MEASURE THEORY

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Here are few short comings of Riemann Integration.

i. The class of Riemann Integration function is relatively small.
ii. Riemann Integral does not satisfy limit properties $\{f_n\}_1^\infty$ of Riemann Integration functions on [a,b] such that $\lim_{n\to\infty} f_n = f$, then it is not necessarily true that $\lim_{n\to\infty} f_n = f$, then it is not necessarily true that $\lim_{n\to\infty}\int_a^b f(x)dx = \int_a^b f(x)dx = \int_a^b \left(\lim_{n\to\infty} f(x)\right)dx$
iii. ℓ^p space except ℓ^∞ fail to be complete under the integral norm.

The main aim of this course is to develop a more satisfactory theory of integration to overcome above mentioned drawbacks.

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The Riemann integral, dealt with in calculus courses, is well suited for computations but less suited for dealing with limit processes. In this course we will introduce the so called Lebesgue integral, which keeps the advantages of the Riemann integral and eliminates its drawbacks.

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Algebras

Let X be an arbitrary non – empty set. A collection $\mathcal{A}$ of subsets of X is an algebra on X if

(a) $X \in \mathcal{A}$
(b) For each set E that belongs to $\mathcal{A}$,the set E^c belongs to $\mathcal{A}$
(c) For each finite sequence $E_1, E_2, \ldots, E_n$ of sets that belong to $\mathcal{A}$,the set $\cup_{i=1}^{n} E_i$ belongs to $\mathcal{A}$
(d) For each finite sequence $E_1, E_2, \ldots, E_n$ of sets that belong to $\mathcal{A}$,the set $\cap_{i=1}^{n} E_i$ belongs to $\mathcal{A}$.

Of course, in conditions (b), (c), and (d),we have required that $\mathcal{A}$ be closed under complementation, under the formation of finite unions, and under the formation of finite intersections. It is easy to check that closure under complementation and closure under the formation of finite unions together imply closure under the formation of finite intersections (use that fact that $\cap_{i=1}^{n} E_i = (\cup_{i=1}^{n} E_i^c)^c$). Thus we could have defined an algebra using only conditions (a), (b), and (c). A similar argument shows that we could have used only conditions (a), (b), and (d).

Property: If $\mathcal{A}$ is algebra then $\varphi, X \in \mathcal{A}$

Solution: Since $\mathcal{A} \neq \varphi$ therefore $E \in \mathcal{A}$ implies $E^c \in \mathcal{A}$

Now if $E, E^c \in \mathcal{A}$ this implies $E \cup E^c = X \in \mathcal{A}$. Thus $X \in \mathcal{A}$

Also as $X \in \mathcal{A}$ implies $X^c = \varphi \in \mathcal{A}$. Thus $\varphi \in \mathcal{A}$

Hence $\varphi, X \in \mathcal{A}$

Property: If a finite sequence $E_1, E_2, \ldots, E_n$ of sets that belong to $\mathcal{A}$,the set $\cap_{i=1}^{n} E_i$ belongs to $\mathcal{A}$.

Solution: If $E_1, E_2, \ldots, E_n \in \mathcal{A}$ implies $E_1^c, E_2^c, \ldots, E_n^c \in \mathcal{A}$

Then by definition $\cup_{i=1}^{n} E_i^c \in \mathcal{A}$

Implies by using de – Morgan's law $\cap_{i=1}^{n} E_i = (\cup_{i=1}^{n} E_i^c)^c \in \mathcal{A}$

Property: If $A, B \in \mathcal{A}$ then $A/B \in \mathcal{A}$
Solution: $B \in \mathcal{A} \Rightarrow B^c \in \mathcal{A}$ then $A, B^c \in \mathcal{A}$ implies $A \cap B^c = A/B \in \mathcal{A}$

Sigma-Algebras (σ-Algebras)

Let X be an arbitrary non – empty set. A collection $\mathcal{A}$ of subsets of $P(X)$ is a σ-algebra on X if

(a) $X \in \mathcal{A}$
(b) For each set E that belongs to $\mathcal{A}$,the set E^c belongs to $\mathcal{A}$
(c) For each infinite sequence $E_1, E_2 ,..., E_n, E_{n+1},\ldots$ of sets that belong to $\mathcal{A}$,the set $\cup_{i=1}^{\infty} E_i$ belongs to $\mathcal{A}$
(d) For each infinite sequence $E_1, E_2 ,..., E_n, E_{n+1},\ldots$ of sets that belong to $\mathcal{A}$, the set $\cap_{i=1}^{\infty} E_i$ belongs to $\mathcal{A}$.

Thus a σ-algebra on X is a family of subsets of X that contains X and is closed under complementation, under the formation of countable unions, and under the formation of countable intersections. Note that, as in the case of algebras, we could have used only conditions (a), (b), and (c), or only conditions (a), (b), and (d), in our definition.

Each σ-algebra on X is an algebra on X since, for example, the union of the finite sequence $E_1, E_2 ,..., E_n$ is the same as the union of the infinite sequence $E_1, E_2 ,..., E_n, E_{n+1}$ ....

If X is a set and $\mathcal{A}$ is a family of subsets of X that is closed under complementation, then X belongs to $\mathcal{A}$ if and only if φ belongs to $\mathcal{A}$. Thus in the definitions of algebras and σ-algebras given above, we can replace condition (a) with the requirement that φ be a member of $\mathcal{A}$ .Furthermore, if $\mathcal{A}$ is a family of subsets of X that is nonempty, closed under complementation, and closed under the formation of finite or countable unions, then $\mathcal{A}$ must contain X: if the set A belongs to ,then X, since it is the union of E and E^c, must also belong to $\mathcal{A}$. Thus in our definitions of algebras and σ-algebras, we can replace condition (a) with the requirement that $\mathcal{A}$ be nonempty.

If $\mathcal{A}$ is a σ-algebra on the set X, it is sometimes convenient to call a subset of X, $\mathcal{A}$ *-measurable* if it belongs to $\mathcal{A}$. Also if algebra $\mathcal{A}$ is a finite collection of subsets of a set X, then it is σ-algebra. Actually this follows from the fact that countable union of members of $\mathcal{A}$ is actually finite union of members of $\mathcal{A}$.

Remember that the smallest σ-algebra on X is { φ,X} and called **trivial σ −algebra**. Also $P(X)$ is the largest σ −algebra on X

Examples: (Some Families of Sets That Are Algebras or σ-algebras, and Some That Are Not).

- Let X be a set, and let $\mathcal{A}$ be the collection of all subsets of X. Then $\mathcal{A}$ is a σ-algebra on X.
- Let X be a set, and let $\mathcal{A}$ = { φ,X}.Then $\mathcal{A}$ is a σ-algebra on X.
- Let X be an infinite set, and let $\mathcal{A}$ be the collection of all finite subsets of X. Then $\mathcal{A}$ does not contain X and is not closed under complementation; hence it is not an algebra (or a σ-algebra) on X.
- Let X be an infinite set, and let $\mathcal{A}$ be the collection of all subsets E of X such that either E or E^c is finite. Then $\mathcal{A}$ is an algebra on X (check this) but is not closed under the formation of countable unions; hence it is not a σ-algebra.
- Let X be an uncountable set, and let $\mathcal{A}$ be the collection of all countable (i.e., finite or countably infinite) subsets of X. Then $\mathcal{A}$ does not contain X and is not closed under complementation; hence it is not an algebra.
- Let X be a set, and let $\mathcal{A}$ be the collection of all subsets E of X such that either E or E^c is countable. Then $\mathcal{A}$ is a σ-algebra.
- Let $\mathcal{A}$ be the collection of all subsets of ℝ that are unions of finitely many intervals of the form (a,b], (a,+∞),or (−∞,b]. It is easy to check that each set that belongs to A is the union of a finite disjoint collection of intervals of the types listed above, and then to check that $\mathcal{A}$ is an algebra on ℝ (the empty set belongs to $\mathcal{A}$, since it is the union of the empty, and hence finite, collection of intervals). The algebra $\mathcal{A}$ is not a σ-algebra; for example, the bounded open subintervals of ℝ are unions of sequences of sets in $\mathcal{A}$ but do not themselves belong to $\mathcal{A}$.
- The collection of intervals in [0,1] forms a semi algebra.

Proposition: If $\mathcal{A}$ is sigma algebra then for $\{E_i\}_1^\infty$ in $\mathcal{A}$ we have $\cap_{i=1}^\infty E_i \in \mathcal{A}$

Solution:

Since $\{E_i\}_1^\infty$ is a sequence in sigma algebra $\mathcal{A}$ therefore $\{E_i^c\}_1^\infty$ is also in $\mathcal{A}$.

Then by definition $\cup_{i=1}^\infty E_i^c \in \mathcal{A}$

Implies by using de – Morgan's law $\cap_{i=1}^\infty E_i = (\cup_{i=1}^\infty E_i^c)^c \in \mathcal{A}$

Question: Let X be a non – empty set. Then the collection $\mathcal{A} = \{E \subseteq X : E\ or E^c\ is\ countable\}$ is a σ-algebra on X

Solution: Let $\{E_i\}_1^\infty$ be a sequence in $\mathcal{A}$ then two cases arise;

Case – I: If each E_i is countable then $\cup_{i=1}^\infty E_i$ is countable. Because countable union of countable set is countable.

Case – II: Suppose $\mathcal{A} = \{E_k : k \epsilon N\ and\ E_k^c\ is\ countable\}$

Now $E_k \subseteq \cup_{i=1}^\infty E_i$ implies $(\cup_{i=1}^\infty E_i)^c \subseteq E_k^c$ $\quad \because A \subseteq B \Rightarrow B^c \subseteq A^c$

This means $(\cup_{i=1}^\infty E_i)^c$ is countable. $\quad \because E_k^c$ is countable

Implies $\cup_{i=1}^\infty E_i \in \mathcal{A}$. This prove that $\mathcal{A}$ is a σ-algebra on X

Proposition: Let X be a set. Then the intersection of an arbitrary nonempty collection of σ-algebras on X is a σ-algebra on X.

Proof: Let $\{\mathcal{A}_i\}_1^\infty$ be a nonempty collection of σ-algebras on X, we have to prove $\cap_{i=1}^\infty \mathcal{A}_i$ is σ-algebra.

For this let $E_i \in \cap_{i=1}^\infty \mathcal{A}_i$ then $E_i \in \mathcal{A}_i$; $\forall\ i \epsilon N$

Then $E_i^c \in \mathcal{A}_i$; $\forall\ i \epsilon N$ so that $E_i^c \in \cap_{i=1}^\infty \mathcal{A}_i$

Let $\{E_i\}_1^\infty$ be the sequence in $\cap_{i=1}^\infty \mathcal{A}_i$ then $\{E_i\}_1^\infty$ will be the sequence in $\mathcal{A}_i$; $\forall\ i \epsilon N$. Then $\cup_{i=1}^\infty E_i \in \mathcal{A}_i$; $\forall\ i \epsilon N$. So that $\cup_{i=1}^\infty E_i \in \cap_{i=1}^\infty \mathcal{A}_i$; $\forall\ i \epsilon N$

Hence $\cap_{i=1}^\infty \mathcal{A}_i$ is a σ-algebra.

Remark: The reader should note that the union of a family of σ-algebras can fail to be a σ-algebra. For example; let $X = \{a, b, c, d\}$ and $\mathcal{A}_1 = \{\varphi, X, \{a\}, \{b, c, d\}\}$ and $\mathcal{A}_2 = \{\varphi, X, \{b\}, \{a, c, d\}\}$ then $\mathcal{A}_1 \cup \mathcal{A}_2 = \{\varphi, X, \{a\}, \{b\}, \{a, c, d\}, \{b, c, d\}\}$

Now as $\{a\}, \{b\} \epsilon \mathcal{A}_1 \cup \mathcal{A}_2$ but $\{a, b\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$

Hence $\mathcal{A}_1 \cup \mathcal{A}_2$ is not σ-algebra.

Another similar example is for the sets $X = \{1,2,3,4\}$ and $\mathcal{A}_1 = \{\varphi, X, \{1\}, \{2,3,4\}\}$ and $\mathcal{A}_2 = \{\varphi, X, \{2\}, \{1,3,4\}\}$

Property: Every Algebra is a σ-algebra.

Proof: Consider $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{n} E_i + \cup_{i=n+1}^{\infty} E_i$ …………..(i)

Take $E_i = \varphi \qquad \forall\ i = n+1, n+2, \dots$

$\Rightarrow \cup_{i=1}^{\infty} E_i = \cup_{i=1}^{n} E_i \in \mathcal{A} \Rightarrow \cup_{i=1}^{\infty} E_i \in \mathcal{A} \Rightarrow \mathcal{A}$ is a σ-algebra.

Interesting to Remember:

- Every algebra is a topology.
- Topology needs not to be algebra.
- Sigma algebra is not a topology.

Sequence

A function whose domain is set of natural numbers is called sequence.

Sequence of Sets

Let $\{A_n\}_1^{\infty}$ be a sequence of subsets of a set X. We say that $\{A_n\}_1^{\infty}$ is increasing if $A_n \subseteq A_{n+1}\ \forall\ n \epsilon N$ i.e. $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$

Similarly

We say that $\{A_n\}_1^{\infty}$ is decreasing if $A_n \supseteq A_{n+1}\ \forall\ n \epsilon N$ i.e. $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$

Monotone Sequence:

A sequence is called a monotone sequence if it is either increasing or decreasing.

Limit of Sequence of Sets

If $\{A_n\}_1^{\infty}$ is increasing then $\lim_{n\to\infty} A_n = \cup_{n=1}^{\infty} A_n$

If $\{A_n\}_1^{\infty}$ is decreasing then $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$

Remark:

- For a monotone sequence $\lim_{n\to\infty} A_n$ always exists although it may be φ
- If $\{A_n\}_1^{\infty}$ is increasing then $\lim_{n\to\infty} A_n = \varphi \Leftrightarrow A_n = \varphi\ \ \forall\ n \epsilon N$
- If $\{A_n\}_1^{\infty}$ is decreasing then we may have $\lim_{n\to\infty} A_n = \varphi$ even $A_n \neq \varphi$ for all $n \epsilon N$. For example if $A_n = \left(0, \frac{1}{n}\right)$; $n = 1,2,3, \dots$. Then $\{A_n\}_1^{\infty}$ is decreasing and $\lim_{n\to\infty} A_n = \varphi$ also if $A_n = [0, \frac{1}{n})$ then $\lim_{n\to\infty} A_n = \{0\}$

How do we find limit of an arbitrary sequence $\{A_k\}_1^\infty$ of subset of a set X?

Let $\{A_k\}_1^\infty$ be an arbitrary sequence of set X, then define two new sequences $\underline{A_k} = \cap_{n \geq k} A_n$ and $\overline{A_k} = \cup_{n \geq k} A_n$

i.e. $\underline{A_1} = \cap_{n \geq 1} A_n, \underline{A_2} = \cap_{n \geq 2} A_n, \ldots$ $(A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots)$

also $\overline{A_1} = \cup_{n \geq 1} A_n, \overline{A_2} = \cup_{n \geq 2} A_n, \ldots$ $(A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots)$

obviously $\underline{A_k}$ is increasing and $\overline{A_k}$ is decreasing then

limit inferior of the sequence $\{A_k\}_1^\infty$ defined as $\lim_{k\to\infty} inf A_k = \cup_{k \geq 1} (\cap_{n \geq k} A_n)$

i.e. $\lim_{k\to\infty} inf A_k = (\cap_{n \geq 1} A_n) \cup (\cap_{n \geq 2} A_n) \cup \ldots = \underline{A_1} \cup \underline{A_2} \cup \ldots$

limit Superior of sequence $\{A_k\}_1^\infty$ defined as $\lim_{k\to\infty} Sup A_k = \cap_{k \geq 1} (\cup_{n \geq k} A_n)$

i.e. $\lim_{k\to\infty} Sup A_k = (\cup_{n \geq 1} A_n) \cap (\cup_{n \geq 2} A_n) \cap \ldots = \overline{A_1} \cap \overline{A_2} \cap \ldots$

Remember:

If the limit superior and the limit inferior become equal then we say that limit exists. Then we can use as needed following;

$\lim_{k\to\infty} inf A_k = \lim_{k\to\infty} Sup A_k = \lim_{k\to\infty} A_k$

Theorem:

Let $\mathcal{A}$ be a σ-algebra of subsets of X, then $lmt\ inf A_k$ and $lmt\ Sup A_k$ are in $\mathcal{A}$.

Proof: Since $\{A_k\}_1^\infty$ is in $\mathcal{A}$ therefore;

$\cap_{n \geq k} A_n \in \mathcal{A}$ $\because \mathcal{A}$ is closed under countable intersection.

Then $\cup_{k \geq 1} (\cap_{n \geq k} A_n) \in \mathcal{A}$ $\because \mathcal{A}$ is σ-algebra

Implies $\lim_{k\to\infty} inf A_k \in \mathcal{A}$

Similarly $\cap_{k \geq 1} (\cup_{n \geq k} A_n) \in \mathcal{A}$ Implies $\lim_{k\to\infty} Sup A_k \in \mathcal{A}$

If $\lim_{k\to\infty} A_k$ exists then $\lim_{k\to\infty} inf A_k = \lim_{k\to\infty} Sup A_k \in \mathcal{A}$

Smallest Sigma Algebra

Let ε be an arbitrary collection of subsets of X that are sigma algebras, then smallest sigma algebra is defined as

$$\boldsymbol{\sigma(\varepsilon) = \sigma - algebra\ generated\ by\ \varepsilon = \cap_{i=1}^{\infty} \mathcal{A}_i\ ;\ \mathcal{A}_i\ contains\ \varepsilon}$$

By the phrase 'smallest σ-algebra on X that contains ε', we mean a σ-algebra on X that includes ε and every σ-algebra on X that includes ε also includes it.

Remark:

- $\varepsilon \subseteq \sigma(\varepsilon)$
  Solution: Since $\varepsilon \subseteq \mathcal{A}_i \ \forall\, i \ \Rightarrow \varepsilon \subseteq \cap_{i=1}^{\infty} \mathcal{A}_i \ \Rightarrow \varepsilon \subseteq \sigma(\varepsilon)$

- If ε_1 and ε_2 are collections of subsets of X, and $\varepsilon_1 \subseteq \varepsilon_2$ then $\sigma(\varepsilon_1) \subseteq \sigma(\varepsilon_2)$
  Solution: Since $\varepsilon_2 \subseteq \sigma(\varepsilon_2)\ \ also\ \varepsilon_1 \subseteq \varepsilon_2 \subseteq \sigma(\varepsilon_2) \ \Rightarrow\ \varepsilon_1 \subseteq \sigma(\varepsilon_2)$
  But $\sigma(\varepsilon_1)$ is smallest for collection ε_1
  $\Rightarrow\ \varepsilon_1 \subseteq \sigma(\varepsilon_1) \subseteq \sigma(\varepsilon_2) \Rightarrow\ \boldsymbol{\sigma(\varepsilon_1) \subseteq \sigma(\varepsilon_2)}$

- If $\mathcal{A}$ be a σ-algebra of subsets of X, then $\sigma(\mathcal{A}) = \mathcal{A}$
  Solution: Since $\mathcal{A}$ is smallest subcollection of subsets of X, therefore by definition of smallest sigma algebra $\boldsymbol{\sigma(\mathcal{A}) = \mathcal{A}}$

- $\sigma\big(\sigma(\varepsilon)\big) = \sigma(\varepsilon)$
  Solution: Since $\sigma(\varepsilon)$ is smallest σ-algebra on X, then by using $\sigma(\mathcal{A}) = \mathcal{A}$ and putting $\mathcal{A} = \sigma(\varepsilon)$ we get $\boldsymbol{\sigma\big(\sigma(\varepsilon)\big) = \sigma(\varepsilon)}$

Corollary: Let X be a set, and let ε be a family of subsets of X. Then there is a smallest σ-algebra on X that includes ε.

Or Let ε be an arbitrary collection of subsets of X, then there exists smallest sigma algebra $\mathcal{A}_0$ of subsets of X containing ε.

(Smallest in the sense that if $\mathcal{A}$ is a σ-algebra of subsets of X containing ε then $\mathcal{A}_0 \subset \mathcal{A}$)

Proof: Let $\{A_i\}_1^{\infty}$ be a collection of σ-algebras containing ε. This collection is non – empty. Since it contains at least $P(X)$ then $\mathcal{A}_0 = \cap_{i=1}^{\infty} A_i$ is the σ-algebras containing ε. Let $\mathcal{A}$ be another σ-algebras containing ε, then $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$ so that $\cap_{i=1}^{\infty} A_i = \mathcal{A}_0 \subset \mathcal{A}$. This implies $\mathcal{A}_0 \subset \mathcal{A}$

Hence $\mathcal{A}_0$ is the smallest σ-algebras containing ε.

Recall:

If X and Y are two sets and $f: X \to Y$ then

- $f(X) \subseteq Y$
- If $E \subset Y$ then E needs not to be subset of $f(X)$ and $f^{-1}(E) = \{x\epsilon X: f(x)\epsilon E\}$ thus if $E \cap f(x) = \varphi$ then $f^{-1}(E) = \varphi$
- If $E \subseteq Y$ then $f\left(f^{-1}(E)\right) \subseteq E$
- $f^{-1}(Y) = X$
- $f^{-1}(E^c) = \left(f^{-1}(E)\right)^c$
- $f^{-1}(E^c) = f^{-1}(Y/E) = f^{-1}(Y)/f^{-1}(E) = X/f^{-1}(E) = \left(f^{-1}(E)\right)^c$
- $f^{-1}(\cup_1^\infty E_i) = \cup_1^\infty f^{-1}(E_i)$ also $f^{-1}(\cap_1^\infty E_i) = \cap_1^\infty f^{-1}(E_i)$
- If ε is an arbitrary collection of subsets of Y then $f^{-1}(\varepsilon) = \{f^{-1}(E): E\epsilon\varepsilon\}$

Preposition: Let $f: X \to Y$ if B is a σ-algebra of subsets of Y then $f^{-1}(B)$ will be a σ-algebra of subsets of X.

Proof:

Since $Y\epsilon B$ therefore $f^{-1}(Y) = X\epsilon f^{-1}(B)$

Now suppose $A\epsilon f^{-1}(B)$ then $A = f^{-1}(E)$ for some $E\epsilon B$

Since B is a σ-algebra, therefore $E^c\epsilon B$ so that

$f^{-1}(E^c) = \left(f^{-1}(E)\right)^c = A^c\epsilon\, f^{-1}(B)$

Let $\{A_n\}_1^\infty$ be a sequence in $f^{-1}(B)$ then $A_n = f^{-1}(E_n)$ for some $E_n\epsilon B$

So that $\cup_1^\infty E_n\epsilon\, B$ $\quad\therefore B$ is a σ-algebra

Then $f^{-1}(\cup_1^\infty E_n) = \cup_1^\infty f^{-1}(E_n) = \cup_1^\infty A_n\epsilon\, f^{-1}(B)$

This proves that $f^{-1}(B)$ is a σ-algebra.

Preposition: Prove for a function $f: X \to Y$ and an arbitrary collection ε of subsets of X, $\boldsymbol{\sigma\left(f^{-1}(\varepsilon)\right) = f^{-1}(\sigma(\varepsilon))}$

Proof: Since $\sigma(\varepsilon)$ is a σ-algebra on subsets of Y, therefore $f^{-1}(\sigma(\varepsilon))$ is a σ-algebra on subsets of X, so that $\sigma\left(f^{-1}(\sigma(\varepsilon))\right) = f^{-1}(\sigma(\varepsilon))$ ……..(i)

And since $\varepsilon \subseteq \sigma(\varepsilon)$ therefore $f^{-1}(\varepsilon) \subseteq f^{-1}(\sigma(\varepsilon))$ ……..(ii)

Implies $\sigma(f^{-1}(\varepsilon)) \subseteq \sigma\left(f^{-1}(\sigma(\varepsilon))\right)$ $\because \varepsilon_1 \subseteq \varepsilon_2$ then $\sigma(\varepsilon_1) \subseteq \sigma(\varepsilon_2)$

So that $\sigma(f^{-1}(\varepsilon)) \subseteq f^{-1}(\sigma(\varepsilon))$ ……..(iii)

To prove the inverse inclusion, let $\mathcal{A}_1$ be an arbitrary σ-algebra of subsets of X, then we claim that $\mathcal{A}_2 = \{A \subseteq Y: f^{-1}(A) \epsilon\, \mathcal{A}_1\}$ is σ-algebra

Let $E \epsilon\, \mathcal{A}_2$ then $f^{-1}(E) \epsilon\, \mathcal{A}_1$ so that $f^{-1}(E^c) = \left(f^{-1}(E)\right)^c \epsilon\, \mathcal{A}_1$ $\because \mathcal{A}_1$ σ-algebra

Implies $E^c \epsilon\, \mathcal{A}_2$ by definition

Let $\{E_i\}_1^\infty \epsilon\, \mathcal{A}_2$ then $\{f^{-1}(E_i)\}_1^\infty \epsilon\, \mathcal{A}_1$ so that $f^{-1}(\cup_1^\infty E_i) = \cup_1^\infty f^{-1}(E_i) \epsilon\, \mathcal{A}_1$

Since $\mathcal{A}_1$ is a σ-algebra

Then $\cup_1^\infty E_i \epsilon\, \mathcal{A}_2$. This proves that $\mathcal{A}_2$ is σ-algebra

In particular; If we choose $\mathcal{A}_1 = \sigma(f^{-1}(\varepsilon))$ then
$\mathcal{A}_2 = \left\{A \subseteq Y: f^{-1}(A) \epsilon\, \sigma(f^{-1}(\varepsilon))\right\}$ is a σ-algebra.

Now $\varepsilon \subseteq \mathcal{A}_2$ $\because A \epsilon\, \varepsilon$ then $f^{-1}(A) \epsilon\, f^{-1}(\varepsilon) \subseteq \sigma(f^{-1}(\varepsilon))$

$\Rightarrow \sigma(\varepsilon) \subseteq \sigma(\mathcal{A}_2) = \mathcal{A}_2$

So $f^{-1}(\sigma(\varepsilon)) \subseteq f^{-1}(\mathcal{A}_2) \subseteq \sigma(f^{-1}(\varepsilon))$

$\Rightarrow f^{-1}(\sigma(\varepsilon)) \subseteq \sigma(f^{-1}(\varepsilon))$ ……..(iv)

From (iii) and (iv) we get $\boldsymbol{\sigma\left(f^{-1}(\varepsilon)\right) = f^{-1}(\sigma(\varepsilon))}$

Borel Subsets of $\mathbb{R}$

Let $\mathfrak{B}$ be the intersection of all the σ-algebra of subsets of $\mathbb{R}$ containing every open subset of $\mathbb{R}$. Then the member of $\mathfrak{B}$ are called Borel subsets of $\mathbb{R}$.

The Borel σ-algebra

Let $(X, \mathfrak{B})$ be a topological space then $\sigma(\mathfrak{B})$ is called Borel σ-algebra of open subsets of topological space X. It is denoted by $\mathfrak{B}(X)$ or $\mathfrak{B}_X$. Then the members of $\mathfrak{B}(X)$ are called Borel Sets.

Or The Borel σ-algebra on $\mathbb{R}^d$ is the σ-algebra on $\mathbb{R}^d$ generated by the collection of open subsets of $\mathbb{R}^d$; it is denoted by $\mathfrak{B}(\mathbb{R}^d)$.The Borel subsets of $\mathbb{R}^d$ are those that belong to $\mathfrak{B}(\mathbb{R}^d)$. In case d = 1, one generally writes $\mathfrak{B}(\mathbb{R})$ in place of $\mathfrak{B}(\mathbb{R}^1)$.

Lemma: Let ρ be the collection of all closed sets in a topological space $(X, \mathfrak{B})$ then $\boldsymbol{\sigma(\rho) = \sigma(\mathfrak{B})}$

Proof: Let $E \epsilon\ \rho$ a closed set then E^c will be open. Then $E^c \epsilon\ \mathfrak{B} \subseteq \sigma(\mathfrak{B})$

$\Rightarrow E^c \epsilon\ \sigma(\mathfrak{B}) \Rightarrow (E^c)^c \epsilon\ \sigma(\mathfrak{B}) \Rightarrow E \epsilon\ \sigma(\mathfrak{B})$ $\qquad$ $\because \sigma(\mathfrak{B})$ is σ-algebra

Thus $\rho \subseteq \sigma(\mathfrak{B})$

$\Rightarrow \sigma(\rho) \subseteq \sigma\big(\delta(\mathfrak{B})\big) = \sigma(\mathfrak{B})$ $\qquad$ $\because \varepsilon_1 \subseteq \varepsilon_2$ then $\sigma(\varepsilon_1) \subseteq \sigma(\varepsilon_2)$

$\Rightarrow \sigma(\rho) \subseteq \sigma(\mathfrak{B})$ ……………(i)

For inverse inclusion let $F \epsilon\ \mathfrak{B}$ an open set then F^c will be closed

$\Rightarrow F^c \epsilon \rho \subseteq \sigma(\rho) \Rightarrow (F^c)^c \epsilon\ \sigma(\rho) \Rightarrow F \epsilon\ \sigma(\rho)$ $\qquad$ $\because \sigma(\rho)$ is σ-algebra

Thus $\mathfrak{B} \subseteq \sigma(\rho)$

$\Rightarrow \sigma(\mathfrak{B}) \subseteq \sigma\big(\sigma(\rho)\big) = \sigma(\rho)$ $\qquad$ $\because \varepsilon_1 \subseteq \varepsilon_2$ then $\sigma(\varepsilon_1) \subseteq \sigma(\varepsilon_2)$

$\Rightarrow \sigma(\mathfrak{B}) \subseteq \sigma(\rho)$ ……………(ii)

From (i) and (ii) $\qquad$ $\boldsymbol{\sigma(\rho) = \sigma(\mathfrak{B})}$

Proposition (Just Read): The σ-algebra $\mathfrak{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ is generated by each of the following collections of sets:

(a) The collection of all closed subsets of $\mathbb{R}$
(b) The collection of all subintervals of $\mathbb{R}$ of the form $(-\infty,b]$
(c) The collection of all subintervals of $\mathbb{R}$ of the form $(a,b]$

Proof: Let $\mathfrak{B}_1, \mathfrak{B}_2$,and $\mathfrak{B}_3$ be the σ-algebras generated by the collections of sets in parts (a), (b), and (c) of the proposition. We will show that $\mathfrak{B}(\mathbb{R}) \supseteq \mathfrak{B}_1 \supseteq \mathfrak{B}_2 \supseteq \mathfrak{B}_3$ and then that $\mathfrak{B}_3 \supseteq \mathfrak{B}(\mathbb{R})$; this will establish the proposition.

Since $\mathfrak{B}(\mathbb{R})$ includes the family of open subsets of $\mathbb{R}$ and is closed under complementation, it includes the family of closed subsets of $\mathbb{R}$; thus it includes the σ-algebra generated by the closed subsets of $\mathbb{R}$, namely $\mathfrak{B}_1$. The sets of the form $(-\infty,b]$ are closed and so belong to $\mathfrak{B}_1$; consequently $\mathfrak{B}_1 \supseteq \mathfrak{B}_2$.

Since $(a, b] = (-\infty, b] \cap (-\infty, a]^c$, each set of the form $(a,b]$ belongs to $\mathfrak{B}_2$; thus $\mathfrak{B}_2 \supseteq \mathfrak{B}_3$. Finally, note that each open subinterval of $\mathbb{R}$ is the union of a sequence of sets of the form $(a,b]$ and that each open subset of $\mathbb{R}$ is the union of a sequence of open intervals. Thus each open subset of $\mathbb{R}$ belongs to $\mathfrak{B}_3$, and so $\mathfrak{B}_3 \supseteq \mathfrak{B}(\mathbb{R})$

As we proceed, the reader should note the following properties of the σ-algebra $\mathfrak{B}(\mathbb{R})$:

(a) It contains virtually every subset of $\mathbb{R}$ that is of interest in analysis.
(b) It is small enough that it can be dealt with in a fairly constructive manner.

It is largely these properties that explain the importance of $\mathfrak{B}(\mathbb{R})$.

Proposition (Just Read): The σ-algebra $\mathfrak{B}(\mathbb{R}^d)$ of Borel subsets of $\mathbb{R}^d$ is generated by each of the following collections of sets:

(a) the collection of all closed subsets of $\mathbb{R}^d$
(b) the collection of all closed half-spaces in $\mathbb{R}^d$ that have the form $\{(x_1, \ldots, x_d) : x_i \le b\}$ for some index i and some b in $\mathbb{R}$;
(c) the collection of all rectangles in $\mathbb{R}^d$ that have the form $\{(x_1, \ldots, x_d) : a_i < x_i \le b_i \; for \; i = 1, \ldots, d\}$.

Let us look in more detail at some of the sets in $\mathfrak{B}(\mathbb{R}^d)$.Let G be the family of all open subsets of $\mathbb{R}^d$,and let F be the family of all closed subsets of $\mathbb{R}^d$. (Of course G and F depend on the dimension d, and it would have been more precise to write $G(\mathbb{R}^d)$ and $F(\mathbb{R}^d)$.) Let G_δ be the collection of all intersections of sequences of sets in G ,and let F_σ be the collection of all unions of sequences of sets in F. Sets in G_δ are often called $G_{\delta\text{'s}}$, and sets in F_σ are often called $F_{\sigma\text{'s}}$. The letters G and F presumably stand for the German word Gebiet and the French word fermè, and the letters σ and δ for the German words Summe and Durchschnitt. Now we properly define above discussed terms.

G_δ Set: Let $(X, \mathfrak{B})$ be a topological space. A set E of X is called G_δ set if E is an intersection of countably many open sets. i.e. $E = \cap_1^\infty G_i$ where G_i are open.

F_σ Set: Let $(X, \mathfrak{B})$ be a topological space. A set E of X is called F_σ set if E is a union of countably many closed sets. i.e. $E = \cup_1^\infty F_i$ where F_i are closed.

Remark:

- If E is G_δ set then E^c is F_σ set and vice versa.
- If E is G_δ set then there exists a sequence $\{E_i\}_1^\infty$ of open sets such that $E = \cap_1^\infty E_i$
- G_δ set is the limit of decreasing sequence. i.e. if $\{G_n\}_1^\infty$ be a sequence of open sets then $\lim_{n\to\infty} G_n = \cap_1^\infty G_n = G$
- F_σ set is the limit of increasing sequence. i.e. if $\{F_n\}_1^\infty$ be a sequence of closed sets then $\lim_{n\to\infty} F_n = \cup_1^\infty F_n = F$

Proposition (Just Read): Each closed subset of $\mathbb{R}^d$ is a G_δ , and each open subset of $\mathbb{R}^d$ is an F_σ.

Proof: Suppose that F is a closed subset of $\mathbb{R}^d$. We need to construct a sequence $\{U_n\}$ of open subsets of $\mathbb{R}^d$ such that $F = \cap_n U_n$. For this define U_n by

$$U_n = \left\{x \in \mathbb{R}^d : \|x - y\| < \frac{1}{n} \text{ } for\ some\ y\ in\ F\right\}$$

(Note that U_n is empty if F is empty.) It is clear that each U_n is open and that $F \subseteq \cap_n U_n$. The reverse inclusion follows from the fact that F is closed (note that each point in $\cap_n U_n$ is the limit of a sequence of points in F). Hence each closed subset of $\mathbb{R}^d$ is a G_δ .

If U is open, then U^c is closed and so is a G_δ . Thus there is a sequence $\{U_n\}$ of open sets such that $U^c = \cap_n U_n$.The sets U_n^c are then closed, and $U = \cup_n U_n^c$ hence U is an F_σ.

Lemma: Let $\{E_n\}_1^\infty$ be an arbitrary sequence in σ-algebra $\mathcal{A}$ of subsets of X. Then there exists a disjoint sequence $\{F_n\}_1^\infty$ in $\mathcal{A}$ such that $\boldsymbol{\cup_1^\infty F_n = \cup_1^\infty E_n}$

Proof: Given $\{E_n\}_1^\infty$ be an arbitrary sequence in σ-algebra $\mathcal{A}$ of subsets of X.

We now define a new sequence $\{F_n\}_1^\infty$ in $\mathcal{A}$ such that

$F_1 = E_1$

$F_2 = E_2/E_1 \subseteq E_2$

$F_3 = E_3/E_1 \cup E_2 \subseteq E_3$

$\vdots \qquad \vdots \qquad \vdots$

$F_n = E_n/E_1 \cup E_2 \cup \ldots \cup E_{n-1} \subseteq E_n$

$\Rightarrow F_n = E_n \cap (E_1 \cup E_2 \cup \ldots \cup E_{n-1})^c \qquad \because A/B = A \cap B^c$

$\Rightarrow F_n = E_n \cap ({E_1}^c \cap {E_2}^c \cap \ldots \cap {E_{n-1}}^c)$ by De – Morgan's Law

Since $\{E_n\}_1^\infty$ be an arbitrary sequence in σ-algebra $\mathcal{A}$ therefore

$E_n \cap ({E_1}^c \cap {E_2}^c \cap \ldots \cap {E_{n-1}}^c) \epsilon \mathcal{A} \Rightarrow \qquad F_n \epsilon \mathcal{A} \qquad \forall n \epsilon N$

Implies $\{F_n\}_1^\infty$ is a sequence in $\mathcal{A}$

Now we have to show that $\{F_n\}_1^\infty$ is a **disjoint** sequence in $\mathcal{A}$.i.e. $F_m \cap F_n = \varphi$

Let $m < n$ then by definition of F_n we have $F_m \subseteq E_m$

$\Rightarrow F_m \cap F_n \subseteq E_m \cap F_n$ ……….(i)

Consider $E_m \cap F_n = E_m \cap (E_n \cap {E_1}^c \cap {E_2}^c \cap \ldots \cap {E_m}^c \cap \ldots \cap {E_{n-1}}^c)$

$\Rightarrow E_m \cap F_n = (E_m \cap {E_m}^c) \cap (E_n \cap {E_1}^c \cap {E_2}^c \cap \ldots \cap E_{m-1}^c \cap E_{m+1}^c \cap \ldots \cap E_{n-1}^c)$

$\Rightarrow E_m \cap F_n = \varphi \cap (E_n \cap {E_1}^c \cap {E_2}^c \cap \ldots \cap E_{m-1}^c \cap E_{m+1}^c \cap \ldots \cap E_{n-1}^c)$

$\Rightarrow E_m \cap F_n = \varphi$

Thus $(i) \Rightarrow F_m \cap F_n = \varphi$

Hence proved that $\{F_n\}_1^\infty$ is a **disjoint** sequence in $\mathcal{A}$

Now we have to prove that $\cup_1^\infty F_n = \cup_1^\infty E_n$

Since $F_n \subseteq E_n \Rightarrow \cup_1^\infty F_n \subseteq \cup_1^\infty E_n$ ……….(ii)

Let $x \epsilon \cup_1^\infty E_n \Rightarrow x \epsilon E_n$ for some $n \epsilon N$

If 'm' is the smallest positive integer then $x \epsilon E_m$ but $x \notin E_1, E_2, \ldots, E_{m-1}$

$\Rightarrow x \epsilon E_n / E_1 \cup E_2 \cup \ldots \cup E_{m-1} \Rightarrow x \epsilon E_n \cap (E_1 \cup E_2 \cup \ldots \cup E_{m-1})^c$

$\Rightarrow x \epsilon E_m \cap ({E_1}^c \cap {E_2}^c \cap \ldots \cap {E_{m-1}}^c) \Rightarrow x \epsilon F_m \Rightarrow x \epsilon F_n \Rightarrow x \epsilon \cup_1^\infty F_n$

$\Rightarrow \cup_1^\infty E_n \subseteq \cup_1^\infty F_n$ ....(iii)

From (ii) and (iii) $\cup_{\mathbf{1}}^\infty \boldsymbol{F_n} = \cup_{\mathbf{1}}^\infty \boldsymbol{E_n}$

Proposition (Just Read): Let X be a set, and let $\mathcal{A}$ be algebra on X. Then $\mathcal{A}$ is a σ-algebra if either

(a) $\mathcal{A}$ is closed under the formation of unions of increasing sequences of sets, or
(b) $\mathcal{A}$ is closed under the formation of intersections of decreasing sequences of sets.

Proof: First suppose that condition (a) holds. Since $\mathcal{A}$ is an algebra, we can check that it is a σ-algebra by verifying that it is closed under the formation of countable unions. Suppose that $\{E_i\}$ is a sequence of sets that belong to . For each *n* let $B_n = \cup_1^n E_i$. The sequence $\{B_n\}$ is increasing, and, since $\mathcal{A}$ is an algebra, each B_n belongs to ; thus assumption (a) implies that $\cup_n B_n$ belongs to $\mathcal{A}$ .However, $\cup_i E_i$ is equal to $\cup_n B_n$ and so belongs to $\mathcal{A}$. Thus $\mathcal{A}$ is closed under the formation of countable unions and so is a σ-algebra.

Now suppose that condition (b) holds. It is enough to check that condition (a) holds. If $\{E_i\}$ is an increasing sequence of sets that belong to $\mathcal{A}$,then $\{E_i^c\}$ is a decreasing sequence of sets that belong to $\mathcal{A}$, and so condition (b) implies that $\cap_i E_i^c$ belongs to $\mathcal{A}$ .Since $\cup_i E_i = (\cap_i E_i^c)^c$, it follows that $\cup_i E_i$ belongs to $\mathcal{A}$. Thus condition (a) follows from condition (b), and the proof is complete.

Set of extended Real Numbers

A set $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is called set of extended real numbers.

Set Function

Let $X \neq \varphi$ and ε be an arbitrary collection of subsets of X, then the function $\mu: \varepsilon \to [0, \infty]$ is called the set function.

Properties of Set Function

- **Additive Set Function:** A set function $\mu: \varepsilon \to [0, \infty]$ is said to be additive if $E_1, E_2 \epsilon\ \varepsilon$ and $E_1 \cap E_2 = \varphi$ such that $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$
- **Sub – Additive Set Function:** A set function $\mu: \varepsilon \to [0, \infty]$ is said to be additive if $E_1, E_2 \epsilon\ \varepsilon$ and $E_1 \cup E_2 \epsilon\ \varepsilon$ having no need to $E_1 \cap E_2 = \varphi$ such that $\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2)$
- **Monotone Property:** A set function $\mu: \varepsilon \to [0, \infty]$ is said to be monotone if $E_1, E_2 \epsilon\ \varepsilon$ such that $E_1 \subseteq E_2 \Rightarrow \mu(E_1) \leq \mu(E_2)$
- **Finitely Additive Property:** A set function $\mu: \varepsilon \to [0, \infty]$ is said to be finitely additive if for every disjoint sequence $\{E_n\}_1^\infty$
  We have $\mu(\cup_1^n E_i) = \sum_1^n \mu(E_i)$
- **Finitely Sub – Additive Property:** A set function $\mu: \varepsilon \to [0, \infty]$ is said to be finitely sub – additive if for every disjoint sequence $\{E_n\}_1^\infty$
  We have $\mu(\cup_1^n E_i) \leq \sum_1^n \mu(E_i)$
- **Countably Additive Property:** A set function $\mu: \varepsilon \to [0, \infty]$ is said to be finitely additive if for every disjoint sequence $\{E_n\}_1^\infty$
  We have $\mu(\cup_1^\infty E_i) = \sum_1^\infty \mu(E_i)$
- **Countably Sub – Additive Property:** A set function $\mu: \varepsilon \to [0, \infty]$ is said to be finitely sub – additive if for every disjoint sequence $\{E_n\}_1^\infty$
  We have $\mu(\cup_1^\infty E_i) \leq \sum_1^\infty \mu(E_i)$
- **Signed measure:** If μ is countably additive and satisfies μ(∅)= 0, then it is a signed measure. Thus signed measures are the functions that result if in the definition of measures the requirement of non – negativity is removed.

Rough Sketch of Measures

Roughly speaking a measure is a weight distribution on a set X. For example, if we toss a coin, the sample space is $S = \{H, T\}$ so $W(H) = \frac{1}{2} = W(T)$. Also noted that measure is a set function.

Pre – Measure

Let $\mathcal{A}$ be sigma algebra of subsets of a non – empty set X, then a non – negative extended real valued function $\mu: \mathcal{A} \to [0, \infty]$ is called a measure if $\mu(\varphi) = 0$

Measures

Let $\mathcal{A}$ be sigma algebra of subsets of a non – empty set X, then a non – negative extended real valued function $\mu: \mathcal{A} \rightarrow [0, \infty]$ is called a measure if;

i. $\mu(\varphi) = 0$
ii. If $\{E_i\}_1^\infty$ is a disjoint sequence in $\mathcal{A}$ then $\mu(\cup_1^\infty E_i) = \sum_1^\infty \mu(E_i)$

Or Let X be a set, and let $\mathcal{A}$ be a σ-algebra on X. A measure (or a countably additive measure) on $\mathcal{A}$ is a function $\mu: \mathcal{A} \rightarrow [0, \infty]$ that satisfies μ(φ)= 0 and is countably additive.

Remember:

- If X is a set, if $\mathcal{A}$ is a σ-algebra on X, and if $\mu: \mathcal{A} \rightarrow [0, \infty]$ is a measure on , then the triplet (X, $\mathcal{A}$,μ) is often called a **measure space**.
- If X is a set and if $\mathcal{A}$ is a σ- algebra on X and if $\mu: \mathcal{A} \rightarrow [0, \infty]$ is a measure on $\mathcal{A}$ then the pair (X, $\mathcal{A}$) is often called a **measurable space**.
- If X is a set, if $\mathcal{A}$ is a σ-algebra on X, and if $\mu: \mathcal{A} \rightarrow [0, \infty]$ is a finite measure on , then the triplet (X, $\mathcal{A}$,μ) is often called a **finite measure space**. i.e. $\mu(X) < \infty$
- If X is a set, if $\mathcal{A}$ is a σ-algebra on X, and if $\mu: \mathcal{A} \rightarrow [0, \infty]$ is a σ-finite measure on , then the triplet (X, $\mathcal{A}$,μ) is often called a **σ- finite measure space**. i.e. there exists a sequence $\{E_i\}_1^\infty$ in $\mathcal{A}$ such that $X = \cup_1^\infty E_i$ with $\mu(E_i) < \infty \;\forall\, i \in N$
- Let (X,) be a measurable space then the members of $\mathcal{A}$ are called **$\boldsymbol{\mathcal{A}}$ – measurable sets**.
- Let (X, $\mathcal{A}$,μ) is a measure space a set $D \epsilon \mathcal{A}$ is called **σ- finite set** if there

exists a sequence $\{D_n\}_1^\infty$ in $\mathcal{A}$ such that $D = \cup_1^\infty D_n$ with $\mu(D_n) < \infty \;\forall\, n \in N$

- If (X, $\mathcal{A}$,μ) is a measure space, then one often says that μ is a measure on (X, $\mathcal{A}$), or, if the σ-algebra $\mathcal{A}$ is clear from context, a measure on X.

Examples:

(a) Let X be an arbitrary set, and let $\mathcal{A}$ be a σ-algebra on X.
Define a function μ : $\mathcal{A}$ → [0,+∞] by letting μ(E) be n if E is a finite set with n elements and letting μ(E) be +∞ if E is an infinite set. Then μ is a measure; it is often called counting measure on (X,).

(b) Let X be a nonempty set, and let $\mathcal{A}$ be a σ-algebra on X. Let x be a member of X. Define a function δ_x : $\mathcal{A}$ →[0,+∞] by letting $\delta_x(E)$ be 1 if x ∈ E and letting $\delta_x(E)$ be 0 if x ∉ E. Then δ_x is a measure; it is called a point mass concentrated at x.

(c) Let X be the set of all positive integers, and let $\mathcal{A}$ be the collection of all subsets E of X such that either E or E^c is finite. Then $\mathcal{A}$ is an algebra, but not a σ-algebra .Define a function μ : $\mathcal{A}$ →[0,+∞] by letting μ(E) be 1 if E is infinite and letting μ(E) be 0 if E is finite. It is easy to check that μ is a finitely additive measure; however, it is impossible to extend μ to a countably additive measure on the σ-algebra generated by $\mathcal{A}$ (if E_k = {k} for each k, then $\mu(\cup_1^\infty E_k)$ = μ(X) = 1, while $\sum_1^\infty \mu(E_k) = 0$).

(d) Let X be an arbitrary set, and let $\mathcal{A}$ be an arbitrary σ-algebra on X. Define a function μ : $\mathcal{A}$ →[0,+∞] by letting μ(E) be +∞ if E ≠ φ, and letting μ(E) be 0 if E = φ.Then μ is a measure.

(e) Let X be a set that has at least two members, and let $\mathcal{A}$ be the σ-algebra consisting of all subsets of X. Define a function μ : $\mathcal{A}$ → [0,+∞] by letting μ(E) be 1 if E ≠ φ and letting μ(E) be 0 if E = φ.Then μ is not a measure, nor even a finitely additive measure, for if E_1 and E_2 are disjoint nonempty subsets of X, then $\mu(E_1 \cup E_2)$ = 1, while $\mu(E_1) + \mu(E_2)$ = 2.

(f) The set function $\mu: \mathfrak{B}_{\mathbb{R}} \to [0,\infty]$ where $X = \mathbb{R}$ define as $\mu(E) = |E|$ is a measure on $\mathfrak{B}_{\mathbb{R}}$.

(g) The set function $\mu: \mathfrak{B}_{\mathbb{R}} \to [0,\infty]$ define as $\mu(E) = \begin{cases} 0 & if\ 2 \notin E \\ 1 & if\ 2 \epsilon E \end{cases}$ is a measure on $\mathfrak{B}_{\mathbb{R}}$.

Solution: Clearly $\mu(\varphi) = 0$ because $2 \notin \varphi$

Let $\{E_i\}_1^\infty$ be a disjoint sequence in $\mathfrak{B}_{\mathbb{R}}$ then

Case – I: If $2 \notin E_i \ \forall i$ then $\mu(\cup_1^\infty E_i) = 0 = \sum_1^\infty \mu(E_i)$

Case – II: If $2 \epsilon E_k \ for\ k \in N$ then $\mu(\cup_1^\infty E_i) = 1 = \sum_1^\infty \mu(E_i)$

Thus μ is a measure on $\mathfrak{B}_{\mathbb{R}}$.

(h) Give an example of set function which is not a measure.

Or The set function $\mu: P(\mathbb{R}) \to [0, \infty]$ where $X = \mathbb{R}$ define as $\mu(E) = \begin{cases} 0 & if\ E\ is\ finite \\ 1 & if\ E\ is\ infinite \end{cases}$ is not a measure on $P(\mathbb{R})$.

Solution: Clearly $\mu(\varphi) = 0$ because φ is finite.

Let $\{\{n\}\}_1^\infty$ be a disjoint sequence in $P(\mathbb{R})$ then

$\mu(\{n\}) = 0\ \ \forall n \in \mathbb{R}\ \ and\ since\ \{n\}\ is\ finite \Rightarrow \boldsymbol{\sum_1^\infty \mu(\{n\}) = 0}$

But $\mu(\cup_1^\infty \{n\}) = 1 = \mu(N)$ since N is infinite.

Implies $\mu(\cup_1^\infty \{n\}) \neq \sum_1^\infty \mu(\{n\})$

Hence μ is not countably additive. Not a measure.

Lemma: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure on a σ-algebra $\mathcal{A}$, then prove that μ has finitely additive property.

i.e. for a disjoint sequence $\{E_i\}_1^n$ in $\mathcal{A}$ we have $\mu(\cup_1^n E_i) = \sum_1^n \mu(E_i)$

Proof:

Let $\{E_i\}_1^\infty$ be a disjoint sequence in $\mathcal{A}$ such that $E_i = \varphi\ \ \forall\ i = n+1, n+2, \ldots$

Since μ is a measure therefore $\mu(\varphi) = 0$ and $\mu(\cup_1^\infty E_i) = \sum_1^\infty \mu(E_i)$

$\Rightarrow \mu[(\cup_1^n E_i) \cup (\cup_{n+1}^\infty E_i)] = \sum_1^n \mu(E_i) + \sum_{n+1}^\infty \mu(E_i)$

$\Rightarrow \mu[(\cup_1^n E_i) \cup (\cup_{n+1}^\infty \varphi)] = \sum_1^n \mu(E_i) + \sum_{n+1}^\infty \mu(\varphi)$

$\Rightarrow \mu[(\cup_1^n E_i) \cup (\varphi)] = \sum_1^n \mu(E_i) + \sum_{n+1}^\infty (0)$ $\qquad \because \mu(\varphi) = 0$

$\Rightarrow \boldsymbol{\mu(\cup_1^n E_i) = \sum_1^n \mu(E_i)}$

Lemma: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure on a σ-algebra , then prove that μ has monotonicity property.

Or Let (X, $\mathcal{A}$,μ) be a measure space, and let E_1 and E_2 be subsets of X that belong to $\mathcal{A}$ and satisfy $E_1 \subseteq E_2$. Then μ(E_1) ≤ μ(E_2).

Or Let μ be a signed measure on the measurable space (X, $\mathcal{A}$), and let E_2 be a subset of X that belongs to $\mathcal{A}$ and satisfies $-\infty < \mu(E_2) < 0$. Then there is a negative set E_1 that is included in E_2 and satisfies $\mu(E_1) \leq \mu(E_2)$

Proof: Let $E_1, E_2 \in \mathcal{A}$ and $E_1 \subseteq E_2$ then $E_2 = E_1 \cup (E_2/E_1)$

$\Rightarrow \mu(E_2) = \mu\big(E_1 \cup (E_2/E_1)\big) = \mu(E_1) + \mu(E_2/E_1)$ $\quad\because \mu$ is finitely additive

$\Rightarrow \mu(E_2) - \mu(E_1) = \mu(E_2/E_1) \geq 0 \quad \because E_1, E_2 \in \mathcal{A}, E_2/E_1 \in \mathcal{A},\ \mu(E_2/E_1) \geq 0$

$\Rightarrow \mu(E_2) - \mu(E_1) \geq 0 \Rightarrow \mu(E_2) \geq \mu(E_1) \Rightarrow \boldsymbol{\mu(E_1) \leq \mu(E_2)}$

Lemma: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure on a σ-algebra , also if $E_1 \subseteq E_2$. Then prove that μ(E_2/E_1) = $\mu(E_2) - \mu(E_1)$

Proof: Let $E_1, E_2 \in \mathcal{A}$ and $E_1 \subseteq E_2$ then $E_2 = E_1 \cup (E_2/E_1)$

$\Rightarrow \mu(E_2) = \mu\big(E_1 \cup (E_2/E_1)\big) = \mu(E_1) + \mu(E_2/E_1)$ $\quad\because \mu$ is finitely additive

$\Rightarrow \boldsymbol{\mu(E_2/E_1) = \mu(E_2) - \mu(E_1)}$ $\quad \because E_1 \cap (E_2/E_1) = \varphi$

Lemma: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure on a σ-algebra $\mathcal{A}$, then prove that μ has Countably Sub – additive property.

i.e. for a disjoint sequence $\{E_i\}_1^\infty$ in $\mathcal{A}$ we have $\mu(\cup_1^\infty E_i) \leq \sum_1^\infty \mu(E_i)$

Proof:

Let $\{E_i\}_1^\infty$ be a sequence in $\mathcal{A}$ then $\cup_1^\infty E_i = E_1 \cup (E_2/E_1) \cup (E_3/E_1 \cup E_2) \cup ...$

$\Rightarrow \mu(\cup_1^\infty E_i) = \mu(E_1 \cup (E_2/E_1) \cup (E_3/E_1 \cup E_2) \cup ...)$

$\Rightarrow \mu(\cup_1^\infty E_i) = \mu(E_1) + \mu(E_2/E_1) + \mu(E_3/E_1 \cup E_2) + \cdots$

$\Rightarrow \mu(\cup_1^\infty E_i) \leq \mu(E_1) + \mu(E_2) + \mu(E_3) + \cdots = \sum_1^\infty \mu(E_i)$ $\quad$ monotonicity property

$\Rightarrow \boldsymbol{\mu(\cup_1^\infty E_i) \leq \sum_1^\infty \mu(E_i)}$

Lemma: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure on a σ-algebra $\mathcal{A}$, then prove that μ has finitely Sub – additive property.

i.e. for a finite sequence $\{E_i\}_1^n$ in $\mathcal{A}$ we have $\mu(\cup_1^n E_i) \leq \sum_1^n \mu(E_i)$

1st Proof: Let $\{E_i\}_1^n$ be a sequence in $\mathcal{A}$ define $\{F_k\}_1^\infty$ by $F_k = E_i$; $k = 1,2, \dots, n$ and $F_k = \varphi \ \forall \ k = n+1, n+2, \dots$ then $\cup_1^n E_i = \cup_1^\infty F_k$

$\Rightarrow \mu(\cup_1^n E_i) = \mu(\cup_1^\infty F_k) \leq \sum_1^\infty \mu(F_k) = \sum_1^n \mu(F_k) + \sum_{n+1}^\infty \mu(F_k)$

$\Rightarrow \mu(\cup_1^n E_i) \leq \sum_1^n \mu(F_k) + \sum_{n+1}^\infty \mu(\varphi)$ $\qquad \because F_k = \varphi \ \forall \ k \geq n+1$

$\Rightarrow \mu(\cup_1^n E_i) \leq \sum_1^n \mu(F_k) + \sum_{n+1}^\infty (0)$ $\qquad \because \mu(\varphi) = 0$

$\Rightarrow \boldsymbol{\mu(\cup_1^n E_i) \leq \sum_1^n \mu(E_i)}$ $\qquad \because F_k = E_i$

2nd Proof: Let $\{E_i\}_1^\infty$ be a sequence in $\mathcal{A}$ such that $E_i = \varphi \ \forall \ k = n+1, n+2, \dots$ then $\cup_1^\infty E_i = (\cup_1^n E_i) \cup (\cup_{n+1}^\infty E_i) = (\cup_1^n E_i)$ $\qquad \because E_i = \varphi \ \forall \ k \geq n+1$

$\Rightarrow \mu(\cup_1^n E_i) = \mu(\cup_1^\infty E_i) \leq \sum_1^\infty \mu(E_i) = \sum_1^n \mu(E_i) + \sum_{n+1}^\infty \mu(E_i)$

$\Rightarrow \mu(\cup_1^n E_i) \leq \sum_1^n \mu(E_i) + \sum_{n+1}^\infty \mu(\varphi)$ $\qquad \because E_i = \varphi \ \forall \ k \geq n+1$

$\Rightarrow \mu(\cup_1^n E_i) \leq \sum_1^n \mu(E_i) + \sum_{n+1}^\infty (0)$ $\qquad \because \mu(\varphi) = 0$

$\Rightarrow \boldsymbol{\mu(\cup_1^n E_i) \leq \sum_1^n \mu(E_i)}$

Finite measure

Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, then $\mu: \mathcal{A} \to [0, \infty]$ is called a finite measure if $\mu(X) < \infty$.

σ- Finite measure

Let $X \neq \varphi$ and $\mathcal{A}$ is a σ- algebra on X, then $\mu: \mathcal{A} \to [0, \infty]$ is called a σ- finite measure if there exists a sequence $\{E_i\}_1^\infty$ in $\mathcal{A}$ such that $X = \cup_1^\infty E_i$ and $\mu(E_i) < \infty$

Note

If $\{E_i\}_1^\infty$ is a monotone sequence in $\mathcal{A}$ then $\{\mu(E_n)\}_1^\infty$ is also a monotone sequence. So $\lim_{n\to\infty} \mu(E_n)$ exists in $[0, \infty]$

Question: Give an example of measure which is σ- finite but not finite measure.

Solution:

Let $X = N$ and $\mathcal{A} = P(N)$ then define a function $\mu: P(N) \to [0, \infty]$ by $\mu(E) = |E|$ this measure is σ- finite but not finite measure because $\mu(N) = \infty$ which is not finite. But for a sequence $\{\{n\}\}_1^\infty$ such that $N = \cup_1^\infty (\{n\})$ and $\mu(\{n\}) = |\{n\}| = 1 < \infty$ for each $n \epsilon N$.

This implies μ is σ- finite but not finite measure.

Question: Give an example of measure which is finite measure.

Solution:

Let $X = \{1,2\}$ and $\mathcal{A} = \{\varphi, X, \{1\}, \{2\}\}$ is a σ- algebra on X then define a function $\mu: \mathcal{A} \to [0, \infty]$ by $\mu(\varphi) = 0, \mu(X) = 1, \mu(\{1\}) = \frac{1}{2} = \mu(\{2\})$ then μ is a measure on $\mathcal{A}$ and $\mu(\{1\} \cup \{2\}) = \mu(\{1,2\}) = 1$

Also $\mu(\{1\}) + \mu(\{2\}) = \frac{1}{2} + \frac{1}{2} = 1$ and $\mu(X) = 1 < \infty$

Therefore μ is a finite measure.

MONOTONE CONVERGENCE THEOREM FOR MONOTONE SEQUENCE OF MEASUREABLE SETS (The Continuity of Measure)

Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure then

a) If $\{E_n\}_1^\infty$ is an increasing sequence then
$$\mu(\cup_1^\infty E_n) = \lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)$$
b) If $\{E_n\}_1^\infty$ is a decreasing sequence with $\mu(E_1) < \infty$ then
$$\mu(\cap_1^\infty E_n) = \lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)$$

Proved on next page

Theorem: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure then if $\{E_n\}_1^\infty$ is an increasing sequence then $\boldsymbol{\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)}$

Proof: Note that if $\{E_n\}_1^\infty$ is a monotone sequence in $\mathcal{A}$ then $\{\mu(E_n)\}_1^\infty$ is also monotone sequence. So that $\lim_{n\to\infty} \mu(E_n)$ exists in $[0, \infty]$. Now suppose that $\{E_n\}_1^\infty$ is increasing then $\{\mu(E_n)\}_1^\infty$ is increasing. Here we discuss two cases;

Case – I: If $\mu(E_{n_0}) = \infty$ for some $n_0 \epsilon N$ then $\lim_{n\to\infty} \mu(E_n) = \infty$ ……..(i)

Now $E_{n_0} \subseteq \cup_1^\infty E_n = \lim_{n\to\infty} E_n$ $\quad\because \{E_n\}_1^\infty$ is increasing

$\Rightarrow \mu(E_{n_0}) \leq \mu\left(\lim_{n\to\infty} E_n\right)$ by monotonicity of μ

$\Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) \geq \mu(E_{n_0}) = \infty \Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) \geq \infty \Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) = \infty$ ……..(ii)

From (i) and (ii) we get $\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)$

Case – II: If $\mu(E_n) < \infty\ \forall n \epsilon N$ then taking $E_n = \varphi$ we define a disjoint sequence $\{F_n\}_1^\infty$ as

$F_1 = E_1/E_0$

$F_2 = E_2/E_1$

$\vdots \qquad \vdots$

$F_n = E_n/E_{n-1}\ \forall n \epsilon N$

Obviously $\cup_1^\infty E_n = \cup_1^\infty F_n$ ……..(iii)

Since we know that fact that for increasing sequence $\lim_{n\to\infty} E_n = \cup_1^\infty E_n$

$\Rightarrow \lim_{n\to\infty} E_n = \cup_1^\infty F_n$ from (iii)

$\Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) = \mu(\cup_1^\infty F_n) = \sum_1^\infty \mu(F_n) \Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) = \sum_1^\infty \mu(E_n/E_{n-1})$

$\Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) = \sum_1^\infty [\mu(E_n) - \mu(E_{n-1})] = \lim_{k\to\infty} \sum_{n=1}^k [\mu(E_n) - \mu(E_{n-1})]$

$\Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) = \lim_{k\to\infty} [\{\mu(E_1) - \mu(E_0)\} + \{\mu(E_2) - \mu(E_1)\} + \cdots + \{\mu(E_k) - \mu(E_{k-1})\}]$

$\Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) = \lim_{k\to\infty}[\mu(E_k) - \mu(E_0)] = \lim_{k\to\infty}[\mu(E_k) - 0]$

Here we use the fact $\mu(E_0) = \mu(\varphi) = 0$

$\Rightarrow \mu\left(\lim_{n\to\infty} E_n\right) = \lim_{k\to\infty}\mu(E_k) = \lim_{n\to\infty}\mu(E_n) \Rightarrow \boldsymbol{\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty}\mu(E_n)}$

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Theorem: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0,\infty]$ is a measure then if $\{E_n\}_1^\infty$ is a decreasing sequence then $\boldsymbol{\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)}$ where $\boldsymbol{\mu(E_1)} < \infty$

Proof:

Suppose that $\{E_n\}_1^\infty$ is decreasing with $\mu(E_1) < \infty$ then $\lim_{n\to\infty} E_n = \cap_1^\infty E_n$

Consider $E_1/\cap_1^\infty E_n = E_1 \cap (\cap_1^\infty E_n)^c$ $\qquad \because A/B = A \cap B^c$

$\Rightarrow E_1/\cap_1^\infty E_n = E_1 \cap (\cup_1^\infty E_n^c) = \cup_1^\infty (E_1 \cap E_n^c) = \cup_1^\infty (E_1/E_n)$

$\Rightarrow \mu(E_1/\cap_1^\infty E_n) = \mu\left(\cup_1^\infty (E_1/E_n)\right)$

Since $\{E_1/E_n\}_1^\infty$ is increasing therefore $\lim_{n\to\infty}(E_1/E_n) = \cup_1^\infty (E_1/E_n)$

$\Rightarrow \mu(E_1/\cap_1^\infty E_n) = \mu(E_1) - \mu(\cap_1^\infty E_n) = \mu(\lim_{n\to\infty}(E_1/E_n))$

$\Rightarrow \mu(E_1) - \mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty}(\mu(E_1/E_n)) \qquad \because \mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty}\mu(E_n)$

$\Rightarrow \mu(E_1) - \mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty}(\mu(E_1)) - \lim_{n\to\infty}(\mu(E_n))$

$\Rightarrow \mu(E_1) - \mu(\lim_{n\to\infty} E_n) = \mu(E_1) - \lim_{n\to\infty}(\mu(E_n)) \qquad \because \mu(E_1) < \infty$

$\Rightarrow \boldsymbol{\mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty}(\mu(E_n))}$

Hahn Decomposition Theorem

Let $(X, \mathcal{A})$ be a measurable space, and let μ be a signed measure on $(X, \mathcal{A})$. Then there are disjoint subsets P and N of X such that P is a positive set for μ, N is a negative set for μ, and X = P∪N.

Proof:

Since the signed measure μ cannot include both +∞ and −∞ among its values, we can for definiteness assume that −∞ is not included.

Let $L = inf\{\mu(A) : A \text{ is a negative set for } \mu\} \neq \varphi$ …………..(i)

Choose a sequence $\{A_n\}$ of negative sets for μ for which $L = lim\mu(A_n)$, and let $N = \cup_1^\infty A_n$

Where N is a negative set for μ (each $\mathcal{A} -$ measurable subset of N is the union of a sequence of disjoint $\mathcal{A}$ −measurable sets, each of which is included in some A_n). Hence $L \leq \mu(N) \leq \mu(A_n)$ holds for each n, and so $L = \mu(N)$.

Furthermore, since μ does not attain the value −∞, $\mu(N)$ must be finite.

Let $P = N^c$. Our only remaining task is to check that P is a positive set for μ.

If P included an $\mathcal{A}$ −measurable set A such that $\mu(A) < 0$, then it would include a negative set B such that $\mu(B) < 0$ by the following result;

(Let μ be a signed measure on the measurable space $(X, \mathcal{A})$, and let A be a subset of X that belongs to $\mathcal{A}$ and satisfies $-\infty < \mu(A) < 0$. Then there is a negative set B that is included in A and satisfies $\mu(B) \leq \mu(A)$),

And N ∪B would be a negative set such that

$\mu(N \cup B) = \mu(N) + \mu(B) < \mu(N) = L$ where μ(N) is finite

However this contradicts (i), and so P must be a positive set for μ.

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Hahn decomposition

A Hahn decomposition of a signed measure μ is a pair (P,N) of disjoint subsets of X such that P is a positive set for μ, N is a negative set for μ,and X = P ∪N.

Jordan Decomposition Theorem : Every signed measure is the difference of two positive measures, at least one of which is finite.

Proof: Let μ be a signed measure on $(X, \mathcal{A})$. Choose a Hahn decomposition (P,N) for μ, and then define functions μ^+ and μ^- on A by

$\mu^+(A) = \mu(A \cap P)$ and $\mu^-(A) = -\mu(A \cap N)$.

It is clear that μ^+ and μ^- are positive measures such that $\mu = \mu^+ - \mu^-$.Since $+\infty$ and $-\infty$ cannot both occur among the values of μ, at least one of the values $\mu(P)$ and $\mu(N)$, and hence at least one of the measures $\mu+$ and $\mu-$, must be finite.

…………………………………………

- The **variation** of the signed measure μ is the positive measure $|\mu|$ defined by $|\mu| = \mu^+ - \mu^-$. It is easy to check that $|\mu(A)| \leq |\mu|(A)$ holds for each A in $\mathcal{A}$ and in fact that $|\mu|$ is the smallest of those positive measures ν that satisfy $|\mu(A)| \leq \nu(A)$ for each A in $\mathcal{A}$.
- The **total variation** $\|\mu\|$ of the signed measure μ is defined by $\|\mu\| = |\mu|(X)$
- **Complex Measure:** Let $(X, \mathcal{A})$ be a measurable space, then μ be a complex measure if $\mu(\varphi) = 0$ and $\mu(\cup_1^\infty E_i) = \sum_1^\infty \mu(E_i)$ for disjoint sequence E_i

…………………………………………

Proposition: Let $(X, \mathcal{A})$ be a measurable space, and let μ be a complex measure on $(X, \mathcal{A})$. Then the variation $|\mu|$of μ is a finite measure on $(X, \mathcal{A})$.

Proof: The relation $|\mu|(\varphi) = 0$ is immediate.

We can check the finite additivity of $|\mu|$ by showing that if B_1 and B_2 are disjoint sets that belong to $\mathcal{A}$,then $|\mu|(B_1 \cup B_2) = |\mu|(B_1) + |\mu|(B_2)$.

For this, note that if $\{A_j\}_1^n$ is a finite partition of $B_1 \cup B_2$ into $\mathcal{A} -$ measurable sets, then $\sum_j |\mu(A_j)| \leq \sum_j |\mu(A_j \cap B_1)| + \sum_j |\mu(A_j \cap B_2)| \leq |\mu|(B_1) + |\mu|(B_2)$

Since $|\mu|(B_1 \cup B_2)$ is the suprimum of the numbers that can appear on the left side of the inequality, it follows that $|\mu|(B_1 \cup B_2) \leq |\mu|(B_1) + |\mu|(B_2)$

A similar argument, based on partitioning B_1 and B_2, shows that

$|\mu|(B_1) + |\mu|(B_2) \leq |\mu|(B_1 \cup B_2)$

Thus $|\mu|(B_1 \cup B_2) = |\mu|(B_1) + |\mu|(B_2)$, and the finite additivity of $|\mu|$ is proved.

Lemma: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure on a σ-algebra $\mathcal{A}$, then for an arbitrary sequence $\{E_n\}_1^n$ in $\mathcal{A}$ we have

$\boldsymbol{\mu(lmt\, Inf E_n) \leq lmt\, Inf \mu(E_n)}$

Proof: We have $lmt\, Inf E_n = \cup_{n\geq 1} (\cap_{k\geq n} E_k)$ where $\{\cap_{k\geq n} E_k\}_1^n$ is an increasing sequence, then $lmt\, Inf E_n = \cup_{n\geq 1} (\cap_{k\geq n} E_k) = \lim_{n\to\infty}(\cap_{k\geq n} E_k)$

$\Rightarrow \mu(lmt\, Inf E_n) = \mu(\lim_{n\to\infty}(\cap_{k\geq n} E_k))$

$\Rightarrow \mu(lmt\, Inf E_n) = \lim\limits_{n\to\infty}(\mu(\cap_{k\geq n} E_k))$ by monotone convergence theorem

$\Rightarrow \mu(lmt\, Inf E_n) = \lim\limits_{n\to\infty} Inf(\mu(\cap_{k\geq n} E_k))$ as limit exists

$\Rightarrow \mu(lmt\, Inf E_n) \leq \lim\limits_{n\to\infty} Inf(\mu(E_n))$ $\because \cap_{k\geq n} E_k \subseteq E_n \therefore \mu(\cap_{k\geq n} E_k) \leq \mu(E_n)$

$\Rightarrow \boldsymbol{\mu(lmt\, Inf E_n) \leq lmt\, Inf \mu(E_n)}$

Lemma: Let $X \neq \varphi$ and $\mathcal{A}$ is a σ-algebra on X, also $\mu: \mathcal{A} \to [0, \infty]$ is a measure on a σ-algebra $\mathcal{A}$, then there exists a set $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $E_n \subseteq A$ for all $n \in N$ then we have

$\boldsymbol{\mu(lmt\, Sup E_n) \geq lmt\, Sup \mu(E_n)}$

Proof: We have $lmt\, Sup E_n = \cap_{n\geq 1} (\cup_{k\geq n} E_k)$ where $\{\cup_{k\geq n} E_k\}_1^n$ is a decreasing sequence, then $lmt\, Sup E_n = \cap_{n\geq 1} (\cup_{k\geq n} E_k) = \lim_{n\to\infty}(\cup_{k\geq n} E_k)$ ……..(i)

Since $E_n \subseteq A \in \mathcal{A}$ for all $n \in N$ therefore $\cup_{k\geq 1} E_k \subseteq A$

$\Rightarrow \mu(\cup_{k\geq 1} E_k) \leq \mu(A) < \infty$ by monotonicity of μ

$\Rightarrow \mu(\cup_{k\geq 1} E_k) < \infty$

$(i) \Rightarrow \mu(lmt\, Sup E_n) = \mu(\lim_{n\to\infty}(\cup_{k\geq n} E_k))$

$\Rightarrow \mu(lmt\, Sup E_n) = \lim\limits_{n\to\infty}(\mu(\cup_{k\geq n} E_k))$ by monotone convergence theorem

$\Rightarrow \mu(lmt\, Sup E_n) = \lim\limits_{n\to\infty} Sup(\mu(\cup_{k\geq n} E_k))$ as limit exists

$\Rightarrow \mu(lmt\, Sup E_n) \geq \lim\limits_{n\to\infty} Sup(\mu(E_n))$ $\because \cup_{k\geq n} E_k \supseteq E_n \therefore \mu(\cup_{k\geq n} E_k) \geq \mu(E_n)$

$\Rightarrow \boldsymbol{\mu(lmt\, Sup E_n) \geq lmt\, Sup \mu(E_n)}$

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space. If $D \in \mathcal{A}$ is a σ- finite set then there exists an increasing sequence $\{F_n\}_1^\infty$ in $\mathcal{A}$ such that $\lim_{n\to\infty} F_n = D$ and $\mu(F_n) < \infty$ also there exists a disjoint sequence $\{G_n\}_1^\infty$ in $\mathcal{A}$ such that $\cup_1^\infty G_n = D$ and $\mu(G_n) < \infty$ for all $n \in N$.

Proof: Suppose $D \in \mathcal{A}$ is a σ- finite set then there exists $\{D_n\}_1^\infty$ in $\mathcal{A}$ such that $D = \cup_1^\infty D_n$ and $\mu(D_n) < \infty$ for all $n \in N$. Now define a sequence $\{F_n\}_1^\infty$ as $F_n = \cup_1^n D_i$ …………(i)

Then clearly the sequence $\{F_n\}_1^\infty$ is increasing in $\mathcal{A}$, then $\lim_{n\to\infty} F_n = \cup_1^\infty F_n$ and $\cup_1^\infty F_n = \cup_1^\infty (\cup_1^n D_i) = \cup_1^\infty D_n = D$ implies $\lim_{n\to\infty} F_n = D$

Now $(i) \Rightarrow \mu(F_n) = \mu(\cup_1^n D_i) = \sum_1^n \mu(D_i) < \infty(i) \Rightarrow \mu(F_n) < \infty$

Now we define a sequence by $\{G_n\}_1^\infty$ by

$G_1 = F_1$, $G_2 = F_2/F_1$, …….. , $G_n = F_n/F_{n-1}$ $\forall n \geq 2$

Then $\{G_n\}_1^\infty$ is a disjoint sequence such that $\cup_1^\infty F_n = \cup_1^\infty G_n = D$

Thus $\mu(G_1) = \mu(F_1) < \infty(i) \Rightarrow \mu(G_1) < \infty$

Now $\mu(G_n) = \mu(F_n/F_{n-1}) = \mu(F_n) - \mu(F_{n-1}) \leq \mu(F_n) < \infty$

$\Rightarrow \mu(G_n) < \infty$ $\forall n \geq 2$

Lemma: If $(X, \mathcal{A}, \mu)$ is a σ- finite measured space then every $D \in \mathcal{A}$ is a σ- finite.

Proof: Let $(X, \mathcal{A}, \mu)$ is a σ- finite measured space then there exists $\{E_n\}_1^\infty$ in $\mathcal{A}$ such that $\cup_1^\infty E_n = X$ and $\mu(E_n) < \infty$ $\forall\, n \in N$

Let $D \in \mathcal{A}$ define a sequence $\{D_n\}_1^\infty$ such that $D_n = D \cap E_n$ then

$\cup_1^\infty D_n = \cup_1^\infty (D \cap E_n) = D \cap (\cup_1^\infty E_n) = D \cap X = D \Rightarrow \cup_1^\infty D_n = D$

Now $D_n \subseteq E_n$ $\forall\, n \in N$

$\Rightarrow \mu(D_n) \leq \mu(E_n) < \infty \Rightarrow \mu(D_n) < \infty$

Hence $D \in \mathcal{A}$ is a σ- finite.

Remarks

- **Null Set:** Let $(X, \mathcal{A}, \mu)$ be a measure space, and $E \subseteq X$. Then E is called Null Set with respect to μ if $\mu(E) = 0$. This also sometime called $\mu -$ negligible.
- φ is Null Set in every measure space but a Null Set need not to be φ

Lemma: Countable union of null sets is null set.

Proof: Let $\{G_i\}_1^\infty$ be a collection of null sets in $(X, \mathcal{A}, \mu)$. We need to show that $\cup_1^\infty G_i$ is a null set. i.e. $\mu(\cup_1^\infty G_i) = 0$

Since $\mu(\cup_1^\infty G_i) = \sum_1^\infty \mu(G_i) = 0$ as $\mu(G_i) = 0 \ \forall\, i \in N$

Therefore $\mu(\cup_1^\infty G_i) = 0$ implies $\cup_1^\infty G_i$ is a null set.

Hence Countable union of null sets is null set.

Complete σ- Algebra: Let $(X, \mathcal{A}, \mu)$ be a measure space, then σ- Algebra $\mathcal{A}$ is called Complete σ- Algebra with respect to measure μ if every subset E_0 of a null set E is a member of $\mathcal{A}$.

In other words, $E_0 \subseteq E$ implies $\mu(E_0) \leq \mu(E)$. Remember since $\mu(E) = 0$ therefore $\mu(E_0) \leq 0$ but $\mu(E_0) \geq 0$. Hence $\mu(E_0) = 0$

Complete Measure Space: A measure space $(X, \mathcal{A}, \mu)$ is called a complete measure space if σ- Algebra $\mathcal{A}$ is a complete σ- Algebra with respect to measure μ

For example: Let $X = \{a, b, c\}$ and $\mathcal{A} = \{\varphi, X, \{a\}, \{b, c\}\}$ is a σ- Algebra on $\mathcal{A}$ the n define μ on $\mathcal{A}$ by $\mu(\varphi) = 0$, $\mu(X) = 1$, $\mu(\{b, c\}) = 0$, $\mu(\{a\}) = 1$ then μ is a measure on $\mathcal{A}$, $(\{b, c\})$ is a null set in $\mathcal{A}$, but $\{b\} \subseteq \{b, c\}$ is not a member of σ- Algebra. So $(X, \mathcal{A}, \mu)$ is not a complete measure space.

Outer Measures: Let X be a set, and let P(X) be the collection of all subsets of X .An outer measure on X is a function $\mu^* : P(X) \to [0, +\infty]$ such that

(a) $\mu^*(\varphi) = 0$
(b) if $E_1 \subseteq E_2 \in P(X)$, then $\mu^*(E_1) \le \mu^*(E_2)$ (monotonicity)
(c) if $\{E_n\}_1^\infty$ is an infinite sequence of subsets of X, then
$\mu^*(\cup_1^\infty E_i) \le \sum_1^\infty \mu^*(E_i)$ (countably sub – additive)

Remember

- An outer measure on X is a monotone and countably sub – additive function from P(X) to [0 ,+∞] whose value at φ is 0.
- In general, an outer measure does not satisfy additivity condition on P(X) and so fails to be a measure but we will prove later that there exists a σ-Algebra $\mathcal{A} \subseteq P(X)$ such that outer measure when restricted to $\mathcal{A}$ satisfy the additivity condition and hence becomes a measure.
- A measure can fail to be an outer measure; in fact, a measure on X is an outer measure if and only if its domain is P(X).
- For each outer measure μ^* on X there is a relatively natural σ-algebra $\mathcal{M}_{\mu^*}$ on X such that the restriction of μ^* to $\mathcal{M}_{\mu^*}$ is countably additive, and hence a measure.

Examples:

(a) Let X be an arbitrary set, and define μ^* on P(X) by $\mu^*(A) = 0$ if A = φ and $\mu^*(A) = 1$otherwise.Then μ^* is an outer measure.
(b) Let X be an arbitrary set, and define μ^* on P(X) by $\mu^*(A) = 0$if A is countable, and $\mu^*(A)= 1$if A is uncountable. Then μ^* is an outer measure.
(c) Let X be an infinite set, and define μ^* on P(X) by $\mu^*(A) = 0$ if A is finite, and $\mu^*(A) = 1$if A is infinite. Then μ^* fails to be countably subadditive and so is not an outer measure.
(d) Let $X = \{1,2\}$ then $P(X) = \{\varphi, X, \{1\}, \{2\}\}$. Define P(X) by $\mu^*(\varphi) = 0$, $\mu^*(\{1\}) = 5, \mu^*(\{2\}) = 7, \mu^*(X) = 8$ then μ^* is an outer measure.
(e) Let $X = \{1,2\}$ then $P(X) = \{\varphi, X, \{1\}, \{2\}\}$. Define P(X) by $\mu^*(\varphi) = 0$, $\mu^*(\{1\}) = 5, \mu^*(\{2\}) = 10, \mu^*(X) = 16$ then μ^* is not an outer measure, because not countably sub – additive.
i.e. $\mu^*(\{1\} \cup \{2\}) \nleq \mu^*(\{1\}) + \mu^*(\{2\})$
(f) For a set function $\mu^*: 2^\propto = P(X) \to [0, \infty]$, following functions are outer measures.

$$\mu^*(A) = |A| \,,\ \mu^*(A) = \begin{cases} 0 & if\ A = \varphi \\ 1 & if\ A \neq \varphi \end{cases} \,,\ \mu^*(A) = \begin{cases} 0 & if\ A\ is\ countable \\ 1 & if\ A\ is\ uncountable \end{cases}$$

Property: Sum of two outer measures is outer measure.

Proof: let *'f'* and *'g'* be two outer measures from $2^{\propto} \rightarrow [0, \infty]$ then define $(f+g): 2^{\propto} = P(X) \rightarrow [0, \infty]$ by $(f+g)(A) = f(A) + g(A)$

- Now if $A = \varphi$ then $(f+g)(\varphi) = f(\varphi) + g(\varphi) = 0 + 0 = 0$
  $\Rightarrow (f+g)(\varphi) = 0 \qquad \because f \text{ and } g$ are outer measure.
- Let $A_1, A_2 \in 2^{\propto} = P(X)$ such that $A_1 \subseteq A_2$ then
  $(f+g)(A_1) \leq (f+g)(A_2)$
- And let $A_1, A_2, \dots, A_i \in 2^{\propto} = P(X)$ and $A \subseteq \cup A_i$ then
  $(f+g)(A) = f(A) + g(A) \leq \sum_1^{\infty} f(A_i) + \sum_1^{\infty} g(A_i)$
  $(f+g)(A) = f(A) + g(A) = \sum_1^{\infty}[f(A_i) + g(A_i)] = \sum_1^{\infty}(f+g)A_i$
  $(f+g)(A) \leq \sum_1^{\infty}(f+g)A_i$
  Hence $f+g$ or Sum of two outer measures is outer measure.

Property: Difference of two outer measures needs not to be outer measure.

Proof: let $\mu : P(X) \rightarrow [0, +\infty]$ and $\mu' : P(X) \rightarrow [0, +\infty]$ be two outer measure defined respectively by

$$\mu(A) = \begin{cases} 0 & if\ A = \varphi \\ 1 & if\ A \neq \varphi \end{cases} \quad \mu'(A) = \begin{cases} 0 & if\ A\ is\ countable \\ 1 & if\ A\ is\ uncountable \end{cases}$$

Then define $\mu - \mu' : P(X) \rightarrow [0, +\infty]$ by $(\mu - \mu')(A) = \mu(A) - \mu'(A)$

Clearly $(\mu - \mu')(\varphi) = 0$ but if A is uncountable and $A = \varphi$ then

$(\mu - \mu')(A) = \mu(A) - \mu'(A) = 0 - 1$

$\Rightarrow (\mu - \mu')(A) = -1$

$\Rightarrow (\mu - \mu')(A) \notin [0, \infty]$

Thus $\mu - \mu' : P(X) \rightarrow [0, +\infty]$ is not an outer measure. Hence the result.

Property: Scalar multiplication of outer measures is an outer measure.

Proof: let $\mu : P(X) \to [0, +\infty]$be an outer measure and 'c' be a non – negative real number then define $c\mu : P(X) \to [0, +\infty]$ by $(c\mu)A = c\mu(A)$

- Now if $A = \varphi$ then $(c\mu)\varphi = c\mu(\varphi) = 0$
  $\Rightarrow (c\mu)\varphi = 0$
- Let $A_1, A_2 \in 2^{\propto} = P(X)$ such that $A_1 \subseteq A_2$ then
  $\mu(A_1) \le \mu(A_2) \Rightarrow c\mu(A_1) \le c\mu(A_2) \Rightarrow (c\mu)A_1 \le (c\mu)A_2$
- And let $A_1, A_2, \dots, A_i \in 2^{\propto} = P(X)$ and $A \subseteq \cup_1^{\infty} A_i$ then
  $(c\mu)A = c\mu(A) \le c\sum_1^{\infty} \mu(A_i) \le \sum_1^{\infty} c\mu(A_i)$
  $(c\mu)A \le \sum_1^{\infty} c\mu(A_i)$
  Hence $c\mu$ or Scalar multiplication of an outer measures is an outer measure.

Remark

Let $E \in P(X)$ then for any $A \in P(X)$ we have;

$(A \cap E) \cap (A \cap E^c) = \varphi$ and $(A \cap E) \cup (A \cap E^c) = A$

μ^* - Measurable Set: Let μ^* be an outer measure on P(X) we say that $E \in P(X)$ is μ^* - Measurable or additive if for all $A \in P(X)$ we have carethedory condition as follows;

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Where A is called testing set for E. Remember that the collection of all μ^* - Measurable sets is denoted by $\mathcal{m}(\mu^*)$

Remember:

- If $A = (A \cap E) \cup (A \cap E^c)$ then by sub – additivity of μ^* we have
  $\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$
  In order to prove carethedory condition i.e.
  $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$
  We need only to verify that $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$
- A μ^*-measurable subset of X is one that divides each subset of X in such a way that the sizes (as measured by μ^*) of the pieces add properly. A Lebesgue measurable subset of $\mathbb{R}$ or of $\mathbb{R}^d$ is of course one that is measurable with respect to Lebesgue outer measure.

Property: φ and X are μ^* - Measurable. Or $\varphi, X \in m(\mu^*)$.

Proof: By carethedory condition for any $A \in P(X)$

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ ………(i)

Let $E = \varphi$ in (i) $\mu^*(A) = \mu^*(A \cap \varphi) + \mu^*(A \cap \varphi^c)$

$\mu^*(A) = \mu^*(\varphi) + \mu^*(A \cap X) = 0 + \mu^*(A)$

$\mu^*(A) = \mu^*(A)$ implies $\varphi \in m(\mu^*)$

Now let $E = X$ in (i) $\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \cap X^c)$

$\mu^*(A) = \mu^*(A) + \mu^*(A \cap \varphi) = \mu^*(A) + \mu^*(\varphi) = \mu^*(A) + 0$

$\mu^*(A) = \mu^*(A)$ implies $X \in m(\mu^*)$

Thus φ and X are μ^* - Measurable.

Property: If E is μ^* - Measurable then E^c is μ^* - Measurable. Or if $E \in m(\mu^*)$ then $E^c \in m(\mu^*)$.

Proof: If $E \in m(\mu^*)$ then for all $A \in P(X)$ we have

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$

$\mu^*(A) = \mu^*(A \cap E^c) + \mu^*(A \cap (E^c)^c)$ by taking $E = E^c$

Implies $E^c \in m(\mu^*)$

Remark: if $E \notin m(\mu^*)$ then $E^c \notin m(\mu^*)$. i.e. if $E \notin m(\mu^*)$ then E^c is not Measurable. For example, consider $X = \{1,2\}$ then $P(X) = \{\varphi, X, \{1\}, \{2\}\}$. Define P(X) by μ* ($\varphi$) = 0 , $\mu^*(\{1\}) = 5, \mu^*(\{2\}) = 7, \mu^*(X) = 10$ then μ^* is not an outer measure, because not countably sub – additive.

i.e. $\mu^*(\{1\} \cup \{2\}) \nleq \mu^*(\{1\}) + \mu^*(\{2\})$

Take $A = X$ and $E = \{1\}$ then

$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap \{1\}) + \mu^*(A \cap \{2\}) = \mu^*(\{1\}) + \mu^*(\{2\})$

$\mu^*(A \cap E) + \mu^*(A \cap E^c) = 5 + 7 = 12 \neq \mu^*(X)$

If $\{1\}$ is not μ^* - Measurable then $\{1\}^c = \{2\}$ is also not μ^* - Measurable.

Similarly for $A = X$ and $E = \{2\}$

Lemma: Let $X \neq \varphi$ and $\mu^*: P(X) \to [0, \infty]$ be an outer measure on X. If $E_1, E_2 \in \mathcal{m}(\mu^*)$ then $E_1 \cup E_2 \in \mathcal{m}(\mu^*)$ for $E_1, E_2 \in P(X)$. i.e. The union of a finite collection of measurable sets is measurable.

Proof: Since $E_1 \in \mathcal{m}(\mu^*)$ then for a testing set $A \in P(X)$ we have

$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap {E_1}^c)$ ……….(i)

If we take a particular testing set $A \cap {E_1}^c \in P(X)$ for $E_2 \in \mathcal{m}(\mu^*)$ we have

$$\mu^*(A \cap {E_1}^c) = \mu^*\big((A \cap {E_1}^c) \cap E_2\big) + \mu^*\big((A \cap {E_1}^c) \cap {E_2}^c\big)$$

$$\mu^*(A \cap {E_1}^c) = \mu^*\big((A \cap {E_1}^c) \cap E_2\big) + \mu^*\big(A \cap ({E_1}^c \cap {E_2}^c)\big)$$

$\mu^*(A \cap {E_1}^c) = \mu^*\big((A \cap {E_1}^c) \cap E_2\big) + \mu^*(A \cap (E_1 \cup E_2)^c)$ ……….(ii)

Using (ii) in (i)

$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*\big((A \cap {E_1}^c) \cap E_2\big) + \mu^*(A \cap (E_1 \cup E_2)^c)$ ……….(iii)

To prove the required $E_1 \cup E_2 \in \mathcal{m}(\mu^*)$ consider first two terms of above

$$(A \cap E_1) \cup \big((A \cap {E_1}^c) \cap E_2\big) = (A \cap E_1) \cup \big(A \cap ({E_1}^c \cap E_2)\big)$$

$$(A \cap E_1) \cup \big((A \cap {E_1}^c) \cap E_2\big) = A \cap \big(E_1 \cup (E_2 \cap {E_1}^c)\big)$$

$$(A \cap E_1) \cup \big((A \cap {E_1}^c) \cap E_2\big) = A \cap \big(E_1 \cup (E_2 / E_1)\big)$$

$$(A \cap E_1) \cup \big((A \cap {E_1}^c) \cap E_2\big) = A \cap (E_1 \cup E_2)$$

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup \big((A \cap {E_1}^c) \cap E_2\big)$$

$$\Rightarrow \mu^*\big(A \cap (E_1 \cup E_2)\big) = \mu^*\Big((A \cap E_1) \cup \big((A \cap {E_1}^c) \cap E_2\big)\Big)$$

$$\Rightarrow \mu^*\big(A \cap (E_1 \cup E_2)\big) \leq \mu^*(A \cap E_1) + \mu^*\big((A \cap {E_1}^c) \cap E_2\big)$$

$\Rightarrow \mu^*(A \cap E_1) + \mu^*\big((A \cap {E_1}^c) \cap E_2\big) \geq \mu^*\big(A \cap (E_1 \cup E_2)\big)$ ……….(iv)

Using (iv) in (iii) $\mu^*(A) \geq \mu^*\big(A \cap (E_1 \cup E_2)\big) + \mu^*(A \cap (E_1 \cup E_2)^c)$

And $\mu^*(A) \leq \mu^*\big(A \cap (E_1 \cup E_2)\big) + \mu^*(A \cap (E_1 \cup E_2)^c)$

Hence $\mu^*(A) = \mu^*\big(A \cap (E_1 \cup E_2)\big) + \mu^*(A \cap (E_1 \cup E_2)^c)$

Implies $E_1 \cup E_2 \in \mathcal{m}(\mu^*)$

Remark:

- If $E_1, E_2, \dots, E_n \in m(\mu^*)$ then $\cup_1^n E_i \in m(\mu^*)$
- Every subset of a null set is μ^* - Measurable.
- Measure always positive.

Lemma: Let $X \neq \varphi$ and $\mu^*: P(X) \to [0, \infty]$ be an outer measure on X. If $E_1, E_2 \in m(\mu^*)$ then $E_1 \cap E_2 \in m(\mu^*)$ for $E_1, E_2 \in P(X)$.

Proof: Let $E_1, E_2 \in m(\mu^*)$ then ${E_1}^c, {E_2}^c \in m(\mu^*)$

$\Rightarrow {E_1}^c \cup {E_2}^c = (E_1 \cap E_2)^c \in m(\mu^*) \Rightarrow ((E_1 \cap E_2)^c)^c = E_1 \cap E_2 \in m(\mu^*)$

Lemma: Let $X \neq \varphi$ and $\mu^*: P(X) \to [0, \infty]$ be an outer measure on X. If $E \in P(X)$ such that $\mu^*(E) = 0$ then every $E_0 \subseteq E$ is μ^* - Measurable. In particular, E itself is μ^* - Measurable.

Or Prove that every subset of a null set is μ^* - Measurable. In particular, a null set is μ^* - Measurable.

Or Any set of outer measure zero is measurable. In particular, any countable set is measurable.

Proof: Let E is a null set then $\mu^*(E) = 0$. And $E_0 \subseteq E$ then by monotonicity of μ^* we have $\mu^*(E_0) \leq \mu^*(E) = 0$ implies $\mu^*(E_0) \leq 0$, but $\mu^*(E_0) \geq 0$

Thus $\mu^*(E_0) = 0$

For $A \in P(X)$ we have $A \cap E_0 \subseteq E_0$ and $A \cap {E_0}^c \subseteq A$

Then $\mu^*(A \cap E_0) \leq \mu^*(E_0)$ ……(i) and $\mu^*(A \cap {E_0}^c) \leq \mu^*(A)$ ………(ii)

Adding (i) and (ii) we get $\mu^*(A \cap E_0) + \mu^*(A \cap {E_0}^c) \leq \mu^*(E_0) + \mu^*(A)$

$\mu^*(A \cap E_0) + \mu^*(A \cap {E_0}^c) \leq \mu^*(A)$ since $\mu^*(E_0) = 0$

$\mu^*(A) \geq \mu^*(A \cap E_0) + \mu^*(A \cap {E_0}^c)$

But obviously we have $\mu^*(A) \leq \mu^*(A \cap E_0) + \mu^*(A \cap {E_0}^c)$

Then $\mu^*(A) = \mu^*(A \cap E_0) + \mu^*(A \cap {E_0}^c)$ Implies E_0 is μ^* - Measurable.

By the similar argument we can show that E is μ^* - Measurable.

i.e. $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Preposition (Countably Sub –Additive Property):

Outer measure is countably subadditive, that is, if $\{A_n\}_1^\infty$ is any countable collection of sets, disjoint or not, then

$\mu^*(\cup_1^\infty A_n) \leq \sum_1^\infty \mu^*(A_n)$

Proof:

Let $\{A_n\}_1^\infty$ be a sequence in $P(X)$, for which we have to show that;

$\mu^*(\cup_1^\infty A_n) \leq \sum_1^\infty \mu^*(A_n)$

Let $\{E_i{}'\}_1^\infty$ be a sequence in ε and cover of A_1

i.e. $A_1 \subseteq \cup_1^\infty E_i{}'$ then $\mu^*(A_1) \leq \mu^*(\cup_1^\infty E_i{}')$

Then by hypothesis $\mu^*(A_1) \leq \sum_1^\infty \rho(E_i{}')$ $\Rightarrow \sum_1^\infty \rho(E_i{}') \geq \mu^*(A_1)$

Let for any $\epsilon > 0$ we have $\sum_1^\infty \rho(E_i{}') \leq \mu^*(A_1) + \epsilon/2$

Similarly $\sum_1^\infty \rho\left(E_i^2\right) \leq \mu^*(A_2) + \epsilon/2^2$ continuingly $\sum_1^\infty \rho\left(E_i^k\right) \leq \mu^*(A_k) + \epsilon/2^k$

For this $A_k \subseteq \cup_1^\infty E_i^k$ implies $\cup_{n=1}^\infty A_n \subseteq \cup_{n=1}^\infty (\cup_{i=1}^\infty E_i^n)$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \leq \sum_1^\infty \rho(\cup_{i=1}^\infty E_i^n)$ ……..(i)

Since $\{E_i\}_1^\infty$ is a sequence for which $\rho(\cup_{i=1}^\infty E_i) \leq \sum_1^\infty \rho(E_i)$ then

$(i) \Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \leq \sum_1^\infty(\sum_1^\infty \rho(E_i)^n)$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \leq \sum_1^\infty(\mu^*(A_n) + \epsilon/2^n)$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \leq \sum_1^\infty \mu^*(A_n) + \sum_1^\infty \frac{\epsilon}{2^n}$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \leq \sum_1^\infty \mu^*(A_n) + \epsilon$

Since $\epsilon > 0$ was an arbitrary positive real number therefore inequality true for all $\epsilon > 0$ and we get

$\mu^*(\cup_{n=1}^\infty A_n) \leq \sum_1^\infty \mu^*(A_n)$ and μ^* is countably sub – additive.

Hence the result. i.e. $\mu^*(E)$ is an outer measure.

Lemma (Finitely Additive Property): Let $X \neq \varphi$ and $\mu^*: P(X) \to [0, \infty]$ be an outer measure on $P(X)$. If $E_1, E_2 \in m(\mu^*)$ and $E_1 \cap E_2 = \varphi$ then

$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$

Proof: Since $E_1 \in m(\mu^*)$ then for any $A \in P(X)$ we have

$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap {E_1}^c)$

In particular if $E_1 \cup E_2 = A$ then

$\mu^*(E_1 \cup E_2) = \mu^*\big((E_1 \cup E_2) \cap E_1\big) + \mu^*\big((E_1 \cup E_2) \cap {E_1}^c\big)$

$\mu^*(E_1 \cup E_2) = \mu^*\big((E_1 \cap E_1) \cup (E_1 \cap E_2)\big) + \mu^*\big((E_1 \cup E_2)/E_1\big)$

$\mu^*(E_1 \cup E_2) = \mu^*(E_1 \cup \varphi) + \mu^*(E_2) \qquad \because E_1 \cap E_2 = \varphi \;\; and \; (E_1 \cup E_2)/E_1 = E_2$

$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$

Lemma: Let $X \neq \varphi$ and $\mu^*: P(X) \to [0, \infty]$ be an outer measure on X. If $E_1, E_2 \in P(X)$ and $\mu^*(E_2) = 0$ then $\mu^*(E_1 \cup E_2) = \mu^*(E_1)$

Proof: By finitely sub - additive property we have,

$\mu^*(E_1 \cup E_2) \leq \mu^*(E_1) + \mu^*(E_2)$

$\mu^*(E_1 \cup E_2) \leq \mu^*(E_1)$ …….(i) $\qquad \because \mu^*(E_2) = 0$

Since $E_1 \subseteq E_1 \cup E_2$ then by monotonicity of μ^*

We have $\mu^*(E_1) \leq \mu^*(E_1 \cup E_2)$ …….(ii)

Combining (i) and (ii) $\quad \mu^*(E_1 \cup E_2) = \mu^*(E_1)$

Lemma: If $A, B \in m(\mu^*)$ and $B \subseteq A$ then $A/B \in m(\mu^*)$

Proof: Since $B \in m(\mu^*)$ then $B^c \in m(\mu^*)$ also then $A, B^c \in m(\mu^*)$

$\Rightarrow A \cap B^c \in m(\mu^*)$

$\Rightarrow A/\mathrm{B} \in m(\mu^*) \qquad \because A \cap B^c = A/\mathrm{B}$

Caratheodory Theorem:

Let μ∗ be an outer measure in X. Then, $m(\mu^*)$ is a σ–algebra, and μ∗ is σ–additive on $m(\mu^*)$.

Proof: We will split the reasoning into four steps.

Step – I: $m(\mu^*)$ is an algebra.

Proof: If $E \in m(\mu^*)$ then $E^c \in m(\mu^*)$

Then also $E, E^c \in m(\mu^*)$ and $E \cup E^c \in m(\mu^*)$

Implies $m(\mu^*)$ is an algebra, because it closed under compliment and finite union.

Step – II: $m(\mu^*)$ is a σ- Algebra.

We need to show that $m(\mu^*)$ is closed under compliment and closed under countable union.

P – I: Let $E \in m(\mu^*)$ then for all $A \in P(X)$ we have;

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap (E^c)^c) + \mu^*(A \cap E^c)$

$\mu^*(A) = \mu^*(A \cap E^c) + \mu^*(A \cap (E^c)^c)$

Implies $E^c \in m(\mu^*)$. Then $m(\mu^*)$ is closed under compliment.

P – II: Let $\{E_i\}_1^\infty$ be a sequence in $m(\mu^*)$ we have to prove $\cup_1^\infty E_i \in m(\mu^*)$

Since $m(\mu^*)$ is closed under finite union. i.e. $\cup_1^n E_i \in m(\mu^*)$

Then for $A \in P(X)$ we have $\mu^*(A) = \mu^*\big(A \cap (\cup_1^n E_i)\big) + \mu^*(A \cap (\cup_1^n E_i)^c)$

Applying limit approaches to infinity;

$\lim_{n\to\infty} \mu^*(A) = \lim_{n\to\infty} \mu^*\big(A \cap (\cup_1^n E_i)\big) + \lim_{n\to\infty} \mu^*(A \cap (\cup_1^n E_i)^c)$

$\mu^*(A) = \mu^*\big(A \cap (\cup_1^\infty E_i)\big) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$

Implies $\cup_1^\infty E_i \in m(\mu^*)$. Thus $m(\mu^*)$ is a σ- Algebra.

Step – III: $\boldsymbol{m}(\boldsymbol{\mu}^*)$ is a Additive.

Since $E_1 \in m(\mu^*)$ then for any $A \in P(X)$ we have

$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap {E_1}^c)$

In particular if $E_1 \cup E_2 = A$ then

$\mu^*(E_1 \cup E_2) = \mu^*\big((E_1 \cup E_2) \cap E_1\big) + \mu^*\big((E_1 \cup E_2) \cap {E_1}^c\big)$

$\mu^*(E_1 \cup E_2) = \mu^*\big((E_1 \cap E_1) \cup (E_1 \cap E_2)\big) + \mu^*\big((E_1 \cup E_2)/E_1\big)$

$\mu^*(E_1 \cup E_2) = \mu^*(E_1 \cup \varphi) + \mu^*(E_2) \qquad \because E_1 \cap E_2 = \varphi \;\; and \; (E_1 \cup E_2)/E_1 = E_2$

$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$

Step – IV: μ∗is σ–additive on $\boldsymbol{m}(\boldsymbol{\mu}^*)$

Since μ∗ is countably sub–additive, and additive, then by result "additive function is σ–additive if and only if it is countably sub - additive" gives the conclusion.

Additive and σ–additive functions

Let $\mathcal{A} \subset$ P(X) be an algebra. and μ : $\mathcal{A} \to [0, +\infty]$ be such that μ$(\varphi) = 0$.

- We say that μ is additive if, for any family $A_1, A_2, \ldots, A_n \in \mathcal{A}$ of mutually disjoint sets, we have

  $$\mu(\cup_1^n A_i) = \sum_1^n \mu(A_i)$$

- We say that μ is σ–additive if, for any sequence $(A_n) \subset \mathcal{A}$ of mutually disjoint sets such that $\cup_1^\infty A_i \in \mathcal{A}$ we have

  $$\mu(\cup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$$

Lemma: Let A be any set and $\{E_i\}_1^n$ be a finite disjoint collection of measureable sets then $\mu^*(A \cap \cup_1^n E_i) = \sum_1^n \mu^*(A \cap E_i)$ and particularly $\mu^*(\cup_1^n E_i) = \sum_1^n \mu^*(E_i)$

Or Let $X \neq \varphi$ and $\mu^*: P(X) \to [0, \infty]$ be an outer measure on X. Let $\{E_i\}_1^n$ be a disjoint sequence in $m(\mu^*)$ then for all $A \in P(X)$;

$\mu^*(A \cap \cup_1^n E_i) = \sum_1^n \mu^*(A \cap E_i)$

Proof: We prove it by induction method;

For n = 1, the result is true. i.e $\mu^*(A \cap E_1) = \mu^*(A \cap E_1)$

Suppose that result is true for n = k;

$\mu^*\left(A \cap \cup_1^k E_i\right) = \sum_1^k \mu^*(A \cap E_i)$

Now we check at n = k + 1; using the fact that $E_n = E_{k+1}$ is μ^* - Measurable.

$\mu^*\left(A \cap \cup_1^{k+1} E_i\right) = \mu^*\left(\left(A \cap \cup_1^{k+1} E_i\right) \cap E_{k+1}\right) + \mu^*\left(\left(A \cap \cup_1^{k+1} E_i\right) \cap E_{k+1}{}^c\right)$

$\mu^*\left(A \cap \cup_1^{k+1} E_i\right) = \mu^*\left(A \cap \left(\cup_1^{k+1} E_i \cap E_{k+1}\right)\right) + \mu^*\left(A \cap \left(\cup_1^{k+1} E_i \cap E_{k+1}{}^c\right)\right)$

$\mu^*\left(A \cap \cup_1^{k+1} E_i\right) = \mu^*(A \cap E_{k+1}) + \mu^*\left(A \cap \left(\cup_1^k E_i\right)\right)$ $\because \{E_i\}_1^n$ be a disjoint sequence.

$\mu^*\left(A \cap \cup_1^{k+1} E_i\right) = \mu^*(A \cap E_{k+1}) + \sum_1^k \mu^*(A \cap E_i)$

$\mu^*\left(A \cap \cup_1^{k+1} E_i\right) = \sum_1^{k+1} \mu^*(A \cap E_i)$

Induction is true for n = k + 1.Hence the result. i.e. $\mu^*(A \cap \cup_1^n E_i) = \sum_1^n \mu^*(A \cap E_i)$

And $\mu^*(\cup_1^n E_i) = \sum_1^n \mu^*(E_i)$ for $A = \varphi$

…………………………………………

Lemma: If μ^* is an outer measure, then $m(\mu^*)$ is an algebra.

Proof: If $E \in m(\mu^*)$ then $E^c \in m(\mu^*)$

Then also $E, E^c \in m(\mu^*)$ and $E \cup E^c \in m(\mu^*)$

Implies $m(\mu^*)$ is an algebra, because it closed under compliment and finite union.

Lemma: If μ^* is an outer measure, then $m(\mu^*)$ is a σ- Algebra.

Proof: We need to show that $m(\mu^*)$ is closed under compliment and closed under countable union.

P – I: Let $E \in m(\mu^*)$ then for all $A \in P(X)$ we have;

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap (E^c)^c) + \mu^*(A \cap E^c)$

$\mu^*(A) = \mu^*(A \cap E^c) + \mu^*(A \cap (E^c)^c)$

Implies $E^c \in m(\mu^*)$. Then $m(\mu^*)$ is closed under compliment.

P – II (1st method): The union of a countable collection of measurable sets is measurable.

Let $\{E_i\}_1^\infty$ be a sequence in $m(\mu^*)$ we have to prove $\cup_1^\infty E_i \in m(\mu^*)$

Since $m(\mu^*)$ is closed under finite union. i.e. $\cup_1^n E_i \in m(\mu^*)$

Then for $A \in P(X)$ we have $\quad \mu^*(A) = \mu^*\big(A \cap (\cup_1^n E_i)\big) + \mu^*(A \cap (\cup_1^n E_i)^c)$

Applying limit approaches to infinity;

$\lim_{n\to\infty} \mu^*(A) = \lim_{n\to\infty} \mu^*\big(A \cap (\cup_1^n E_i)\big) + \lim_{n\to\infty} \mu^*(A \cap (\cup_1^n E_i)^c)$

$\mu^*(A) = \mu^*\big(A \cap (\cup_1^\infty E_i)\big) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$

Implies $\cup_1^\infty E_i \in m(\mu^*)$. Thus $m(\mu^*)$ is a σ- Algebra.

P – II (2nd method): Let $\{E_i\}_1^\infty$ be a sequence in $m(\mu^*)$ then there exists a disjoint sequence $\{F_n\}_1^\infty$ in $m(\mu^*)$ such that $\cup_1^\infty F_n = \cup_1^\infty E_i$

Suppose that $\cup_1^\infty F_n = \cup_1^\infty E_i = E$ then $\cup_1^n F_i \subseteq E$ implies $E^c \subseteq (\cup_1^n F_i)^c$

Then for $A \in P(X)$ we have $\quad A \cap E^c \subseteq A \cap (\cup_1^n F_i)^c$

Then $\mu^*(A \cap E^c) \leq \mu^*(A \cap (\cup_1^n F_i)^c)$ ……..(i)

Since $\cup_1^n F_i$ is μ^* - Measurable, as $m(\mu^*)$, then for all $A \in P(X)$ we have;

$\mu^*(A) = \mu^*\big(A \cap (\cup_1^n F_i)\big) + \mu^*(A \cap (\cup_1^n F_i)^c)$

$\mu^*(A) \geq \mu^*\big(A \cap (\cup_1^n F_i)\big) + \mu^*(A \cap E^c) \qquad \because E^c \subseteq (\cup_1^n F_i)^c$

$\mu^*(A) \geq \mu^*\big(A \cap (\cup_1^n F_i)\big) + \mu^*(A \cap E^c) = \sum_1^n \mu^*(A \cap F_i) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$

$\mu^*(A) \geq \sum_1^n \mu^*(A \cap F_i) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$ ……..(ii)

Since L.H.S. of (ii) is independent of 'n' therefore R.H.S. is also independent of 'n' then (ii) implies

$\mu^*(A) \geq \sum_1^\infty \mu^*(A \cap F_i) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$

$\mu^*(A) \geq \mu^*\big(\cup_1^\infty (A \cap F_i)\big) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$ $\quad \because \mu^*(\cup_1^\infty E_i) \leq \sum_1^\infty \mu^*(E_i)$

$\mu^*(A) \geq \mu^*\big(A \cap (\cup_1^\infty F_i)\big) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$

$\mu^*(A) \geq \mu^*\big(A \cap (\cup_1^\infty E_i)\big) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$ $\quad \because \cup_1^\infty F_n = \cup_1^\infty E_i$

But $\mu^*(A) \leq \mu^*\big(A \cap (\cup_1^\infty E_i)\big) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$

Then $\mu^*(A) = \mu^*\big(A \cap (\cup_1^\infty E_i)\big) + \mu^*(A \cap (\cup_1^\infty E_i)^c)$

Implies $\cup_1^\infty E_i \in \mathcal{m}(\mu^*)$

Thus $\mathcal{m}(\mu^*)$ is a σ- Algebra.

Remark

- Since σ- Algebra is closed under countable intersection therefore $\mathcal{m}(\mu^*)$ is closed under countable intersection.
  i.e. if $E_1, E_2, \dots, E_n, .. \in \mathcal{m}(\mu^*)$ then $\cap_1^\infty E_i \in \mathcal{m}(\mu^*)$
- If μ^* is zero on X then subset of X is μ^* - Measurable.

Symmetric difference (Δ)

Symmetric difference is given as follows;

$F\Delta G = (F/G) \cup (G/F) = (F \cap G^c) \cup (F^c \cap G) = F \cup G/F \cap G$

Explanation:

Let $F, G \subseteq X$ then symmetric difference will be as follows;

$F\Delta G = (F/G) \cup (G/F) = (F \cap G^c) \cup (F^c \cap G)$

$F\Delta G = \big(F \cup (F^c \cap G)\big) \cap \big(G^c \cup (F^c \cap G)\big)$

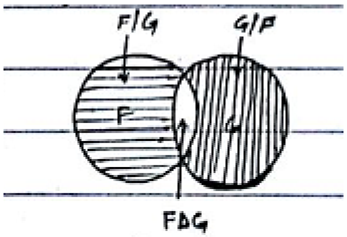
$F\Delta G = \big((F \cup G) \cap (F \cup F^c)\big) \cap \big((G^c \cap G) \cap (F^c \cup G^c)\big)$

$F\Delta G = \big((F \cup G) \cap X\big) \cap (X \cap (F \cap G)^c) = (F \cup G) \cap (F \cap G)^c$

$F\Delta G = F \cup G/F \cap G$

Theorem: If $F \in m(\mu^*)$ and $\mu^*(F\Delta G) = 0$ then $G \in m(\mu^*)$.

Proof:(1st method)



Since $\mu^*(F\Delta G) = 0$ and $F/G, G/F \subseteq F\Delta G$

Implies $F/G, G/F \in m(\mu^*)$ because every subset of null set is μ^*-measurable

$\Rightarrow (F/G)^c, (G/F)^c \in m(\mu^*)$

Now $F \cap G = F \cap (F/G)^c$ ……….(i)

And since $F, (F/G)^c \in m(\mu^*)$ implies $F \cap (F/G)^c \in m(\mu^*)$

$\Rightarrow F \cap G \in m(\mu^*)$ using (i)

From figure $G = (F \cap G) \cup (G/F)$ ……….(ii)

And since $F \cap G, G/F \in m(\mu^*)$ implies $(F \cap G) \cup (G/F) \in m(\mu^*)$

$\Rightarrow G \in m(\mu^*)$ using (ii)

Proof:(2nd method) Since $\mu^*(F\Delta G) = 0$ and $F/G, G/F \subseteq F\Delta G$

Implies $F/G, G/F \in m(\mu^*)$ because every subset of null set is μ^*-measurable

$\Rightarrow (F/G)^c, (G/F)^c \in m(\mu^*)$

Now $F \cap G = F \cap (F/G)^c$ be intersection of two μ^*-measurable sets is μ^*-measurable then $G = (F \cap G) \cup (G/F)$ being union of two μ^*-measurable is μ^*-measurable.

Theorem: If $E, F \in m(\mu)$ and $\mu(E\Delta F) = 0$ then $\mu(E) = \mu(F)$.

Proof: Since $\mu(E\Delta F) = \mu(E/F) + \mu(F/E)$ and $\mu(E\Delta F) = 0$

We have $\mu(E/F) = \mu(F/E) = 0$ Also writing $E = (E/F) \cup (E \cap F)$ and $F = (F/E) \cup (F \cap E)$ We have $\mu(E) = \mu(E/F) + \mu(E \cap F)$
and $\mu(F) = \mu(F/E) + \mu(F \cap E)$
And hence $\mu(E) = \mu(E \cap F) = \mu(F)$

Theorem:

Let $\varepsilon \subseteq P(X)$ such that $\varphi, X \in \varepsilon$ and $\rho: \varepsilon \to [0,\infty]$ given by $\rho(\varphi) = 0$ and ρ is countably sub – additive i.e. $\rho(E_i \cup_1^\infty) \leq \sum_1^\infty \rho(E_i)$ then for any $E \in P(X)$;

$\mu^*(E) = inf\{\sum_1^\infty \rho(E_i) : E \subseteq \cup_1^\infty E_i;\ E_i \in \varepsilon\}$ is an outer measure.

Proof: To prove μ^* is an outer measure we will prove $\mu^*(\varphi) = 0$, monotone property as well as countable sub – additive property.

P – I: $\boldsymbol{\mu^*(\varphi) = 0}$: Since $\varphi \subseteq \cup_1^\infty \varphi$ and $\rho(\varphi) = 0$ therefore $\sum_1^\infty \rho(\varphi_i) = 0$, this implies $\mu^*(\varphi) = inf\{\sum_1^\infty \rho(\varphi_i) : \varphi \subseteq \cup_1^\infty \varphi_i;\ \varphi_i \in \varepsilon\} \Rightarrow \mu^*(\varphi) = 0$

P – II: Monotone Property:

Suppose $A, B \in P(X)$ and $A \subseteq B$ then every covering sequence for B is covering sequence for A but every covering sequence for A needs not to be covering sequence for B

Then $\{\sum_1^\infty \rho(E_i) : B \subseteq \cup_1^\infty E_i;\ E_i \in \varepsilon\} \subseteq \{\sum_1^\infty \rho(F_i) : A \subseteq \cup_1^\infty F_i;\ F_i \in \varepsilon\}$

Since every covering sequence for B is covering sequence for A therefore;

$inf\{\sum_1^\infty \rho(F_i) : A \subseteq \cup_1^\infty F_i;\ F_i \in \varepsilon\} \leq inf\{\sum_1^\infty \rho(E_i) : B \subseteq \cup_1^\infty E_i;\ E_i \in \varepsilon\}$

Implies $\mu^*(A) \leq \mu^*(B)$ $\quad\quad \because A \subseteq B \therefore inf(B) \leq inf(A)$

P – III: Countably Sub - Additive Property:

Let $\{A_n\}_1^\infty$ be a sequence in $P(X)$, for which we have to show that;

$\mu^*(\cup_1^\infty A_n) \leq \sum_1^\infty \mu^*(A_n)$

Let $\{E_i{}'\}_1^\infty$ be a sequence in ε and cover of A_1

i.e. $A_1 \subseteq \cup_1^\infty E_i{}'$ then $\mu^*(A_1) \leq \mu^*(\cup_1^\infty E_i{}')$

Then by hypothesis $\mu^*(A_1) \leq \sum_1^\infty \rho(E_i{}')$ $\quad\quad \Rightarrow \sum_1^\infty \rho(E_i{}') \geq \mu^*(A_1)$

Let for any $\epsilon > 0$ we have $\sum_1^\infty \rho(E_i{}') \leq \mu^*(A_1) + \epsilon/2$

Similarly $\sum_1^\infty \rho\left(E_i^2\right) \leq \mu^*(A_2) + \epsilon/2^2$ continuingly $\sum_1^\infty \rho\left(E_i^k\right) \leq \mu^*(A_k) + \epsilon/2^k$

For this $A_k \subseteq \cup_1^\infty E_i^k$ implies $\cup_{n=1}^\infty A_n \subseteq \cup_{n=1}^\infty (\cup_{i=1}^\infty E_i^n)$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \leq \sum_1^\infty \rho(\cup_{i=1}^\infty E_i^n)$ ……..(i)

Since $\{E_i\}_1^\infty$ is a sequence for which $\rho(\cup_{i=1}^\infty E_i) \le \sum_1^\infty \rho(E_i)$ then

$(i) \Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \le \sum_1^\infty(\sum_1^\infty \rho(E_i)^n)$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \le \sum_1^\infty(\mu^*(A_n) + \epsilon/2^n)$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \le \sum_1^\infty \mu^*(A_n) + \sum_1^\infty \frac{\epsilon}{2^n}$

$\Rightarrow \mu^*(\cup_{n=1}^\infty A_n) \le \sum_1^\infty \mu^*(A_n) + \epsilon$

Since $\epsilon > 0$ was an arbitrary positive real number therefore inequality true for all $\epsilon > 0$ and we get

$\mu^*(\cup_{n=1}^\infty A_n) \le \sum_1^\infty \mu^*(A_n)$ and μ^* is countably sub – additive.

Hence the result. i.e. $\mu^*(E)$ is an outer measure.

Theorem: Let μ^* be an outer measure on X and $m(\mu^*)$ be the collection of all μ^* -measurable subsets. Prove that μ^* when restricted to $m(\mu^*)$ is a measure.

Furthermore $(X, m(\mu^*), \mu)$ is complete measure space.

Proof: Since $\mu^*: P(X) \to [0,\infty]$ is countably sub – additive so its restriction $\mu^* \mid_{m(\mu^*)}: m(\mu^*) \to [0,\infty]$ is also countably sub – additive. We are to show that $\mu^* \mid_{m(\mu^*)}: m(\mu^*) \to [0,\infty]$ is a measure on $m(\mu^*)$.

I: Since $\mu^*(\varphi) = 0$ therefore $\mu^* \mid_{m(\mu^*)}(\varphi) = 0$

II: For countably additive suppose $\{E_i\}_1^\infty$ be a sequence in $m(\mu^*)$ therefore $\{E_i\}_1^\infty$ is a disjoint sequence in $m(\mu^*)$ and

$\mu^*(\cup_1^\infty E_i) \le \sum_1^\infty \mu^*(E_i)$ $\quad \because \mu^*$ is an outer measure

$\Rightarrow \mu^* \mid_{m(\mu^*)}(\cup_1^\infty E_i) \le \sum_1^\infty \mu^* \mid_{m(\mu^*)}(E_i)$ ………(i)

Now for $n \epsilon N$ we have $\cup_1^n E_i \subseteq \cup_1^\infty E_i$

$\Rightarrow \mu^* \mid_{m(\mu^*)}(\cup_1^n E_i) \le \mu^* \mid_{m(\mu^*)}(\cup_1^\infty E_i)$

$\because$ If $\{E_i\}_1^n \epsilon m(\mu^*)$ and disjoint then $\mu^* \mid_{m(\mu^*)}(\cup_1^n E_i) = \sum_1^n \mu^* \mid_{m(\mu^*)}(E_i)$ therefore

$\Rightarrow \sum_1^n \mu^* \mid_{m(\mu^*)}(E_i) \le \mu^* \mid_{m(\mu^*)}(\cup_1^\infty E_i)$

$\Rightarrow \mu^* \mid_{m(\mu^*)}(\cup_1^\infty E_i) \geq \sum_1^n \mu^* \mid_{m(\mu^*)}(E_i)$

Since this is true for all $n \epsilon N$ therefore we have

$\Rightarrow \mu^* \mid_{m(\mu^*)}(\cup_1^\infty E_i) \geq \sum_1^\infty \mu^* \mid_{m(\mu^*)}(E_i)$ ....(ii)

Combining (i) and (ii) $\mu^* \mid_{m(\mu^*)}(\cup_1^\infty E_i) = \sum_1^\infty \mu^* \mid_{m(\mu^*)}(E_i)$

$\mu^* \mid_{m(\mu^*)}$ is countably additive.

Hence $\mu^* \mid_{m(\mu^*)}$ is a measure on $m(\mu^*)$

Now let $E \epsilon m(\mu^*)$ be a null set then $\mu^*(E) = 0$ implies $\mu^* \mid_{m(\mu^*)}(E) = 0$ then every subset of E is μ^* - measurable. Implies every subset of E is a member of $m(\mu^*)$ and hence $(X, m(\mu^*), \mu)$ is complete measure space.

Interval

An interval is the set of real numbers. e.g. $I = [a,b] = \{x: x \in \mathbb{R} \wedge a \leq x \leq b\}$ and length of interval $I = l(I) = b - a$, also keep in mind $[a,b], (a,b)$ have same length.

Notations: Let ℝ is the set of real numbers then;

- τ_o = Collection of φ and all open intervals on ℝ.
- τ_c = Collection of φ and all close intervals on ℝ.
- τ_{oc} = Collection of φ and all open close intervals on ℝ. i.e. $(a,b]$
- τ_{co} = Collection of φ and all close open intervals on ℝ. i.e. $[a,b)$
- $\tau = \tau_o \cup \tau_{oc} \cup \tau_{co} \cup \tau_c$ = Collection of φ and all intervals on ℝ.

Remember: Let ℝ is the set of real numbers then;

- $[a,\infty] = [a,\infty)$ and $(-\infty,b] = [-\infty,b]$
- For extended real valued function $l: \tau \to [0,\infty]$ and $\forall I \in \tau$ we have $l(I) = b - a$ and $l(\varphi) = 0$
- For an arbitrary disjoint sequence $\{I_n\}_1^\infty$ in τ we have

  $$l(\cup_1^\infty I_n) = \sum_1^\infty l(I_n)$$

Theorem: Open interval (a,∞) is $\mu^* -$ measurable. We may use jut measurable.

Or A monotone function that is defined on an interval is measurable.

Or Every Borel set in $\mathbb{R}$ is measurable.

Proof:

Actually we have to prove <u>every interval is measurable</u>. Using the fact $m(\mu^*)$ is $\sigma -$Algebra.

Let A be any set of real numbers then $E = (a,\infty), E^c = \mathbb{R} - (a,\infty) = (-\infty, a]$

For $\mu^* -$ measurable set $E = (a,\infty)$ we have to prove

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$

$\mu^*(A) = \mu^*\big(A \cap (a,\infty)\big) + \mu^*(A \cap (-\infty, a])$

Suppose $A_1 = A \cap (a,\infty)$ and $A_2 = A \cap (-\infty, a]$ then

$\mu^*(A) = \mu^*(A_1) + \mu^*(A_2)$

Obviously true that $\mu^*(A) \leq \mu^*(A_1) + \mu^*(A_2)$ ……………(i)

Therefore we are to show that $\mu^*(A) \geq \mu^*(A_1) + \mu^*(A_2)$

Now if $\mu^*(A) = \infty$ then there is nothing to prove.

But if $\mu^*(A) < \infty$ then for any $\epsilon > 0$ there exists a countable collection $\{I_n\}_1^\infty$ of open intervals such that

$\sum_1^\infty l(I_n) \leq \mu^*(A) + \epsilon$ ……………(ii) $\because \mu^*(A) \leq \sum_1^\infty l(I_n)$

Let $I'_n = I_n \cap (a,\infty)$ and $I''_n = I_n \cap (-\infty, a]$ then I'_n, I''_n are intervals and $I_n = I'_n \cup I''_n$

$\Rightarrow \sum_1^\infty l(I_n) = \sum_1^\infty l(I'_n) + \sum_1^\infty l(I''_n)$ ……………(iii)

Now $A_1 \subseteq \cup_1^\infty (I'_n)$ and $A_2 \subseteq \cup_1^\infty (I''_n)$

$\Rightarrow \mu^*(A_1) \leq \mu^*\big(\cup_1^\infty (I'_n)\big) = \sum_1^\infty l(I'_n)$

$\Rightarrow \mu^*(A_1) \leq \sum_1^\infty l(I'_n)$ ……………(a)

Similarly $\mu^*(A_2) \leq \sum_1^\infty l(I''_n)$ ……………(b)

From (ii) $\sum_1^\infty l(I_n) \le \mu^*(A) + \epsilon$

$\Rightarrow \mu^*(A) + \epsilon \ge \sum_1^\infty l(I_n)$

$\Rightarrow \mu^*(A) + \epsilon \ge \sum_1^\infty l(I'_n) + \sum_1^\infty l(I''_n)$ using (iii)

$\Rightarrow \mu^*(A) + \epsilon \ge \mu^*(A_1) + \mu^*(A_2)$ using (a) and (b)

Since $\epsilon > 0$ was an arbitrary, so $\epsilon \to 0$

$\Rightarrow \mu^*(A) \ge \mu^*(A_1) + \mu^*(A_2)$ ……………(iv)

From (i) and (iv) we get $\mu^*(A) = \mu^*(A_1) + \mu^*(A_2)$

$\Rightarrow \mu^*(A) = \mu^*\big(A \cap (a,\infty)\big) + \mu^*(A \cap (-\infty, a])$

Hence Open interval (a,∞) is $\mu^* -$ measurable. We may use jut measurable.

Hence in view of above all discussion, the σ −Algebra $\mathcal{m}(\mu^*)$ contains all the open sets in $\mathbb{R}$. Since $\mathfrak{B}$ is the smallest σ −Algebra containing all the open sets, we conclude that Borel set in $\mathbb{R}$ is measurable.

Lebesgue Outer measure

The set function $\mu_L^*: P(\mathbb{R}) \to [0,\infty]$ defined by for all $E \in P(\mathbb{R})$

$\mu_L^*(E) = inf\{\sum_1^\infty l(I_n) : E \subseteq \cup_1^\infty I_n;\ I_n \in \tau_o\}$

Is called Lebesgue outer measure on $P(\mathbb{R})$ where $l: \tau \to [0,\infty]$ such that $l(\varphi) = 0$ and $l(I) = b - a$

Lebesgue Sigma Algebra

The collection $\mathcal{m}(\mu_L^*)$ of all $\mu_L^* -$ measureable sets is denoted by $\mathcal{m}_L$ is called the Lebesgue $\sigma -$ Algebra. Member of $\mathcal{m}_L$ are called $\mathcal{m}_L -$ measureable or Lebesgue measurable set s, the pair $(\mathbb{R}, \mathcal{m}_L)$ is called Lebesgue measurable space and the triplet $(\mathbb{R}, \mathcal{m}_L, \mu_L)$ is called Lebesgue measure space, where μ_L is measure on $\mathcal{m}_L$.

Remark:

$\mathcal{m}(\mu^*)$ in general is $\sigma -$ Algebra, so $\mathcal{m}(\mu_L^*)$ is $\sigma -$ Algebra in particular case.

Lemma: Singleton are null sets in Lebesgue measure space.

Or for every $x\epsilon\mathbb{R}$ we have $\mu_L^*\{x\} = 0$ and $x\epsilon\mathcal{m}_L$

Or Prove that Lebesgue outer measure of singleton sets is zero i.e. $\mu_L^*\{x\} = 0$

Proof:

Let $x\epsilon\mathbb{R}$ then for all $\epsilon > 0$ we have $(x-\in, x+\in) \in \tau_o$ so that $(x-\in, x+\in), \varphi, \varphi, \ldots$ is an open cover for $\{x\}$ then $\mu_L^*\{x\} \leq l(x-\in, x+\in) + l(\varphi) + l(\varphi) + \cdots = 2 \in$

$\because l(x-\in, x+\in) = 2 \in$ and $l(\varphi) = 0$

$\Rightarrow \mu_L^*\{x\} \leq 2 \in$ for all $\epsilon > 0$

Since $\in$ was an arbitrary therefore we have $\mu_L^*\{x\} = 0$

$\Rightarrow x\epsilon\mathcal{m}_L$ Since if $\mu_L^*(E) = 0$ then $E\epsilon\mathcal{m}_L$

Lemma: Prove that every countable subset of $\mathbb{R}$ is a null set in Lebesgue measure space. i.e. $(\mathbb{R}, \mathcal{m}_L, \mu_L)$

Proof: Let E be a countable subset of $\mathbb{R}$ then E is countable union of singletons. i.e. $E = \cup_{x\in E} \{x\}$.

Then $\mu_L(E) = \mu_L(\cup_{x\in E} \{x\}) = \sum_{x\in E} \mu_L\{x\} = 0$

$\Rightarrow \mu_L(E) = 0$ Since $\mu_L\{x\} = 0 \;\; \forall x \in \mathbb{R}$

Question: Prove that the set of rational numbers Q is null set and $Q\epsilon\mathcal{m}_L$

Solution: Since Q is countable subset of $\mathbb{R}$ then Q is countable union of singletons. i.e. $Q = \cup_{x\in Q} \{x\}$.

Then $\mu_L(Q) = \mu_L\big(\cup_{x\in Q} \{x\}\big) = \sum_{x\in Q} \mu_L\{x\} = 0$

$\Rightarrow \mu_L(Q) = 0$ i.e. Q is null set Since $\mu_L\{x\} = 0 \;\; \forall x \in \mathbb{R}$

$\Rightarrow Q\epsilon\mathcal{m}_L$

Question: Prove that the set of irrational numbers $\mu_L(Q' = Q^c) = \infty$ but $Q' \epsilon \mathcal{m}_L$

Solution: Since $\mathbb{R} = Q \cup Q'$ and $Q' = \mathbb{R} - Q = \mathbb{R}/Q$

Then $\mu_L(Q') = \mu_L(\mathbb{R}/Q) = \mu_L(\mathbb{R}) - \mu_L(Q) = \infty - 0 \Rightarrow \mu_L(Q') = \infty$

Since $\mathbb{R} \epsilon \mathcal{m}_L$ and $Q \epsilon \mathcal{m}_L$ then $\mathbb{R}/Q \epsilon \mathcal{m}_L$ as $\mathcal{m}_L$ is sigma algebra.

$\Rightarrow Q' \epsilon \mathcal{m}_L$

Dense Subset of X: Let (X, τ) be a topological space a subset E of X is called dense in X if for all open sets O in X we have $O \cap E \neq \varphi$ or $\bar{E} = X$

Preposition: If E is a null set in $(\mathbb{R}, \mathcal{m}_L, \mu_L)$ then E^c is dense in $\mathbb{R}$.

Proof: Suppose $E \epsilon \mathcal{m}_L$ is a null set in $(\mathbb{R}, \mathcal{m}_L, \mu_L)$ i.e. $\mu_L^*(E) = 0$ and let $I \subseteq E$ is an open interval then $\mu_L^*(I) \leq \mu_L^*(E) = 0 \Rightarrow \mu_L^*(I) = 0$

But in fact, $\mu_L^*(I) > 0$

Hence $I \nsubseteq E$ implies $E^c \cap I \neq \varphi$. Thus E^c is dense in $\mathbb{R}$.

Lemma: Prove that Lebesgue outer measure of an interval is its length. i.e. $\mu_L^*(I) = l(I)$ where I is an interval in $\mathbb{R}$.

Proof: Case – I: If I is a finite closed interval i.e. $I = [a, b]$ where $a, b \in \mathbb{R}$ such that $a < b$ for every $\in > 0$ we have $[a, b] \subseteq (a - \in, b + \in)$ so that $(a - \in, b + \in), \varphi, \varphi, \ldots$ is covering sequence of open intervals that cover $[a, b]$ then by definition of μ_L^* we have $\mu_L^*([a, b]) \leq \sum_1^\infty l(I_i)$

$\mu_L^*([a, b]) \leq l\big((a - \in, b + \in)\big) + l(\varphi) + l(\varphi) + \cdots$

$\mu_L^*([a, b]) \leq (\mathrm{b} - \mathrm{a}) + 2 \in + 0 + 0 + \cdots$

Since this is true for all $\in > 0$ therefore $\mu_L^*([a, b]) \leq (\mathrm{b} - \mathrm{a}) = l(I)$

$\mu_L^*([a, b]) \leq l(I)$ ……..(i)

Now we will prove the reverse inequality $\mu_L^*([a, b]) \geq l(I)$ but this is equivalent to $\sum_1^\infty l(I_i) \geq l(I)$ ……..(ii)

For any countable cover $\{I_i\}_1^\infty$ in τ_o of the I i.e. $I \subseteq \cup_1^\infty I_i$ it its sufficient to prove inequality (ii) by using Hein Borel Theorem, according to which "every countable cover of closed interval can be reduced to finite sub – cover"

For a finite sub – cover i.e. if $\{I_i\}_1^n$ is the finite sub – cover of the interval $[a, b]$ then we are to prove $\sum_1^n l(I_i) \geq l(I)$ ……..(iii)

Since $I \subseteq \cup_1^n I_i$ and as for $a \in \sum_1^n l(I_i)$ there exists an open interval $(a_1, b_1) \in \{I_i\}_1^n$ so $a \in (a_1, b_1)$ then $a_1 < a < b_1 < b$

If $b_1 < b$ then $b_1 \in [a, b]$ but $b_1 \notin (a_1, b_1)$ then there exists an open interval $(a_2, b_2) \in \{I_i\}_1^n$ such that $b_1 \in (a_2, b_2)$ then $a_2 < b_1 < b_2$

Preceding in this manner, we get an open interval $(a_k, b_k) \in \{I_i\}_1^n$ such that $a_k < b < b_k$ i.e. $b \in (a_k, b_k)$, so we obtain a sub – sequence of $\{I_i\}_1^n$ that will be $\{(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)\} \subseteq \{I_i\}_1^n$ therefore $\sum_1^n l(I_i) \geq \sum_1^k l(a_i, b_i)$

$\sum_1^n l(I_i) \geq l(a_1, b_1) + l(a_2, b_2) + \cdots + l(a_k, b_k)$

$\sum_1^n l(I_i) \geq b_1 - a_1 + b_2 - a_2 + \cdots + b_k - a_k$

$\sum_1^n l(I_i) \geq (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1)$

$\sum_1^n l(I_i) > b_k - a_1 \geq b - a$

$\sum_1^n l(I_i) \geq l(I)$

$\mu_L^*([a, b]) \geq l(I)$ ……..(iv)

$a_i \leq b_{i-1}$
$a_i - b_{i-1} \leq 0$
$b_k \geq b, a_1 \leq a$
$b_k \geq b, -a_1 \geq -a$
$b_k - a_1 \geq b - a$

From (i) and (iv) $\mu_L^*([a, b]) = l(I)$

i.e. $\mu_L^*(I) = l(I)$

Case – II: If $I = (a, b)$ then $(a, b) \subseteq [a, b]$

Then by monotone property $\mu_L^*\big((a, b)\big) \leq \mu_L^*([a, b]) = l([a, b]) = b - a$

$\mu_L^*\big((a, b)\big) \leq b - a$ ……..(v)

For $\in > 0$ we have $\mu_L^*\big((a, b)\big) \geq b - a - \in$

Since $\in > 0$ was an arbitrary therefore $\mu_L^*\big((a, b)\big) \geq b - a$ ……..(vi)

Combining (v) and (vi) $\mu_L^*\big((a, b)\big) = b - a$

Thus $\mu_L^*(I) = l(I)$

Case – III: If $I = (a, b]$ then $(a, b] = (a, b) \cup \{b\}$

Then $\mu_L^*\big((a, b]\big) = \mu_L^*\big((a, b)\big) + \mu_L^*(\{b\}) = b - a + 0 \qquad \because \mu_L^*(\{b\}) = 0$

$\mu_L^*\big((a, b]\big) = b - a$. Hence $\mu_L^*(I) = l(I)$

Case – IV: If $I = [a, b)$ then $[a, b) = \{a\} \cup (a, b)$

Then $\mu_L^*([a, b)) = \mu_L^*(\{a\}) + \mu_L^*\big((a, b)\big) = 0 + b - a \qquad \because \mu_L^*(\{a\}) = 0$

$\mu_L^*([a, b)) = b - a$. Hence $\mu_L^*(I) = l(I)$

Case – V: If $I = (a, \infty)$ then $(a, n) \subseteq (a, \infty)$

Then $\mu_L^*\big((a, n)\big) \leq \mu_L^*\big((a, \infty)\big) \Rightarrow n - a \leq \mu_L^*\big((a, \infty)\big)$

Since this hold for all $n \epsilon N$ we must have $\mu_L^*\big((a, \infty)\big) = \infty = l\big((a, \infty)\big)$

Thus $\mu_L^*(I) = l(I)$

Case – VI: If $I = (-\infty, b)$ then $(-n, b) \subseteq (-\infty, b)$

Then $\mu_L^*\big((-n, b)\big) \leq \mu_L^*\big((-\infty, b)\big) \Rightarrow b - (-n) \leq \mu_L^*\big((-\infty, b)\big)$

Since this hold for all $n \epsilon N$ we must have $\mu_L^*\big((-\infty, b)\big) = \infty = l\big((-\infty, b)\big)$

Thus $\mu_L^*(I) = l(I)$

Theorem:

Prove that every Borel set is Lebesgue measurable. Or $\mu_L -$ measurable.

Or Prove that $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{m}_L = \mathcal{m}(\mu_L^*)$

Proof: As every interval in $\mathbb{R}$ is $\mathcal{m}_L -$ measurable. And since every open set in $\mathbb{R}$ is countable union of open intervals in $\mathbb{R}$ therefore it is member of $\mathcal{m}_L$. If $\mathfrak{B}$ be a collection of open sets in $\mathbb{R}$ then $\mathfrak{B} \in \mathcal{m}_L$

Implies $\sigma(\mathfrak{B}) \subseteq \sigma(\mathcal{m}_L) = \mathcal{m}_L$

i.e. $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{m}_L = \mathcal{m}(\mu_L^*)$

Remark: The condition $\mu_L^*(A) = \mu_L^*(A \cap E) + \mu_L^*(A \cap E^c) \;\; \forall A \in P(\mathbb{R})$ is equivalent to $\mu_L^*(I) = \mu_L^*(I \cap E) + \mu_L^*(I \cap E^c) \;\; \forall I \in \tau_o$

Lemma:

Prove that every interval in $\mathbb{R}$ is Lebesgue measurable. Or $\mu_L^* -$ measurable.

Or Prove that $\tau \subseteq \mathcal{m}_L = \mathcal{m}(\mu_L^*)$

Proof: Note that a subset E of $\mathbb{R}$ is $\mu_L^* -$ measurable, if for all $I \in \tau_o$ we have $\mu_L^*(I) = \mu_L^*(I \cap E) + \mu_L^*(I \cap E^c)$

Case – I: If $I = (a, \infty) \in \tau_o$ & $a \in \mathbb{R}$ then $I = I \cap \mathbb{R}$

$I = I \cap [(a, \infty) \cup (a, \infty)^c]$ $\quad \because X = A \cup A^c$ so $\mathbb{R} = (a, \infty) \cup (a, \infty)^c$

$I = [I \cap (a, \infty)] \cup [I \cap (a, \infty)^c]$

Since $I \cap (a, \infty)$ and $I \cap (a, \infty)^c$ are disjoint so that;

$l(I) = l\big(I \cap (a, \infty)\big) + l(I \cap (a, \infty)^c)$

$\mu_L^*(I) = \mu_L^*\big(I \cap (a, \infty)\big) + \mu_L^*(I \cap (a, \infty)^c)$ $\quad \because \mu_L^*(I) = l(I)$

Implies $\quad (a, \infty) \in \mathcal{m}_L$

Similarly $\quad (-\infty, b) \in \mathcal{m}_L$

Case – II: If $I = (a, b) \in \tau_o$ then $(a, b) = (-\infty, b) \cap (a, \infty) \in \mathcal{m}_L$

Implies $\quad (a, b) \in \mathcal{m}_L$ $\quad \because \mathcal{m}_L$ is $\sigma -$algebra

Case – III: If $I = (-\infty, \infty) \in \tau_o$ then $(-\infty, \infty) = (-\infty, b) \cup (a, \infty) = \mathbb{R} \in \mathcal{m}_L$

Implies $\quad (-\infty, \infty) \in \mathcal{m}_L$ $\quad \because \mathcal{m}_L$ is $\sigma -$algebra

Case – IV: If $I = [a, b]$ then $[a, b] = \{a\} \cup (a, b) \cup \{b\} \in \mathcal{m}_L$

Implies $\quad [a, b] \in \mathcal{m}_L$ $\quad \because \mathcal{m}_L$ is $\sigma -$algebra

Case – V: If $I = [a, b)$ then $[a, b) = \{a\} \cup (a, b) \in \mathcal{m}_L$

Implies $\quad [a, b) \in \mathcal{m}_L$ $\quad \because \mathcal{m}_L$ is $\sigma -$algebra

Case – VI: If $I = (a, b]$ then $(a, b] = (a, b) \cup \{b\} \in \mathcal{m}_L$

Implies $\quad (a, b] \in \mathcal{m}_L$ $\quad \because \mathcal{m}_L$ is $\sigma -$algebra

Hence every interval in $\mathbb{R}$ is $\mu_L^* -$ measurable.

Theorem: Prove that the Lebesgue measure space $(\mathbb{R}, \mathcal{m}_L, \mu_L)$ is σ −finite but not finite.

Proof: Since $\mathbb{R} = (-\infty, \infty)$ therefore $\mu_L(\mathbb{R}) = l(\mathbb{R}) = \infty$,

so $(\mathbb{R}, \mathcal{m}_L, \mu_L)$ is not finite.

Now consider the sequence $\{(-n, n)\}_1^\infty$ in $\mathcal{m}_L$ then $\cup_1^\infty (-n, n) = \mathbb{R}$ and $\mu_L(-n, n) = l(-n, n) = n - (-n) = 2n < \infty$

Hence $(\mathbb{R}, \mathcal{m}_L, \mu_L)$ is σ −finite space.

Translation of a Set

For each element x_0 and subset **E** of $\mathbb{R}^d$ we will denote by $E + x_0$ the subset of $\mathbb{R}^d$ defined by $E + x_0 = \{y \in \mathbb{R}^d : y = x + x_0 \; for \; some \; x \; in \; E\}$

The set $E + x_0$ is called the ***translate of*** **E** by x_0. We turn to the invariance of Lebesgue measure under such translations.

Or Let X be a linear vector space over a field $\mathbb{R}$ then for $E \subseteq$ X and $x_0 \in X$ we have $E + x_0 = \{x + x_0 : x \in E\}$ and call it $\boldsymbol{x_0}$ **translate of** ***E***

Dialation of a Set

Let X be a linear vector space over a field $\mathbb{R}$ then for $E \subseteq$ X and $\propto \in \mathbb{R}$ we have $\propto E = \{\propto x : x \in E\}$ and is call **dialation of *E*by** $\propto$

Remark:

- For a collection ε of subsets of $X(\mathbb{R})$ and for all $x_0 \in X$ we have $\varepsilon + x_0 = \{E + x_0 : E \in \varepsilon\}$ and $\propto \varepsilon = \{\propto x : E \in \varepsilon\}$
- $(E + x_1) + x_2 = E + (x_1 + x_2)$
- $(E + x)^c = (E^c + x)$
- If $E_1 \subseteq E_2$ then $E_1 + \text{x} \subseteq E_2 + x$
- $\cup_1^\infty E_i + \text{x} = \cup_1^\infty (E_i + \text{x})$
- $\cap_1^\infty E_i + \text{x} = \cap_1^\infty (E_i + \text{x})$
- $\propto (\beta E) = (\propto \beta) E$
- $(\propto E)^c = \propto E^c$
- $E - x$ is translation invariant.

Translation Invariant: Let (X, $\mathcal{A}$,μ) be a measure space where X is a linear vector space over a field ℝ then;

- $\sigma-$ algebra $\mathcal{A}$ is called translation invariant if for all $E \in \mathcal{A}$ and $x \in X$ we have $E + x \in \mathcal{A}$
- The measure μ is said to be translation invariant if for all $E \in \mathcal{A}$ and $x \in X$ we have $E + x \in \mathcal{A}$ and $\mu(E + x) = \mu(E)$
- The measure space (X, $\mathcal{A}$,μ) is called translation invariant if $\mathcal{A}$ and μ both are translation invariant.

Theorem: Prove that Lebesgue outer measure is translation invariant.

Or Prove that for all $E \in P(\mathbb{R})$ and $x \in \mathbb{R}$ we have $\mu_L^*(E + x) = \mu_L^*(E)$

Proof:

First we will show that $l: \tau_o \to [0, \infty]$ i.e. $l(I) = b - a$ for $I = (a, b) \in \tau_o$ is translation invariant.

If $I = (a, b) \in \tau_o$ then $I + x = (a + x, b + x) \in \tau_o$ and $l(I + x) = l(I)$

i.e. $l(I + x) = l(a + x, b + x) = b + x - a - x = b - a = l(I)$

if $I = (a, \infty)$ or $I = (-\infty, b)$ or $I = (-\infty, \infty)$

then $I + x = (a + x, \infty)$ or $I + x = (-\infty, b + x)$ or $I + x = (-\infty, \infty)$

and for all $l(I + x) = \infty = l(I)$

hence for all $I \in \tau_o$ and $x \in \mathbb{R}$ we have $I + x \in \tau_o$ and $l(I + x) = l(I)$ ……(i)

so length of I is translation invariant.

Now let $\{I_n\}_1^\infty$ be an arbitrary sequence in τ_o such that $E \subseteq \cup_1^\infty I_n$. Then for an arbitrary $x \in \mathbb{R}$ we have $\{I_n + x\}_1^\infty$ in τ_o with $l(I_n + x) = l(I_n)$; $\forall n \in N$

Now $E + x \subseteq (\cup_1^\infty I_n) + x \subseteq \cup_1^\infty (I_n + x)$ implies $E + x \subseteq \cup_1^\infty (I_n + x)$

$\mu_L^*(E + x) \leq \mu_L^*\left(\cup_1^\infty (I_n + x)\right)$ by monotonicity property

$\mu_L^*(E + x) \leq \sum_1^\infty \mu_L^*(I_n + x) = \sum_1^\infty l(I_n + x) = \sum_1^\infty l(I_n)$

$\mu_L^*(E + x) \leq \sum_1^\infty l(I_n)$ from (i)

Since $\mu_L^*(E) = inf\{\sum_1^\infty l\,(I_n): E \subseteq \cup_1^\infty I_n\,, I_n \in \tau_o\}$

Therefore $\mu_L^*(E+x) \leq \mu_L^*(E)$ ………..(ii)

Applying (ii) to $(E+x)$ and its translation by $(-x)$ i.e. $(E+x)+(-x)$ we obtain

$\mu_L^*(E+x) \geq \mu_L^*\big((E+x)+(-x)\big) = \mu_L^*(E)$

Implies $\mu_L^*(E+x) \geq \mu_L^*(E)$ ………..(iii)

From (ii) and (iii) we obtain $\mu_L^*(E+x) = \mu_L^*(E)$

This shows that Lebesgue outer measure is translation invariant.

Theorem: Prove that Lebesgue measure space $(\mathbb{R}, m_L, \mu_L)$ is translation invariant.

Or Prove that for all $E \in m_L$ and $x \in \mathbb{R}$ we have $E + x \in m_L$

and $\mu_L(E+x) = \mu_L(E)$ furthermore $m_L + x = m_L$

Or The translate of a measurable set is measurable.

Proof: Let $E \in m_L$ and $x \in \mathbb{R}$ we are to show that $E + x \in m_L$

For this we have to show that for all $A \in P(\mathbb{R})$ following phenomenon;

$\mu_L^*(A) = \mu_L^*\big(A \cap (E+x)\big) + \mu_L^*(A \cap (E+x)^c)$

For this we will solve R.H.S. and equate it with L.H.S

$\mu_L^*\big(A \cap (E+x)\big) + \mu_L^*(A \cap (E+x)^c) = \mu_L^*(\{A \cap (E+x)\} - x) + \mu_L^*(\{A \cap (E+x)^c\} - x)$

$= \mu_L^*\big((A-x) \cap \{(E+x)-x\}\big) + \mu_L^*\big((A-x) \cap \{(E+x)^c - x\}\big)$

$= \mu_L^*\big((A-x) \cap \{(E+x)-x\}\big) + \mu_L^*\big((A-x) \cap \{(E^c+x) - x\}\big)$

$= \mu_L^*\big((A-x) \cap E\big) + \mu_L^*\big((A-x) \cap E^c\big)$

$= \mu_L^*(A-x)$ considering $(A-x)$ a testing set for $E \in m_L$

$= \mu_L^*(A)$ since μ_L^* is translation invariant

So $\mu_L^*(A) = \mu_L^*\big(A \cap (E+x)\big) + \mu_L^*(A \cap (E+x)^c)$ for all $A \in P(\mathbb{R})$

Implies $E + x \in m_L$

Since restriction of μ_L^* to $\mathcal{m}_L$ become measure, mean outer measure become measure. i.e. $\mu_L^* = \mu_L$ therefore

For $\mu_L^*(E+x) = \mu_L^*(E)$ we have $\mu_L(E+x) = \mu_L(E)$

Now for $E \in \mathcal{m}_L$ and $x \in \mathbb{R}$ we have $E + x \in \mathcal{m}_L$ implies $\mathcal{m}_L + x \subseteq \mathcal{m}_L$ …….(i)

Let $E \in \mathcal{m}_L$ and $x \in \mathbb{R}$ then we have $E - x \in \mathcal{m}_L$

$\Rightarrow (E-x) + x \in \mathcal{m}_L + x \Rightarrow E + (-x+x) \in \mathcal{m}_L + x \Rightarrow E \in \mathcal{m}_L + x$

So $\quad \mathcal{m}_L \subseteq \mathcal{m}_L + x$ …….(ii)

Combining (i) and (ii) $\quad \mathcal{m}_L + x = \mathcal{m}_L$

Addition modulo *1*: For $x, y \in I = [0,1)$ in $\mathbb{R}$ we define addition modulo *1* by

$$x \dotplus y = \begin{cases} x+y & if\ x+y<1 \\ x+y-1 & if\ x+y \geq 1 \end{cases}$$

The operation $\dotplus$ takes a pair of elements from $[0,1)$ to an element of $[0,1)$.

The operation $\dotplus$ is commutative as well as associative. i.e.

$x \dotplus y = y \dotplus x \quad$ and $\quad x \dotplus (y \dotplus z) = (x \dotplus y) \dotplus z$

Translation of E modulo *1*: Let $E \subseteq I = [0,1)$ and $y \in I = [0,1)$ we define translation of E modulo *1* by $E \dotplus y = \{x \dotplus y : x \in E\}$ and call it ***y translate of E modulo 1***.

Lemma: Lebesgue measure is translation invariant modulo *1*

Let $E \subseteq [0,1)$ and if $E \in \mathcal{m}_L$ then for every $y \in [0,1)$ we have $E \dotplus y \in \mathcal{m}_L$ and $\mu_L(E \dotplus y) = \mu_L(E)$

Proof: Let $E \subseteq [0,1)$ and $y \in [0,1)$ then the intervals $[0,1-y)$ and $[1-y,1)$ are disjoint. i.e. $[0,1-y) \cap [1-y,1) = \varphi$

Now we define two subsets of E as follows;

$E_1 = E \cap [0,1-y)$ and $E_2 = E \cap [1-y,1)$ where $E_1 \cap E_2 = \varphi$, $E_1 \cup E_2 = E$

Since $E, [0,1-y), [1-y,1) \in \mathcal{m}_L$ therefore $E_1, E_2 \in \mathcal{m}_L$ as $\mathcal{m}_L$ is $\sigma-$algebra.

Then using $E = E_1 \cup E_2$ we have $\mu_L(E) = \mu_L(E_1 \cup E_2)$

$\mu_L(E) = \mu_L(E_1) + \mu_L(E_2)$ ………..(i)

Since $E_1 \dotplus y = \{x \dotplus y : x \in E_1\} = \{x + y : x \in E_1 \wedge x + y < 1 \;\; i.e. x < 1 - y\}$

Implies $E_1 \dotplus y = E_1 + y \in \mathcal{m}_L$ $\because \mathcal{m}_L$ is translation invariant

Now $\mu_L(E_1 \dotplus y) = \mu_L(E_1 + y)$

$\mu_L(E_1 \dotplus y) = \mu_L(E_1)$ ………..(ii) $\because \mu_L$ is translation invariant

Also Since $E_2 \dotplus y = \{x \dotplus y : x \in E_2\}$

$E_2 \dotplus y = \{x + y - 1 : x \in E_2 \wedge x + y \geq 1 \;\; i.e. x \geq 1 - y\}$

Implies $E_2 \dotplus y = E_2 + (y - 1) \in \mathcal{m}_L$ $\because \mathcal{m}_L$ is translation invariant

Now $\mu_L(E_2 \dotplus y) = \mu_L\big(E_2 + (y - 1)\big)$

$\mu_L(E_2 \dotplus y) = \mu_L(E_2)$ ………..(iii) $\because \mu_L$ is translation invariant

Now $E \dotplus y = (E_1 \cup E_2) \dotplus y \in \mathcal{m}_L$

$E \dotplus y = (E_1 \dotplus y) \cup (E_2 \dotplus y) \in \mathcal{m}_L$ $\because \mathcal{m}_L$ is $\sigma -$algebra.

Then $\mu_L(E \dotplus y) = \mu_L(E_1 \dotplus y) + \mu_L(E_2 \dotplus y)$

$\mu_L(E \dotplus y) = \mu_L(E_1) + \mu_L(E_2)$ by (ii) and (iii)

$\mu_L(E \dotplus y) = \mu_L(E)$ by (i)

Hence proved Lebesgue measure it translation invariant modulo *1*

Theorem:

There is a subset of $\mathbb{R}$, and in fact of the interval (0,1), that is not Lebesgue measurable.

Or The interval $(0,1) \in \mathcal{m}_L$ contains a non - Lebesgue measurable set

Proof:

First we define a relation $\sim$ on $(0,1) \in \mathbb{R}$ by letting $x \ \sim \ y$ hold if and only if $x - y$ is rational. The relation $\sim$ partition $(0,1)$ into equal classes $\{E_\propto\}$ any two numbers of $(0,1)$ which are in same equivalence class differ by a rational. i.e. $x, y \in E_k$ for some 'k' if $x - y$ is rational. And any two numbers of $(0,1)$ which are not in same equivalence class differ by an irrational. i.e. $x \in E_i, y \in E_j$ for some 'i , j' if $x - y$ is irrational. Since these equivalence classes are disjoint, and since each intersects the interval $(0,1)$, we can use the axiom of choice to form a subset P of $(0,1)$ that contains exactly one element from each equivalence class. We will prove that the set P is not Lebesgue measurable

By axiom of choice, construct a set $P \subseteq (0,1)$ by picking exactly an element from each equivalence class. Also let $\{r_n : n\epsilon Z'_+\}$ be rational in $(0,1)$ with $r_0 = 0$ and for each $n\epsilon Z'_+$ let $P_n = p\dot{+}r_n$ we will check that

(a) the sets P_n are disjoint
(b) the interval $(0,1)$ is included in $\cup_{n\epsilon Z'_+} P_n$
(c) $P \notin \mathcal{m}_L$

a) Contrarily suppose that $P_n = \{P_n : P_n = p\dot{+}r_n ; n\epsilon Z'_+\}$ are not disjoint i.e. $P_m \cap P_n \neq \varphi$ for $m \neq n$ then $x \in P_m \cap P_n$ implies $x \in P_m$ and $x \in P_n$ then there exists $p_m, p_n \in P$ such that $x = p_m\dot{+}r_m$ and $x = p_n\dot{+}r_n$ then $p_m\dot{+}r_m = p_n\dot{+}r_n$

Since $p_m\dot{+}r_m$ is either $p_m + r_m$ or $p_m + (r_m - 1)$ also $p_n\dot{+}r_n$ is either $p_n + r_n$ or $p_n + (r_n - 1)$ therefore in either case $p_m - p_n$ is rational number, so $p_m, p_n \in E_\propto$ for some $\propto$.

Now since $p_m, p_n \in E_\propto$ for some $\propto$ therefore Pcontains exactly one element from each equivalence class, thus for $p_m = p_n$ we have $m = n$, which is contradiction. Hence $P_m \cap P_n = \varphi$ for $m \neq n$

b) Now we claim that $(0,1) = \cup_{n\epsilon Z'_+} P_n$

Since $P_n \subseteq (0,1)$ for all $n\epsilon Z'_+$ implies $\cup_{n\epsilon Z'_+} P_n \subseteq (0,1)$ ………(i)

Now let $x \in (0,1)$ then $x \in E_\propto$ for some $\propto$ and since Pcontains exactly one element from each equivalence class therefore there exists $p \in P$ such that $p \in E_\propto$ so $x - p$ is rational in $(0,1)$

Implies $x - p \in \{r_n : n\epsilon Z'_+\}$ therefore $x - p = r_n$ for some $n\epsilon Z'_+$

Here we discuss two cases;

i. If $x \geq p$ then $x = p + r_n \in P_n$ $\quad\because x - p \geq 0 \in (0,1)$
ii. If $x \geq p$ then $p - x = r'_n$

Let $r_m = 1 - r'_n \in (0,1)$ then $x = p - r'_n$

Implies $x = p - r'_n = p - (1 - r_m) = p - r_m + 1 \in P_n$

Hence $x \in \cup_{n\epsilon Z'_+} P_n$ implies $(0,1) \subseteq \cup_{n\epsilon Z'_+} P_n$ ………(ii)

Combining (i) and (ii) $\quad (0,1) = \cup_{n\epsilon Z'_+} P_n$

c) Now we will show that $P \notin \mathcal{m}_L$

Contrarily suppose that $P \in \mathcal{m}_L$ then from $(0,1) = \cup_{n\epsilon Z'_+} P_n$ we obtain

$\mu_L\big((0,1)\big) = \mu_L\big(\cup_{n\epsilon Z'_+} P_n\big)$

$$l\big((0,1)\big) = \sum_{n\epsilon Z'_+} \mu_L\,(P_n)$$

$1 - 0 = \sum_{n\epsilon Z'_+} \mu_L\,(P \dot{+} r_n) = \sum_{n\epsilon Z'_+} \mu_L\,(P)$ $\quad\because \mu_L$ is translation invariant

$1 = \sum_{n\epsilon Z'_+} \mu_L\,(P)$ ………(iii) $\quad$ where $\mu_L(P) \geq 0$

Since $P \in \mathcal{m}_L$ then $\mu_L(P) \geq 0$ but if $\mu_L(P) = 0$ then equation (iii) reduces to $0 = 1$, also if $\mu_L(P) > 0$ then equation (iii) reduces to $1 = \infty$ which is contradiction.

Thus $P \notin \mathcal{m}_L$.

Hence the result.

Functions and Integrals

This chapter is devoted to the definition and basic properties of the Lebesgue integral. We first introduce measurable functions—the functions that are simple enough that the integral can be defined for them if their values are not too large. After a brief look at properties that hold almost everywhere (that is, that may fail on some set of measure zero, as long as they hold everywhere else), we turn to the definition of the Lebesgue integral and to its basic properties. The chapter ends with a sketch of how the Lebesgue integral relates to the Riemann integral and then with a few more details about measurable functions.

In this section we introduce measurable functions and study some of their basic properties. We begin with the following elementary result.

Proposition: Let $(X, \mathcal{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathcal{A}$ i.e. $A \epsilon \mathcal{A}$. For a function $f : \mathcal{A} \to [-\infty, +\infty]$ the following conditions are equivalent for all $t \in \mathbb{R}$.

(a) $\{x \in A : f(x) \leq t\} = f^{-1}([-\infty, t])$ belongs to $\mathcal{A}$
(b) $\{x \in A : f(x) > t\} = f^{-1}((t, \infty])$ belongs to $\mathcal{A}$
(c) $\{x \in A : f(x) \geq t\} = f^{-1}([t, \infty])$ belongs to $\mathcal{A}$
(d) $\{x \in A : f(x) < t\} = f^{-1}([-\infty, t))$ belongs to $\mathcal{A}$

Proof: $(\boldsymbol{a}) \Leftrightarrow (\boldsymbol{b})$

Let $t \in \mathbb{R}$ and let $A_1 = \{x \in A: f(x) \leq t\}$ and $A_2 = \{x \in A : f(x) > t\}$ then $A_1 \cap A_2 = \varphi$ and $A_1 \cup A_2 = A$

Let $A_1 \in \mathcal{A}$ then $A_2 = A/A_1 \in \mathcal{A}$ $\quad \because \mathcal{A}$ is sigma algebra

And $A_2 \in \mathcal{A}$ then $A_1 = A/A_2 \in \mathcal{A}$ $\quad \because \mathcal{A}$ is sigma algebra

Implies $\quad (a) \Leftrightarrow (b)$

$(\boldsymbol{c}) \Leftrightarrow (\boldsymbol{d})$

Let $t \in \mathbb{R}$ and let $A_1 = \{x \in A: f(x) \geq t\}$ and $A_2 = \{x \in A : f(x) < t\}$ then $A_1 \cap A_2 = \varphi$ and $A_1 \cup A_2 = A$

Let $A_1 \in \mathcal{A}$ then $A_2 = A/A_1 \in \mathcal{A}$ $\quad \because \mathcal{A}$ is sigma algebra

And $A_2 \in \mathcal{A}$ then $A_1 = A/A_2 \in \mathcal{A}$ $\quad \because \mathcal{A}$ is sigma algebra

Implies $\quad (c) \Leftrightarrow (d)$

$(\boldsymbol{d}) \Rightarrow (\boldsymbol{a})$

Suppose (d) is true then $\{x \in A : f(x) < t\}$ belongs to $\mathcal{A}$ then for every $x \in A$ and $t \in \mathbb{R}$ we have $f(x) \leq t$ if and only if $f(x) < t + \frac{1}{n}$ for all $n \in N$ and since $\mathcal{A}$ is sigma algebra therefore

$\{x \in A : f(x) \leq t\} = \cap_1^\infty \left\{x \in A : f(x) < t + \frac{1}{n}\right\}$ belongs to $\mathcal{A}$

$\{x \in A : f(x) \leq t\}$ belongs to $\mathcal{A}$ which is (a)

Implies $(d) \Rightarrow (a)$

$(\boldsymbol{b}) \Rightarrow (\boldsymbol{c})$

Suppose (b) is true then $\{x \in A : f(x) > t\}$ belongs to $\mathcal{A}$ then for every $x \in A$ and $t \in \mathbb{R}$ we have $f(x) \geq t$ if and only if $f(x) < t - \frac{1}{n}$ for all $n \in N$ and since $\mathcal{A}$ is sigma algebra therefore

$\{x \in A : f(x) \geq t\} = \cap_1^\infty \left\{x \in A : f(x) > t - \frac{1}{n}\right\}$ belongs to $\mathcal{A}$

$\{x \in A : f(x) \geq t\}$ belongs to $\mathcal{A}$ which is (c)

Implies $(b) \Rightarrow (c)$

Hence all given conditions are equivalent.

Measurable functions

Let $(X, \mathcal{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathcal{A}$. An extended real valued function $f: A \to \overline{\mathbb{R}}$ or $f: A \to [-\infty, +\infty]$ is measurable with respect to A if

$\{x \in A : f(x) < t\} = \{x \in A : f(x) \in [-\infty, t)\} = f^{-1}\big(([-\infty, t))\big) \in \mathcal{A}$

Remark:

- Above definition requires that we must be able to measure inverse image of intervals of the type$[-\infty, t)$ for $t \in \mathbb{R}$
- A function that is measurable with respect to $\mathcal{A}$ is sometimes called $\mathcal{A}$ - measurable or, if the σ-algebra $\mathcal{A}$ is clears from context, simply measurable
- In case $X = \mathbb{R}^d$, a function that is measurable with respect to $\mathfrak{B}(\mathbb{R}^d)$is called Borel measurable or a Borel function.

Examples:

(a) Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous. Then for each real number *'t'* the set $\{x \in \mathbb{R}^d : f(x) < t\}$ is open and so is a Borel set. Thus f is Borel measurable.

(b) Let I be a subinterval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be non-decreasing. Then for each real number *'t'* the set $\{x \in I : f(x) < t\}$ is a Borel set (it is either an interval, a set consisting of only one point, or the empty set). Thus f is Borel measurable.

(c) With $\mathcal{A} = P(X)$ every extended real valued function defined on X is $\mathcal{A}$ – measurable.

(d) Let $(X, \mathcal{A})$ be a measurable space, and let B be a subset of X. Then χ_B,the characteristic function of B, is $\mathcal{A}$ -measurable if and only if B $\in \mathcal{A}$.

Question: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ are sigma algebras such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then every $\mathcal{A}_1 -$ measurable function is $\mathcal{A}_2 -$ measurable.

Solution: Suppose f is $\mathcal{A}_1 -$ measurable function then for all $x \in A$ and $t \in \mathbb{R}$ we have $\{x \in A: f(x) < t\} \in \mathcal{A}_1$ and since $\mathcal{A}_1 \subseteq \mathcal{A}_2$ therefore we have $\{x \in A: f(x) < t\} \in \mathcal{A}_2$ and this implies f is $\mathcal{A}_2 -$ measurable.

Result: If $\mathcal{A} = \{\varphi, X\}$ is the smallest $\sigma -$algebra on X then $f: \mathcal{A} \to [-\infty, +\infty]$ an extended real valued function on X is $\mathcal{A}$ –measurable if and only if f is constant function.

Proof:

Suppose f is $\mathcal{A}$ –measurable then for $t \in \mathbb{R}$ we have $\{x \in X: f(x) < t\} \in \mathcal{A}$

If $\{x \in X: f(x) < t\} = \varphi$ then $f(x) = constant = c \geq t; \forall x \in X$ and $t \in \mathbb{R}$

If $\{x \in X: f(x) < t\} = X$ then $f(x) = constant = c < t; \forall x \in X$ and $t \in \mathbb{R}$

Conversely: Suppose f is constant function. i.e. $f(x) = c\ ; \forall x \in X$ and let $t \in \mathbb{R}$ then $\{x \in X: f(x) < t\} = \begin{cases} X & if\ c < t \\ \varphi & if\ c \geq t \end{cases}$

In each case; $\{x \in X: f(x) < t\} \in \mathcal{A}; \forall t \in \mathbb{R}$

Hence f is $\mathcal{A}$ –measurable.

Result: With $\mathcal{A} = P(X)$ every extended real valued function defined on X i.e. $f: X \to \overline{\mathbb{R}}$ is $\mathcal{A}$ –measurable.

Proof: For every subset of X and $t \in \mathbb{R}$ we have $\{x \in A: f\,(x) < t\} \in P(X)$.
So f is $\mathcal{A}$ –measurable.

Proposition: Let (X, $\mathcal{A}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{A}$, and let *f* and *g* be $[-\infty, +\infty] -$ valued measurable functions on A. then

a) $\{x \in A : f\,(x) < g(x)\}$ belong to $\mathcal{A}$
b) $\{x \in A : g\,(x) < f(x)\}$ belong to $\mathcal{A}$
c) $\{x \in A: f\,(x) \le g(x)\}$ belong to $\mathcal{A}$
d) $\{x \in A : f\,(x) = g(x)\}$ belong to $\mathcal{A}$

Proof: Note that the inequality $f(x) < g(x)$ holds if and only if there is a rational number *r* such that $f(x) < r < g(x)$. Thus

a) As

$\{x \in A : f(x) < g(x)\} = \cup_{r \in Q} (\{x \in A : f(x) < r\} \cap \{x \in A : r < g(x)\})$

So $\{x \in A : f(x) < g(x)\}$, as the union of a countable collection of sets that belong to $\mathcal{A}$, itself belongs to $\mathcal{A}$.

b) Similarly

$\{x \in A : g(x) < f(x)\} = \cup_{r \in Q} (\{x \in A : g(x) < r\} \cap \{x \in A : r < f(x)\})$
So $\{x \in A : g(x) < f(x)\}$ as the union of a countable collection of sets that belong to , itself belongs to $\mathcal{A}$.

c) As we know that
$\{x \in A : f(x) \le g(x)\} = A - \{x \in A : g(x) < f\,(x)\}$
Imply that $\{x \in A: f\,(x) \le g(x)\}$ belongs to .

d) $\{x \in A: f(x) = g(x)\}$ is the difference of $\{x \in A: f(x) \le g(x)\}$ and $\{x \in A: f(x) < g(x)\}$ and so belongs to $\mathcal{A}$.

Characteristic Function: For an arbitrary subset E of X then characteristic function of E is defined as $\chi_E: X \to [0,1]$ by $\chi_E(x) = \begin{cases} 0 & if\ x \notin E \\ 1 & if\ x \epsilon E \end{cases}$

Note: In measure theory χ_E is replaced by 1_E.

Result: Let (X, $\mathcal{A}$) be a measurable space and let $E \epsilon P(X)$ then 1_E is $\mathcal{A} -$ measurable if and only if $E \epsilon\ \mathcal{A}$.

Proof: Suppose $E \epsilon\ \mathcal{A}$ and let $t \in \mathbb{R}$ be fixed then we are to show that 1_E is $\mathcal{A} -$ measurable, then $\{x \epsilon\ X: 1_E(x) \leq t\} = \begin{cases} \varphi & if\ t < 0 \\ E^c & if\ 0 \leq t < 1 \\ X & if\ t \geq 1 \end{cases}$

In each case $\{x \epsilon\ X: 1_E(x) \leq t\} \epsilon\ \mathcal{A}$ and so $1_E \epsilon\ \mathcal{A}$

Conversely: Suppose that 1_E is $\mathcal{A} -$ measurable then for all $t \in \mathbb{R}$ we have $\{x \epsilon\ X: 1_E(x) \leq t\} \epsilon\ \mathcal{A}$ and in particular if $t = 0.5$ then $\{x \epsilon\ X: 1_E(x) \leq t\} = E^c$ so that $E \epsilon\ \mathcal{A}$ as $\mathcal{A}$ is sigma algebra.

Proposition: Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let f be $\mathcal{A} -$ measurable functions on D. then

a) $f^{-1}([c,d)) = \{x \in\ D : c \leq f\ (x) < d\}$ belong to $\mathcal{A}$
b) $f^{-1}((c,d]) = \{x \in\ D : c < f\ (x) \leq d\}$ belong to $\mathcal{A}$
c) $f^{-1}((c,d)) = \{x \in\ D : c < f\ (x) < d\}$ belong to $\mathcal{A}$
d) $f^{-1}([\infty]) = \{x \in\ D : f\ (x) = \infty\}$belong to $\mathcal{A}$
e) $f^{-1}([-\infty]) = \{x \in\ D : f\ (x) = -\infty\}$belong to $\mathcal{A}$
f) $f^{-1}([c]) = \{x \in\ D : f\ (x) = c\}$belong to $\mathcal{A}$

Proof: Here for all case we will use the following result

$\mathcal{A}$ is $\sigma -$ algebra , f^{-1} is $\mathcal{A} -$ measurable and $f^{-1}(\cap_1^\infty E_i) = \cap_1^\infty f^{-1}(E_i)$

a) As
$f^{-1}([c,d)) = f^{-1}([c,\infty] \cap [-\infty,d)) = f^{-1}([c,\infty]) \cap f^{-1}([-\infty,d)) \epsilon\ \mathcal{A}$
$\Rightarrow f^{-1}([c,d)) = \{x \in\ D : c \leq f\ (x) < d\} \epsilon\ \mathcal{A}$

b) As
$f^{-1}((c,d]) = f^{-1}((c,\infty] \cap [-\infty,d]) = f^{-1}((c,\infty]) \cap f^{-1}([-\infty,d]) \epsilon\ \mathcal{A}$
$\Rightarrow f^{-1}((c,d]) = \{x \in\ D : c < f\ (x) \leq d\} \epsilon\ \mathcal{A}$

c) As
$f^{-1}((c,d)) = f^{-1}((c,\infty] \cap [-\infty,d)) = f^{-1}((c,\infty]) \cap f^{-1}([-\infty,d)) \epsilon\ \mathcal{A}$
$\Rightarrow f^{-1}((c,d)) = \{x \in\ D : c < f\ (x) < d\} \epsilon\ \mathcal{A}$

d) As
$f^{-1}([\infty]) = \{x \in\ D : f\ (x) = \infty\} = \cap_1^\infty \{x \in\ D : f\ (x) > k\} \epsilon\ \mathcal{A}$
$\Rightarrow f^{-1}([\infty]) = \{x \in\ D : f\ (x) = \infty\} \epsilon\ \mathcal{A}$

e) As

$f^{-1}([-\infty]) = \{x \in D : f(x) = -\infty\} = \cap_1^\infty \{x \in D : f(x) < -k\} \epsilon \mathcal{A}$

$\Rightarrow f^{-1}([-\infty]) = \{x \in D : f(x) = -\infty\} \epsilon \mathcal{A}$

f) As

$f^{-1}([c]) = \{x \in D : f(x) = c\}$

$f^{-1}([c]) = \{x \in D : f(x) \geq c\} \cap \{x \in D : f(x) \leq c\} \epsilon \mathcal{A}$

$\Rightarrow f^{-1}([c]) = \{x \in D : f(x) = c\} \epsilon \mathcal{A}$

Question:

Let G be an open set in $\mathbb{R}$, and let $(\mathbb{R}, \mathcal{m}_L)$ be a measurable space and $f: D \to \overline{\mathbb{R}}$ is $\mathcal{m}_L -$ measurable on $D \epsilon \mathcal{m}_L$ then show that $f^{-1}(G) \epsilon \mathcal{m}_L$

Proof:

Since G is open subset of $\mathbb{R}$ therefore there exists disjoint collection of open intervals in $\mathbb{R}$ such that $G = \cup_1^\infty I_n$ then

$f^{-1}(G) = f^{-1}(\cup_1^\infty I_n) = \cup_1^\infty f^{-1}(I_n) \epsilon \mathcal{m}_L$ implies $f^{-1}(G) \epsilon \mathcal{m}_L$

Here we use the following results;

f is $\mathcal{m}_L$- measurable and $\mathcal{m}_L$ is $\sigma -$ algebra

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Proposition:

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $f: D \to \overline{\mathbb{R}}$ be an extended real valued $\mathcal{A} -$ measurable functions on D. then for all $\propto \in \overline{\mathbb{R}} = \mathbb{R} \cup [-\infty, \infty]$ we have $\{x \in D : f(x) = \propto\}$ belong to $\mathcal{A}$

Proof:

If $\propto \in \mathbb{R}$ then $f^{-1}([\propto]) = \{x \in D : f(x) = \propto\} \epsilon \mathcal{A}$

If $\propto = \infty$ then $f^{-1}([\infty]) = \{x \in D : f(x) = \infty\} \epsilon \mathcal{A}$

If $\propto = -\infty$ then $f^{-1}([-\infty]) = \{x \in D : f(x) = -\infty\} \epsilon \mathcal{A}$

Hence $\{x \in D : f(x) = \propto\} \epsilon \mathcal{A}$ for all $\propto \in \overline{\mathbb{R}}$

Note: Converse of this proposition not holds. See next.

Proposition: An extended real valued function $f: D \to \overline{\mathbb{R}}$ defined on $D \epsilon \mathcal{A}$ satisfying $\{x \in D : f(x) = \propto\} \epsilon \mathcal{A}$ needs not to be $\mathcal{A} -$ measurable.

Proof: Consider the Lebesgue measurable space $(\mathbb{R}, m_L)$. And also we know that there exists a non – Lebesgue measurable subset of $(0,1)$ i.e. $P \subseteq (0,1)$ then let define a function $f: (0,1) \to \{x, -x\}$ by $f(x) = \begin{cases} x & if\ x \epsilon P \\ -x & if x \epsilon (0,1)/P\ or\ x \notin P \end{cases}$
then for every $\propto \in \overline{\mathbb{R}}$ the set $\{x \in (0,1) : f(x) = \propto\}$ is either singleton or empty set. In each case it is member of m_L but if we choose $\propto = 0$

then $\{x \in (0,1) : f(x) \geq 0\} = P \notin m_L$. So that f is not $\mathcal{A} -$ measurable.

Theorem :

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $f: X \to \overline{\mathbb{R}}$ be an extended real valued measurable functions on D. then for every $D_0 \subseteq$ D such that $D_0 \epsilon \mathcal{A}$ then restriction of f on D_0 is $\mathcal{A} -$ measurable.

Or For a measurable subset D of E, f is measurable on E if and only if the restrictions of f to D and E ~D are measurable.

Proof: Since f is measurable functions on D and $\mathcal{A}$ is sigma algebra then for $D_0 \subseteq$ D$\epsilon \mathcal{A}$ we have $\{x \in D_0 : f(x) \leq \propto\} = \{x \in D : f(x) \leq \propto\} \cap D_0 \epsilon \mathcal{A}$

Implies f on D_0 is $\mathcal{A} -$ measurable.

Theorem:

Let (X, $\mathcal{A}$) be a measurable space, and $\{D_i\}_1^\infty$ be a sequence in $\mathcal{A}$ also $D = \cup_1^\infty D_i$. Let $f: X \to \overline{\mathbb{R}}$ be an extended real valued measurable functions on D. if the restriction on D_n is $\mathcal{A} -$ measurable for all $n \in N$ then f is $\mathcal{A} -$ measurable on D.

Proof:

Since f is $\mathcal{A}$ −measurable functions on D_i and $\mathcal{A}$ is sigma algebra then for $\propto \in \mathbb{R}$ consider

$\{x \in D : f(x) \leq \propto\} = \{x \in \cup_1^\infty D_i : f(x) \leq \propto\} = \cup_1^\infty \{x \in D_i : f(x) \leq \propto\} \epsilon \mathcal{A}$

Implies f is $\mathcal{A} -$ measurable on D.

Theorem: Let (X, $\mathcal{A}$) be a measurable space, and $D\epsilon\ \mathcal{A}$ then every constant function on D is $\mathcal{A}$ −measurable.

Proof: Let $f(x) = c\ \forall x\epsilon D$ then for all $\propto\in \mathbb{R}$ we have

$$\{x\epsilon\ D: f(x) \leq\propto\} = \begin{cases} D & if\ c \leq\propto \\ \varphi & if\ c >\propto \end{cases}$$

In each case $\{x\epsilon\ D: f(x) \leq\propto\}\epsilon\ \mathcal{A}$ for all $\propto\in \mathbb{R}$. So that f is $\mathcal{A} -$ measurable on D.

In the following proposition we deal with arithmetic operations on $[0, +\infty]$ −valued measurable functions and on $\mathbb{R}$ −valued measurable functions. Arithmetic operations on $[-\infty, +\infty] -$ valued functions are trickier and are seldom needed.

Proposition: Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, let f and g be $[0, +\infty]$ −valued measurable functions or extended real valued measurable functions on D, and let c be any real number. Then $f + c, cf, f + g\ , f - g, f^2, fg$ and $\frac{f}{g}$; $(g \neq 0)$ are $\mathcal{A} -$ measurable.

Proof: Let $\propto\in \mathbb{R}$ also using f measurable function then

$$\{x\epsilon\ D: f + c(x) \leq\propto\} = \{x\epsilon\ D: f(x) + c \leq\propto\} = \{x\epsilon\ D: f(x) \leq\propto - c\}\epsilon\ \mathcal{A}$$

Implies $\boldsymbol{f + c}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A} -$ measurable.

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If $c = 0$ then $cf = 0$ so f being constant function is $\mathcal{A} -$ measurable.

If $c > 0$ then for all $\propto\in \mathbb{R}$ we have

$$\{x\epsilon\ D: (cf)x \leq\propto\} = \{x\epsilon\ D: cf(x) \leq\propto\} = \left\{x\epsilon\ D: f(x) \leq \frac{\propto}{c}\right\}\epsilon\ \mathcal{A}$$

If $c < 0$ then for all $\propto\in \mathbb{R}$ we have

$$\{x\epsilon\ D: (cf)x \leq\propto\} = \{x\epsilon\ D: cf(x) \leq\propto\} = \left\{x\epsilon\ D: f(x) \geq \frac{\propto}{c}\right\}\epsilon\ \mathcal{A}$$

Implies $\boldsymbol{cf}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A} -$ measurable.

Particularly $-\boldsymbol{f}$ is $\mathcal{A} -$ measurable for $\boldsymbol{c = -1}$

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Now we are to show that $\boldsymbol{f} + \boldsymbol{g}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable function, equivalently we are to show that the set $\{x \epsilon\, D: f + g(x) > \propto\} \epsilon\, \mathcal{A}$

Let $\propto \in \mathbb{R}$ also using f and g measurable functions then consider the set

$\{x \epsilon\, D: (f + g)x > \propto\} = \{x \epsilon\, D: f(x) + g(x) > \propto\} = \{x \epsilon\, D: f(x) > \propto - g(x)\}$

Since $f(x), \propto - g(x) \in \mathbb{R}$ and set of rational number $\mathbb{Q}$ is dense in $\mathbb{R}$ therefore $f(x) > r > \propto - g(x)$ where $r \in \mathbb{Q}$. And we claim that;

$\{x \epsilon\, D: (f + g)x > \propto\} = \cup_{r \in Q} (\{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\})$

To show this let $y \in \{x \epsilon\, D: (f + g)x > \propto\}$

Then $(f + g)y > \propto \Rightarrow f(y) + g(y) > \propto \Rightarrow f(y) > \propto - g(y)$

$\Rightarrow f(y) > r > \propto - g(y)$ where $r \in \mathbb{Q}$ then

$y \in \{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\}$

$\Rightarrow y \in \cup_{r \in Q} (\{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\})$

$\Rightarrow \{x \epsilon\, D: (f + g)x > \propto\} \subseteq \cup_{r \in Q} (\{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\})$

………………(i)

Now suppose that $y \in \cup_{r \in Q} (\{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\})$

$\Rightarrow y \in \{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\}$ where $r \in \mathbb{Q}$

$\Rightarrow f(y) > r > \propto - g(y)$ where $r \in \mathbb{Q}$

$\Rightarrow f(y) > \propto - g(y) \Rightarrow f(y) + g(y) > \propto \Rightarrow (f + g)y > \propto$ where $r \in \mathbb{Q}$

$\Rightarrow \cup_{r \in Q} (\{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\}) \subseteq \{x \epsilon\, D: (f + g)x > \propto\}$

………………(ii)

Combining (i) and (ii)

$\{x \epsilon\, D: (f + g)x > \propto\} = \cup_{r \in Q} (\{x \in D : f(x) > r\} \cap \{x \in D : \propto - g(x) < r\})$

Implies $\{x \epsilon\, D: f + g(x) > \propto\} \epsilon\, \mathcal{A}$ as is $\mathcal{A}$ sigma algebra

Implies $\boldsymbol{f} + \boldsymbol{g}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable.

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Now we are to show that $\boldsymbol{f-g}: \boldsymbol{D} \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable function on D

Since $\boldsymbol{g}$ is $\mathcal{A}-$ measurable function on D then $-\boldsymbol{g}$ is also $\mathcal{A}-$ measurable function on D. Now Since $\boldsymbol{f}$ and $-\boldsymbol{g}$ are $\mathcal{A}-$ measurable function on D therefore their sum $\boldsymbol{f}+(-\boldsymbol{g})=\boldsymbol{f}-\boldsymbol{g}$ is also $\mathcal{A}-$ measurable function on D

Implies $\boldsymbol{f-g}: \boldsymbol{D} \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable.

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Let $\boldsymbol{f^2}: \boldsymbol{D} \rightarrow \overline{\mathbb{R}}$ is extended real valued function defined on D such that for all $x \epsilon D$
$f^2(x)=[f(x)]^2$

Now we are to show that $\boldsymbol{f^2}: \boldsymbol{D} \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable function on D, then consider the set $\{x\epsilon D: f^2(x)>\propto\}$

If $\propto\in\mathbb{R}<0$ then we have $\{x\epsilon D: f^2(x)>\propto\}=D\epsilon\mathcal{A}$

If $\propto\in\mathbb{R}>0$ then we have $\{x\epsilon D: f^2(x)\leq\propto\}$

$=\{x\epsilon D: [f(x)]^2\leq\propto\}=\left\{x\epsilon D: f(x)\leq\pm\sqrt{\propto}\right\}$

$=\left\{x\epsilon D: f(x)\leq+\sqrt{\propto}\right\}\cup\left\{x\epsilon D: f(x)\geq-\sqrt{\propto}\right\}\epsilon\mathcal{A}$

So $\{x\epsilon D: f^2(x)\leq\propto\}=D\epsilon\mathcal{A}$

Implies $\boldsymbol{f^2}: \boldsymbol{D} \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable.

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Now we are to show that $\boldsymbol{fg}: \boldsymbol{D} \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable function on D

Since $fg=\frac{1}{4}[(f+g)^2-(f-g)^2]$

Also $f, g, f^2, g^2, f+g, f-g, (f+g)^2, (f-g)^2$ are $\mathcal{A}-$ measurable

Therefore $\frac{1}{4}[(f+g)^2-(f-g)^2]=fg$ is $\mathcal{A}-$ measurable

Implies $\boldsymbol{fg}: \boldsymbol{D} \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}-$ measurable.

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Now we are to show that $\frac{\boldsymbol{f}}{\boldsymbol{g}}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A} -$ measurable function on D. For this first we show that $\frac{\mathbf{1}}{\boldsymbol{g}}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A} -$ measurable function on D. Let $\frac{\mathbf{1}}{\boldsymbol{g}}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is extended real valued function defined on D such that for all $x \epsilon D$; $\frac{1}{g}(x) = \frac{1}{g(x)}$

Consider the set $\left\{x \epsilon D: \frac{1}{g(x)} > \propto\right\}$ and $\propto \in \mathbb{R}$ then discuss following assumptions;

If $\propto = 0$ then assuming g as $\mathcal{A} -$ measurable

$$\left\{x \epsilon D: \frac{1}{g}(x) > 0\right\} = \left\{x \epsilon D: \frac{1}{g(x)} > 0\right\} = \{x \epsilon D: g(x) < 0\} \epsilon \mathcal{A}$$

If $\propto > 0$ then assuming g as $\mathcal{A} -$ measurable

$$\left\{x \epsilon D: \frac{1}{g}(x) > \propto\right\} = \left\{x \epsilon D: \frac{1}{g(x)} > \propto\right\} = \left\{x \epsilon D: g(x) < \frac{1}{\propto}\right\} \epsilon \mathcal{A}$$

If $\propto < 0$ then assuming g as $\mathcal{A} -$ measurable and $\mathcal{A}$ as sigma algebra

$$\left\{x \epsilon D: \frac{1}{g}(x) > \propto\right\} = \left\{x \epsilon D: \frac{1}{g(x)} > \propto\right\}$$

$$= \left\{x \epsilon D: g(x) > \frac{1}{\propto}; g(x) > 0\right\} \cup \left\{x \epsilon D: g(x) > \frac{1}{\propto}; g(x) < 0\right\} \epsilon \mathcal{A}$$

$$\left\{x \epsilon D: \frac{1}{g}(x) > \propto\right\} = \left\{x \epsilon D: g(x) > \frac{1}{\propto}\right\} \cup \left\{x \epsilon D: g(x) < \frac{1}{\propto}\right\} \epsilon \mathcal{A}$$

Hence in each case $\frac{\mathbf{1}}{\boldsymbol{g}}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A} -$ measurable function on D.

Now using the fact *"If $\boldsymbol{f}$ and $\boldsymbol{g}$ are $\mathcal{A} -$ measurable then $\boldsymbol{fg}$ is $\mathcal{A} -$ measurable"*

Implies $\frac{\boldsymbol{f}}{\boldsymbol{g}} = \boldsymbol{f}.\frac{\mathbf{1}}{\boldsymbol{g}}: \boldsymbol{D} \to \overline{\mathbb{R}}$ is $\mathcal{A} -$ measurable.

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Larger and Smaller of two functions or Maximum and Minimum of f and g

Let f and g be $[-\infty, +\infty]$ −valued functions or extended real valued functions having a common domain A. The maximum and minimum of f and g, written $f \vee g$ and $f \wedge g$, are the functions from A to $[-\infty, +\infty]$ defined by

$(f \vee g)(x) = max(f(x), g(x))$ and $(f \wedge g)(x) = min(f(x), g(x))$

Equivalently, we can define $f \vee g$ by $(f \vee g)(x) = \begin{cases} f(x) & if\ f(x) > g(x) \\ g(x) & otherwise \end{cases}$ and

$(f \wedge g)(x) = \begin{cases} f(x) & if\ f(x) < g(x) \\ g(x) & otherwise \end{cases}$

Limit inferior and limit superior of a sequence

If $\{ f_n \}$ is a sequence of $[-\infty,+\infty]$-valued functions on A, then

- $sup\ f_n : A \to [-\infty, +\infty]$ is defined by
  $(sup\ f_n)(x) = sup\{ f_n(x) : n = 1, 2, .. \}$
- $inf\ f_n : A \to [-\infty, +\infty]$ is defined by
  $\left(inf\ f_n(x)\right) = inf\{ f_n(x) : n = 1, 2, .. \}$
- $\lim_{n\to\infty} sup\ f_n = \lim_{n\to\infty}(sup\{ f_k(x) : k \geq n \})$
- $\lim_{n\to\infty} inf\ f_n = \lim_{n\to\infty}(inf\{ f_k(x) : k \geq n \})$

The domain of $\lim\limits_{n} f_n$ consists of those points in A at which $lmt sup f_n$ and $lmt(inf f_n)$ agree.

Remember

- For an increasing sequence i.e. $\underline{x_n} \leq \underline{x_{n+1}}$ we have
  $\lim_{n\to\infty} inf\ x_n = \lim_{n\to\infty}(inf\{ x_k(x) : k \geq n \}) = \lim_{n\to\infty}(inf_{k\geq n}\{x_k\})$
- For a decreasing sequence i.e. $\overline{x_n} \geq \overline{x_{n+1}}$ we have
  $\lim_{n\to\infty} sup\ x_n = \lim_{n\to\infty}(sup\{ x_k(x) : k \geq n \}) = \lim_{n\to\infty}(sup_{k\geq n}\{x_k\})$
- $\{f_n\}_1^\infty$ is a sequence of functions and $\{f_n(x)\}_1^\infty$ is a sequence of real numbers.
- $\left(\min_{n=1,2,\dots,N} f_n\right)(x) = \min_{n=1,2,\dots,N}(f_n(x))$
- $\left(\max_{n=1,2,\dots,N} f_n\right)(x) = \max_{n=1,2,\dots,N}(f_n(x))$
- $(lmt\ inf\ f_n)(x) = lmt\ inf\ \left(f_n(x)\right)$ and
  $(lmt\ sup\ f_n)(x) = lmt\ sup\ \left(f_n(x)\right)$
- $(\lim_{n\to\infty} f_n)(x) = \lim\limits_{n\to\infty}\left(f_n(x)\right)$
- $(\inf_{n\in N} f_n)(x) = \inf_{n\in N}(f_n(x))$ and $(\sup_{n\in N} f_n)(x) = \sup_{n\in N}(f_n(x))$

Proposition: Let (X, $\mathcal{A}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{A}$, and let f and g be [−∞,+∞]-valued measurable or extended real valued functions on A. Then f ∨g and f ∧g are measurable.

Proof: The measurability of $f \vee g$ follows from the identity

$\{x \in A : (f \vee g)(x) \leq \propto\} = \{x \in A : f(x) \leq \propto\} \cap \{x \in A : g(x) \leq \propto\}$

And the measurability of $f \wedge g$ follows from the identity

$\{x \in A : (f \wedge g)(x) \leq \propto\} = \{x \in A : f(x) \leq \propto\} \cup \{x \in A : g(x) \leq \propto\}$

Proposition:

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $\lim_{n\to\infty} f_n$ exists on D and is $\mathcal{A} -$ measurable.

Proof:

Since $\{f_n\}_1^\infty$ is a monotone sequence on D, therefore $\{f_n(x)\}_1^\infty$ is a monotone sequence of extended real valued numbers so that $\lim_{n\to\infty} f_n(x)$ in $\overline{\mathbb{R}}$ for all $x \in D$. And hence $\lim_{n\to\infty} f_n$ exists on D.

Now we are to show that $\lim_{n\to\infty} f_n = f$ is $\mathcal{A} -$ measurable on D.

If f_n is increasing then $\lim_{n\to\infty} f_n = \cup_1^\infty f_n$ using $\mathcal{A}$ as sigma algebra for every $\propto \in \mathbb{R}$ we have; $\{x \in D: (\lim_{n\to\infty} f_n)(x) > \propto\} = \{x \in D: \lim_{n\to\infty} f_n(x) > \propto\}$

$\Rightarrow \lim_{n\to\infty} f_n(x) > \propto \Leftrightarrow f_n(x) > \propto$ for some n

$\Rightarrow \{x \in D: (\lim_{n\to\infty} f_n)(x) > \propto\} = \cup_{n\in N} \{x \in D: f_n(x) > \propto\} \epsilon \mathcal{A}$

$\Rightarrow \lim_{n\to\infty} f_n$ is $\mathcal{A} -$ measurable on D

If f_n is decreasing then $-f_n$ **is increasing**

So that $\lim_{n\to\infty} (-f_n)$ is $\mathcal{A} -$ measurable on D

$\Rightarrow -(\lim_{n\to\infty} f_n)$ is $\mathcal{A} -$ measurable on D

$\Rightarrow \lim_{n\to\infty} f_n$ is $\mathcal{A} -$ measurable on D

Note: We may write the above phenomenon as follows with a different proof.

Proposition:

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $lmt\ \ f_n$ is $\mathcal{A} -$ measurable on D.

Proof: Let $\propto \in \mathbb{R}$ and $x \in D$ also using $\mathcal{A}$ as sigma algebra and each f_n is $\mathcal{A} -$ measurable and Let $D_0 = \{x \in D: lmt\, sup\ f_n = lmt\, inf\ \ f_n\}$ be the domain of $lmt\ \ f_n$ then;

$\{x \in D_0: lmt\ \ f_n \leq \propto\} = D_0 \cap \{x \in D: lmt\, sup\ f_n \leq \propto\} \epsilon\ \mathcal{A}$

$\Rightarrow \{x \in D_0: lmt\ \ f_n \leq \propto\} \epsilon\ \mathcal{A}$

$\Rightarrow lmt\ \ f_n$ is $\mathcal{A} -$ measurable on D

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Proposition:

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $\min_{n=1,2,\dots,N} f_n$ is $\mathcal{A} -$ measurable on D.

Proof:

Let $\propto \in \mathbb{R}$ and $x \in D$ also using $\mathcal{A}$ as sigma algebra and each f_n is $\mathcal{A} -$ measurable then

$\min_{n=1,2,\dots,N}\{f_n(x)\} < \propto \Leftrightarrow f_n(x) < \propto$ for some $n = 1,2, \dots, N$

$\Rightarrow \left\{x \in D: \left(\min_{n=1,2,\dots,N} f_n\right)(x) < \propto\right\} = \left\{x \in D: \min_{n=1,2,\dots,N} f_n(x) < \propto\right\}$

$\Rightarrow \left\{x \in D: \left(\min_{n=1,2,\dots,N} f_n\right)(x) < \propto\right\} = \cup_{n=1,2,\dots,N} \{x \in D: f_n(x) < \propto\} \epsilon\ \mathcal{A}$

$\Rightarrow \min_{n=1,2,\dots,N} f_n$ is $\mathcal{A} -$ measurable on D

Proposition:

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $\max_{n=1,2,\dots,N} f_n$ is $\mathcal{A}-$ measurable on D.

Proof:

Let $\propto\in \mathbb{R}$ and $x \in D$ also using $\mathcal{A}$ as sigma algebra and each f_n is $\mathcal{A}-$ measurable then

$\max_{n=1,2,\dots,N}\{f_n(x)\} >\propto \Leftrightarrow f_n(x) >\propto$ for some $n = 1,2,\dots,N$

$\Rightarrow \left\{x \in D: \left(\max_{n=1,2,\dots,N} f_n\right)(x) >\propto\right\} = \left\{x \in D: \max\limits_{n=1,2,\dots,N} f_n(x) >\propto\right\}$

$\Rightarrow \left\{x \in D: \left(\max_{n=1,2,\dots,N} f_n\right)(x) >\propto\right\} = \cup_{n=1,2,\dots,N} \{x \in D: f_n(x) >\propto\} \epsilon\ \mathcal{A}$

$\Rightarrow \max_{n=1,2,\dots,N} f_n$ is $\mathcal{A}-$ measurable on D

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Proposition:

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $\inf_{n\in N} f_n$ is $\mathcal{A}-$ measurable on D.

Proof:

Let $\propto\in \mathbb{R}$ and $x \in D$ also using $\mathcal{A}$ as sigma algebra and each f_n is $\mathcal{A}-$ measurable then

$\inf_{n\in N}\{f_n(x)\} <\propto \Leftrightarrow f_n(x) <\propto$ for some $n \in N$

$\Rightarrow \{x \in D: (\inf_{n\in N} f_n)(x) <\propto\} = \left\{x \in D: \inf\limits_{n\in N} f_n(x) <\propto\right\}$

$\Rightarrow \{x \in D: (\inf_{n\in N} f_n)(x) <\propto\} = \cup_{n\in N} \{x \in D: f_n(x) <\propto\} \epsilon\ \mathcal{A}$

$\Rightarrow \inf_{n\in N} f_n$ is $\mathcal{A}-$ measurable on D

Proposition: Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $\sup_{n\in N} f_n$ is $\mathcal{A} -$ measurable on D.

Proof: Let $\propto\in \mathbb{R}$ and $x \in D$ also using $\mathcal{A}$ as sigma algebra and each f_n is $\mathcal{A} -$ measurable then

$\sup_{n\in N}\{f_n(x)\} >\propto \Leftrightarrow f_n(x) >\propto$ for some $n \in N$

$\Rightarrow \{x \in D: (\sup_{n\in N} f_n)(x) >\propto\} = \left\{x \in D: \sup_{n\in N} f_n(x) >\propto\right\}$

$\Rightarrow \{x \in D: (\sup_{n\in N} f_n)(x) >\propto\} = \cup_{n\in N} \{x \in D: f_n(x) >\propto\} \epsilon\ \mathcal{A}$

$\Rightarrow \sup_{n\in N} f_n$ is $\mathcal{A} -$ measurable on D

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Proposition: Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $lmt\ inf\ f_n$ is $\mathcal{A} -$ measurable on D.

Proof: We know that $lmt\ inf\ f_n = \lim_{n\to\infty}(inf_{k\geq n}\{f_k\})$ where $\{inf_{k\geq n}\{f_k\}\}_1^\infty$ is an increasing sequence and since $\inf_{k\geq n} f_n$ is $\mathcal{A} -$ measurable for all $n \in N$ therefore

$lmt\ inf\ f_n = \lim_{n\to\infty}(inf_{k\geq n}\{f_k\})$ is $\mathcal{A} -$ measurable on D

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Proposition:

Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let $\{f_n\}_1^\infty$ be a monotone sequence of extended real valued measurable functions on D. Then $lmt\ sup\ f_n$ is $\mathcal{A} -$ measurable on D.

Proof:

We know that $lmt\ sup\ f_n = \lim_{n\to\infty}(sup_{k\geq n}\{f_k\})$ where $\{sup_{k\geq n}\{f_k\}\}_1^\infty$ is a decreasing sequence and since $\sup_{k\geq n} f_n$ is $\mathcal{A} -$ measurable for all $n \in N$

Therefore $lmt\ sup\ f_n = \lim_{n\to\infty}(sup_{k\geq n}\{f_k\})$ is $\mathcal{A} -$ measurable on D

The positive part f^+ and the negative part f^- of f

Let D be a set, and let f be an extended real-valued function on D. The positive part f^+ and the negative part f^- of f are the extended real-valued functions defined by

$$f^+ = (f \vee 0)(x) = max(f(x),0) \text{ and } f^- = -(f \wedge 0)(x) = -min(f(x),0)$$

The absolute value of f

Let D be a set, and let f be an extended real-valued function on D. The absolute valued of f is the extended real-valued functions defined by $|f|(x) = |f(x)| \geq 0$
or $|f|(x) = max(f(x), -f(x))$

Remember

- $|f| = f^+ + f^-$
- For any $x \in D$ at least one of $f^+$and f^- is zero. So that $f^+ - f^-$ is well defined and we have $f = f^+ - f^-$

Proposition: Let (X, $\mathcal{A}$) be a measurable space, let D be a subset of X that belongs to $\mathcal{A}$, and let f be an extended real valued measurable functions on D. Then $f^+, f^-, |f|$ are $\mathcal{A} -$ measurable on D.

Proof: Since we know that $f^+ = max(f(x),0)$ and f and 0 are $\mathcal{A} -$ measurable on $D \in \mathcal{A}$. Also using the fact "$\{f_n\}_1^\infty$ is $\mathcal{A} -$ measurable and is $\max_{n=1,2,\ldots,N} f_n$" Therefore f^+ is $\mathcal{A} -$ measurable on $D \in \mathcal{A}$.

Also we know that $f^- = -min(f(x),0)$ and f and 0 are $\mathcal{A} -$ measurable on $D \in \mathcal{A}$. Also using the fact "$\{f_n\}_1^\infty$ is $\mathcal{A} -$ measurable and is $\min_{n=1,2,\ldots,N} f_n$" Therefore f^- is $\mathcal{A} -$ measurable on $D \in \mathcal{A}$.

Now $|f| = f^+ + f^-$ being addition of $\mathcal{A} -$ measurable on $D \in \mathcal{A}$ is $\mathcal{A} -$ measurable on $D \in \mathcal{A}$

Proposition: Let f be a measurable function on E. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.

Proof: Assume f^+ and $f^-$are integrable nonnegative functions. By the linearity of integration for nonnegative functions, $|f| = f^+ + f^-$ is integrable over E.
Conversely, suppose$|f|$ is integrable over E. Since $0 \leq f^+, f^- \leq |f|$ on E, we infer from the monotonicity of integration for nonnegative functions that both f^+ and f^- are integrable over E.

Properties That Hold Almost Everywhere

Let $(X, \mathcal{A}, \mu)$ be a measure space. A property of points of X is said to hold μ *−almost everywhere* if the set of points in X at which it fails to hold is μ *−negligible*. In other words, a property holds μ −almost everywhere if there is a set N that belongs to $\mathcal{A}$, satisfies μ(N)=0, and contains every point at which the property fails to hold. More generally, if E is a subset of X, then a property is said to hold μ*-almost everywhere* on E if the set of points in E at which it fails to hold is μ*-negligible*. The expression μ-almost everywhere is often abbreviated to $\mu - a.e.$ or to $a.e.[\mu]$. In cases where the measure μ is clear from context, the expressions almost everywhere and $a.e.$ are also used.

Consider a property that holds almost everywhere, and let F be the set of points in X at which it fails. Then it is not necessary that F belong to ; it is only necessary that there be a set N that belongs to A , includes F, and satisfies μ(N)= 0. Of course, if μ is complete, then F will belong to $\mathcal{A}$.

In short we can define <u>Almost Everywhere</u> as

A property is said to be *almost everywhere* if the set of points where does not hold is a set of measure zero.

Or Let $(X, \mathcal{A}, \mu)$ be a measured space, a property P *holds almost everywhere* in X, iff a set $N \in \mathcal{A}$ such that $\mu(N) = 0$ and property P is hold for all $x \in X/N$.

Equal almost Everywhere

For a given complete measure space $(X, \mathcal{A}, \mu)$, we say that two extended real valued $\mathcal{A} -$ measurable functions f and g defined on $D \in \mathcal{A}$ are equal almost everywhere $(i.e.\ f = g\ \ a.e.)$ if there exists a null set N in $(X, \mathcal{A}, \mu)$ such that $N \subset D$ and $f(x) = g(x)$ for all $x \in X/N$.

Remember

- In above definition $f = g\ \ a.e.$ on D if $f = g$ outside a null set $N \subset D$ This does not exclude the possibility that $f(x) = g(x)$ for some and indeed for every $x \in N$
- Every subset of a null set is null set and belongs to Sigma algebra.

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a complete measure space, then every extended real-valued function defined on a null set N in $(X, \mathcal{A}, \mu)$ is $\mathcal{A} -$ measurable.

Proof:

Let $\propto \in \mathbb{R}$ and f be an extended real-valued function defined on a null set N in $(X, \mathcal{A}, \mu)$ then

$\{x \in N: f(x) \leq \propto\} \subset N$

$\because \mu(N) = 0$ and every subset of a null set is null set and belongs to $\mathcal{A}$.

$\Rightarrow \{x \in N: f(x) \leq \propto\} \subset N \in \mathcal{A}$

Implies f is $\mathcal{A} -$ measurable on the null set N.

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a complete measure space, and let f and g be extended real-valued functions on $D \in \mathcal{A}$ that are equal almost everywhere. If f is $\mathcal{A} -$ measurable on D, then g is $\mathcal{A} -$ measurable on D.

Proof:

Suppose $f = g \;\; a.e.$ on D then there exists a null set N in $(X, \mathcal{A}, \mu)$ such that $N \subset D$ and $f(x) = g(x)$ for all $x \in X/N$.

Since f is $\mathcal{A} -$ measurable on D then it is $\mathcal{A} -$ measurable on D/N by theorem "if f is $\mathcal{A} -$ measurable on D then f is $\mathcal{A} -$ measurable on $D_0 \subseteq D$"

But $f = g$ on D/N so that g is $\mathcal{A} -$ measurable on $D/N \subseteq D$.

Since $(X, \mathcal{A}, \mu)$ is a complete measure space then by theorem **"Let $(X, \mathcal{A}, \mu)$ be a complete measure space, then every extended real-valued function defined on a null set N in $(X, \mathcal{A}, \mu)$ is $\mathcal{A} -$ measurable."** g is $\mathcal{A} -$ measurable on $N \subset D$.

So g is $\mathcal{A} -$ measurable on $D = (D/N) \cup N$.

Remark

If f is $\mathcal{A} -$ measurable on $\{D_i\}_1^n$ then it is $\mathcal{A} -$ measurable on $\cup_1^n D_i$

Existence of limit Almost Everywhere

Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\{f_n\}_1^\infty$ be a sequence of extended real valued $\mathcal{A} -$ measurable function on $D \in \mathcal{A}$. Then we say that $\lim_{n\to\infty}(f_n)$ exists almost everywhere on $D \in \mathcal{A}$ if there exists a null set N such that $\lim_{n\to\infty}(f_n)$ exists on D/N.

Equivalently we say that $\{f_n(x)\}_1^\infty$ converges a.e. on D if $\{f_n(x)\}_1^\infty$ converges on D/N where $\mu(N) = 0$.

Note that the convergence of the sequence $\{f_n\}_1^\infty$ depends on the convergence of $\{f_n(x)\}_1^\infty$ for $x \in D$

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Lemma:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\{f_n\}_1^\infty$ be a sequence of extended real valued $\mathcal{A} -$ measurable function on $D \in \mathcal{A}$. If for every $n > 0$ there exists a $\mathcal{A} -$ measurable subset E of D ($E \subseteq D$) with $\mu(E) < \frac{1}{n}$ such that $\lim_{n\to\infty} f_n(x)$ exists for all $x \in D/E$ then $\lim_{n\to\infty} f_n(x)$ exists almost everywhere on D.

Proof:

From the condition for all $n \in N$ there exists a $\mathcal{A} -$ measurable subset E_n of D ($E_n \subseteq D$) such that $\mu(E_n) < \frac{1}{n}$ and $\lim_{n\to\infty} f_n(x)$ exists for all $x \in D/E_n$.

Now we have to prove $\lim_{n\to\infty} f_n(x)$ almost everywhere on D.

Define $N = \cap_1^\infty E_n$ then $N \subseteq D$ such that $\mu(N) = \mu(\cap_1^\infty E_n) \leq \mu(E_n) < \frac{1}{n}$; $\forall n$
i.e. $\mu(N) = 0$, so N is null set in $(X, \mathcal{A}, \mu)$

$D/N = D \cap N^c = D \cap (\cap_1^\infty E_n)^c = D \cap (\cup_1^\infty E_n^c) = \cup_1^\infty (D \cap E_n^c) = \cup_1^\infty (D/E_n)$

$\Rightarrow x \in D/N \Leftrightarrow x \in D/E_k$ for $k \in N$

Hence $\lim_{n\to\infty} f_n(x)$ exists for all $x \in D/E_n$

Implies $\lim_{n\to\infty} f_n(x)$ exists for all $x \in D/N$

That is $\lim_{n\to\infty} f_n(x)$ exists almost everywhere on D.

The Integral

In this section we construct the integral and study some of its basic properties. The construction will take place in three stages.

- We begin with the simple functions
- As our next step, we define the integral of an arbitrary $[0,+\infty]-$ valued $\mathcal{A}-$measurable function on X
- Finally, let f be an arbitrary $[-\infty,+\infty]-$valued $\mathcal{A}-$ measurable function on X. If $\int f^{+}d\mu$ and $\int f^{-}d\mu$ are both finite, then f is called *integrable* (or $\mu-$ *integrable* or *summable*), and its integral $\int f d\mu$ is defined by $\int f d\mu = \int f^{+}d\mu - \int f^{-}d\mu$

The integral of f is said to exist if at least one of $\int f^{+}d\mu$ and $\int f^{-}d\mu$ is finite, and again in this case, $\int f d\mu$ is defined to be $\int f^{+}d\mu - \int f^{-}d\mu$. In either case one sometimes writes $\int f(x)\mu\,(dx)$ or $\int f(x)\,d(\mu x)$ in place of $\int f d\mu$

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Step Function

Let $I=[a,b]$ in $\mathbb{R}$ and $P=\{a=x_0,x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_n=b\}$ is a partition of interval $I=[a,b]$ such that $I=\cup_1^n I_k$ where $I_k=(x_{k-1},x_{k+1})$ then a real valued function $f: I\to\mathbb{R}$ is called step function that is defined as follows;

$$f(x)=\begin{cases} c_k & if\ x\in I_k \\ d_k & if\ x=x_k \end{cases} \qquad ; k = 0 , 1 , 2 , \dots , n$$

Or

A real valued function φ with $I=[a,b]$ is called a step function if there exists a partition $a=x_0,x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_n=b$ of the interval $I=[a,b]$ such that f is constant on each sub – interval $I_k=(x_{k-1},x_{k+1})$. i.e.

$$f(x)=\begin{cases} c_k & if\ x\in I_k \\ d_k & if\ x=x_k \end{cases} \qquad ; k = 0 , 1 , 2 , \dots , n$$

Simple Function:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f: D \in \mathcal{A} \to \mathbb{R}$ is called simple function if

- $D(f) \in \mathcal{A}$ (domain of f)
- $R(f)$ is finite subset of $\mathbb{R}$. i.e. Range of f is only finitely many reals.
- f is $\mathcal{A} -$ measurable function on $D(f)$

We will denote the collection of all simple $\mathcal{A} -$ measurable real valued functions with $\mathcal{S}$ and $\mathcal{S}_+$ the collection of all non – negative functions in $\mathcal{S}$

Question:

Every step function is simple but Simple function needs not to be a step function.

Answer: Consider the following function;

$$f(x) = \begin{cases} 1 & if\ x \in Q \\ 0 & if\ x \in Q' \end{cases}$$

This is simple function but not a step function.

Keep in mind a simple function is linear combination of characteristic functions.

……………………………………..

Canonical Representation: Let f be a simple function on $D \in \mathcal{A}$ in a measure space $(X, \mathcal{A}, \mu)$, also let $c_1, c_2, \ldots, c_n$ are distinct values assumed by f on D and let $D_i = \{x \in D: f(x) = c_i\}$ then $\{D_i\}_1^n$ is a distinct collection with $D = \cup_1^n D_i$ then the representation $f(x) = \sum_1^n c_i 1_{D_i}(x)$; $\forall\ x \in D$ is called the canonical representation of f on D

Remark

- In above definition the collection $\{D_i\}_1^n$ is a partition of set $D \in \mathcal{A}$
  i.e. $D = \cup_1^n D_i$ and $D_i \cap D_j = \varphi$; $\forall i, j = 1,2, \ldots, n$
- If $D_i \in \mathcal{A}$ and $c_i \in \mathbb{R}$ for *i = 1, 2,…., n* and if we have $D = \cup_1^n D_i$ then if $f(x) = \sum_1^n c_i 1_{D_i}(x)$; $\forall\ x \in D$ is a simple function on D then the expression $\sum_1^n c_i 1_{D_i}(x)$ may not be a canonical representation of f. Since $\{D_i\}_1^n$ may not be disjoint collection and $c_{i's}$ may not be distinct.

Integrable Function

Suppose that $f: X \to [-\infty, +\infty]$ is $\mathcal{A}-$ measurable and that $D \in \mathcal{A}$ .Then f is *integrable over D* if the function f_{χ_D} is integrable, and in this case $\int_D f d\mu$, the integral of f over D, is defined to be $\int_D f_{\chi_D} d\mu$. Likewise, if $D \in \mathcal{A}$ and if f is a measurable function whose domain is D (rather than the entire space X), then the integral of f over D is defined to be the integral (if it exists) of the function on X that agrees with f on D and vanishes on D^c.In case $\mu(D^c) = 0$, one often writes $\int f d\mu$ in place of $\int_D f d\mu$ and calls f integrable, rather than integrable over D.

Lebesgue Integral of a Simple Function

Let $f(x) = \sum_1^n c_i 1_{D_i}(x)$ be a canonical representation of a simple function f on $D \in \mathcal{A}$ in a measure space $(X, \mathcal{A}, \mu)$ then Lebesgue Integral of f on D with respect to μ is defined as; $\int_D f d\mu = \sum_1^n c_i \mu(D_i)$

Lebesgue Semi – Integrable Function

If Lebesgue Integral of f exists in $\overline{\mathbb{R}}$ then we say that f is *Lebesgue Semi integrable* on D

Or A simple function f on a set $D \in \mathcal{A}$ is called $\mu - Semi\ integrable\ on\ D$ if $\int_D f dx \in \overline{\mathbb{R}}$

Lebesgue Integrable Function

If Lebesgue Integral of a simple function f on $D \in \mathcal{A}$ exists in $\mathbb{R}$ then we say that f is *Lebesgue Integrable* on D

Or A simple function f on a set $D \in \mathcal{A}$ is *Lebesgue Integrable* on D

Or $\mu -\ integrable\ on\ D$ if $\int_D f dx \in \mathbb{R}$

Question:

Give an example of a simple function which is semi – Lebesgue integrable.

Solution: Consider $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu_L)$ be a Borel measureable space and let a simple function $f: \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = \begin{cases} 0 & if\ x \in Q \\ 1 & if\ x \in Q' \end{cases}$, then canonical representation of f is $f(x) = 0.1_Q(x) + 1.1_{Q'}(x)$

And so its Lebesgue integral is $\int_{\mathbb{R}} f d\mu_L = 0.\mu_L(Q) + 1.\mu_L(Q')$

$\int_{\mathbb{R}} f d\mu_L = 0 + \mu_L(\mathbb{R}/Q)$

$\int_{\mathbb{R}} f d\mu_L = \mu_L(\mathbb{R}) - \mu_L(Q) = \infty - 0 = \infty \in \overline{\mathbb{R}}$

$\int_{\mathbb{R}} f d\mu_L \in \overline{\mathbb{R}}$ Implies f is semi – Lebesgue integrable.

Set of rational is countable union of singletons. i.e. $\mu_L(Q) = 0$

……………………………………..

Question: Give an example of a simple function which is Lebesgue integrable.

Solution: Consider $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu_L)$ be a Borel measureable space and let a simple function $f: [0,1] \to \mathbb{R}$ is defined as $f(x) = \begin{cases} 0 & if\ x \in Q \cap [0,1] \\ 1 & if\ x \in Q' \cap [0,1] \end{cases}$ then canonical representation of f is $f(x) = 0.1_{Q\cap[0,1]}(x) + 1.1_{Q'\cap[0,1]}(x)$

And so its Lebesgue integral is $\int_{[0,1]} f d\mu_L = 0.\mu_L(Q \cap [0,1]) + 1.\mu_L(Q' \cap [0,1])$

$\int_{\mathbb{R}} f d\mu_L = 0 + \mu_L(Q' \cap [0,1]) = \mu_L([0,1] \cap Q') = \mu_L([0,1]/Q)$

$\int_{\mathbb{R}} f d\mu_L = \mu_L([0,1]) - \mu_L(Q) = 1 - 0 = 1 \in \mathbb{R}$

$\int_{\mathbb{R}} f d\mu_L \in \mathbb{R}$ Implies f is Lebesgue integrable.

Question:

Give an example of a simple function which is not Lebesgue integrable.

Solution:

Consider $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu_L)$ be a Borel measureable space and let a simple function $f:[0,\infty) \to \mathbb{R}$ is defined as $f(x) = \begin{cases} -1 & if\ x \in \cup_{k\in Z'_+} [2k+1, 2k+2] \\ 1 & if\ x \in \cup_{k\in Z'_+} [2k, 2k+1] \end{cases}$ then canonical representation of f is

$f(x) = (-1).1_{\cup_{k\in Z'_+}[2k+1,2k+2]}(x) + 1.1_{\cup_{k\in Z'_+}[2k,2k+1]}(x)$

And so its Lebesgue integral is

$\int f d\mu_L = (-1).\mu_L\left(\cup_{k\in Z'_+} [2k+1, 2k+2]\right) + 1.\mu_L\left(\cup_{k\in Z'_+} [2k, 2k+1]\right)$

$\int f d\mu_L = (-1).\sum_{k\in Z'_+} \mu_L[2k+1, 2k+2] + 1.\sum_{k\in Z'_+} \mu_L[2k, 2k+1]$

$\int f d\mu_L = -\infty + \infty =$ Undefined and Lebesgue Integral is not exists

Implies f is not Lebesgue integrable.

……………………………………..

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f belongs to $\mathcal{S}_+$ is a simple function defined on a set $D \in \mathcal{A}$ with finite members i.e $\mu(D) < \infty$ and let $k \in \mathbb{R}$. Then kf is simple on $D \in \mathcal{A}$ and $\int_D kf d\mu = k\int_D f d\mu$

Proof:

Since f is simple function on D therefore there exists a disjoint sequence $\{E_i\}_1^n$ and distinct numbers $\{c_i\}_1^n$ such that $D = \cup_1^n E_i$ and its canonical representation is $f(x) = \sum_1^n c_i 1_{E_i}(x)$ then

$kf(x) = k\sum_1^n c_i 1_{E_i}(x) \Rightarrow (kf)x = \sum_1^n (kc_i) 1_{E_i}(x)$ this is canonical representation of kf and it is a simple function.

Now $\int_D kf d\mu = \sum_1^n kc_i\mu(E_i) = k\sum_1^n c_i\mu(E_i) = k\int_D f d\mu$

Proposition (Linearity of Integration):

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f & g belongs to $\mathcal{S}_+$ are simple function defined on a set $D \in \mathcal{A}$ with finite members i.e $\mu(D) < \infty$ then $f + g$ is simple on $D \in \mathcal{A}$ and $\int_D (\propto f + \beta g) d\mu = \propto \int_D f d\mu + \beta \int_D g d\mu$

Proof:

Since f & g are simple function on D therefore there exists disjoint sequences $\{E_i\}_1^n$ and $\{F_j\}_1^m$ and distinct numbers $\{c_i\}_1^n$ and $\{d_j\}_1^m$ such that $D = \cup_1^n E_i$ and $D = \cup_1^m F_j$ and their canonical representations are $f(x) = \sum_1^n c_i 1_{E_i}(x)$ and $g(x) = \sum_1^m d_j 1_{F_j}(x)$.

Now we define $G_{ij} = E_i \cap F_j$ then $\{G_{ij}: i = 1,2, \dots, n; j = 1,2, \dots, m\}$ is a disjoint collection such that $D = \cup_{i=1}^n \cup_{j=1}^m G_{ij}$ then $(f + g)x = \sum_{i=1}^n \sum_{j=1}^m (c_i + d_j) 1_{G_{ij}}(x)$

This implies $f + g$ and also in similar way $\propto f + \beta g$ is simple on $D \in \mathcal{A}$

Then $\int_D (\propto f + \beta g) d\mu = \sum_{i=1}^n \sum_{j=1}^m (\propto c_i + \beta d_j) \mu(G_{ij})$

$$\int_D (\propto f + \beta g) d\mu = \sum_{i=1}^n \sum_{j=1}^m \propto c_i \mu(G_{ij}) + \sum_{i=1}^n \sum_{j=1}^m \beta d_j \mu(G_{ij})$$

$$\int_D (\propto f + \beta g) d\mu = \propto \sum_{i=1}^n \sum_{j=1}^m c_i \mu(E_i \cap F_j) + \beta \sum_{i=1}^n \sum_{j=1}^m d_j \mu(E_i \cap F_j)$$

$$\int_D (\propto f + \beta g) d\mu = \propto \sum_{i=1}^n c_i \left[\sum_{j=1}^m \mu(E_i \cap F_j)\right] + \beta \sum_{j=1}^m d_j \left[\sum_{i=1}^n \mu(E_i \cap F_j)\right]$$

$$\int_D (\propto f + \beta g) d\mu = \propto \sum_{i=1}^n c_i \mu\left[\cup_{j=1}^m (E_i \cap F_j)\right] + \beta \sum_{j=1}^m d_j \mu\left[\cup_{i=1}^n (E_i \cap F_j)\right]$$

$$\int_D (\propto f + \beta g) d\mu = \propto \sum_{i=1}^n c_i \mu\left[E_i \cap \left(\cup_{j=1}^m F_j\right)\right] + \beta \sum_{j=1}^m d_j \mu\left[\left(\cup_{i=1}^n E_i\right) \cap F_j\right]$$

$$\int_D (\propto f + \beta g) d\mu = \propto \sum_{i=1}^n c_i \mu[E_i \cap D] + \beta \sum_{j=1}^m d_j \mu[D \cap F_j]$$

$$\int_D (\propto f + \beta g) d\mu = \propto \sum_{i=1}^n c_i \mu(E_i) + \beta \sum_{j=1}^m d_j \mu(F_j)$$

$$\int_D (\propto f + \beta g) d\mu = \propto \int_D f d\mu + \beta \int_D g d\mu$$

Proposition (Monotone Property): Let $(X, \mathcal{A}, \mu)$ be a measure space, let f & g belongs to $\mathcal{S}_+$ are simple function defined on a set $D \in \mathcal{A}$ with finite members i.e $\mu(D) < \infty$ and $f \leq g$ on $D \in \mathcal{A}$ then $\int_D f d\mu \leq \int_D g d\mu$

Proof: If $f \leq g$ then $g - f \geq 0$ so that $\int_D (g - f) d\mu \geq 0$

$\Rightarrow \int_D g d\mu + \left(-\int_D f d\mu\right) \geq 0$ $\qquad \because \int_D (f + g) d\mu = \int_D f d\mu + \int_D g d\mu$

$\Rightarrow \int_D g d\mu - \int_D f d\mu \geq 0$ $\qquad \because \int_D k f d\mu = k \int_D f d\mu$

$\Rightarrow \int_D g d\mu \geq \int_D f d\mu \Rightarrow \int_D f d\mu \leq \int_D g d\mu$

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Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f belongs to $\mathcal{S}_+$ is simple function defined on a set $D \in \mathcal{A}$ with finite members i.e $\mu(D) < \infty$ and if D_1, D_2 are disjoint measurable subsets on $D \in \mathcal{A}$ with $D = D_1 \cup D_2$ then

$\int_D f d\mu = \int_{D_1} f d\mu + \int_{D_2} f d\mu$

Proof: Since f is simple function on D therefore there exists a disjoint sequence $\{E_i\}_1^n$ and distinct numbers $\{c_i\}_1^n$ such that $D = \cup_1^n E_i$ and its canonical representation is $f(x) = \sum_1^n c_i 1_{E_i}(x)$.

Now if $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \varphi$ then $1_D(x) = 1_{D_1}(x) + 1_{D_2}(x)$

Now $\int_D f d\mu = \sum_1^n c_i \mu(E_i) = \sum_1^n c_i \mu(E_i \cap D) = \sum_1^n c_i \mu[E_i \cap (D_1 \cup D_2)]$

$\int_D f d\mu = \sum_1^n c_i \mu[(E_i \cap D_1) \cup (E_i \cap D_2)] = \sum_1^n c_i [\mu(E_i \cap D_1) + \mu(E_i \cap D_2)]$

$\int_D f d\mu = \sum_1^n c_i \mu(E_i \cap D_1) + \sum_1^n c_i \mu(E_i \cap D_2)$ …………(i)

Now $\{E_i \cap D_1\}_1^n$ and $\{E_i \cap D_2\}_1^n$ are disjoint sequences

And $\cup_1^n (E_i \cap D_1) = \cup_1^n E_i \cap D_1 = D \cap D_1 = D_1$

Also $\cup_1^n (E_i \cap D_2) = \cup_1^n E_i \cap D_2 = D \cap D_2 = D_2$

$(i) \Rightarrow \int_D f d\mu = \int_{D_1} f d\mu + \int_{D_2} f d\mu$

Proposition: Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a simple function defined on a set $D \in \mathcal{A}$ then if $\mu(D) = 0$ then $\int_D f d\mu = 0$

Proof: Since f is simple function on D therefore there exists a disjoint sequence $\{E_i\}_1^n$ and distinct numbers $\{c_i\}_1^n$ such that $D = \cup_1^n E_i$ and its canonical representation is $f(x) = \sum_1^n c_i 1_{E_i}(x)$.

Let $\mu(D) = 0$ and $E_i \subseteq D$ then $\mu(E_i) \leq \mu(D) = 0$ implies $\mu(E_i) = 0$

So $\int_D f d\mu = \sum_1^n c_i \mu(E_i) = 0$ implies $\int_D f d\mu = 0$

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Proposition: Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a simple function defined on a set $D \in \mathcal{A}$ then if $f = 0$ then $\int_D f d\mu = 0$

Proof: Since f is simple function on D therefore there exists a disjoint sequence $\{E_i\}_1^n$ and distinct numbers $\{c_i\}_1^n$ such that $D = \cup_1^n E_i$ and its canonical representation is $f(x) = \sum_1^n c_i 1_{E_i}(x)$.

Let $f = 0$ then $c_i = 0 \ \forall i = 1,2, \dots, n$

So $\int_D f d\mu = \sum_1^n c_i \mu(E_i) = 0$ implies $\int_D f d\mu = 0$

……………………………………………….

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a simple function defined on a set $D \in \mathcal{A}$ then if $f \geq 0$ then $\int_D f d\mu \geq 0$

Proof:

Since f is simple function on D therefore there exists a disjoint sequence $\{E_i\}_1^n$ and distinct numbers $\{c_i\}_1^n$ such that $D = \cup_1^n E_i$ and its canonical representation is $f(x) = \sum_1^n c_i 1_{E_i}(x)$.

Let $f \geq 0$ then $f(x) = \sum_1^n c_i 1_{E_i}(x) \geq 0$

So $\int_D f d\mu = \sum_1^n c_i \mu(E_i) \geq 0$ implies $\int_D f d\mu \geq 0$

Proposition: Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a simple function defined on a set $D \in \mathcal{A}$ then if $f \leq 0$ then $\int_D f d\mu \leq 0$

Proof: Since f is simple function on D therefore there exists a disjoint sequence $\{E_i\}_1^n$ and distinct numbers $\{c_i\}_1^n$ such that $D = \cup_1^n E_i$ and its canonical representation is $f(x) = \sum_1^n c_i 1_{E_i}(x)$.

Let $f \leq 0$ then $-f \geq 0$ then $-f(x) = \sum_1^n c_i 1_{E_i}(x) \geq 0$

So $-\int_D f d\mu = -\sum_1^n c_i \mu(E_i) \geq 0$ implies $-\int_D f d\mu \geq 0$

Hence $\int_D f d\mu \leq 0$

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Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a simple function defined on a set $D \in \mathcal{A}$ then f is μ −integrable on D iff $\mu(\{x \in D: f(x) \neq 0\}) < \infty$

Proof:

Since f is simple function on D therefore there exists a disjoint sequence $\{E_i\}_1^n$ and distinct numbers $\{c_i\}_1^n$ such that $D = \cup_1^n E_i$ and its canonical representation is $f(x) = \sum_1^n c_i 1_{E_i}(x)$.

Let f is μ −integrable on D then $\int_D f d\mu = \sum_1^n c_i \mu(E_i) < \infty$

Now $\mu(\{x \in D: f(x) \neq 0\}) = \mu(\{x \in \cup_1^n E_i: f(x) \neq 0\})$

$\mu(\{x \in D: f(x) \neq 0\}) = \mu(\{x \in E_i: f(x) \neq 0\}) < \infty \quad \because \mu(E_i) < \infty$

Conversely suppose that $\mu(\{x \in D: f(x) \neq 0\}) < \infty$

$\Rightarrow \mu(D) < \infty \quad \forall x \in D$

Since $E_i \subseteq D$ therefore $\sum_1^n c_i \mu(E_i) < \infty$

$\Rightarrow \int_D f d\mu < \infty$

Hence f is μ −integrable on D

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a simple function defined on a set $D \in \mathcal{A}$. Let $\{E_1, E_2, \dots, E_n\}$ be a disjoint collection in $\mathcal{A}$ such that $D = \cup_1^n E_i$ then f is simple function on $E_i \in \mathcal{A}$; $i = 1,2, \dots, n$ and $\int_D f d\mu = \sum_{i=1}^n \int_{E_i} f d\mu$

Proof:

Since f is simple function on D therefore there exists a disjoint sequence $\{D_j\}_1^n$ and distinct numbers $\{c_j\}_1^n$ such that $D = \cup_1^n D_j$ and its canonical representation is $f(x) = \sum_1^n c_j 1_{D_j}(x)$. Also the Lebesgue integral of f on $D \in \mathcal{A}$ will be

$\int_D f d\mu = \sum_{j=1}^n c_j \mu(D_j)$ ……………..(i)

Since f assumes finitely many values on D so its restriction to E_i; $i = 1,2, \dots, n$ assumes only finitely many values with $D = \cup_1^n E_i$. Hence f is simple function on $E_j \in \mathcal{A}$; $j = 1,2, \dots, n$. Then we have a disjoint sequence $\{D \cap E_i\}_1^n$ and distinct numbers $\{c_j\}_1^n$ such that $E_i = \cup_1^n (D_j \cap E_i)$; $i, j = 1,2, \dots, n$ and its canonical representation is $f(x) = \sum_1^n c_j 1_{D_j \cap E_i}(x)$

Then from (i) we have $\int_D f d\mu = \sum_{j=1}^n c_j \mu(D_j)$

$\int_D f d\mu = \sum_{j=1}^n c_j \mu(D_j \cap D) = \sum_{j=1}^n c_j \mu\left(D_j \cap (\cup_1^n E_i)\right)$ $\because D = \cup_1^n E_i$

$\int_D f d\mu = \sum_{j=1}^n c_j \mu\left(\cup_1^n (D_j \cap E_i)\right)$ by distributive property

$\int_D f d\mu = \sum_{j=1}^n c_j \sum_{i=1}^n \mu(D_j \cap E_i)$ by definition of measure

$\int_D f d\mu = \sum_{i=1}^n \left[\sum_{j=1}^n c_j \mu(D_j \cap E_i)\right]$

$\int_D f d\mu = \sum_{i=1}^n \int_{E_i} f d\mu$

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f belong to $\mathcal{S}_+$, and let $\{f_n\}$ be a nondecreasing sequence of functions in $\mathcal{S}_+$ such that $f\,(x) = \ lim_{n\to\infty} f_n(x)$ holds at each x in X. Then $\int f d\mu = \ lim_{n\to\infty}\int f_n d\mu$.

(This proposition is a ***weak version*** of one of the fundamental properties of the Lebesgue integral, ***the monotone convergence theorem***. We need this weakened version now for use as a tool in completing the definition of the integral)

Proof: Since from Proposition we know that

"Let $(X, \mathcal{A}, \mu)$ be a measure space, let f & g belongs to $\mathcal{S}_+$ are simple function defined on a set $D \in \mathcal{A}$ with finite members i.e $\mu(D) < \infty$ and $f \leq g$ on $D \in \mathcal{A}$ then $\int_D f d\mu \leq \int_D g d\mu$"

Therefore $\int f_1 d\mu \leq \int f_2 d\mu \leq \cdots \leq \int f d\mu$

Hence $lim_{n\to\infty} f_n d\mu$ exists and satisfies $lim_{n\to\infty}\int f_n d\mu \leq \int f d\mu$ ……..(i)

Conversely: Let ε be a number such that $0 < \varepsilon < 1$.We will construct a nondecreasing sequence $\{g_n\}$ of functions in $\mathcal{S}_+$ such that $g_n \leq f_n$ holds for each *n* and such that $lim_{n\to\infty}\int g_n d\mu = (1-\varepsilon)\int f d\mu$

$\Rightarrow\ lim_{n\to\infty}\int g_n d\mu = (1-\varepsilon)\int f d\mu \leq\ lim_{n\to\infty}\int f_n d\mu \quad \because \int g_n d\mu \leq \int f_n d\mu$

$\Rightarrow\ lim_{n\to\infty}\int g_n d\mu = \int f d\mu \leq\ lim_{n\to\infty}\int f_n d\mu \quad \because \varepsilon$ is arbitrary

$\Rightarrow \int f d\mu \leq\ lim_{n\to\infty}\int f_n d\mu$ ……..(ii)

From (i) and (ii) $\int f d\mu = \ lim_{n\to\infty}\int f_n d\mu$

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Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f_1, f_2, g_1 and g_2 be nonnegative real-valued integrable functions on X such that

$f_1 - f_2 = g_1 - g_2$.Then $\int f_1 d\mu - \int f_2 d\mu = \int g_1 d\mu - \int g_2 d\mu$

Proof: Since the functions f_1, f_2, g_1 and g_2 satisfy $f_1 - f_2 = g_1 - g_2$,they also satisfy $f_1 + g_2 = g_1 + f_2$ and so satisfy $\int f_1 d\mu + \int g_2 d\mu = \int g_1 d\mu + \int f_2 d\mu$

since all the integrals involved are finite, this implies that

$\int f_1 d\mu - \int f_2 d\mu = \int g_1 d\mu - \int g_2 d\mu$

Examples of Integrable Functions

a) If μ is a finite measure, then every bounded measurable function on $(X, \mathcal{A}, \mu)$ is integrable.
b) In particular, every bounded Borel function, and hence every continuous function, on [a,b] is Lebesgue integrable.
c) Suppose that $\mathcal{A}$ is the σ −algebra on N containing all subsets of N and that μ is counting measure on . It follows that a nonnegative function f on N is $\mu -$ integrable if and only if the infinite series $\sum f(n)$ is convergent, and that in that case the integral and the sum of the series agree. Since a not necessarily nonnegative function f is integrable if and only if f^+ and f^- are integrable, it follows that f is integrable if and only if the infinite series $\sum f(n)$ is absolutely convergent. Once again, the integral and the sum of the series have the same value.
d) Note that a simple measurable function that vanishes almost everywhere is integrable, with integral 0.

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Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a $[-\infty, +\infty]$ −valued or extended real valued $\mathcal{A} -$ measurable function on X. Then f is integrable if and only if $|f|$ is integrable. If these functions are integrable, then $\left|\int f d\mu\right| \leq \int |f| d\mu$

Proof:

Recall that by definition f is integrable if and only if f^+ and f^- are integrable.

since $|f| = f^+ + f^-$

$\int_D |f| d\mu = \int_D f^+ d\mu + \int_D f^- d\mu$ $\quad\quad \because \int_D (f+g) d\mu = \int_D f d\mu + \int_D g d\mu$

Thus the integrability of f is equivalent to the integrability of $|f|$.

Now $\left|\int f d\mu\right| = \left|\int_D f^+ d\mu + \int_D f^- d\mu\right| \leq \int_D f^+ d\mu + \int_D f^- d\mu \leq \int |f| d\mu$

Hence $\left|\int f d\mu\right| \leq \int |f| d\mu$

Proposition: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f and g be $[-\infty, +\infty]$ −valued or extended real valued $\mathcal{A} -$ measurable function on X that agree almost everywhere. If either $\int_D f d\mu$ or $\int_D g d\mu$ exists, then both exist, and $\int_D f d\mu = \int_D g d\mu$

Or Let $(X, \mathcal{A}, \mu)$ be a measure space, let f & g are simple function defined on a set $\in \mathcal{A}$. Assume that f & g are μ −integrable on D. If $f = g$ almost everywhere on $D \in \mathcal{A}$ then $\int_D f d\mu = \int_D g d\mu$

Proof:

Given that If $f = g$ almost everywhere on $D \in \mathcal{A}$ then there exists a null set $N \subseteq D$ in $(X, \mathcal{A}, \mu)$ such that $f(x) = g(x) \quad \forall x \in D/N$

Since $D = (D/N) \cup N$ therefore

$\int_D f d\mu = \int_{D/N} f d\mu + \int_N f d\mu = \int_{D/N} f d\mu \qquad \because \mu(N) = 0 \Rightarrow \int_N f d\mu = 0$

$\int_D f d\mu = \int_{D/N} g d\mu \qquad \because f = g$ almost everywhere

$\int_D f d\mu = \int_{D/N} g d\mu + 0 = \int_{D/N} g d\mu + \int_N g d\mu = \int_D g d\mu$

$\int_D f d\mu = \int_D g d\mu$

……………………………………………….

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a $[0, \infty]$ −valued $\mathcal{A} -$ measurable function on X. If t is a positive real number and if $D_t = \{x \in X : f(x) \geq t\}$, then $\mu(D_t) \leq \frac{1}{t} \int_{D_t} f d\mu \leq \frac{1}{t} \int f d\mu$

Proof:

Since $0 \leq t\chi_{D_t} \leq f_{\chi_{D_t}} \leq f$ and $f \leq g$ implies $\int_D f d\mu \leq \int_D g d\mu$ then

$\int t\chi_{D_t} d\mu \leq \int_{D_t} f d\mu \leq \int f d\mu$

$t\mu(D_t) \leq \int_{D_t} f d\mu \leq \int f d\mu \qquad \because t\mu(D_t) = \int t\chi_{D_t} d\mu$

$\mu(D_t) \leq \frac{1}{t} \int_{D_t} f d\mu \leq \frac{1}{t} \int f d\mu$

Corollary: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be $[-\infty, +\infty]$ −valued or extended real valued $\mathcal{A}-$ measurable function on X. Then $\{x \in X: f(x) \neq 0\}$ is σ −finite under μ.

Proof: Consider a sequence $A_1, A_2, \ldots$ such that $A_n = \left\{x \in X: |f(x)| \geq \frac{1}{n}\right\}$

Now using the result "Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a $[0, \infty]$ −valued $\mathcal{A}-$ measurable function on X. If t is a positive real number and if $D_t = \{x \in X: f(x) \geq t\}$, then $\mu(D_t) \leq \frac{1}{t}\int_{D_t} f d\mu \leq \frac{1}{t}\int f d\mu$"

If we replace f with $|f|$ we conclude that $\mu(A_n) = \mu\left(\{x \in X: |f(x)| \geq \frac{1}{n}\}\right)$ is finite under μ. Then $\cup_1^\infty A_n = \{x \in X: f(x) \neq 0\}$ is σ −finite under μ.

……………………………………………

Corollary: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be $[-\infty, +\infty]$ −valued or extended real valued $\mathcal{A}-$ measurable function on X that satisfies $\int |f| d\mu = 0$. Then f vanishes μ −almost everywhere.

Proof: Consider a sequence $A_1, A_2, \ldots$, such that $A_n = \left\{x \in X: |f(x)| \geq \frac{1}{n}\right\}$

Now using the result "Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a $[0, \infty]$ −valued $\mathcal{A}-$ measurable function on X. If t is a positive real number and if $D_t = \{x \in X: f(x) \geq t\}$, then $\mu(D_t) \leq \frac{1}{t}\int_{D_t} f d\mu \leq \frac{1}{t}\int f d\mu$"

If we replace f with $|f|$ we conclude that

$\mu(A_n) = \mu\left(\{x \in X: |f(x)| \geq \frac{1}{n}\}\right) \leq n\int |f| d\mu = 0$ hold for each $n \in Z^+$

Since $\cup_1^\infty A_n = \cup_1^\infty \left\{x \in X: |f(x)| \geq \frac{1}{n}\right\} = \{x \in X: f(x) \neq 0\}$

Therefore $\mu(\cup_1^\infty A_n) = \mu\left(\cup_1^\infty \left\{x \in X: |f(x)| \geq \frac{1}{n}\right\}\right) = \mu(\{x \in X: f(x) \neq 0\})$

$\mu(\{x \in X: f(x) \neq 0\}) = \mu\left(\cup_1^\infty \left\{x \in X: |f(x)| \geq \frac{1}{n}\right\}\right)$ Rearranging

$\mu(\{x \in X: f(x) \neq 0\}) = \sum_1^\infty \mu\left\{x \in X: |f(x)| \geq \frac{1}{n}\right\} \leq n\int |f| d\mu = 0$

Implies $\mu(\{x \in X: f(x) \neq 0\}) = 0$. Thus f vanishes μ −almost everywhere.

Corollary:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be $[-\infty, +\infty]$ $-$valued or extended real valued $\mathcal{A} -$ measurable function on X such that $\int_A f d\mu \geq 0$ holds for all A in $\mathcal{A}$ (or even just for all A in the smallest σ-algebra on X that makes f measurable). Then $f \geq 0$ holds μ $-$almost everywhere.

Proof: Let $A = \{x \in X: f(x) < 0\}$.Then $\int f\chi_A d\mu = \int_A f d\mu = 0$ (since $f < 0$ on A, yet we are assuming that $\int_A f d\mu \geq 0$). It follows from Corollary

"Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be $[-\infty, +\infty]$ $-$valued or extended real valued $\mathcal{A} -$ measurable function on X that satisfies $\int |f| d\mu = 0$. Then f vanishes μ $-$almost everywhere"

That $f\chi_A$ vanishes almost everywhere and hence that $f \geq 0$ holds almost everywhere.

…………………………………………………

Corollary:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be $[-\infty, +\infty]$ $-$valued or extended real valued $\mathcal{A} -$ measurable function on X. Then $|f(x)| < \infty$ holds at μ $-$almost every x in X.

Proof: Using the result

"Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a $[0, \infty]$ $-$valued $\mathcal{A} -$ measurable function on X. If t is a positive real number and if $D_t = \{x \in X: f(x) \geq t\}$, then $\mu(D_t) \leq \frac{1}{t}\int_{D_t} f d\mu \leq \frac{1}{t}\int f d\mu$"

If we replace f with $|f|$ we conclude that

$\mu(A_n) = \mu(\{x \in X: |f(x)| \geq n\}) \leq \frac{1}{n}\int |f| d\mu = 0$ holds for each $n \in Z^+$

Thus $\mu(\{x \in X: |f(x)| = \infty\}) \leq \mu(\{x \in X: |f(x)| \geq n\}) \leq \frac{1}{n}\int |f| d\mu = 0$ holds for each n

And so $\mu(\{x \in X: |f(x)| = \infty\}) = 0$

Limit Theorems

In this section we prove the basic limit theorems of integration theory. These results are extremely important and account for much of the power of the Lebesgue integral.

non – Negative Functions

Let $(X, \mathcal{A}, \mu)$ be a measure space, a real valued function $f: D \to \mathbb{R}$ on $D \in \mathcal{A}$ is said to be non – negative if $f(x) \geq 0$; $\forall x \in D$ with $\mu(D) < \infty$

Lebesgue Integral of non – Negative Functions

Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a non – negative extended real valued $\mathcal{A}$ −measurable function on $D \in \mathcal{A}$ with $\mu(D) < \infty$. Then Lebesgue Integral of non – negative function f on D with respect to μ is defined as;

$$\int_D f d\mu = Sup_{0 \leq g \leq f} \int_D g d\mu$$

Where suprimum is taken over all non – negative simple function g on D such that $g \leq f$

Remark: A non – negative extended real valued function need not to be bounded and therefore there exists may not be simple function ψ such that $f \leq \psi$ then the equality $\int_D f d\mu = Inf_{f \leq \psi} \int_D \psi d\mu$ for a bounded real valued $\mathcal{A}$ −measurable function does not exists for a non – negative extended real valued $\mathcal{A}$ −measurable function f.

This fact has the consequence that while the integral of a non – negative extended real valued $\mathcal{A}$ −measurable function can be approximated by integrals of simple functions from below, it cannot be approximated by integrals of simple functions from above.

Lemma (Without Proof): Let $(X, \mathcal{A}, \mu)$ be a measure space, let f, f_1, f_2 be non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ then

- If $\int_D f d\mu = 0$ then $f = 0$ almost everywhere on $D \in \mathcal{A}$
- If $D_0 \subseteq D$ is $\mathcal{A}$ −measurable then $\int_{D_0} f d\mu \leq \int_D f d\mu$
- If $f \geq 0$ almost everywhere on $D \in \mathcal{A}$ and $\int_D f d\mu = 0$ then $\mu(D) = 0$
- If $f_1 \leq f_2$ on $D \in \mathcal{A}$ then $\int_D f_1 d\mu \leq \int_D f_2 d\mu$
- If $f_1 = f_2$ almost everywhere on $D \in \mathcal{A}$ then $\int_D f_1 d\mu = \int_D f_2 d\mu$

Preposition

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non – negative simple functions on X. Then show that a set function $\upsilon: \mathcal{A} \rightarrow [0, \infty]$ defined as $\upsilon(A) = \int_A f d\mu$ for all $A \in \mathcal{A}$ is a measure on $\mathcal{A}$.

Proof: To show that $\upsilon(A) = \int_A f d\mu$ is a measure we have to show;

- $\upsilon(\varphi) = 0$
- For a disjoint sequence $\{E_j\}_1^\infty$ we have $\upsilon(\cup_1^\infty E_j) = \sum_1^\infty \upsilon(E_j)$

Let $f(x) = \sum_1^n c_i 1_{D_i}(x)$be a canonical representation of f on X then $\cup_1^n D_i = X$ then the restriction of f on $A \in \mathcal{A}$ is given by $f(x) = \sum_1^n c_i 1_{D_i \cap A}(x)$ then

$\upsilon(A) = \int_A f d\mu = \sum_1^n c_i \mu(D_i \cap A) \in [0, \infty]$ for all $A \in \mathcal{A}$

Particularly if $A = \varphi$ then

$\upsilon(\varphi) = \int_\varphi f d\mu = \sum_1^n c_i \mu(D_i \cap \varphi) = \sum_1^n c_i \mu(\varphi) = 0 \qquad \because \mu(\varphi) = 0$

$\upsilon(\varphi) = 0$

Now let $\{E_j\}_1^\infty$ be a disjoint sequence in $\mathcal{A}$ then;

$$\upsilon(\cup_1^\infty E_j) = \int_{\cup_1^\infty E_j} f d\mu = \sum_1^n c_i \mu\left(D_i \cap (\cup_1^\infty E_j)\right)$$

$$\upsilon(\cup_1^\infty E_j) = \sum_1^n c_i \mu\left(\cup_1^\infty (D_i \cap E_j)\right)$$

$$\upsilon(\cup_1^\infty E_j) = \sum_1^n c_i \sum_{j=1}^\infty \mu(D_i \cap E_j)$$

$$\upsilon(\cup_1^\infty E_j) = \sum_{j=1}^\infty \left[\sum_{i=1}^n c_i \mu(D_i \cap E_j)\right]$$

$\upsilon(\cup_1^\infty E_j) = \sum_1^\infty \upsilon(E_j)$

Hence $\upsilon(A) = \int_A f d\mu$ is a measure.

Theorem: (The Monotone Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_n\}_1^\infty$ be an increasing sequence of $[0, \infty]$ $-$valued or non – negative extended real valued $\mathcal{A}$ $-$measurable functions on $D \in \mathcal{A}$ and $f(x) = lim_{n\to\infty} f_n(x)$ on $D \in \mathcal{A}$. Then $\int_D f d\mu = lim_{n\to\infty} \int_D f_n d\mu$

In this theorem the functions f and $f_1, f_2, \ldots$ are only assumed to be nonnegative and measurable; there are no assumptions about whether they are integrable.

Proof:

Since $\{f_n\}_1^\infty$ is an increasing sequence of non – negative extended real valued $\mathcal{A}$ $-$measurable functions on $D \in \mathcal{A}$ then $f(x) = lim_{n\to\infty} f_n(x)$ exists in $[0, \infty]$ for all $x \in D$ so that $f(x) = lim_{n\to\infty} f_n(x)$ is a non – negative extended real valued $\mathcal{A}$ $-$measurable functions on $D \in \mathcal{A}$, also f is $\mathcal{A}$ $-$measurable function (if $\{f_n\}_1^\infty$ is $\mathcal{A}$ $-$measurable so its limit exists)

Since $f_n \leq f$ therefore $\int_D f_n d\mu \leq \int_D f d\mu \quad \forall n \in N$

Also $f_n \leq f_{n+1}$ therefore $\int_D f_n d\mu \leq \int_D f_{n+1} d\mu$

So $\left\{\int_D f_n d\mu : n \in N\right\}$ is an increasing sequence of non – negative extended real valued numbers bounded above by $\int_D f d\mu$

Hence $\quad lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu \quad$ …………..(i)

Now let g be an arbitrary non – negative simple functions on $D \in \mathcal{A}$ such that $0 \leq g \leq f$ with $\propto \in (0,1)$arbitrarily fixed as $0 \leq \propto g \leq g \leq f$ on $D \in \mathcal{A}$

Define a sequence $\{E_n\}_1^\infty$ of subset of D by setting as follows for all $n \in N$

$E_n = \{x \in D : f_n(x) \geq \propto g(x)\}$ …………..(ii)

Since f_n and $\propto g$ are $\mathcal{A}$ $-$measurable therefore for all $n \in N$ we have $E_n \in \mathcal{A}$

Now $f_n \leq f_{n+1}$ implies $E_n \subseteq E_{n+1}$

And this shows that $\{E_n\}_1^\infty$ is an increasing sequence in $\mathcal{A}$

Since $E_n \subseteq D$ therefore $\cup_1^\infty E_n \subseteq D$ …………..(iii)

We claim that $\cup_1^\infty E_n = D$

To see this let $x \in D$

If $\boldsymbol{f(x) = 0}$ then since $0 \leq g \leq f$ therefore $g(x) = 0$ and since $0 \leq g \leq f_n \leq f$ therefore $f_n(x) = 0$

This implies $f_n(x) = 0 = \propto g(x)$ and $x \in E_n$

Implies $D \subseteq E_n$ for some $n \in N$

Then $D \subseteq \cup_1^\infty E_n$ …………..(iv)

If $\boldsymbol{f(x) > 0}$ then since $0 \leq g \leq f$ and $\propto \in (0,1)$ we have $f(x) > \propto g(x)$

Since f_n is an increasing function therefore there exists $n \in N$ such that

$f_n(x) > \propto g(x)$ and so $x \in E_n$

Implies $D \subseteq E_n$ for some $n \in N$

Then $D \subseteq \cup_1^\infty E_n$ …………..(v)

Using (iii),(iv),(v) we have $\cup_1^\infty E_n = D$

Now define a set υ on $\mathcal{A}$ by setting $\upsilon(D) = \int_D g d\mu$ then υ is a measure

Now $\int_D f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} \propto g d\mu = \propto \int_{E_n} g d\mu = \propto \upsilon(E_n)$

Implies $\int_D f_n d\mu \geq \propto \upsilon(E_n)$ or $\propto \upsilon(E_n) \leq \int_D f_n d\mu$

$\Rightarrow \propto \ lim_{n\to\infty} \upsilon(E_n) \leq \ lim_{n\to\infty} \int_D f_n d\mu$

$\Rightarrow \propto \upsilon(\ lim_{n\to\infty} E_n) \leq \ lim_{n\to\infty} \int_D f_n d\mu$

$\Rightarrow \propto \upsilon(\cup_1^\infty E_n) \leq \ lim_{n\to\infty} \int_D f_n d\mu$ $\because E_n$ is increasing

$\Rightarrow \propto \upsilon(D) \leq \ lim_{n\to\infty} \int_D f_n d\mu$

$\Rightarrow \propto \int_D g d\mu \leq \ lim_{n\to\infty} \int_D f_n d\mu$

Since this holds for arbitrary non – negative simple function g on D such that $0 \leq g \leq f$ we have $\propto \int_D f d\mu \leq \ lim_{n\to\infty} \int_D f_n d\mu$

$\Rightarrow \int_D f d\mu \leq \ lim_{n\to\infty} \int_D f_n d\mu$ $\because \propto \in (0,1);\ \propto \to 1$ …………..(vi)

Hence from (i) and (vi) $lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu$

Note: This theorem is not valid for decreasing sequence. (See Next)

Theorem:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_n\}_1^\infty$ be a decreasing sequence of $[0, \infty]$ −valued or non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ and $f(x) = lim_{n\to\infty} f_n(x)$ on $D \in \mathcal{A}$. Then $\int_D f d\mu \neq lim_{n\to\infty} \int_D f_n d\mu$

Proof:

Consider a Lebesgue measure space $(\mathbb{R}, m_L, \mu_L)$ and $\{f_n\}_1^\infty$ be a decreasing sequence of $[0, \infty]$ −valued or non – negative extended real valued functions on $\mathbb{R}$ defined by $f_n = 1_{[n,\infty)}$; $\forall n \in N$ then we have

$$\int_D f_n d\mu_L = \int_D 1_{[n,\infty)}(x) d\mu_L = \mu_L([n, \infty)) = \infty$$

$$\Rightarrow lim_{n\to\infty} \int_D f_n d\mu = \infty$$

Now $f(x) = lim_{n\to\infty} f_n(x) = 0$ for decreasing sequence $\{f_n\}_1^\infty$ i.e. $\{f_n\}_1^\infty$converges to infimum then

$$f(x) = 0 \Rightarrow \int_D f d\mu = 0$$

Hence $\int_D f d\mu \neq lim_{n\to\infty} \int_D f_n d\mu$

…………………………………………………

Lemma (Without Proof)

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non – negative extended real valued functions on X then there exists a sequence of non – negative simple functions $\{g_n\}_1^\infty$ on X such that;

- g_n approaches f on X
- g_n approaches f uniformly on an arbitrary subset E of X on which f is bounded.
- $\int_D f d\mu = lim_{n\to\infty} \int_D g_n d\mu$

Corollary: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\sum_1^n f_k$ be a finite series whose terms are $[0, +\infty]$ −valued or non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$. Then $\int \sum_1^n f_k \, d\mu = \sum_1^n \int f_k d\mu$

Proof: Let f_1, f_2 be two non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ then there exists two increasing sequences $\{g_{n_1}\}_1^\infty$ and $\{g_{n_2}\}_1^\infty$ on X such that $g_{n_1} \to f_1$ and $g_{n_2} \to f_2$ then the $\{g_{n_1} + g_{n_2}\} \to f_1 + f_2$ as $n \to \infty$ and clearly is a non – negative increasing sequence of simple functions on X. Then by monotone convergence theorem we have

$$lim_{n\to\infty}\int_D (g_{n_1} + g_{n_2}) d\mu = \int_D (f_1 + f_2) d\mu \qquad \text{..........(i)}$$

Now consider $lim_{n\to\infty}\int_D (g_{n_1} + g_{n_2}) d\mu = lim_{n\to\infty}\left(\int_D g_{n_1} d\mu + \int_D g_{n_2} d\mu\right)$

$$lim_{n\to\infty}\int_D (g_{n_1} + g_{n_2}) d\mu = lim_{n\to\infty}\int_D g_{n_1} d\mu + lim_{n\to\infty}\int_D g_{n_2} d\mu$$

$$lim_{n\to\infty}\int_D (g_{n_1} + g_{n_2}) d\mu = \int_D f_1 d\mu + \int_D f_2 d\mu \qquad \text{..........(ii)}$$

Using (i) and (ii) $\int_D (f_1 + f_2) d\mu = \int_D f_1 d\mu + \int_D f_2 d\mu$ ....(iii)

By repeated application of (iii) to the sequence $\sum_1^n f_k$ we obtain

$$\int \sum_1^n f_k \, d\mu = \sum_1^n \int f_k d\mu$$

……………………………………………….

Beppo Levi's Theorem: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\sum_1^\infty f_k$ be an infinite series whose terms are $[0, +\infty]$ −valued or non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$. Then $\int \sum_1^\infty f_k \, d\mu = \sum_1^\infty \int f_k d\mu$

Proof: If $\{f_n\}_1^\infty$ is a sequence of non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ then for $\sum_1^n f_k \; ; n \in N$ we have

$$\int \sum_1^n f_k \, d\mu = \sum_1^n \int f_k d\mu$$

Now the sum of the series $\sum_1^\infty f_k$ is the limit of the sequence of partial sums $\{\sum_1^n f_k \; ; n \in N\}$ and since $\{f_n\}$ is non – negative therefore $\{\sum_1^n f_k \; ; n \in N\}$ is increasing with $lim_{n\to\infty} S_k = lim_{n\to\infty} \sum_1^n f_k \, d\mu = \sum_1^\infty f_k$

Then by monotone convergence theorem $lim_{n\to\infty}\int \sum_1^n f_k \, d\mu = \sum_1^\infty \int f_k d\mu$

$lim_{n\to\infty} \sum_1^n \int f_k d\mu = \sum_1^\infty \int f_k d\mu$ i.e. $\int \sum_1^\infty f_k \, d\mu = \sum_1^\infty \int f_k d\mu$

Corollary: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$. Suppose $A, B \in \mathcal{A}$ such that $A \cup B = D$ and $A \cap B = \varphi$. If $f = 0$ on B Then $\int_D f d\mu = \int_A f d\mu$

Proof: Since f is non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ then there exists an increasing sequences $\{g_n\}_1^\infty$ of non – negative simple function such that $g_n \to f$ and since $0 \leq g_n \leq f$ also $f = 0$ on B therefore $g_n = 0$ on B for all $n \in N$ and $\int_B g_n d\mu = 0$ then

$$\int_D g_n d\mu = \int_A g_n d\mu + \int_B g_n d\mu = \int_A g_n d\mu \qquad \because \int_B g_n d\mu = 0$$

Now $g_n \to f$ on D implies $g_n \to f$ on A

Then by monotone convergence theorem

$$\int_D f d\mu = lim_{n\to\infty} \int_D g_n d\mu = lim_{n\to\infty} \int_A g_n d\mu = \int_A f d\mu$$

Hence $\int_D f d\mu = \int_A f d\mu$

……………………………………………….

Corollary:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$. If $\{D_i\}_1^n$ is a disjoint sequence in $\mathcal{A}$ such that $\cup_1^n D_i = D$ Then $\int_{\cup_1^n D_i} f d\mu = \sum_1^n \int_{D_i} f d\mu$

Proof: Let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ and $\{D_i\}_1^n$ be a disjoint sequence in $\mathcal{A}$ such that $\cup_1^n D_i = D$ Then define a function f_{D_i} on D by setting $f_{D_i}(x) = \begin{cases} f(x) & if\ x \in D_i \\ 0 & if\ x \in D/D_i \end{cases}$ then $\{f_{D_i}\}_1^n$ is a non – negative extended real valued $\mathcal{A}$ −measurable sequence of functions on $D \in \mathcal{A}$ and $\sum_1^n f_{D_i} = f$ then

$$\int_D f d\mu = \int_D \sum_1^n f_{D_i} d\mu = \sum_1^n \int_D f_{D_i} d\mu$$

$$\int_D f d\mu = \sum_1^n \int_{D_i} f_{D_i} d\mu \qquad \because \int_D f d\mu = \int_A f d\mu \text{ when } f = 0 \text{ on B}$$

$$\int_D f d\mu = \sum_1^n \int_{D_i} f d\mu \qquad \because f = f_{D_i} \text{ on } D_i \text{ where } \cup_1^n D_i = D \text{ and } D_i \cap D_j = \varphi$$

Corollary:

Let f be a bounded measurable function on a set of finite measure E. Suppose A and B are disjoint measurable subsets of E. Then

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$$

Proof:

Both $f.\chi_A$ and $f.\chi_B$ are bounded measurable functions on E. Since A and B disjoint,

$$f.\chi_{A\cup B} = f.\chi_A + f.\chi_B$$

Furthermore, for any measurable subset E_1 of E

$$\int_{E_1} f d\mu = \int_E f.\chi_E d\mu$$

Therefore, by the linearity of integration,

$$\int_{A\cup B} f d\mu = \int_E f.\chi_{A\cup B} d\mu = \int_E f.\chi_A d\mu + \int_E f.\chi_B d\mu = \int_A f d\mu + \int_B f d\mu$$

Hence $\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$

Corollary:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$. If $\{D_i\}_1^\infty$ is an increasing sequence in $\mathcal{A}$ such that $lim_{n\to\infty} D_i = D$ Then $\int_{D=\cup_1^\infty D_i} f d\mu = lim_{n\to\infty} \int_{D_i} f d\mu$

Proof:

Let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ and $\{D_i\}_1^\infty$ be a disjoint sequence in $\mathcal{A}$ such that $lim_{n\to\infty} D_i = D$ Then define a function f_{D_i} on D by setting $f_{D_i}(x) = \begin{cases} f(x) & if\ x \in D_i \\ 0 & if\ x \in D/D_i \end{cases}$ then $\{f_{D_i}\}_1^\infty$ is an increasing sequence with $lim_{n\to\infty} f_{D_i} = f$ on D.

So by monotone convergence theorem we have $lim_{n\to\infty} \int_D f_{D_i} d\mu = \int_D f d\mu$

$lim_{n\to\infty} \int_{D_i} f_{D_i} d\mu = \int_D f d\mu$

$lim_{n\to\infty} \int_{D_i} f d\mu = \int_D f d\mu$

……………………………………………

Corollary:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$. If $\{D_i\}_1^\infty$ is a disjoint sequence in $\mathcal{A}$ such that $\cup_1^\infty D_i = D$ Then $\int_{D=\cup_1^\infty D_i} f d\mu = \sum_1^\infty \int_{D_i} f d\mu$

Proof: Let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$ and $\{D_i\}_1^\infty$ be a disjoint sequence in $\mathcal{A}$ such that $\cup_1^\infty D_i = D$ Then define an increasing sequence $\{E_n\}_1^\infty$ such that;
$E_n = \cup_1^n D_i$ and $lim_{n\to\infty} E_n = \cup_1^\infty E_n = \cup_1^\infty D_i = D$ then by theorem

"Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$. If $\{D_i\}_1^\infty$ is an increasing sequence in $\mathcal{A}$ such that $lim_{n\to\infty} D_i = D$ Then $\int_{D=\cup_1^\infty D_i} f d\mu = lim_{n\to\infty} \int_{D_i} f d\mu$"

We have $\int_D f d\mu = lim_{n\to\infty} \int_{E_n} f d\mu$ or $\int_D f d\mu = lim_{n\to\infty} \int_{\cup_1^n D_i} f d\mu$

$\Rightarrow \int_D f d\mu = lim_{n\to\infty} \sum_1^n \int_{D_i} f d\mu$

$\Rightarrow \int_D f d\mu = \sum_1^\infty \int_{D_i} f d\mu$

The next result is often used to show that a function is integrable or to provide an upper bound for the value of an integral.

Fatou's Lemma

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_n\}_1^{\infty}$ be a sequence of $[0, +\infty]$ −valued A -measurable functions on X. Then $\int \underline{lmt} f_n d\mu \leq \underline{lmt} \int f_n d\mu$

Or Let $(X, \mathcal{A}, \mu)$ be a measure space, and then for every sequence $\{f_n\}_1^{\infty}$ of non – negative extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$, we have $\int lmtInf f_n d\mu \leq lmtInf \int f_n d\mu$

Proof: We have $\lim_{n\to\infty} inf f_n = \lim_{n\to\infty}(inf_{k\geq n} f_k)$ where $\{inf_{k\geq n} f_k\}_1^{\infty}$ is an increasing sequence of extended real valued $\mathcal{A}$ −measurable functions on $D \in \mathcal{A}$, then by monotone convergence theorem;

$$\int lmtInf f_n d\mu = \lim_{n\to\infty} \int inf_{k\geq n} f_k d\mu \qquad \text{…………….(i)}$$

Since $\{inf_{k\geq n} f_k\}_1^{\infty}$ is an increasing sequence in $\overline{\mathbb{R}}$ therefore its limit exists in $\overline{\mathbb{R}}$ and is equal to '$\lim\limits_{n\to\infty}(inf f_n)$' so that from (i)

$$\int lmtInf f_n d\mu = lmtInf \int (inf_{k\geq n} f_k) d\mu$$

$$\int lmtInf f_n d\mu \leq lmtInf \int f_n d\mu \qquad \because inf_{k\geq n} f_k \leq f_n ; \forall n \in N$$

Hence $\int lmtInf f_n d\mu \leq lmtInf \int f_n d\mu$

………………………………………………

Lebesgue's Dominated Convergence Theorem (Without Proof)

Let $(X, \mathcal{A}, \mu)$ be a measure space, let g be a $[0, +\infty]$ −valued integrable function on X, and let f and $f_1, f_2, \ldots$ be $[0, +\infty]$ −valued $\mathcal{A}$ − measurable functions on X such that $f(x) = lim_{n\to\infty} f_n(x)$ and $|f_n(x)| \leq g(x), n = 1, 2, \ldots$ hold at μ −almost every x in X. Then f and $f_1, f_2, \ldots$ are integrable, and

$$\int f d\mu = lim_{n\to\infty} \int f_n d\mu$$

The Riemann Integral

This section contains the standard facts that relate the Lebesgue integral to the Riemann integral. We begin by recalling Darboux's definition of the Riemann integral, as given in the Introduction (we use it as our basic definition), and then we give a number of details that we omitted earlier. We also give the standard characterization of the Riemann integrable functions on a closed bounded interval as the bounded functions on that interval that are almost everywhere continuous.

Partition of an Interval:

Let [a,b] be a closed bounded interval. A ***partition*** of [a,b] is a finite sequence $\{a_i\}_0^n$ of real numbers such that $a = a_0 < a_1 < \cdots < a_n = b$ and We will generally denote a partition by a symbol such as $\mathcal{p}$ or $\mathcal{p}_n$.

Refinement of a Partition

If $\{a_i\}_0^n$ and $\{b_i\}_0^m$are partitions of [a,b] and if each term of $\{a_i\}_0^n$ appears among the terms of $\{b_i\}_0^m$,then $\{b_i\}_0^m$ is a ***refinement of*** or is ***finer than*** $\{a_i\}_0^n$

Lower Sum and Upper Sum

Let f be a bounded real-valued function on [a,b].If $\mathcal{p}$ is the partition $\{a_i\}_0^n$ of [a,b] and if $m_i = inf\{ f(x): x \in [a_{i-1}, a_i]\}$ and $M_i = Sup\{ f(x): x \in [a_{i-1}, a_i]\}$ for $i = 1, \ldots, n$, then

- The ***lower sum*** $l(f, \mathcal{p})$ corresponding to f and $\mathcal{p}$ is defined to be $\sum_1^n m_i(a_i - a_{i-1})$
- The ***upper sum*** $u(f, \mathcal{p})$ corresponding to f and $\mathcal{p}$ is defined to be $\sum_1^n M_i(a_{i-1} - a_i)$
- It is easy to check that if $\mathcal{p}$ is an arbitrary partition of [a,b],then $l(f, \mathcal{p}) \leq u(f, \mathcal{p})$
- If $\mathcal{p}_1$ and $\mathcal{p}_2$ are partitions of [a,b] such that $\mathcal{p}_2$ is a refinement of $\mathcal{p}_1$, then $l(f, \mathcal{p}_1) \leq l(f, \mathcal{p}_2)$ and $u(f, \mathcal{p}_2) \leq u(f, \mathcal{p}_1)$
- If $\mathcal{p}_1$ and $\mathcal{p}_2$ are arbitrary partitions of [a,b],then $l(f, \mathcal{p}_1) \leq u(f, \mathcal{p}_2)$
- Let $\mathcal{p}_3$ be a partition of [a,b] that is a refinement of both $\mathcal{p}_1$ and $\mathcal{p}_2$ and note that $l(f, \mathcal{p}_1) \leq l(f, \mathcal{p}_3) \leq u(f, \mathcal{p}_3) \leq u(f, \mathcal{p}_2)$

Hence the set of all lower sums for f is bounded above by each of the upper sums for .

Lower Integral of a Function

Let $(X, \mathcal{A}, \mu)$ be a measure space and f is bounded function defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$ then the suprimum of the set of lower sums $l(f, \wp)$ is the lower integral of f over [a,b] and is denoted by the formula $\int_{-a}^{b} f dx = Sup_{g \leq f} \int g(x)dx$ where $g(x)$ is simple function.

The lower integral satisfies b a $\int_{-a}^{b} f dx \leq u(f, \wp)$for each upper sum $u(f, \wp)$ and so is a lower bound for the set of all upper sums for .

Upper Integral of a Function

Let $(X, \mathcal{A}, \mu)$ be a measure space and f is bounded function defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$ then the infimum of the set of upper sums is the upper integral of f over [a,b] and is denoted by $\int_{a}^{-b} f dx = Inf_{f \leq g} \int g(x)dx$ where $g(x)$ is simple function.

It follows immediately that $\int_{-a}^{b} f dx \leq \int_{a}^{-b} f dx$ and $\int_{-a}^{b} f dx = \int_{a}^{-b} f dx$,then f is Riemann integrable on [a,b], and the common value of $\int_{-a}^{b} f dx$ and $\int_{a}^{-b} f dx$ is called the Riemann integral of f over [a,b] and is denoted by $\int_{a}^{b} f dx$ or $\int_{a}^{b} f(x)dx$

Riemann Integral

Let f be a step function on [a,b] then Riemann Integral of f on [a,b] is defined by

$$\int_{a}^{b} f(x)dx = \sum_{1}^{n} c_i \Delta(x_i) = \sum_{1}^{n} c_i (x_i - x_{i-1})$$

Remark

- We can write the step function as $f(x) = \sum_{1}^{n} c_i 1_{(x_{i-1}.x_i)}(x) + \sum_{0}^{n} d_i 1_{(x_i)}(x)$
- The step function's value at the end points of the sub – intervals have no bearing on existence or value of Riemann Integral the step function $f(x)$(Since d_i does not appear in the definition of integral)
- The value of Riemann Integral of step function is independent of choice of partition of [a,b] as long as step function is constant on the sub – interval of the partition.

The following reformulation of the definition of Riemann integrability is often useful.

- **Lemma:** A bounded function $f: [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every positive ε there is a partition $\wp$ of [a,b] such that $u(f, \wp) - l(f, \wp) < \varepsilon$
- **Theorem:** Let [a,b] be a closed bounded interval, and let f be a bounded real-valued function on [a,b].Then
  (a) f is Riemann integrable if and only if it is continuous at almost every point of [a,b], and
  (b) if f is Riemann integrable, then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.
- The ***mesh*** or norm $\|\wp\|$ of a partition (or a ***tagged*** partition) $\wp$ is defined by $\wp = max(a_i - a_{i-1})$, where $\{a_i\}$ is the sequence of division points for $\wp$. In other words, the ***mesh of a partition*** is the length of the longest of its subintervals.
- **The Riemann sum $\mathfrak{R}(f, \wp)$** corresponding to the function f and the tagged partition $\wp$ is defined by $\mathfrak{R}(f, \wp) = \sum_1^n f(x_i)(a_i - a_{i-1})$
- **Proposition:** A function $f: [a, b] \to \mathbb{R}$ is Riemann integrable if and only if there is a real number L such that $lmt_{\wp}\mathfrak{R}(f, \wp) = L$ where the limit is taken as the mesh of the tagged partition $\wp$ approaches 0.Ifthis limit exists, then it is equal to the Riemann integral $\int_a^b f(x)dx$

…………………………………………………

Remember

- Let $(X, \mathcal{A}, \mu)$ be a measure space and f is simple function defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$ then f will be Lebesgue Integrable.
- **Bounded Function:** Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be a function defined on $D \in \mathcal{A}$ then f is said to be bounded if there exists a real number $M > 0$ such that $|f(x)| \leq M \ ; \ \forall x \in D$
- Let g and h be simple functions on $D \in \mathcal{A}$ such that $g(x) \leq f(x) \leq h(x)$ then such pairs of simple functions always exists when $f(x)$ is bounded. Such pairs always exists for instance $g(x) = -M$ and $h(x) = M$ will do.

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space and f_1, f_2 are bounded real valued measurable functions defined on $D \in \mathcal{A}$ with $(D) < \infty$, if $f_1 = f_2$ almost everywhere on D then $\int_D f_1 d\mu = \int_D f_2 d\mu$

Proof: Let $\Phi_i; i = 1,2,3, \ldots$ be the collection of all simple functions g_i on D such that $g_i \leq f_i$ then $\int_D f_1 d\mu = Sup_{g_1 \leq f_1}\{\int_D g_1 d\mu : g_1 \in \Phi_1\}$

And $\int_D f_2 d\mu = Sup_{g_2 \leq f_2}\{\int_D g_2 d\mu : g_2 \in \Phi_2\}$

Firstly we will show that for every $g_1 \in \Phi_1$ and $g_2 \in \Phi_2$ such that

$\int_D g_1 d\mu = \int_D g_2 d\mu$

Since $f_1 = f_2$ almost everywhere on D then there exists a null set $D_0 \subseteq D$ such that $f_1(x) = f_2(x)$ almost everywhere on D/D_0

Since f_1 and f_2 are bounded on D then there exists $M > 0$ such that

$f(x_1), f(x_2) \in [-M, M]$ implies $-M \leq f(x_1), f(x_2) \leq M$

Define a simple function g_2 on D by setting $g_2(x) = \begin{cases} g_1(x) & ; x \in D/D_0 \\ -M & ; x \in D_0 \end{cases}$ then

$g_2 \leq f_2$ $\because g_1 \leq f_1$ and $f_1 = f_2$ a.e. on D/D_0 so that $g_1 \leq f_2$; $-M \leq f_2$

Hence $g_2 \in \Phi_2$

Then $\int_D g_1 d\mu = \int_{D/D_0} g_1 d\mu + \int_{D_0} g_1 d\mu = \int_{D/D_0} g_1 d\mu$ $\because \mu(D_0) = 0$

Since $\mu(D_0) = 0 \Rightarrow \int_{D_0} g_1 d\mu = \int_{D_0} g_2 d\mu = 0$ therefore $\int_D g_1 d\mu = \int_{D/D_0} g_2 d\mu$

$\int_D g_1 d\mu = \int_{D/D_0} g_2 d\mu + \int_{D_0} g_2 d\mu$

$\int_D g_1 d\mu = \int_D g_2 d\mu$

Thus $\{\int_D g_1 d\mu : g_1 \in \Phi_1\} \subseteq \{\int_D g_2 d\mu : g_2 \in \Phi_2\}$

$Sup_{g_1 \leq f_1}\{\int_D g_1 d\mu : g_1 \in \Phi_1\} \leq Sup_{g_2 \leq f_2}\{\int_D g_2 d\mu : g_2 \in \Phi_2\}$

$\int_D f_1 d\mu \leq \int_D f_2 d\mu$ …………(i)

Interchanging the roles of functions we arrive $\int_D f_2 d\mu \leq \int_D f_1 d\mu$ …………(ii)

From (i) and (ii) $\int_D f_1 d\mu = \int_D f_2 d\mu$

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space and f be a bounded real valued $\mathcal{A}-$ measurable function defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$, then for any real constant 'c' we have then $\int_D cf d\mu = c\int_D f d\mu$

Proof: Since $c \in \mathbb{R}$ then there will be the three cases;

Case – I: If $c = 0$ then $cf = 0$ on $D \in \mathcal{A}$ and $\int_D cf d\mu = 0$

Also since $\int_D f d\mu \in \mathbb{R}$ and $c = 0$ then $c\int_D f d\mu = 0$

Hence $\int_D cf d\mu = c\int_D f d\mu$

Case – II: If $c > 0$ then $\int_D cf d\mu = Sup_{g \le cf}\int_D g d\mu = Sup_{\frac{1}{c}g \le f}\int_D g d\mu$

$\int_D cf d\mu = Sup_{\frac{1}{c}g \le f} c\int_D \frac{1}{c} g d\mu = cSup_{\frac{1}{c}g \le f}\int_D \frac{1}{c} g d\mu = c\int_D f d\mu$

Hence $\int_D cf d\mu = c\int_D f d\mu$

Case – III: If $c < 0$ then $-c < 0$ and so

$\int_D cf d\mu = \int_D -|c| f d\mu$ where $|c| = \begin{cases} -c & ; c < 0 \\ c & ; c > 0 \end{cases}$ ………..(i)

Now if $c = -1$ then $\int_D - f d\mu = Sup_{g \le -f}\int_D g d\mu = -Inf_{f \le -g}\int_D - g d\mu$

$\int_D - f d\mu = -\int_D f d\mu$ ………..(ii)

$(1) \Rightarrow \int_D cf d\mu = \int_D -|c| f d\mu = -\int_D |c| f d\mu$

$\Rightarrow \int_D cf d\mu = -|c|\int_D f d\mu$ ∵Reseult for $c > 0$

Hence $\int_D cf d\mu = c\int_D f d\mu$

Remember

$Sup(D) = -Inf(-D)$

Let $D = \{1,2,3,4\}$ then $-D = \{-1, -2, -3, -4\}$

Then $Sup(D) = 4$ also $Inf(-D) = -4$

And $Sup(D) = 4 = -Inf(-D)$

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space and f, f_1, f_2 be bounded real valued $\mathcal{A}-$ measurable functions defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$ then

$\int_D (f_1 + f_2) d\mu = \int_D f_1 d\mu + \int_D f_2 d\mu$

Proof: Let g_1, g_2 be simple functions on $D \in \mathcal{A}$ such that $g_1 \leq f_1$ and $g_2 \leq f_2$ then $g_1 + g_2$ be simple functions on $D \in \mathcal{A}$ and

$\int_D g_1 d\mu + \int_D g_2 d\mu = \int_D (g_1 + g_2) d\mu$

$\int_D g_1 d\mu + \int_D g_2 d\mu = \int_D g d\mu$ where we use $g = g_1 + g_2$

Also f_1, f_2 are bounded therefore their sum $f_1 + f_2 = f$ (Say) is bounded and $g_1 + g_2 \leq f_1 + f_2$ i.e. $g \leq f$

$\Rightarrow Sup_{g_1 \leq f_1} \int_D g_1 d\mu + \int_D g_2 d\mu \leq Sup_{g \leq f} \int_D f d\mu$

$\Rightarrow \int_D f_1 d\mu + Sup_{g_2 \leq f_2} \int_D g_2 d\mu \leq \int_D f d\mu$

$\Rightarrow \int_D f_1 d\mu + \int_D f_2 d\mu \leq \int_D f d\mu$ ………..(i)

Similarly h_1, h_2 be simple functions on $D \in \mathcal{A}$ such that $f_1 \leq h_1$ and $f_2 \leq h_2$ then $h_1 + h_2$ be simple functions on $D \in \mathcal{A}$ and

$\int_D (h_1 + h_2) d\mu = \int_D h_1 d\mu + \int_D h_2 d\mu$

$\int_D h d\mu = \int_D h_1 d\mu + \int_D h_2 d\mu$ Where we use $h = h_1 + h_2$

Let $f_1 \leq h_1$ and $f_2 \leq h_2$ therefore $f_1 + f_2 \leq h_1 + h_2$ i.e. $f \leq h$ then

$\Rightarrow \int_D f d\mu \leq inf_{f_1 \leq h_1} \int_D h_1 d\mu + \int_D h_2 d\mu$

$\Rightarrow \int_D f d\mu \leq \int_D f_1 d\mu + inf_{f_2 \leq h_2} \int_D h_2 d\mu$

$\Rightarrow \int_D f d\mu \leq \int_D f_1 d\mu + \int_D f_2 d\mu$ ………..(ii)

From (i) and (ii) we have $\int_D f d\mu \leq \int_D f_1 d\mu + \int_D f_2 d\mu$

Where $f = f_1 + f_2$

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space and f be a bounded real valued $\mathcal{A}-$ measurable function defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$, Let $\{D_n\}_1^\infty$ be a disjoint sequence in $\mathcal{A}$ such that $\cup_1^\infty D_n = D$ then $\int_D f d\mu = \sum_1^\infty \int_{D_n} f d\mu$

Proof: Let g be an arbitrary simple function defined on $D \in \mathcal{A}$ such that $g \leq f$ on $D \in \mathcal{A}$. Also consider $g(x) = \sum_1^k c_i 1_{E_i}(x)$ be canonical representation of g. If we consider g_n be a restriction of g to D_n then its canonical representation will be $g_n(x) = \sum_1^k c_i 1_{E_i \cap D_n}(x)$. Noting that $\cup_1^k (E_i \cap D_n) = D_n$ then

$\int_D g d\mu = \sum_1^k c_i \mu(E_i) = \sum_1^k c_i \mu(E_i \cap D)$ $\quad \because E_i \subseteq D \Rightarrow E_i \cap D = E_i$

$\int_D g d\mu = \sum_1^k c_i \mu(E_i \cap (\cup_1^\infty D_n)) = \sum_1^k c_i \mu[\cup_1^\infty (E_i \cap D_n)]$

$\int_D g d\mu = \sum_{i=1}^k c_i \sum_{n=1}^\infty \mu(E_i \cap D_n) = \sum_{n=1}^\infty \left[\sum_{i=1}^k c_i \mu(E_i \cap D_n)\right]$

$\int_D g d\mu = \sum_{n=1}^\infty \int_{D_n} g_n d\mu$ …………………..(i)

$\because g \leq f \Rightarrow g_n \leq f$ on D

Then $\int_{D_n} g_n d\mu \leq Sup_{g \leq f} \int_{D_n} g d\mu \Rightarrow \int_{D_n} g_n d\mu \leq \int_{D_n} f d\mu$

$(i) \Rightarrow \int_D g d\mu \leq \sum_{n=1}^\infty \int_{D_n} f d\mu$ …………………..(ii)

Where the least inequality is from the fact that g_n is simple function on D and $g_n \leq f$ on D. So that $\int_{D_n} g_n d\mu \leq Sup_{g \leq f} \int_{D_n} g_n d\mu = \int_{D_n} f d\mu$

$\Rightarrow \int_{D_n} g_n d\mu \leq \int_{D_n} f d\mu \Rightarrow \int_D g d\mu \leq \sum_{n=1}^\infty \int_{D_n} f d\mu$

$\Rightarrow Sup_{g \leq f} \int_D g d\mu \leq \sum_{n=1}^\infty \int_{D_n} f d\mu$ $\quad \because g$ is arbitrary

$\Rightarrow \int_D f d\mu \leq \sum_{n=1}^\infty \int_{D_n} f d\mu$ …………………..(iii)

Similarly by starting with a simple function h such that $f \leq h$ on D we obtain $\sum_{n=1}^\infty \int_{D_n} f d\mu \leq inf_{f \leq h} \int_D h d\mu$

$\Rightarrow \sum_{n=1}^\infty \int_{D_n} f d\mu \leq \int_D f d\mu$ …………………..(iv)

Combining (iii) and (iv) $\quad \int_D f d\mu = \sum_1^\infty \int_{D_n} f d\mu$

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space and f be a bounded real valued $\mathcal{A} -$ measurable function defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$, if $f \geq 0$ almost everywhere on $D \in \mathcal{A}$ and $\int_D f d\mu = 0$ then $f = 0$ almost everywhere on $D \in \mathcal{A}$.

Proof: Consider the first case that $f \geq 0$ on $D \in \mathcal{A}$ and let $D_0 = \{x \in D: f = 0\}$ and $D_1 = \{x \in D: f > 0\}$ then $D_0 \cap D_1 = \varphi$ and $D_0 \cup D_1 = D$

We claim that $f = 0$ almost everywhere on $D \in \mathcal{A}$ iff $\mu(D_1) = 0$ ……….(i)

Suppose $f = 0$ almost everywhere on $D \in \mathcal{A}$ then there exists a null set $E \subseteq D$ in $(X, \mathcal{A}, \mu)$ such that $f = 0$ on D/E

Then $D/E \subseteq D_0 \Rightarrow D/E \subseteq D/D_1$ $\qquad \because D_0 \cup D_1 = D \Rightarrow D_0 = D/D_1$

$\Rightarrow D_1 \subseteq E \Rightarrow \mu(D_1) \leq \mu(E)$ $\qquad \because$ by monotonicity of μ

$\Rightarrow \mu(D_1) = 0$ $\qquad \because \mu(E) = 0$ as E is null.

Conversely Suppose that $\mu(D_1) = 0$ then D_1 is a null set in $(X, \mathcal{A}, \mu)$ but $f = 0$ on $D_0 = D/D_1$ i.e. $f = 0$ almost everywhere on $D \in \mathcal{A}$

If $\mu(D) = 0$ then from $D_1 \subseteq D$ we have $\mu(D_1) \leq \mu(D) = 0 \Rightarrow \mu(D_1) = 0$

$\Rightarrow f = 0$ almost everywhere on $D \in \mathcal{A}$ $\qquad$ by (i)

Now consider $\mu(D) \epsilon (0, \infty)$ then we have to show that $f = 0$ almost everywhere on $D \in \mathcal{A}$. But contrarily suppose that $f = 0$ almost everywhere on $D \in \mathcal{A}$ is false then by (i) we have $\mu(D_1) > 0$.

Now $D_1 = \{x \in D: f > 0\} = D_1 = \cup_{k=1}^{n} \left\{x \in D: f \geq \frac{1}{k}\right\}$ then

$0 < \mu(D_1) < \sum_{k=1}^{n} \left\{x \in D: f \geq \frac{1}{k}\right\}$ then there exists $k_0 \in N$ such that

$\mu\left(\left\{x \in D: f \geq \frac{1}{k_0}\right\}\right) > 0$

Define a simple function g on D by $g(x) = \begin{cases} \frac{1}{k_0} & ; x \in \left\{x \in D: f \geq \frac{1}{k_0}\right\} \\ 0 & ; x \in D/\left\{x \in D: f \geq \frac{1}{k_0}\right\} \end{cases}$ then

$g(x) \leq f(x)$ on D

$\Rightarrow \int_D f(x) d\mu \geq \int_D g(x) d\mu = \frac{1}{k_0} \mu\left(\left\{x \in D: f \geq \frac{1}{k_0}\right\}\right) > 0$

$\Rightarrow \int_D f(x)d\mu > 0$ contradiction to $\int_D f(x)d\mu = 0$

Hence $f = 0$ almost everywhere on $D \in \mathcal{A}$

Now consider if $f \geq 0$ on $D \in \mathcal{A}$ and $\int_D f d\mu = 0$ then

$f = 0$ almost everywhere on $D \in \mathcal{A}$ ………………(ii)

Now consider if $f \geq 0$ almost everywhere on $D \in \mathcal{A}$ and $\int_D f d\mu = 0$ then there exists a null set E in $(X, \mathcal{A}, \mu)$ such that $f \geq 0$ on D/E then

$0 = \int_D f d\mu = \int_{D/E} f d\mu + \int_E f d\mu = \int_{D/E} f d\mu$

$\Rightarrow \int_{D/E} f d\mu = 0$

Now $f \geq 0$ on D/E and $\int_{D/E} f d\mu = 0$ implies $f = 0$ a.e. on $D \in \mathcal{A}$ by (ii) then there exists a null set F in $(X, \mathcal{A}, \mu)$ such that $F \in D/E$ and $f = 0$ on $(D/E)/F$

Implies $f = 0$ on $D/E \cup F$

Hence $f = 0$ almost everywhere on $D \in \mathcal{A}$ $\because E \cup F$ is null set

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Lemma:

Let $(X, \mathcal{A}, \mu)$ be a measure space and f and g be bounded real valued $\mathcal{A}-$ measurable functions defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$, if $f \leq g$ almost everywhere on $D \in \mathcal{A}$ and $\int_D f d\mu = \int_D g d\mu$ then $f = g$ almost everywhere on $D \in \mathcal{A}$.

Proof: If $f \leq g$ almost everywhere on $D \in \mathcal{A}$ then $g - f \geq 0$ almost everywhere on $D \in \mathcal{A}$ and in addition $\int_D f d\mu = \int_D g d\mu$ then $\int_D (g - f) d\mu = 0$ then

By theorem "if $f \geq 0$ almost everywhere on $D \in \mathcal{A}$ and $\int_D f d\mu = 0$ then $f = 0$ almost everywhere on $\in \mathcal{A}$ " we have

$g - f = 0$ almost everywhere on $D \in \mathcal{A}$

Implies $f = g$ almost everywhere on $D \in \mathcal{A}$.

The Simple Approximation Theorem

An extended real-valued function *f* on a measurable set E is measurable if and only if there is a sequence $\{g_n\}$ of simple functions on E which converges pointwise on E to *f* and has the property that $|g_n| \leq |f|$ for all n.

If *f* is nonnegative, we may choose $\{g_n\}$ to be increasing.

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Theorem

Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.

Proof

Let *n* be a natural number. By the Simple Approximation Lemma,

"An extended real-valued function *f* on a measurable set E is measurable if and only if there is a sequence $\{g_n\}$ of simple functions on E which converges pointwise on E to *f* and has the property that $|g_n| \leq |f|$ for all n."

With $\in = \frac{1}{n}$ there are two simple functions g_n and h_n defined on E for which $g_n \leq f \leq h_n$ on E, and $0 \leq h_n - g_n \leq \frac{1}{n}$ on E.

By the monotonicity and linearity of the integral for simple functions,

$$0 \leq \int h_n d\mu - \int g_n d\mu = \int (h_n - g_n) d\mu \leq \frac{1}{n}\mu(E)$$

However,

$$0 \leq inf\left\{\int hd\mu : h\ simple, h \geq f\right\} - sup\left\{\int gd\mu : g\ simple, g \leq f\right\}$$

$$0 \leq \int h_n d\mu - \int g_n d\mu \leq \frac{1}{n}\mu(E)$$

This inequality holds for every natural number *n* and $\mu(E)$ is finite. Therefore the upper and lower Lebesgue integrals are equal and thus the function f is integrable over E.

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a complex valued function on X that is measurable with respect to $\mathcal{A}$ and $\mathfrak{B}$(C).Then f is integrable if and only if $|f|$ is integrable. If these functions are integrable, then $\left|\int f d\mu\right| \leq \int |f| d\mu$

Or Let f be a bounded measurable function on a set of finite measure E. Then $\left|\int_E f d\mu\right| \leq \int_E |f| d\mu$

Proof:

Let $\Re(f)$ and $\Im(f)$ be the real and imaginary parts of f .If f is integrable, then the integrability of $|f|$ follows from the inequality $|f| \leq |\Re(f)| + |\Im(f)|$,

while if $|f|$ is integrable, then the integrability of f follows from the inequalities $|\Re(f)| \leq |f|$ and $|\Im(f)| \leq |f|$

Now suppose that f is integrable. Write the complex number $\int f d\mu$ in its polar form, letting w be a complex number of absolute value 1 i.e. $|w| = 1$ such that

$\int f d\mu = w\left|\int f d\mu\right|$

$\Rightarrow \left|\int f d\mu\right| = w^{-1}\int f d\mu$

$\Rightarrow \left|\int f d\mu\right| = \int w^{-1} f d\mu$

$\Rightarrow \left|\int f d\mu\right| = \int \Re(w^{-1} f) d\mu$ $\quad\because |w^{-1}| = 1$

$\Rightarrow \left|\int f d\mu\right| \leq \int |f| d\mu$ $\quad\because |\Re(f)| \leq |f|$

Hence $\left|\int f d\mu\right| \leq \int |f| d\mu$

2nd Method

The function $|f|$ is measurable and bounded.

Now $-|f| \leq f \leq |f|$ on E.

By the linearity and monotonicity of integration,

$-\int_E |f| \leq \int_E f \leq \left|\int_E f d\mu\right| \leq \int_E |f|$

Hence $\left|\int_E f d\mu\right| \leq \int_E |f| d\mu$

Convergence

In this chapter we look in some detail at the convergence of sequences of functions.

Uniform Convergence

Let $(X, \mathcal{A}, \mu)$ be a measure space, A sequence of extended real valued functions $\{f_n\}_1^\infty$ converges uniformly on a set $D \in \mathcal{A}$ to an extended real valued function f if for every $\in > 0$ there exists $n_0 \in N$ depending upon $\in$ but not on $x \in D$ such that

$|f_n(x) - f(x)| < \in$ for all $x \in D$ whenever $n \geq n_0 \in N$

Or equivalently for all $m \in N$ such that

$|f_n(x) - f(x)| < \frac{1}{m}$ for all $x \in D$ whenever $n \geq n_0 \in N$

Almost Uniform Convergence

Let $(X, \mathcal{A}, \mu)$ be a measure space, A sequence of extended real valued functions $\{f_n\}_1^\infty$ converges almost uniformly on a set $D \in \mathcal{A}$ to an extended real valued function f if for every $\eta > 0$ there exists a $\mathcal{A}$ −measurable subset E of D such that $\mu(E) < \eta$ and $\{f_n\}_1^\infty$ converges uniformly on a set D/E

Examples:

We should note that in general convergence in measure neither implies nor is implied by convergence almost everywhere.

(a) To see that convergence almost everywhere does not imply convergence in measure, consider the space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda)$ and the sequence whose nth term is the characteristic function of the interval $[n, +\infty)$. This sequence clearly converges to the zero function almost everywhere (in fact, everywhere) but not in measure.

(b) Consider the interval [0,1), together with the σ-algebra of Borel subsets of [0,1) and Lebesgue measure. Let $\{f_n\}$ be the sequence whose first term is the characteristic function of [0,1), whose next two terms are the characteristic functions of [0,1/2) and [1/2,1), whose next four terms are the characteristic functions of [0,1/4), [1/4,1/2), [1/2,3/4),and [3/4,1), and so on. Then $\{f_n\}$ converges to the zero function in measure, but for each x in [0,1) the sequence $\{f_n(x)\}$ contains infinitely many ones and infinitely many zeros and so is not convergent.

Remark

- **Proposition:** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \ldots$ be real valued $\mathcal{A} -$ measurable functions on X. If μ is finite and if $\{f_n\}$ converges to f almost everywhere, then $\{f_n\}$converges to f in measure.
- **Proposition:** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \ldots$ be real valued $\mathcal{A} -$measurable functions on X. If $\{f_n\}$ converges to f in measure, then there is a subsequence of $\{f_n\}$that converges to f almost everywhere.

Proposition: (Egoroff's Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \ldots$ be real valued $\mathcal{A} -$ measurable functions on X. If μ is finite and if $\{f_n\}$ converges to f almost everywhere, then for each positive number ε there is a subset B of X that belongs to $\mathcal{A}$, satisfies $\mu(B^c) < \varepsilon$, and is such that $\{f_n\}$ converges to f uniformly on B.

Or Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_n\}_1^\infty$ be a sequence of extended real valued $\mathcal{A} -$ measurable functions on $D \in \mathcal{A}$ with $\mu(D) < \infty$. Let f be a real valued $\mathcal{A} -$ measurable function on $D \in \mathcal{A}$. If $\{f_n\}_1^\infty$ converges to f almost everywhere then if $\{f_n\}_1^\infty$ converges to f almost uniformly on $D \in \mathcal{A}$.

Proof: Let ε be a positive number, and for each *n* let $g_n = sup_{j \geq n} \mid f_j - f \mid$. It is easy to check that each g_n is finite almost everywhere. The sequence $\{g_n\}$ converges to 0 almost everywhere, and so in measure. Hence for each positive integer *k* we can choose a positive integer n_k such that

$$\mu(\left\{x \epsilon X : g_{n_k}(x) > \frac{1}{k}\right\}) < \frac{\varepsilon}{2^k}$$

Define sets $B_1, B_2, \ldots$ by $B_k = \left\{x \epsilon X : g_{n_k}(x) \leq \frac{1}{k}\right\}$ and let $B = \cap_k B_k$. Then the set B satisfies $\mu(B^c) = \mu(\cup_k B_k^c) \leq \sum_k \mu(B_k^c) < \sum_k \frac{\varepsilon}{2^k} = \varepsilon$

Implies $\mu(B^c) < \varepsilon$

If δ is a positive number and if k is a positive integer such that $1/k < \delta$, then, since $B \subseteq B_k$ then $|f_n(x) - f(x)| \leq g_{n_k}(x) \leq \frac{1}{k} < \delta$ holds for all x in B and all positive integers n such that $n \geq n_k$; thus $\{f_n\}$ converges to f uniformly on B.

Bounded Convergence Theorem

Let $(X, \mathcal{A}\ , \mu)$ be a measure space, and let $\{f_n\}_1^\infty$ be a bounded sequence of real valued $\mathcal{A} -$ measurable functions on $D \in \mathcal{A}$ with $\mu(D) < \infty$. Let f be a bounded real valued $\mathcal{A} -$ measurable function on $D \in \mathcal{A}$. If $\{f_n\}_1^\infty$ converges to f almost everywhere on $D \in \mathcal{A}$ then $\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0$

And in particular $\lim_{n\to\infty} \int_D f_n d\mu = \int_D \lim\limits_{n\to\infty} f_n d\mu = \int_D f d\mu$

Proof:

Since $\{f_n\}_1^\infty$ is bounded on $D \in \mathcal{A}$ therefore there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in D$ and for all $n \in N$. Since f is also bounded on $D \in \mathcal{A}$ then we can assume that there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$

Now $\{f_n\}_1^\infty$ converges to f almost everywhere on $D \in \mathcal{A}$ and $\mu(D) < \infty$ therefore by Egoroff's Theorem "$\{f_n\}_1^\infty$ converges to f almost uniformly on $D \in \mathcal{A}$" then for all $\eta > 0$ there exists a $\mathcal{A}$ −measurable subset E of D such that $\mu(E) < \eta$ and $\{f_n\}_1^\infty$ converges uniformly on a set D/E. Then for all $\in > 0$ there exists $n_0 \in N$ depending upon $\in$ but not on x such that

$|f_n(x) - f(x)| < \in$ for all $x \in D/E$ whenever $n \geq n_0 \in N$

Now for $n \geq n_0$ we have

$$\int_D |f_n - f| d\mu = \int_{D/E} |f_n - f| d\mu + \int_E |f_n - f| d\mu$$

$$\int_D |f_n - f| d\mu \leq \int_{D/E} \in d\mu + \int_E 2M d\mu \qquad \because |f_n - f| \leq |f_n| + |f| \leq M + M$$

$$\int_D |f_n - f| d\mu \leq \in \int_{D/E} d\mu + 2M \int_E d\mu = \in \mu(D/E) + 2M\mu(E)$$

$$\int_D |f_n - f| d\mu \leq \in \mu(D) + 2M\eta \qquad \text{Hence this holds for all } n \geq n_0$$

We have $lmtSup \int_D |f_n - f| d\mu \leq \in \mu(D) + 2M\eta$

Hence this is true for every $\in > 0$ and $\eta > 0$ therefore $lmtSup \int_D |f_n - f| d\mu = 0$

Also we have $\int_D |f_n - f| d\mu \geq 0 \qquad \because |f_n - f| \geq 0$

Therefore $0 \leq lmtInf \int_D |f_n - f| d\mu \leq lmtSup \int_D |f_n - f| d\mu$

Implies $lmtInf \int_D |f_n - f| d\mu = 0$ and hence $\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0$

Now we have to prove $\lim_{n\to\infty}\int_D f_n d\mu = \int_D \lim_{n\to\infty} f_n d\mu = \int_D f d\mu$

Since $\int_D (f_1 + f_2) d\mu = \int_D f_1 d\mu + \int_D f_2 d\mu$

Therefore $\left|\int_D f_n d\mu - \int_D f d\mu\right| = \left|\int_D (f_n - f) d\mu\right| \le \int_D |f_n - f| d\mu$

$\left|\int_D f_n d\mu - \int_D f d\mu\right| \le \int_D |f_n - f| d\mu$

$\lim\limits_{n\to\infty}\left|\int_D f_n d\mu - \int_D f d\mu\right| \le \lim\limits_{n\to\infty}\int_D |f_n - f| d\mu$

$\lim\limits_{n\to\infty}\left|\int_D f_n d\mu - \int_D f d\mu\right| = 0$ $\qquad \because \lim_{n\to\infty}\int_D |f_n - f| d\mu = 0$

$\lim\limits_{n\to\infty}\left(\int_D f_n d\mu - \int_D f d\mu\right) = 0$ Hence $\lim_{n\to\infty}\int_D f_n d\mu = \int_D \lim\limits_{n\to\infty} f_n d\mu = \int_D f d\mu$

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ℓ^p Spaces: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let p satisfy $1 \le p < \infty$. Then $\ell^p(X, \mathcal{A}, \mu, \mathbb{R})$ is the set of all $\mathcal{A}-$ measurable functions $f: X \to \mathbb{R}$ such that $|f|^p$ is integrable, and $\ell^p(X, \mathcal{A}, \mu, \mathbb{C})$ is the set of all $\mathcal{A}-$measurable functions $f: X \to \mathbb{C}$ such that $|f|^p$ is integrable.

In discussions that are valid for both real- and complex-valued functions we will often use $\ell^p(X, \mathcal{A}, \mu)$ to represent either $\ell^p(X, \mathcal{A}, \mu, \mathbb{R})$ or $\ell^p(X, \mathcal{A}, \mu, \mathbb{C})$.

ℓ^∞ Spaces: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $p = \infty$ then $\ell^\infty(X, \mathcal{A}, \mu, \mathbb{R})$ be the set of all bounded real valued $\mathcal{A}-$measurable functions on X, and $\ell^\infty(X, \mathcal{A}, \mu, \mathbb{C})$ be the set of all bounded complex-valued $\mathcal{A}-$measurable functions on X.

Remember: (This is already we have done in previous classes)

- **Lemma:** Let p satisfy $1 < p < \infty$, let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$, and let x and y be nonnegative real numbers. Then $xy < \frac{x^p}{p} + \frac{y^q}{q}$
- **Holder's Inequality:** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let p and q satisfy $1 \le p \le +\infty, 1 \le q \le +\infty$, and $\frac{1}{p} + \frac{1}{q} = 1$.If $f \in \ell^p(X, \mathcal{A}, \mu)$ and $g \in \ell^q(X, \mathcal{A}, \mu)$, then $fg \in \ell^1(X, \mathcal{A}, \mu)$ and satisfies $\int |fg| d\mu \le \|f\|_p \|g\|_q$
- **Minkowski's Inequality:** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let p satisfy $1 \le p \le +\infty$. If $f, g \in \ell^p(X, \mathcal{A}, \mu)$,then $f + g \in \ell^p(X, \mathcal{A}, \mu)$ and $\|f + g\|_p \le \|f\|_p + \|g\|_p$

Product measures

Let (X, $\mathcal{A}$) and (Y, $\mathcal{B}$) be measurable spaces. Consider the Cartesian product $X \times Y$. A subset of $X \times Y$ is said to be a rectangle with measurable sides if it is of the form $A \times B$ for some A ∈ $\mathcal{A}$ and B ∈ $\mathcal{B}$.

The σ-algebra on $X \times Y$ generated by the collection of all rectangles with measurable sides is called the **product** of the σ-algebras $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \times \mathcal{B}$. That is,

$\mathcal{A} \times \mathcal{B} := \sigma(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\})$.

- **Sections** Suppose that X and Y are sets and that E is a subset of X ×Y. Then for each x in X and each y in Y the sections E_x and E^y are the subsets of Y and X given by $E_x = \{y \in Y : (x,y) \in E\}$ and $E^y = \{x \in X : (x,y) \in E\}$.
- If f is a function on X ×Y, then the sections f_x and f^y are the functions on Y and X given by $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$

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Theorem

If (X, $\mathcal{A}$) and (Y, $\mathcal{B}$) are measurable spaces, then $(X \times Y, \mathcal{A} \times \mathcal{B})$ is a measurable space.

The measurable space $(X \times Y, \mathcal{A} \times \mathcal{B})$ is the Cartesian product of the two given measurable spaces.

Proof:

If $(x, y) \in X \times Y$, then there exist sets A and B such that

$x \in A \in \mathcal{A}$ and $y \in B \in \mathcal{B}$

it follows that $(x, y) \in A \times B \in \mathcal{A} \times \mathcal{B}$

Lemma: Let (X, $\mathcal{A}$) and (Y, $\mathcal{B}$) be measurable spaces. If E is a subset of $X \times Y$ that belongs to $\mathcal{A} \times \mathcal{B}$, then each section E_x belongs to $\mathcal{B}$ and each section E^y belongs to $\mathcal{A}$. **Or** Every section of a measurable set is a measurable set.

1st Proof: Suppose that x belongs to X, and let $\mathfrak{I} = \{E: E \subseteq X \times Y, E_x \in \mathcal{B}\}$.

Then $\mathfrak{I}$ contains all rectangles A×B for which $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

In particular, $X \times Y \in \mathfrak{I}$.

Furthermore, the identities $(E^c)_x = (E_x)^c$ and $(\cup_n E_n)x = \cup_n ((E_n)_x)$ imply that $\mathfrak{I}$ is closed under complementation and under the formation of countable unions; thus $\mathfrak{I}$ is a σ-algebra. It follows that $\mathfrak{I}$ includes the σ-algebra $\mathcal{A} \times \mathcal{B}$ and hence that E_x belongs to $\mathcal{B}$ whenever E belongs to $\mathcal{A} \times \mathcal{B}$.

A similar argument shows that E^y belongs to $\mathcal{A}$ whenever E belongs to $\mathcal{A} \times \mathcal{B}$.

2nd Proof: Let $\mathfrak{I}$ be the class of all those subsets of X ×Y which have the property that each of their sections is measurable. If $E = A \times B$ is a measurable rectangle, then every section of E is either empty or else equal to one of the sides, (A or B according as the section is a Y-section or an X-section), and therefore $E \in \mathfrak{I}$. Since it is easy to verify that $\mathfrak{I}$ is a σ-ring, it follows that $\mathcal{A} \times \mathcal{B} \in \mathfrak{I}$.

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Lemma: Let (X, $\mathcal{A}$) and (Y, $\mathcal{B}$) be measurable spaces. If f is an extended real-valued (or a complex-valued) $\mathcal{A} \times \mathcal{B}$ -measurable function on X×Y, then each section f_x is $\mathcal{B}$-measurable and each section f^y is $\mathcal{A}$-measurable.

Or Every section of a measurable function is a measurable function.

Proof: If f is a measurable function on X×Y if x is a point of X and if M is any Borel set on the real line, then the measurability of $N(f_x) \cap f_x^{-1}(M)$ follows from previous Theorem and the relations

$$f_x^{-1}(M) = \{y: f_x(y) \in M\} = \{y: f(x,y) \in M\}$$

$$f_x^{-1}(M) = \{y: f_x(y) \in M\} = \{y: f(x,y) \in M\}$$

(Observe that $N(f_x) = N(f)_x$.) The proof of the measurability of an arbitrary Y-section is similar.

Remember

- **Proposition:** Let (X, $\mathcal{A}$,μ) and (Y, $\mathcal{B}$,ν) be σ-finite measure spaces. If E belongs to the σ-algebra A ×B, then the function $x \longmapsto \nu(E_x)$ is $\mathcal{A}$ – measurable and the function $y \longmapsto \mu(E^y)$ is $\mathcal{B}$ -measurable.
- **Theorem:** Let (X, $\mathcal{A}$,μ) and (Y, $\mathcal{B}$,ν) be σ-finite measure spaces then there is a unique measure $\mu \times \nu$ on the σ-algebra $\mathcal{A} \times \mathcal{B}$ such that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ holds for each A in $\mathcal{A}$ and B in $\mathcal{B}$. Furthermore, the measure under $\mu \times \nu$ arbitrary set E in $A \times B$ is given by
$$(\mu \times \nu)(E) = \int_X \nu(E_x)\mu(dx) = \int_Y \mu(E^y)\nu(dy)$$
The measure $\mu \times \nu$ is called the product of μ and ν.
- **Tonelli's Theorem:** Let (X, $\mathcal{A}$,μ) and (Y, $\mathcal{B}$,ν) be σ-finite measure spaces, and let $f: X \times Y \to [0, +\infty]$ be $\mathcal{A} \times \mathcal{B}$ -measurable. Then
(a) the function $x \longmapsto \int_Y f_x d\nu$ is $\mathcal{A}$ -measurable and the function $y \longmapsto \int_X f^y d\mu$ is $\mathcal{B}$ -measurable.
(b) f satisfies
$$\int_{X\times Y} f(x,y) d(\mu \times \nu) = \int_X \left(\int_Y f_x d\nu\right)\mu(dx) = \int_y \left(\int_X f^y d\mu\right)\nu(dy)$$
- **Fubini's Theorem:** Let (X, $\mathcal{A}$,μ) and (Y, $\mathcal{B}$,ν) be σ-finite measure spaces, and let $f: X \times Y \to [-\infty, +\infty]$ be $\mathcal{A} \times \mathcal{B}$-measurable and $\mu \times \nu$ -integrable. Then

a) for μ −almost every x in X the section f_x is ν −integrable and for ν −almost everyy in Y the section f^y is μ −integrable,

b) the functions I_f and J_f defined by
$$I_f = \begin{cases} \int_Y f_x d\nu & ;\text{if } f_x \text{ is } \nu - \text{integrable} \\ 0 & otherwise \end{cases}$$
and $$J_f = \begin{cases} \int_X f^y d\mu & ;\text{if } f^y \text{ is } \mu - \text{integrable} \\ 0 & otherwise \end{cases}$$
belong to $\mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{R})$ and $\mathcal{L}^1(Y, \mathcal{B}, \nu, \mathbb{R})$, respectively

c) The relation
$\int_{X\times Y} f d(\mu \times \nu) = \int_X I_f d\mu = \int_y J_f d\nu$ holds.

Recommended Books

- Roydon H.L. Real Analysis.
- Barra G. De. Measure Theory and Integration.
- Philip E.R. An introduction to Analysis and Integration Theory.
- W.Rudin, Real & Complex Analysis.
- Bartle R.G, The Elements of Integration and Lebesgue Measure.
- Paul R. Halmos, Measure Theory.

حرفِ آخر (27-12-2020)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالٰی آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (اٰمین)

محمد عثمان حامد

چک نمبر 105 شمالی (گودھے والا) سرگودھا

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