MECHANICS III

03 Credit Hours

ANALYTIC DYNAMICS II

Dr. Babar Ahmad

A Textbook for Undergraduate
Science and Engineering Programs

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To My Parents

Mr. & Mrs. Rana Muhammad Hanif (*late*)

(*May ALLAH bless them*)

PREFACE

Mechanics is one of the most important course in maximum disciplines of science and engineering. No matter what your interest in science or engineering, mechanics will be important for you.
Mechanics is a branch of physics which deals with the bodies at rest and in motion. During the early modern period, scientists such as Galileo, Kepler, and Newton laid the foundation for what is now known as classical mechanics. Hence there is an extensive use of mathematics in its foundation.
Mechanics is core course for undergraduate Mathematics, Physics and many engineering disciplines. It appears under different names as Analytical/Classical Mechanics, Theoretical Mechanics, Mechanics I, Mechanics II, Mechanics III, Analytical Dynamics.
This textbook is designed to support teaching activities in Theoretical Mechanics specially Dynamics. It covers the contents of “Mechanics” for many undergraduate science and engineering programs. It presents simply and clearly the main theoretical aspects of mechanics.
It is assumed that the students have completed their courses in Calculus, Linear Algebra and Differential Equations and Mechanics II. This book also lay the foundations for further studies in physics, physical sciences, and engineering.
For each concept a number books, documents and lecture notes are consulted. I wish to express my gratitude to the authors of such works.
In chapter 1, Lagrangian and Hamiltonian Mechanics are discussed. The concepts of constraints, degree of freedom, generalized coordinates, ignorable coordinates, generalized momenta, Lagrangian, Lagrange’s equations of motion, law of conservation of energy, Hamiltonian, Hamilton’s equations of motion are given by considering the motion of a particle in one dimensional, two dimensional, and three dimensional rectangular coordinate systems and polar, cylindrical, and spherical coordinate systems. Routhian Mechanics is also included in this chapter. Lagrange’s equations of motion and Hamiltonian, Hamilton’s equations of motion are also derived by using Hamilton’s principle.
In chapter 2, the concepts of exact/canonical transformations are discussed. In your differential equation course, you have studied exact differential equation. The same concept is here. The generating function and its four types are also given in this chapter. In chapter 3, Lagrange and Poisson Brackets are given. This chapter is in progress. Hopefully will be complete in its next edition.

In a book of this concept, level and size, there may be a possibility that some misprint might have remained uncorrected. If you find such misprints or want to give some suggestions for its improvement, please write me at: babar.sms@gmail.com

Dr. Babar Ahmad

Islamabad, Pakistan
June, 2020

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Chapter 1

Lagrangian Mechanics

In Newtonian Mechanics, we have different vector equations for the motions of objects in the form of forces. The simplest example is, a point like particle moving under the influence of some force, is governed by the vector equation.

$$\vec{F} = \dot{\vec{p}}$$

In 1788 Lagrange did something different. He examined energy, by using generalized coordinates, and thats what we will do in this chapter.
dynamical system In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. In Ordinary Differential Equations course, If you have followed the book by D. G. Zill, 8th edition, In chapter 3, you have studied "Modeling with First-Order Differential Equations" and in chapter 5, "Modeling with Higher-Order Differential Equations" The mathematical models in these chapters are all Dynamical systems.

1.1 System Configurations and Coordinates

1.1.1 Constraints

Any thing that resists the motion of a particle is known as constraint. *e.g* gas molecules contained in a cylinder are constrained by the walls of the cylinder to move only inside the cylinder.

Holonomic Constraints

If the constrains are relation between coordinates (and possibly time) only are called holonomic constraints. If $x_1, x_2, ..., x_N$ are generalized coordinates then holonomic constraints can be expressed as following

$$f(x_1, x_2, ..., x_N, t) = 0$$

Examples

1. The motion of simple pendulum is in two dimensional system. We consider polar coordinate system. Its length l (radial component r)is fixed is the constraint, mathematically is given as

$$r^2 - l^2 \quad = \quad 0$$

2. The motion of a particle constrained to lie on the surface of a sphere of radius a is holonomic constraint. Its equation will be

$$r^2 - a^2 \quad = \quad 0$$

Where r is the distance of the particle from the centre of a sphere of radius a.

3. A rigid body is a holonomic system, as the distance between any two points is fixed. Consider a rigid body in $3-space$ rectangular coordinate system. Let $P_i = P_i\,(x_i, y_i, z_i)$ and $P_j = P_j\,(x_j, y_j, z_j)$ two points of it having position vectors r_i and r_j relative to some reference point O, then the distance between them is

$$|r_i - r_i| \quad = \quad d_{ij}$$

or

$$\text{or}\,(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \quad = \quad d_{ij}^2$$

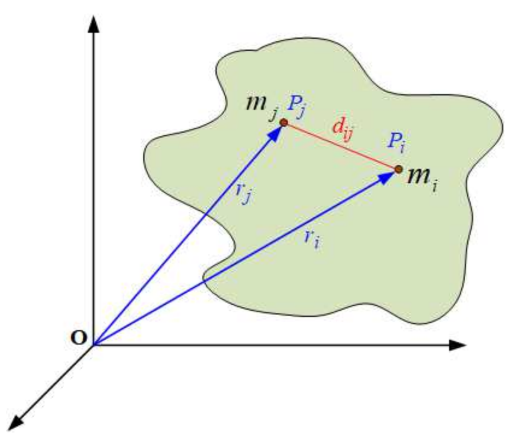

Figure 1.1: Rigid body

Non-Holonomic Constraints

Any constraints which is not holonomic is called non-holonomic constraint, hence a non-holonomic constraint can not be expressed as

$$f(x_1, x_2, ..., x_N, t) \quad = \quad 0$$

Such constrains are relation between coordinates, velocities or higher order derivatives (and possibly time). If $\dot{x}_1, \dot{x}_2, ..., \dot{x}_N$ are velocities with respect to $x_1, x_2, ..., x_N$ generalized coordinates then non-holonomic constraints can be expressed as following

$$f(x_1, x_2, ..., x_N, \dot{x}_1, \dot{x}_2, ..., \dot{x}_N, t) \quad = \quad 0$$

Examples

1. A particle moving on the surface of a sphere of radius a may fall off under the influence of gravity is non-holonomic constraint. Its equation will be

$$r^2 - a^2 \quad \geq \quad 0$$

   Where r is the distance of the particle from the centre of a sphere of radius a.

2. A particle of mass m moves within a cylinder of radius a and height h. The motion

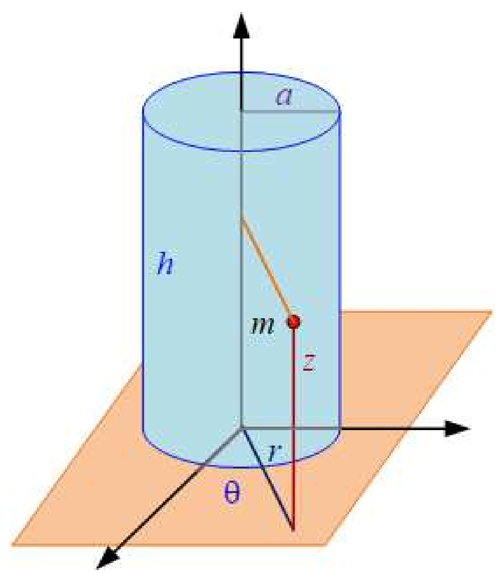

Figure 1.2: Rigid body

is constrained by the following relations

$$
\begin{aligned}
0 &\leq r \leq a \\
0 &\leq \theta \leq 2\pi \\
0 &\leq z \leq h
\end{aligned}
$$

These constraints are non-holonomic.

1.1.2 Degrees of Freedom

A most fundamental property of a physical system is its number of degrees of freedom. This is the minimal number of variables needed to completely specify the positions of all particles and bodies that are part of the system, i.e. its configuration.
For a general system, the number of degrees of freedom is denoted by $NDOF$. We usually thus need $NDOF$ numbers (called coordinates) to describe the system. If there are N coordinates and r constraints then

$$NDOF = N - r$$

Critical Point: The number of DOF is a characteristic of the system and does not depend on the particular set of coordinates used to describe the configuration.
Examples

(1) A point particle moving on a line has one degree of freedom. A generalized coordinate can be taken as x, the coordinate along the line.

(2) A particle moving in three dimensions has three degrees of freedom. Examples of generalized coordinates are the usual rectilinear ones, $\vec{r} = (x, y, z)$, and the spherical ones, $r = (r, \theta, \phi)$, where

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
$$

Here $r = 0$ and $N = 3$
Hence $DOF = N - r = 3$

(3) A rigid body in two dimensions has three degrees of freedom - two "translational" which give the position of some specified point on the body and one "rotational" which gives the orientation of the body. An example, the most common one, of generalized coordinates is (x_c, y_c, ϕ), where x_c and y_c are rectilinear components of the position of the center of mass of the body, and ϕ is the angle from the $xaxis$ to a line from the center of mass to another point (x_1, y_1) on the body.

(4) A rigid body in three dimensions has six degrees of freedom (Three due to its position, and three due to its orientation). Three of these are translational and correspond to the degrees of freedom of the center of mass. The other three are rotational and give the orientation of the rigid body. We will not discuss how to assign generalized coordinates to the rotational degrees of freedom (one way is the so called Euler angles), but the number should be clear from the fact that one needs a vector ω with three components to specify the rate of change of the orientation.

1.1.3 Transforming Coordinates

Once a problem is described in certain generalized coordinates, it can also be described in other coordinate systems. For this, we use coordinate transformations, like

$$q_i = q_i(x_1, x_2, ..., x_N, t) \tag{1.1.1}$$

similarly

$$x_i = x_i(q_1, q_2, ..., q_N, t) \tag{1.1.2}$$

is known as the inverse transformation.

1.1.4 Generalized Coordinates

Describing the configuration of a system can be done in many ways. (We could use many kinds of coordinate systems.) However, we want to be able to work with any description of the system. To accomplish this, we define generalized coordinates q_i as the coordinates that describe the configuration of the system relative to some reference configuration. These coordinates must uniquely define the configuration of the system relative to the reference configuration. If the number of degrees of freedom of a system is N, any set of variables $(q_1, q_2, ..., q_N)$ specifying the configuration is called a set of generalized coordinates. Mathematically consider the transformation

$$T : (x_1, x_2, ..., x_N) \rightarrow (q_1, q_2, ..., q_N) \tag{1.1.3}$$

If T is invertible transformation $i.e$

$$J = \frac{\partial\left(x_1, x_2, ..., x_N\right)}{\partial\left(q_1, q_2, ..., q_N\right)} \neq 0 \tag{1.1.4}$$

Then

$$\begin{aligned} q_1 &= q_1\left(x_1, x_2, ..., x_N\right) \\ q_2 &= q_2\left(x_1, x_2, ..., x_N\right) \\ &. \\ &. \\ &. \\ q_N &= q_N\left(x_1, x_2, ..., x_N\right) \end{aligned}$$

are called generalized coordinates.

Example 1.1.1. *The cylindrical coordinates (r, θ, z) are generalized coordinates.*

By transformation

$$T : (x, y, z) \quad \rightarrow \quad (r, \theta, z)$$

the cartesian coordinates in terms of cylindrical coordinates are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Then

$$\begin{aligned} J &= \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \\ &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \end{aligned}$$

If $r = 0$, then there is no transformation. Hence $r \neq 0$, so the cylindrical coordinates

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \\ z &= z \end{aligned}$$

are generalized coordinates.
Note: The formulation of dynamics problems in terms of generalized coordinates is known as Lagrangian dynamics.

1.1.5 Generalized Velocities

Generalized velocities are defined from the generalized coordinates exactly as ordinary velocity from ordinary coordinates:

$$v_i = \dot{q}_i \tag{1.1.5}$$

Note that the dimension of a generalized velocity depends on the dimension of the corresponding generalized coordinate, so that *e.g.* the dimension of a generalized velocity for an angular coordinate is $(time)^{-1}$ it is an angular velocity. In general, $(v_1, ..., v_N)$ is not the velocity vector.

Example 1.1.2. *With polar coordinates (r, θ) as generalized coordinates, the generalized velocities are $(\dot{r}, \dot{\theta})$, while the velocity vector is $(\dot{r}\hat{r}, r\dot{\theta}\hat{\theta})$.*

1.1.6 Generalized Forces

Generalized Forces are obtained from the applied forces, $F_i, i = 1, ..., n$, acting on a system that has its configuration defined in terms of generalized coordinates. Consider a system, consisting of N point particles with coordinates $(x_1, ..., x_N)$, and that the configuration of the system also is described by the set of generalized coordinates $(q_1, ..., q_N)$. Since both sets of coordinates specify the configuration, there must be a relation between them:

$$\begin{aligned} x_1 &= x_1\,(q_1, q_2, ..., q_N) = x_1(q) \\ x_2 &= x_2\,(q_1, q_2, ..., q_N) = x_2(q) \\ &. \\ &. \\ &. \\ x_N &= x_3\,(q_1, q_2, ..., q_N) = x_N(q) \end{aligned}$$

compactly written as $x_i = x_i(q)$. To make the relation between the two sets of variable specifying the configuration completely general, the functions x_i could also involve an explicit time dependence.We choose not to include it here. If we make a small (infinitesimal) displacement dq_i in the variables q_i, the chain rule implies that the corresponding displacement in x_i is

$$dx_i = \sum_{i=1}^{N} \frac{\partial x_i}{\partial q_i} dq_i \tag{1.1.6}$$

The infinitesimal work done by a force during such a displacement is the sum of terms of the type $\vec{F} \cdot \vec{r}$, *i.e.*

$$dW_i = \sum_{i=1}^{N} F_i dx_i$$

Using (1.1.6), the infinitesimal work done

$$dW_i = \sum_{i=1}^{N} F_i \frac{\partial x_i}{\partial q_i} dq_i \tag{1.1.7}$$

or we can write

$$dW_i = \sum_{j=1}^{N} Q_j dq_j \tag{1.1.8}$$

where Q_j is the generalized force associated to the generalized coordinate q_j hence is given as

$$Q_j = \sum_{i=1}^{N} F_i \frac{\partial x_i}{\partial q_j} \tag{1.1.9}$$

Note: As was the case with the generalized velocities, the dimensions of the Q_j's need not be those of ordinary forces.
Example: Consider a mathematical pendulum with length l, the generalized coordinate being ϕ, the angle from the vertical. Suppose that the mass moves an angle $d\phi$ under the influence of a force $\vec{F}$. The displacement of the mass is

$$d\vec{r} = ld\phi\hat{\phi}$$

and the infinitesimal work becomes

$$dW_i = \vec{F} \cdot \vec{r} = F_\phi l d\phi$$

The generalized force associated with the angular coordinate ϕ obviously is

$$Q_\phi = F_\phi l \tag{1.1.10}$$

which is exactly the torque of the force.
If the force is conservative, we may get it from a potential U as

$$Q_i = -\frac{\partial U}{\partial x_i} \tag{1.1.11}$$

Using this expression in (1.1.9), the generalized force are

$$\begin{aligned} Q_j &= \sum_{i=1}^{N} -\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_j} \\ &= -\frac{\partial U}{\partial q_i} \end{aligned} \tag{1.1.12}$$

The relation between the potential and the generalized force looks the same whatever generalized coordinates one uses.

1.2 Virtual Displacement and Virtual Work

The concepts of virtual displacement and virtual work are very useful and are given next.

1.2.1 Virtual Displacement

A hypothetical displacement of a system in which the forces and constraints remain unchanged and which takes place during infinitesimal time interval is called virtual displacement. It is denoted by δr_i for the ith particle.
Note: During this displacement, the forces of constraints do not do work.

1.2.2 Real and Virtual Displacement

Let $\vec{r_i}$ be the position vector of the ith particle having generalized coordinates q_i at time t. Then

$$r_i = r_i\left(q_i, t\right) \tag{1.2.1}$$

and the quantity

$$dr_i = \frac{\partial r_i}{\partial q_i} dq_i + \frac{\partial r_i}{\partial t} dt \tag{1.2.2}$$

is called the real displacement. If t is fixed then $dt = 0$ and the quantity

$$\delta r_i = \frac{\partial r_i}{\partial q_i} \delta q_i \tag{1.2.3}$$

is called the virtual displacement.

1.2.3 Virtual Work

The work done by a force in virtual displacement.

Example 1.2.1. *A particle of mass m moves under the central force $F = -\mu \frac{m}{r^2}$, where μ is some constant. Find virtual work done.*

The particle moves in polar coordinates, so r and θ are the generalized coordinates. Then r, θ, $\dot{r}$ and $\dot{\theta}$ are linearly independent. The force acting on the particle is

$$F = -\mu \frac{m}{r^2}$$

The generalized force F can be written in polar components as

$$F_r = -\mu \frac{m}{r^2}$$
$$F_\theta = 0$$

As the system is Holonomic, the virtual work done is given by

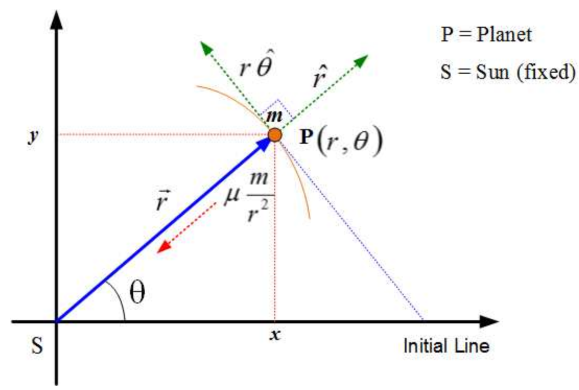

Figure 1.3: Polar motion

$$\begin{aligned} \delta W &= \sum_{i=1}^{2} F_i.\delta q_i \\ &= F_1.\delta q_1 + F_2.\delta q_2 \\ &= F_r.\delta r + F_\theta.\delta\theta \\ &= F_r.\delta r \end{aligned}$$

1.2.4 Principle of Virtual Work

The necessary and sufficient condition for a system of N particles to be in equilibrium is the total virtual work done by applied forces is zero.
Proof: Consider a system of N particles. Let Q_i be the force acting on the ith particle. Then

$$Q_i = F_i + f_i \tag{1.2.4}$$

Where F_i are external applied forces and f_i are constraint forces. If δr_i is the virtual displacement of the ith particle. Then the virtual work is

$$\delta W_i = (F_i + f_i).\delta r_i \tag{1.2.5}$$

Let the system be in equilibrium, then

$$Q_i = 0 \quad \forall i \tag{1.2.6}$$

$$\Rightarrow \delta W_i = 0 \quad \forall i \tag{1.2.7}$$

And for the whole system

$$\begin{aligned}\sum_{i=1}^{N} \delta W_i &= \sum_{i=1}^{N} Q_i.\delta r_i = 0\\ &= \sum_{i=1}^{N} \left(F_i + f_i\right).\delta r_i = 0\\ &= \sum_{i=1}^{N} F_i.\delta r_i + \sum_{i=1}^{N} f_i.\delta r_i = 0\end{aligned}$$

Since the work done by the constraint forces is zero, we have

$$\sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} (F_i.\delta r_i) = 0 \tag{1.2.8}$$

Conversely suppose that the total work done by applied forces is zero. Then

$$\sum_{i=1}^{N} \delta W_i = 0$$

If δr_i is the virtual displacement for the applied force F_i, then we have

$$\sum_{i=1}^{N} F_i.\delta r_i = 0$$

or

$$\sum_{i=1}^{N} F_i = 0$$

Hence the system is in equilibrium.

1.3 D Alembert's Principle

The virtual work done by the applied forces acting on a system in equilibrium is zero.
Let us consider a system of N particles. Let $\vec{r_i}$ be the position vector of the ith particle P_i of mass m_i of the system, at any time t. Let $\dot{\vec{r_i}}$, $\ddot{\vec{r_i}}$ be the velocity and acceleration of particle P_i and $\vec{F_i}$ be the external force acting on it. Let $\vec{R_i}$ be the force of constraints on particle P_i and $\vec{F_{ij}}$ be the mutual force exerted by particle P_i on particle P_j and $\vec{F_{ji}}$ be the mutual force exerted by particle P_j on particle P_i. (see Fig. 1.4) Then by Newton's second law of motion, the equation of motion of the ith particle are given by

$$m_i\ddot{\vec{r_i}} = \vec{F_i} + \vec{R_i} + \vec{F_{ij}} + \vec{F_{ji}}; \quad i, j = 1 \ . \ . \ . \ N \tag{1.3.1}$$

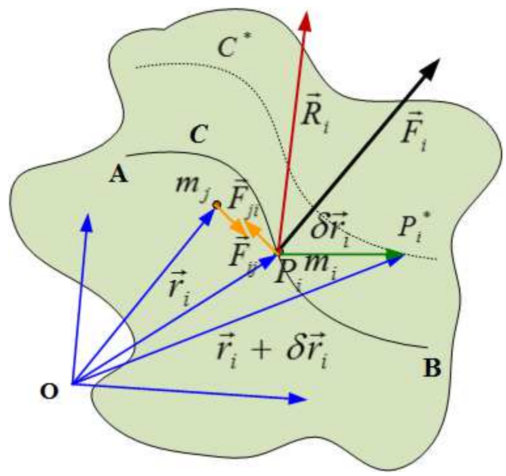

Figure 1.4: virtual motion

For the whole system of particles, we sum this equation (1.3.1) over $i, \ j$ from 1 to N

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i = \sum_{i=1}^{N} \vec{F}_i + \sum_{i=1}^{N} \vec{R}_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\vec{F_{ij}} + \vec{F_{ji}}\right) \tag{1.3.2}$$

We assume that the mutual forces $\vec{F_{ij}}$ and $\vec{F_{ji}}$ are equal in magnitude and opposite in direction. Then by Newton's law of actions and reactions

$$\vec{F_{ij}} = -\vec{F_{ji}}$$

which gives

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\vec{F_{ij}} + \vec{F_{ji}}\right) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\vec{F_{ij}} - \vec{F_{ij}}\right) = 0 \tag{1.3.3}$$

Using (1.3.3) in (1.3.2), we have

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i = \sum_{i=1}^{N}\left(\vec{F}_i + \vec{R}_i\right) \tag{1.3.4}$$

Let us now consider a virtual displacement $\delta\vec{r}_i$ and dot multiplication of (1.3.1) with $\delta\vec{r}_i$ to give

$$m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \left[\vec{F}_i + \vec{R}_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\vec{F_{ij}} + \vec{F_{ji}}\right)\right] \cdot \delta\vec{r}_i \qquad i, j = 1 \ . \ . \ . \ N$$

This can be written for the whole system of N particles as

$$\sum_{i=1}^{N}\left(\vec{F_i} - m_i\ddot{\vec{r_i}}\right)\cdot\delta\vec{r_i} \quad = \quad -\left[\sum_{i=1}^{N}\vec{R_i} + \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\vec{F_{ij}} + \vec{F_{ji}}\right)\right]\cdot\delta\vec{r_i} \tag{1.3.5}$$

We assume that the condition (1.3.3) hold, also suppose that the constraints are ideal *i.e.* the work done by all the forces of constraints along a virtual displacement is zero.

$$\sum_{i=1}^{N}\vec{R_i}\cdot\delta\vec{r_i} \quad = \quad 0 \tag{1.3.6}$$

Using (1.3.3) and (1.3.6), then (1.3.2) takes the form

$$\sum_{i=1}^{N}\left(\vec{F_i} - m_i\ddot{\vec{r_i}}\right)\cdot\delta\vec{r_i} \quad = \quad 0 \tag{1.3.7}$$

(1.3.7) is known as D Alembert's Principle. If the system is at rest, then $\dot{\vec{r_i}} = 0$ and $\ddot{\vec{r_i}} = 0$ and (1.3.7) takes the form

$$\sum_{i=1}^{N}\vec{F_i}\cdot\delta\vec{r_i} \quad = \quad 0 \tag{1.3.8}$$

Also for uniform motion, $\dot{\vec{r_i}}$ is constant and $\ddot{\vec{r_i}} = 0$ and the result is (1.3.8).

1.4 Euler Lagrange's Equation of Motion

Consider a system of N particles whose configuration at any time t is specified by N Lagrangian coordinates $(q_1, q_2, q_3 \cdots q_N)$, then

$$\begin{aligned}\vec{r_i} \quad &= \quad \vec{r_i}\left(q_1, q_2, q_3 \cdots q_N, t\right)\\ &= \quad \vec{r_i}\left(q_s\right) \qquad s = 1 \;.\;.\;.\; N\end{aligned} \tag{1.4.1}$$

(Note: i is fixed while s is free index)
Let $\vec{F_i}$ be the applied force on the ith particle P_i of mass m_i of the system as shown in Fig. 1.12. Then by D Alembert's Principle, the virtual work done by the applied forces acting on a system in equilibrium is zero. So we transform (1.3.7) in terms of generalized coordinates. We have to find expressions for $\delta\vec{r_i}$ and $\ddot{\vec{r_i}}$ from (1.4.1).

$$\delta\vec{r_i} \quad = \quad \sum_{s=1}^{N}\frac{\partial\vec{r_i}}{\partial q_s}\delta q_s \tag{1.4.2}$$

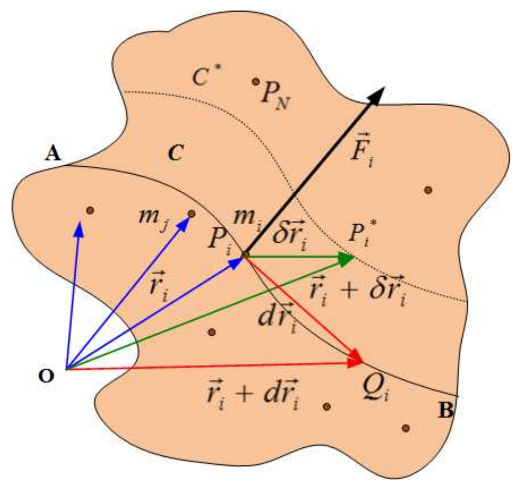

Figure 1.5: virtual motion

Using (1.4.2) in (1.3.7), we have

$$\sum_{s=1}^{N}\sum_{i=1}^{N}\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\cdot\frac{\partial\vec{r}_i}{\partial q_s}\delta q_s = 0$$

$$\sum_{s=1}^{N}\left(\sum_{i=1}^{N}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_s} - \sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_s}\right)\delta q_s = 0 \tag{1.4.3}$$

we write

$$Q_s = \sum_{s=1}^{N}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_s} \tag{1.4.4}$$

generalized forces corresponding to coordinates $q_s \quad s = 1\ \ldots\ N$
Next consider

$$\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_s}\right) = m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_s} + m_i\dot{\vec{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_s}\right)$$

or we can write

$$m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_s} = \frac{d}{dt}\left(m_i\dot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_s}\right) - m_i\dot{\vec{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_s}\right) \tag{1.4.5}$$

Using chain rule, the time derivative of (1.4.1) is

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \sum_{s=1}^{N}\frac{\partial\vec{r}_i}{\partial q_s}\dot{q}_s \tag{1.4.6}$$

and differentiate (1.4.6) partially with respect to $\dot{q}_s$,

$$\begin{aligned} \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q_s}} &= \frac{\partial}{\partial \dot{q_s}} \left(\sum_{s=1}^{N} \frac{\partial \vec{r_i}}{\partial q_s} \dot{q_s} \right) \\ &= \sum_{s=1}^{N} \frac{\partial \vec{r_i}}{\partial q_k} \frac{\partial \dot{q_k}}{\partial \dot{q_s}} \end{aligned} \tag{1.4.7}$$

Since all q_k for $k = 1 \cdot \cdot \cdot N$ are linearly independent and so does $\dot{q_k}$, then we have

$$\frac{\partial \dot{q_k}}{\partial \dot{q_s}} = \begin{cases} 1 & k = s \\ 0 & k \neq s \end{cases} \tag{1.4.8}$$

In view of (1.4.8), (1.4.7) becomes

$$\frac{\partial \dot{\vec{r_i}}}{\partial \dot{q_s}} = \frac{\partial \vec{r_i}}{\partial q_s} \tag{1.4.9}$$

Next the term $\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_s}\right)$ can be calculated as by using contraction property

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_s}\right) &= \frac{\partial}{\partial q_s}\left(\frac{d\vec{r_i}}{dt}\right) \\ &= \frac{\partial \dot{\vec{r_i}}}{\partial q_s} \end{aligned} \tag{1.4.10}$$

Using (1.4.9) and (1.4.10) in (1.4.5)

$$\begin{aligned} m_i\ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_s} &= \frac{d}{dt}\left(m_i\dot{\vec{r_i}} \cdot \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q_s}} \right) - m_i\dot{\vec{r_i}} \cdot \frac{\partial \dot{\vec{r_i}}}{\partial q_s} \\ &= \frac{d}{dt}\left(\frac{1}{2} m_i \frac{\partial}{\partial \dot{q_s}} \left(\dot{\vec{r_i}}\right)^2 \right) - \left(\frac{1}{2} m_i \frac{\partial}{\partial q_s} \left(\dot{\vec{r_i}}\right)^2 \right) \end{aligned} \tag{1.4.11}$$

Summing (1.4.11) over i from 1 to N

$$\sum_{i=1}^{N} m_i\ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_s}} \left(\sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^2 \right) \right) - \frac{\partial}{\partial q_s} \left(\sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^2 \right) \tag{1.4.12}$$

If

$$T_i = \frac{1}{2} m_i \dot{r}_i^2 \tag{1.4.13}$$

is the kinetic energy of the ith particle, then summing (1.4.13) over i from 1 to N

$$T = \sum_{i=1}^{N} T_i = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^2 \tag{1.4.14}$$

is the kinetic energy of the whole system. Using (1.4.14) in (1.4.12)

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_s} \quad = \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} \tag{1.4.15}$$

Using (1.4.4) and (1.4.15) in (1.4.3)

$$\left[Q_s - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) + \frac{\partial T}{\partial q_s}\right] \delta q_s \quad = \quad 0$$

or

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s\right] \delta q_s \quad = \quad 0 \tag{1.4.16}$$

Since all q_s for $s = 1 \cdot \cdot \cdot N$ are linearly independent and so does δq_s, and consequently the coefficient of each δq_s must be equal to zero. Then from (1.4.16), we can write

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s \quad = \quad 0, \qquad s = 1 \cdot \cdot \cdot N \tag{1.4.17}$$

(1.4.17) are known as Lagrange's Equations of Motion.
If the given forces are conservative, then there exist a function $U(q_s)$, $\quad s = 1 \cdot \cdot \cdot N$ is at least of C^1,

$$U \quad = \quad U(q_1, q_2, \ldots q_N) \tag{1.4.18}$$

then Q_s is expressible in the form

$$Q_s \quad = \quad -\frac{\partial U}{\partial q_s}, \qquad s = 1 \cdot \cdot \cdot N \tag{1.4.19}$$

Using (1.4.19), (1.4.17) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial U}{\partial q_s} \quad = \quad 0, \qquad s = 1 \cdot \cdot \cdot N \tag{1.4.20}$$

Since $U(q_s)$, it follows that $\frac{\partial U}{\partial \dot{q}_s} = 0$, then we can write (1.4.20) as

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial U}{\partial \dot{q}_s}\right) - \frac{\partial}{\partial q_s}(T - U) \quad &= \quad 0, \qquad s = 1 \cdot \cdot \cdot N \\ \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_s}(T - U)\right) - \frac{\partial}{\partial q_s}(T - U) \quad &= \quad 0, \qquad s = 1 \cdot \cdot \cdot N \end{aligned} \tag{1.4.21}$$

We now introduce Lagrangian L by the relation

$$L \quad = \quad T - U \tag{1.4.22}$$

Using (1.4.22), (1.4.21) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0, \quad s = 1 \cdot \cdot \cdot N \tag{1.4.23}$$

(1.4.23) are called Euler Lagrange equations (Lagrange equations) of motion for conservative forces (field) in terms of any generalized coordinates. The Lagrangian function can be written as

$$L = L(q_i, \dot{q}_i) \tag{1.4.24}$$

1.4.1 Free Particle Motion

In this case, we have

$$\begin{aligned} T &= \frac{1}{2}m\dot{r}^2 \\ \text{and} \quad U &= 0 \\ \text{then} \quad L &= T = \frac{1}{2}m\dot{r}^2 \end{aligned}$$

then

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad \frac{\partial L}{\partial r} = 0$$

then Lagrange's equations of motion are

$$\begin{aligned} \frac{d}{dt}(m\dot{r}) &= 0 \\ m\ddot{r} &= 0 \end{aligned}$$

Since $m \neq 0$ then $\ddot{r} = 0$ or $\dot{r} = v$ is constant.

1.4.2 Expression for Kinetic Energy in terms of Generalized Coordinates

The kinetic energy of a system of N particles is given by (1.4.14)

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^2$$

Using (1.4.6),

$$
\begin{aligned}
T &= \sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial q_s} \dot{q}_s \right)^2 \\
&= \sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial q_s} \dot{q}_s \right) \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right) \\
&= \sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial q_s} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_s \dot{q}_k
\end{aligned}
$$

Let

$$
a_{sk} = \sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial q_s} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right) = a_{ks}
$$

$$
T = \frac{1}{2} a_{sk} \dot{q}_s \dot{q}_k = T(q_s, \dot{q}_s) \tag{1.4.25}
$$

(1.4.25) is the expression for kinetic energy in terms of generalized coordinates.

1.5 One Dimensional Lagrange's Equations of Motion

Consider a particle of mass m moves in one dimensional conservative system. At any time

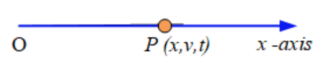

Figure 1.6: One dimensional motion

t it is at P having position x relative to origin as shown in Fig. 1.6. Then $\dot{x}$ be its velocity and $\ddot{x}$ be its acceleration at P. Its kinetic energy is

$$
T = \frac{1}{2} m \dot{x}^2 \tag{1.5.1}
$$

and potential energy is

$$
U = U(x) \tag{1.5.2}
$$

Using (1.5.1) and (1.5.2) in (1.4.22), the Euler Lagrange function is

$$
L = \frac{1}{2} m \dot{x}^2 - U(x) \tag{1.5.3}
$$

Here x is the only generalized coordinate and x and $\dot{x}$ are independent variables, hence the Euler Lagrange equation for x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{1.5.4}$$

From (1.5.3) the quantity

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = m\dot{x}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \tag{1.5.5}$$

and the quantity

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} \tag{1.5.6}$$

Using (1.5.5), (1.5.6), (1.5.4) becomes

$$m\ddot{x} + \frac{\partial U}{\partial x} = 0$$

As U is a function of one variable so we can write

$$m\ddot{x} + \frac{dU}{dx} = 0 \tag{1.5.7}$$

(1.5.7) is the one dimensional Euler Lagrange equation of motion.

1.5.1 Energy in One Dimensional Motion is conserved

Multiplying (1.5.7) by $\dot{x} = \frac{dx}{dt}$, we have

$$m\ddot{x}\dot{x} + \frac{dU}{dx}\frac{dx}{dt} = 0 \tag{1.5.8}$$

The quantity $m\ddot{x}\dot{x} = \frac{1}{2}m\dot{x}^2$ and by chain rule the quantity $\frac{dU}{dx}\frac{dx}{dt} = \frac{dU}{dt}$, then (1.5.8) can be written as

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + U(x)\right] &= 0 \\ \frac{d}{dt}\left[T + U\right] &= 0 \\ \frac{dE}{dt} &= 0 \end{aligned}$$

Integrating we have

$$E = constant$$

Hence the energy of the system is conserved.

Example 1.5.1. *Find Euler lagrange equation of motion of free fall body.*

Solution In free fall motion, a body of mass m is dropped (at rest) from a height of h meters. Since it is one dimensional motion, the reference axis may be $z - axis$ only. At

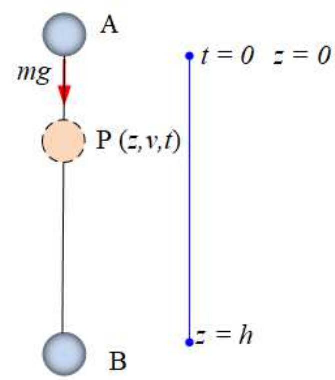

Figure 1.7: Free fall motion

time t the body is at P with position z relative to point A as shown in Fig. 1.18. Then $\dot{z}$ be its velocity and $\ddot{z}$ be its acceleration at P. Its kinetic energy at P is

$$T = \frac{1}{2}m\dot{z}^2 \tag{1.5.9}$$

Taking A as the reference point its potential energy is

$$U = -mgz \tag{1.5.10}$$

Using (1.5.9) and (1.5.10) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\dot{z}^2 + mgz \tag{1.5.11}$$

Here z is the only generalized coordinate and z and $\dot{z}$ are independent variables, hence the Euler Lagrange equation for z is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \tag{1.5.12}$$

From (1.5.11) the quantity

$$\left(\frac{\partial L}{\partial \dot{z}}\right) = m\dot{z}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z} \tag{1.5.13}$$

and the quantity

$$\frac{\partial L}{\partial z} = mg \tag{1.5.14}$$

Using (1.5.13), (1.5.14), (1.5.12) becomes

$$m\ddot{z} - mg = 0 \tag{1.5.15}$$

(1.5.15) is the Euler Lagrange equation of free fall motion.

Corollary 1.5.1. *Energy in free fall motion is conserved*

Multiplying (1.5.15) by $\dot{z}$, we have

$$m\dot{z}\ddot{z} - mg\dot{z} = 0 \tag{1.5.16}$$

we can write (1.5.16) as

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}m\dot{z}^2 - mgz\right] &= 0\\ \frac{d}{dt}[T+U] &= 0\\ \frac{dE}{dt} &= 0\end{aligned}$$

Integrating we have

$$E = constant$$

Hence the energy of the system is conserved.

1.5.2 Two Dimensional Euler Lagrange Equations of Motion

Consider a particle of mass m moves in two dimensional conservative system. At any time t it is at P having position $P(x, y)$ relative to origin as shown in Fig. 1.19. Then its velocity at P is

$$v = \langle \dot{x}, \dot{y} \rangle$$

and square of its magnitude is

$$v^2 = \dot{x}^2 + \dot{y}^2$$

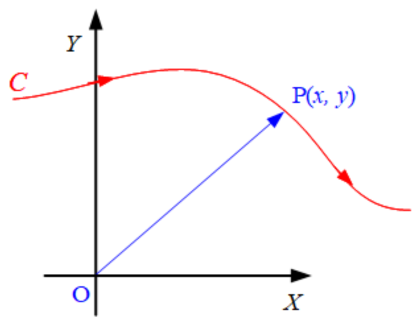

Figure 1.8: Two dimensional motion

Its kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \end{aligned} \tag{1.5.17}$$

and potential energy is

$$U = U(x,y) \tag{1.5.18}$$

Using (1.5.17) and (1.5.18) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - U(x,y) \tag{1.5.19}$$

Here x and y are the generalized coordinate. The Euler Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0, \qquad s = 1,2$$

Set $q_1 = x$ and $q_2 = y$, then Euler Lagrange equation for x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{1.5.20}$$

and for y is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{1.5.21}$$

From (1.5.19) for x we have

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = m\dot{x}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \tag{1.5.22}$$

and the quantity

$$\frac{\partial L}{\partial x} = \frac{\partial U}{\partial x} \tag{1.5.23}$$

Using (1.5.22), (1.5.23), (1.5.20) becomes

$$m\ddot{x} - \frac{\partial U}{\partial x} = 0 \tag{1.5.24}$$

From (1.5.19) for y we have

$$\left(\frac{\partial L}{\partial \dot{y}}\right) = m\dot{y}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \tag{1.5.25}$$

and the quantity

$$\frac{\partial L}{\partial y} = \frac{\partial U}{\partial y} \tag{1.5.26}$$

Using (1.5.25), (1.5.26), (1.5.21) becomes

$$m\ddot{y} - \frac{\partial U}{\partial y} = 0 \tag{1.5.27}$$

(1.5.24) and (1.5.27) are Euler Lagrange equations of motion for two dimensional system.
Special Case In above if we consider OX axis as reference line, then y will be the height of the body (see Fig 1.20) and potential energy function is

$$U = mgy \tag{1.5.28}$$

The Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{1.5.29}$$

then

$$\frac{\partial L}{\partial x} = 0 \tag{1.5.30}$$

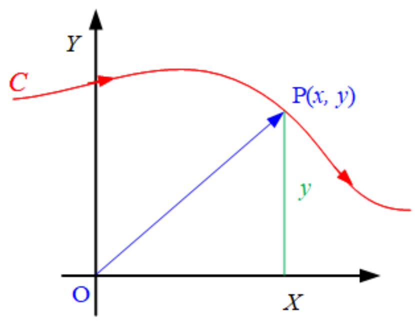

Figure 1.9: Two dimensional motion

and Euler Lagrange equation for x becomes

$$m\ddot{x} = 0 \tag{1.5.31}$$

and the quantity

$$\frac{\partial L}{\partial y} = mg$$

then Euler Lagrange equation for y is

$$m\ddot{y} - mg = 0 \tag{1.5.32}$$

1.6 Lagrange's Equations of Motion in terms of Polar Coordinates

Consider a particle of mass m moves in polar coordinates. At any time t, it be at $P = P(r, \theta)$. Then its velocity in polar coordinate is

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

and

$$v^2 = (\dot{r})^2 + (r\dot{\theta})^2$$

Then its kinetic energy at P is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \end{aligned} \tag{1.6.1}$$

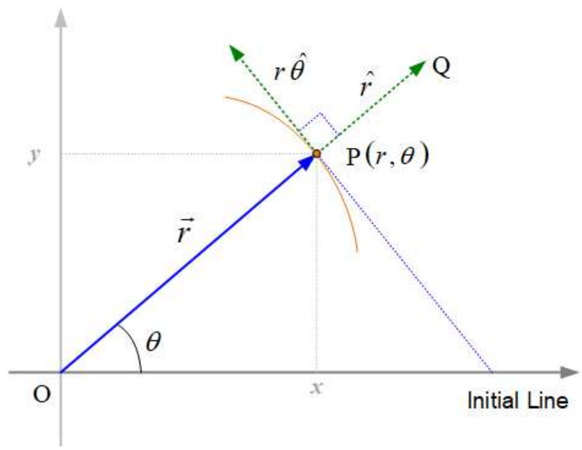

Figure 1.10: Polar motion

The potential energy can be written as

$$U = U(r,\theta) \tag{1.6.2}$$

Using (1.6.1) and (1.6.2) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r,\theta) \tag{1.6.3}$$

Here r and θ are the generalized coordinates. Then r, θ, $\dot{r}$ and $\dot{\theta}$ are linearly independent variables. The Euler Lagrange equations for r and θ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{1.6.4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{1.6.5}$$

From (1.6.3) the quantities

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

and

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

then (1.6.4) becomes

$$m\ddot{r} - \left(mr\dot{\theta}^2 - \frac{\partial U}{\partial r}\right) = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0 \tag{1.6.6}$$

From (1.6.3) the quantities

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

and

$$\frac{\partial L}{\partial \theta} = -\frac{\partial U}{\partial \theta}$$

then (1.6.5) becomes

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + \frac{\partial U}{\partial \theta} = 0 \tag{1.6.7}$$

(1.6.6) and (1.6.7) are the Euler Lagrange equations of motion in terms of polar coordinates.

Example 1.6.1. *Lagrangian and the Lagrange's equations of motion for a simple pendulum.*

Consider OXY a cartesian coordinate system. Let a particle of m is attached with a massless string of length l, with other end fixed at O, forming a simple pendulum, as shown in Fig. 1.11 At any time t, the particle be at $P(r, \theta)$. Clearly

$$l = r$$

is the constraint
Here

$$N = 2$$

and

$$r = 1$$

Hence degree of freedom of this system is

$$DOF = 2 - 1 = 1$$

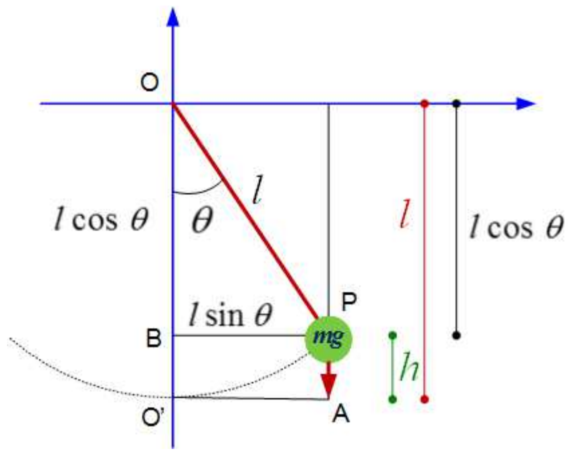

Figure 1.11: Simple Pendulum

And the only generalized coordinate is θ. The velocity of particle P is

$$\vec{v} = l\dot{\theta}\hat{\theta}$$

then

$$v^2 = (l\dot{\theta})^2$$

and the kinetic energy is

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \tag{1.6.8}$$

At P, the particle has height h, given by

$$h = l - l\cos\theta$$

Hence the potential energy of the particle is

$$\begin{aligned} U &= mgh \\ &= mgl\,(1-\cos\theta) \end{aligned} \tag{1.6.9}$$

Using (1.13.3) and (1.6.9) the lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1-\cos\theta) \end{aligned} \tag{1.6.10}$$

Using (1.4.17), the Euler - Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{1.6.11}$$

Differentiate (1.6.10) with respect to θ and $\dot{\theta}$,

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \tag{1.6.12}$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \tag{1.6.13}$$

Next time derivative of (1.6.13) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} \tag{1.6.14}$$

Using (1.6.12) and (1.6.14), (1.6.11) becomes

$$\begin{aligned} ml^2\ddot{\theta} + mgl\sin\theta &= 0 \\ \ddot{\theta} + \frac{g}{l}\sin\theta &= 0 \\ \ddot{\theta} + \omega^2\sin\theta &= 0 \qquad (1.6.15) \end{aligned}$$

$$\text{with } \omega = \sqrt{\frac{g}{l}} \quad \text{is the frequency of oscillation.}$$

By Euler - Lagrange's equation of motion, (1.6.15) is the equation of motion of a simple pendulum.

Example 1.6.2. *A particle of mass m moves under the central force $F = -\mu\frac{m}{r^2}$, where μ is some constant, describing planetary motion. Find its Euler Lagrange equations of motion.*

Solution For central force motion, the particle moves in polar coordinates. At any time t, it be at $P = P(r,\theta)$. Then its kinetic energy at P is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

The force acting on the particle is

$$F = -\mu\frac{m}{r^2}$$

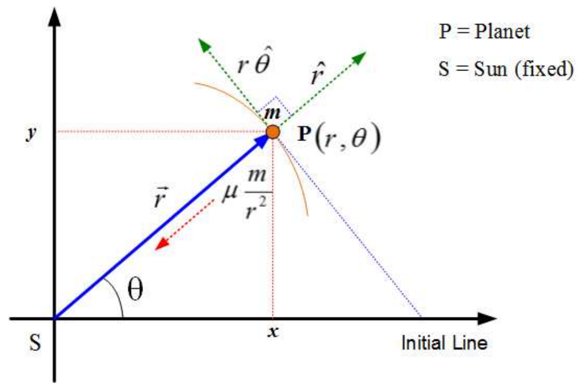

Figure 1.12: Polar motion

The potential energy function is

$$\begin{aligned} U &= -\int F dr = -\int \left(-\mu \frac{m}{r^2}\right) dr \\ &= \mu \frac{m}{r} \end{aligned}$$

The Lagrangian is

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) - \mu \frac{m}{r}$$

Here r and θ are the generalized coordinates. Then r, θ, $\dot{r}$ and $\dot{\theta}$ are linearly independent variables. Using (1.4.17), the Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{1.6.16}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{1.6.17}$$

Differentiate (1.13.11) with respect to r, θ, $\dot{r}$ and $\dot{\theta}$, we have

$$\begin{aligned} \frac{\partial L}{\partial r} &= m r \dot{\theta}^2 + \mu \frac{m}{r^2} \\ \frac{\partial L}{\partial \dot{r}} &= m \dot{r} \\ \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= m r^2 \dot{\theta} \end{aligned}$$

Next

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right)$$

Using above results, (1.6.16) becomes

$$m\ddot{r} - mr\dot{\theta}^2 + \mu\frac{m}{r^2} = 0$$
$$\ddot{r} = r\dot{\theta}^2 - \mu\frac{1}{r^2} \tag{1.6.18}$$

and (1.6.16) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) = 0$$
$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$
$$mr^2\dot{\theta} = c \ \ \text{(constant)}$$

which shows that the angular momentum is conserved (Keplers Second Law)

$$\dot{\theta} = \frac{c}{mr^2} \tag{1.6.19}$$

(1.6.18) and (1.6.19) can be regarded as Lagrange's equations of motion.
In view of (1.6.19), (1.6.18) becomes

$$\ddot{r} = \frac{c^2}{m^2r^3} - \mu\frac{1}{r^2} \tag{1.6.20}$$

(1.6.20) is the equation of motion under central force.

1.7 Lagrange's Equations of Motion in terms of $3-space$ Cartesian Coordinates

Consider a particle of mass m moves in $3-space$ cartesian coordinates system. At any time t, it be at $P = P(x, y, z)$, see Fig. 1.24 . Its velocity at P is

$$\vec{v} = \left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right)$$

then

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

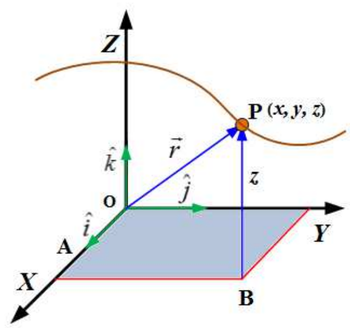

Figure 1.13: Cylindrical motion

the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) \end{aligned} \tag{1.7.1}$$

and the potential energy is

$$U = U(x,y,z) \tag{1.7.2}$$

Using (1.7.1) and (1.7.2) in (1.4.22), the Euler Lagrange function is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-U(x,y,z) \qquad (1.7.3) \\ L &= L(x,y,z,\dot{x},\dot{y},\dot{z}) \end{aligned}$$

Here x, y and z are the generalized coordinate. The Euler Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-\frac{\partial L}{\partial q_s} = 0, \qquad s=1,2,3$$

Set $q_1 = x, q_2 = y$ and $q_3 = z$, then Euler Lagrange equation for x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x} = 0 \tag{1.7.4}$$

for y is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y} = 0 \tag{1.7.5}$$

and for z is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \tag{1.7.6}$$

From (1.7.3) for x we have

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = m\dot{x}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \tag{1.7.7}$$

and the quantity

$$\frac{\partial L}{\partial x} = \frac{\partial U}{\partial x} \tag{1.7.8}$$

Using (1.7.7) and (1.7.8) in (1.7.4) we have

$$m\ddot{x} - \frac{\partial U}{\partial x} = 0 \tag{1.7.9}$$

From (1.7.3) for y we have

$$\left(\frac{\partial L}{\partial \dot{y}}\right) = m\dot{y}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \tag{1.7.10}$$

and the quantity

$$\frac{\partial L}{\partial y} = \frac{\partial U}{\partial y} \tag{1.7.11}$$

Using (1.7.10) and (1.7.11) in (1.7.5) we have

$$m\ddot{y} - \frac{\partial U}{\partial y} = 0 \tag{1.7.12}$$

From (1.7.3) for z we have

$$\left(\frac{\partial L}{\partial \dot{z}}\right) = m\dot{z}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z} \tag{1.7.13}$$

and the quantity

$$\frac{\partial L}{\partial z} = \frac{\partial U}{\partial z} \tag{1.7.14}$$

Using (1.7.13) and (1.7.14) in (1.7.7) we have

$$m\ddot{z} - \frac{\partial U}{\partial z} = 0 \tag{1.7.15}$$

(1.7.9), (1.7.12)and (1.7.15) are Euler Lagrange equations of motion for three dimensional system.

Special Case In above if XOY plane be the zero level for potential energy of the particle, then clearly P is at height z above the XOY plane. Then potential energy function is

$$U = mgz \tag{1.7.16}$$

Using (1.4.22), the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - mgz \qquad (1.7.17)\\ L &= L(x, y, z, \dot{x}, \dot{y}, \dot{z}) \end{aligned}$$

Here x, y and z are the generalized coordinates. Using (1.4.23), the Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{1.7.18}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{1.7.19}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \tag{1.7.20}$$

Differentiate (1.7.17) with respect to $x,\ y,\ z,\ \dot{x},\ \dot{y}$ and $\dot{z}$, we have

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0\\ \frac{\partial L}{\partial \dot{r}} &= m\dot{x}\\ \frac{\partial L}{\partial y} &= 0\\ \frac{\partial L}{\partial \dot{y}} &= m\dot{y}\\ \frac{\partial L}{\partial z} &= mg\\ \frac{\partial L}{\partial \dot{z}} &= m\dot{z} \end{aligned}$$

Next

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= m\ddot{x}\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) &= m\ddot{y}\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) &= m\ddot{z}\end{aligned}$$

Using above results, (1.7.18) becomes

$$\begin{aligned}m\ddot{x} &= 0\\ \ddot{x} &= 0 \qquad (1.7.21)\end{aligned}$$

Next (1.7.19) becomes

$$\begin{aligned}m\ddot{y} &= 0\\ \ddot{y} &= 0 \qquad (1.7.22)\end{aligned}$$

and (1.7.20) becomes

$$\begin{aligned}m\ddot{z} + mg &= 0\\ \ddot{z} &= -g \qquad (1.7.23)\end{aligned}$$

(1.7.21), (1.7.22) and (1.7.23) are the Lagrange equations of motion.

Example 1.7.1. *A particle of mass m moves in space with Lagrangian*

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + A\dot{x} + B\dot{y} + C\dot{z} - U \qquad (1.7.24)$$

where $A,\ B,\ C$ and U are functions of $x,\ y,\ z$. Then show that the particle has equations of motion as

$$\begin{aligned}m\ddot{x} &= -\frac{\partial U}{\partial x} + \dot{y}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) + \dot{z}\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right)\\ m\ddot{y} &= -\frac{\partial U}{\partial y} + \dot{x}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) + \dot{z}\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)\\ m\ddot{z} &= -\frac{\partial U}{\partial z} + \dot{x}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + \dot{y}\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right)\end{aligned}$$

Solution From (1.7.24) the Lagrangian can be written as

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right) \qquad (1.7.25)$$

with $q_1 = x,\ q_2 = y,\ q_3 = z$ as generalized coordinates. Then by (1.4.23), the Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{1.7.26}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{1.7.27}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \tag{1.7.28}$$

From (1.7.24), we can write

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + A \tag{1.7.29}$$

Time derivative of (1.7.29) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} \tag{1.7.30}$$

Again considering (1.7.24), we have

$$\left(\frac{\partial L}{\partial x}\right) = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} - \frac{\partial U}{\partial x} \tag{1.7.31}$$

Using (1.7.30) and (1.7.31) in (1.7.26)

$$m\ddot{x} + \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} - \left(\frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} - \frac{\partial U}{\partial x}\right) = 0$$

or

$$m\ddot{x} = -\frac{\partial U}{\partial x} + \dot{y}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) + \dot{z}\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right) \tag{1.7.32}$$

Similarly from (1.7.27) and (1.7.28), we have

$$m\ddot{y} = -\frac{\partial U}{\partial y} + \dot{x}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) + \dot{z}\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) \tag{1.7.33}$$

$$m\ddot{z} = -\frac{\partial U}{\partial z} + \dot{x}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + \dot{y}\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) \tag{1.7.34}$$

Hence (1.7.32), (1.7.33) and (1.7.34) are the required equations of motion.

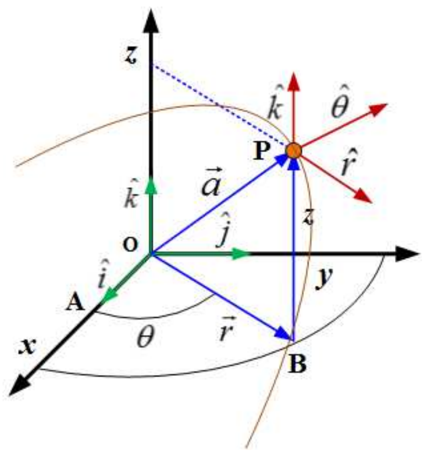

Figure 1.14: Cylindrical motion

1.8 Lagrange's Equations of Motion in terms of Cylindrical Polar Coordinates

Consider a particle of mass m moves in cylindrical polar coordinates. At any time t, it be at $P = P(r, \theta, z)$. Then its velocity at P is

$$\vec{v} = \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}\right)$$

then

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2$$

and the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \end{aligned} \tag{1.8.1}$$

The potential energy can be written as

$$U = U(r, \theta, z) \tag{1.8.2}$$

Using (1.8.1) and (1.8.2) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - U(r, \theta, z) \tag{1.8.3}$$

$$L = L(r, \theta, z, \dot{r}, \dot{\theta}, \dot{z})$$

Here r, θ and z are the generalized coordinates. Then $r,\ \theta,\ z,\ \dot{r},\ \dot{\theta}$, and $\dot{z}$ are linearly independent variables. The Euler Lagrange equations for r, θ and z are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{1.8.4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{1.8.5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \tag{1.8.6}$$

From (1.8.3) the quantities

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

and

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

then (1.8.4) becomes

$$m\ddot{r} - \left(mr\dot{\theta}^2 - \frac{\partial U}{\partial r}\right) = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0 \tag{1.8.7}$$

From (1.8.3) the quantities

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

and

$$\frac{\partial L}{\partial \theta} = -\frac{\partial U}{\partial \theta}$$

then (1.8.5) becomes

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + \frac{\partial U}{\partial \theta} = 0 \tag{1.8.8}$$

From (1.8.3) the quantities

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}$$

and

$$\frac{\partial L}{\partial z} = -\frac{\partial U}{\partial z}$$

then (1.8.6) becomes

$$m\ddot{z} + \frac{\partial U}{\partial z} = 0 \tag{1.8.9}$$

(1.8.7) (1.8.8) and (1.8.9) are the Euler Lagrange equations of motion in terms of cylindrical polar coordinates.
Special Case In above if XOY plane be the zeroth level, then the potential energy of the particle is

$$U = mgz \tag{1.8.10}$$

Using (1.4.22), the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \qquad (1.8.11) \\ L &= L(r, z, \dot{r}, \dot{\theta}, \dot{z}) \end{aligned}$$

Here r, θ and z are the generalized coordinates. Using (1.4.23), the Lagrange's equations of motion for r, θ and z are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{1.8.12}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{1.8.13}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \tag{1.8.14}$$

Differentiate (1.8.11) with respect to $r,\ \theta,\ z,\ \dot{r},\ \dot{\theta}$ and $\dot{z}$, we have

$$\begin{aligned}
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 \\
\frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\
\frac{\partial L}{\partial \theta} &= 0 \\
\frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \\
\frac{\partial L}{\partial z} &= -mg \\
\frac{\partial L}{\partial \dot{z}} &= m\dot{z}
\end{aligned}$$

Next

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= m\ddot{r} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= m\ddot{z}
\end{aligned}$$

Using above results, (1.8.12) becomes

$$\begin{aligned}
m\ddot{r} - mr\dot{\theta}^2 &= 0 \\
\ddot{r} &= r\dot{\theta}^2 \qquad (1.8.15)
\end{aligned}$$

Next (1.8.13) becomes

$$\begin{aligned}
m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) &= 0 \\
\ddot{\theta} &= -2\frac{1}{r}\dot{r}\dot{\theta} \qquad (1.8.16)
\end{aligned}$$

and (1.8.14) becomes

$$\begin{aligned}
m\ddot{z} + mg &= 0 \\
\ddot{z} &= -g \qquad (1.8.17)
\end{aligned}$$

(1.8.15), (1.8.16) and (1.8.17) are the Lagrangian equations of motion for this system.

1.9 Lagrange's Equations of Motion in terms of Spherical Polar Coordinates

Consider a particle of mass m moves in spherical polar coordinates. At any time t, it be at $P = P(r, \theta, \phi)$. Then its velocity at P is

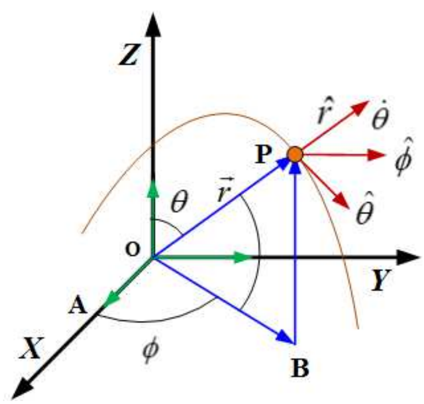

Figure 1.15: Cylindrical motion

$$\vec{v} = \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right)$$

then

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 + \left(r\dot{\phi}\sin\theta\right)^2$$

and the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \end{aligned} \tag{1.9.1}$$

The potential energy can be written as

$$U = U(r, \theta, \phi) \tag{1.9.2}$$

Using (1.9.1) and (1.9.2) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r, \theta, \phi) \tag{1.9.3}$$

$$L = L(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi})$$

Here r, θ and ϕ are the generalized coordinates. Then $r,\ \theta,\ \phi,\ \dot{r},\ \dot{\theta}$, and $\dot{\phi}$ are linearly independent variables. The Euler Lagrange equations for r, θ and ϕ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r} = 0 \tag{1.9.4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta} = 0 \tag{1.9.5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi} = 0 \tag{1.9.6}$$

Differentiate (1.9.3) with respect to r and $\dot{r}$, we have

$$\begin{aligned}\frac{\partial L}{\partial r} &= mr\left(r\dot{\theta}^2+r\sin^2\theta\dot{\phi}^2\right)-\frac{\partial U}{\partial r}\\ \frac{\partial L}{\partial \dot{r}} &= m\dot{r}\end{aligned}$$

Next

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

Using above results, (1.9.4) becomes

$$m\ddot{r}-mr\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)-\frac{\partial U}{\partial r} = 0 \tag{1.9.7}$$

(1.9.7) is the Euler-Lagrange equation of motion in the radial direction.
Next differentiate (1.9.3) with respect to θ and $\dot{\theta}$, we have

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= mr^2\sin\theta\cos\theta\dot{\phi}^2-\frac{\partial U}{\partial \theta}\\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}\end{aligned}$$

Next

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left(2r\dot{r}\dot{\theta}+r^2\ddot{\theta}\right)$$

Using above results, (1.9.5) becomes

$$m\left(2r\dot{r}\dot{\theta}+r^2\ddot{\theta}\right)-mr^2\sin\theta\cos\theta\dot{\phi}^2-\frac{\partial U}{\partial \theta} = 0 \tag{1.9.8}$$

(1.9.8), is the Euler-Lagrange equation of motion in the polar direction.
Next Differentiate (1.9.3) with respect to ϕ and $\dot{\phi}$, we have

$$\begin{aligned}\frac{\partial L}{\partial \phi} &= -\frac{\partial U}{\partial \phi}\\ \frac{\partial L}{\partial \dot{\phi}} &= mr^2\sin^2\theta\dot{\phi}\end{aligned}$$

Next

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{d}{dt}\left(mr^2 \sin^2\theta\dot{\phi}\right) \\ &= m\left(2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + r^2\sin^2\theta\ddot{\phi}\right) \end{aligned}$$

Using above results, (1.9.6) becomes

$$m\left(2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + r^2\sin^2\theta\ddot{\phi}\right) - \frac{\partial U}{\partial \phi} = 0 \tag{1.9.9}$$

(1.9.9), is the Euler Lagrange equation of motion in the azimuthal direction.
Special Case If xy plane is the zeroth level, then its potential energy is

$$U = mgr\cos\theta \tag{1.9.10}$$

Using (1.9.10) in (1.9.3), the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - mgr\cos\theta \qquad (1.9.11) \\ L &= L(r,\theta,\dot{r},\dot{\theta},\dot{\phi}) \end{aligned}$$

Here r, θ and ϕ are the generalized coordinates. Differentiate (1.9.11) with respect to r, θ, ϕ, $\dot{r}$, $\dot{\theta}$ and $\dot{\phi}$, we have

$$\begin{aligned} \frac{\partial L}{\partial r} &= mr\left(r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2\right) - mg\cos\theta \\ \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\ \frac{\partial L}{\partial \theta} &= mr^2\sin\theta\cos\theta\dot{\phi}^2 + mgr\sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \\ \frac{\partial L}{\partial \phi} &= 0 \\ \frac{\partial L}{\partial \dot{\phi}} &= mr^2\sin^2\theta\dot{\phi} \end{aligned}$$

Next

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= m\ddot{r} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\right) \end{aligned}$$

Using above results, (1.9.4) becomes

$$
\begin{aligned}
m\ddot{r} - mr\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mg\cos\theta &= 0 \\
\ddot{r} - r\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) &= -g\cos\theta \qquad (1.9.12)
\end{aligned}
$$

(1.9.12) is the Euler-Lagrange equation in the radial direction.
Next Using (1.9.11), the Euler-Lagrange equation in the polar direction is

$$
\begin{aligned}
m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) - mr^2\sin\theta\cos\theta\dot{\phi}^2 + mgr\sin\theta &= 0 \\
\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) - r\sin\theta\cos\theta\dot{\phi}^2 + g\sin\theta &= 0 \qquad (1.9.13)
\end{aligned}
$$

and by (1.9.11), in the azimuthal direction is

$$
\begin{aligned}
\frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\right) &= 0 \qquad (1.9.14) \\
\left(mr^2\sin^2\theta\dot{\phi}\right) &= C \qquad \text{(constant)} \\
r^2\dot{\phi} &= \frac{C}{m\sin^2\theta} = A \qquad \text{(constant)} \qquad (1.9.15)
\end{aligned}
$$

with $m \neq 0$ and $\sin^2\theta \neq 0$
(1.9.12), (1.9.13) and (1.9.15) are the Lagrangian equations of motion.
From (1.9.14), in the azimuthal direction, we have another form as

$$
\begin{aligned}
r^2\sin^2\theta\ddot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 2r\dot{r}\sin^2\theta\ddot{\phi} &= 0 \\
r\sin\theta\ddot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + 2\dot{r}\sin\theta\ddot{\phi} &= 0 \qquad (1.9.16)
\end{aligned}
$$

with $r \neq 0$ and $\sin\theta \neq 0$
If the particle of mass m moves under the influence of a potential $U(r)$ (potential independent on velocity) and assume for the time being that the kinetic energy is given as the sum of quadratic terms of the time derivatives of the independent coordinates, then the kinetic energy is (1.9.1) and the potential energy is

$$
U = U(r) \qquad (1.9.17)
$$

Using (1.4.22), the Lagrangian is

$$
\begin{aligned}
L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r) \qquad (1.9.18) \\
L &= L(r, \theta, z, \dot{r}, \dot{\theta}, \dot{z})
\end{aligned}
$$

The Lagrange's equations of motion are same as above. For r coordinate, the force component F_r which (for a conservative force) can be derived from the potential energy function as

$$
F_r = -\frac{\partial U}{\partial r}
$$

and the Lagrange's equations of motion is

$$mr\left(r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2\right) + \frac{\partial U}{\partial r} = 0$$

Example 1.9.1. *A particle of mass m moves on the surface of a sphere of radius a. Find its Euler Lagrange equations of motion.*

Solution Consider $OXYZ$ a cartesian coordinate system and a sphere of radius a, with center at the origin. Let a particle of m is moving on a sphere as shown in Fig. 1.26. Under the gravitational force, its configuration at any time t is $P(x, y, z) = P(r, \theta, \phi)$. Clearly

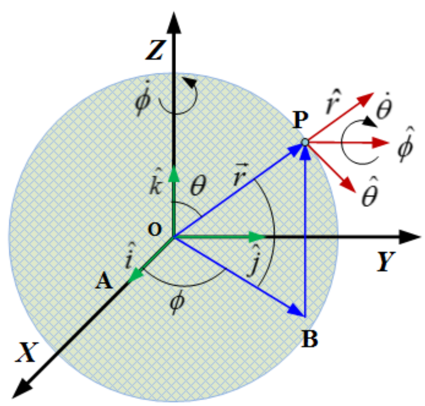

Figure 1.16: Cylindrical motion

$$r = a = \sqrt{x^2 + y^2 + z^2}$$

is the constraint
The number of coordinates is

$$N = 3$$

and the number of constraints is

$$r = 1$$

Hence degree of freedom of this system is

$$DOF = 3 - 1 = 2$$

The radius of the sphere is fixed, so the generalized coordinates are θ and ϕ. The velocity of particle P is

$$\vec{v} = \left(0\hat{r} + a\dot{\theta}\hat{\theta} + a\dot{\phi}\sin\theta\hat{\phi}\right)$$

then

$$v^2 = (a\dot{\theta})^2 + \left(a\dot{\phi}\sin\theta\right)^2$$

and the kinetic energy is

$$T = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)$$

Let xy plane be the zero level for potential energy of the particle, then

$$U = mga\cos\theta$$

Using (1.4.22), the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mga\cos\theta \qquad (1.9.19) \\ L &= L(\theta, \dot{\theta}, \dot{\phi}) \end{aligned}$$

Here θ and ϕ are the generalized coordinates. Using (1.4.23), the Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (1.9.20)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \qquad (1.9.21)$$

Differentiate (1.9.19) with respect to θ, ϕ, $\dot{\theta}$ and $\dot{\phi}$, we have

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= ma^2\sin\theta\cos\theta\dot{\phi}^2 + mga\sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= ma^2\dot{\theta} \\ \frac{\partial L}{\partial \phi} &= 0 \\ \frac{\partial L}{\partial \dot{\phi}} &= ma^2\sin^2\theta\dot{\phi} \end{aligned}$$

Next

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= ma^2\ddot{\theta} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{d}{dt}\left(ma^2\sin^2\theta\dot{\phi}\right) \end{aligned}$$

Next Using (1.9.20), the Euler-Lagrange equation in the polar direction is

$$ma^2\ddot{\theta} - ma^2 \sin\theta\cos\theta\dot{\phi}^2 - mga\sin\theta = 0 \tag{1.9.22}$$
$$a\ddot{\theta} - a\sin\theta\cos\theta\dot{\phi}^2 - g\sin\theta = 0 \tag{1.9.23}$$

and by (1.9.21), in the azimuthal direction is

$$\frac{d}{dt}\left(ma^2\sin^2\theta\dot{\phi}\right) = 0 \tag{1.9.24}$$
$$\left(ma^2\sin^2\theta\dot{\phi}\right) = C \qquad \text{(constant)} \tag{1.9.25}$$

(1.9.23) and (1.9.25) are the Lagrangian equations of motion.

Example 1.9.2. *A particle of mass m moves on the surface of a sphere of radius a. Show that energy of the system is conserved.*

Solution Multiplying (1.9.22) by $\dot{\theta}$, the result is

$$ma^2\dot{\theta}\ddot{\theta} - ma^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}^2 - mga\sin\theta\dot{\theta} = 0 \tag{1.9.26}$$

As θ is a function of t, the terms in (1.9.26) can be written in derivative form. First the term $\dot{\theta}\ddot{\theta}$ can be written as

$$\dot{\theta}\ddot{\theta} = \frac{1}{2}\frac{d}{dt}\left(\dot{\theta}^2\right)$$

the therm $\sin\theta\cos\theta\dot{\theta}$ can be written as

$$\sin\theta\cos\theta\dot{\theta} = \frac{1}{2}\frac{d}{dt}\left(\sin^2\theta\right)$$

and the therm $\sin\theta\dot{\theta}$ can be written as

$$-\sin\theta\dot{\theta} = \frac{d}{dt}\left(\cos\theta\right)$$

Then (1.9.26) becomes

$$\frac{1}{2}ma^2\frac{d}{dt}\left(\dot{\theta}^2\right) - \frac{1}{2}\frac{d}{dt}\left(ma^2\sin^2\theta\right)\dot{\phi}^2 + \frac{d}{dt}\left(mga\cos\theta\right) = 0 \tag{1.9.27}$$

(1.9.24) indicates that $\dot{\phi}$ is independent of t, so the term $\frac{d}{dt}\left(ma^2\sin^2\theta\right)\dot{\phi}^2$ can be written as $\frac{d}{dt}\left(ma^2\sin^2\theta\dot{\phi}\right)\dot{\phi}$ and the quantity $\frac{d}{dt}\left(ma^2\sin^2\theta\dot{\phi}\right) = 0$ as given in (1.9.24). Then (1.9.27) can be written as

$$\frac{1}{2}ma^2\frac{d}{dt}\left(\dot{\theta}^2\right) + \frac{1}{2}\frac{d}{dt}\left(ma^2\sin^2\theta\right)\dot{\phi}^2 + \frac{d}{dt}\left(mga\cos\theta\right) = 0$$
$$\frac{d}{dt}\left[\frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mga\cos\theta\right] = 0 \tag{1.9.28}$$

Note the quantity $\frac{1}{2}ma^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)$ is the kinetic energy of the system and the quantity $mga\cos\theta$ is the potential energy of the system, hence (1.9.28) becomes

$$\begin{aligned}\frac{d}{dt}\left[T+U\right] &= 0\\ \frac{dE}{dt} &= 0\end{aligned}$$

Integrating we have

$$E = constant$$

Hence the energy of the system is conserved.

1.10 Ignorable or Cyclic Coordinates

Sometimes it may be happened that some of the generalized coordinates say $q_1,\ q_2,\ q_3, ...q_k, (k \leq N;\ N$ is the degree of freedom of the system) are not present in the Lagrangian function but their corresponding generalized velocities are present in it. Such coordinates are known as ignorable or cyclic coordinates.

Example 1.10.1. *The Lagrangian for a particle of mass m moving under the central force $F = -\mu \frac{m}{r^2}$, where μ is some constant, is*

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \mu\frac{m}{r}$$

The generalized coordinate θ is not present in the Lagrangian but its corresponding velocity $\dot{\theta}$ is present in it. Hence θ is ignorable coordinate.

Example 1.10.2. *The Lagrangian for a particle of mass m moving on a sphere of radius a is*

$$\begin{aligned} L &= \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mga\cos\theta \\ &= L(\theta, \dot{\theta}, \dot{\phi}) \end{aligned}$$

The generalized coordinate ϕ is not present in the Lagrangian but its corresponding velocity $\dot{\phi}$ is present in it. Hence ϕ is ignorable coordinate.

1.11 Generalized Momentum

The generalized momentum p_s associated with the generalized coordinate q_s is defined as

$$p_s = \frac{\partial L}{\partial \dot{q}_s} \tag{1.11.1}$$

1.12 Canonical Conjugate Momentum

If a system has ignorable coordinates then Euler Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = 0$$

then the generalized momentum

$$p_s = c$$

is constant or conserved, such momentum is termed as canonical conjugate momentum. In this motion, the momentum is same specified by initial conditions.
In simple cases, the canonical momentum is constant multiple of its corresponding generalized velocity. For example the generalized momentum p_z for free fall motion is

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

But generally it is not true. For example a particle of mass m moving under the central force $F = -\mu \frac{m}{r^2}$, where μ is some constant, the generalized momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Then we can get very useful information between coordinates and/or velocities. The Euler Lagrange equation of motion for generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

which implies that

$$\frac{\partial L}{\partial \dot{\theta}} = c \quad \text{(constant)}$$

then the generalized momentum

$$p_\theta = mr^2\dot{\theta} = c$$

is constant and we have a very useful fact that angular velocity is inversely proportional to the square of the radius of the circle. Mathematically can be written as

$$\dot{\theta} \propto r^{-2}$$

As the particle moves inward towards origin, its angular velocity must increase in a specific way. In general, a cyclic coordinate results in a conserved momentum that simplifies the dynamics in the cyclic coordinate.
The above definition of canonical momentum also holds for non-cyclic coordinates.

1.13 Routh's Function

Let us consider a system of N particles whose configuration at any time t is specified by N Lagrangian coordinates $(q_1, q_2, ..., q_N)$. Let $(q_1, q_2, ..., q_m)\,, m \leq N$ be ignorable coordinates. Then we have

$$\frac{\partial L}{\partial q_k} = 0, \quad k = 1 \cdot \cdot \cdot m \leq N \tag{1.13.1}$$

Then the Lagrangian (1.4.23) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = 0, \quad s = 1 \cdot \cdot \cdot N \tag{1.13.2}$$

Using (1.11.1), (1.13.2) becomes

$$\dot{p}_s = 0, \quad s = 1 \cdot \cdot \cdot N \tag{1.13.3}$$

The generalized forces corresponding to cyclic coordinates are all zero. The Lagrangian for cyclic coordinates can be written as

$$\begin{aligned} L &= L\left(q_{m+1},\ q_{m+2} \ . \ . \ . \ q_N,\ \dot{q}_1,\ \dot{q}_2,\ . \ . \ . \ \dot{q}_N,\ t\right) \\ &= L(q_k,\ \dot{q}_s,\ t) \quad s = 1 \cdot \cdot \cdot N, \quad k = m+1 \cdot \cdot \cdot N \end{aligned}$$

Ruth's define a function as

$$\begin{aligned} R &= L - \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s, \quad s = 1 \cdot \cdot \cdot N \\ &= L(q_k, \dot{q}_s, t) - \dot{q}_s p_s, \quad k = m+1 \cdot \cdot \cdot N \end{aligned} \tag{1.13.4}$$

or we can write

$$R = R\left(q_k, \dot{q}_k,\ p_s,\ t\right) \quad s = 1 \cdot \cdot \cdot N, \quad k = m+1 \cdot \cdot \cdot N \tag{1.13.5}$$

1.13.1 Rouths Equations of Motion for Non-cyclic Coordinates

This equation of motion can be derived by taking total differential of (1.13.4) and (1.13.5). First take total differential of (1.13.4)

$$\begin{aligned} dR &= dL(q_k, \dot{q}_s, t) - d(\dot{q}_s\ p_s) \\ &= \frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial \dot{q}_s}d\dot{q}_s + \frac{\partial L}{\partial t}dt - d\dot{q}_s\ p_s - \dot{q}_s\ dp_s \end{aligned}$$

Using (1.11.1) and (1.13.2)

$$\begin{aligned} dR &= \frac{\partial L}{\partial q_k}dq_k + p_s d\dot{q}_s + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k + \frac{\partial L}{\partial t}dt - d\dot{q}_s\ p_s - \dot{q}_s\ dp_s \\ &= \frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k - \dot{q}_s\ dp_s + \frac{\partial L}{\partial t}dt \end{aligned} \tag{1.13.6}$$

Next taking total differential of (1.13.5)

$$\begin{aligned} dR &= dR\,(q_k, \dot{q}_k,\ p_s,\ t) \\ &= \frac{\partial R}{\partial q_k} dq_k + \frac{\partial R}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial R}{\partial p_s}\ dp_s + \frac{\partial R}{\partial t} dt \end{aligned} \tag{1.13.7}$$

From (1.13.6) and (1.13.7), we can write

$$\begin{aligned} \frac{\partial L}{\partial q_k} &= \frac{\partial R}{\partial q_k} \\ \frac{\partial L}{\partial \dot{q}_k} &= \frac{\partial R}{\partial \dot{q}_k} \\ -\ \dot{q}_s &= \frac{\partial R}{\partial p_s} \\ \frac{\partial L}{\partial t} &= \frac{\partial R}{\partial t} \end{aligned}$$

The Lagrangian's equations of motion for non-cyclic coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \qquad k = m+1\ \cdot\ \cdot\ \cdot N \tag{1.13.8}$$

Using above results, we can write

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_k}\right) - \frac{\partial R}{\partial q_k} = 0, \qquad k = m+1\ \cdot\ \cdot\ \cdot N \tag{1.13.9}$$

(1.13.9) are known as Ruth's equations of motion for non-cyclic coordinates.
Consider

$$\dot{q}_s = -\frac{\partial R}{\partial p_s}$$

Integrating with respect to t, we have

$$q_s = -\int \frac{\partial R}{\partial p_s} dt \tag{1.13.10}$$

(1.13.10) gives the generalized coordinates.

1.13.2 Routh's Equations of Motion for Circular Orbit

OR

Routh's Equations of Motion in terms of Polar Coordinates

It can be continued from Lagrange's equation of motion for planetary motion.
The kinetic energy is

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right)$$

Under the action of inverse square law of attraction, the force acting on the particle is

$$F = -\mu\frac{m}{r^2}$$

This force has a relation with potential energy function as

$$\begin{aligned} F &= -\frac{\partial U}{\partial q_s} \\ -\mu\frac{m}{r^2} &= -\frac{dU}{dr} \\ \frac{dU}{dr} &= \mu\frac{m}{r^2} \end{aligned}$$

is separable first order differential equation and can be solved as

$$\begin{aligned} U &= \mu m \int_{-\infty}^{r} \frac{1}{r^2} dr \\ &= \mu\frac{m}{r} \end{aligned}$$

is the potential energy of the system The Lagrangian function is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \mu\frac{m}{r} \\ &= L\left(r,\ \dot{r},\ \dot{\theta}\right) \end{aligned}$$

Here θ is the cyclic coordinates. The Routh's function is

$$\begin{aligned} R &= L - \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s, \qquad s = 1 \cdot \cdot \cdot N \\ &= L - \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} \end{aligned} \tag{1.13.11}$$

Differentiate (1.18.9) with respect to $\dot{\theta}$, we have

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \\ p_\theta &= mr^2\dot{\theta} \end{aligned}$$

or

$$\dot{\theta} = \frac{p_\theta}{mr^2} \tag{1.13.12}$$

Using above results, (1.13.11) can be written as

$$R = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \mu\frac{m}{r} - \left(mr^2\dot{\theta}\right)\dot{\theta}$$

Using, (1.13.12)

$$
\begin{aligned}
R &= \frac{1}{2}m\left(\dot{r}^2 + r^2\frac{p_\theta^2}{m^2r^4}\right) - \mu\frac{m}{r} - mr^2\frac{p_\theta^2}{m^2r^4} \\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{p_\theta^2}{mr^2} - \mu\frac{m}{r} - \frac{p_\theta^2}{mr^2} \\
&= \frac{1}{2}m\dot{r}^2 - \frac{1}{2}\frac{p_\theta^2}{mr^2} - \mu\frac{m}{r} \qquad (1.13.13)
\end{aligned}
$$

(1.13.13) is the Ruth's function. For generalized coordinate r the Routh's equation of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0, \qquad (1.13.14)$$

Differentiate (1.13.13) with respect to r and $\dot{r}$

$$
\begin{aligned}
\frac{\partial R}{\partial r} &= \frac{p_\theta^2}{mr^3} + \mu\frac{m}{r^2} \\
\frac{\partial R}{\partial \dot{r}} &= m\dot{r}
\end{aligned}
$$

Next

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) = m\ddot{r}$$

(1.13.14) becomes

$$m\ddot{r} - \frac{\partial R}{\partial r} - \frac{p_\theta^2}{mr^3} - \mu\frac{m}{r^2} = 0 \qquad (1.13.15)$$

(1.13.15) is the Routh's equation of motion for circular orbit.

1.14 Hamiltons Dynamics

Hamiltonian mechanics is a theory developed as a reformulation of classical mechanics and predicts the same outcomes as non-Hamiltonian classical mechanics. It uses a different mathematical formalism, providing a more abstract understanding of the theory. Historically, it was an important reformulation of classical mechanics, which later contributed to the formulation of quantum mechanics.
Hamiltonian mechanics was first formulated by William Rowan Hamilton in 1833, starting from Lagrangian mechanics, a previous reformulation of classical mechanics introduced by Joseph Louis Lagrange in 1788.

1.14.1 Hamilton's Function

Let us consider a system of N particles whose configuration at any time t is specified by N Lagrangian coordinates $(q_1, q_2, ..., q_N)$. In Lagrangian formulation, the independent variables are the generalized coordinates q_i and the generalized velocities $\dot{q}_i$. *i.e.*

$$L = L(q_s, \dot{q}_s) \tag{1.14.1}$$

Its time derivative is

$$\frac{dL}{dt} = \sum \left(\frac{\partial L}{\partial q_s} \dot{q}_s + \frac{\partial L}{\partial \dot{q}_s} \ddot{q}_s \right) \tag{1.14.2}$$

Lagrange's equation of motion (1.4.23) can be written as

$$\frac{\partial L}{\partial q_s} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_s} \right), \qquad s = 1 \ \cdot \ \cdot \ \cdot N \tag{1.14.3}$$

Using (1.11.1)

$$\frac{\partial L}{\partial q_s} = \frac{d}{dt} (p_s) = \dot{p}_s, \qquad s = 1 \ \cdot \ \cdot \ \cdot N \tag{1.14.4}$$

Using (1.11.1) and (1.14.4) in (1.14.2), we have

$$\begin{aligned} \frac{dL}{dt} &= \sum (\dot{p}_s \dot{q}_s + p_s \ddot{q}_s) \\ &= \frac{d}{dt} \left(\sum p_s \dot{q}_s \right) \end{aligned}$$

or

$$\frac{d}{dt} \left(\sum p_s \dot{q}_s - L \right) = 0$$

Integrating,

$$H = \sum \dot{q}_s p_s - L(q_s, \dot{q}_s), \qquad s = 1 \ \cdot \ \cdot \ \cdot N \tag{1.14.5}$$

(1.14.5) is known as Hamilton's function.

1.14.2 Physical Significance of Hamilton's Function

For a conservative system, potential energy function is

$$U = U(q_s)$$

then

$$\frac{\partial U}{\partial \dot{q}_s} = 0 \tag{1.14.6}$$

Using (1.11.1) in (1.14.5)

$$H = \sum \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L(q_s, \dot{q}_s), \qquad s = 1 \cdot \cdot \cdot N \tag{1.14.7}$$

Since $L = T - U$, then (1.14.7) can be written as

$$\begin{aligned} H &= \sum \frac{\partial}{\partial \dot{q}_s}(T - U)\,\dot{q}_s - L(q_s, \dot{q}_s), \qquad s = 1 \cdot \cdot \cdot N \\ &= \sum \left(\frac{\partial T}{\partial \dot{q}_s}\dot{q}_s - \frac{\partial U}{\partial \dot{q}_s}\dot{q}_s\right) - L(q_s, \dot{q}_s) \end{aligned} \tag{1.14.8}$$

Since U is independent of $\dot{q}_s$, so (1.14.8) can be written as

$$H = \sum \frac{\partial T}{\partial \dot{q}_s}\dot{q}_s - L(q_s, \dot{q}_s), \qquad s = 1 \cdot \cdot \cdot N \tag{1.14.9}$$

Using (1.4.14), (1.14.9) can be written as

$$H = \sum \frac{\partial}{\partial \dot{q}_s}\left(\sum_{i=1}^{N} \frac{1}{2} m_i \dot{q}_i^2\right)\dot{q}_s - L(q_s, \dot{q}_s), \qquad s, i = 1 \cdot \cdot \cdot N \tag{1.14.10}$$

Since all q_s and q_i for $s, i = 1 \cdot \cdot \cdot N$ are linearly independent and so does $\dot{q}_s$ and $\dot{q}_i$, then we have

$$\frac{\partial \dot{q}_i}{\partial \dot{q}_s} = \begin{cases} 1 & i = s \\ 0 & i \neq s \end{cases} \tag{1.14.11}$$

In view of (1.14.11), (1.14.10) becomes

$$\begin{aligned} H &= \sum \left(\frac{1}{2} m_s\,(2\dot{q}_s)\right)\dot{q}_s - L(q_s, \dot{q}_s), \qquad s, i = 1 \cdot \cdot \cdot N \\ &= 2\sum \left(\frac{1}{2} m_s \dot{q}_s^2\right) - L \\ &= 2T - (T - U) \\ &= T + U \end{aligned} \tag{1.14.12}$$

This means Hamiltonian is the total energy of the system.

1.14.3 Hamiltonian is Time independent Function

For this we will show only

$$\frac{dH}{dt} = 0$$

Differentiate (1.14.12) with respect to t

$$\frac{dH}{dt} = \frac{d}{dt}(T+U) = \frac{dE}{dt}$$

From law of conservation of energy we can use (**??**) in above expression, then

$$\frac{dH}{dt} = 0$$

Hence Hamiltonian is time independent function.

1.15 Hamiltons Equations of Motion

In Lagrangian formulation, the independent variables are the generalized coordinates q_s and the generalized velocities $\dot{q}_s$, whereas in Hamiltonian formulation, the independent variables are the generalized coordinates q_s and the generalized momenta p_s. *i.e.*

$$\begin{aligned} L &= L(q_s, \dot{q}_s) \\ H &= H(q_s, p_s) \end{aligned} \tag{1.15.1}$$

The Hamiltonian's equations of motion can be derived by taking total differential of (1.15.1) and (1.14.6). First take total differential of (1.15.1)

$$\begin{aligned} dH &= dH(q_s, p_s) \\ &= \sum \frac{\partial H}{\partial q_s} dq_s + \sum \frac{\partial H}{\partial p_s}\ dp_s \end{aligned} \tag{1.15.2}$$

Next taking total differential of (1.14.6)

$$\begin{aligned} dH &= \sum d(\dot{q}_s\ p_s) - \sum dL(q_s, \dot{q}_s) \\ &= \sum d\dot{q}_s\ p_s + \sum \dot{q}_s\ dp_s - \sum \left(\frac{\partial L}{\partial q_s} dq_s + \frac{\partial L}{\partial \dot{q}_s} d\dot{q}_s \right) \end{aligned}$$

Using (1.11.1)

$$\begin{aligned} dH &= \sum d\dot{q}_s\ p_s + \sum \dot{q}_s\ dp_s - \sum \frac{\partial L}{\partial q_s} dq_s - \sum p_s d\dot{q}_s \\ &= \sum \dot{q}_s\ dp_s - \sum \frac{\partial L}{\partial q_s} dq_s \end{aligned} \tag{1.15.3}$$

From (1.4.23), we can write

$$\frac{\partial L}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right), \qquad s = 1 \cdot \cdot \cdot N$$

Using (1.11.1)

$$\frac{\partial L}{\partial q_s} = \frac{d}{dt}(p_s) = \dot{p}_s, \qquad s = 1 \cdot \cdot \cdot N \tag{1.15.4}$$

Using (1.15.4), (1.15.3) becomes

$$dH = \sum \dot{q}_s \, dp_s - \sum \dot{p}_s dq_s \tag{1.15.5}$$

From (1.15.2) and (1.15.5), we can write

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \tag{1.15.6}$$

$$\dot{p}_s = -\frac{\partial H}{\partial q_s} \tag{1.15.7}$$

are Hamilton's equations of motion.

1.16 One Dimensional Hamilton's Equations of Motion

Consider a particle of mass m moves in one dimensional conservative system. At any time

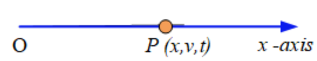

Figure 1.17: One dimensional motion

t it is at P having position x relative to origin as shown in Fig. 1.17. Then $\dot{x}$ be its velocity and $\ddot{x}$ be its acceleration at P. Its kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 \tag{1.16.1}$$

and potential energy is

$$U = U(x) \tag{1.16.2}$$

Using (1.16.1) and (1.16.2) in (1.4.22), the Euler Lagrange's function is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m\dot{x}^2 - U(x) \end{aligned}$$

The Hamilton's function is

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2}m\dot{x}^2 + U(x) \end{aligned} \tag{1.16.3}$$

Since Hamiltonian is function of coordinates and momentum, so the velocity coordinate will be replaced by momentum coordinate. Here x is the only generalized coordinate and x and $\dot{x}$ are independent variables. Its corresponding generalized momentum p_x can be calculated as

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} \\ &= m\dot{x} \end{aligned}$$

The velocity in momentum can be expressed as

$$\dot{x} = \frac{p_x}{m} \tag{1.16.4}$$

Using (1.16.4) in (1.16.10), the Hamiltonian is

$$H = \frac{1}{2}\frac{p_x^2}{m} + U(x) \tag{1.16.5}$$

Here x and p_x are independent variables. The Hamilton's equations of motion are

$$\dot{q}_s = \frac{\partial H}{\partial p_s}$$

and

$$\dot{p}_s = -\frac{\partial H}{\partial q_s}$$

Here $s = 1$ with $q_1 = x$. Hence the Hamilton's equations of motion are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} \\ \dot{p}_x &= -\frac{\partial H}{\partial x} \end{aligned}$$

the equation for x is obtained by differentiating (1.16.5) $w.r.t. p_x$

$$\dot{x} = \frac{p_x}{m} \tag{1.16.6}$$

and for p the equation is obtained by differentiating (1.16.5) $w.r.t.x$

$$\dot{p}_x = -\frac{\partial U}{\partial x} \tag{1.16.7}$$

(1.16.6) and (1.16.7) are one dimensional Hamilton's equations of motion.

Example 1.16.1. *Find Hamilton's equations of motion of free fall body.*

Solution In free fall motion, a body of mass m is dropped (at rest) from a height of h meters. Since it is one dimensional motion, the reference axis may be $z - axis$ only. At

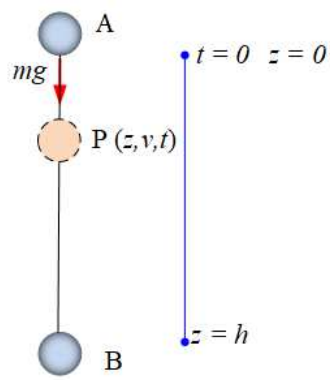

Figure 1.18: Free fall motion

time t the body is at P with position z relative to point A as shown in Fig. 1.18. Then $\dot{z}$ be its velocity and $\ddot{z}$ be its acceleration at P. Its kinetic energy is

$$T = \frac{1}{2}m\dot{z}^2 \tag{1.16.8}$$

Taking A as the reference point its potential energy is

$$U = -mgz \tag{1.16.9}$$

Using (1.16.8) and (1.16.9) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\dot{z}^2 + mgz$$

Here z is the only generalized coordinate and z and $\dot{z}$ are independent variables. The Hamiltonian is

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2}m\dot{z}^2 + U(z) \end{aligned} \tag{1.16.10}$$

Since Hamiltonian is function of coordinates and momentum, so the velocity coordinate will be replaced by momentum coordinate. The generalized momentum p_z is

$$\begin{aligned} p_z &= \frac{\partial L}{\partial \dot{z}} \\ &= m\dot{z} \end{aligned}$$

The velocity in momentum can be expressed as

$$\dot{z} = \frac{p_z}{m} \tag{1.16.11}$$

Then the Hamiltonian is

$$H = \frac{1}{2}\frac{p_z^2}{m} - mgz \tag{1.16.12}$$

Here z and p_z are independent variables. The Hamilton's equations of motion are

$$\begin{aligned} \dot{z} &= \frac{\partial H}{\partial p_z} \\ \dot{p}_z &= -\frac{\partial H}{\partial z} \end{aligned}$$

the equation for z is

$$\dot{z} = \frac{p_z}{m} \tag{1.16.13}$$

and for p is

$$\dot{p}_z = -(-mg) = mg \tag{1.16.14}$$

(1.16.13) and (1.16.14) are Hamilton's equations of motion for free fall motion.

1.17 Two Dimensional Hamilton's Equations of Motion

Consider a particle of mass m moves in two dimensional conservative system. At any time t it is at P having position $P(x, y)$ relative to origin as shown in Fig. 1.19. Set $q_1 = x$ and $q_2 = y$. Then its velocity at P is

$$v = \langle \dot{x}, \dot{y} \rangle$$

and square of its magnitude is

$$v^2 = \dot{x}^2 + \dot{y}^2$$

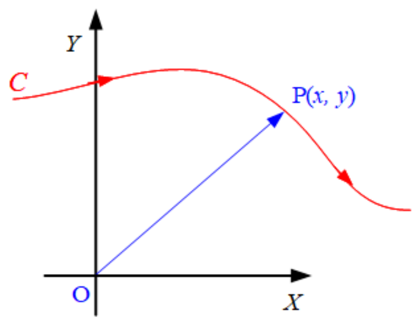

Figure 1.19: Two dimensional motion

Its kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \end{aligned} \tag{1.17.1}$$

and potential energy is

$$U = U(x,y) \tag{1.17.2}$$

Using (1.17.1) and (1.17.2) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - U(x,y) \tag{1.17.3}$$

Here x and y are the generalized coordinate. The Hamilton's function is

$$\begin{aligned} H &= T+U \\ &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) + U(x,y) \end{aligned} \tag{1.17.4}$$

Since Hamiltonian is function of coordinates and momentum, so the velocity coordinates will be replaced by momentum coordinate. Here x and y are the generalized coordinate. For x, the corresponding generalized momentum p_x can be calculated as

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} \\ &= m\dot{x} \end{aligned}$$

The velocity in momentum can be expressed as

$$\dot{x} = \frac{p_x}{m} \tag{1.17.5}$$

For y, the corresponding generalized momentum p_y can be calculated as

$$\begin{aligned} p_y &= \frac{\partial L}{\partial \dot{y}} \\ &= m\dot{y} \end{aligned}$$

The velocity in momentum can be expressed as

$$\dot{y} = \frac{p_y}{m} \tag{1.17.6}$$

Then the Hamiltonian is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + U(x,y) \tag{1.17.7}$$

The Hamilton's equations of motion for x are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} \\ &= \frac{p_x}{m} && (1.17.8) \\ \text{and } \dot{p}_x &= -\frac{\partial H}{\partial x} \\ &= -\frac{\partial U}{\partial x} && (1.17.9) \end{aligned}$$

The Hamilton's equations of motion for y are

$$\begin{aligned} \dot{y} &= \frac{\partial H}{\partial p_y} \\ &= \frac{p_y}{m} && (1.17.10) \\ \text{and } \dot{p}_y &= -\frac{\partial H}{\partial y} \\ &= -\frac{\partial U}{\partial y} && (1.17.11) \end{aligned}$$

Special Case In above if we consider OX axis as reference line, then y will be the height of the body (see Fig 1.20) and potential energy function is

$$U = mgy \tag{1.17.12}$$

The Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{1.17.13}$$

And the Hamiltonian is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy \tag{1.17.14}$$

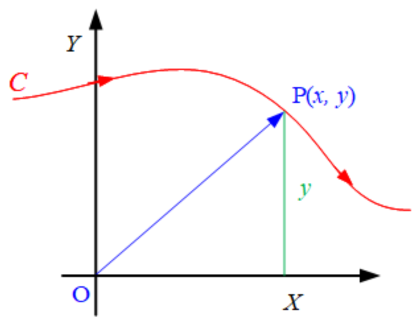

Figure 1.20: Two dimensional motion

The Hamilton's equations of motion are

$$\dot{q}_s = \frac{\partial H}{\partial p_s}$$

and

$$\dot{p}_s = -\frac{\partial H}{\partial q_s}$$

Here $s = 2$ with $q_1 = x$ and $q_2 = y$.
The Hamilton's equations of motion for x are

$$\dot{x} = \frac{p_x}{m} \tag{1.17.15}$$

and for p_x is

$$\dot{p}_x = -\frac{\partial U}{\partial x} = 0 \tag{1.17.16}$$

The Hamilton's equations of motion for y are

$$\dot{y} = \frac{p_y}{m} \tag{1.17.17}$$

and for p_y is

$$\dot{p}_y = -\frac{\partial U}{\partial y} = -mg \tag{1.17.18}$$

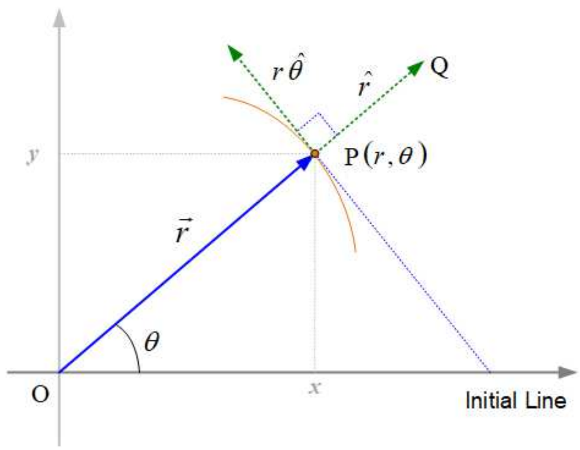

Figure 1.21: Planetary Motion

1.18 Hamilton's Equations of Motion in terms of Polar Coordinates

Consider a particle of mass m moves in polar coordinates. At any time t, it be at $P = P(r, \theta)$. Then its velocity in polar coordinate is

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

and

$$v^2 = (\dot{r})^2 + (r\dot{\theta})^2$$

Then its kinetic energy at P is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \end{aligned} \tag{1.18.1}$$

The potential energy can be written as

$$U = U(r, \theta) \tag{1.18.2}$$

Using (1.18.1) and (1.18.2) in (1.4.22), the Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r, \theta) \tag{1.18.3}$$

Here r and θ are the generalized coordinates. Then $r,\ \theta,\ \dot{r}$ and $\dot{\theta}$ are linearly independent variables. The Hamilton's function is

$$\begin{aligned} H &= \sum \dot{q}_s p_s - L(q_s, \dot{q}_s, t), \qquad s = 1\ \cdot\ \cdot\ \cdot N \\ &= p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r,\theta) \end{aligned}$$

Next

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

Here r is non-cyclic coordinate, $\dot{r}$ can be written as

$$\dot{r} = \frac{p_r}{m}$$

and

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

also $\dot{\theta}$ can be written as

$$\begin{aligned} r^2\dot{\theta} &= \frac{p_\theta}{m} \\ \dot{\theta} &= \frac{p_\theta}{r^2 m} \end{aligned}$$

Using all above results, Hamilton's function becomes

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r,\theta) \\ &= p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{r^2 m} - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 + \frac{1}{2}m\left(\frac{p_\theta}{mr}\right)^2 + U(r,\theta) \\ &= \frac{p_r^2}{m} + \frac{p_\theta^2}{r^2 m} - \frac{1}{2}\left(\frac{p_r^2}{m}\right) - \frac{1}{2}\left(\frac{p_\theta^2}{mr^2}\right) + U(r,\theta) \\ &= \frac{1}{2}\left(\frac{p_r^2}{m}\right) + \frac{1}{2}\left(\frac{p_\theta^2}{r^2 m}\right) + U(r,\theta) \qquad (1.18.4) \end{aligned}$$

(1.18.4) is the Hamiltonian.
The Hamilton's equations of motion are

$$\dot{q}_s = \frac{\partial H}{\partial p_s}$$

and

$$\dot{p}_s = -\frac{\partial H}{\partial q_s}$$

Here $s = 2$ with $q_1 = r$ and $q_2 = \theta$.
For generalized coordinate r, the Hamilton's equations of motion are

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} \\ &= \frac{p_r}{m} \end{aligned} \tag{1.18.5}$$

and

$$\begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} \\ &= \left(\frac{p_\theta^2}{mr^3}\right) - \frac{\partial U(r)}{\partial r} \end{aligned} \tag{1.18.6}$$

For generalized coordinate θ, the Hamilton's equations of motion are

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} \\ &= \frac{p_\theta}{r^2 m} \end{aligned} \tag{1.18.7}$$

and

$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \\ &= -\frac{\partial U}{\partial \theta} \end{aligned} \tag{1.18.8}$$

Special Case If potential energy is a function of r coordinate only, can be considered as motion under central force. The potential energy function is

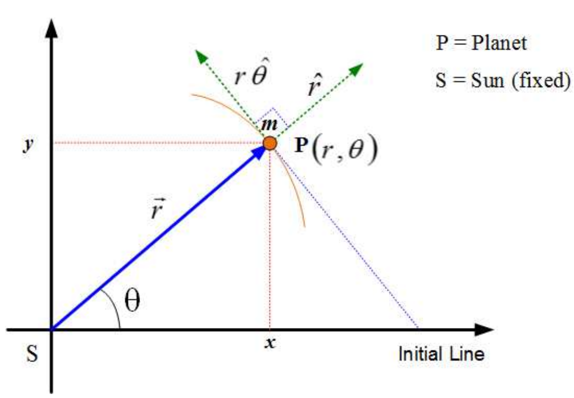

Figure 1.22: Planetary Motion

$$U = U(r)$$

The Lagrangian function is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \\ &= L\left(r,\ \dot{r},\ \dot{\theta}\right) \end{aligned}$$

Here θ is the cyclic coordinates. Using (1.14.6), the Hamilton's function is

$$\begin{aligned} H &= \sum \dot{q}_s p_s - L(q_s, \dot{q}_s, t), \qquad s = 1 \ \cdot\ \cdot\ \cdot N \\ &= p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) \end{aligned}$$

Next

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

Here r is non-cyclic coordinate, $\dot{r}$ can be written as

$$\dot{r} = \frac{p_r}{m}$$

and

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

also $\dot{\theta}$ can be written as

$$\begin{aligned} r^2\dot{\theta} &= \frac{p_\theta}{m} \\ \dot{\theta} &= \frac{p_\theta}{r^2 m} \end{aligned}$$

Using all above results, Hamilton's equation of motion becomes

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) \\ &= p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{r^2 m} - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 + \frac{1}{2}m\left(\frac{p_\theta}{mr}\right)^2 + U(r) \\ &= \frac{p_r^2}{m} + \frac{p_\theta^2}{r^2 m} - \frac{1}{2}\left(\frac{p_r^2}{m}\right) - \frac{1}{2}\left(\frac{p_\theta^2}{mr^2}\right) + U(r) \\ &= \frac{1}{2}\left(\frac{p_r^2}{m}\right) + \frac{1}{2}\left(\frac{p_\theta^2}{r^2 m}\right) + U(r) \end{aligned} \tag{1.18.9}$$

(1.18.9) is the Hamiltonian.
For generalized coordinate r, the Hamilton's equations of motion are

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} \\ &= \frac{p_r}{m} \end{aligned} \tag{1.18.10}$$

and

$$\begin{aligned}\dot{p}_r &= -\frac{\partial H}{\partial r}\\ &= \left(\frac{p_\theta^2}{mr^3}\right) - \frac{\partial U(r)}{\partial r}\end{aligned} \tag{1.18.11}$$

Since θ is cyclic variable, so it is dropped off by itself.

Example 1.18.1. *Hamiltonian and the Hamilton's equations of motion for a simple pendulum.*

Consider OXY a cartesian coordinate system. Let a particle of m is attached with a massless string of length l, with other end fixed at O, forming a simple pendulum, as shown in Fig. 1.23 At any time t, the particle be at $P(r, \theta)$. From (1.6.10) the lagrangian is

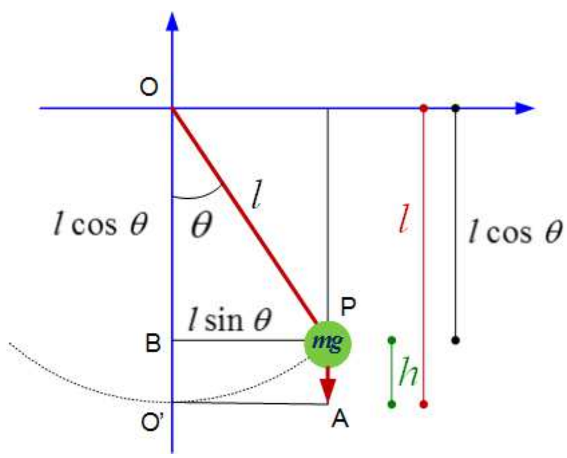

Figure 1.23: Simple Pendulum

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1 - \cos\theta)$$

Here θ is the only generalized coordinate. From (1.11.1)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

Then

$$\dot{\theta} = \frac{p_\theta}{ml^2} \tag{1.18.12}$$

Using (1.14.6), the Hamiltonian is

$$\begin{aligned} H &= \dot{q}_s p_s - L(q_s, \dot{q}_s, t), \qquad s = 1 \ \cdot \ \cdot \ \cdot N \\ &= \dot{\theta} p_\theta - \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\,(1 - \cos\theta) \end{aligned} \tag{1.18.13}$$

Using (1.18.12), (1.18.13)

$$\begin{aligned} H &= \frac{p_\theta}{ml^2} p_\theta - \frac{1}{2} m l^2 \left(\frac{p_\theta}{ml^2}\right)^2 + mgl\,(1 - \cos\theta) \\ &= \frac{1}{2} \frac{p_\theta^2}{ml^2} + mgl\,(1 - \cos\theta) \end{aligned} \tag{1.18.14}$$

Here θ is the only generalized coordinate, its Hamilton's equation of motion is

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} \\ &= \frac{p_\theta}{ml^2} \end{aligned}$$

or

$$p_\theta = ml^2 \dot{\theta} \tag{1.18.15}$$

time derivative of (1.18.15)

$$\dot{p}_\theta = ml^2 \ddot{\theta} \tag{1.18.16}$$

$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \\ &= -mgl \sin\theta \end{aligned} \tag{1.18.17}$$

Using (1.18.16), (1.18.17) becomes

$$\begin{aligned} ml^2 \ddot{\theta} &= -mgl \sin\theta \\ ml^2 \ddot{\theta} + mgl \sin\theta &= 0 \\ \ddot{\theta} + \frac{g}{l} \sin\theta &= 0 \\ \ddot{\theta} + \omega^2 \sin\theta &= 0 \end{aligned} \tag{1.18.18}$$

$$\text{with } \omega = \sqrt{\frac{g}{l}} \qquad \text{is the frequency of oscillation.}$$

Same as (1.6.15) (equation of motion of a simple pendulum.) By Hamilton's equation of motion, (1.18.18) is the equation of motion of a simple pendulum.

1.19 Hamilton's Equations of Motion in terms of $3 - space$ Cartesian Coordinates

Consider a particle of mass m moves in $3 - space$ cartesian coordinates system. At any time t, it be at $P = P(x, y, z)$, see Fig. 1.24 . Its velocity at P is

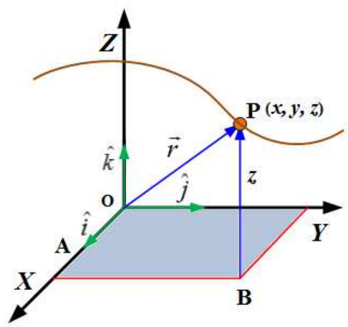

Figure 1.24: Cylindrical motion

$$\vec{v} = \left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right)$$

then

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \end{aligned} \tag{1.19.1}$$

and the potential energy is

$$U = U(x, y, z) \tag{1.19.2}$$

Using (1.19.1) and (1.19.2) in (1.4.22), the Euler Lagrange function is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(x, y, z) \\ L &= L(x, y, z, \dot{x}, \dot{y}, \dot{z}) \end{aligned}$$

The Hamilton's function is

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + U(x, y, z) \end{aligned} \tag{1.19.3}$$

Since Hamiltonian is function of coordinates and momentum, so the velocity coordinates will be replaced by momentum coordinate. Here x, y and z are the generalized coordinate. For x, the corresponding generalized momentum p_x can be calculated as

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} \\ &= m\dot{x} \end{aligned}$$

The velocity component $\dot{x}$ in momentum can be expressed as

$$\dot{x} = \frac{p_x}{m} \tag{1.19.4}$$

For y, the corresponding generalized momentum p_y can be calculated as

$$\begin{aligned} p_y &= \frac{\partial L}{\partial \dot{y}} \\ &= m\dot{y} \end{aligned}$$

The velocity component $\dot{y}$ in momentum can be expressed as

$$\dot{y} = \frac{p_y}{m} \tag{1.19.5}$$

For z, the corresponding generalized momentum p_z can be calculated as

$$\begin{aligned} p_z &= \frac{\partial L}{\partial \dot{z}} \\ &= m\dot{z} \end{aligned}$$

The velocity component $\dot{z}$ in momentum can be expressed as

$$\dot{z} = \frac{p_z}{m} \tag{1.19.6}$$

Then the Hamiltonian is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + U(x, y, z) \tag{1.19.7}$$

The Hamilton's equation of motion for x is

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} \\ &= \frac{p_x}{m} \end{aligned} \tag{1.19.8}$$

and for p_x is

$$\begin{aligned} \dot{p}_x &= -\frac{\partial H}{\partial x} \\ &= -\frac{\partial U}{\partial x} \end{aligned} \tag{1.19.9}$$

The Hamilton's equation of motion for y is

$$\begin{aligned} \dot{y} &= \frac{\partial H}{\partial p_y} \\ &= \frac{p_y}{m} \end{aligned} \tag{1.19.10}$$

and for p_y is

$$\begin{aligned} \dot{p}_y &= -\frac{\partial H}{\partial y} \\ &= -\frac{\partial U}{\partial y} \end{aligned} \tag{1.19.11}$$

The Hamilton's equation of motion for z is

$$\begin{aligned} \dot{z} &= \frac{\partial H}{\partial p_z} \\ &= \frac{p_z}{m} \end{aligned} \tag{1.19.12}$$

and for p_z is

$$\begin{aligned} \dot{p}_z &= -\frac{\partial H}{\partial z} \\ &= -\frac{\partial U}{\partial z} \end{aligned} \tag{1.19.13}$$

Special Case In above if XOY plane be the zero level for potential energy of the particle, then clearly P is at height z above the XOY plane. Then potential energy function is

$$U = mgz$$

and the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \\ L &= L(z, \dot{x}, \dot{y}, \dot{z}) \end{aligned}$$

Here x, y and z are the generalized coordinate. The Hamilton's function is

$$\begin{aligned} H &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + mgz \\ H &= H(z, \dot{x}, \dot{y}, \dot{z}) \end{aligned}$$

Replacing coordinates with their corresponding momenta, the Hamiltonian is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz \tag{1.19.14}$$

Here x and y are cyclic coordinates, so are dropped and are left with z coordinate only. The Hamilton's equationsof motion for z is

$$\begin{aligned} \dot{z} &= \frac{\partial H}{\partial p_z} \\ &= \frac{p_z}{m} \end{aligned} \tag{1.19.15}$$

and for p_z is

$$\begin{aligned} \dot{p}_z &= -\frac{\partial H}{\partial z} \\ &= -\frac{\partial U}{\partial z} = -mg \end{aligned} \tag{1.19.16}$$

1.20 Hamilton's Equations of Motion in terms of Cylindrical Polar Coordinates

Consider a particle of mass m moves in cylindrical polar coordinates. At any time t, it be at $P = P(r, \theta, z)$. Then its velocity at P is

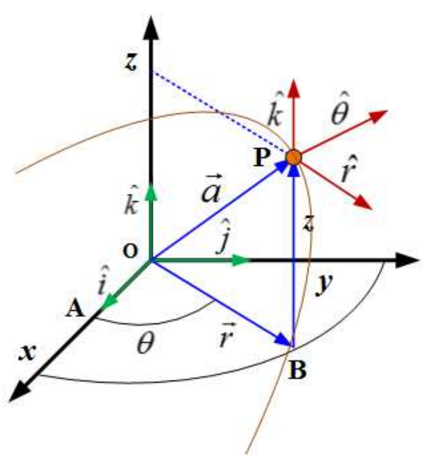

Figure 1.25: Cylindrical motion

$$\vec{v} = \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}\right)$$

then

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2$$

and the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \end{aligned}$$

The potential energy can be written as

$$U = U(r, \theta, z)$$

The Euler Lagrange function is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - U(r, \theta, z)$$

The Hamilton's function is

$$\begin{aligned} H &= \sum \dot{q}_s p_s - L(q_s, \dot{q}_s, t), \qquad s = 1 \cdot \cdot \cdot N \\ &= p_r\dot{r} + p_\theta\dot{\theta} + +p_z\dot{z} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + U(r, \theta, z) \end{aligned} \tag{1.20.1}$$

Here r, θ and z are the generalized coordinates. Then r, θ, z, $\dot{r}$, $\dot{\theta}$, and $\dot{z}$ are linearly independent variables, hence are non-cyclic coordinates. The velocities components will be replaced by generalized momentum corresponding to generalized coordinate. First for r is

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

then $\dot{r}$ can be written as

$$\dot{r} = \frac{p_r}{m}$$

and

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

also $\dot{\theta}$ can be written as

$$\begin{aligned} r^2\dot{\theta} &= \frac{p_\theta}{m} \\ \dot{\theta} &= \frac{p_\theta}{r^2 m} \end{aligned}$$

For z, the corresponding generalized momentum p_z can be calculated as

$$\begin{aligned} p_z &= \frac{\partial L}{\partial \dot{z}} \\ &= m\dot{z} \end{aligned}$$

The velocity component $\dot{z}$ in momentum can be expressed as

$$\dot{z} = \frac{p_z}{m}$$

Using all above results, Hamilton's equation of motion becomes

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} + p_z\dot{z} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + U(r,\theta,z) \\ &= p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{r^2m} + p_z\frac{p_z}{m} - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 - \frac{1}{2}m\left(\frac{p_\theta}{mr}\right)^2 - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 - \frac{1}{2}m\left(\frac{p_z}{m}\right)^2 + U(r,\theta,z) \\ &= \frac{p_r^2}{m} + \frac{p_\theta^2}{r^2m} + \frac{p_z^2}{m} - \frac{1}{2}\left(\frac{p_r^2}{m}\right) - \frac{1}{2}\left(\frac{p_\theta^2}{mr^2}\right) - \frac{1}{2}\left(\frac{p_z^2}{m}\right) + U(r,\theta,z) \\ &= \frac{1}{2}\left(\frac{p_r^2}{m}\right) + \frac{1}{2}\left(\frac{p_\theta^2}{r^2m}\right) + \frac{1}{2}\left(\frac{p_z^2}{m}\right) + U(r,\theta,z) \end{aligned} \tag{1.20.2}$$

(1.20.10) is the Hamiltonian.
The Hamilton's equations of motion are

$$\begin{aligned} \dot{q}_s &= \frac{\partial H}{\partial p_s} \\ \dot{p}_s &= -\frac{\partial H}{\partial q_s} \end{aligned}$$

Here $s = 1,2,3$, set $q_1 = r, q_2 = \theta$ and $q_3 = z$ For generalized coordinate r, the Hamilton's equations of motion are

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} \\ &= \frac{p_r}{m} \end{aligned} \tag{1.20.3}$$

and

$$\begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} \\ &= \left(\frac{p_\theta^2}{mr^3}\right) - \frac{\partial U(r)}{\partial r} \end{aligned} \tag{1.20.4}$$

For generalized coordinate θ, the Hamilton's equations of motion are

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} \\ &= \frac{p_\theta}{r^2m} \end{aligned} \tag{1.20.5}$$

and

$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \\ &= -\frac{\partial U}{\partial \theta} \end{aligned} \tag{1.20.6}$$

For generalized coordinate z, the Hamilton's equations of motion are

$$\begin{aligned} \dot{z} &= \frac{\partial H}{\partial p_z} \\ &= \frac{p_z}{m} \end{aligned} \tag{1.20.7}$$

and

$$\begin{aligned} \dot{p}_z &= -\frac{\partial H}{\partial z} \\ &= -\frac{\partial U}{\partial z} \end{aligned} \tag{1.20.8}$$

Special Case In above if XOY plane be the zeroth level, then the potential energy of the particle is

$$U = mgz \tag{1.20.9}$$

Using (1.4.22), the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \\ L &= L(r, z, \dot{r}, \dot{\theta}, \dot{z}) \end{aligned}$$

and the Hamiltonian is Using all above results, Hamilton's equation of motion becomes

$$H = \frac{1}{2}\left(\frac{p_r^2}{m}\right) + \frac{1}{2}\left(\frac{p_\theta^2}{r^2 m}\right) + \frac{1}{2}\left(\frac{p_z^2}{m}\right) + mgz \tag{1.20.10}$$

(1.20.10) is the Hamiltonian.
The Hamilton's equations of motion are For generalized coordinate r, the Hamilton's equations of motion are

$$\dot{r} = \frac{p_r}{m}$$

and

$$\dot{p}_r = \left(\frac{p_\theta^2}{mr^3}\right)$$

For generalized coordinate θ, the Hamilton's equations of motion are

$$\dot{\theta} = \frac{p_\theta}{r^2 m}$$

and

$$\begin{aligned}\dot{p}_\theta &= -\frac{\partial H}{\partial \theta}\\ &= -\frac{\partial U}{\partial \theta}\end{aligned} \tag{1.20.11}$$

For generalized coordinate z, the Hamilton's equations of motion are

$$\begin{aligned}\dot{z} &= \frac{\partial H}{\partial p_z}\\ &= \frac{p_z}{m}\end{aligned} \tag{1.20.12}$$

and

$$\begin{aligned}\dot{p}_z &= -\frac{\partial H}{\partial z}\\ &= -\frac{\partial U}{\partial z}\end{aligned} \tag{1.20.13}$$

1.21 Hamilton's Equations of Motion in terms of Spherical Polar Coordinates

Consider a particle of mass m moves in spherical polar coordinates. At any time t, it be at $P = P(r, \theta, \phi)$. Then its velocity at P is

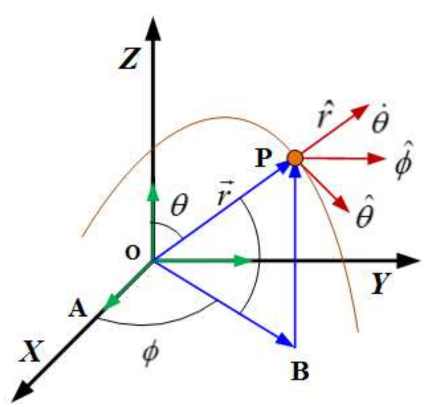

Figure 1.26: Cylindrical motion

$$\vec{v} = \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right)$$

then

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 + \left(r\dot{\phi}\sin\theta\right)^2$$

and the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \end{aligned}$$

The potential energy can be written as

$$U = U(r,\theta,\phi)$$

The Euler Lagrange function is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r,\theta,\phi) \\ L &= L(r,\theta,\phi,\dot{r},\dot{\theta},\dot{\phi}) \end{aligned}$$

The Hamilton's function is

$$\begin{aligned} H &= \sum \dot{q}_s p_s - L(q_s,\dot{q}_s,t), \qquad s = 1 \ \cdot \ \cdot \ \cdot N \\ &= p_r\dot{r} + p_\theta\dot{\theta} + +p_\phi\dot{\phi} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + U(r,\theta,\phi) \qquad (1.21.1) \end{aligned}$$

Here r, θ and ϕ are the generalized coordinates. Then r, θ, ϕ, $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ are linearly independent variables, hence are non-cyclic coordinates. The velocities components will be replaced by generalized momentum corresponding to generalized coordinate. First for r is

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

then $\dot{r}$ can be written as

$$\dot{r} = \frac{p_r}{m}$$

For generalized coordinate θ, the corresponding generalized momentum p_θ can be calculated as

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

also $\dot{\theta}$ can be written as

$$\begin{aligned} r^2\dot{\theta} &= \frac{p_\theta}{m} \\ \dot{\theta} &= \frac{p_\theta}{r^2 m} \end{aligned}$$

For generalized coordinate ϕ, the corresponding generalized momentum p_ϕ can be calculated as

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial \dot{\phi}} \\ &= mr^2 \sin^2\theta \dot{\phi} \end{aligned}$$

The velocity component $\dot{\phi}$ in momentum can be expressed as

$$\dot{\phi} = \frac{p_\phi}{mr^2 \sin^2\theta} \tag{1.21.2}$$

Using all above results, Hamilton's function becomes

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + U(r,\theta,\phi) \\ &= p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{r^2m} + p_\phi\frac{p_\phi}{mr^2\sin^2\theta} - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 - \frac{1}{2}m\left(\frac{p_\theta}{mr}\right)^2 - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 \\ &- \frac{1}{2}mr^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2 + U(r,\theta,\phi) \\ &= \frac{p_r^2}{m} + \frac{p_\theta^2}{r^2m} + \frac{p_\phi^2}{m} - \frac{1}{2}\left(\frac{p_r^2}{m}\right) - \frac{1}{2}\left(\frac{p_\theta^2}{mr^2}\right) - \frac{1}{2}\left(\frac{p_\phi^2}{mr^2\sin^2\theta}\right) + U(r,\theta,\phi) \\ &= \frac{1}{2}\left(\frac{p_r^2}{m}\right) + \frac{1}{2}\left(\frac{p_\theta^2}{r^2m}\right) + \frac{1}{2}\left(\frac{p_\phi^2}{mr^2\sin^2\theta}\right) + U(r,\theta,\phi) \end{aligned} \tag{1.21.3}$$

(1.21.3) is the Hamiltonian.
The Hamilton's equations of motion are

$$\begin{aligned} \dot{q}_s &= \frac{\partial H}{\partial p_s} \\ \dot{p}_s &= -\frac{\partial H}{\partial q_s} \end{aligned}$$

Here $s = 1, 2, 3$, set $q_1 = r, q_2 = \theta$ and $q_3 = \phi$ For generalized coordinate r, the Hamilton's equations of motion are

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} \\ &= \frac{p_r}{m} \end{aligned} \tag{1.21.4}$$

and

$$\begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} \\ &= \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3\sin^2\theta} - \frac{\partial U}{\partial r} \end{aligned} \tag{1.21.5}$$

For generalized coordinate θ, the Hamilton's equations of motion are

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} \\ &= \frac{p_\theta}{r^2 m} \end{aligned} \tag{1.21.6}$$

and

$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \\ &= \frac{p_\phi^2 \csc^2\theta \cot\theta}{mr^2} - \frac{\partial U}{\partial \theta} \end{aligned} \tag{1.21.7}$$

For generalized coordinate ϕ, the Hamilton's equations of motion are

$$\begin{aligned} \dot{\phi} &= \frac{\partial H}{\partial p_\phi} \\ &= \frac{p_\phi}{mr^2 \sin^2\theta} \end{aligned} \tag{1.21.8}$$

and

$$\begin{aligned} \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} \\ &= -\frac{\partial U}{\partial \phi} \end{aligned} \tag{1.21.9}$$

1.22 Hamilton's Principle

In many physical systems we are interested to minimize certain physical quantities. Hamilton formalized this minimization under the principle known as Hamilton's Principle which states:
" Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies. "
If T is kinetic energy and U is potential energy, this principle in terms of the calculus of variations is

$$\delta I = \delta \int_{t_1}^{t_2} (T - U)\, dt = \delta \int_{t_1}^{t_2} L dt \tag{1.22.1}$$

The quantity $T - U$ is the Lagrangian L.

1.22.1 Lagrange's Equation of motion from Hamilton's Principle

1.22.2 Lagrange's Equation of motion from Hamilton's Principle

Let us consider a system of N particles whose configuration at any time t is specified by N Lagrangian coordinates $(q_1, q_2, ..., q_N)$. Let the particle moves along curve C with A and B be terminal points with time t_1 and t_2 respectively. Let it be at $P(q_s, \dot{q}_s)$ as shown in Fig. 1.28. We can write the integral along C as

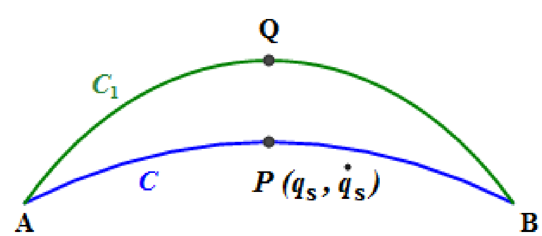

Figure 1.27: Two neighbouring paths with same end points

$$I_1 = \int_{t_1}^{t_2} L(q_s, \dot{q}_s)\, dt \tag{1.22.2}$$

Let C_1 be a neighbouring curve with same end points A and B. Let position of the particle at time $t+\delta t$ is $Q\left(q_s+\delta q_s,\dot{q}_s+\delta\dot{q}_s\right)$ as shown in Fig. 1.28. We can write the integral along C_1 as

$$I_2 = \int_{t_1}^{t_2} L\left(q_s+\delta q_s,\dot{q}_s+\delta\dot{q}_s\right)dt \qquad (1.22.3)$$

The change in integrals is

$$\delta I = \int_{t_1}^{t_2} \left[L\left(q_s+\delta q_s,\dot{q}_s+\delta\dot{q}_s\right)-L\left(q_s,\dot{q}_s\right)\right]dt \qquad (1.22.4)$$

This integral can be simplified by using "Increment theorem for functions of several variables". Then the result is

$$\begin{aligned}\delta I &= \int_{t_1}^{t_2} \left[L\left(q_s,\dot{q}_s\right)+\left(\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial \dot{q}_s}\delta\dot{q}_s\right)-L\left(q_s,\dot{q}_s\right)\right]dt\\ &= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial \dot{q}_s}\delta\dot{q}_s\right)dt \qquad (1.22.5)\end{aligned}$$

Integrating second term by parts

$$\delta I = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_s}\delta q_s\right)dt+\left|\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right)dt$$

Since both curves have same end points, then the variation in coordinates at time t_1 and t_2 is zero. This means

$$\delta q_s\left(t_1\right) = 0=\delta q_s\left(t_2\right)$$

Then we are left with

$$\delta I = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_s}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right)\delta q_s dt \qquad (1.22.6)$$

By Hamilton's principle, the variation of the action integral for fixed time t_1 and t_2 must be zero.

$$\delta I = \delta\int_{t_1}^{t_2} L dt = 0$$

Using (1.22.12), we can write

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \right) \right) \delta q_s dt = 0 \tag{1.22.7}$$

This could be satisfied only if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \right) - \frac{\partial L}{\partial q_s} = 0 \quad s = 1, 2, ...N$$

is the Lagrange equations of motion.

Let us consider a system of N particles whose configuration at any time t is specified by N Lagrangian coordinates $(q_1, q_2, ..., q_N)$. Let the particle moves along curve C with A and B be terminal points with time t_1 and t_2 respectively. Let it be at $P(q_s, \dot{q}_s)$ as shown in Fig. 1.28. We can write the integral along C as

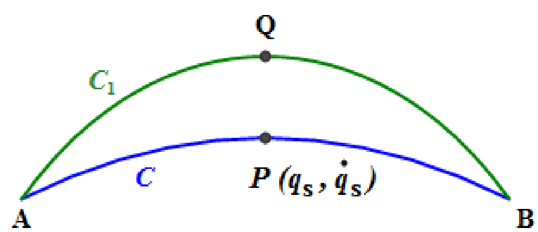

Figure 1.28: Two neighbouring paths with same end points

$$I_1 = \int_{t_1}^{t_2} L(q_s, \dot{q}_s)\, dt \tag{1.22.8}$$

Let C_1 be a neighbouring curve with same end points A and B. Let position of the particle at time $t + \delta t$ is $Q(q_s + \delta q_s, \dot{q}_s + \delta \dot{q}_s)$ as shown in Fig. 1.28. We can write the integral along C_1 as

$$I_2 = \int_{t_1}^{t_2} L(q_s + \delta q_s, \dot{q}_s + \delta \dot{q}_s)\, dt \tag{1.22.9}$$

The change in integrals is

$$\delta I = \int_{t_1}^{t_2} \left[L(q_s + \delta q_s, \dot{q}_s + \delta \dot{q}_s) - L(q_s, \dot{q}_s) \right] dt \tag{1.22.10}$$

This integral can be simplified by using "Increment theorem for functions of several variables". Then the result is

$$\begin{aligned}
\delta I &= \int_{t_1}^{t_2} \left[L(q_s, \dot{q}_s) + \left(\frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s \right) - L(q_s, \dot{q}_s) \right] dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s \right) dt \qquad (1.22.11)
\end{aligned}$$

Integrating second term by parts

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s}\delta q_s \right) dt + \left| \frac{\partial L}{\partial \dot{q}_s}\delta q_s \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\delta q_s \right) dt$$

Since both curves have same end points, then the variation in coordinates at time t_1 and t_2 is zero. This means

$$\delta q_s(t_1) = 0 = \delta q_s(t_2)$$

Then we are left with

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \right) \right) \delta q_s dt \qquad (1.22.12)$$

Using Lagrange's equation of motion we can write

$$\delta I = 0$$

Hence the integral I has stationary value along the actual curve C as comparing with the neighbouring trajectory.

1.22.3 Lagrange's Equation of motion from Hamilton's Principle

By Hamilton's principle, the variation of the action integral for fixed time t_1 and t_2 must be zero.

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0$$

Using (1.22.12), we can write

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \right) \right) \delta q_s dt = 0 \qquad (1.22.13)$$

This could be satisfied only if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} \quad = \quad 0 \quad s = 1, 2, ...N$$

is the Lagrange equations of motion.

1.22.4 Hamilton's Equation of motion from Hamilton's Principle

From (1.14.6) Hamilton's function is

$$\begin{aligned} H &= \sum \dot{q}_s p_s - L(q_s, \dot{q}_s, t), \qquad s = 1 \cdot \cdot \cdot N \\ \text{or} \quad L &= \sum \dot{q}_s p_s - H \end{aligned} \tag{1.22.14}$$

Using (1.22.14) in (1.22.1), change in integral is

$$\delta I = \delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} \left(\sum \dot{q}_s p_s - H \right) dt$$

By Hamilton's principle, the variation of the action integral for fixed time t_1 and t_2 must be zero.

$$\begin{aligned} \delta I &= 0 \\ \delta \int_{t_1}^{t_2} \left(\sum \dot{q}_s p_s - H \right) dt &= 0 \\ \int_{t_1}^{t_2} \left(\sum \dot{q}_s \delta p_s + \sum p_s \delta \dot{q}_s - \delta H \right) dt &= 0 \end{aligned} \tag{1.22.15}$$

Since

$$\delta H = \sum \frac{\partial H}{\partial q_s} \delta q_s + \sum \frac{\partial H}{\partial p_s} \, \delta p_s \tag{1.22.16}$$

Using (1.22.16) in (1.22.15), we have

$$\int_{t_1}^{t_2} \left(\sum p_s \delta \dot{q}_s + \sum \dot{q}_s \delta p_s - \left(\sum \frac{\partial H}{\partial q_s} \delta q_s + \sum \frac{\partial H}{\partial p_s} \, \delta p_s \right) \right) dt = 0 \tag{1.22.17}$$

Integrate first term by parts

$$\sum \left| p_s \delta q_s \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum \dot{p}_s \delta q_s dt + \int_{t_1}^{t_2} \left(\sum \dot{q}_s \delta p_s - \sum \frac{\partial H}{\partial q_s} \delta q_s - \sum \frac{\partial H}{\partial p_s} \, \delta p_s \right) dt = 0$$

Since

$$\delta q_s \left(t_1 \right) = 0 = \delta q_s \left(t_2 \right)$$

Then we are left with

$$\sum \int_{t_1}^{t_2} \left(\left(\dot{q}_s - \frac{\partial H}{\partial p_s} \right) \delta p_s - \left(\dot{p}_s + \frac{\partial H}{\partial q_s} \right) \delta q_s \right) dt \quad = \quad 0$$

Since all q_s and p_s are linearly independent and so does δq_s and δp_s, which is possible only if

$$\begin{aligned} \dot{q}_s - \frac{\partial H}{\partial p_s} &= 0 \\ \text{and} \quad \dot{p}_s + \frac{\partial H}{\partial q_s} &= 0 \end{aligned}$$

which gives us

$$\begin{aligned} \dot{q}_s &= \frac{\partial H}{\partial p_s} \\ \text{and} \quad \dot{p}_s &= -\frac{\partial H}{\partial q_s} \end{aligned}$$

the Hamilton's equations of motion.

1.23 The method of Lagrange multiplier

The method of Lagrange multiplier is used in classical mechanics in handling situations where the number of dynamical variables happens to be more than the number of degrees of freedom.

In the presence of non-holonomic constraints, the generalised coordinates are not independent as their number is greater than the number of degrees of freedom and consequently (1.4.23) is not valid. In such situations (and also in case care is not taken to reduce the number of generalised coordinates using the holonomic constraints), the method of undetermined multiplier is useful. (The method has limitations, for instance, it cannot be used in cases where the constraints are stated as inequalities)

Suppose there are m number of non-holonomic constraints involving the generalised coordinates in differential form

$$\sum_{s=1}^{n} a_{r,s} dq_s + b_{r,s} dt \quad = \quad 0 \tag{1.23.1}$$

$a_{r,s}$ and $b_{r,s}$ may depend on s and t. Here r is an index which runs from 1 to m and (1.23.1) is actually m equations, one for each value of r. We can get correct equations motion if the varied paths are virtual displacements from actual motion in which case the constraint is $\sum_{s=1}^{n} a_{r,s} dq_s$ because vital displacements take place over constant time. We can then rewrite the principle of least action as

$$\sum_{s=1}^{n} s \int dt \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{m} \lambda_r a_{r,s} \right] \delta q_s \;=\; 0$$

Note that the additional term $\sum_{s=1}^{n} \sum_{r=1}^{m} \lambda_r a_{r,s} \delta q_s$ is actually zero and hence we can put it inside the integral. δq_s are not independent and satisfy (1.23.1). Since we have m undetermined multipliers λ_r, we can choose them such that the first m terms, *i.e.*$s = 1$ to m is each zero. Suppose we choose λ_s such that for $s = 1, 2, \; . \; . \; .m$, the equation to be satisfied is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{m} \lambda_r a_{r,s} \;=\; 0 \tag{1.23.2}$$

Note that the last term in the above equation is no longer zero as the sum over s is missing and we have simply redistributed the term which was zero in various ways. With the λ_r determined by (1.23.2), we are left with

$$\sum_{s=m+1}^{n} \int dt \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{m} \lambda_r a_{r,s} \right] \delta q_s \;=\; 0$$

However, now our q_s are independent and we have, as a consequence, for $s = m + 1, \; . \; . \; .m$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{m} \lambda_r a_{r,s} \;=\; 0 \tag{1.23.3}$$

(1.23.2) and (1.23.3) allows us to write a single equation for $s = 1, 2, \; . \; . \; .m$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{m} \lambda_r a_{r,s} \;=\; 0 \tag{1.23.4}$$

(1.23.2) determines the m values of λ and (1.23.3) gives us n equations of motion. Define

$$\sum_{r=1}^{m} \lambda_r a_{r,s} \;=\; -Q_s \tag{1.23.5}$$

as generalised force corresponding to the constraint conditions. Equation (1.23.3) can now be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} \;=\; Q_s \tag{1.23.6}$$

Exercises

1. Are (q_1, q_2, q_3) are generalized coordinates defined as

$$\begin{aligned} x &= q_1 + q_2 + q_3 \\ y &= 2q_1 + 3q_2 - q_3 \\ z &= 4q_1 - q_2 + 4q_3 \end{aligned}$$

2. Find Routh's equation of motion for spherical pendulum.

3. Find Hamilton's equation of motion in example 1.7.1.

4. Find Hamilton's equation of motion for the following Hamilton's function

$$H = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2$$

   Also show the following relations

   (a) q_1q_2 is constant.

   (b) $\ln q_1 = t + C$, where C is some constant.

   (c) $\frac{p_2 - bq_2}{q_1}$ is constant.

5. Find Hamilton's equations of motion of a particle moving on a Sphere of radius a. (Use Lagrangian from article 1.9)

Chapter 2

Exact or Canonical Transformation

2.1 Exact or Canonical Transformation

In Hamiltonian mechanics, a canonical transformation is a change of canonical coordinates $(q, p, t) \to (Q, P, t)$ that preserves the form of Hamilton's equations (that is, the new Hamilton's equations resulting from the transformed Hamiltonian may be simply obtained by substituting the new coordinates for the old coordinates), although it might not preserve the Hamiltonian itself. This is sometimes known as form invariance.

2.1.1 Canonical Conjugate Variables

In mathematics conjugate means the change of sign in the middle of two terms like this

$$\begin{array}{ccc} x & + & 2 \\ x & - & 2 \end{array}$$

Physical conjugation of variables was invented by Hamilton in 1833 to reduce the second degree $n-$dimensional partial differential equations of Lagrange into the first degree $2n-$dimensional partial differential equations in phase space. Clearly, the implementation of phase space is only effective if there are pairs of conjugated variables. The traditional pair of conjugated variables was the position vector and the linear momentum vector of analytical mechanics. The two canonical forms of Hamiltons equations are written as the time derivative of the $s-th$ position vector set equal to the partial derivative of the Hamiltonian energy function with respect to the $s-th$ momentum vector and the time derivative of the $s-th$ linear momentum vector set equal to the negative of the partial derivative of the Hamiltonian energy function with respect to the $s-th$ position vector, the dummy index s takes on value from 1 to $2n$, where for infinitely dimensional spaces, $n = 2n$ is equal to infinity. In phase space, none of the two conjugate variables takes the value of zero. Since these possible values are positive definite, always greater than zero, they are useful for quantifying the uncertainty principle of quantum mechanics.

Coming to our subject, we can say that the variables satisfying Hamilton's equation of motion

$$\begin{aligned} \dot{q}_s &= \frac{\partial H}{\partial p_s} \\ \dot{p}_s &= -\frac{\partial H}{\partial q_s} \end{aligned}$$

are called canonical conjugate variables.

2.1.2 Exact or Canonical Transformation

Let $q_1, q_2, ...q_N \,\&\, p_1, p_2, ...p_N$ be independent variables and $Q_1, Q_2, ...Q_N \,\&\, P_1, P_2, ...P_N$ be another set of independent variables connected by the transformation

$$\left.\begin{aligned} Q_s = &\quad Q_s\left(q_1, q_2, ...q_N, p_1, p_2, ...p_N, t\right) \\ P_s = &\quad P_s\left(q_1, q_2, ...q_N, p_1, p_2, ...p_N, t\right) \end{aligned}\right\} \tag{2.1.1}$$

such that the new variables $Q_s \,\&\, P_s$ are canonical conjugate variables. Then

$$\begin{aligned} \dot{Q}_s &= \frac{\partial K}{\partial P_s} \\ \dot{P}_s &= -\frac{\partial K}{\partial Q_s} \end{aligned}$$

Where K play's the role of Hamilton's to the variables $Q_s \,\&\, P_s$. (2.1.1) contains $4n+1$ variables out of which only $2n+1$ are independent variables. The inverse transformation of (2.1.1) can be obtained if

$$J = \frac{\partial\left(Q_s, P_s\right)}{\partial\left(q_s, p_s\right)} \neq 0$$

The Hamilton's principle

$$\delta I = \delta \int_{t_1}^{t_2} L dt$$

Using Hamiltonian we can write

$$\delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} \left(\dot{q}_s p_s - H\right) dt \tag{2.1.2}$$

Hence for transformed Hamiltonian, Hamilton's principle is

$$\delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} \left(\dot{Q}_s P_s - K\right) dt \tag{2.1.3}$$

In both cases the variation at the end points is zero. (2.1.2) and (2.1.3) does not mean that the two integrals are equal but can differ at the most by the total time derivative of an arbitrary function. If F is an arbitrary function then we can write

$$\begin{aligned}
\dot{q}_s p_s - H &= \dot{Q}_s P_s - K + \frac{dF}{dt} \\
\dot{q}_s p_s - H - \left(\dot{Q}_s P_s - K\right) &= \frac{dF}{dt} \\
\frac{dq_s}{dt} p_s - H - \left(\frac{dQ_s}{dt} P_s - K\right) &= \frac{dF}{dt} \\
p_s dq_s - H dt - (P_s dQ_s - K dt) &= dF
\end{aligned}$$

If Hamiltonian is time independent, then

$$H = K$$

then we have

$$\frac{dF}{dt} = 0$$

and we are left with

$$p_s dq_s - P_s dQ_s = dF \tag{2.1.4}$$

For very small increment we can write

$$p_s \delta q_s - P_s \delta Q_s = \delta F$$

2.1.3 Criteria for a transformation to be Canonical Transformation

A transformation from $(q_s, p_s, t) \to (Q_s, P_s, t)$ is said to be contact if the following differential

$$p_s \delta q_s - P_s \delta Q_s = 0 \tag{2.1.5}$$

is an exact differential and its solution is a generating function F. For exactness we first transform (2.1.5) in the form

$$M(q_s, p_s)\,\delta q_s + N(q_s, p_s)\,\delta p_s = 0$$

and then show that the following relation holds.

$$\frac{\partial}{\partial p_s} M(q_s, p_s) = \frac{\partial}{\partial q_s} N(q_s, p_s) \tag{2.1.6}$$

2.1.4 Method to find Generating Function

If (2.1.6) holds then there exist a function $F(q_s, p_s)$ such that

$$\frac{\partial F}{\partial q_s} = M(q_s, p_s)$$

We can find F by integrating $M(q_s, p_s)$ with respect to q_s while holding p_s constant:

$$F(q_s, p_s) = \int M(q_s, p_s)\, dq_s + g(p_s) \tag{2.1.7}$$

where the arbitrary function $g(p_s)$ is the constant of integration. We assume that

$$\frac{\partial F}{\partial p_s} = N(q_s, p_s) \tag{2.1.8}$$

Next differentiate (2.1.7) with respect to p_s and equate it with (2.1.8)

$$\frac{\partial F}{\partial p_s} = \frac{\partial}{\partial p_s}\int M(q_s, p_s)\, dq_s + \frac{d}{dp_s} g(p_s) = N(q_s, p_s)$$

This gives

$$\frac{d}{dp_s} g(p_s) = N(q_s, p_s) - \frac{\partial}{\partial p_s}\int M(q_s, p_s)\, dq_s \tag{2.1.9}$$

Finally, integrate (2.1.9) with respect to p_s and substitute the result in (2.1.7). The generating function is

$$F(q_s, p_s) = C \tag{2.1.10}$$

It is important to realize that the expression (2.1.9) is independent of p_s because

$$\begin{aligned}\frac{\partial}{\partial q_s}\left[N(q_s, p_s) - \frac{\partial}{\partial p_s}\int M(q_s, p_s)\, dq_s\right] &= \frac{\partial N}{\partial q_s} - \frac{\partial}{\partial p_s}\left(\frac{\partial}{\partial q_s}\int M(q_s, p_s)\, dq_s\right) \\ &= \frac{\partial N}{\partial q_s} - \frac{\partial M}{\partial p_s} \\ &= 0\end{aligned}$$

We can also start the foregoing procedure with the assumption that

$$\frac{\partial F}{\partial p_s} = N(q_s, p_s)$$

Next integrate N with respect to p_s

$$F(q_s, p_s) = \int N(q_s, p_s)\, dp_s + h(q_s) \tag{2.1.11}$$

and then differentiating that result

$$\frac{d}{dq_s} h(q_s) = M(q_s, p_s) - \frac{\partial}{\partial q_s}\int N(q_s, p_s)\, dp_s \tag{2.1.12}$$

Finally, integrate (2.1.12) with respect to q_s and substitute the result in (2.1.11). The generating function is $F(q_s, p_s) = C$. Sometimes we ignore this C.

2.1.5 Types of Generating Functions

There are four types of generating functions, namely

$$\begin{aligned} F_1 &= F_1(q_s, Q_s) \\ F_2 &= F_2(q_s, P_s) \\ F_3 &= F_3(p_s, Q_s) \\ F_4 &= F_4(p_s, P_s) \end{aligned}$$

The generating function given by (2.1.10) has equivalence relation with above generating functions in the following way.

$$\begin{aligned} F_1(q_s, Q_s) &= F(q_s, p_s) \\ F_2(q_s, P_s) &= F(q_s, p_s) + PQ \\ F_3(p_s, Q_s) &= F(q_s, p_s) - pq \\ F_4(p_s, P_s) &= F(q_s, p_s) - pq + PQ \end{aligned}$$

Example 2.1.1. *Show that the transformation*

$$\begin{aligned} Q &= \frac{1}{2}\left(p^2 + q^2\right) \\ P &= -\arctan\left(\frac{q}{p}\right) \end{aligned}$$

is exact. Find its generating function and then transform it into its four types.

Solution From given transformation, we first formulate the expression

$$\begin{aligned} p\delta q - P\delta Q &= 0 \\ p\delta q + \arctan\left(\frac{q}{p}\right)\frac{1}{2}(2p\delta p + 2q\delta q) &= 0 \\ p\delta q + \arctan\left(\frac{q}{p}\right)(p\delta p + q\delta q) &= 0 \\ \left(p + q\arctan\left(\frac{q}{p}\right)\right)\delta q + \left(p\arctan\left(\frac{q}{p}\right)\right)\delta p &= 0 \end{aligned}$$

$$\begin{aligned} M(q_s, p_s) &= \left(p + q\arctan\left(\frac{q}{p}\right)\right) \\ \text{and} \quad N(q_s, p_s) &= p\left(\arctan\left(\frac{q}{p}\right)\right) \end{aligned}$$

Following (2.1.5) the above equation is exact if

$$\begin{aligned}
\frac{\partial}{\partial p_s}\left(p+q\arctan\left(\frac{q}{p}\right)\right) &= \frac{\partial}{\partial q_s}\left(p\arctan\left(\frac{q}{p}\right)\right) \\
1+q\frac{1}{1+\left(\frac{q}{p}\right)^2}\left(-\frac{q}{p^2}\right) &= p\frac{1}{1+\left(\frac{q}{p}\right)^2}\left(\frac{1}{p}\right) \\
1-\frac{q^2}{q^2+p^2} &= \frac{p^2}{q^2+p^2} \\
\frac{p^2}{q^2+p^2} &= \frac{p^2}{q^2+p^2}
\end{aligned}$$

Since (2.1.5) is satisfied, hence the given transformation is exact or canonical transformation. We can find generating function F by integrating $M(q,p)$ with respect to q while holding p constant:

$$\begin{aligned}
F(q,p) &= \int M(q,p)\,dq + g(p) \\
&= \int\left(p+q\arctan\left(\frac{q}{p}\right)\right)dq + g(p)
\end{aligned}$$

Where the arbitrary function $g(p)$ is the constant of integration. Integrate second term in first expression by parts

$$\begin{aligned}
F(q,p) &= pq+\frac{q^2}{2}\arctan\left(\frac{q}{p}\right)-\int\frac{q^2}{2}\frac{p^2}{q^2+p^2}\frac{1}{p}dq+g(p) \\
&= pq+\frac{q^2}{2}\arctan\left(\frac{q}{p}\right)-\frac{p}{2}\int\frac{q^2}{q^2+p^2}dq+g(p) \\
&= pq+\frac{q^2}{2}\arctan\left(\frac{q}{p}\right)-\frac{p}{2}\int\left(1-\frac{p^2}{q^2+p^2}\right)dq+g(p) \\
&= pq+\frac{q^2}{2}\arctan\left(\frac{q}{p}\right)-\frac{pq}{2}+\frac{p^3}{2}\int\left(\frac{1}{q^2+p^2}\right)dq+g(p) \\
&= \frac{pq}{2}+\frac{q^2}{2}\arctan\left(\frac{q}{p}\right)+\frac{p^3}{2}\frac{1}{p}\arctan\left(\frac{q}{p}\right)+g(p) \\
&= \frac{pq}{2}+\frac{1}{2}\left(q^2+p^2\right)\arctan\left(\frac{q}{p}\right)+g(p) \qquad (2.1.13)
\end{aligned}$$

We assume that

$$\frac{\partial F}{\partial p} = N(q_s,p_s) = p\left(\arctan\left(\frac{q}{p}\right)\right) \qquad (2.1.14)$$

Next differentiate (2.1.13) with respect to p and equate it with (2.1.14)

$$\begin{aligned}
\frac{\partial F}{\partial p} &= \frac{q}{2} + \frac{1}{2}(2p)\arctan\left(\frac{q}{p}\right) + \frac{1}{2}\left(q^2+p^2\right)\frac{-q/p^2}{1+(q^2/p^2)} + g^{'}(p) \\
p\left(\arctan\left(\frac{q}{p}\right)\right) &= \frac{q}{2} + p\left(\arctan\left(\frac{q}{p}\right)\right) + \frac{1}{2}\left(q^2+p^2\right)\frac{-q/p^2}{\left(q^2+p^2\right)/p^2} + g^{'}(p) \\
&= \frac{q}{2} + p\left(\arctan\left(\frac{q}{p}\right)\right) - \frac{q}{2} + g^{'}(p)
\end{aligned}$$

This gives

$$g^{'}(p) = 0 \tag{2.1.15}$$

Finally, integrate (2.1.16) with respect to p

$$g(p) = C \tag{2.1.16}$$

and substitute the result in (2.1.13).

$$F(q,p) = \frac{pq}{2} + \frac{1}{2}\left(q^2+p^2\right)\arctan\left(\frac{q}{p}\right) + C$$

Ignoring C the generating function is

$$F(q,p) = \frac{pq}{2} + \frac{1}{2}\left(q^2+p^2\right)\arctan\left(\frac{q}{p}\right)$$

Its corresponding four types of generating functions are as following. For F_1, we have to transform $(q_s,p_s) \to (q_s,Q_s)$.

$$\begin{aligned}
F_1(q_s,Q_s) &= F(q_s,p_s) \\
&= \frac{pq}{2} + \frac{1}{2}\left(q^2+p^2\right)\arctan\left(\frac{q}{p}\right)
\end{aligned}$$

From

$$Q = \frac{1}{2}\left(q^2+p^2\right)$$

We can write

$$p = \sqrt{2Q-q^2}$$

Then

$$F_1(q_s,Q_s) = \left(\frac{q}{2}\sqrt{2Q-q^2}\right) + Q\arctan\left(\frac{q}{\sqrt{2Q-q^2}}\right)$$

To calculate F_2 we proceed as

$$\begin{aligned} F_2\left(q_s, P_s\right) &= F\left(q_s, p_s\right) + PQ \\ &= \frac{pq}{2} + \frac{1}{2}\left(q^2 + p^2\right)\arctan\left(\frac{q}{p}\right) + PQ \end{aligned}$$

Using given transformation we can write

$$\begin{aligned} F_2\left(q_s, P_s\right) &= \frac{pq}{2} + Q\left(-P\right) + PQ \\ &= \frac{pq}{2} \end{aligned} \tag{2.1.17}$$

Since

$$\begin{aligned} P &= -\arctan\left(\frac{q}{p}\right) \\ \Rightarrow \frac{q}{p} &= -\tan\left(P\right) \\ \text{or } p &= -q\cot\left(P\right) \end{aligned}$$

Using above result in (2.1.17), F_2 can be written as

$$F_2 = -\frac{1}{2}q^2\cot\left(P\right)$$

For F_3, consider

$$\begin{aligned} F_3\left(p_s, Q_s\right) &= F\left(q_s, p_s\right) - pq \\ &= \frac{pq}{2} + \frac{1}{2}\left(q^2 + p^2\right)\arctan\left(\frac{q}{p}\right) - pq \\ &= -\frac{pq}{2} + \frac{1}{2}\left(q^2 + p^2\right)\arctan\left(\frac{q}{p}\right) \end{aligned}$$

From

$$Q = \frac{1}{2}\left(q^2 + p^2\right)$$

We can write

$$q = \sqrt{2Q - p^2}$$

Then

$$F_3\left(p_s, Q_s\right) = -\frac{1}{2}p\sqrt{2Q - p^2} + Q\arctan\left(\frac{\sqrt{2Q - p^2}}{p}\right)$$

For F_4, consider

$$\begin{aligned} F_4(p_s, P_s) &= F(q_s, p_s) - pq + PQ \\ &= \frac{pq}{2} + \frac{1}{2}\left(q^2 + p^2\right)\arctan\left(\frac{q}{p}\right) - pq + PQ \end{aligned}$$

Using given transformation, F_4 becomes

$$\begin{aligned} F_4(p_s, P_s) &= -\frac{pq}{2} + (Q)\arctan(-P) + PQ \\ &= -\frac{pq}{2} \end{aligned} \tag{2.1.18}$$

Since

$$\begin{aligned} P &= -\arctan\left(\frac{q}{p}\right) \\ \Rightarrow \frac{q}{p} &= -\tan(P) \\ \text{or } q &= -p\tan(P) \end{aligned}$$

Using above result in (2.1.18), F_4 can be written as

$$F_4 = \frac{1}{2}p^2\tan(P)$$

2.2 Canonical Transformation for these Four Types of Generating Functions

If we are given a particular generating function, then we have the canonical transformation as following:

2.2.1 Canonical Transformation for Generating Functions $F_1(q_s, Q_s, t)$

If the generating function is $F_1(q_s, Q_s, t)$, then we can write

$$F = F_1$$

then total time derivative on both sides gives

$$dF = dF_1$$

Since dF is given by (2.1.4), and

$$dF_1 = \frac{\partial F_1}{q_s}dq_s + \frac{\partial F_1}{Q_s}dQ_s + \frac{dF_1}{dt}dt$$

so we have

$$p_s dq_s - P_s dQ_s + (K - H)\, dt = \frac{\partial F_1}{q_s} dq_s + \frac{\partial F_1}{Q_s} dQ_s + \frac{dF_1}{dt} dt$$

Comparing the coefficients of dq_s, dQ_s and dt we have

$$p_s = \frac{\partial F_1}{q_s} \tag{2.2.1}$$

$$P_s = -\frac{\partial F_1}{Q_s} \tag{2.2.2}$$

$$K - H = \frac{dF_1}{dt} \tag{2.2.3}$$

If Hamiltonian is time independent, then

$$H = K$$

and we have

$$\frac{dF_1}{dt} = 0$$

Then (2.2.1) and (2.2.2) are known as canonical transformation for generating function F_1.

2.2.2 Canonical Transformation for Generating Functions $F_2(q_s, P_s, t)$

If the generating function is $F_2(q_s, P_s, t)$, then we can write

$$F + P_s Q_s = F_2$$

then total time derivative on both sides gives

$$dF + d(P_s Q_s) = dF_2$$

Since dF is given by (2.1.4), and

$$dF_2 = \frac{\partial F_2}{q_s} dq_s + \frac{\partial F_2}{P_s} dP_s + \frac{dF_2}{dt} dt$$

so we have

$$p_s dq_s - P_s dQ_s + (K - H)\, dt + P_s dQ_s + Q_s dP_s = \frac{\partial F_2}{q_s} dq_s + \frac{\partial F_2}{P_s} dP_s + \frac{dF_2}{dt} dt$$

or

$$p_s dq_s + Q_s dP_s + (K - H)\, dt = \frac{\partial F_2}{q_s} dq_s + \frac{\partial F_2}{P_s} dP_s + \frac{dF_2}{dt} dt$$

Comparing the coefficients of dq_s, dP_s and dt we have

$$p_s = \frac{\partial F_2}{q_s} \tag{2.2.4}$$

$$Q_s = -\frac{\partial F_2}{P_s} \tag{2.2.5}$$

$$K - H = \frac{dF_2}{dt} \tag{2.2.6}$$

If Hamiltonian is time independent, then

$$H = K$$

and we have

$$\frac{dF_2}{dt} = 0$$

Then (2.2.4) and (2.2.5) are known as canonical transformation for generating function F_2.

2.2.3 Canonical Transformation for Generating Functions $F_3\left(p_s, Q_s, t\right)$

If the generating function is $F_3\left(p_s, Q_s, t\right)$, then we can write

$$F - p_s q_s = F_3$$

then total time derivative on both sides gives

$$dF - d\left(p_s q_s\right) = dF_3$$

Since dF is given by (2.1.4), and

$$dF_3 = \frac{\partial F_3}{p_s} dp_s + \frac{\partial F_3}{Q_s} dQ_s + \frac{dF_3}{dt} dt$$

so we have

$$p_s dq_s - P_s dQ_s + (K - H)\, dt - p_s dq_s - q_s dp_s = \frac{\partial F_3}{p_s} dp_s + \frac{\partial F_3}{Q_s} dQ_s + \frac{dF_3}{dt} dt$$

or

$$-q_s dp_s - P_s dQ_s + (K - H)\, dt = \frac{\partial F_3}{p_s} dp_s + \frac{\partial F_3}{Q_s} dQ_s + \frac{dF_3}{dt} dt$$

Comparing the coefficients of dp_s, dQ_s and dt we have

$$q_s = -\frac{\partial F_3}{p_s} \tag{2.2.7}$$

$$P_s = -\frac{\partial F_3}{Q_s} \tag{2.2.8}$$

$$K - H = \frac{dF_3}{dt} \tag{2.2.9}$$

If Hamiltonian is time independent, then

$$H = K$$

and we have

$$\frac{dF_3}{dt} = 0$$

Then (2.2.7) and (2.2.8) are known as canonical transformation for generating function F_3.

2.2.4 Canonical Transformation for Generating Functions $F_4\left(p_s, P_s, t\right)$

If the generating function is $F_4\left(p_s, P_s, t\right)$, then we can write

$$F + P_s Q_s - p_s q_s = F_4$$

then total time derivative on both sides gives

$$dF + d\left(P_s Q_s\right) - d\left(p_s q_s\right) = dF_4$$

Since dF is given by (2.1.4), and

$$dF_4 = \frac{\partial F_4}{p_s} dp_s + \frac{\partial F_4}{P_s} dP_s + \frac{dF_4}{dt} dt$$

so we have

$$p_s dq_s - P_s dQ_s + (K - H)\, dt + P_s dQ_s - Q_s dP_s - p_s dq_s - q_s dp_s = \frac{\partial F_4}{p_s} dp_s + \frac{\partial F_4}{P_s} dP_s + \frac{dF_4}{dt} dt$$

or

$$Q_s dP_s - q_s dp_s + (K - H)\, dt = \frac{\partial F_4}{p_s} dp_s + \frac{\partial F_4}{P_s} dP_s + \frac{dF_4}{dt} dt$$

Comparing the coefficients of dp_s, dQ_s and dt we have

$$q_s = -\frac{\partial F_4}{p_s} \tag{2.2.10}$$

$$Q_s = \frac{\partial F_4}{P_s} \tag{2.2.11}$$

$$K - H = \frac{dF_4}{dt} \tag{2.2.12}$$

If Hamiltonian is time independent, then

$$H = K$$

and we have

$$\frac{dF_4}{dt} = 0$$

Then (2.2.10) and (2.2.11) are known as canonical transformation for generating function F_4.

Example 2.2.1. *Consider a one dimensional linear harmonic oscillator oscillates about its mean position. Let this mean position be origin. The generating function F_1 for this system is*

$$F_1 = \frac{m\omega}{2}q^2 \cot Q,$$

with m is mass and $\omega = \sqrt{\frac{k}{m}}$ is frequency of oscillator and k is spring constant. Find all possible transformations in terms of q, p, Q and P

Solution:
The transformation for generating function F_1 are given by (2.2.1) and (2.2.2). So we have

$$\begin{aligned} p &= \frac{\partial F_1}{q} = \frac{\partial}{q}\left(\frac{m\omega}{2}q^2 \cot Q\right) \\ &= m\omega q \cot Q \qquad (2.2.13) \\ P &= -\frac{\partial F_1}{Q} = -\frac{\partial}{Q}\left(\frac{m\omega}{2}q^2 \cot Q\right) \\ &= \frac{m\omega}{2}q^2 \csc^2 Q \qquad (2.2.14) \end{aligned}$$

From (2.2.13), q can be written as

$$q = \frac{p}{m\omega}\tan Q \qquad (2.2.15)$$

and from (2.2.14) q can be written as

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q \qquad (2.2.16)$$

Using (2.2.16) in (2.2.13) p is

$$p = \sqrt{2Pm\omega}\cos Q \qquad (2.2.17)$$

Using (2.2.15) in (2.2.14) P is

$$P = \frac{1}{2m\omega}p^2 \sec^2 Q \qquad (2.2.18)$$

From (2.2.13), $\tan Q$ can be written as

$$\tan Q = \frac{qm\omega}{p}$$

then Q is

$$Q = \arctan\left(\frac{m\omega q}{p}\right) \qquad (2.2.19)$$

Next using trigonometric relation

$$\sec^2 Q = 1 + \tan^2 Q$$

From (2.2.18), P can also be written as

$$P = \frac{1}{2m\omega}\left(p^2 + m^2\omega^2 q^2\right) \tag{2.2.20}$$

From (2.2.20), p is

$$p = \sqrt{2m\omega P - m^2\omega^2 q^2} \tag{2.2.21}$$

From (2.2.20), p is

$$q = \frac{1}{m\omega}\left(\sqrt{2m\omega P - p^2}\right) \tag{2.2.22}$$

From (2.2.14), Q can be written as

$$Q = \arcsin\left(\sqrt{\frac{m\omega}{2P}}q\right) \tag{2.2.23}$$

From (2.2.17), Q can be written as

$$Q = \arccos\left(\frac{p}{\sqrt{2Pm\omega}}\right) \tag{2.2.24}$$

Any two from above transformation containing all 4 coordinates can be taken as canonical transformation.

Example 2.2.2. *If*

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q$$
$$p = \sqrt{2Pm\omega}\cos Q$$

(with m is mass and $\omega = \sqrt{\frac{k}{m}}$ is frequency of oscillator and k is spring constant) are canonical transformation of one dimensional linear harmonic oscillator. Express its position in terms of energy.

Solution:
Kinetic energy for one dimensional linear harmonic oscillator is

$$T = \frac{1}{2}m\dot{q}^2$$

Since

$$p = m\dot{q}$$

so the kinetic energy in terms of p is

$$T = \frac{1}{2m}p^2$$

And the potential energy for one dimensional linear harmonic oscillator is

$$U = \frac{k}{2}q^2$$

The Hamiltonian is the total energy of the system

$$E = H = T + U$$

Exercises

1. Check whether the following transformations are exact, if yes find their corresponding generating functions and then transform them into their four types.

   (a)
$$\begin{aligned} p &= m\omega q \cot Q, \quad m \,\&\, \omega \text{ are constants} \\ P &= \frac{m\omega q^2}{2\sin^2 Q} \end{aligned}$$

   (b)
$$\begin{aligned} Q &= \ln\left(\frac{\sin p}{q}\right) \\ P &= q\cot p \end{aligned}$$

   (c)
$$\begin{aligned} Q &= \sqrt{2q}e^{k}\cos p, \quad k \text{ is constant} \\ P &= \sqrt{2q}e^{-k}\sin p \end{aligned}$$

   (d)
$$\begin{aligned} Q &= \ln\left(1+\sqrt{q}\cos p\right) \\ P &= 2\left(1+\sqrt{q}\cos p\right)\sqrt{q}\sin p \end{aligned}$$

   (e)
$$\begin{aligned} Q &= \sqrt{q}\cos 2p \\ P &= \sqrt{q}\sin 2p \end{aligned}$$

2. For what values of α and β the transformation
$$\begin{aligned} Q &= q^{\alpha}\cos\beta p \\ P &= q^{\alpha}\sin\beta p \end{aligned}$$
   is exact.

3. Prove that the following relations hold for canonical transformations

   (a)
$$\frac{\partial q_s}{\partial Q_s} = \frac{\partial P_s}{\partial p_s}$$

(b)

$$\frac{\partial p_s}{\partial Q_s} = -\frac{\partial P_s}{\partial q_s}$$

(c)

$$\frac{\partial q_s}{\partial P_s} = -\frac{\partial Q_s}{\partial p_s}$$

(d)

$$\frac{\partial p_s}{\partial P_s} = \frac{\partial Q_s}{\partial q_s}$$

Chapter 3

Lagrange and Poisson Brackets

3.1 Lagrange Brackets

Lagrange brackets are certain expressions closely related to Poisson brackets that were introduced by Joseph Louis Lagrange in $1808 - 1810$ for the purposes of mathematical formulation of classical mechanics, but unlike the Poisson brackets, have fallen out of use. Suppose that $(q_1, , q_n, p_1, , p_n)$ is a system of canonical coordinates on a phase space. If each of them is expressed as a function of two variables, u and v, then the Lagrange bracket of u and v is defined by the formula

$$[u, v]_{p,q} = \sum_{i=1}^{n} \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right)$$

3.2 Poisson Brackets

If u and v are two functions defined on phase space, we can define a new function on phase space, called the Poisson bracket of the two functions:

$$[u, v]_{p,q} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

It turns out that for the variables (q, p) themselves, the Poisson bracket takes on particularly simple values:

$$[q_j, q_k]_{q,p} = 0 = [p_j, p_k]_{p,q}$$

$$[q_j, p_k]_{p,q} = \delta_{jk} = [p_j, q_k]_{p,q}$$

These relationships are called the fundamental Poisson brackets. It turns out that Poisson brackets are invariant under Canonical transformations. This means that a necessary

and sufficient condition for a transformation to be a Canonical transformation, is that the transformation functions satisfy the fundamental Poisson brackets.
The invariance of Poisson brackets under canonical transformations allows us to write all time evolution as follows:

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

A special case are the canonical equations of Hamilton:

$$\begin{aligned}\dot{q} &= [q_i, H] \\ \dot{p} &= [p_i, H]\end{aligned}$$

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