{2}\right|_0^\pi + SinxSiny\int_0^\pi \left(\frac{1+Cos2t}{2}\right)dt + CosxCosy\int_0^\pi \left(\frac{1-Cos2t}{2}\right)dt$

$\Rightarrow K_2(x,y) = Sin(x+y)(0) + \frac{1}{2}SinxSiny\ \left|t + \frac{Sin2t}{2}\right|_0^\pi + \frac{1}{2}CosxCosy\left|t -\right.$

$\left.\frac{Sin2t}{2}\right|_0^\pi$

$\Rightarrow K_2(x,y) = \frac{1}{2}SinxSiny\ \left|t + \frac{Sin2t}{2}\right|_0^\pi + \frac{1}{2}CosxCosy\left|t - \frac{Sin2t}{2}\right|_0^\pi$

$\Rightarrow K_2(x,y) = \frac{\pi}{2}SinxSiny\ + \frac{\pi}{2}CosxCosy = \frac{\pi}{2}(SinxSiny\ + CosxCosy)$

$\Rightarrow K_2(x,y) = \frac{\pi}{2}Cos(x-y)$

Similarly $\Rightarrow K_3(x,y) = \int_0^\pi K_1(x,t)K_2(t,y)dt$

$\Rightarrow K_3(x,y) = \int_0^\pi (SinxCost + CosxSint)\left(\frac{\pi}{2}Cos(t-y)\right)dt$

$\Rightarrow K_3(x,y) = \frac{\pi}{2}\int_0^\pi (SinxCost + CosxSint)(SintSiny + CostCosy)dt$

$\Rightarrow K_3(x,y) =$
$\frac{\pi}{2}\int_0^{\pi}(SinxSinyCostSint + SinxCosyCos^2t + CosxSinySin^2t +$
$CosxCosyCostSint)dt$

$\Rightarrow K_3(x,y) =$
$\frac{\pi}{2}\left[(SinxSiny + CosxCosy)\int_0^{\pi} CostSintdt + SinxCosy\int_0^{\pi} Cos^2tdt +\right.$
$\left. CosxSiny\int_0^{\pi} Sin^2tdt\right]$

$\Rightarrow K_3(x,y) = \frac{\pi}{2}\left[Cos(x-y)\left|\frac{Cos^2t}{2}\right|_0^{\pi} + SinxCosy\int_0^{\pi}\left(\frac{1+Cos2t}{2}\right)dt +\right.$
$\left. CosxSiny\int_0^{\pi}\left(\frac{1-Cos2t}{2}\right)dt\right]$

$\Rightarrow K_3(x,y) = \frac{\pi}{2}\left[Cos(x-y)(0) + \frac{1}{2}SinxCosy\left|t + \frac{Sin2t}{2}\right|_0^{\pi} + \frac{1}{2}CosxSiny\left|t +\right.\right.$
$\left.\left.\frac{Sin2t}{2}\right|_0^{\pi}\right]$

$\Rightarrow K_3(x,y) = \frac{\pi}{2}\left[\frac{\pi}{2}SinxCosy + \frac{\pi}{2}CosxSiny\right] = \frac{\pi^2}{4}(SinxCosy + CosxSiny)$

$\Rightarrow K_3(x,y) = \frac{\pi^2}{4}Sin(x+y) = \frac{\pi^2}{4}K_1(x,y)$

Similarly $K_4(x,y) = \frac{\pi^3}{8}Cos(x-y) = \frac{\pi^2}{4}K_2(x,y)$

And $K_5(x,y) = \frac{\pi^4}{16}Sin(x+y) = \frac{\pi^4}{16}K_1(x,y)$

Since

$\Gamma(x,y;\lambda) = \sum_{m=1}^{n}\lambda^{m-1}K_m(x,y) = K_1 + \lambda K_2 + \lambda^2K_3 + \lambda^3K_4 + \cdots$

$\Gamma(x,y;\lambda) = K_1 + \lambda K_2 + \frac{\lambda^2\pi^2}{4}K_1 + \frac{\lambda^3\pi^2}{4}K_2 + \frac{\lambda^4\pi^4}{16}K_1 + \cdots$

$\Gamma(x,y;\lambda) = K_1\left(1 + \frac{\lambda^2\pi^2}{4} + \frac{\lambda^4\pi^4}{16} + \cdots\right) + \lambda K_2\left(1 + \frac{\lambda^2\pi^2}{4} + \frac{\lambda^4\pi^4}{16} + \cdots\right)$

$\Gamma(x,y;\lambda) = (K_1 + \lambda K_2)\left(1 + \frac{\lambda^2\pi^2}{4} + \frac{\lambda^4\pi^4}{16} + \cdots\right) = (K_1 + \lambda K_2)\left(1 + \frac{\lambda^2\pi^2}{4}\right)^{-1}$

$\Gamma(x,y;\lambda) = \left[Sin(x+y) + \frac{\lambda\pi}{2}Cos(x-y)\right]\left(1 + \frac{\lambda^2\pi^2}{4}\right)^{-1}$

So the solution is given by $u(x) = 1 + \lambda \int_0^{\pi} \Gamma(x, y; \lambda) f(y) dy$

$$u(x) = 1 + \frac{\lambda}{1+\frac{\lambda^2\pi^2}{4}} \int_0^{\pi} \left[Sin(x+y) + \frac{\lambda\pi}{2} Cos(x-y) \right] f(y) dy$$

EXAMPLE:

Prove that the m th iterate kernel $K_m(x, y)$ satisfies the relation

$$K_m(x, y) = \int_a^b K_r(x, t) K_{m-r}(t, y) dt, \quad 1 \leq r < m$$

Solution

By definition

$$K_m(x, y) = \int_a^b K_1(x, t_1) K_{m-1}(t_1, y) dt_1$$

and

$$K_{m-1}(t_1, y) = \int_a^b K_1(t_1, t_2) K_{m-2}(t_2, y) dt_2$$

Therefore on substitution

$$\begin{aligned} K_m(x, y) &= \int_a^b K_1(x, t_1) \int_a^b K_1(t_1, t_2) K_{m-2}(t_2, y) dt_2 dt_1 \\ &= \int_a^b \left[\int_a^b K_1(x, t_1) K_1(t_1, t_2) dt_1 \right] K_{m-2}(t_2, y) dt_2 \\ &= \int_a^b K_2(x, t_2) K_{m-2}(t_2, y) dt_2 \end{aligned}$$

where

$$K_2(x, t_2) = \int_a^b K_1(x, t_1) K_1(t_1, t_2) dt_1$$

Repeating the above process r times, we obtain

$$K_m(x, y) = \int_a^b K_r(x, t) K_{m-r}(t, y) dt$$

EXAMPLE:

Solve the integral equation by the method of successive approximations to the third order.

$$u(x) = 2x + \lambda \int_0^1 (x+y)\, u(y)\, dy$$

Solution

For the zeroth order approximation, we have $u_0(x) = 2x$. Therefore first order approximate solution is

$$\begin{aligned} u_1(x) &= 2x + \lambda \int_0^1 (x+y)(2y)\, dy \\ &= 2x + \lambda \int_0^1 (2xy + 2y^2)\, dy = 2x + \lambda(x + 2/3) \end{aligned}$$

Again

$$\begin{aligned} u_2(x) &= 2x + \lambda \int_0^1 (x+y)\, u_1(y)\, dy \\ &= 2x + \lambda \int_0^1 (x+y)(2y + \lambda y + 2\lambda/3)\, dy \\ &= 2x + \lambda \int_0^1 \left(2xy + \lambda xy + \frac{2}{3}\lambda x + 2y^2 + \lambda y^2 + \frac{2}{3}\lambda y\right) dy \\ &= 2x + \lambda \left(x + \frac{\lambda}{2}x + \frac{2\lambda}{3}x + \frac{2}{3} + \frac{2\lambda}{3}\right) \\ &= 2x + \lambda \left(x + \frac{7\lambda}{6} + \frac{2}{3} + \frac{2\lambda}{3}\right) \end{aligned}$$

and similarly

$$\begin{aligned} u_3(x) &= 2x + \lambda \int_0^1 (x+y)\, u_2(y)\, dy \\ &= 2x + \lambda \int_0^1 (x+y) \left[2x + \lambda\{(1 + 7\lambda/6)x + \frac{2}{3}(1+\lambda)\}\right] dy \\ &= 2x + \lambda \int_0^1 \left[2x^2 + \lambda\left(1 + \frac{7\lambda}{6}\right)x^2 + \frac{2}{3}\lambda(1+\lambda)x + 2xy \right. \\ &+ \left. \lambda\left(1 + \frac{7\lambda}{6}\right)xy + \frac{2\lambda}{3} + (1+\lambda)y\right] dy \\ &= 2x + \lambda\left[2x^2 + \lambda\left(1 + \frac{7\lambda}{6}\right)x^2 + \frac{2}{3}\lambda(1+\lambda)x \right. \\ &+ \left. x + (\lambda/2)(1 + 7\lambda/6)x + \frac{\lambda}{3}(1+\lambda)\right] \\ &= 2x + \lambda\left[\left(2 + \lambda + \frac{7\lambda^2}{6}\right)x^2 + \left(\frac{2}{3}\lambda + \frac{2}{3}\lambda^2 + 1 + \frac{1}{2}\lambda + \frac{7}{12}\lambda^2\right)x \right. \\ &+ \left. \frac{1}{3}\lambda(1+\lambda)\right] \\ &= 2x + \lambda\left[\left(\frac{7}{6}\lambda^2 + \lambda + 2\right)x^2 + \left(\frac{15}{12}\lambda^2 + \frac{7}{6}\lambda + 1\right)x + \frac{1}{3}\lambda(1+\lambda)\right] \end{aligned}$$

VOLTERA IE AND THE METHOD OF SUCCESSIVE APOROXIMATION

This method can be applied to Volterra I.E. of the second type, *viz.*

$$u(x) = f(x) + \lambda \int_a^x K(x,\ y)\, u(y)\, dy$$

The Neumann series solution can in this case be written as

$$u(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^s K_m(x,\ y)\, f(y)\, dy$$

and the resolvent kernel can be written as

$$\Gamma(x,\ y,\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,\ y)$$

where K_m is the m th iterate kernel. Note that in this case

$$K_2(x,\ y) = \int_y^x K_1(x,\ t)\, K_1(t,\ y)\, dt,\ \ K_3(x,\ y) = \int_y^x K_1(x,\ t)\, K_2(t,\ y)\, dt,\ \text{etc.}$$

EXAMPLE:

Find the Neumann series solution of the I.E.

$$u(x) = 1 + x + \lambda \int_0^x (x - y)\, u(y)\, dy$$

i.e. by volterra method
variable of limit

Solution

Here $K(x, y) \equiv K_1(x, y) = x - y$, and

$$\begin{aligned}
K_2(x, y) &= \int_y^x K_1(x, t)\, K_1(t, y)\, dt \\
&= \int_y^x (x - t)(t - y)\, dt \\
&= \int_y^x (xt - t^2 - xy + ty)\, dt \\
&= \left(xt^2/2 - t^3/3 - xyt + yt^2/2\right)\Big]_y^x \\
&= \frac{1}{6}\left(x^3 - y^3 + 3xy^2 - 3yx^2\right) \\
&= \frac{1}{6}(x - y)^3 = \frac{(x - y)^3}{3!}
\end{aligned}$$

Similarly

$$K_3 = \int_y^x (x - t)\, \frac{(t - y)^3}{3!}\, dt$$

$$\begin{aligned}
&= \frac{x - t}{3!}\, \frac{(t - y)^4}{4}\Bigg]_y^x + \frac{1}{3!}\int_y^x \frac{(t - y)^4}{4}\, dt \\
&= 0 + \frac{1}{4 \cdot 3!}\, \frac{(t - y)^5}{5}\Bigg]_y^x = \frac{(x - y)^5}{5!}
\end{aligned}$$

Similarly

$$K_5(x, y) = \frac{(x - y)^9}{9!}, \quad \cdots \quad K_m(x, y) = \frac{(x - y)^{2m-1}}{(2m - 1)!}$$

The Neumann series for the solution of the equation is

$$\begin{aligned}
u(x) &= f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x, y) f(y)\, dy \\
&= (1 + x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x \frac{(x - y)^{2m-1}}{(2m - 1)!} f(y)\, dy
\end{aligned}$$

EXAMPLE:

Find the Neumann series for the solution of I.E.

$$u(x) = f(x) + \lambda \int_0^x e^{x-y} u(y)dy$$

Solution

Here $K(x, y) = e^{x-y} = K_1(x, y)$. Therefore

$$\begin{aligned} K_2(x, y) &= \int_y^x K_1(x, t)K_1(t, y)dt \\ &= \int_y^x e^{x-t}e^{t-y}dt \\ &= e^{x-y}\int_y^x dt = e^{x-y}(x-y) \end{aligned}$$

Similarly

$$\begin{aligned} K_3(x, y) &= \int_y^x K_1(x, t)K_2(t, y)dt \\ &= \int_y^x e^{x-t}e^{t-y}(t-y)dt \end{aligned}$$

$$
\begin{aligned}
&= e^{x-y}\int_y^x (t-y)dt \\
&= e^{x-y}\left(\frac{t^2}{2}-yt\right)^2\Big]_y^x \\
&= e^{x-y}\left(\frac{x^2}{2}-\frac{y^2}{2}\right) = e^{x-y}\left(\frac{x^2}{2}+\frac{1}{2}y^2-yx\right) \\
&= \frac{1}{2}e^{s-t}\left(s^2+t^2-2ts\right) = \frac{e^{x-y}}{2!}(x-y)^2
\end{aligned}
$$

and

$$
\begin{aligned}
K_4(x,\ y) &= \int_y^x K_1(x,\ t)K_3(t,\ y)dt \\
&= \frac{1}{2!}\int_y^x e^{x-t}e^{t-y}(t-y)^2dt \\
&= \frac{1}{2!}e^{x-y}\cdot\int_y^x (t-y)^2dt \\
&= \frac{e^{x-y}}{3!}(t-y)^3\Big]_y^x = \frac{e^{x-y}}{3!}(x-y)^3
\end{aligned}
$$

By similar calculation we obtain

$$K_5(x,\ y) = \frac{e^{x-y}}{4!}(x-y)^4$$

and

$$K_m(x,\ y) = \frac{e^{x-y}}{(m-1)!}(x-y)^{m-1}$$

The Neumann series for the solution of the equation is

$$u(x) = f(x) + \sum_{m=1}^{\infty}\lambda^m\int_0^x K_m(x,\ y)\,f(y)\,dy$$

i.e.

$$u(x) = f(x) + \sum_{m=1}^{\infty}\lambda^m\int_0^x \frac{e^{x-y}}{(m-1)!}(x-y)^{m-1}f(y)\,dy$$

EXAMPLE:

Solve the Volterra I.E.

$$u(x) = 1 + \int_0^x x\, y\, u(y)\, dy$$

Solution

Here $K_1(x,\ y) \equiv K(x,\ y) = xy$, and

$$\begin{aligned} K_2(x,\ y) &= \int_y^x K_1(x,\ t)K_1(t,\ y)dt = \int_y^x xt \cdot t \cdot ydt \\ &= xy\frac{t^3}{3}\Big]_y^x = \frac{1}{3}xy(x^3 - y^3) \end{aligned}$$

Similarly

$$\begin{aligned} K_3(x,\ y) &= \int_y^x K_1(x,\ t)\, K_2(t,\ y)dt \\ &= \frac{1}{3}\int_y^x xt\, ty\, (t^3 - y^3)dt \\ &= \frac{1}{3}\int_y^x (xyt^5 - xt^2y^4)dt \\ &= \frac{1}{3}xy\left(\frac{t^6}{6} - \frac{t^3}{3}y^3\right)\Big]_y^x \\ &= \frac{1}{3}xy\left(\frac{x^6}{6} - \frac{y^6}{6} - \frac{x^3y^3}{3} + \frac{y^6}{3}\right) \\ &= \frac{1}{3}xy\left[\frac{x^6 - y^6}{6} - \frac{y^3(x^3 - y^3)}{3}\right] \\ &= \frac{1}{3\cdot 6}xy\left(x^6 - y^6 - 2y^3x^3 + 2y^6\right) \\ &= \frac{xy}{3\cdot 6}\left(x^6 + y^6 - 2y^3x^3\right) = \frac{xy}{18}\left(x^3 - y^3\right)^2 \end{aligned}$$

Therefore

$$\begin{aligned} \Gamma(x,\ y;\lambda) &= K_1 + \lambda K_2 + \lambda^2 K_3 + \lambda^3 K_4 + \cdots \\ &= xy + \lambda xy\left(\frac{x^3 - y^3}{3}\right) + \lambda^2\frac{xy}{18}(x^3 - y^3)^2 + \cdots \\ &= xy\left[1 + \frac{\lambda}{3}(x^3 - y^3) + \frac{\lambda^2}{18}(x^3 - y^3)^2 + \cdots\right] \end{aligned}$$

Therefore the solution is

$$u(x) = 1 + \lambda\int_0^x \Gamma(x,\ y;\ \lambda)f(y)dy$$

where $\Gamma(x,\ y;\ \lambda)$ is given by (1).

EXAMPLE:

Example

Find an approximate solution by the iterative method

$$u(x) = \sinh x + \int_0^x e^{y-x}\, u(y)\, dy$$

Solution

Here $f(x) = 1$, $K(x,\ y) = x - y$, and $\lambda = 1$.

The solution is given by

$$u(x) = f(x) + \lambda \int_0^x \Gamma(x,\ y,\ \lambda)\, f(y)\, dy$$

where in this case

$$\Gamma(x,\ y,\ \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,\ y) == \sum_{m=1}^{\infty} K_m(x,\ y), \quad \lambda = 1$$

with $K_1(x,\ y) = K(x,\ y) = x - y$.

$$\begin{aligned} K_2(x,\ y) &= \int_y^x K_1(x,\ t)K_1(t,\ y)\, dt = \int_y^x (x-t)(t-y)\, dt \\ &= \int_y^x (xt - ty - x^2 + ty)\, dt = x/2 - xy - 1/3 + y/2 \end{aligned}$$

and

$$\begin{aligned} K_3(x,\ y) &= \int_y^x K_1(x,\ t)K_2(t,\ y)\, dt = \int_y^x (x-t)(t/2 - ty - 1/3 + y/2)\, dt \\ &= \frac{x^3}{3} - \frac{yx^3}{6} - \frac{1}{6}x^2 + \frac{1}{4}yx^2 \qquad) \end{aligned}$$

(UoS, Past Paper)

Justify that Laplace Transformation could be used to solve I.E.

Consider the IE (Voltera IE of 1st kind)

$$f(x) = \int_0^x K(x-y)u(y)dy$$

$\Rightarrow f(x) = K * u$ **using Convolution law.**

$\Rightarrow \mathcal{L}\{f(x)\} = \mathcal{L}\{K * u\}$

$\Rightarrow \tilde{f}(p) = \widetilde{K}(p)\tilde{u}(p)$ **using Convolution law.**

$$\Rightarrow \tilde{u}(p) = \frac{\tilde{f}(p)}{\widetilde{K}(p)} \Rightarrow \mathcal{L}^{-1}\{\tilde{u}(p)\} = \mathcal{L}^{-1}\left\{\frac{\tilde{f}(p)}{\widetilde{K}(p)}\right\} \Rightarrow u(x) = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{px}\tilde{f}(p)}{\widetilde{K}(p)}dp$$

after solving we get the result.

Justify that Fourier Transformation could be used to solve I.E.

Consider the IE

$$u(x) = f(x) + \lambda\int_{-\infty}^{\infty} K(x-y)u(y)dy$$

$\Rightarrow u(x) = f(x) + \sqrt{2\pi}\lambda K * u$ **using Convolution law.**

$\Rightarrow \mathcal{F}\{u(x)\} = \mathcal{F}\{f(x) + \sqrt{2\pi}\lambda K * u\}$

$\Rightarrow \tilde{u}(k) = \tilde{f}(k) + \sqrt{2\pi}\lambda\widetilde{K}(k)\tilde{u}(k)$ **using Convolution law.**

$$\Rightarrow \tilde{u}(k) = \frac{\tilde{f}(k)}{1-\lambda\widetilde{K}(k)} \Rightarrow \mathcal{F}^{-1}\{\tilde{u}(k)\} = \mathcal{F}^{-1}\left\{\frac{\tilde{f}(k)}{1-\lambda\widetilde{K}(k)}\right\}$$

$\Rightarrow u(x) = \frac{1}{\sqrt{2}}\int_{\infty}^{\infty}\frac{e^{-ikx}\tilde{f}(k)}{1-\lambda\widetilde{K}(k)}dk$ **after solving we get the result.**

Example 1

Solve the Volterra I.E.

$$\int_0^x \sin\alpha(x-y)u(y)dy = 1-\cos\beta x \tag{1}$$

Solution

It can be seen that the I.E. is consistent. With $x=0$ we obtain $0=0$.

Making use of the definition of the convolution (of functions $\sin\alpha x$ and $u(x)$) we can write (1) as

$$(\sin\alpha x) * u(x) = 1-\cos\beta x \tag{2}$$

Taking the Laplace transform of both sides of (2) and using the convolution theorem we have

$$L\{\sin\alpha x\}L\{u(x)\} = L\{1-\cos\beta x\}$$

or

$$\frac{\alpha}{\alpha^2+p^2}\bar{u}(p) = \frac{1}{p}-\frac{p^2}{p^2+\beta^2} = \frac{\beta^2}{p(p^2+\beta^2)}$$

wherefrom,

$$\bar{u}(p) = \frac{\beta^2}{\alpha}\frac{\alpha^2+p^2}{p(\beta^2+p^2)} \tag{3}$$

Finally on taking inverse Laplace transform, we have

$$u(x) = \frac{\beta^2}{\alpha}L^{-1}\left\{\frac{\alpha^2+p^2}{p(\beta^2+p^2)}\right\}$$

Now

$$\begin{aligned}\frac{\alpha^2+p^2}{p(\beta^2+p^2)} &= \frac{\alpha^2}{p(\beta^2+p^2)}+\frac{p}{\beta^2+p^2}\\ &= \frac{\alpha^2}{\beta^2}\left(\frac{1}{p}-\frac{p}{\beta^2+p^2}\right)+\frac{p}{\beta^2+p^2}\end{aligned}$$

Therefore from (3)

$$\begin{aligned}u(x) &= \frac{\beta^2}{\alpha}L^{-1}\left\{\frac{\alpha^2+p^2}{p(\beta^2+p^2)}\right\}\\ &= \alpha L^{-1}\left\{\frac{1}{p}-\frac{p}{\beta^2+p^2}\right\}+\frac{\beta^2}{\alpha}L^{-1}\left\{\frac{p}{\beta^2+p^2}\right\}\\ &= \alpha-\alpha\cos\beta x+\frac{\beta^2}{\alpha}\cos\beta x\\ &= \alpha+\frac{\beta^2-\alpha^2}{\alpha}\cos\beta x\end{aligned}$$

Example 2

Solve the Volterra I.E.

$$u(x) = x^3 + \int_0^x e^{3(x-y)}\, u(y)\, dy \tag{1}$$

Solution

This is Volterra I.E. of the second kind, with convolution-type kernel. It can be solved by the L.T. method.

Equation (1) can be written as

$$u(x) = x^3 + e^{3x} * u(x)$$

Therefore taking Laplace transforms of both sides, we obtain

$$L\{u(x)\} = L\{x^3\} + L\{e^{3x} * u(x)\}$$

or

$$\begin{aligned} \bar{u}(p) &= \frac{3!}{p^4} + L\{e^{3x}\}L\{u(x)\} \\ &= \frac{6}{p^4} + \frac{1}{p-3} \cdot \bar{u}(p) \end{aligned}$$

or on simplification

$$\begin{aligned} \bar{u}(p) &= \frac{6(p-3)}{p^4(p-4)} = \frac{6(p-4)+6}{p^4(p-4)} \\ &= \frac{3!}{p^4} + \frac{3!}{p^4(p-4)} \\ &= L\{x^3\} + L\{x^3\}L\{e^{4x}\} \end{aligned}$$

Hence

$$\begin{aligned} u(x) &= L^{-1}\{\bar{u}(p)\} = x^3 + x^3 * e^{4x} \\ &= x^3 + \int_0^x x'^3 e^{4(x-x')} dx' \end{aligned}$$

On performing the integration on the right side, we obtain

$$u(x) = x^3 - \left(\frac{1}{4}x^3 + \frac{3}{16}x^2 + \frac{3}{32}x + \frac{3}{128} - \frac{3}{128}e^{4x}\right)$$

Example 3

Solve the Volterra I.E. of the second kind using Laplace transforms.

$$u(x) = \cos x - \int_0^x (x-y)\cos(x-y)u(y)dy \tag{1}$$

Solution

The I.E. (1) can be written as

$$u(x) = \cos x - (x\cos x) * u(x) \tag{2}$$

Taking the Laplace transform of both sides of (2) and using the convolution theorem, we have

$$\begin{aligned}
\bar{u}(p) &= \frac{p}{p^2+1} - L\{x\cos x\}\,\bar{u}(p) \\
&= \frac{p}{p^2+1} - \left[\left(-\frac{d}{dp}\right)\frac{p}{p^2+1}\right]\bar{u}(p) \\
&= \frac{p}{p^2+1} + \left[\frac{d}{dp}\frac{p}{p^2+1}\right]\bar{u}(p) \\
&= \frac{p}{p^2+1} + \frac{1-p^2}{(p^2+1)^2}\,\bar{u}(p)
\end{aligned}$$

which on simplification can be reduced to

$$\frac{p^4+3p^2}{(p^2+1)^2}\,\bar{u}(p) = \frac{p}{p^2+1}$$

On further simplification we obtain

$$\bar{u}(p) = \frac{p^2+1}{p(p^3+3}$$

$$\begin{aligned}
&= \frac{p}{p^3+3} + \frac{1}{p(p^2+3)} \\
&= \frac{p}{p^3+3} + \frac{1}{3}\left(\frac{1}{p} - \frac{p}{p^2+3}\right)
\end{aligned}$$

On taking inverse Laplace transform, we obtain

$$u(x) = \cos\sqrt{3}\,x + \frac{1}{3}\left(1 - \cos\sqrt{3}x\right) = \frac{1}{3} + \frac{2}{3}\cos\sqrt{3}x$$

Example 4

Solve the integral equation

$$y(x) = f(x) + \lambda \int_{-\infty}^{+\infty} e^{-ixz}\, y(z)\, dz \tag{1}$$

Solution

We will use the definitions of Fourier transform and its inverse as given in chapter 7. Denoting the Fourier transform of $f(x)$ by $\bar{f}(k)$, (1) can be written as

$$y(x) = f(x) + \lambda\sqrt{2\pi}\,\bar{y}(-x) \tag{2}$$

where $\bar{y}(x)$ is the Fourier transform of $y(z)$.

Taking the Fourier transform of both sides of (2), we have

$$\begin{aligned}
\bar{y}(k) &= \bar{f}(k) + \lambda\sqrt{2\pi}\,\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx}\bar{y}(-x)\, dx \\
&= \bar{f}(k) + \lambda\sqrt{2\pi}\cdot\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx}\bar{y}(x)\, dx \qquad (3)
\end{aligned}$$

Using the definition of inverse Fourier transform, (3) becomes

$$\bar{y}(k) = \bar{f}(k) + \sqrt{2\pi}\lambda y(k) \tag{4}$$

From (4)

$$\bar{y}(-x) = \bar{f}(-x) + \sqrt{2\pi}\lambda\ y(-x) \tag{5}$$

Substituting for $\bar{y}(-x)$ from (5) into (2), we have

$$\begin{aligned}
y(x) &= f(x) + \lambda\sqrt{2\pi}\left[\bar{f}(-x) + \sqrt{2\pi}\lambda\ y(-x)\right] \\
&= f(x) + \lambda\sqrt{2\pi}\bar{f}(-x) + 2\pi\lambda^2\ y(-x) \qquad (6)
\end{aligned}$$

From (6)

$$y(-x) = f(-x) + \sqrt{2\pi}\lambda\bar{f}(x) + 2\pi\lambda^2 y(x)$$

Substituting for $y(-x)$ into (6), we finally obtain

$$\begin{aligned}
y(x) &= f(x) + \sqrt{2\pi}\lambda\bar{f}(-x) + 2\pi\lambda^2 \times \\
&\times \left[f(-x) + \sqrt{2\pi}\lambda\bar{f}(x) + 2\pi\lambda^2\ y(x)\right]
\end{aligned}$$

From the last equation

$$y(x)(1 - 4\pi^2\lambda^4) = f(x) + \sqrt{2\pi}\lambda\bar{f}(-x) + 2\pi\lambda^2 f(-x) + (2\pi)^{3/2}\lambda^3\bar{f}(x)$$

or

$$y(x) = \frac{1}{1 - 4\pi^2\lambda^4}\left[f(x) + \sqrt{2\pi}\lambda\bar{f}(-x) + 2\pi\lambda^2 f(-x) + (2\pi)^{3/2}\lambda^2\bar{f}(x)\right]$$

The solution given above is unique as long as $1 - 4\pi^2\lambda^4 \neq 0$, i.e λ is not equal to one of the eigenvalues of the associated homogeneous IE.

AN IMPORTANT RESULT:

$$\int_0^x \int_0^{x_2} \varphi(x_1)dx_1dx_2 = \int_0^x (x - x_1)\varphi(x_1)dx_1$$

Example 1: (UoS, 2019 – I)

Redue the IVP $y''(x) + \lambda y(x) = F(x)$ **with** $y(0) = 1, y'(0) = 0$

Solution:

Given that $y''(x) + \lambda y(x) = F(x)$

$\Rightarrow \int_0^x [y''(x) + \lambda y(x)]\, dx = \int_0^x F(x)\, dx$

$\Rightarrow |y'(x)|_0^x + \lambda \int_0^x y(x)\, dx = \int_0^x F(x)\, dx$

$\Rightarrow y'(x) - y'(0) + \lambda \int_0^x y(x)\, dx = \int_0^x F(x)\, dx$

$\Rightarrow y'(x) + \lambda \int_0^x y(x)\, dx = \int_0^x F(x)\, dx$ **since** $y'(0) = 0$

$\Rightarrow \int_0^x y'(x)\, dx + \lambda \int_0^x \int_0^{x_2} y(x_1)\, dx_1dx_2 = \int_0^x \int_0^{x_2} F(x_1)\, dx_1dx_2$

$\Rightarrow y(x) - y(0) + \lambda \int_0^x \int_0^{x_2} y(x_1)\, dx_1dx_2 = \int_0^x \int_0^{x_2} F(x_1)\, dx_1dx_2$

$\Rightarrow y(x) - 1 + \lambda \int_0^x \int_0^{x_2} y(x_1)\, dx_1dx_2 = \int_0^x \int_0^{x_2} F(x_1)\, dx_1dx_2$

$\Rightarrow y(x) - 1 + \lambda \int_0^x (x - x_1)y(x_1)dx_1 = \int_0^x (x - x_1)F(x_1)dx_1$ **using result**

$\Rightarrow y(x) = 1 - \lambda \int_0^x (x - x_1)y(x_1)dx_1 + \int_0^x (x - x_1)F(x_1)dx_1$

$\Rightarrow y(x) = 1 - \lambda \int_0^x (x - t)y(t)dt + \int_0^x (x - t)F(t)dt$

$\Rightarrow y(x) = 1 + \int_0^x (x - t)F(t)dt + \lambda \int_0^x (t - x)y(t)dt$

$\Rightarrow y(x) = f(x) + \lambda \int_0^x K(x, t)y(t)dt$ **required Voltera IE**

Where $f(x) = 1 + \int_0^x (x - t)F(t)dt$ **and** $K(x, t) = t - x$

Example 2

Reduce the boundary-value problem

$$y''(x) + \lambda P(x)y = Q(x) \tag{1}$$

$$y(a) = 0, \quad y(b) = 0 \tag{2}$$

to a Fredholm integral equation.

Solution

Integrating both sides of (1) from a to x.

$$y'(x) - y'(a) + \lambda \int_a^x P(x_1)y(x_1)dx_1 = \int_a^x Q(x_1)dx_1 \tag{3}$$

$y'(a)$ can be replaced by a constant c. Integrating again from a to x and using (2),

$$y(x) - y(a) - c(x-a) + \lambda \int_a^x \left\{ \int_a^{x_2} P(x_1)y(x_1)dx_1 \right\} dx_2$$
$$= \int_a^x \left\{ \int_a^{x_2} Q(x_1)dx_1 \right\} dx_2$$

Or making use of the first initial condition of (2) and of the result (10.5.1), we have

$$y(x) - 0 - c(x-a) + \lambda \int_a^x (x - x_1)P(x_1)\, y(x_1)\, dx_1$$
$$= \int_a^x (x - x_1)\, Q(x_1)\, dx_1 \tag{4}$$

To determine the constant c, we put $x = b$ in (4) and use the condition $y(b) = 0$. This gives

$$0 - 0 - c(b-a) + \lambda \int_a^b (b-t)P(t)\, y(t)\, dt = \int_a^b (b-t)\, Q(t)\, dt$$

Wherefrom

$$c = \frac{\lambda}{b-a} \int_a^b (b-t)P(t)\, y(t)\, dt - \frac{1}{b-a} \int_a^b (b-t)\, Q(t)\, dt \tag{5}$$

Substituting for c in (4), we get

$$y(x) + \frac{\lambda(x-a)}{b-a} \int_a^b (t-b)\, P(t)\, y(t)\, dt + \frac{x-a}{b-a} \int_a^b (b-t)\, Q(t)\, dt$$
$$+ \quad \lambda \int_a^x (x-t)\, P(t)\, y(t)\, dt = \int_a^x (x-t)\, Q(t)\, dt$$

or

$$y(x) \quad = \quad \int_a^x (x-t)\, Q(t)\, dt + \frac{x-a}{b-a} \int_a^b (t-b)\, Q(t)\, dt$$

$$+ \quad \lambda \int_a^x \left[\frac{(x-a)(b-t)}{b-a} P(t) + (t-x) P(t) \right] y(t)\, dt$$

$$+ \quad \lambda \int_x^b \frac{(x-a)(b-t)}{b-a} P(t)\, y(t)\, dt$$

which can also be rewritten as

$$y(x) = f(x) + \lambda \int_a^b K(x,\ t)\, f(t)\, dt$$

where

$$f(x) = \int_a^x (x-t)\, Q(t)\, dt + \frac{x-a}{b-a} \int_a^b (t-b)\, Q(t)\, dt$$

and

$$K(x,\ t) = \begin{cases} [(x-a)(b-t)/(b-a) + (t-x)]\ P(t), & t \le x \\ [(x-a)(b-t)/(b-a)]P(t), & t > x \end{cases}$$

Example 3

Convert the D.E.

$$y''(x) + a_2 y(x) = f(x) \quad \text{on} \quad [0, b] \tag{1}$$

(where a_2 and b are constants), subject to the I.C's

$$y(0) = y'(0) = 1 \tag{2}$$

into an equivalent IE, where $f(x)$ is defined by

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & 1 < x \le b \end{cases} \tag{3}$$

Solution

Integrating (1) w.r.t. x from 0 to x,

$$y'(x) - y'(0) + a_2 \int_0^x y(x_1) dx_1 = \int_0^x f(x_1) dx_1$$

or using the second I.C. of (2),

$$y'(x) - 1 + a_2 \int_0^x y(x_1) dx_1 = \int_0^x f(x_1) dx_1 \tag{4}$$

Integrating (4) w.r.t. x from 0 to x,

$$y(x) - y(0) - x + a_2 \int_0^x \int_0^{x_2} y(x_1)dx_1dx_2 = \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2$$

Using the first I.C. of (2) and the result (10.5.1), we have

$$y(x) - 1 - x + a_2 \int_0^x (x-t)y(t)dt = \int_0^x (x-t)f(t)dt \qquad (5)$$

Now to substitute for $f(x)$ in (5), we will make use of (3), and consider the cases (i) $x < 1$ and (ii) $x > 1$.

(i) When $x < 1$,

$$\int_0^x (x-t)\, f(t)\, dt = \int_0^x (x-t)\, dt = x^2 - x^2/2 = x^2/2$$

In this case (5) becomes

$$y(x) = 1 + x + \frac{x^2}{2} + a_2 \int_0^x (t-x)\, y(t)\, dt$$

(ii) When $x > 1$,

$$\int_0^x (x-t)\, f(t)\, dt = 0$$

In this case (5) becomes

$$y(x) = 1 + x + \int_0^x (t-x)\, y(t)\, dt$$

10.3 Eigenvalues and Eigensolutions of Fredholm IEs and the Alternative Theorem

10.3.1 Solution of homogeneous Fredholm integral equation

When the kernel is separable, the method of solution for homogeneous Fredholm IE of the second kind, *viz*

$$u(x) = \lambda \int_a^b K(x,\ y)\, u(y)\, dy \tag{10.3.1}$$

is similar to that discussed above for the nonhomogeneous Fredholm IE. However there is one important difference as far as the existence of solution corresponding to a particular value of λ is concerned. We have observed that if $\lambda \neq \lambda_i$, $i = 1,\ 2, \cdots$, (where λ_i is an eigenvalue of the homogeneous IE), then the nonhomogeneous Fredholm IE has a unique solution. On the other hand the homogeneous Fredholm IE will, in general, have a solution only when $\lambda = \lambda_i$. This can be seen from the following discussion.

We repeat the steps from equations (10.2.3), (10.2.4) and (10.2.6 a, b) with $f(x) = 0$ and obtain the following equation for the constants c_i:

$$\sum_k (\delta_{ik} - \lambda a_{ik})\, c_k = 0 \tag{10.3.2}$$

When written in full this equation is the same as equations (10.2.7)′ with $f_i = 0$, $i = 1, 2, \cdots$, *viz.*

$$\left.\begin{array}{rcrcrcrcl} (1-\lambda a_{11})c_1 & - & \lambda a_{12}c_2 & - \cdots - & \lambda a_{1n}c_n & = & 0 \\ -\lambda a_{21}c_1 & + & (1-\lambda a_{12})c_2 & - \cdots - & \lambda a_{2n}c_n & = & 0 \\ -\lambda a_{31}c_1 & - & \lambda a_{32}c_2 & - \cdots - & \lambda a_{3n}c_n & = & 0 \\ & & & \vdots & & & \\ -\lambda a_{n1}c_1 & - & \lambda a_{n2}c_2 & - \cdots + & (1-\lambda a_{nn})c_n & = & 0 \end{array}\right\} \quad (10.3.3)$$

This system of equations will have a non-trivial solution if

$$D(\lambda) \equiv |\delta_{ik} - \lambda a_{ik}| = 0 \qquad (10.3.4)$$

Equation (10.3.4) is a polynomial of degree n and therefore has n roots, which may be real, repeated or complex. These roots are called *eigenvalues* or *characteristic* values of the IE (10.3.1) or of the kernel $K(x, y)$. The corresponding solutions are called eigenfunctions of the IE (10.3.1) or of the kernel $K(x, y)$ and can be found from the equation

$$u(x) = \lambda \sum_i c_i A_i(x) \qquad (10.3.5)$$

where c_i are to be determined from the system of equations

$$\sum_k (\delta_{ik} - \lambda a_{ik})\, c_k = 0 \qquad (10.3.6)$$

for each value of $\lambda = \lambda_i$.

It is clear that the homogeneous IE (10.3.1) will have a nontrivial solution only when the parameter λ equals an eigenvalue. On the contrary, the nonhomogeneous IE will have a unique solution corresponding to those values of λ which do not equal eigenvalues.

V.V. Imp

10.3.2 The Fredholm alternative theorem

This theorem relates to the existence of solutions of a nonhomogeneous and related homogeneous Fredholm IEs of the second kind, *viz.*.

$$u(x) = f(x) + \lambda \int_a^b K(x, y)\, u(y)\, dy \qquad (10.3.7)$$

and

$$u(x) = \lambda \int_a^b K(x,\ y)\, u(y)\, dy \tag{10.3.8}$$

Statement of the theorem

Either

the IE (10.3.7) with fixed λ, possesses one and only one solution $u(x)$ for the arbitrary square integrable functions $f(x)$ and $K(x,\ y)$, in particular $u = 0$ for $f = 0$.

Or

the homogeneous I.E. (10.3.8) possesses a finite number of linearly independent solutions u_{0i}, $i = 1,\ 2,\ \cdots,\ r$.

In the first case the transposed equation

$$v(x) = f(x) + \lambda \int_a^b K(y,\ x)\, v(y)\, dy \tag{10.3.9}$$

also possesses a unique solution. In the second case the transposed homogeneous equation

$$w(x) = \lambda \int_a^b K(y,\ x)\, w(y)\, dy \tag{10.3.10}$$

also has r linearly independent solutions, w_{0i}, $i = 1, 2, \cdots r$.

Moreover the inhomogeneous integral equation (10.3.7) has a solution corresponding to $\lambda = \lambda_i$, an eigenvalue, if and only if the function $f(x)$ satisfies the r conditions

$$\int_a^b w_{0i}(x)\, f(x)\, dx = 0, \quad i = 1,\ 2,\ \cdots r$$

10.3.3 Summary of results on homogeneous IEs

1. The homogeneous IE

$$u(x) = \lambda \int_a^b K(x, y)\, u(y)\, dy$$

has only the trivial solution $u = 0$ if $D(\lambda) \neq 0$. If $D(\lambda) = 0$ it may have non-trivial solutions.

2. It is clear that $\lambda = 0$ is *not* an eigenvalue, because it gives $u(x) \equiv 0$.

3. If the kernel $K(x, y)$ is continuous in the square $a \leq x, y \leq b$; a, b being finite, or the kernel is quadratically integrable then to every eigenvalue λ there corresponds a finite number of linearly independent eigenfunctions $w_1, w_2, \cdots, w_r$. The number r is called the *index* or *multiplicity* of the eigenvalue λ. Different eigenvalues can have different multiplicities.

4. In general the solution of the Fredholm IE of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x, y)\, u(y)\, dy$$

can be written as

$$u(x) = f(x) + \lambda \int_a^b \Gamma(x, y; \lambda)\, f(y)\, dy$$

where

$$\Gamma(x, y; \lambda) = \frac{D(x, y, \lambda)}{D(\lambda)}$$

and $D(x, y, \lambda)$ and $D(\lambda)$ are Fredholm determinants; (the kernel K may not be separable).

5. In the case of an arbitrary (non-separable) kernel, the eigenvalues are zeros of the Fredholm determinant $D(\lambda)$ i.e. poles of $\Gamma(x, y, \lambda)$.

10.3.4 Illustrative examples

The method of determining eigenvalues and eigenvectors of a homogeneous Fredholm I.E. as well as the applications of Fredholm alternative theorem are illustrated below with examples.

Example 1

Solve the homogeneous Fredholm IE and find its eigenvalues.

$$u(x) = \lambda \int_0^1 e^{x+y}\, u(y)\, dy$$

Solution

Here the kernel $K(x, y) = \exp(x + y) = \exp(x)\exp(y)$ is separable. Following the standard procedure we have

$$u(x) = \lambda\, e^x \int_0^1 e^y\, u(y)\, dy = \lambda\, e^x\, c \tag{1}$$

where

$$c = \int_0^1 e^y u(y)\, dy \tag{2}$$

Substituting for $u(y)$ from (1) into (2), we have

$$c = c\,\lambda \int_0^1 e^{2y}\, dy = \frac{1}{2}\, c\, \lambda\, (e^2 - 1)$$

which can also be written as

$$c\left[1 - \frac{\lambda}{2}(e^2 - 1)\right] = 0$$

Either $c = 0$ which leads to trivial solution or $c \neq 0$. This gives

$$1 - \frac{\lambda}{2}(e^2 - 1) = 0 \quad \text{or} \quad \lambda = \frac{2}{e^2 - 1}$$

which is an eigenvalue of the given IE, and the required eigensolution is given by

$$u(x) = \frac{2c}{e^2 - 1}\, e^x$$

where $c \neq 0$ is an arbitrary constant.

Example 2

Show that the IE

$$u(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x + y)\, u(y)\, dy \tag{1}$$

possesses no solution for $f(x) = x$, but it possesses infinitely many solutions when $f(x) = 1$.

Solution

We will make use of the Fredholm alternative theorem. First we calculate the eigenvalues of the homogeneous IE

$$u(x) = \lambda \int_0^{2\pi} \sin(x + y)\, u(y)\, dy \tag{1'}$$

where $\lambda = \frac{1}{\pi}$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

Following the standard procedure for a separable kernel, we can write

$$u(x) = \lambda \sin x \, c_1 + \lambda \cos x \, c_2 \tag{2}$$

where

$$c_1 = \int_0^{2\pi} \cos y \, u(y) \, dy = \int_0^{2\pi} \cos y [\lambda \sin y \, c_1 + \lambda \cos y \, c_2] \, dy \tag{3}$$

and

$$c_2 = \int_0^{2\pi} \sin y \, u(y) \, dy = \int_0^{2\pi} \lambda \sin y \cos y \, c_1 + \lambda c_2 \int_0^{2\pi} \cos^2 y \, dy = \frac{c_1}{2}\lambda \int_0^{2\pi} \sin 2y \, dy + \frac{\lambda c_2}{2} \int_0^{2\pi} (1 + \cos 2y) \, dy \tag{4}$$

Substituting for $u(y)$ from (2) into (3) and (4), we obtain, on simplification

$$c_1 = \lambda \pi c_2 \quad \text{or} \quad c_2 = \lambda \pi c_1$$

These equations form a system of simultaneous equations and can be written as

when $\lambda = \frac{1}{\pi}$ (3) $\Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2 \Rightarrow U(x) = \lambda c_1 (\sin x + \cos x) = \frac{c_1}{\pi}(\sin x + \cos x)$

$$\left.\begin{array}{rcrcl} c_1 & - & \lambda \pi c_2 & = & 0 \\ -\lambda \pi c_1 & + & c_2 & = & 0 \end{array}\right\} \tag{5}$$

$\int_0^{2\pi} \sin y \, u(y) \, dy = \int_0^{2\pi} \sin y (\lambda \sin y \, c_1 + \lambda \cos y \, c_2) \, dy$

$\lambda c_1 \int_0^{2\pi} \sin^2 y \, dy + \lambda c_2 \int_0^{2\pi} \sin y \cos y \, dy$

For eigenvalues of the IE we have the characteristic polynomial

$$D(\lambda) = \begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = 1 - \lambda^2 \pi^2$$

The characteristic equation therefore is

$$D(\lambda) = 0 \quad \text{or} \quad 1 - \lambda^2 \pi^2 = 0$$

which gives $\lambda_1 = 1/\pi$ and $\lambda_2 = -(1/\pi)$ as eigenvalues.

From the Fredholm alternative theorem it follows that corresponding to $\lambda = \lambda_1, \lambda_2$ the given nonhomogeneous IE (1) in general has no solution. To discuss the special case in which the same has a solution, as guaranteed by the said theorem, we need the eigensolution of the transposed IE of (1′), which is the same as (1′) itself. The eigensolution of (1′) with $\lambda = \lambda_1 = 1/\pi$ can be found from (2), where c_1, c_2 are to be found from equations (5). In this case $(\lambda_1 = 1/\pi)$, they reduce to

$$c_1 = c_2 = c, \quad \text{(say)}$$

Therefore eigensolution of (1′) can be obtained from (2) as

$$\psi_1(x) = u_{01}(x) = \lambda \sin x \, c + \lambda \cos x \, c = c \lambda (\sin x + \cos x) \tag{6}$$

According to Fredholm alternative theorem, the nonhomogeneous (1) will have a solution if

$$\int_0^{2\pi} u_{01}\, f(x)\, dx = 0 \tag{7}$$

When $f(x) = x$, the integral on L.H.S. of (7) becomes

$$\frac{c}{\pi}\int_0^{2\pi} (\sin x + \cos x)\, x\, dx = -2c \neq 0$$

Hence we conclude that when $f(x) = x$ the IE (1) has no solution corresponding to $\lambda = 1/\pi$.

When $f(x) = 1$, the same integral becomes → integration by Parts

$$\frac{c}{\pi}\int_0^{2\pi} (\sin x + \cos x)\, dx = 0$$

which shows that (1′) has a solution in this case.

Example 3

Find the eigenvalues of the homogeneous IE associated with the IE

$$u(x) = f(x) + \lambda \int_0^1 (1 - 3xy)\, u(y)\, dy \tag{1}$$

and discuss its solution.

Solution

We first determine the eigenvalues of IE (1). Using the standard method for a separable kernel $K(x,\ y) = 1 - 3xy$, we have

$$u(x) = \lambda \int_0^1 u(y)\, dy - 3x \int_0^1 y\, u(y)\, dy = \lambda c_1 - 3x\,\lambda c_2 \tag{2}$$

where

$$c_1 = \int_0^1 u(y)\, dy \quad \text{and} \quad c_2 = \int_0^1 y\, u(y)\, dy \tag{3}$$

On substituting for $u(y)$ from (2) into (3), we obtain, after simplification, the set of equations

$$\left.\begin{aligned} (1-\lambda)\, c_1 &+ (3/2)\,\lambda c_2 = 0 \\ -(\lambda/2)\, c_1 &+ (1+\lambda)\, c_2 = 0 \end{aligned}\right\} \tag{4}$$

when $\lambda = 2$ (4)''' $\Rightarrow -c_1 + 3c_2 = 0 \Rightarrow c_1 = 3c_2$
(2) $\Rightarrow u(x) = 6c_2(1-x)$

from fredholm A. Theorem we know that the Non. Homo IE

The characteristic polynomial in this case is

$$D(\lambda) = \begin{vmatrix} 1-\lambda & 3\lambda/2 \\ -\lambda/2 & 1+\lambda \end{vmatrix} = 1 - \lambda^2/4$$

The characteristic equation $D(\lambda) = 0$ gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$.

From the Fredholm alternative theorem we know that the nonhomogeneous IE (1) will in general have no solution when $\lambda = \lambda_1, \lambda_2$. For all other values the same will have a unique solution.

Next we find the solution corresponding to $\lambda \neq \lambda_1, \lambda_2$. The required solution is given by

$$u(x) = f(x) + \lambda c_1 - 3x\lambda c_2 \qquad (5)$$

which is different from the equation (2) in having the nonhomogeneous term $f(x)$.

The constants c_1 and c_2 in (5) are the same as given by (3). On substituting for $u(y)$ from (5) into (3), we obtain

$$\left.\begin{aligned} (1-\lambda)c_1 &+ (3/2)\lambda c_2 = f_1 \\ -(\lambda/2)c_1 &+ (1+\lambda)c_2 = f_2 \end{aligned}\right\} \qquad (6)$$

Here

$$f_1 = \int_0^1 f(y)\,dy, \quad f_2 = \int_0^1 y f(y)\,dy$$

On solving equations (6) for c_1 and c_2, we obtain

$$\frac{c_1}{(-3/2)\lambda f_2 + (1+\lambda)f_1} = \frac{c_2}{(1/2)\lambda f_1 + (1-\lambda)f_2} = \frac{1}{(1-\lambda^2) + (3/4)\lambda^2}$$

which gives

$$c_1 = \frac{(-3/2)\lambda f_2 + (1+\lambda)f_1}{(1-\lambda^2) + (3/4)\lambda^2}$$

and

$$c_2 = \frac{(1/2)\lambda f_1 + (1-\lambda)f_2}{(1-\lambda^2) + (3/4)\lambda^2}$$

Therefore on putting the values of c_1 and c_2 in (1) we obtain the required solution.

Example 4

Find the eigenvalues and eigenfunctions of the integral equation

$$u(x) = \lambda \int_0^\pi K(x,\ y)\, u(y)\, dy \qquad (1)$$

where

$$K(x,\ y) = \begin{cases} \cos x \sin y, & 0 \le x \le y \\ \cos y \sin x, & y < x \le \pi \end{cases} \qquad (2)$$

Solution

Here the separable kernel is not defined by a single expression but two expressions. We have to split the integral into two parts, corresponding to the subintervals $(0, x]$ and $(x, \pi]$. The method discussed in example 3 above and other examples cannot be applied here. Instead we follow a different method whereby we transform the given IE into a boundary value problem.

Rewriting (1) as

$$u(x) = \lambda \int_0^x K(x,y)\, u(y)\, dy + \lambda \int_x^\pi K(x,\ y)\, u(y)\, dy$$

and using (2), we have

$$\begin{aligned} u(x) &= \lambda \textstyle\int_0^x \cos y \sin x\, u(y)\, dy + \lambda \int_x^\pi \cos x \sin y\, u(y)\, dy \\ &= \lambda \sin x \textstyle\int_0^x \cos y\, u(y)\, dy + \lambda \cos x \int_x^\pi \sin y\, u(y)\, dy \end{aligned} \qquad (3)$$

To obtain the equivalent initial-value problem, we differentiate both sides of (3) w.r.t. x. Using Leibniz rule of differentiation under the integral sign, *viz.*

$$\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} F(x,\ y)\, dy = \int_{\alpha(x)}^{\beta(x)} F_x\, dy + F(x,\ \beta)\, \beta' - F(x,\ \alpha)\alpha'$$

we obtain

$$\begin{aligned} u'(x) &= \lambda \cos x \textstyle\int_0^x \cos y\ u(y)dy + \lambda \sin x\ \cos x\, u(x) \\ &- \lambda \sin x \textstyle\int_x^\pi \sin y u(y)dy - \lambda \cos x\ \sin x u(x) \\ &= \lambda \cos x \textstyle\int_0^x \cos y u(y)dy - \lambda \sin x \int_x^\pi \sin y u(y)dy \end{aligned} \qquad (4)$$

Differentiating again

$$\begin{aligned} u''(x) &= -\lambda \sin x \int_0^x \cos y\, u(y)dy + \lambda \cos^2 x u(x) \\ &- \lambda \cos x \int_x^\pi \sin y\, u(y)\, dy + \lambda \sin^2 x\, u(x) \\ &= -\lambda\, u(x) + \lambda\, u(x) \end{aligned}$$

or

$$u'' + (1-\lambda)\, u = 0 \quad \text{or} \quad u'' + \gamma^2 u = 0 \tag{5}$$

From (3) and (4)

$$u(\pi) = 0, \quad \text{and} \quad u'(0) = 0 \tag{6}$$

DE (5) alongwith B.C's (6) is equivalent to the given IE

It can be shown that the problem (5), (6) has trivial solution only when $\lambda = 1$ and $\lambda > 1$. Therefore we consider the case $\lambda < 1$. In this case we put $1 - \lambda = \gamma^2$.

$$u = c_1 \cos \gamma x + c_2 \sin \gamma x$$

It will satisfy the given B.C's if $c_2 = 0$ is arbitrary and $\cos \gamma \pi = 0$ which gives $\gamma = n - 1/2$, where n is a positive integer.

Therefore

$1 - \lambda_n = \gamma_n^2 = (n - 1/2)^2$ or $\lambda_n = 1 - (n - 1/2)^2$ are the eigenvalues and

$u_n = c_{1n} \cos \gamma_n x \equiv d_n \cos(n - 1/2)x$ are the corresponding eigenfunctions.

Since $\int_0^\pi \cos^2(n-1/2)\, x\, dx = \pi/2$, the normalized eigenfunctions will be

$$u_n = \sqrt{\frac{2}{\pi}} \cos\left(n - \frac{1}{2}\right) x, \quad n = 1, 2, 3, \cdots$$

10.8 Fredholm's Theory of Integral Equations

Fredholm, a Swedish mathematician, was founder of the theory of integral equations. Here we will not discuss the details of his theory but briefly state his main results on the solution of Fredholm I.Es of the second kind. His analysis gives a method for resolvent kernel $\Gamma(x, y, \lambda)$ of the I.E. Whereas in the method of successive approximations, the solution is approximate, here it is exact. The resolvent kernel is in the form of an infinite series which must be convergent for the solution to be valid.

In Fredholm's method we write

$$\Gamma(x, y, \lambda) = \frac{D(x, y, \lambda)}{D(\lambda)} \tag{10.8.1}$$

where

$$D(x, y; \lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} D_k(x, y)\lambda^k \tag{10.8.2}$$

and

$$D(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} dk\, \lambda^k \tag{10.8.3}$$

The functions $D_k(x,\ y)$ and the constants d_k can be determined by recurrence relations as explained below.

We start with

$$D_0(x,\ y) = K(x,\ y) \text{ and } d_0 = 1$$

where $K(x, y)$ is the kernel appearing in the original I.E. The remaining d_k and $D_k(x,y)$ are given by

$$d_k = \int_a^b D_{k-1}(x,x)dx, \qquad k = 1,2,\cdots \tag{10.8.4}$$

$$D_k(x,y) = K(x,y)d_k - \int_a^b K(x,y_1)\, D_{k-1}(y_1,y)\, dy_1 \tag{10.8.5}$$

A special feature of the Fredholm method is that the power series (10.8.2) and (10.8.3) are both guaranteed to converge for all values of λ, unlike the Neumann series, which converges when the condition $|\lambda|B < 1$ is satisfied, (see section 10.4).

The Fredholm method, therefore, leads to a unique, non-singular solution provided $D(\lambda) \neq 0$. The solution $D(\lambda) = 0$ gives the eigenvalues of the associated homogeneous I.E.

Example

Use the Fredholm power series method to solve the I.E.

$$u(x) = x + \lambda \int_0^1 x\, y\, u(y)\, dy \tag{1}$$

Solution

Here $f(x) = x$, $K(x, y) = xy$, and $D_0(x, y) = xy$. Also $d_0 = 1$

From (10.8.4) and (10.8.5)

$$d_1 = \int_0^1 D_0(x,x)dx = \int_0^1 x^2 dx = \frac{1}{3}$$

$$\begin{aligned} D_1(x, y) &= K(x,y)d_1 - \int_0^1 K(x, y_1)D_0(y_1, y)dy_1 \\ &= xy\left(\frac{1}{3}\right) - \int_0^1 (xy_1)(y_1y)dy_1 \\ &= \frac{1}{3}xy - xy\,\frac{1}{3} = 0 \end{aligned}$$

Again

$$d_2 = \int_0^1 D_1(x,x)dx = 0$$

$$
\begin{aligned}
D_2(x,y) &= K(x,y)d_2 - 2\int_0^1 K(x,y_1)D_1(y_1,y)dy_1 \\
&= 0
\end{aligned}
$$

Similarly $d_3 = d_4 = \cdots = 0$, and $D_3 = D_4 = \cdots = 0$.

Also

$$D(\lambda) = d_0 - d_1\lambda + d_2\lambda^2 + \cdots = 1 - \frac{\lambda}{3}$$

$$D(x,y;\lambda) = D_0(x,y) - \lambda D_1(x,y) = D_0(x,y)$$

Therefore

$$\Gamma(x,y,\lambda) = \frac{D_0(x,y)}{1-(\lambda/3)} = \frac{3xy}{3-\lambda}$$

Hence the solution is given by

$$
\begin{aligned}
u(x) &= f(x) + \lambda\int_0^1 \Gamma(x,\ y,\ \lambda)\,f(y)\,dy \\
&= x + \lambda\int_0^1 \frac{3xy}{3-\lambda}y\ dy \\
&= x + \frac{3\lambda x}{3-\lambda}\times\frac{1}{3} \\
&= x + \frac{\lambda x}{3-\lambda} = \frac{3x}{3-\lambda}
\end{aligned}
$$

10.9 Hilbert-Schmidt Theory of Symmetric Kernels

This theory deals with IEs with Hermitian and real symmetric kernels. It is similar to the S-L theory for DEs. It is based on the following facts about a real-symmetric kernel:

(i) Eigenvalues of such a kernel are real.

(ii) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

(iii) The set of eigenvectors corresponding to a given eigenvalue is complete.

Let $u_i(x),\ i = 1, 2, \cdots$ denote orthonormal eigenfunctions of the homogeneous Fredholm IE

$$u(x) = \lambda \int_a^b K(x,\ y)\, u(y)\, dy \tag{10.9.1}$$

corresponding to an eigenvalue λ_i of (10.9.1), where $K(x,\ y)$ is real-symmetric.

With this information about a homogeneous Fredholm IE with symmetric kernel, we can obtain a unique solution of the Fredholm IE of the second kind, *viz.*

$$u(x) = f(x) + \lambda \int_a^b K(x,\ y)\, u(y)\, dy \tag{10.9.2}$$

Because of property (iii) of the eigenfunctions $u_i(x)$, we can expand the solution $u(x)$, as

$$u(x) = \sum_i a_i\, u_i(x) \tag{10.9.3}$$

where a_i are constant coefficients.

Substituting from (10.9.3) into (10.9.2), we have

$$\begin{aligned} \sum_i a_i u_i(x) &= f(x) + \lambda \int_a^b K(x,y) \sum_i a_i u_i(y)\, dy \\ &= f(x) + \lambda \sum_i a_i \int_a^b K(x,y) u_i(y) dy \end{aligned} \tag{10.9.4}$$

Now from (10.9.1)

$$\int_a^b K(x,y)u_i(y)dy = \frac{1}{\lambda_i}u_i(x) \tag{10.9.5}$$

(Note that $\lambda_i \neq 0$). Substituting from (10.9.5) into (10.9.4), we

$$\sum_i a_i u_i(x) = f(x) + \lambda \sum_i \frac{a_i}{\lambda_i} u_i(x)$$

or

$$\sum_i a_i u_i(x)\left(1 - \frac{\lambda}{\lambda_i}\right) = f(x) \tag{10.9.6}$$

Now we will make use of orthonormality property of the eigenfunctions $u_i(x)$ and express a_i in (10.9.6) in term of known quantities.

On multiplying both sides of (10.9.6) with $u_j(x)$ and integrating w.r.t. x from a to b, we have

$$\sum_i a_i \left(1 - \frac{\lambda}{\lambda_i}\right) \int_a^b u_i(x)u_j(x)\, dx = \int_0^b f(x)u_j(x)\, dx$$

or

$$\sum_i a_i \left(1 - \frac{\lambda}{\lambda_i}\right) \delta_{ij} = \int_a^b f(x)\, u_j(x)\, dx$$

or

$$a_j \left(1 - \frac{\lambda}{\lambda_j}\right) = < f, u_j >$$

or

$$a_j = \frac{\lambda_j}{\lambda_j - \lambda} < f, u_j > \tag{10.9.7}$$

On substituting in (10.9.3), we obtain

$$\begin{aligned} u(x) &= \sum_i \frac{\lambda_i}{\lambda_i - \lambda} < f, u_i > u_i(x) \\ &= \sum_i \lambda_i \frac{< f, u_i >}{\lambda_i - \lambda} u_i(x) \end{aligned} \tag{10.9.8}$$

The above solution is unique as long as $\lambda \neq \lambda_i$, an eigenvalue of the homogeneous I.E.

If $\lambda = \lambda_i$, then the coefficients a_j in (10.9.7) will become singular and no solution will exist. However if $< f, u_i >$ also becomes 0, then the coefficients in (10.9.8) may be finite, and a non-singular solution is possible.

Thus if the condition $< f,\ u_i >= 0$ i.e. $\int_a^b f(x)u_i\ dx = 0$ is satisfied, then a solution of the inhomogeneous IE corresponding to $\lambda = \lambda_i$ can be obtained.

Example

Using the method based on Hilbert-Scmidt theory to solve the I.E.

$$u(x) = \sin(x+\alpha) + \lambda \int_0^\pi \sin(x+y)\ u(y)dy \tag{1}$$

Solution

The eigenvalues and eigenfunctions of the homogeneous I.E.

$$u(x) = \lambda \int_0^\pi \sin(x+y)u(y)\ dy \tag{2}$$

are found to be $\lambda_1 = 2/\pi$, $\lambda_2 = -2/\pi$ with orthonormalized eigenfunctions

$$u_1(x) = \frac{(\sin x + \cos x)}{\sqrt{\pi}}, \quad u_2(x) = \frac{(\sin x - \cos x)}{\sqrt{\pi}}$$

The solution to the I.E. (2) is given by

$$\begin{aligned} u(x) &= \sum_i \lambda_i \frac{< f,\ u_i >}{\lambda_i - \lambda} u_i(x) \\ &= \lambda_1 \frac{< f,\ u_1 >}{\lambda_1 - \lambda} u_1 + \lambda_2 \frac{< f,\ u_2 >}{\lambda_2 - \lambda} u_2 \end{aligned}$$

Here $f = \sin(x+\alpha)$. Therefore

$$\begin{aligned} < f,\ u_1 > &= \int_0^\pi \sin(x+\alpha)\, \frac{\sin x + \cos x}{\sqrt{\pi}}\, dx \\ &= \frac{1}{\sqrt{\pi}} \int_0^\pi [\sin x(\sin x\ \cos\alpha + \cos x\ \sin\alpha) \\ &\quad + \cos x(\sin x \cos\alpha + \cos x \sin\alpha)]\ dx \\ &= \frac{1}{\sqrt{\pi}} \left(\cos\alpha\, \frac{\pi}{2} + \sin\alpha\, \frac{\pi}{2}\right) = \frac{\sqrt{\pi}}{2}(\cos\alpha + \sin\alpha) \end{aligned}$$

Similarly

$$\begin{aligned}
< f,\ u_2 > &= \int_0^{\pi} \sin(x+\alpha)\frac{\sin x - \cos x}{\sqrt{\pi}}dx \\
&= \frac{1}{\sqrt{\pi}}\int_0^{\pi} [\sin x(\sin x \cos\alpha + \cos x \alpha) \\
&\quad - \cos x\ (\sin x \cos\alpha + \cos x \sin\alpha)]\, dx \\
&= \frac{1}{\sqrt{\pi}}\left(\cos\alpha\,\frac{\pi}{2} - \sin\alpha\,\frac{\pi}{2}\right) = \frac{\sqrt{\pi}}{2}(\cos\alpha - \sin\alpha)
\end{aligned}$$

Therefore the solution will be

$$\begin{aligned}
u(x) &= \frac{(2/\pi)}{(2/\pi)-\lambda}\frac{\sqrt{\pi}}{2}(\cos\alpha + \sin\alpha)\frac{\sin x + \cos x}{\sqrt{\pi}} \\
&+ \frac{-2/\pi}{-2/\pi - \lambda}\frac{\sqrt{\pi}}{2}(\cos\alpha - \sin\alpha)\frac{\sin x - \cos x}{\sqrt{\pi}} \\
&= \frac{1}{2-\lambda\pi}(\cos\alpha\sin x + \cos\alpha\cos x + \sin\alpha\sin x + \sin\alpha\cos x) \\
&+ \frac{1}{2+\lambda\pi}(\cos\alpha\sin x - \cos\alpha\cos x - \sin\alpha\sin x + \sin\alpha\cos x) \\
&= \frac{1}{2-\lambda\pi}[\sin(x+\alpha) + \cos(x-\alpha)] \\
&+ \frac{1}{2+\lambda\pi}[\sin(x+\alpha) - \cos(x-\alpha)] \\
&= \sin(x+\alpha)\frac{4}{4-\lambda^2\pi^2} + \cos(x-\alpha)\frac{2\lambda\pi}{4-\lambda^2\pi^2} \\
&= \frac{4}{4-\lambda^2\pi^2}\left[\sin(x+\alpha) + \frac{1}{2}\lambda\pi\cos(x-\alpha)\right]
\end{aligned}$$

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خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (آمین)

محمد عثمان حامد

چک نمبر 105 شمالی (گودھے والا) سرگودھا

UNIVERSITY OF SARGODHA

PUNJAB, PAKISTAN