Measure Theory: Handwritten Notes

by

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Partial Contents

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CHAPTER : 1

SET THEORY

Equivalent Set:-

If A and B are two non-empty sets then A is said to be equivalent to B if there is bijective function from A to B.

If A is equivalent to B then we write as $A \approx B$ or $A \sim B$.

e.g: $A = \{1, 2, 3\}$
$B = \{a, b, c\}$
So $A \sim B$

<u>Note</u>: If A and B are finite then the number of elements of A and B are finite.

Infinite Set:-

A set which is equivalent to its proper subset is called the infinite set.

e.g:

$N = \{1, 2, 3, \ldots\}$ is infinite because N is equivalent to its proper subset $E = \{2, 4, 6, 8, \ldots\}$ that is if we define the function

$f : N \longrightarrow E$ by $f(n) = 2n \ ; \forall \ n \in N$

then f is clearly bijective.

Denumerable set:-

A set which is equivalent to natural number $\mathbb{N}$.

e.g:

The sets of $\mathbb{N}, \mathbb{Z}, E, \mathbb{Q}$,
$A = \{2^n, n \in N\}$, $B = \{1/n, n \in N\}$ etc
are denumerable sets

Non-Denumerable set:-

An infinite set is said to be non-denumerable set if it is not denumerable.

e.g:

$[1, 2]$, $R = (-\infty, \infty)$ are non-denumerable sets.

Countable set:-

A set which is finite or denumerable is called countable set.

e.g:

The sets of $\mathbb{N}, E, \mathbb{Q}$,
$A = \{1, 2, 3\}$ and $B = \{-1/2, 0, 1/2\}$ etc
are countable sets.

$A = \{1, 2, 3, ---, 1000\}$
$B = \{a, b, c, ----, z\}$
$C = \{a, e, i, o, u\}$ all are countable sets because these are finite.

Note: → Every interval is uncountable.
→ A is said to countable set if there exists injective (one-one) function

from S to N.

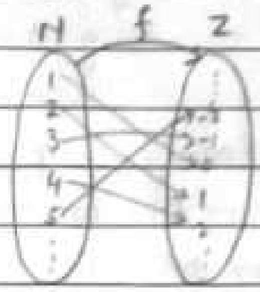
Examples:-

Q1. Show that the sets $Z = \{\text{all integers}\}$ and $X = \{2^n ; n \in N\}$ are countable.

Sol:-

Given that $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ is the set of all integers.

we have to show that Z is countable.



Let us define

$f : N \to Z$ by

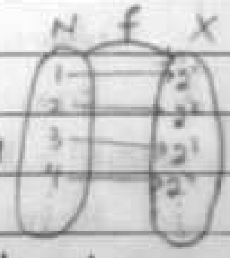
$$f(n) = \begin{cases} n/2 & \text{if } n = \text{even} \\ \dfrac{-(n-1)}{2} & \text{if } n = \text{odd} \end{cases}$$

Clearly f is bijective and hence $Z \sim N$ or $N \sim Z$.

$\Rightarrow$ Z is denumerable and hence Z is countable.

Next, $X = \{2^n ; n \in \mathbb{N}\}$

we will show that X is countable.



Let us define

$g : \mathbb{N} \longrightarrow X$ by

$g(n) = 2^n \quad ; \forall\ n \in \mathbb{N}$

clearly g is bijective and hence $\mathbb{N} \sim X$ or $X \sim \mathbb{N}$.

$\Rightarrow$ X is denumerable and hence X is countable.

Q2:- Show that the set $M = \{1/n ; n \in \mathbb{N}\}$ is countable.

Sol:-

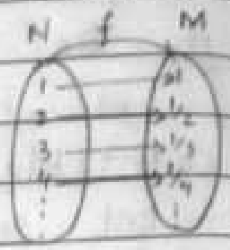
Given that

$$M = \{1/n ; n \in \mathbb{N}\}$$

we will show that M is countable.

let us define

$$g : N \longrightarrow M \text{ by}$$
$$g(n) = 1/n \quad ; \forall\, n \in \mathbb{N}$$



Clearly g is bijective and hence $\mathbb{N} \sim M$ or $M \sim \mathbb{N}$.

$\Rightarrow M$ is denumerable and hence M is countable.

Theorem:-

Every subset of denumerable set is either finite or denumerable

Proof:-

Let A be any denumerable set, and let B be any subset of A.

we need to prove that B is finite or denumerable.

Now since A is denumerable, so A can be written as $A = \{a_1, a_2, a_3, \dots\}$ where $A \sim N$.

Here the function $f : N \longrightarrow A$ is defined as $f(k) = a_k \ \forall\, k \in \mathbb{N}$; which is obviously bijective.

Now since $B \subseteq A$.

If $B = \phi$ then B is finite.

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If $B \neq \phi$ then since $B \subseteq A$, so we can find an element say $a_{n_1} \in A$ such that $a_{n_1} \in B$. If $B = \{a_{n_1}\}$ then B is finite. If $B \neq \{a_{n_1}\}$, then there is $a_{n_2} \in A$ such that $a_{n_2} \in B$.

If $B = \{a_{n_1}, a_{n_2}\}$ then B is finite, but if $B \neq \{a_{n_1}, a_{n_2}\}$ then we can have $a_{n_3} \in A$ such that $a_{n_3} \in B$.

If $B = \{a_{n_1}, a_{n_2}, a_{n_3}\}$ then B is finite. But if $B \neq \{a_{n_1}, a_{n_2}, a_{n_3}\}$ then we continue the above process and as such we can have $B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ where $n_i \in Z^+$.

If the indexing set $\{n_1, n_2, n_3, \ldots\}$ is bounded then it is finite otherwise it is denumerable because we can find a function $g: N \to B$ defined by $g(i) = a_n$ where $i \in N$.

then g as defined above is obviously bijective.

Thus B is not finite, but in this case B is denumerable i-e. $B \sim N$

Hence every subset of denumerable set is either finite or denumerable.

Theorem:-

Every subset of countable set is countable.

Proof:-

Let A be a countable set. We need to show that each subset of A is countable.

Now since A is countable, so by def: of countable set A is either finite or countably infinite (denumerable).

So two cases arises:

(i) **When A is finite:-**

In this case each subset of A is finite and we know that every finite set is countable.

So in this case each subset of A is countable.

(ii) **When A is denumerable:-**

let A is denumerable and let B be a subset of A then B can either be

(a) $B = \phi$ or

(b) B = finite and non-empty. or

(c) B is infinite.

(a) and (b) cases are clear because in these cases B is finite and so B is countable.

(c) Here B is a subset of A. we will show that B is countable.

For this let "n_1" be the least +ve integer such that $a_{n_1} \in B$ then $B \neq \{a_{n_1}\}$, because B is infinite.

let n_2 be the least +ve integer

such that $n_2 > n_1$ and $a_{n_2} \in B$, then $B = \{a_{n_1}, a_{n_2}\}$ because B is infinite.
Continuing this process we get A subset
$$B = \{a_{n_1}, a_{n_2}, a_{n_3}, ----\}$$
clearly B is countable because if we define $f: N \to B$ by
$f(K) = a_{n_K}$, then f is bijective.

$\Rightarrow B \sim N \Rightarrow B$ is denumerable and hence B is countable.
So in each case each subset of the countable set A is countable, which is the required result.

2^ND^ method:-
Let A is any countable set. Then A is either finite or denumerable.
If B is any subset of A then B will also be finite if A is finite.
Now if B is finite, and we know that every finite set is countable. So B being a subset of A is countable.
Now if A is countable, but not finite means that A is denumerable. Now A is denumerable and B is a subset of A.
$\Rightarrow$ B is also denumerable.
(i.e. by a result 'the subset of a

denumerable set is again denumerable.
So in both cases "B" being a subset of A is countable.
Hence every subset of countable set is countable.

Theorem:-
Show that $[0,1]$ is uncountable.

Proof:-
Let us suppose that $I=[0,1]$ is countable, then we can write as
$I=[0,1]=\{x_1, x_2, x_3, ----\}$.
To prove the required result we construct a sequence of nested intervals of I.
Divide $[0,1]$ in three equal parts as $[0, 1/3], [1/3, 2/3], [2/3, 1]$

Since $x_1 \in I$, so there exists one interval say $[a_1, b_1]$ such that $x_1 \notin [a_1, b_1] = I_1$, $l(I_1) = 1/3$

Divide $[a_1, b_1]$ in three equal parts
$[a_1, a_1+1/9], [a_1+1/9, a_1+2/9], [a_1+2/9, b_1]$

So there exists one interval say $I_2 = [a_2, b_2]$ such that
$x_2 \notin I_2$, $l(I_2) = 1/3^2$

Similarly $x_3 \notin I_3$, $l(I_3) = 1/3^3$
$\vdots \qquad \vdots$

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$x_n \notin I_n$, $l(I_n) = 1/3^n$

$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n$

$\lim_{n\to\infty} l(I_n) = 0$

By using Nested Interval Theorem there exists only one element (say) $x \in \bigcap_{n=1}^{\infty} I_n$.

So this result, $\exists$ x_m (say) in I such that $x_m \in \bigcap_{n=1}^{\infty} I_n$

$\Rightarrow x_m \in I_m$, but according to our conclusion $x_m \notin I_m$.

Which is contradiction to our supposition. So our supposition was wrong and this contradiction arises due to our wrong supposition that $[0,1]$ is countable.

Thus $[0,1]$ is uncountable.

Note $\rightarrow$ If $A \subseteq B$ and A is countable then B is countable.

$\rightarrow$ If $A \subseteq B$ and A is uncountable then B is uncountable.

Theorem:-

Show that $\mathbb{R}$ is uncountable.

Proof:-

Let us suppose that $\mathbb{R}$ is countable, then we can write $\mathbb{R}$ in the form

$\mathbb{R} = \{x_1, x_2, x_3, \ldots\}$

Now let $I_1 = \left(x_1 - \frac{1}{2^2}, x_1 + \frac{1}{2^2}\right)$

$x_1 \in I_1$, $l(I_1) = \frac{1}{2}$

and $I_2 = \left(x_2 - \frac{1}{2^3}, x_2 + \frac{1}{2^3}\right)$

$x_2 \in I_2$, $l(I_2) = \frac{1}{2^2}$

$\vdots \qquad \vdots$

$I_n = \left(x_n - \frac{1}{2^{n+1}}, x_n + \frac{1}{2^{n+1}}\right)$

$x_n \in I_n$, $l(I_n) = \frac{1}{2^n}$

$\Rightarrow x_n \in I_n \quad \forall n$

$\Rightarrow \{x_n\} \subset I_n \quad \forall n$

$\Rightarrow \bigcup_{n=1}^{\infty}\{x_n\} \subseteq \bigcup_{n=1}^{\infty} I_n \quad \forall n$

$\Rightarrow \mathbb{R} \subseteq \bigcup_{n=1}^{\infty} I_n$ —— (1) (Geometric series $\frac{a}{1-r}$)

$l(I_1) + l(I_2) + \cdots = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

$\mathbb{R}$ is contained in the union of interval where sum of length is 1. But the length of $\mathbb{R}$ is infinite, which is contradiction.

Hence $\mathbb{R}$ is uncountable.

Q:- Show that $[-1, 5] \sim [0, 1]$.

Sol:-

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Since we know that $[a,b] \sim [c,d]$ iff $f:[a,b] \longrightarrow [c,d]$ is defined as

$$f(x) = \left(\frac{b-x}{b-a}\right)c + \left(\frac{x-a}{b-a}\right)d \quad \text{—} (1)$$

Here let us define $f:[-1,5] \longrightarrow [0,1]$

i.e. putting $a=-1$, $b=5$, $c=0$, $d=1$ in eq(1

$$f(x) = \left(\frac{5-x}{5-(-1)}\right)(0) + \left(\frac{x-(-1)}{5-(-1)}\right)(1)$$

$$= 0 + \left(\frac{x+1}{5+1}\right)$$

$$f(x) = \frac{x+1}{6}$$

$$f(x) = x/6 + 1/6 \quad \forall \; x \in [-1,5]$$

Clearly f is linear polynomial and we can easily show that "f" is bijective.

Thus we have defined bijective function between $[-1,5]$ and $[0,1]$

$\Rightarrow [-1,5] \sim [0,1]$.

→ **Bijective function b/w the interval:-**

Q1:- Show that $[0,1] \sim [a,b]$.

Sol:-

Define a function $f:[0,1] \longrightarrow [a,b]$ by

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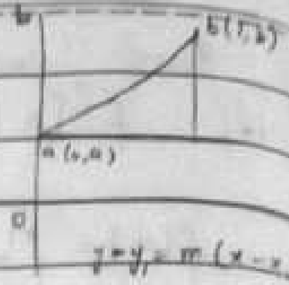
$$f(x) = a + (b-a)x$$



$$f(x) = a + bx - ax$$
$$f(x) = bx + a - ax$$
$$f(x) = bx + a(1-x)$$

$y - y_1 = m(x - x_1)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - a = \frac{b-a}{1-0}(x - 0)$

$y - a = (b-a)x$

$y = a + (b-a)x$

$$f(0) = a$$
$$f(1) = b$$

$$f'(x) = b - a > 0$$

Thus f is bijective function.

Q2:- Find bijective function from $[a,b]$ to $[0,1]$.

Sol:-

Since $f(x) = bx + (1-x)a$ is bijective function from $[0,1]$ to $[a,b]$.

let $y = bx + a(1-x)$

$\Rightarrow \quad bx + a(1-x) = y$

$\Rightarrow \quad bx + a - ax = y$

$\Rightarrow \quad bx - ax = y - a$

$\Rightarrow \quad x(b-a) = y - a$

$\Rightarrow \quad x = \dfrac{y-a}{b-a}$

$\Rightarrow \quad f(x) = \dfrac{y-a}{b-a}$

Thus f is bijective function from $[a,b]$ to $[0,1]$.

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Q3:- Find a bijective function from $[a,b]$ to $[c,d]$.

Sol:-

Since $f:[a,b] \longrightarrow [0,1]$ is define by $f(x) = \dfrac{x-a}{b-a}$ is bijective.

And $g:[0,1] \longrightarrow [c,d]$ is define by $g(x) = dx + (1-x)c$ is bijective.

$$h = g\circ f : [a,b] \longrightarrow [c,d]$$

$$h(x) = g\circ f(x) : [a,b] \longrightarrow [c,d]$$

$$h(x) = g\left(\frac{x-a}{b-a}\right) = d\left(\frac{x-a}{b-a}\right) + \left(1 - \left(\frac{x-a}{b-a}\right)\right)c$$

$$= d\left(\frac{x-a}{b-a}\right) + \left(\frac{b-\cancel{a}-x+\cancel{a}}{b-a}\right)c$$

$$h(x) = d\left(\frac{x-a}{b-a}\right) + \left(\frac{b-x}{b-a}\right)c$$

Hence this is bijective.

Q4:- Show a bijective function $[1,3] \rightarrow [4,5]$.

Sol:-

Since we know that the function $f:[a,b] \longrightarrow [c,d]$ is defined as

$$f(x) = \left(\frac{b-x}{b-a}\right)c + \left(\frac{x-a}{b-a}\right)d \quad \text{--- } (1) \quad \forall\ x \in [a,b]$$

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Here let us define $f:[1,3]\to[4,5]$

i.e.

putting $a=1, b=3, c=4, d=5$ in eq

$$f(x)=\left(\frac{3-x}{3-1}\right)4+\left(\frac{x-1}{3-1}\right)5$$

$$f(x)=\left(\frac{3-x}{\cancel{2}}\right)\overset{2}{\cancel{4}}+\left(\frac{x-1}{2}\right)5$$

$$f(x)=(3-x)2+(x-1)\frac{5}{2}$$

$$f(x)=6-2x+\frac{5x}{2}-\frac{5}{2}$$

$$f(x)=\frac{5x}{2}-2x+6-\frac{5}{2}$$

$$f(x)=\frac{5x-4x}{2}+\frac{12-5}{2}$$

$$f(x)=\frac{x}{2}+\frac{7}{2}\quad\forall\ x\in[1,3]$$

Clearly f is linear polynomial. Thus f is bijective function b/w $[1,3]$ and $[4,5]$.

Q5:- Show a bijective function $[-1,5]\longrightarrow[\frac{1}{4},\frac{1}{2}]$.

Sol:-

Since we know that $f:[a,b]\longrightarrow[c,d]$ is defined as

$$f(x)=\left(\frac{b-x}{b-a}\right)c+\left(\frac{x-a}{b-a}\right)d \quad \text{--- (1)}\quad \forall\ x\in[a,b]$$

Here let us define $f : [-1, 5] \to [1/4, 1/2]$

i.e. putting $a = -1$, $b = 5$, $c = 1/4$, $d = 1/2$ in eq ①

$$f(x) = \left(\frac{5-x}{5-(-1)}\right)\frac{1}{4} + \left(\frac{x-(-1)}{5-(-1)}\right)\frac{1}{2}$$

$$= \left(\frac{5-x}{5+1}\right)\frac{1}{4} + \left(\frac{x+1}{5+1}\right)\frac{1}{2}$$

$$= \left(\frac{5-x}{6}\right)\frac{1}{4} + \left(\frac{x+1}{6}\right)\frac{1}{2}$$

$$= \frac{5-x}{24} + \frac{x+1}{12}$$

$$= \frac{5-x+2x+2}{24}$$

$$= \frac{x+7}{24}$$

$$f(x) = \frac{x}{24} + \frac{7}{24} \quad \forall \; x \in [-1, 5]$$

clearly f is linear polynomial. Thus f is bijective function b/w $[-1, 5]$ and $[1/4, 1/2]$.

Q6: Show a bijective function $[3, 4] \sim [1, 100]$

Sol:-

Since we know that $[a, b] \sim [c, d]$ iff $f : [a, b] \to [c, d]$ is defined as

$$f(x) = \left(\frac{b-x}{b-a}\right)c + \left(\frac{x-a}{b-a}\right)d \quad \text{①} \quad \forall \; x \in [a, b]$$

Here let us define $f:[3,4]\to[1,100]$

i.e. putting $a=3, b=4, c=1, d=100$ in eq (i)

$$f(x)=\left(\frac{4-x}{4-3}\right)(1)+\left(\frac{x-3}{4-3}\right)(100)$$

$$f(x)=\frac{4-x}{1}+\frac{(x-3)}{1}100$$

$$f(x)=4-x+100x-300$$

$$f(x)=99x-296 \quad \forall\ x\in[3,4]$$

Clearly 'f' is linear polynomial and we can easily show that "f" is bijective.

Thus we have defined bijective function between $[3,4]$ and $[1,100]$

Theorem:-

Let A be an uncountable set and B be countable (or denumerable) subset of A then prove that $A\setminus B\sim A$.

Proof:-

Let B is a countable subset of A, so $B=\{b_1, b_2, b_3, \dots\}$ and let $C=\{c_1, c_2, c_3, \dots\}$ be a countable subset of $A\setminus B$.

Define a function

$f: A\setminus B\longrightarrow A$ by

$f(x) = x$ if $x \notin C$

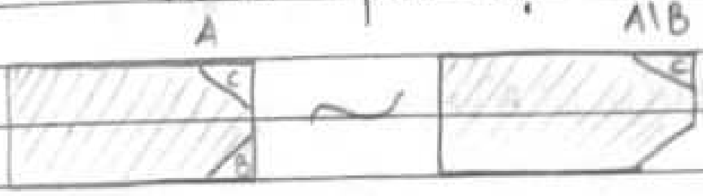
$f(C_{2n+1}) = C_{n+1}$ if $n = 0, 1, 2, 3, \ldots$

$f(C_{2n}) = b_n$ if $n = 1, 2, 3, \ldots$

OR (see figure below)

$$f(x) = \begin{cases} x & \text{if } x \notin C \\ c_n & \text{if } x \in C_{2n-1} \\ b_n & \text{if } x \in C_{2n} \end{cases}$$

$\Rightarrow f$ is bijective and so $A \setminus B \sim A$.
Hence proved!



$C \subseteq A \setminus B$

and $C \longrightarrow B \cup C$

$C = \{c_1, c_2, c_3, c_4, c_5, c_6, \ldots\}$

$B \cup C = \{b_1, c_1, b_2, c_2, b_3, c_3, \ldots\}$

Examples:-

Q1:- Show that $[0,1] \sim [0,1)$.

Sol:-

We have to show that $[0,1] \sim [0,1)$.

let $A = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$

then clearly $A \subseteq [0,1]$.

and $B = \{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\}$

then clearly $B \subseteq [0,1)$

Since $[0,1]\setminus A = [0,1)\setminus B$

let us define a function,
$f:[0,1] \longrightarrow [0,1)$ by

$f(x) = x$ if $x \notin A$.
$f(0) = 0$
$f\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$ if $n = 0,1,2,\dots$

Clearly f is bijective function or this relation of 1-1 correspondence can already shown as

$[0,1] = A\cup([0,1]\setminus A) = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots$

$[0,1) = B\cup([0,1)\setminus B) = \{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots\}$

Hence $[0,1] \sim [0,1)$.

Q:- 2 Show that $[0,1] \sim (0,1)$.

Sol:-

We need to show that $[0,1] \sim (0,1)$
let $A = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$

then clearly $A \subseteq [0,1]$

and let $B = \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \dots\}$

then clearly $B \subseteq (0,1)$.

Since $[0,1]\setminus A = (0,1)\setminus B$

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let us define a function
$f : [0,1] \to (0,1)$ by

$f(x) = x$ if $x \notin A$.

$f(0) = 1/2$

$f\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+2}}$ if $n = 0, 1, 2, 3, \ldots$

Clearly f is bijective function or this can already shown as

$[0,1] = A \cup ([0,1] \setminus A) = \{0, 1, 1/2, 1/2^2, 1/2^3, \ldots\}$

$(0,1) = B \cup ((0,1) \setminus B) = \{1/2, 1/2^2, 1/2^3, 1/2^4, \ldots\}$

Hence $[0,1] \sim (0,1)$.

OR let $A = \{0, 1, 1/2, 1/3, 1/4, \ldots\} \subseteq [0,1]$

and $B = \{1/2, 1/3, 1/4, 1/5, 1/6, \ldots\} \subseteq (0,1)$

Since $[0,1] \setminus A = (0,1) \setminus B$
Define a function
$f : [0,1] \to (0,1)$ by

$f(x) = x$ if $x \notin A$
$f(0) = 1/2$

$f(1/n) = 1/(n+2)$ if $n = 1, 2, 3, \ldots$

Clearly f is bijective, so $[0,1] \sim (0,1)$.

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Q3:- Show that $[1,0) \sim (1,0]$.

Sol:-

we need to show that $[1,0) \sim (1,0]$.

Let us define a function $f: [1,0) \longrightarrow (1,0]$ by

$$f(x) = 1-x \quad \forall \quad x \in [1,0).$$

Here f is a polynomial, so continuous and hence f is bijective function.

Thus $[1,0) \sim (1,0]$.

Note:-

Since $[a,b] \sim [0,1] \sim [0,1) \sim [c,d)$

$$\Rightarrow [a,b] \sim [c,d)$$

Also $[a,b] \sim [0,1] \sim (0,1) \sim (c,d)$

$$\Rightarrow [a,b] \sim (c,d).$$

Q4:- Show that $[0,1] \setminus \{1/2, 1/3, 1/4\} \sim [0,1]$

Sol:-

let $A = \{0, 1, \frac{1}{5}, \frac{1}{6}, \frac{1}{7} \dots\}$

then clearly $A \subseteq [0,1] \setminus \{1/2, 1/3, 1/4\}$

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and let $B = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$

then clearly $B \subseteq [0,1]$.

Since $([0,1] \setminus \{1/2, 1/3, 1/4\}) \setminus A = [0,1] \setminus B$.

let us define a function
$f : [0,1] \setminus \{1/2, 1/3, 1/4\} \longrightarrow [0,1]$ by

$f(x) = x \quad \forall \; x \notin A$
$f(0) = 0$
$f(1) = 1$

$f\left(1/_{n+3}\right) = \frac{1}{n}$ if $n = 2, 3, 4, \ldots$

clearly f is bijective function.
Thus $[0,1] \setminus \{1/2, 1/3, 1/4\} \sim [0,1]$.

Q5:- Show that $[2, 10] \sim [2, 10)$.

Sol:-
let $A = \{2, 10, 2\frac{1}{2}, 2\frac{1}{2^2}, \ldots\}$

then clearly $A \subseteq [2, 10]$

and let $B = \{2, 2\frac{1}{2}, 2\frac{1}{2^2}, 2\frac{1}{2^3}, \ldots\}$

then clearly $B \subseteq [2, 10)$

Since $[2, 10] \setminus A = [2, 10) \setminus B$

let us define a function
$f : [2, 10] \longrightarrow [2, 10)$ by

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$f(x) = x$ if $x \notin A$

$f(2) = 2$

$f(10) = 2\frac{1}{2}$

$f\left(2\frac{1}{2^n}\right) = 2\frac{1}{2^{n+1}}$ if $n = 1, 2, 3, \ldots$

Clearly f is bijective function.
Thus $[2, 10] \sim [2, 10)$.

Q6:- Show that $[5, 10] \sim [5, 10] \setminus \{6, 7, 8, 9\}$.

Sol:-

let $A = \{5, 10, 6, 7, 8, \ldots\}$

then clearly $A \subseteq [5, 10]$

and $B = \{5, 10, 5\frac{1}{2}, 5\frac{1}{2^2}, 5\frac{1}{2^3}, \ldots\}$

then clearly $B \subseteq [5, 10] \setminus \{6, 7, 8, 9\}$

Since $[5, 10] \setminus A = ([5, 10] \setminus \{6, 7, 8, 9\}) \setminus B$

let us define a function.
$f : [5, 10] \to [5, 10] \setminus \{6, 7, 8, 9\}$ by

$f(x) = x$ if $x \notin A$
$f(5) = 5$
$f(10) = 10$

$f(n+5) = 5\frac{1}{2^n}$ if $n = 1, 2, 3, \ldots$

Clearly f is bijective function or this relation of 1-1 correspondence can already shown as

$[5,10] = A \cup ([5,10] \setminus A) = \{5, 10, 6, 7, 8, ---\}$

$[5,10] \setminus \{6,7,8,9\} = B \cup ([5,10] \setminus \{6,7,8,9\}) = \{5, 10, 5\frac{1}{2}, 5\frac{1}{2^2}, 5\frac{1}{2^3}, \ldots\}$

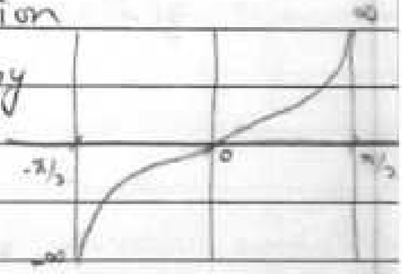
Hence $[5,10] \sim [5,10] \setminus \{6,7,8,9\}$.

Q7. Show that $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim \mathbb{R}$.
means $(-\infty, \infty)$

Sol:-
we need to show that $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim \mathbb{R}$

let us define a function
$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$ by

$f(x) = \tan x$



→ $\tan x$ $\forall$ $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is an increasing function because $f'(x) = \sec^2 x \geq 0$; $\forall$ $x \in \text{Dom} f'$.
So this can be increased upto $\mathbb{R}$.
Hence f is bijective function.
Thus $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim \mathbb{R}$.

Q8:- Show that $(a,b) \sim \mathbb{R}$.

Sol:-
Since $(a,b) \sim \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim \mathbb{R}$ —— (1)

we need to show that $(a,b) \sim \mathbb{R}$.

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the function will be for eq ① is defined as the composition of "f" and "g", where

$f : (a,b) \longrightarrow (-\pi/2, \pi/2)$ defined by

$$f(x) = \left(\frac{b-x}{b-a}\right)\left(\frac{-\pi}{2}\right) + \left(\frac{x-a}{b-a}\right)\pi/2$$

and $g : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ defined by

$$g(x) = \tan x$$

$\Rightarrow$ we will show that $g \circ f : (a,b) \longrightarrow \mathbb{R}$

$$\Rightarrow g(f(x)) = g\left[\left(\frac{b-x}{b-a}\right)(-\pi/2) + \left(\frac{x-a}{b-a}\right)\pi/2\right]$$

$$= \tan\left[\left(\frac{b-x}{b-a}\right)(-\pi/2) + \left(\frac{x-a}{b-a}\right)(\pi/2)\right]$$

$$= \tan\left[\pi/2\left(\frac{-(b-x) + x-a}{b-a}\right)\right]$$

$$= \tan\left[\pi/2\left(\frac{-b+x+x-a}{b-a}\right)\right]$$

$$= \tan\left[\pi/2\left(\frac{2x-b-a}{b-a}\right)\right]$$

$$g \circ f = \tan\left[\pi/2\left(\frac{2x-b-a}{b-a}\right)\right]$$

$$h(x) = \tan\left[\pi/2\left(\frac{2x-b-a}{b-a}\right)\right] \quad \forall x \in (a,b)$$

So $h(x)$ is bijective function.

Thus $(a,b) \sim \mathbb{R}$.

Q9:- Find a bijective function $(0,1) \sim \mathbb{R}$.

Sol:-

Since we know that $(a,b) \sim \mathbb{R}$ iff $f:(a,b) \to \mathbb{R}$ defined by

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-b-a}{b-a}\right)\right] \quad \text{—— (A)}$$

Here let us define $f:(0,1) \to \mathbb{R}$.
i.e. putting $a=0$, $b=1$ in eq (A), we get

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-1-0}{1-0}\right)\right]$$

$$= \tan\left[\frac{\pi}{2}\left(\frac{2x-1}{1}\right)\right]$$

$$= \tan\left(\frac{\pi}{2}(2x-1)\right)$$

$$= \tan\left(\frac{\cancel{2}x\pi}{\cancel{2}} - \frac{\pi}{2}\right)$$

$$f(x) = \tan\left(x\pi - \frac{\pi}{2}\right)$$

Thus f is bijective function.
Hence $(0,1) \sim \mathbb{R}$.

Note:- $(-\infty,\infty) = \mathbb{R}$ is an unbounded interval which is equivalent to bounded interval.

Q10:- Show that $(-1,1) \sim \mathbb{R}$.

Sol:-

Since we know that $(a,b) \sim \mathbb{R}$ iff $f:(a,b) \to \mathbb{R}$ defined by

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$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-b-a}{b-a}\right)\right] \text{ --- (A)}$$

Here let us define a function $f:(-1,1) \to \mathbb{R}$ i.e. putting $a=-1$, $b=1$ in eq (A), we get

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-1-(-1)}{1-(-1)}\right)\right]$$

$$= \tan\left[\frac{\pi}{2}\left(\frac{2x-\cancel{1}+\cancel{1}}{1+1}\right)\right]$$

$$= \tan\left[\frac{\pi}{2}\left(\frac{\cancel{2}x}{\cancel{2}}\right)\right]$$

$$f(x) = \tan\left(\frac{\pi x}{2}\right) \quad \forall\ x \in (-1,1)$$

Thus f is bijective function.
Hence $(-1,1) \sim \mathbb{R}$.

Q:- Show that $(1,100) \sim \mathbb{R}$.

Sol:-
Since we know that $(a,b) \sim \mathbb{R}$ iff $f:(a,b) \to \mathbb{R}$ defined by

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-b-a}{b-a}\right)\right] \text{ --- (A)}$$

Here let us define a function $f:(1,100) \to \mathbb{R}$ i.e. putting $a=1$, $b=100$ in eq (A), we get

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-100-1}{100-1}\right)\right]$$

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-101}{99}\right)\right] \quad \forall\ x \in (1,100)$$

Thus f is bijective function.
Hence $(1,100) \sim \mathbb{R}$.

Q12:- Show that $[0,\infty) \sim (0,\infty)$.

Sol:-
We need to show that $[0,\infty) \sim (0,\infty)$.

let $A = \{0,1,2,3,----\}$
then clearly $A \subseteq [0,\infty)$

and $B = \{1,2,3,4,----\}$
then clearly $B \subseteq (0,\infty)$.

Here $[0,\infty) \setminus A = (0,\infty) \setminus B$.

let us define a function
$f : [0,\infty) \to (0,\infty)$ by

$f(x) = x$ if $x \notin A$

$f(n) = n+1$ if $n = 0,1,2,3,---$

Clearly f is bijective function.
Thus $[0,\infty) \sim (0,\infty)$.

Q13:- Show that $(0,\infty) \sim (-\infty,0)$.

Sol:-
let $A = \{1,2,3,4,----\}$
then $A \subseteq (0,\infty)$
and $B = \{-1,-2,-3,-4,----\}$
then $B \subseteq (-\infty,0)$

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Since $(0,\infty)\setminus A = (-\infty,0)\setminus B$

Let us define a function
$f:(0,\infty)\to(-\infty,0)$ by

$f(n) = -n$ if $n = 1, 2, 3, 4, ---$

Clearly f is bijective function.
Thus $(0,\infty)\sim(-\infty,0)$.

Q14:- Show that $(0,\infty)\sim(-\infty,\infty)$.

Sol:-
we need to show that $(0,\infty)\sim(-\infty,\infty)$

let us define a function.
$f:(0,\infty)\longrightarrow(-\infty,\infty)$ by

$f(x) = x - 1/x$

if $x\to 0$ then $f\to -\infty$
if $x\to\infty$ then $f\to\infty$

So clearly f is bijective function
Thus $(0,\infty)\sim(-\infty,\infty)$.

Q15:- Show that $[a,\infty)\sim[0,\infty)$.

Sol:-
we need to show that $[a,\infty)\sim[0,\infty)$

let us define a function
$f:[a,\infty)\longrightarrow[0,\infty)$ by

$f(x) = x - a$

Clearly f is bijective function.
Thus $[a,\infty) \sim [0,\infty)$.

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CHAPTER # 2

Semi-ring:-

let X be a non-empty set i.e. $X \neq \phi$ and S be the collection of subsets of X, then S is said to be semi-ring if it satisfies the following conditions:

(i) $\phi \in S$

(ii) For any $A, B \in S \Rightarrow A \cap B \in S$.

(iii) For any $A, B \in S$

$$\Rightarrow A \setminus B = \bigcup_{i=1}^{n} C_i \quad , \quad C_i \in S \; ; \; i = 1, 2, 3, \dots, n$$

$$\text{and } C_i \cap C_j = \phi \quad ; \; j \neq i.$$

We say that (X, S) is semi-ring.

Sigma set (σ-set):-

let (X, S) is a semi-ring and $A \subseteq X$ (not necessary that $A \in S$) then A is said to be a sigma set if $A = \bigcup_{i=1}^{\infty} C_i$, $C_i \in S$, $C_i \cap C_j = \phi$; $i \neq j$.

Theorem:-

let (X, S) is a semi-ring and $A_1, A_2, A_3, \dots, A_n \in S$, then show that $A \setminus \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} C_i$, $C_i \cap C_j = \phi$, $C_i \in S$. is σ-set.

Proof:-

We prove that by Mathematical induction:

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For $n=1$:-

$A \setminus A_1$ is a σ-set. (proved)

Let us suppose that

$A \setminus \bigcup_{i=1}^{n} A_i$ is σ-set.

And we need to prove that

$A \setminus (\bigcup_{i=1}^{n+1} A_i)$ is σ-set.

Now; $A \setminus (\bigcup_{i=1}^{n+1} A_i) = A \setminus (\bigcup_{i=1}^{n} A_i \cup A_{n+1})$

$= (A \setminus \bigcup_{i=1}^{n} A_i) \setminus A_{n+1}$

$= (\bigcup_{j=1}^{\infty} B_j) \setminus A_{n+1}$

where $B_j \in S$, $B_j \cap B_k = \phi$; $j \neq k$.

By hypothesis $A \setminus (\bigcup_{i=1}^{n} A_i)$ is a σ-set

$= \bigcup_{j=1}^{\infty} (B_j \setminus A_{n+1})$ —— (1)

$\Rightarrow B_j \in S$, $A_{n+1} \in S$

$\Rightarrow B_j \setminus A_{n+1}$ is a σ-set.

$\Rightarrow B_j \setminus A_{n+1} = \bigcup_{i=1}^{\infty} C_{ji}$; $C_{ji} \in S$

; $C_{ji} \cap C_{jk} = \phi$, $i \neq$

eq (1) $\Rightarrow A \setminus (\bigcup_{i=1}^{n+1} A_i) = \bigcup_{j=1}^{\infty} (\bigcup_{i=1}^{\infty} C_{ji})$; $C_{ji} \in S$

$C_{ji} \cap C_{jk} = \phi$; $i \neq$

So, $A \setminus (\bigcup_{i=1}^{n+1} A_i)$ is a σ-set.

Thus all the result of mathematical induction are satisfied.

Hence $A \setminus (\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{\infty} C_i$; $C_i \in S$, $C_i \cap C_j = \phi$; $i \neq j$

is a δ-set.

Theorem:-

If (X, S) is semi-ring and $A_1, A_2, A_3, \dots \in S$ then $\bigcup_{i=1}^{\infty} A_i$ is δ-set.

OR

If (X,S) is semi-ring and $\{A_i\}_{i=1}^{\infty}$ be a sequence of sets from "S". Then show that $\bigcup_{i=1}^{\infty} A_i$ is a δ-set.

Proof:-

let (X,S) be a semi-ring such that $A_1, A_2, A_3, \dots \in S$.

we need to show that $\bigcup_{i=1}^{\infty} A_n$ is a δ-set.

let us denote $\bigcup_{i=1}^{\infty} A_i$ by A i.e. $A = \bigcup_{i=1}^{\infty} A_i$.

So we will show that A is a δ-set.

let us construct a sequence of sets as;

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus A_1 \cup A_2$$
$$B_4 = A_4 \setminus A_1 \cup A_2 \cup A_3$$
$$\vdots \qquad \vdots$$
$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

Now $A = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i \quad ; B_i \cap B_j = \phi \quad ; i \neq j$

Since we know that "if (X,S) is a semi-ring and $A_1, A_2, A_3, \dots, A_n \in S$ then

$A \setminus \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} C_i \; ; \; C_i \cap C_j = \phi \; ; i \neq j$"

So this theorem is also applicable here, because each $A_i \in S \; \forall \; i$ and each B_i is in the form of "$A_i \setminus \bigcup_j A_j$", so we can say that

$$B_i = \bigcup_{j=1}^{n} C_{ij} \; ; \; C_{ij} \cap C_{ik} = \phi \, , \, C_{ij} \in S$$

So $A = \bigcup_{i=1}^{\infty}\left(\bigcup_{j=1}^{n} C_{ij}\right) \; ; \; C_{ij} \cap C_{kl} = \phi$
$C_{ij} \in S$

So A is a σ-set.
Thus $\bigcup_{i=1}^{\infty} A_i$ is σ-set.

Algebra:-

If $X \neq \phi$ and F is the collection of subsets of X, then F is said to be algebra.

If the following conditions are satisfied.

(i) If $A \in F$ then $A^c \in F$
(ii) for any $A, B \in F \Rightarrow A \cap B \in F$

2nd definition:-

If $X \neq \phi$ and F is the collection of subsets of X, then F is said to be algebra

If the following conditions are satisfied.

(i) If $A \in F$ then $A^c \in F$
(ii) For any $A, B \in F \Rightarrow A \cup B \in F$

Theorem:-

Show that Def:(1) and Def:(2) are equivalent.

Proof:-

Since Def:(1) ⇒ " let $X \neq \phi$ and F be any collection of subsets of X, satisfying the two conditions i.e

$\left.\begin{array}{l} \text{(i) for any } A, B \in F \Rightarrow A \cap B \in F \\ \text{(ii) if } A \in F \text{ then } A^c \in F \end{array}\right\}$ — (A)

Def:(2) ⇒ " let $X \neq \phi$ and F be any collection of subsets of X, satisfying the two conditions i.e.

$\left.\begin{array}{l} \text{(i) for any } A, B \in F \Rightarrow A \cup B \in F \\ \text{(ii) If } A \in F \text{ then } A^c \in F \end{array}\right\}$ — (B)

let Def:(1) is true.

we need to prove ~~that~~ Def:(2).

let $A, B \in F$, we prove that $A \cup B \in F$

$\Rightarrow A, B \in F \Rightarrow A^c, B^c \in F$ (by eq (A) (ii))

$\Rightarrow A^c \cap B^c \in F$ (by eq (A) (i))

By Demorgan's law

$\Rightarrow (A \cup B)^c \in F$

$\Rightarrow ((A \cup B)^c)^c \in F$ (by eq (A) (ii))

$\Rightarrow A \cup B \in F$

Now, let Def:(2) is true.

we need to prove Def:(1)

let $A, B \in F$, we prove that $A \cap B \in F$

$\Rightarrow A, B \in F \Rightarrow A^c, B^c \in F$ (by eq (B) (ii))

$\Rightarrow A^c \cup B^c \in F$ (by eq (B) (i))

By Demorgan's law
$\Rightarrow (A \cap B)^c \in F$
$\Rightarrow ((A \cap B)^c)^c \in F$ (by eq (B) (iii))
$\Rightarrow A \cap B \in F$
which is the required result.

Note:-
From the definition of algebra it is clear that $\phi, X \in F$ and $A \setminus B = A \cap B^c \in F$ always.

↳ Consequences from the Def: of Algebra

Theorem:-
If (X, F) is an algebra then
(i) $\phi, X \in F$
(ii) Intersection and union of finite number of sets from F belong to F.
(iii) (X, F) is a semi-ring.

Proof:-
(i) Since $X \neq \phi$, So let $A \in F$.
by def: of algebra $\Rightarrow A^c \in F$
$\Rightarrow A \cup A^c \in F$ ($\because$ by def: of algebra)
$\Rightarrow \boxed{X \in F}$

Now Since $X \in F$, so $X^c \in F$
$\Rightarrow \boxed{\phi \in F}$

So $\phi, X \in F$

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ii) Next let $A_1, A_2, A_3, \ldots, A_n \in F$, then for

$A_1, A_2 \in F \Rightarrow A_1 \cap A_2 \in F$ (by def. of algebra)

$\Rightarrow A_1 \cap A_2 \cap A_3 \in F$

$\Rightarrow A_1 \cap A_2 \cap A_3 \cap A_4 \in F$

Continuing on the same way we can write that $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \in F$

$\Rightarrow \bigcap_{i=1}^{n} A_i \in F$

Similarly, let $A_1, A_2, A_3, \ldots, A_n \in F$, then for $A_1, A_2 \Rightarrow A_1 \cup A_2 \in F$ (by def. of algebra)

$\Rightarrow A_1 \cup A_2 \cup A_3 \in F$

$\Rightarrow A_1 \cup A_2 \cup A_3 \cup A_4 \in F$

Continuing on the same way we can write that $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \in F$

$\Rightarrow \bigcup_{i=1}^{n} A_i \in F$.

iii) We show that (X, F) is semi-ring.

(a) $\phi \in F$ (proved)

(b) Also by def. of algebra, for any $A, B \in F \Rightarrow A \cap B \in F$.

(c) Next we show that for any $A, B \in F$;

$A \setminus B = \bigcup_{i=1}^{n} C_i$; $C_i \in F$; $C_i \cap C_j = \phi$; $i \neq j$

Now, since $A \setminus B = A \cap B^c$

$\Rightarrow A, B^c \in F$

$\Rightarrow A \cap B^c \in F$ (by def. of algebra)

$\Rightarrow A \setminus B \in F$

$\Rightarrow A\setminus B = A \cap B^c$

$= A \cap B^c \cup \phi \cup \phi \cup \cdots \cup \phi$

$= C_1 \cup C_2 \cup \cdots \cup C_n$; where $C_1 = A \cap B^c$ and $C_i = \phi$ $\forall\ i = 2, 3, \ldots, n$

$A \setminus B = \dot{\bigcup}_{i=1}^{n} C_i$; $C_i \in F$; $C_i \cap C_j = \phi$; $i \neq j$

Thus (X, F) is a semi-ring.

Examples:-

1, let $X \neq \phi$ and $F = \{\phi, X\}$, show that F is an algebra.

Sol:-

Since $X \in F$

$\Rightarrow X^c \in F$ (by def. of algebra)

$\Rightarrow \boxed{\phi \in F}$

Now since $\phi \in F$

$\Rightarrow \phi^c \in F$ (by def. of algebra)

$\Rightarrow \boxed{X \in F}$

$\Rightarrow \phi \cap X = \phi \in F$

$\Rightarrow \phi \cup X = X \in F$

Thus F is an algebra.
And this is the smallest algebra of all possible algebras defined on X.

2, let $X \neq \phi$ and $A \subseteq X$ and $F = \{\phi, X, A, A^c\}$, then show that

F is an algebra.

Sol:-

Since $\phi \in F$

$\Rightarrow \phi^c \in F$ (by def of algebra)

$\Rightarrow \boxed{X \in F}$

Now since $X \in F$

$\Rightarrow X^c \in F$ (by def of algebra)

$\Rightarrow \boxed{\phi \in F}$

And $\boxed{A \in F}$

$\Rightarrow \boxed{A^c \in F}$

$\Rightarrow X \cup \phi = X \in F$ $\Rightarrow X \cap \phi = \phi \in F$

$\Rightarrow X \cup A = X \in F$ $\Rightarrow X \cap A = A \in F$

$\Rightarrow X \cup A^c = X \in F$ $\Rightarrow X \cap A^c = A^c \in F$

$\Rightarrow A \cup \phi = A \in F$ $\Rightarrow A \cap \phi = \phi \in F$

$\Rightarrow \phi \cup A^c = A^c \in F$ $\Rightarrow \phi \cap A^c = \phi \in F$

$\Rightarrow A \cup A^c = X \in F$ $\Rightarrow A \cap A^c = \phi \in F$.

Thus F is an algebra.
And F is the smallest algebra containing A.

3) $P(X) = \{$All possible subsets of $X\}$ and $X \neq \phi$, then show that $P(X)$ is an algebra.

Sol:-

$P(X)$ is also an algebra and this is the largest algebra of all possible algebras defined on X.

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4) Let $X \neq \phi$ and F be the collection of pairwise disjoint subset of X. Show that F is semi-ring.

Sol:-

i) Since for any set $A \in F$, we can write $A \cap \phi = \phi \quad \forall A \subseteq X$.

So $\phi \in F$.

ii) Next, we show that for any $A, B \in F$ $A \cap B \in F$.
For this, since $A, B \in F$, so we can write

$$A \cap B = A \in F \text{ if } A = B$$
$$A \cap B = \phi \in F \text{ if } A \neq B.$$

So in both cases "$A \cap B \in F$".

iii) Let $A, B \in F \Rightarrow A \setminus B = \phi$ if $A = B$.
$\Rightarrow A \setminus B = A$ if $A \neq B$.

So, $\phi, A \in F$

So in both cases "$A \setminus B \in F$".

Since $A \setminus B = A \setminus B \cup \phi \cup \phi \cup \cdots \cup \phi$
$= C_1 \cup C_2 \cup \cdots \cup C_m$

$$A \setminus B = \bigcup_{i=1}^{m} C_i \quad ; C_i \in F, C_i \cap C_j = \phi.$$

Hence (X, F) is a semi-ring.

which completes the proof.

5) Give an example of semi-ring, which is not algebra.

Sol:-

let $X = \mathbb{R}$ and

let $S = \{[a,b) : a,b \in \mathbb{R} \,;\, a<b\} \cup \{\phi\}$

First we show that S is a semi-ring.

i) $\phi \in S$ (given)

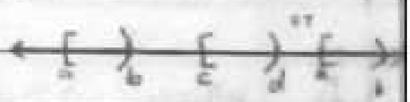
ii) Next we show that the intersection of any two sets from S is again in S.

For this let $A, B \in S$, then

$A = [a,b)$, $B = [c,d)$ for some $a,b,c,d \in \mathbb{R}$

Now there are cases i-e.

(a) when $b<c$ or $a>d$ i-e.



then $A \cap B = [a,b) \cap [c,d)$

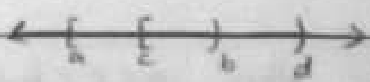
$A \cap B = \phi \in S$.

(b) when $c<a<d<b$ i-e.

then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [a,d) \in S$

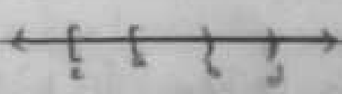
(c) when $a<c<b<d$ i-e.



then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [c,b) \in S$

(d) when $[a,b) \subseteq [c,d)$ i-e



then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [a,b) \in S$

(e) when $[c,d) \subseteq [a,b)$ i-e.

then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [c,d) \in S$

So all the cases we see that $A \cap B \in S$, so "S" is closed under finite intersection.

(iii) Next we show that the difference of any two sets from S can be written as the finite union of disjoint sets from S.

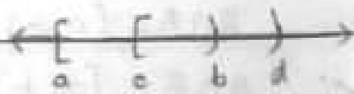
For this let $A, B \in S$
$\Rightarrow A = [a, b)$; $B = [c, d)$
where $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$.
Now there are cases i.e.

(f) When $a > d$ or $c > b$ i.e.


then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = [a, b) \in S$

(g) When $c < a < d < b$ i.e.


then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = [d, b) \in S$

(h) When $a < c < b < d$ i.e.


then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = [a, c) \in S$

(i) When $[a, b) \subseteq [c, d)$ i.e.


then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = \phi \in S$

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(j) when $[c,d) \subseteq [a,b)$ i.e. $\leftarrow [\; [\;) \;) \rightarrow$ ($c \; a \; b \; d$)

then $A \setminus B = [a,b) \setminus [c,d)$

$A \setminus B = [a,b) \in S$

So $\forall$ cases we see that $A \setminus B \in S$

So $A \setminus B = A \setminus B \cup \phi \cup \phi \cup \cdots \cup \phi$

$= C_1 \cup C_2 \cup \cdots \cup C_n$; $C_1 = A \setminus B$, $C_i = \phi$; $i = 2, 3, \ldots$

$$A \setminus B = \bigcup_{i=1}^{m} C_i \; ; \; C_i \cap C_j = \phi \; ; \; C_i \in S \; ; \; i \neq j$$

This shows that S is a semi-ring.

↳ Now we will show that S is not algebra.

For this let $\phi \in S$.

So $\phi^c = \mathbb{R} = (-\infty, \infty) \notin S$

this means that S is not closed under complementation. So "S" does not satisfy the property of algebra. Thus "S" is not an algebra.

OR e.g $\Rightarrow [1,2) \in S$

$\Rightarrow [7,9) \in S$

$\Rightarrow [1,2) \cup [7,9) \notin S$ $\leftarrow [\;) \; [\;) \rightarrow$ ($1 \; 2 \; 7 \; 9$)

So "S" is not an algebra.

σ-Algebra:-

An algebra F is said to be a σ-Algebra, if for any sequence

$A_1, A_2, A_3, \ldots \in F$, we have $\bigcup_{i=1}^{\infty} A_i \in F$

OR

for any $A_1, A_2, A_3, \ldots \in F \Rightarrow \bigcap_{i=1}^{\infty} A_i \in F$

Remark:-

If F is a σ- Algebra, then $\bigcap_{i=1}^{\infty} A_i \in F$.

Proof:-

If $A_1, A_2, A_3, \ldots \in F$

$\Rightarrow A_1^c, A_2^c, A_3^c, \ldots \in F$

$\Rightarrow A_1^c \cup A_2^c \cup A_3^c \cup \ldots \in F$ (by def. of σ-al

By Demorgan's law

$\Rightarrow (A_1 \cap A_2 \cap A_3 \cap \ldots)^c \in F$

$\Rightarrow \left(\bigcap_{i=1}^{\infty} A_i\right)^c \in F$

$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in F$

Example:-

Let X be an uncountable set and F be the collection as
$F = \{E \subseteq X : E \text{ or } E^c \text{ is countable}\}$
then show that "F" is σ-algebra

Sol:-

i) Let $A \in F$, we show that $A^c \in F$.

$\Rightarrow A$ or A^c is countable

$\Rightarrow (A^c)^c$ or A^c is countable.

Let $A^c = B$

$\Rightarrow B^c$ or B is countable

$\Rightarrow B \in F$ ($\because$ by def. of F)

$\Rightarrow A^c \in F$ ($\because A^c = B$)

$\Rightarrow$ F is closed under complementation.

ii) Now let $A_1, A_2, A_3, \ldots$ be any collection from F.

we will show that $\bigcup_{i=1}^{\infty} A_i \in F$

Case-I:-

If all A_i's are countable, then we know that $\bigcup_{i=1}^{\infty} A_i$ is always countable.

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$

Case-II:-

let there exist one A_i say A_k such that A_k^c is countable, then since

$\Rightarrow \bigcap_{i=1}^{\infty} A_i^c \subseteq A_k^c$

$\Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq A_k^c$

Since A_k^c is countable

$\Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right)^c$ is countable.

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$

Thus F is σ-Algebra.

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Example:-

let X be a topological space and $S = \{C \cap O ; C$ is closed set and O is open set$\}$. Show that S is a semi-ring.

Sol:-

i) Since ϕ is both open and closed such that $\phi = \phi \cap \phi$

$\Rightarrow \phi \in S$

ii) let $A, B \in S$

we will show that $A \cap B \in S$.

For this, since $A, B \in S$

$\Rightarrow A = C_1 \cap O_1$ and $B = C_2 \cap O_2$

where C_1, C_2 are closed sets.

and O_1, O_2 are open sets

Now $A \cap B = (C_1 \cap O_1) \cap (C_2 \cap O_2)$

$= (C_1 \cap C_2) \cap (O_1 \cap O_2) \in S$

So $A \cap B \in S$

iii) let $A, B \in S$, we will show that

$A \setminus B = \bigcup_{i=1}^{n} C_i$; $C_i \in S$ and $C_i \cap C_j = \phi$; $i \neq j$

Here, since $A, B \in S$

$\Rightarrow A = C_1 \cap O_1$ and $B = C_2 \cap O_2$

where C_1, C_2 are closed sets

and O_1, O_2 are open sets.

Now $A \setminus B = (C_1 \cap O_1) \setminus (C_2 \cap O_2)$

$A \setminus B = (C_1 \cap O_1) \cap (C_2 \cap O_2)^c$ $(\because A \setminus B = A \cap B^c)$

By demorgan's law

$= (C_1 \cap O_1) \cap (C_2^c \cup O_2^c)$

$= ((C_1 \cap O_1) \cap C_2^c) \cup ((C_1 \cap O_1) \cap O_2^c)$

$A \setminus B = (C_1 \cap (O_1 \cap C_2^c)) \cup ((C_1 \cap O_2^c) \cap O_1)$ — (*)

Since "C_1" is closed and $(O_1 \cap C_2^c)$ is open.

$\Rightarrow C_1 \cap (O_1 \cap C_2^c) \in S$

Also $(C_1 \cap O_2^c)$ is closed and O_1 is open

$\Rightarrow (C_1 \cap O_2^c) \cap O_1 \in S$

So eq (*) $\Rightarrow$ $A \setminus B = \bigcup_{i=1}^{2} G_i$; where $G_i \in S$

$G_i \cap G_j = \phi$; $i \neq j$

Thus S is a semi-ring.

Example:-

Let S be the collection of all sub-sets of $[0,1)$ that we can written as finite union of subsets of $[0,1)$ of the form $[a,b)$.

Show that S is an algebra of set but not σ-algebra.

Sol:-

i) Let $A \in S$, we will show that $A^c \in S$.

So, here $A \in S$

$\Rightarrow A = \bigcup_{i=1}^{n} [a_i, b_i)$ (by def of S)

$A^c = [0,1) \setminus A$

$A^c = [0,1) \setminus \bigcup_{i=1}^{n} [a_i, b_i)$

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$$A^c = \bigcup_{i=1}^{n} \{ [0,1) \setminus [a_i, b_i) \}$$

$$A^c = \bigcup_{i=1}^{n} \{ [0, a_i) , [b_i, 1) \} \in S$$

So $A^c \in S$.

ii) let $A, B \in S$. we show that $A \cup B \in S$

$\Rightarrow$ $A = \bigcup_{i=1}^{n} [a_i, b_i)$ ($\because$ by def. of S)

and

$B = \bigcup_{i=1}^{n} [c_i, d_i)$

Now $A \cup B = \left[\bigcup_{i=1}^{n} [a_i, b_i) \right] \cup \left[\bigcup_{i=1}^{n} [c_i, d_i) \right]$

$= [a_1, b_1) \cup [a_2, b_2) \cup \cdots \cup [a_n, b_n)$

$\cup$

$[c_1, d_1) \cup [c_2, d_2) \cup \cdots \cup [c_n, d_n)$

$$A \cup B = \bigcup_{i=1}^{n} \left([a_i, b_i) \cup [c_i, d_i) \right) \in S$$

So $A \cup B \in S$.

Thus S is an algebra.

Now we will show that S is not a σ-algebra.

For this, Since $[0, 1/n) \in S \quad \forall n \in \mathbb{N}$

but $\bigcap_{n=1}^{\infty} [0, 1/n) = \{0\} \notin S$

$\Rightarrow$ S is not a σ-algebra.

which completes the proof.

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Problem:-

let S be a semi-ring of subsets of X and $Y \subseteq X$, and $S_Y = \{Y \cap A : A \in S\}$. Show that S_Y form a semi-ring on Y.

Sol:-

i) Since S is a semi-ring, so $\phi \in S$

$\Rightarrow \phi = Y \cap \phi$

$\Rightarrow Y \cap \phi \in S_Y$

$\Rightarrow \phi \in S_Y$

ii) let $A, B \in S_Y$. we need to show that $A \cap B \in S_Y$.

For this, since $A, B \in S_Y$, so by def of S_Y we can write $A = Y \cap A_1$, $B = Y \cap B_1$ where $A_1, B_1 \in S$

Now $A \cap B = (Y \cap A_1) \cap (Y \cap B_1)$

$= Y \cap (A_1 \cap B_1)$, $A_1 \cap B_1 \in S$

$= Y \cap (A_1 \cap B_1) \in S_Y$

So $A \cap B \in S_Y$.

iii) we need to show that $A \setminus B \in S_Y$.

$A \setminus B = (Y \cap A_1) \setminus (Y \cap B_1)$

$A \setminus B = Y \cap (A_1 \setminus B_1)$

Since $A_1, B_1 \in S \Rightarrow A_1 \setminus B_1 = \bigcup_{i=1}^{n} C_i$, $C_i \in S$, $C_i \cap C_j = \phi$, $i \neq j$

$A \setminus B = Y \cap (\bigcup_{i=1}^{n} C_i)$

$= \bigcup_{i=1}^{n} (Y \cap C_i)$; $C_i \in S$; $Y \cap C_i \in S_Y$;

$(Y \cap C_i) \cap (Y \cap C_j) = \phi$

So $A \setminus B \in S_Y$

Thus S_Y is a semi-ring.

Measure:-

Let (X, S) be a semi-ring and $\mu : S \to [0, \infty]$ be a function such that $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ where $A_1, A_2, A_3, \ldots \in S$; $A_i \cap A_j = \phi$ for $i \neq j$

and $\bigcup_{i=1}^{\infty} A_i \in S$. Also $\mu(\phi) = 0$.

Examples:-

1. let $X \neq \phi$ and $S = P(X) = \{$All subset of $X\}$ and $\mu : S \to [0, \infty]$ be defined by
$\mu(A) =$ number of elements in A if A is finite
$\mu(A) = \infty$, if A is infinite set.
Show that μ is a measure.

Sol:-

Clearly $\mu(\phi) = 0$

let $A_1, A_2, A_3, \ldots \in S$ such that $A_i \cap A_j = \phi$

We show that $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$
we discuss the following cases.

Case - 1 :-

If A_i's say A_k is infinite, then $A = \bigcup_{i=1}^{\infty} A_i$ will be infinite and

$$\mu(A) = \infty$$
$$\Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \infty \quad \text{—— (i)}$$

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Also $\sum_{i=1}^{\infty} \mu(A_i) = \mu(A_1) + \mu(A_2) + \dots + \mu(A_k) + \dots$

$= \mu(A_1) + \mu(A_2) + \dots + \infty + \dots$

$= \infty$

$$\sum_{i=1}^{\infty} \mu(A_i) = \infty \text{ ——— (ii)}$$

From eq (i) and eq (ii), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Case-2:-

let $A_{i_1}, A_{i_2}, A_{i_3}, \dots, A_{i_n}$ be non-empty, finite and disjoint sets.

Let $A_{i_j} = \phi$ for $i_j \neq i_1, i_2, i_3, \dots, i_n$

then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{j=1}^{\infty} A_{i_j}\right)$$

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) =$ no of elements in "$\bigcup_{j=1}^{\infty} A_{i_j}$" $\left\{\text{because } \bigcup_{j=1}^{\infty} A_{i_j} \text{ is finite}\right.$

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) =$ no of elements in A_{i_1} + no of elements in A_{i_2} + no of elements in A_{i_3} + ... + no of elements in A_{i_n} + no of elements in ϕ + ... (ϕ is Finite)

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(A_{i_1}) + \mu(A_{i_2}) + \mu(A_{i_3}) + \dots + \mu(A_{i_n}) + \mu(\phi) + \dots$$

$$= \sum_{i=1}^{\infty} \mu(A_i)$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

In each case "μ" satisfies the properties of measure.

So μ is a measure.

This type of measure is called Discrete or or counting measure.

2. Let $X \neq \phi$ and $S = P(X)$ and fix $a \in X$ and define a function $\mu : S \to [0, \infty]$ by

$\mu(A) = 1$ if $a \in A$

$\mu(A) = 0$ if $a \notin A$

Show that μ is a measure.

Sol:-

Let $A_1, A_2, A_3, \ldots \in S$ such that $A_i \cap A_j = \phi$; $i \neq j$

we need to prove that μ is a measure.

For this we need to show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

we discuss the following cases.

Case-1:-

When $a \notin A_i \quad \forall i$

then $\mu(A_i) = 0$

$$\Rightarrow \sum_{i=1}^{\infty} \mu(A_i) = 0 \quad \text{--- (i)}$$

Also $a \notin \bigcup_{i=1}^{\infty} A_i$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 \quad \text{--- (ii)}$$

From eq (i) and (ii), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

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Case-2

When $a \in \bigcup_{i=1}^{\infty} A_i$, then there exist only one A_i, say A_k such that $a \in A_k$ (because A_i's are disjoint).

Now since $a \in \bigcup_{i=1}^{\infty} A_i$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \quad \text{—— (iii)}$$

Also $\sum_{i=1}^{\infty} \mu(A_i) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_k) + \cdots$

$= 0 + 0 + \cdots + 1 + 0 + \cdots$

$= 1$

$$\sum_{i=1}^{\infty} \mu(A_i) = 1 \quad \text{—— (iv)}$$

From eq (iii) and (iv), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

And clearly $\mu(\phi) = 0$. In each case "μ" satisfies the properties of measure.

Thus μ is a measure.

This type of measure is called Dirac measure.

3. let $X \neq \phi$ and "S" be the collection of pair-wise disjoint collection of sub-set of X and for each $A \in S$. let us choose $m_A \in [0, \infty]$ and define $\mu : S \to [0, \infty]$ by $\mu(A) = m_A \; ; \forall A \in S, A \neq \phi$

$\mu(\phi) = 0 \; ; \forall A \in S, A = \phi$

Show that μ is a measure.

Sol:-

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let $A_1, A_2, A_3, \dots \in S$, such that $\bigcup_{i=1}^{\infty} A_i \in S$

we need to prove that "μ" is a measure.

For this we need to show that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Now, Here we see that $A_1, A_2, A_3, \dots$, $A = \bigcup_{i=1}^{\infty} A_i \in S$.

Now since "S" contains disjoint sets. So at a time it is not possible that $A_1, A_2, A_3, \dots \in S$ and $\bigcup_{i=1}^{\infty} A_i \in S$ because these sets (i.e A_i's and $\bigcup_{i=1}^{\infty} A_i$) are not disjoint.

So $A_1, A_2, A_3, \dots$ must be one set say A_k. i.e. $\bigcup_{i=1}^{\infty} A_i = A_k$

So $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(A_k)$

$= m_{A_k}$ (by def of "μ")

$= 0 + 0 + \dots + m_{A_k} + 0 + 0 + \dots$

$= \mu(\phi) + \mu(\phi) + \dots + \mu(A_k) + \mu(\phi) + \dots$

$= \sum_{i=1}^{\infty} \mu(A_i)$; where all A_i's are empty except A_k

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

which shows that "μ" is a measure.

Measure Space:-

If (X, S) is semi-ring and μ is a measure on S then

(X, S, μ) is called measure space.

Theorem:-

If (X, S, μ) is a measure space and $A, B \in S$ such that $A \subseteq B$ then $\mu(A) \leq \mu(B)$.

Proof:-

Since $A \subseteq B$

$\Rightarrow \quad B = A \cup (B \setminus A)$

$\Rightarrow \quad \mu(B) = \mu(A \cup (B \setminus A)) \longrightarrow ①$

But $A, B \in S$

$\Rightarrow \quad B \setminus A = \bigcup_{i=1}^{n} C_i \quad ; \quad C_i \in S \;;\; C_i \cap C_j = \phi \;;\; i \neq j$

$\Rightarrow \quad \mu(B) = \mu\left[A \cup \left(\bigcup_{i=1}^{n} C_i\right)\right]$

$\Rightarrow \quad \mu(B) = \mu(A) + \mu\left(\bigcup_{i=1}^{n} C_i\right)$

by definition of measure.

$\Rightarrow \quad \mu(B) = \mu(A) + \sum_{i=1}^{n} \mu(C_i) \geq \mu(A)$

as $\sum_{i=0}^{n} \mu(C_i) \geq 0$

$\Rightarrow \quad \mu(B) \geq \mu(A)$

$\Rightarrow \quad \mu(A) \leq \mu(B)$.

6- Additive:-

let $\mu : S \longrightarrow [0, \infty]$ is a function such that for $A_1, A_2, A_3, \cdots \in S \;;\; A_i \cap A_j = \phi \;;$

$i \neq j$ for $\bigcup_{i=1}^{\infty} A_i \in S$, if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$, then μ is said to be σ-additive function.

Finitly Additive:-

let $\mu : S \to [0, \infty]$ is a function such that for $A_1, A_2, A_3, \ldots \in S$, where $A_i \cap A_j = \phi$ $\forall$ $i \neq j$ and for $\bigcup_{i=1}^{n} A_i \in S$, if $\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$, then μ is said to be finitly additive function.

σ - Sub Additive:-

let $\mu : S \to [0, \infty]$ is a function such that for $A_1, A_2, A_3, \ldots \in S$ where $A_i \cap A_j = \phi$ $\forall$ $i \neq j$ and for $\bigcup_{i=1}^{\infty} A_i \in S$, if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$, then μ is said to be σ-sub additive function.

Finitly Sub Additive:-

Let $\mu : S \to [0, \infty]$ is a function such that for $A_1, A_2, A_3, \ldots \in S$ where $A_i \cap A_j = \phi$ $\forall$ $i \neq j$ and for $\bigcup_{i=1}^{n} A_i \in S$, If $\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i)$, then μ is said to be finitly sub additive function.

Theorem:-

let (X, S, μ) be a measure space then show that μ is finitly additive i.e. If μ is measure then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

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Proof:-

let μ is measure.
we need to show that μ is finitly additive i.e. $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

For this let us choose disjoint collections from "S" i.e. $A_1, A_2, A_3, \cdots, A_n \in S$ such that $A_i \cap A_j = \phi$ $i \neq j$ and $\bigcup_{i=1}^{n} A_i \in S$

Now $\mu(\bigcup_{i=1}^{n} A_i) = \mu(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \cup \phi \cup \phi \cdots)$

$= \mu(\bigcup_{i=1}^{\infty} A_i)$ where $A_{n+1} = A_{n+2} = \cdots = \phi$

$= \sum_{i=1}^{\infty} \mu(A_i)$ ($\because \mu$ is measure)

$= \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n) + \mu(A_{n+1}) + \cdots$

$= \sum_{i=1}^{n} \mu(A_i) + \mu(\phi) + \mu(\phi) + \cdots$

$= \sum_{i=1}^{n} \mu(A_i) + 0 + 0 + \cdots$

$\Rightarrow \mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$

which shows that "μ" is finitly additive.

Theorem:-

If μ is measure then $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Proof:-

let μ is measure such that $A_1, A_2, A_3, \cdots \in S$ and let $A = \bigcup_{i=1}^{\infty} A_i$.
We need to prove that $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$
let us contruct a sequence of a set as:

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$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1 = A_2 \cap A_1^c \subseteq A_2$$
$$B_3 = A_3 \setminus (A_1 \cup A_2) = A_3 \cap (A_1 \cup A_2)^c \subseteq A_3$$
$$\vdots \qquad \vdots \qquad \vdots$$

Clearly $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \quad ; B_i \cap B_j = \phi \; ; i \neq j$

$$B_1 \subseteq A_1 \Rightarrow \mu(B_1) \leq \mu(A_1)$$
$$B_2 \subseteq A_2 \Rightarrow \mu(B_2) \leq \mu(A_2)$$
$$B_3 \subseteq A_3 \Rightarrow \mu(B_3) \leq \mu(A_3)$$
$$\vdots \qquad \vdots \qquad \vdots$$

Since $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \quad \text{(using property of measure)}$$
$$\leq \sum_{i=1}^{\infty} \mu(A_i)$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

which shows that "μ" is σ- sub additive

Example:-

let $S = \{A \subseteq R : A \text{ is countable}\}$ be a semi-ring and $\mu : S \to [0, \infty]$ be a function defined by

$$\mu(A) = 0 \quad ; \text{if } A \text{ is finite set.}$$
$$= \infty \quad ; \text{if } A \text{ is infinite set}$$

Show that "μ" is finitly additive

but not σ- additive.

Sol:-

let $A_1, A_2, A_3, \ldots, A_n \in S$ such that

$A_i \cap A_j = \phi$; $i \neq j$

we will prove that $\mu(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} \mu(A_i)$

For this there are cases:

Case-1:-

If all A_i's are finite, then obviously "$\bigcup_{i=1}^{n} A_i$" will be finite, so by definition of μ,

$$\Rightarrow \mu(\bigcup_{i=1}^{n} A_i) = 0 \text{ ——— (i)}$$

Also $\sum_{i=1}^{n} \mu(A_i) = \mu(A_1) + \mu(A_2) + \ldots$

$= 0 + 0 + \ldots$

$= 0$

$$\sum_{i=1}^{n} \mu(A_i) = 0 \text{ ——— (ii)}$$

From eq (i) and (ii), we get

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$$

Case-2:-

If one of A_i's is infinite, say A_k is infinite, then $\bigcup_{i=1}^{n} A_i$ is an infinite set.

So by definition of "μ", we can write

$$\mu(\bigcup_{i=1}^{n} A_i) = \infty \text{ ——— (iii)}$$

Also $\sum_{i=1}^{n} \mu(A_i) = \mu(A_1) + \mu(A_2) + \mu(A_3) + \ldots + \mu(A_k) + \ldots$

$= 0 + 0 + 0 + \ldots + \infty + \ldots = \infty$

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$$\sum_{i=1}^{n} \mu(A_i) = \infty \quad \text{—— (iv)}$$

From eq (iii) and eq (iv), we get

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

Hence "μ" is finitly additive.

Next, we need to show that "μ" is not σ-additive.

let us suppose that

$A_1 = \{1\} \Rightarrow \mu(A_1) = 0$

$A_2 = \{2\} \Rightarrow \mu(A_2) = 0$

$A_3 = \{3\} \Rightarrow \mu(A_3) = 0$

$\vdots \quad \vdots \qquad \vdots \quad \vdots$

$$\sum_{i=1}^{\infty} \mu(A_i) = 0 \quad \text{—— (v)}$$

Also $A = \bigcup_{i=1}^{\infty} A_i = \{1, 2, 3, 4, \dots\}$

$\Rightarrow \mu(A) = \infty$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty \quad \text{—— (vi)}$$

From eq (v) and (vi), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \neq \sum_{i=1}^{\infty} \mu(A_i)$$

Thus "μ" is not σ-additive.

Theorem:-

let (X, S) be a semi-ring and

$\mu: S \to [0, \infty]$ is a finitly additive such that "μ" is a σ- sub additive, then prove that "μ" is a measure.

Proof:-

let $A_1, A_2, A_3, \ldots \in S$ such that $A = \bigcup_{i=1}^{\infty} A_i \in S$; $A_i \cap A_j = \phi$ for $i \neq j$.

we need to show that "μ" is measure.

Equivalently we will show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad \text{—— (A)}$$

Now since "μ" is σ- sub additive, so

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) \longleftarrow (1)$$

Now to prove (A), it is enough to show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i)$$

For this let $A = \bigcup_{i=1}^{\infty} A_i$, then

$$A \setminus A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \bigcup_{i=1}^{m} B_i \text{ —— (2) } \forall B_i \in S$$

where $\forall\ i \neq j$, $B_i \cap B_j = \phi$

Now eq (2) $\Rightarrow$ $A = \left[\bigcup_{i=1}^{n} A_i \cup \left(\bigcup_{i=1}^{m} B_i\right)\right]$

$$\mu(A) = \mu\left[\bigcup_{i=1}^{n} A_i \cup \left(\bigcup_{i=1}^{m} B_i\right)\right]$$

$$\mu(A) = \mu\left(\bigcup_{i=1}^{n} A_i\right) + \mu\left(\bigcup_{i=1}^{m} B_i\right) \quad (\because \mu \text{ is finitly additive})$$

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i) + \sum_{i=1}^{m} \mu(B_i)$$

$$\Rightarrow \mu(A) \geq \sum_{i=1}^{n} \mu(A_i) \qquad \text{as } \sum_{i=1}^{m} \mu(B_i) \geq 0$$

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Taking $n \to \infty$, we have

$$\mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)$$

$\left(\text{as } A = \bigcup_{i=1}^{\infty} A_i\right)$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i) \longrightarrow (3)$$

from eq(1) and eq(3), we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Thus "μ" is a measure.

Example:-

let $X \neq \phi$ and $f : X \to [0,\infty]$ be a function and $\mu : P(X) \to [0,\infty]$ be defined by

$$\mu(A) = \begin{cases} \sum_{x \in A} f(x) & ; \text{ if } A \text{ is countable} \\ \infty & ; \text{ if } A \text{ is uncountable} \end{cases}$$

and $\mu(\phi) = 0$.

Show that "μ" is a measure.

Sol:-

Let $A_1, A_2, A_3, \ldots \in P(X)$ such that $A_i \cap A_j = \phi$, for $i \neq j$

we need to show that "μ" is measure.

Equivalently we need to show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

For this there are cases:

Case-1:-

If all A_i's are countable

So, $\bigcup_{i=1}^{\infty} A_i$ is countable.

$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{x \in (\bigcup_{i=1}^{\infty} A_i)} f(x)$ [by definition of function "μ"

$= \sum_{x \in A_1} f(x) + \sum_{x \in A_2} f(x) + \sum_{x \in A_3} f(x) + \cdots$

$= \mu(A_1) + \mu(A_2) + \mu(A_3) + \cdots$

$= \sum_{i=1}^{\infty} \mu(A_i)$

$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

Case-2:-

If one of A_i's is uncountable, say A_k is uncountable $\Rightarrow \mu(A_k) = \infty$

$\Rightarrow \sum_{i=1}^{\infty} \mu(A_i) = \infty$ —— (i)

Also since A_k is uncountable.

$\Rightarrow \bigcup_{i=1}^{\infty} A_i$ is uncountable.

So by def: of μ,

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty$ —— (ii)

From eq (i) and (ii), we get

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

And $\mu(\phi) = 0$

So in all cases "μ" satisfies the

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property of measure.

$\Rightarrow$ Thus "μ" is a measure.

Outer Measure (Carathedory Measure).

let $X \neq \phi$ and $\mu : P(X) \rightarrow [0,\infty]$ is a function such that

i) $\mu(\phi) = 0$.

ii) $\mu(A) \leq \mu(B)$ if $A \subseteq B$.

iii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ where $A_1, A_2, A_3, \dots \in P(X)$

then "μ" is called outer measure.

Measurable Set:-

If $E \subseteq X$ and μ is an outer measure and

$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$ for any $A \subseteq X$.

then E is said to be measurable set.

Note:-

Since $A = (A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \dots$

$\Rightarrow \mu(A) = \mu\left((A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \dots\right)$

Now Since "μ" is outer measure, So by property (iii), since "μ" is σ-sub additive, so

$\Rightarrow \mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + \mu(\phi) + \mu(\phi) + \dots$

$\Rightarrow \mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + 0 + 0 + \dots$

$\Rightarrow \mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c)$ always holds.

So from here we can say that "E" will be measurable, with respect to the outer measure "μ" if

$$\mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c) \quad \text{(i)}$$

if "μ" is measure on $P(X)$ then every set (i.e. subset of X) is measurable.

Because "$\leq$" reduces to "$=$" in above eq (i).

Null set:-

Let "μ" be an outer measure, then any set $E \subseteq X$ is said to be null if $\mu(E) = 0$.

Now, Since $\mu(\phi) = 0$ (by 1st property of outer measure)

So, empty set is a null set.
But not every null set is empty, because there exist non-empty sets, which are null sets.

Theorem:-

Every null set is measurable, under the outer measure μ.

Proof:-

Let "μ" be an outer measure, and E is a null set under μ, i.e. $\mu(E) = 0$.

We need to prove that E is measurable.

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i.e. we will show that for any $A \subseteq X$, $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

For this, since

$$A = (A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \cdots$$

$$\mu(A) = \mu\left((A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \cdots\right)$$

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + \mu(\phi) + \mu(\phi) + \cdots$$

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + 0 + 0 + \cdots$$

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) \quad \text{--- (1)}$$

Now, since $A \cap E \subseteq E$

$$\mu(A \cap E) \leq \mu(E) = 0$$

$$\mu(A \cap E) \leq 0 \quad \text{--- (i)}$$

and $A \cap E^c \subseteq A$

$$\Rightarrow \mu(A \cap E^c) \leq \mu(A) \quad \text{--- (ii)}$$

Adding eq (i) and (ii), we get

$$\mu(A \cap E) + \mu(A \cap E^c) \leq 0 + \mu(A)$$

$$\mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A) \quad \text{--- (2)}$$

From eq (1) and eq (2), we have

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A)$$

$\Rightarrow \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

Hence E is measurable.

Theorem:-
If E is measurable, then prove that E^c is measurable.

Proof:-
Since E is measurable, so
$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$$

$$\Rightarrow \mu(A) = \mu(A \cap (E^c)^c) + \mu(A \cap E^c)$$

$$\Rightarrow \mu(A) = \mu(A \cap E^c) + \mu(A \cap (E^c)^c)$$

Thus E^c is measurable.

Note:- Every complemente of measurable set is measurable.

Theorem:-
If E_1 and E_2 are measurable sets, then prove that $E_1 \cup E_2$ is measurable set.

Proof:-
Since E_1 and E_2 are measurable
let $E = E_1 \cup E_2$
we need to prove that E is measurable
i.e. we prove that $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$.

Now, $E = E_1 \cup E_2$
$E = E_1 \cup (E_1^c \cap E_2)$

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$E = (E_1 \cup E_1^c) \cap (E_1 \cup E_2)$

$E = E_1 \cup (E_1^c \cap E_2)$

Now $\mu(A) = \mu[(A \cap E) \cup (A \cap E^c)]$

$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

put $E = E_1 \cup (E_1^c \cap E_2)$

$\mu(A) = \mu[A \cap (E_1 \cup (E_1^c \cap E_2))] + \mu[A \cap (E_1 \cup (E_1^c \cap E_2))^c]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (E_1^c \cap (E_1^c \cap E_2)^c)]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (E_1^c \cap ((E_1^c)^c \cup E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (E_1^c \cap (E_1 \cup E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap ((E_1^c \cap E_1) \cup (E_1^c \cap E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (\phi \cup (E_1^c \cap E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu(A \cap E_1^c \cap E_2^c)$

$\mu(A) \le \mu(A \cap E_1) + \mu(A \cap E_1^c \cap E_2) + \mu(A \cap E_1^c \cap E_2^c)$

Since E_2 is measurable.

$\mu(A) \le \mu(A \cap E_1) + \mu(A \cap E_1^c)$

$= \mu(A)$ as E_1 is measurable

$\Rightarrow \mu(A) \le \mu(A \cap E) + \mu(A \cap E^c) \le \mu(A)$

$\Rightarrow \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

which shows that "E" is measurable.

So $E_1 \cup E_2 = E$ is measurable.

Note:-

→ Every union set is measurable set

→ Ground set is measurable set.

$\phi^c = X$ is measurable set.

Lemma:-

let $E_1, E_2, E_3, \ldots, E_n$ be pairwise disjoint and measurable sets, then

$$\mu\left(\dot{\bigcup_{i=1}^{n}}(A\cap E_i)\right) = \sum_{i=1}^{n} \mu(A\cap E_i)$$

where "μ" is an outer measure, where $A \subseteq X$.

Proof:-

We will prove this by mathematical induction;

let $n=1$, then

$\mu(A\cap E_1) = \mu(A\cap E_1)$, it is clearly true for $n=1$.

Suppose it is true for $n=k$.

$$\mu\left[\dot{\bigcup_{i=1}^{k}}(A\cap E_i)\right] = \sum_{i=1}^{k} \mu(A\cap E_i) \quad \text{——— (1)}$$

where $E_1, E_2, E_3, \ldots, E_n$ are pairwise disjoint and measurable.

We need to prove for $n=k+1$,

$$\mu\left[\dot{\bigcup_{i=1}^{k+1}}(A\cap E_i)\right] = \sum_{i=1}^{k+1} \mu(A\cap E_i)$$

For this let us consider

$$\mu\left[\dot{\bigcup_{i=1}^{k+1}}(A\cap E_i)\right] = \mu\left[(A\cap E_1)\cup(A\cap E_2)\cup\cdots\cup(A\cap E_k)\cup(A\cap E_{k+1})\right]$$

$$= \mu\left(\left[\dot{\bigcup_{i=1}^{k}}(A\cap E_i)\right]\cup(A\cap E_{k+1})\right)$$

$$= \mu\left[\dot{\bigcup_{i=1}^{k}}(A\cap E_i)\cup(A\cap E_{k+1})\right]$$

$$= \mu\left[\bigcup_{i=1}^{k}(A\cap E_i)\right] + \mu(A\cap E_{k+1})$$

$$= \sum_{i=1}^{k} \mu(A\cap E_i) + \mu(A\cap E_{k+1})$$ (using...

$$\mu\left[\bigcup_{i=1}^{k+1}(A\cap E_i)\right] = \sum_{i=1}^{k+1} \mu(A\cap E_i)$$

it is true for $n=k+1$.
So all the cases of mathematical induction are true.
Thus $\mu\left[\bigcup_{i=1}^{n}(A\cap E_i)\right] = \sum_{i=1}^{n} \mu(A\cap E_i)$.

Hence proved!

Theorem:-
Let $E_1, E_2, E_3, \cdots$ be measurable sets, then prove that $\bigcup_{i=1}^{\infty} E_i$ is measurable set.

Proof:-
Let $E_1, E_2, E_3, \cdots$ be measurable sets.
We need to prove that $\bigcup_{i=1}^{\infty} E_i$ is measurable set.

Let
$G_1 = E_1$
$G_2 = E_2 / E_1$
$G_3 = E_3 / (E_1 \cup E_2)$
$\vdots \quad \vdots$

So let $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} G_i$, clearly $G_i \cap G_j$...

And let $F_n = \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} G_i$

Since E_i's are measurable set, so F_n is measurable set.

So by definition of measurable set for any set $A \subseteq X$.

$\mu(A) = \mu(A \cap F_n) + \mu(A \cap F_n^c)$; —— ① $\forall A \subseteq X$

Since $F_n \subseteq E$

$\Rightarrow F_n^c \supseteq E^c$

$\Rightarrow A \cap F_n^c \supseteq A \cap E^c$

$\Rightarrow \mu(A \cap F_n^c) \geq \mu(A \cap E^c)$

eq ①, becomes

$\mu(A) \geq \mu(A \cap F_n) + \mu(A \cap E^c)$

$\Rightarrow \mu(A) \geq \mu\left[A \cap \left(\bigcup_{i=1}^{n} G_i\right)\right] + \mu(A \cap E^c)$

$\Rightarrow \mu(A) \geq \mu\left(\bigcup_{i=1}^{n} (A \cap G_i)\right) + \mu(A \cap E^c)$

$\Rightarrow \mu(A) \geq \sum_{i=1}^{n} \mu(A \cap G_i) + \mu(A \cap E^c)$ ② (using lemma)

Now $E = \bigcup_{i=1}^{\infty} G_i$

$A \cap E = \bigcup_{i=1}^{\infty} (A \cap G_i)$

$\Rightarrow \mu(A \cap E) = \mu\left[\bigcup_{i=1}^{\infty} (A \cap G_i)\right]$

$\Rightarrow \mu(A \cap E) \leq \sum_{i=1}^{\infty} \mu(A \cap G_i)$ —— ⊛

Taking $n \to \infty$ in eq ②, we get

$\Rightarrow \mu(A) \geq \sum_{i=1}^{\infty} \mu(A \cap G_i) + \mu(A \cap E^c)$

$\Rightarrow \mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c)$ (using eq ⊛)

$\Rightarrow$ E is measurable set.

$\Rightarrow$ $E = \bigcup_{i=1}^{\infty} E_i$ is measurable set.

Theorem:-

let Δ be the class of all measurable set and μ is outer measure, then μ restrickted to Δ is measure $\{(X, \Delta, \mu)$ is measure space$\}$.

Proof:-

let $E_1, E_2, E_3, \ldots$ be measurable sets such that $E_i \cap E_j = \phi$

we need to prove that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Since "μ" is an outer measure. So by definition of outer measure "μ".

$$\mu(E) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i) \quad \text{——— (i)}$$

So to prove the required result, it is enough to show that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

For this consider

$$\sum_{i=1}^{k} \mu(E_i) = \sum_{i=1}^{k} \mu(E \cap E_i)$$

$$= \mu\left(\bigcup_{i=1}^{k} (E \cap E_i)\right)$$

$$\sum_{i=1}^{k} \mu(E_i) = \mu\left[E \cap \left(\bigcup_{i=1}^{k} E_i\right)\right] \quad \text{——— (ii)}$$

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But $E \cap (\bigcup_{i=1}^{k} E_i) \subseteq E$

$$\Rightarrow \mu\left[E \cap (\bigcup_{i=1}^{k} E_i)\right] \leq \mu(E)$$

So eq (ii) $\Rightarrow \sum_{i=1}^{k} \mu(E_i) = \mu\left[E \cap (\bigcup_{i=1}^{k} E_i)\right] \leq \mu(E)$

$$\Rightarrow \sum_{i=1}^{k} \mu(E_i) \leq \mu(E)$$

Taking $k \to \infty$, we get

$$\Rightarrow \sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E) = \mu(\bigcup_{i=1}^{\infty} E_i) \quad \text{—— (iii)}$$

From eq (i) and eq (iii), we get

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$$

Thus μ is measure.

Hence proved!

Question:-

let $A \subseteq \mathbb{R}^2$ and define μ^* by $\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \mu(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i\}$, where A_i's are rectangles and μ represents area and infimum is taken over possible of covers of rectangles.

We show that μ^* is an outer measure.

Sol:-

We have to show that μ^* satisfies three properties of outer measure.

Clearly $\mu^*(A) \geq 0$

Since $\phi \subseteq \phi \cup \phi \cup \phi \cup ---$

$\mu^*(\phi) \leq \mu(\phi) + \mu(\phi) + \mu(\phi) + \cdots$

$\mu^*(\phi) \leq 0 + 0 + 0 + \cdots$

$\mu^*(\phi) \leq 0$

$\mu^*(\phi) = 0$

ii) If $A \subseteq B$, we show that
$\mu^*(A) \leq \mu^*(B)$

let $L = \left\{ \sum_{i=1}^{\infty} \mu(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$

and $M = \left\{ \sum_{i=1}^{\infty} \mu(B_i) : B \subseteq \bigcup_{i=1}^{\infty} B_i \right\}$

If $B \subseteq \bigcup_{i=1}^{\infty} B_i$ and $A \subseteq B$

$\Rightarrow A \subseteq \bigcup_{i=1}^{\infty} B_i$

Every cover of B is also cover of A.

Thus $M \subseteq L$.

$\inf L \leq \inf M$.

$\mu^*(A) \leq \mu^*(B)$.

Hence μ^* is monotonic.

iii) let $A_1, A_2, A_3, \cdots \subseteq \mathbb{R}^2$, we show that

$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

By definition of μ^*, for every $\epsilon > 0$, there exists covers A_i^n of A_i such that

$\mu^*(A_i) + \epsilon 2^{-i} > \sum_{n=1}^{\infty} \mu(A_i^n)$

$$\sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \sum_{i=1}^{\infty} 2^{-i} > \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A^n_i).$$

$$\sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon > \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A^n_i) \quad \text{——— (1)}$$

Since $A_i \subseteq \bigcup_{n=1}^{\infty} A^n_i$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A^n_i$$

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A^n_i)$$

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) < \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \quad \text{(using eq (1))}$$

Since ϵ is an arbitrary, so takes $\epsilon \to 0$.

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + 0$$

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Thus all the conditions of outer measure are satisfied.

Hence μ^* is an outer measure.

Q:- Let $A = \{x\}$ then $\mu^*(A) = 0$.

Sol:-

$$x \in \left(x - \epsilon/2, x + \epsilon/2\right)$$

$$\Rightarrow \{x\} \subseteq \left(x - \epsilon/2, x + \epsilon/2\right)$$

$$\Rightarrow \mu^*\{x\} \leq \mu\left(x - \epsilon/2, x + \epsilon/2\right)$$

$$\Rightarrow \mu^*\{x\} \leq x + \epsilon/2 - \left(x - \epsilon/2\right)$$

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$$\mu^*\{x\} \le \cancel{x}+\frac{\epsilon}{2}-\cancel{x}+\frac{\epsilon}{2}$$

$$\mu^*\{x\} \le \frac{\epsilon}{2}+\frac{\epsilon}{2}$$

$$\mu^*\{x\} \le \frac{2\epsilon}{2}$$

$$\mu^*\{x\} \le \epsilon$$

Since $A=\{x\}$

$$\mu^*(A) \le \epsilon$$

but taking $\epsilon \to 0$

$$\mu^*(A) = 0$$

Q:- Let $A=\{x_1, x_2, x_3, ----\}$ be countable set then $\mu^*(A)=0$.

Sol:-

Given

$$A=\{x_1, x_2, x_3, ----\}$$

$$A=\{x_1\}\cup\{x_2\}\cup\{x_3\}\cup ----$$

$$\mu^*(A)=\mu^*\{x_1\}+\mu^*\{x_2\}+\mu^*\{x_3\}+----$$

$$\mu^*(A)=0+0+0+----$$

$$\mu^*(A)=0$$

Lemma:-

Show that each countable set is measurable.

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Proof:-

let A be any countable set,

so $\mu^*(A) = 0$

$\Rightarrow A$ is null set.

Since we know that "every null set is measurable".

So A is measurable.

Hence each countable set is measurable.

Question:-

Show that (a,∞) is measurable set.

Sol:-

To show that the set $E = (a,\infty)$ is measurable.

we have to show that for any set $A \subseteq \mathbb{R}$, we have

$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$ ——— (*).

For this we discuss the following two cases:

Case-1:-

If $\mu(A) = \infty$, then

$\mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c)$

but

$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c)$

always holds, so

$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$.

Case-2:-

If $\mu(A) < \infty$, then for any cover $\{I_n : n \in N\}$ of A, by def. of Lebsegue measure.

$\mu(A) = \inf\{\sum_{n=1}^{\infty} l(I_n) \quad : \quad A \subseteq \bigcup_{n=1}^{\infty} I_n\}$

$\mu(A) \leq \sum_{n=1}^{\infty} l(I_n) \quad , \quad A \subseteq \bigcup_{n=1}^{\infty} I_n$

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let $\epsilon > 0$ be any arbitrary small +ve value.

So by definition of infimum

$$\mu(A) + \epsilon > \sum_{n=1}^{\infty} l(I_n) \quad , A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\mu(A) + \epsilon \geq \sum_{n=1}^{\infty} l(I_n) \text{ --- (1) (by property of inequal}$$

Now $I_n = (I_n \cap E) \cup (I_n \cap E^c)$

$\because l$ is measurable

$$\Rightarrow l(I_n) = l(I_n \cap E) + l(I_n \cap E^c)$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} [l(I_n \cap E) + l(I_n \cap E^c)]$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} l(I_n \cap E) + \sum_{n=1}^{\infty} l(I_n \cap E^c)$$

therefore eq (1) becomes,

$$\mu(A) + \epsilon \geq \sum_{n=1}^{\infty} l(I_n \cap E) + \sum_{n=1}^{\infty} l(I_n \cap E^c) \text{ --- (A)}$$

Now Since

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$A \cap E \subseteq \bigcup_{n=1}^{\infty} I_n \cap E$$

$$= \bigcup_{n=1}^{\infty} (I_n \cap E)$$

$$\Rightarrow A \cap E \subseteq \bigcup_{n=1}^{\infty} (I_n \cap E)$$

$\Rightarrow \{I_n \cap E : n \in N\}$ form cover for $A \cap E$.

So by def. of infimum and lebesgue measure.

$$\mu(A \cap E) \leq \sum_{n=1}^{\infty} l(I_n \cap E) \text{ --- (2)}$$

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On the similar way, since $A \cap E^c \subseteq \bigcup_{n=1}^{\infty} (I_n \cap E^c)$,

So,

$$\mu(A \cap E^c) \leq \sum_{n=1}^{\infty} l(I_n \cap E^c) \quad \text{——— (3)}$$

Adding eq (2) and (3), we have

$$\mu(A \cap E) + \mu(A \cap E^c) \leq \sum_{n=1}^{\infty} l(I_n \cap E) + \sum_{n=1}^{\infty} l(I_n \cap E^c)$$

$$\mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A) + \epsilon \qquad (\text{using eq (A)})$$

Since ϵ is an arbitrary, so $\epsilon \to 0$

$$\mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A) + 0$$

$$\mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A)$$

$$\Rightarrow \mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c) \quad \text{——— (B)}$$

But $\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c)$ ——— (C) is obviously

From eq (B) and eq (C), we get

$$\Rightarrow \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$$

$\Rightarrow E$ is measurable.

i.e. (a, ∞) is measurable set.

Lebesgue Measure:-

Let X be any non-empty set usually we take $X = \mathbb{R}^n$, and F be the collection of subsets of X, and $\mu : F \to [0, \infty]$, be a set function such that $\mu(\phi) = 0$.

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Let us define $\mu^*: P(X) \rightarrow [0,\infty]$, by

$$\mu^*(A) = \text{Inf}\left\{\sum_{i=1}^{\infty} \mu(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i\right\},$$
where $\{A_i\}$ is a cover for A, $A_i \in \mathcal{F}$ ...
$= \infty$, if there is no cover for A

Where infimum is taken over all possible covers of A.

Q:- Show that $(-\infty, a]$ is measurable set.

Sol:-

Since (a,∞) is measurable set.

$\Rightarrow (a,\infty)^c$ is measurable set.

$\Rightarrow (-\infty, a]$ is measurable set.

Q:- Show that $(-\infty, a)$ is measurable set.

Sol:-

Since $(-\infty, a]$ is measurable, $\forall\, a \in \mathbb{R}$.

$\Rightarrow (-\infty, a-1/n]$ is measurable set for each $n \in N$

let $E_n = (-\infty, a-1/n]$

$\Rightarrow E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$

Since each E_i is measurable,

So $\bigcup_{n=1}^{\infty} E_n$ is measurable

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} (-\infty, a-1/n]$$

$$= (-\infty, a)$$

$\Rightarrow (-\infty, a)$ is measurable set.

Q:- Show that $[a,b]$ is measurable set.

Sol:-

Since $(-\infty,b]$ and $[a,\infty)$ are measurable set, so

$\Rightarrow (-\infty,b] \cap [a,\infty)$ is measurable set.

$\Rightarrow [a,b]$ is measurable set.

Q:- Show that $\mathbb{R}$ is measurable set.

Sol:-

Since $(-\infty,a]$ and $[a,\infty)$ are measurable set

$\Rightarrow (-\infty,a] \cup [a,\infty)$ is measurable set.

$\Rightarrow (-\infty,\infty)$ is measurable set.

$\Rightarrow \mathbb{R}$ is measurable set.

Theorem:-

If A is not null set and $B \subseteq A$. If B is null set, prove that $A \setminus B$ is not null set.

Proof:-

Let us suppose that $A \setminus B$ is null set.

As B is null set.

$\Rightarrow (A \setminus B) \cup B$ is null set.

$\Rightarrow A$ is null set. $(\because B \subseteq A)$

But A is not null set (given)

So which is contradiction.

Hence $A \setminus B$ is not null set.

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Theorem:-

Show that for any set E,
$m(E) = m(E+x)$, where $E+x = \{e+x ; e \in E\}$

Proof:-

By def. of lebesgue measure

$$m(E) = \inf\{\sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n\}$$

$$m(E) \leq \sum_{n=1}^{\infty} l(I_n) \quad , \quad E \subseteq \bigcup_{n=1}^{\infty} I_n$$

for every $\epsilon > 0$

$$m(E) + \epsilon > \sum_{n=1}^{\infty} l(I_n) \quad \text{for some covers } \{I_n\} \text{ of}$$

$$m(E) + \epsilon \geq \sum_{n=1}^{\infty} l(I_n) \text{ --- (i)}$$

Now Since

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$E+x \subseteq \bigcup_{n=1}^{\infty} I_n + x$$

$$E+x \subseteq \bigcup_{n=1}^{\infty} (I_n + x)$$

$$m(E+x) \leq \sum_{n=1}^{\infty} l(I_n + x)$$

$$\Rightarrow m(E+x) \leq \sum_{n=1}^{\infty} l(I_n + x) = \sum_{n=1}^{\infty} l(I_n)$$

$$\Rightarrow m(E+x) \leq \sum_{n=1}^{\infty} l(I_n) \text{ --- (ii)}$$

Combining eq (i) and (ii), we get

$$m(E+x) \leq \sum_{n=1}^{\infty} l(I_n) \leq m(E) + \epsilon$$

$$\Rightarrow m(E+x) \leq m(E) + \epsilon$$

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but ϵ is any arbitrary small +ve number
i.e. $\epsilon \to 0$

$$\mu(E+x) \le \mu(E) \text{ ------ } (A)$$

Since , $E = E + x - x$

$$E = (E+x) - x$$

$$\Rightarrow \mu(E) = \mu((E+x)-x)$$

$$\Rightarrow \mu(E) = \mu((E+x)+(-x))$$

So by eq (A) , we get

$$\mu(E) = \mu((E+x)+(-x)) \le \mu(E+x)$$

$$\Rightarrow \mu(E) \le \mu(E+x)$$

$$\Rightarrow \mu(E+x) \ge \mu(E) \text{ ------ } (B)$$

From eq (A) and (B) , we get

$$\mu(E) = \mu(E+x)$$

which is the required proof.

Theorem:-

let $\{E_n\}$ be a sequence of measurable sets , then the following statements holds.

i) If $E_n \uparrow E$, then $\mu(E_n) \uparrow \mu(E)$.

ii) If $E_n \downarrow E$ and $\mu(E_k) < \infty$ for some k, then $\mu(E_n) \downarrow \mu(E)$.

Proof:-

Since $E_n \uparrow E \Rightarrow E_n$ is increasing sequence approches to E.

$\Rightarrow E_1 \subseteq E_2 \subset E_3 \subset E_4 \subseteq \text{----}$ and $E = \bigcup_{n=1}^{\infty} E_n$

let $B_1 = E_1$, $B_2 = E_2 \setminus E_1$, $B_3 = E_3 \setminus E_2$, $B_4 = E_4 \setminus E_3$ ----

then $B_i \cap B_j = \Phi$ for $i \neq j$.
Further more since each $E_i \ \forall \ i$ is measurable.

$\Rightarrow B_i \ \forall i$ is measurable.

Now since $E_n = \bigcup_{i=1}^{n} B_i$

$$\Rightarrow \lim_{n\to\infty} E_n = \lim_{n\to\infty} \bigcup_{i=1}^{n} B_i$$

But $E_n \uparrow E$. So from above

$$E = \lim_{n\to\infty} \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{\infty} B_i$$

$$\Rightarrow \quad E = \bigcup_{i=1}^{\infty} B_i$$

$$\mu(E) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$$\mu(E) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$\mu(E) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i)$$

$$\mu(E) = \lim_{n\to\infty} \mu(E_n)$$

$$\Rightarrow \lim_{n\to\infty} \mu(E_n) = \mu(E) \quad \text{——— (A)}$$

But $\mu(E_1) \leq \mu(E_2) \leq \mu(E_3) \leq \cdots$
then

$$\mu(E_n) \uparrow \mu(E)$$

ii) Since $E_n \downarrow E$

$\Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq \cdots$.

Since $\mu(E_k) < \infty$, for some K.
So without of loss of generality

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let
$$\mu(E_1) < \infty$$
Now since $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

$$\Rightarrow E_1 \setminus E_1 \subseteq E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq \cdots$$
$$\Rightarrow E_1 \setminus E_n \nearrow E_1 \setminus E$$

by eq (A), we have
$$\lim_{n\to\infty} \mu(E_1 \setminus E_n) = \lim_{n\to\infty} \mu(E_1 \setminus E)$$

$$\lim_{n\to\infty} \mu(E_1 - E_n) = \lim_{n\to\infty} \mu(E_1 - E)$$

$$\Rightarrow \mu(E_1) - \lim_{n\to\infty} \mu(E_n) = \mu(E_1) - \mu(E)$$

$$\Rightarrow -\lim_{n\to\infty} \mu(E_n) = -\mu(E)$$

$$\Rightarrow \lim_{n\to\infty} \mu(E_n) = \mu(E).$$

Thus $\mu(E_n) \downarrow \mu(E)$.

Zermelo's Axiom:-

Consider a family of arbitrary non-empty disjoint sets indexed by set A, $\{E_\alpha : \alpha \in A\}$. Then there exist a set consisting of exactly one element from each E_α, $\alpha \in A$.

Lemma:-

let E be a measurable subset of $\mathbb{R}$ with $\mu(E) > 0$. Then the set of difference $\{d : d = x - y, x \in E, y \in E\}$ contain an interval centre at the origin.

Theorem:- (Vitali): there exists Non-lebesgue measurable subset of $\mathbb{R}$.

Proof:-

Define a relation T on $\mathbb{R}$ by xTy if $x-y=r$ (rational), $x,y\in\mathbb{R}$.

We show that T is an equivalence relation.

i) **Reflexive:-**

let $x\in\mathbb{R}$

$\Rightarrow x-x=0$ (rational)

$\Rightarrow xTx$

$\Rightarrow T$ is reflexive.

ii) **Symmetric:-**

let $x,y\in\mathbb{R}$

there exists xTy

$\Rightarrow x-y=r$ (rational)

$\Rightarrow -(y-x)=r$

$\Rightarrow y-x=-r$ (rational)

$\Rightarrow yTx$

$\Rightarrow T$ is symmetric.

iii) **Transitive:-**

let $x,y,z\in\mathbb{R}$

there exists xTy and yTz

$\Rightarrow x-y=r_1$ and $y-z=r_2$

$\Rightarrow x-\cancel{y}+\cancel{y}-z=r_1+r_2$ (rational)

$\Rightarrow x-z=r_1+r_2$ (rational)

$\Rightarrow xTz$

$\Rightarrow T$ is transitive.

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Hence T is an equivalence relation.

let E_x be an equivalence w.r.t $x \in \mathbb{R}$.

$$E_x = \{y \in \mathbb{R} : yTx\}$$

$$E_x = \{y \in \mathbb{R} : y - x = r, \; r \in \mathbb{Q}\}$$

$$E_x = \{y \in \mathbb{R} : y = x + r, \; r \in \mathbb{Q}\}$$

$$E_x = \{x + r : r \in \mathbb{Q}\}$$

Any two classes are either identical or disjoint.

let E_x and E_y be two classes such that $E_x \cap E_y \neq \phi$

$\Rightarrow z \in E_x \cap E_y$

$\Rightarrow z \in E_x$ and $z \in E_y$

$\Rightarrow zTx$ and zTy

$\Rightarrow xTy$

let $x_1 \in E_x$

$\Rightarrow x_1 T x$

$\Rightarrow x_1 T y$

$\Rightarrow x_1 \in E_y$

$\Rightarrow E_x \subseteq E_y$

Similarly $x_1 \in E_y$

$\Rightarrow x_1 T y \Rightarrow x_1 T x$

$\Rightarrow x_1 \in E_x$

$\Rightarrow E_y \subseteq E_x$

So, $E_x = E_y$

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$E_x = \{x+r : r \in \mathbb{Q}\}$
there exists one class which contain all rational number.
Each class is countable.
As the union of all these classes is equal to $\mathbb{R}$.
So the number of class is uncountable.
By zermelo's axiom, we can find a set E which contain exactly one element from each different class.
Now we show that E is Non-measurable set.
If E is measurable and consider $D = \{d : d = x-y, \ x, y \in E\}$, then D cannot contain rational numbers.
So D cannot contain any interval.
let us suppose that E has lebesgue measure is zero.

$x \in E \quad , \quad y \in E$

$x+r \quad , \quad y+r$

$\bigcup_r (E+r) = \mathbb{R}$

$\mu(\mathbb{R}) = 0$ as $\mu(E) = \mu(E+r) = 0$
which is contradiction and hence E is non-lebesgue measurable subset of $\mathbb{R}$.

Assignment:-
Every interval contain non-lebesgue measurable subset.

Question:-

Show that the lebesgue measure of cantour set C is zero.

Sol:-

uncountable

Since cantour set C is a subset of $[0,1]$ i-e $C \subseteq [0,1]$

$\Rightarrow \mu(C) = l[0,1] -$ total length of removed open interval.

$= \left[1 - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots\right)\right]$

$= 1 - 1$ $\left(S_\infty = \frac{a_1}{1-r}\right)$

$= 0$

$\Rightarrow \boxed{\mu(C) = 0}$

Example:-

$[1,3] \cup [5,7]$

$\Rightarrow \mu[1,3] + \mu[5,7]$

$\Rightarrow (3-1) + (7-5)$

$\Rightarrow 2+2$

$\Rightarrow 4$

Question:-

If μ is an outer measure and A is a null set then show that $\mu(B) = \mu(B \cup A) = \mu(B \setminus A)$. holds for any $B \subseteq X$.

Sol:-

Since $B \subseteq B \cup A$

$\Rightarrow \mu(B) \leq \mu(B \cup A)$ ——— ①

as μ is an outer measure

and $B \cup A = (B \setminus A) \cup A$

$\Rightarrow \mu(B \cup A) = \mu[(B \setminus A) \cup A]$

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$$\mu(B\cup A) = \mu(B\setminus A) + \mu(A)$$

$$\mu(B\cup A) = \mu(B\setminus A) + 0 \quad \text{as } \mu(A) = 0$$

$$\mu(B\cup A) = \mu(B\setminus A)$$

So eq ① $\Rightarrow$

$$\mu(B) \leq \mu(B\cup A) = \mu(B\setminus A) \quad \text{——— ②}$$

Since $B\setminus A \subseteq B$

$$\mu(B\setminus A) \leq \mu(B)$$

So eq ② $\Rightarrow$

$$\mu(B) \leq \mu(B\cup A) = \mu(B\setminus A) \leq \mu(B)$$

$$\Rightarrow \mu(B) = \mu(B\cup A) = \mu(B\setminus A).$$

Question:-

Let μ be an outer measure and if a sequence $\{A_n\}$ satisfies $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then show that the set $E = \{x\in X : x\in A_n \text{ for infinitly many}\}$ is null set.

Sol:-

Let us suppose that

$$E_n = \bigcup_{n=1}^{\infty} A_n$$

$$\mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$\mu(E_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

Clearly $E \subseteq E_n$ for each n.

$$\mu(E) \leq \mu(E_n)$$

Also $\mu(E) \geq 0$

$$\Rightarrow 0 \leq \mu(E) \leq \mu(E_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

$$\Rightarrow 0 \leq \mu(E) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

If $n \to \infty$, $\sum_{n=1}^{\infty} \mu(A_n) \to 0$

$$\Rightarrow 0 \leq \mu(E) \leq 0$$

$$\Rightarrow \mu(E) = 0$$

Hence E is the null set.

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CHAPTER : 3

MEASURABLE FUNCTIONS

Almost everywhere (a.e) :-

We say that a property holds almost everywhere (a.e) if that property for all point except for the set of point having lebesgue measure zero.

Example :-

let $f(x) = x+1$ if $x \in Q$
$= x^2+2$ if $x \in Q'$

and $g(x) = 4x^3+10$ if $x \in Q$
$= x^2+2$ if $x \in Q'$

Clearly $f(x) \neq g(x)$

Now let $A = \{x : f(x) \neq g(x)\}$
$\Rightarrow A = Q$
$\mu^*(A) = \mu^*(Q) = 0$
Hence $f = g$ a.e

Example :-

let $f(x) = 2x^2+1$ if $x \in Q'$
$= x-10$ if $x \in Q$

and

$g(x) = x^2$ if $x \in Q'$
$= x+20$ if $x \in Q$

Clearly $f \ngtr g$

Now let $A = \{x : f \ngtr g\}$
$\Rightarrow A = Q$

But $\mu^*(A) = \mu^*(Q) = 0$
Hence $f > g$ a.e.

Example:-
let $\{f_n(x)\}$ be a sequence of functions defined on $[0,1]$ by
$f_n(x) = x^n$ if $x \in [0,1)$
and let $f : [0,1] \longrightarrow \mathbb{R}$ be defined by $f(x) = 0$ if $x \in [0,1)$
$= 5$ if $x = 1$

then clearly $f_n(x) \not\rightarrow f(x)$

Now let $A = \{x : f_n(x) \not\longrightarrow f(x)\}$
$\Rightarrow A = \{1\}$

$\Rightarrow \mu^*(A) = 0$
Hence $f_n \longrightarrow f$ a.e.

Measurable Function:-
i) $f = g$ a.e if $\mu^*(\{x \in X : f(x) \neq g(x)\}) = 0$
ii) $f \geq g$ a.e if $\mu^*(\{x \in X : f(x) < g(x)\}) = 0$
iii) $f_n \rightarrow f$ a.e if $\mu^*(\{x \in X : f_n(x) \not\rightarrow f(x)\}) = 0$.
iv) $f_n \uparrow f$ a.e if $f_n \leq f_{n+1}$ a.e $\forall n$ and $f_n \rightarrow f$ a.e.
v) $f_n \downarrow f$ a.e if $f_{n+1} < f_n$ a.e. and $f_n \rightarrow f$ a.e.
Let $(X \in \mathbb{R}^n, S, \mu)$ be a measure space.
let f be a real valued function on E

i.e. $f : E \subset \mathbb{R}^n \rightarrow \mathbb{R}$

Then f is lebesgue measurable function on E or simply measurable function, if for every finite a. The set $\{x \in E : f(x) > a\}$ is measurable.

$f : E \rightarrow \mathbb{R}$ if $S \subset \mathbb{R}$ then

$f^{-1}(S) = \{x \in E : f(x) \in S\}$

We know that if S is a set

$f^{-1}(S) = \{x : f(x) \in S\}$

Now

$\{x \in E : f(x) > a\}$

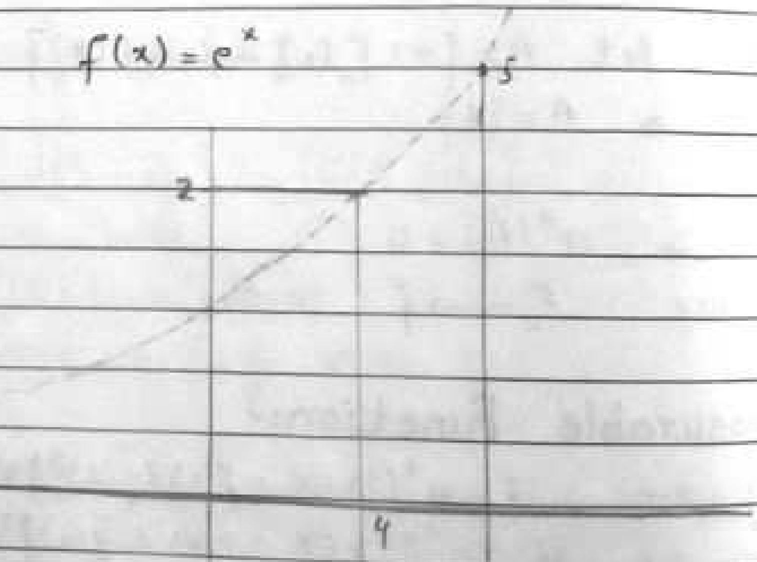
$\Rightarrow \{x \in E : f(x) \in (a, \infty)\}$

$\Rightarrow f^{-1}(a, \infty)$

if $x > 1$
$x \in (1, \infty)$
if $x > -2$
$x \in (-2, \infty)$

So a function f is measurable if $f^{-1}(a, \infty)$ is measurable set.

e.g let $f(x) = e^x$



$f^{-1}(1, \infty) = \{x : f(x) \in (1, \infty)\}$

$= \{x : e^x \in (1, \infty)\}$

$= (0, \infty)$ is measurable.

$f^{-1}(2, \infty) = (x, \infty)$

$f^{-1}(5,\infty) = (x_2,\infty)$ is measurable.

Lemma (i) :-

Show that the sets

$\{x; f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x; f(x) > a - \frac{1}{n}\}$

Proof:-

Let $x_0 \in A = \{x : f(x) \geq a\}$

$\Rightarrow f(x_0) > a > a - \frac{1}{n} \quad \forall n$

$\Rightarrow f(x_0) > a - \frac{1}{n} \quad \forall n$

$\Rightarrow x_0 \in \{x : f(x) > a - \frac{1}{n}\} \quad \forall n$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$

$\Rightarrow A \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\} \quad \forall n$ —— (i)

Conversely; let $x_0 \in \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$

$\Rightarrow x_0 \in \{x : f(x) > a - \frac{1}{n}\} \quad \forall n$

$\Rightarrow f(x_0) > a - \frac{1}{n} \quad \forall n$ —— (*)

We claim that $f(x_0) \geq a$. because

if $f(x_0) < a$, then $f(x_0) < a - \frac{1}{n}$

But $f(x_0) > a - \frac{1}{n}$ for some n.

which is contradiction for eq (*), So

$f(x_0) \geq a$

$\Rightarrow x_0 \in \{x; f(x) \geq a\}$

$\Rightarrow x_0 \in A$

So $\bigcap_{n=1}^{\infty}\{x : f(x) > a - \frac{1}{n}\} \subseteq A$ —— (ii) $\forall$ n

From eq (i) and (ii), we get

$$A = \bigcap_{n=1}^{\infty}\{x ; f(x) > a - \frac{1}{n}\} \quad \forall n$$

Since $A = \{x ; f(x) \geq a\}$

Hence
$$\{x ; f(x) \geq a\} = \bigcap_{n=1}^{\infty}\{x ; f(x) > a - \frac{1}{n}\}$$
proved!

Lemma (ii):-

Show that
$$\{x : f(x) > a\} = \bigcup_{n=1}^{\infty}\{x : f(x) \geq a + \frac{1}{n}\}.$$

Proof:-

let $A = \{x ; f(x) > a\}$
and $B = \bigcup_{n=1}^{\infty}\{x ; f(x) \geq a + \frac{1}{n}\}$

We need to show that $A = B$

let $y \in A$

$\Rightarrow y \in \{x : f(x) > a\}$

$\Rightarrow f(y) > a$

$\Rightarrow f(y) > f(y) - \frac{1}{n} \geq a$ for some "n".

$\Rightarrow f(y) \geq a + \frac{1}{n}$ for some n.

$\Rightarrow y \in \{x : f(x) \geq a + \frac{1}{n}\}$

$\Rightarrow y \in \bigcup_{n=1}^{\infty}\{x : f(x) \geq a + \frac{1}{n}\}$

$y \in B$

Thus $A \subseteq B$ ——— (i)

conversely ; let $y \in B$

$\Rightarrow y \in \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}$

$\Rightarrow f(y) \geq a + \frac{1}{n}$ for some n

$\Rightarrow f(y) \geq a + \frac{1}{n} > a$

$\Rightarrow f(y) > a$

$\Rightarrow y \in \{x : f(x) > a\}$

$\Rightarrow y \in A$

Thus $B \subseteq A$ ——— (ii)

From eq (i) and (ii), we get

$A = B$

Hence $\{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}$

proved!

Theorem:-

let $f : E \to \mathbb{R}$ be a function, where E is measurable ~~function~~ subset of $\mathbb{R}^n$. Then the following statements are equivalent.

f is measurable function.

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ii) $f^{-1}(-\infty, a]$ is measurable set for any $a \in \mathbb{R}$.

iii) $f^{-1}([a, \infty))$ is measurable set for any $a \in \mathbb{R}$.

iv) $f^{-1}(-\infty, a)$ is measurable set for any $a \in \mathbb{R}$.

Proof:-

Let (i) is true.

Since f is measurable function

So $\{x : f(x) > a\}$ is measurable set.

$\Rightarrow \{x : f(x) > a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) \leq a\}$ is measurable set.

$\Rightarrow \{x : f(x) \in (-\infty, a]\}$ is measurable set.

$\Rightarrow f^{-1}(-\infty, a]$ is measurable set.

Now let (ii) is true

Since $f^{-1}(-\infty, a]$ is measurable set

$\Rightarrow \{x : f(x) \leq a\}$ is measurable set

$\Rightarrow \{x : f(x) \leq a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) > a\}$ is measurable set.

$\Rightarrow \{x : f(x) > a - \frac{1}{n}\}$ is measurable set for any $n \in \mathbb{N}$.

$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$ is measurable set

$\Rightarrow \{x : f(x) \geq a\}$ is measurable set

$\Rightarrow \{x : f(x) \in [a, \infty)\}$ is measurable set

$\Rightarrow f^{-1}[a, \infty)$ is measurable set.

Now let (iii) is true.
Since $f^{-1}[a, \infty)$ is measurable set.

$\Rightarrow \{x : f(x) \geq a\}$ is measurable set.

$\Rightarrow \{x : f(x) \geq a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) < a\}$ is measurable set.

$\Rightarrow \{x : f(x) \in (-\infty, a)\}$ is measurable set.

$\Rightarrow f^{-1}(-\infty, a)$ is measurable set.

Now let (iv) is true.
Since $f^{-1}(-\infty, a)$ is measurable set.

$\Rightarrow \{x : f(x) < a\}$ is measurable set.

$\Rightarrow \{x : f(x) < a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) \geq a\}$ is measurable set.

$\Rightarrow \{x : f(x) \geq a + \frac{1}{n}\}$ is measurable set for any $n \in \mathbb{N}$.

$\Rightarrow \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}$ is measurable set.

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$\Rightarrow \{x : f(x) > a\}$ is measurable set. (by lemma)

Thus f is measurable function.

Imp Theorem:-

A function $f : \mathbb{R}^n \to \mathbb{R}$ is measurable iff for every open set G in $\mathbb{R}$ the inverse image $f^{-1}(G)$ is a measurable subset of $\mathbb{R}^n$.

Proof:-

Let $f^{-1}(G)$ is measurable for every open set G in R.

We will prove that f is measurable i.e. we prove that the set $\{x \in \mathbb{R}^n : f(x) > a\}$ is measurable for every finite a.

Since (a, ∞) is open and

$f^{-1}(a, \infty) = \{x \in \mathbb{R}^n : f(x) \in (a, \infty)\}$

i.e. $f^{-1}(a, \infty) = \{x \in \mathbb{R}^n : a < f(x) < \infty\}$ measurable

$\Rightarrow f$ is measurable function.

Conversely, let G be any open set in $\mathbb{R}$.

We prove that $f^{-1}(G)$ is measurable for every open set.

Since G can be written as $\bigcup_k (a_k, b_k)$.

But $f^{-1}(a_k, b_k) = \{x \in \mathbb{R}^n : a_k < f(x) < b_k\}$ is measurable.

$f^{-1}(G) = f^{-1}\left[\bigcup_k (a_k, b_k)\right]$

$f^{-1}(G) = \bigcup_k f^{-1}(a_k, b_k)$ it follows that $f^{-1}(G)$ is measurable.

proved!

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Theorem:-

A function $f:R^n \to R$ is measurable iff $f^{-1}(C)$ is measurable for every closed subset C of R.

Proof:-

Let $f:R^n \longrightarrow R$ is measurable.
we need to prove that for any closed set "C" $f^{-1}(C)$ is measurable.

Let us consider "C" is any arbitrary closed set.

$\Rightarrow$ C^c is open.

Since we know that "$f: \mathbb{R}^n \to \mathbb{R}$ is measurable iff $f^{-1}(G)$ is measurable for any open set G."

So $f^{-1}(C^c)$ is measurable.

$\Rightarrow$ $[f^{-1}(C^c)]^c$ is measurable.
Since $f^{-1}(C) = [f^{-1}(C^c)]^c$

Thus $f^{-1}(C)$ is measurable.

Conversely; Let $f^{-1}(C)$ is measurable for closed C.

Let G be any open set. $\Rightarrow G^c$ is closed
Then $f^{-1}(G^c)$ is measurable.
$\Rightarrow$ $[f^{-1}(G^c)]^c$ is measurable

$\Rightarrow$ $f^{-1}(G)$ is measurable

$\Rightarrow$ f is measurable

Theorem:-

If f is measurable and $g: \mathbb{R}^n \to \mathbb{R}$ such $f = g$ a.e then g is measurable.

Proof:-

If $A = \{x \in \mathbb{R}^n : f(x) \neq g(x)\}$

then $\mu^*(A) = 0$

$\Rightarrow$ A is measurable

Let G be an open set in $\mathbb{R}$

$\Rightarrow f^{-1}(G)$ is measurable as f is measurable

and hence $A^c \cap g^{-1}(G) = A^c \cap f^{-1}(G)$ is measurable set.

Also since $A \cap g^{-1}(G) \subset A$

$\Rightarrow \mu^*(A \cap g^{-1}(G)) = 0$

So $g^{-1}(G) = [A \cap g^{-1}(G)] \cup [A^c \cap g^{-1}(G)]$

Since we know that "the union of two measurable sets are measurable"

So $g^{-1}(G)$ is measurable.

Thus g is measurable

Theorem:-

let f, g are measurable functions on set E, then prove that the sets $\{x \in E ; f(x) > g(x)\}$, $\{x \in E ; f(x) \leq g(x)\}$ and $\{x \in E : f(x) = g(x)\}$ are measurable.

Proof:-

Given that f, g are measurable functions on set E.

We are to prove that

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(i) $\{x \in E;\ f(x) > g(x)\}$ is also measurable on E.
Since for each $x \in E$ we can find a rational number (say) "r_1", such that $f(x) \geq r_1 \geq g(x)$ i.e.

Let $r_1, r_2, r_3, \ldots$ be the enumeration of such rational numbers, then

$$\{x;\ f(x) > g(x)\} = \bigcup_{i=1}^{\infty} \{x \in E;\ f(x) \geq r_i \geq g(x)\}$$

$$= \bigcup_{i=1}^{\infty} \left(\{x \in E;\ f(x) \geq r_i\} \cap \{x \in E;\ g(x) \leq r_i\}\right)$$

$$= \bigcup_{i=1}^{\infty} \left(\{x;\ f(x) \in [r_i, \infty)\} \cap \{x;\ g(x) \in (-\infty, r_i]\}\right)$$

$$= \bigcup_{i=1}^{\infty} \left(f^{-1}[r_i, \infty) \cap g^{-1}(-\infty, r_i]\right)$$

= measurable set (because $f^{-1}[r_i, \infty)$ and $g^{-1}(-\infty, r_i]$ are measurable.)

Hence $\{x \in E:\ f(x) > g(x)\}$ is measurable set.

(ii) We have to show that $\{x \in E;\ f(x) \leq g(x)\}$ is measurable set.
For this since $\{x \in E;\ f(x) > g(x)\}$ is measurable.

$\Rightarrow \{x;\ f(x) > g(x)\}^c$ is measurable.

$\Rightarrow \{x \in E;\ f(x) \leq g(x)\}$ is measurable set

Next we have to show that
$\{x \in E:\ f(x) = g(x)\}$ is measurable.

For this since $A = \{x;\ f(x) \geq g(x)\}$ and
$B = \{x;\ f(x) \leq g(x)\}$ are measurable sets.

$\Rightarrow A \cap B = \{x ; f(x) \geq g(x)\} \cap \{x : f(x) \leq g(x)\}$

$\Rightarrow A \cap B = \{x ; f(x) = g(x)\}$

$\Rightarrow A \cap B$ is measurable

$\Rightarrow \{x ; f(x) = g(x)\}$ is measurable.

Theorem:-

Let f, g be two measurable functions on set E, then show that

(i) $c - f$ (ii) $f + g$ (iii) f^2 (iv) $f \cdot g$

(v) $|f|$ (vi) $f \vee g$ (vii) $f \wedge g$

(viii) f^+ (ix) f^- (x) cf

(xi) f/g ; $g \neq 0$ all are measurable.

where c is any constant.

Proof:-

(i) To show that "$c - f$" is measurable.

Let us consider

$\{x \in E ; c - f(x) \geq a\} = \{x \in E ; f(x) \leq c - a\}$; $a \in \mathbb{R}$

$\{x \in E ; c - f(x) \geq a\} =$ measurable (because "f" is measurable)

$\Rightarrow$ "$c - f$" is measurable.

(ii) To show that "$f + g$" is measurable

Let us consider

$\{x : f + g \geq a\} = \{x ; f(x) \geq a - g(x)\}$; $a \in \mathbb{R}$

Since f is measurable and g is measurable.

$\Rightarrow a - g(x)$ is measurable (by proof (i))

$\Rightarrow$ "f" and "$a-g(x)$" are measurable functions

$\Rightarrow \{x \in E : f(x) \geq a - g(x)\}$ is measurable set.

$\Rightarrow \{x \in E : f(x) + g(x) \geq a\}$ is measurable.

$\Rightarrow$ "$f+g$" is measurable.

→ We will show that "f^2" is measurable.
Let us consider the set
$\{x : f^2(x) \leq a\}$ where $a \in \mathbb{R}$
Now if $a < 0 \Rightarrow \{x \in E ; f^2(x) \leq a\} = \phi =$ measurable

$\Rightarrow$ for $a < 0$, "f^2" is measurable.

if $a \geq 0$, then $\{x ; f^2 < a\} = \{x ; -\sqrt{a} \leq f \leq \sqrt{a}\}$

$$= f^{-1}[-\sqrt{a}, \sqrt{a}]$$

$$= \text{measurable.}$$

So $\forall$ $a \in \mathbb{R}$, the set $\{x \in E, f^2 < a\}$ is measurable.

$\Rightarrow f^2$ is measurable.

→ We have to show that "$f \cdot g$" is measurable.
For this since

$$(f+g)^2 = f^2 + g^2 + 2f.g$$

$$2f.g = (f+g)^2 - f^2 - g^2$$

$$f.g = \frac{1}{2}\left[(f+g)^2 - (f^2 + g^2)\right]$$

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Now since "$f+g$" is measurable

$\Rightarrow (f+g)^2$ is measurable.

Also f and g are measurable.

$\Rightarrow f^2$ and g^2 are measurable.

$\Rightarrow f^2+g^2$ is measurable.

So $(f+g)^2$ and f^2+g^2 are measurable.

$\Rightarrow (f+g)^2-(f^2+g^2)$ is measurable.

Thus

$\Rightarrow f.g$ is measurable.

v) We will show that $|f|$ is measurable.

For this let us consider the set $\{x\in E ; |f(x)| < a\}$ where $a\in\mathbb{R}$.

if $a<0 \Rightarrow \{x : |f(x)| < a\} = \Phi$ = measurable

if $a\geq 0 \Rightarrow \{x\in E : |f(x)| < a\} = \{x : -a < f(x) < a\}$

$= f^{-1}(-a, a)$

$=$ measurable

$\Rightarrow |f|$ is measurable.

vi) We will show that $f\vee g = \max\{f, g\}$ is measurable.

For this since

$\max\{f,g\} \geq f$ —(i) , $\max\{f,g\}\geq g$ —(ii)

Adding (i) and (ii), we get

$2\max\{f,g\} \geq f+g$

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$$\max\{f,g\} \geq \frac{1}{2}\{f+g\}$$

Since f and g are measurable
$\Rightarrow$ "$f+g$" is measurable

So $\max\{f,g\}$ is measurable
Thus
"$f \vee g$" is measurable.

We need to show that $f \wedge g = \min\{f,g\}$ is measurable.
Since
$\min\{f,g\} \leq f$ — (i) and $\min\{f,g\} \leq g$ — (ii)
Adding eq (i) and (ii), we get

$$2\min\{f,g\} \leq f+g$$

$$\min\{f,g\} \leq \frac{1}{2}\{f+g\}$$

Since f and g are measurable.
$\Rightarrow$ $f+g$ is measurable.

$\Rightarrow$ $\min\{f,g\}$ is measurable.

So $f \wedge g$ is measurable.

We have to show that f^+ is measurable.
For this since
$$f^+ = \max\{f,0\}$$
where "f" and "$g=0$" both are measurable
$\Rightarrow$ $\max\{f,0\}$ is also measurable.

$\Rightarrow f^+$ is measurable.

(ix) we have to show that f^- is measurable.

For this since

$f^- = \min\{f, 0\}$

where "f" and "$g=0$" are measurable.

$\Rightarrow \min\{f, 0\}$ is measurable.

$\Rightarrow f^-$ is measurable.

x, we have to show that "cf" is measurable, where "c" is any constant.

For this if $c=0 \Rightarrow cf = 0.f = 0$

$\Rightarrow cf = 0$ = constant function

= measurable

if $c>0$, then consider the set

$\{x; cf(x) \geq a\} = \{x; f(x) \geq a/c\}$

= measurable.

$\Rightarrow cf$ is measurable.

if $c<0$, then

$\{x; cf(x) \geq a\} = \{x; f(x) \leq a/c\}$

= measurable.

$\Rightarrow cf$ is measurable.

So in each case "cf" is measurable.

We have to show that "f/g" is measurable.

If $g(x) \neq 0$, then $\frac{1}{g(x)}$ is measurable.

let $A = \{x : \frac{1}{g(x)} > a\}$

Case-I:- If $a = 0$ then

$$A = \{x ; \frac{1}{g(x)} > 0\}$$

{ if $g(x) > 0$ then $1/g(x) > 0$

$$A = \{x : g(x) > 0\}$$

Since $g(x)$ is measurable.

$\Rightarrow$ A is measurable.

Case-II. If $a < 0$, then

$$A = \{x ; \frac{1}{g(x)} > a\}$$

$$A = \{x : g(x) < \frac{1}{a}\}$$

$$A = \{x : g(x) > 0\} \cup \{x : g(x) > a\}$$

Since g is measurable

$\Rightarrow$ A is measurable

Case-III:- If $a > 0$, then

$$A = \{x : 1/g(x) > a\}$$

$$A = \{x : g(x) < \frac{1}{a}\}$$

$$A = \{x : g(x) < 1/a\} \setminus \{x : g(x) < 0\}$$

Since g is measurable.

$\Rightarrow$ A is measurable.

So $\frac{1}{g(x)}$ is measurable.

and $f(x)$ is measurable.

Thus f/g is measurable.

Theorem:- (a)

let $\{f_n(x)\}$ be a sequence of functions and let $F(x) = \operatorname{Sup}\{f_n(x)\}$, then $\{x : F(x) > k\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

Proof:-

Let $A = \{x : F(x) > k\}$

and $B = \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

we need to show that $A = B$.

For this let $x_0 \in A$

$\Rightarrow F(x_0) > k$ —— (i)

we show that $x_0 \in B$

i.e. we show that $f_n(x_0) > k$ for some n

Assume on contrary that

$f_n(x_0) \leq k \quad \forall\, n.$

$\Rightarrow f_n(x_0) \leq k \quad \forall\, n$

$\Rightarrow F(x_0) \leq k$

which is contradiction to eq (i)

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So $f_n(x_0) > k$ for some n.

$\Rightarrow x_0 \in \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

$\Rightarrow x_0 \in B$

Thus $A \subseteq B$ —— ⓐ

Again let $x_0 \in B$

$\Rightarrow x_0 \in \bigcup_{n=1}^{\infty} \{x : f_n(x_0) > k\}$

$\Rightarrow f_n(x_0) > k \quad \forall\, n$

Since $F(x) = \operatorname{Sup}\{f_n(x)\}$

$\Rightarrow k < f_n(x_0) \leq F(x_0)$

$\Rightarrow k < F(x_0)$

$\Rightarrow F(x_0) > k$

$\Rightarrow x_0 \in \{x : F(x) > k\}$

$\Rightarrow x_0 \in A$

Thus $B \subseteq A$ —— ⓑ

From eq ⓐ and ⓑ, we get

$A = B$

Hence

$\{x : F(x) > k\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

Proved.

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Theorem:-(b)

let $f_n(x)$ be a sequence of functions and let $G(x) = \inf\{f_n(x)\}$, then $\{x : G(x) > k\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$.

Proof:-

Let $A = \{x : G(x) > k\}$

and $B = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

We need to show that $A = B$.

i.e. $\{x : G(x) > k\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

let $x_0 \in A \Rightarrow G(x_0) > k$

$\Rightarrow k < G(x_0) \leq f_n(x_0) \quad \forall n$ (by def of inf)

$\Rightarrow k < f_n(x_0) \quad \forall n$

$\Rightarrow f_n(x_0) > k \quad \forall n$

$\Rightarrow x_0 \in \{x : f_n(x) > k\}$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

$\Rightarrow x_0 \in B$

Thus $A \subseteq B$ ——— (i)

Again let $x_0 \in B$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

$\Rightarrow f_n(x_0) > k \quad \forall n$

$\Rightarrow \inf\{f_n(x_0)\} > \inf(k) \quad \forall n$

$\Rightarrow G(x_0) > k$

$\Rightarrow x_0 \in \{x : G(x) > k\}$

$\Rightarrow x_0 \in A$

Thus $B \subseteq A$ ——— (ii)

From eq (i) and (ii), we get
$A = B$

Hence $\{x : G(x) > k\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$
proved!

Theorem:-
If $\{f_n(x)\}$ is a sequence of measurable function, then prove that $F(x) = \sup\{f_n(x)\}$ and $G(x) = \inf\{f_n(x)\}$ are measurable functions.

Proof:-
Theorem (a), Theorem (b).

Theorem:-
Let $\{f_n(x)\}$ be a monotonic sequence of measurable functions, such that $\lim_{n\to\infty} f_n(x) = f(x)$, then show that $f(x)$ is measurable.

Proof:-
Since we know that
"Supremum of measurable function is measurable" ——— ①
"Infimum of measurable function is measurable" ——— ②
There are two case
Case-I:- when $\{f_n(x)\}$ is monotonic increasing

sequence i.e $f_1(x) \le f_2(x) \le f_3(x) \le \dots$ then we know that a monotonic increasing sequence always converges to its lub (supremum).

$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) =$ supremum of $f_n(x)$

So using eq ① $f(x)$ is measurable.

Case-II:- When $\{f_n(x)\}$ is monotonic decreasing sequence i.e.

$f_1(x) \ge f_2(x) \ge f_3(x) \ge \dots$

then we know that a monotonic decreasing sequence always converges to its glb (infimum).

$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) =$ glb of $f_n(x)$.

So using eq ② $f(x)$ is measurable.

Hence in both cases $f(x)$ is measurable.

Definition:-

Limit Superior of a sequence:-

A number $\bar{l}$ is called the limit superior, greatest limit or upper limit of a sequence $\langle a_n \rangle$ iff infinitly many terms of the sequence are greater than $\bar{l} - \epsilon$, while only a finitly terms are greater than $\bar{l} + \epsilon$ for any

$\epsilon > 0$. We denote the limit superior by $\lim \sup (a_n)$ or $\overline{\lim}\, a_n$.

Limit Inferior of a sequence:-

A number l is called limit inferior, least limit or lower limit of a sequence $\langle a_n \rangle$ iff infinitly many terms of the sequence are less than $l+\epsilon$, while only a finite number of terms are less than the $l-\epsilon$, for any $\epsilon > 0$. We denote the limit inferior of $\langle a_n \rangle$ by $\lim \inf (a_n)$ or $\underline{\lim}\, a_n$.

Note:- If infinitly many terms of $\langle a_n \rangle$ exceeds any positive number M. we write $\overline{\lim}\, a_n = \infty$ and if infinitly many terms are less than $-M$, where M is any positive number. we write $\underline{\lim}\, a_n = -\infty$.

Another Definition:-

Limit Superior:-

If $\{a_n\}^{\infty}$ is a sequence of real numbers, we define a sequence $\{S_n\}_{n=1}^{\infty}$ as;

$$S_n = \sup_{k \geq n} \{a_k\}$$

$$\Rightarrow S_1 = \sup \{a_1, a_2, a_3, \dots\}$$
$$S_2 = \sup \{a_2, a_3, a_4, \dots\}$$
$$S_3 = \sup \{a_3, a_4, a_5, \dots\}$$
$$\vdots \qquad \vdots \qquad \vdots$$

Here $S_1 \geq S_2 \geq S_3 \geq \dots$

So $\langle S_n \rangle$ is decreasing sequence, which is clear from the construction of $\langle S_n \rangle$. Then the $\lim\limits_{n \to \infty} S_n$ is called the limit superior of $\langle a_n \rangle$ and it is denoted by $\overline{\lim}\, a_n$ or $\lim \sup (a_n)$.

"Now note that if we are interested to find the limit supremum of $\langle a_n \rangle$ i.e. $\overline{\lim}\, a_n = ?$ then $\overline{\lim}\, a_n = \lim\limits_{n\to\infty} S_n$ — ①

but since $\langle S_n \rangle$ is a monotonic decreasing sequence and we know that a monotonic decreasing sequence always tends to it glb.

So $\lim\limits_{n \to \infty} S_n = \text{glb}\{S_n\}$

So eq ① $\Rightarrow \overline{\lim}\, a_n = \lim\limits_{n\to\infty} S_n = \text{glb}\{S_n\}$

$$\Rightarrow \boxed{\overline{\lim}\, a_n = \text{glb}\{S_n\}}$$

Limit Inferior:-

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, we define a sequence $\{T_n\}_{n=1}^{\infty}$ as;

$$T_n = \inf_{k \geq n}\{a_k\}$$

$$\Rightarrow \quad T_1 = \inf\{a_1, a_2, a_3, \dots\}$$
$$T_2 = \inf\{a_2, a_3, a_4, \dots\}$$
$$T_3 = \inf\{a_3, a_4, a_5, \dots\}$$
$$\vdots \qquad \vdots \qquad \vdots$$

Clearly from the construction of

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$\langle T_n \rangle$, we see that

$T_1 \leq T_2 \leq T_3 \leq \dots$

$\Rightarrow \langle T_n \rangle$ is an increasing sequence and the "$\lim\limits_{n\to\infty} T_n$" is called the limit inferior of the original sequence $\langle a_n \rangle$ and it is denoted by $\underline{\lim}\, a_n$ or $\lim\inf (a_n)$.

"And note that $\underline{\lim}\, a_n = \lim\limits_{n\to\infty} T_n$ — ②

Now since $\langle T_n \rangle$ is increasing sequence.

So $\lim\limits_{n\to\infty} T_n = \text{lub}\{T_n\}$

eq ② $\Rightarrow$ $\underline{\lim}\, a_n = \lim\limits_{n\to\infty} T_n = \text{lub}\{T_n\}$

$$\Rightarrow \boxed{\underline{\lim}\, a_n = \text{lub}\{T_n\}}$$

Example:- $\langle a_n \rangle = \langle \frac{1}{n} \rangle$; find (i) $\overline{\lim}\, a_n = ?$ and (ii) $\underline{\lim}\, a_n = ?$

Solution:-

i) To find $\overline{\lim}\, a_n = ?$

Let $S_1 = \sup\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = 1$

$S_2 = \sup\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = \frac{1}{2}$

$\vdots$

$S_n = \sup\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots\} = \frac{1}{n}$

So $\overline{\lim}\, a_n = \lim\limits_{n\to\infty} \langle S_n \rangle = \lim\limits_{n\to\infty} \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle$

$= \lim\limits_{n\to\infty} \langle \frac{1}{n} \rangle = 0$

$$\Rightarrow \boxed{\overline{\lim}\, a_n = 0} \quad \text{———} \quad ①$$

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ii) To find $\underline{\lim}\, a_n = \lim\inf(a_n) = ?$

let $T_1 = \inf\{1, 1/2, 1/3, \dots\} = 0$

$T_2 = \inf\{1/2, 1/3, 1/4, \dots\} = 0$

$T_3 = \inf\{1/3, 1/4, 1/5, \dots\} = 0$

$\vdots$

$T_n = \inf\{1/n, \frac{1}{n+1}, \frac{1}{n+2}, \dots\} = 0$

So $\underline{\lim}\, a_n = \lim_{n\to\infty} T_n = \lim_{n\to\infty}\{0\} = 0$

$\Rightarrow \boxed{\underline{\lim}\, a_n = 0}$ —— ②

From eq ① and ②, we get

$\overline{\lim}\, a_n = \underline{\lim}\, a_n = 0$

Thus $\lim_{n\to\infty} a_n = 0$ (exist)

Example:-

let $\langle a_n\rangle = 1-(-1)^n$

$= \langle 2, 0, 2, 0, 2, \dots\rangle$

Find $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

Solution:-

i) To find $\overline{\lim}\, a_n$, let

$S_1 = \sup\{2, 0, 2, 0, 2, \dots\} = 2$

$S_2 = \sup\{0, 2, 0, 2, 0, \dots\} = 2$

$S_3 = \sup\{2, 0, 2, 0, 2, \dots\} = 2$

$\vdots$

$S_n = \sup\{1-(-1)^n\} = 2$

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$\Rightarrow \overline{\lim}\, a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \{2\} = 2$

$\Rightarrow \boxed{\overline{\lim}\, a_n = 2}$ —— (1)

Similarly to find $\underline{\lim}\, a_n$ = .

let $T_1 = \inf\{2,0,2,0,\cdots\} = 0$

$T_2 = \inf\{0,2,0,2,0,\cdots\} = 0$

$T_3 = \inf\{2,0,2,0,\cdots\} = 0$

$\vdots$

$T_n = \inf\{1-(-1)^n\} = 0$

So $\underline{\lim}\, a_n = \lim_{n\to\infty} T_n = \lim_{n\to\infty} (0) = 0$

$\boxed{\underline{\lim}\, a_n = 0}$ —— (2)

From eq (1) and (2), we get

$\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$

$\Rightarrow \lim_{n\to\infty} \langle a_n \rangle$ does not exist.

Example:-

let $\langle a_n \rangle = \{2,-2,-1,1,-1,1,-1,\cdots\}$

Find $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

Solution:-

To find $\overline{\lim}\, a_n$.

let $S_1 = \sup\{2,-2,-1,1,-1,1,-1,\cdots\} = 2$

$S_2 = \sup\{-2,-1,1,-1,1,-1,\cdots\} = 1$

$S_3 = \sup\{-1,1,-1,1,-1,\cdots\} = 1$

$\overline{\lim}\, a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \{2,-2,-1,1,-1,1,-1,\cdots\}$

$$\boxed{\overline{\lim}\ a_n = 1} \quad \text{---} \ (1)$$

Also let $T_1 = \inf\{2,-2,-1,1,-1,1,-1,\ldots\} = -2$
$T_2 = \inf\{-2,-1,1,-1,1,-1,\ldots\} = -2$
$T_3 = \inf\{-1,1,-1,1,-1,\ldots\} = -1$
$T_4 = \inf\{1,-1,1,-1,\ldots\} = -1$
$T_5 = \inf\{-1,1,-1,1,-1,\ldots\} = -1$

$$\Rightarrow \underline{\lim}\ a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\{2,-2,-1,1,-1,1,-1,\ldots\}$$
$$= -1$$

$$\Rightarrow \boxed{\underline{\lim}\ a_n = -1} \quad \text{---} \ (2)$$

From eq. (1) and (2), we get
$\overline{\lim}\ a_n \neq \underline{\lim}\ a_n$

So
$\Rightarrow \lim\limits_{n\to\infty} a_n$ does not exist.

Theorem:-

let $\{f_n(x)\}$ be a sequence of measurable functions then
$\lim \sup f_n = \overline{\lim}\ f_n$ and $\lim \inf f_n = \underline{\lim}\ f_n$
are measurable functions.

Proof:-

We need to show that $\overline{\lim}\ a_n$ is measurable function.

We proceed as:

Let $S_n(x) = \sup\limits_{k\geq n}\{f_k(x)\}$

i.e $S_1(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$
$S_2(x) = \sup\{f_2(x), f_3(x), f_4(x), \ldots\}$
$S_3(x) = \sup\{f_3(x), f_4(x), f_5(x), \ldots\}$
$\vdots$

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$\Rightarrow \quad S_1(x) \geq S_2(x) \geq S_3(x) \geq \cdots$

Since we know that $S_n(x)$ is a decreasing sequence, so it always converges to its glb.

$\Rightarrow \lim_{n\to\infty} S_n(x) = glb\{S_n(x)\}$ —— (i)

But by definition of limit superior i.e. $\overline{\lim} f_n$ we know

$\overline{\lim} f_n = \lim_{n\to\infty}\{S_n(x)\}$ —— (ii)

From eq (i) and (ii), we get

$\overline{\lim} f_n = \lim_{n\to\infty} S_n(x) = glb\{S_n(x)\}$ —— $\circledast$

Since we see that "$S_n(x)$" is a sequence of measurable functions.

$\Rightarrow glb\{S_n(x)\}$ is also measurable.

So eq $\circledast \Rightarrow \overline{\lim} f_n$ is measurable.

Next, we show that $\underline{\lim} f_n$ is measurable

let $T_1 = \inf\{f_1(x), f_2(x), f_3(x), \cdots\}$

$T_2 = \inf\{f_2(x), f_3(x), f_4(x), \cdots\}$

$T_3 = \inf\{f_3(x), f_4(x), f_5(x), \cdots\}$

$\vdots \quad \vdots \quad \vdots$

Since $T_1 \leq T_2 \leq T_3 \leq \cdots$

i.e. $\langle T_n \rangle$ is an increasing sequence.

So it always converges to its lub.

So $\lim_{n\to\infty} T_n(x) = lub\{T_n(x)\}$ —— (iii)

But from definition of $\underline{\lim}\, f_n$, we know that

$$\underline{\lim}\, f_n = \lim_{n\to\infty} T_n(x) \text{ --- (iv)}$$

Combining eq (iii) and (iv), we get

$$\underline{\lim}\, f_n = \lim_{n\to\infty} T_n(x) = lub\{T_n(x)\} \text{ --- } (**)$$

Since we see that $\{T_n(x)\}$ is a sequence of measurable functions
$\Rightarrow$ $lub\{T_n(x)\}$ is also measurable.

So eq $(**) \Rightarrow \underline{\lim}\, f_n$ is measurable.

Hence
$\overline{\lim}\, f_n$ & $\underline{\lim}\, f_n$ are measurable functions.

Theorem:-
Let $\{f_n(x)\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$, then show that $f(x)$ is measurable.

Proof:-
Since it is given that
$\lim_{n\to\infty} f_n(x) = f(x)$ (exists)

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) = \overline{\lim}\, f_n = \underline{\lim}\, f_n \text{ --- } (*)$$

Since we know that for the sequence $\{f_n(x)\}$ $\overline{\lim}\, f_n(x)$ and $\underline{\lim}\, f_n(x)$ are always measurable.

So eq ⊛ $\Rightarrow$ $-f(x)$ is measurable

Characteristic Function:-

Let $A \subset X$, then characteristic function on A (or w.r.t A) is defined as:

$$\chi_A : X \longrightarrow \{0,1\} \text{ by}$$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Theorem:-

A set A is measurable iff χ_A is measurable.

Proof:-

Let A is measurable.

We need to show that χ_A is measurable.

For this let us take any arbitrary open set "G" in $\mathbb{R}$, we will show that $\chi_A^{-1}(G)$ is measurable.

i) if $1 \in G$, $0 \notin G$

$\Rightarrow \chi_A^{-1}(G) = A$ = measurable.

ii) if $1 \notin G$, $0 \in G$

$\Rightarrow \chi_A^{-1}(G) = A^c$ = measurable (because the complement of measurable set is measurable)

iii) if $1, 0 \notin G$

$\Rightarrow \chi_A^{-1}(G) = \phi$ = measurable iv, if $1, 0 \in G$

$$\Rightarrow \chi_A^{-1}(G) = X = \text{measurable}.$$

So in each case the inverse image of an arbitrary open set G ; i-e $\chi_A^{-1}(G)$ is measurable.

$\Rightarrow \chi_A$ is measurable.

Conversely;

let χ_A is measurable.

we need to show that the set A is measurable.

For this let us take any arbitrary open set G in $\mathbb{R}$,

Since χ_A is measurable.

$\Rightarrow \chi_A^{-1}(G)$ is also measurable.

Since $\chi_A^{-1}(G)$ is measurable $\forall$ open set G in $\mathbb{R}$.

So let us take G such an open set, which contains "1" but not "0", then

$$\chi_A^{-1}(G) = \text{measurable}.$$

but $\chi_A^{-1}(G) = A$

$\Rightarrow A$ is measurable.

which is the required result.

Question:-

Give an example of a function f, such that $|f|$ is measurable, but f is not measurable.

Proof:-

Let E be a non-measurable subset of $\mathbb{R}$, consider the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{if } x \notin E \end{cases}$$

Here we can find an open set G in $\mathbb{R}$ such that $1 \in G$ and $-1 \notin G$.

$\Rightarrow f^{-1}(G) = E$ = non-measurable.

$\Rightarrow f^{-1}(G)$ is non-measurable

So we have find an open set G in $\mathbb{R}$ such that $f^{-1}(G)$ is not measurable

$\Rightarrow f$ is not measurable

But consider $|f(x)| = g(x) = 1 \quad \forall \; x \in \mathbb{R}$ which is constant and it will be always measurable.

i.e. $|f|$ is measurable.

Theorem:-

Show that every continuous function is measurable.

Proof:-

Since f is continuous.
Let us take an arbitrary open set G in $\mathbb{R}$.
So by definition of continuity $f^{-1}(G)$ will be open in the domain of f, but we know that every open set is measurable.

$\Rightarrow f^{-1}(G)$ is measurable.

but G was an arbitrary open set.
So for any open set G, $f^{-1}(G)$ is measurable.

$\Rightarrow f$ is measurable.

Theorem:-
Let f be differentiable function then prove that f' is measurable.
Proof:-
Let us define a sequence $\{g_n(x)\}$ of function as;

$$g_n(x) = \frac{f(x+1/n) - f(x)}{1/n}$$

Clearly $\{g_n(x)\}$ is a sequence of measurable function as f is differentiable

$\Rightarrow f$ is continuous.
Since we know that "every continuous function is measurable".

So f is measurable.

Now $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \dfrac{f(x+1/n) - f(x)}{1/n}$

$= f'(x)$

$\Rightarrow f'(x)$ is the limit of sequence of measurable function.

i.e. $f'(x)$ is measurable.

proved!

Theorem:-

If $f: \mathbb{R} \to \mathbb{R}$ is continuous a.e, then show that f is lebesgue measurable function.

Proof:-

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous a.e. We need to show that f is measurable.

Let $E = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$

then

$\mu(E^c) = 0$

$\Rightarrow E$ and E^c are measurable sets.

Let "O" be any open set in $\mathbb{R}$, then $f^{-1}(O) \cap E^c$ is measurable because it is null set

Since $f_{\setminus E} : E \to \mathbb{R}$ is continuous function, so $f^{-1}_{\setminus E}(O)$ is an open set in E.

But

$f^{-1}_{\setminus E}(O) = f^{-1}(O) \cap E$

So $f^{-1}(O) \cap E$ is open in E.
So there exists an open set V of $\mathbb{R}$ such that
$f^{-1}(O) \cap E = V \cap E$
But V and E are measurable, so $V \cap E$ is measurable.

$\Rightarrow f^{-1}(O) \cap E$ is measurable

Hence $f^{-1}(O) \cap E$ and $f^{-1}(O) \cap E^c$ are measurable sets.

Therefore; $f^{-1}(O) = (f^{-1}(O) \cap E) \cup (f^{-1}(O) \cap E^c)$ is measurable.

$\Rightarrow f$ is measurable function.

Question:-

Let (X, S, μ) be a measure space and $f: X \to \mathbb{R}$ is measurable function and $g: \mathbb{R} \to \mathbb{R}$ is continuous function, then show that $g \circ f$ is a measurable function.

Solution:-

We show that $g \circ f : X \to \mathbb{R}$ is measurable function.

Let "O" be any open set in $\mathbb{R}$.
We need to show that $(g \circ f)^{-1}(O)$ is measurable set.

Since "O" is open set in $\mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ is continuous.

So $g^{-1}(O)$ is open set in $\mathbb{R}$.

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But $f: X \to \mathbb{R}$ is measurable.
So $f^{-1}(g^{-1}(0))$ is measurable set.

i.e. $(g \circ f)^{-1}(0)$ is measurable set.
Hence
$\Rightarrow$ $g \circ f$ is measurable function.

Question:-
Let $f: [1, 10] \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 & \text{if } x \in [1, 5] \\ 3 & \text{if } x \in (5, 8) \\ -5 & \text{if } x \in [8, 10] \end{cases}$$

clearly $f(x)$ is not continuous.

Show that $f(x)$ is measurable function.

Solution:-
Let G be any open set in $\mathbb{R}$.
We need to show that $f^{-1}(G)$ is an open set.
We discuss the following cases:

Case-I:-
If $2, 3, -5 \notin G$, then
$f^{-1}(G) = \phi$ = measurable set.

Case-2:-
If $2 \in G$ and $3, -5 \notin G$, then
$f^{-1}(G) = [1, 5]$, which is measurable.

Case-3:-
If $2, 3 \in G$ and $-5 \notin G$, then
$f^{-1}(G) = [1, 5] \cup (5, 8) = [1, 8)$, which is measurable.

Case-4:-

If $2, -5 \in G$ and $3 \notin G$, then

$f^{-1}(G) = [1,5] \cup [8,10]$ = measurable

Case-5:-

If $2, 3 \notin G$ and $-5 \in G$, then

$f^{-1}(G) = [8,10]$ = measurable

Case-6:-

If $2, -5 \notin G$ and $3 \in G$, then

$f^{-1}(G) = (5,8)$ = measurable

Case-7:-

If $2, 3, -5 \in G$, then

$f^{-1}(G) = [1,5] \cup (5,8) \cup [8,10]$

$f^{-1}(G) = [1,10]$ = measurable.

Hence in every case $f^{-1}(G)$ is measurable function.

So f is measurable function.

Question:-

Let $f: R \to \mathbb{R}$ defined by

$f(x) = x^2$ if x is irrational.

$= -5x+2$ if x is rational.

Show that f is measurable function.

Solution:-

Let $g: R \to \mathbb{R}$ be defined by

$g(x) = x^2$

then clearly $g = f$ a.e.

But g is measurable.

g is continuous, So

So f is measurable function

Question:-

Give an example of non-measurable function such that $|f|$ is measurable and $f^{-1}(\{a\})$ is measurable set for any $a \in \mathbb{R}$.

Solution:-

let E be non-measurable subset of $[0,1]$ and $f:[0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in E \\ -x & \text{if } x \in [0,1] \setminus E \end{cases}$$

let $f^{-1}[0,1] = E$

Since E is non-measurable.

$\Rightarrow f^{-1}[0,1]$ is non-measurable.

Thus f is non-measurable.

$\Rightarrow |f| = x$ if $x \in \mathbb{R}$

So $|f|$ is measurable.

Now we check $f^{-1}(\{a\})$ for any $a \in \mathbb{R}$.

If $a \geq 0$, then

$f^{-1}\{a\} = a$ if $a \in E$

$= \phi$ if $a \notin E$

$f\{a\} = a$ if $a \in E$

$\{a\} = f^{-1}\{a\}$

If $a \leq 0$, then

$f^{-1}\{a\} = \{-a\}$ if $-a \in [0,1] \setminus E$

$f^{-1}\{a\} = \phi$ if $-a \notin [0,1] \setminus E$

Hence $f^{-1}(\{a\})$ is measurable set.

Question:-

If f is measurable then the inverse image of interval (a,b) is measurable or not?

Solution:-

Given that f is measurable function and (a,b) is open interval. We need to show that $f^{-1}(a,b)$ is measurable.

Now $(a,b) = (-\infty, b) \cap (a, \infty)$

$\Rightarrow f^{-1}(a,b) = f^{-1}[(-\infty, b) \cap (a, \infty)]$

$\Rightarrow f^{-1}(a,b) = f^{-1}(-\infty, b) \cap f^{-1}(a, \infty)$ by def:

$\Rightarrow f^{-1}(a,b) = \{x : f(x) < b\} \cap \{x : f(x) > a\}$ — ①

Since f is measurable, so $\{x : f(x) > a\}$ and $\{x : f(x) < b\}$ is also measurable. So there intersection is also measurable.

So $f^{-1}(a,b)$ is measurable.

Hence the inverse image of open interval (a,b) is measurable.

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CHAPTER : 4

LEBESGUE INTEGRAL

(for bounded functions)

Explanation:-

For a bounded measurable function $f(x)$ over a measurable set "E" having finite measure, the lebesgue integral is defined as follows;

let "U" is the upper bound of $f(x)$ over E, and "l" is the lower bound of $f(x)$ over E. The interval $[L, U]$ is divided into "n" subintervals by numbers

$$L = y_0 < y_1 < y_2 < \dots < y_{n-1} < y_n = U$$

The set E is divided into sets $E_1, E_2, \dots$, where E_1 is the set of points "x" for which

$$y_0 \leq f(x) < y_1$$

Also

$$E_2 = \{x;\ y_1 \leq f(x) < y_2\}$$

$$E_3 = \{x;\ y_2 \leq f(x) < y_3\}$$

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Generally

$$E_k = \{x;\ y_{k-1} \leq f(x) < y_k\}$$

The Lebesgue measure of the set E_k are measure as $\mu^*(E)$.

Two sum can be formed

$$S = \sum_{k=1}^{n} y_k \mu^*(E_k) \text{ and } s = \sum_{k=1}^{n} y_{k-1} \mu^*(E_k)$$

S is called the upper sum over the partitioning set $\{y_0, y_1, y_2, \dots, y_m\}$ and Similarly s is called the lower sum of the same partition.

Now corresponding to different partition s, we obtain different upper and lower sums.

Thus we get

Upper lebesgue Integral $= I = \inf\{S\}$ or glb

lower Lebesgue Integral $= J = \sup\{s\}$ or lub

If $I = J$, we say that $f(x)$ is lebesgue integrable on E. and denote the common value by

$I = J = \int_E f(x)\,dx$ called the lebesgue definite integral of $f(x)$ over E.

Theorem:-

Let $f: E \to \mathbb{R}$ be bounded and measurable function (mean lebesgue integrable), then for any partition the upper sum has lower bound and the lower sum has upper bound.

Proof:-

let $P = \{y_0, y_1, y_2, \dots, y_n\}$ be the partition and since f is bounded, so

$\alpha \le y_k \le \beta \quad \forall \; k = 0, 1, 2, \dots, n$

Since the upper sum $= S = \sum_{i=1}^{n} y_i \, \mu^*(E_i)$

and Lower sum $= s = \sum_{i=1}^{n} y_{i-1} \, \mu^*(E_i)$

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where $E_i = \{x;\ y_{i-1} \leq f(x) < y_i\}\ \forall\ i = 1, 2, \ldots$

Since f is measurable so each "E_i" is measurable, Also since

$$y_i \geq \alpha \quad \forall\ i = 0, 1, 2, \ldots, n$$

and

$$\mu^*(E_i) \geq 0 \Rightarrow y_i \mu^*(E_i) \geq \alpha \mu^*(E_i)$$

$$\Rightarrow \sum_{i=1}^{n} y_i \mu^*(E_i) \geq \alpha \sum_{i=1}^{n} \mu^*(E_i)$$

$$\Rightarrow S \geq \alpha \mu^*(E)$$

$\because E = \bigcup_{i=1}^{n} E_i \Rightarrow \mu^*(E) = \sum_{i=1}^{n} \mu^*(E_i)$

Since $\alpha \mu^*(E) \in \mathbb{R} \Rightarrow \alpha \mu^*(E)$ is a lower bound of S.

Next since

$$y_{i-1} \leq \beta \quad \forall\ i = 0, 1, 2, \ldots, n$$

$$\Rightarrow y_{i-1} \mu^*(E_i) \leq \beta \mu^*(E_i)$$

$$\Rightarrow \sum_{i=1}^{n} y_{i-1} \mu^*(E_i) \leq \beta \sum_{i=1}^{n} \mu^*(E_i)$$

$$\Rightarrow s \leq \beta \mu^*(E)$$

Since $\beta \mu^*(E) \in \mathbb{R}$

$\Rightarrow s$ has an upper bound.

Theorem:-

Prove that for any partition the upper sum has glb, and the lower sum has lub.

Proof:-

Since in above theorem we have

proved that for any partition the upper sum has lower bound.
⇒ i.e. corresponding to different partitions we get different upper sums and if we make a set of all these upper sum i.e. $\{S\}$
⇒ the set $\{S\}$ has lower bound.
But according to completeness property of $\mathbb{R}$, " every non-empty set of real numbers that has a lower bound also has an infimum in $\mathbb{R}$.
⇒ For any partition the upper sum has glb, and similarly on the same lines replacing upper by lower and infimum by supremum, we can say that the lower sum has lub.

Theorem:-
Prove that the upper sum cannot increase and the lower sum cannot decrease.

OR

If "S" is an upper sum to the given partition P, and S_1 is an upper sum corresponding to refienment of the same partition P, say P_1, then $S_1 \leq S$.
Also if "s" is a lower sum to the given partition P and s_1 is a lower sum corresponding to the refienment of the same partition P, say P_1 then $s_1 \geq s$.

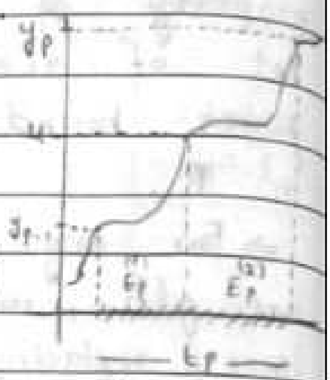
Proof:-

let $P = \{y_0, y_1, y_2, \dots, y_n\}$ be the partition.

And let $P_1 = \{y_0, y_1, \dots, y_{p-1}, u, y_p, \dots, y_n\}$ be the refined partition or refinement of P.

let s, S and s_1, S_1 be the lower and upper sum corresponding to P and P_1 respectively.

Then



$$S = \sum_{i=1}^{n} y_i \mu^*(E_i)$$

$$= y_1 \mu^*(E_1) + y_2 \mu^*(E_2) + \dots + y_n \mu^*(E_n) \quad \text{—— (1)}$$

$$\Rightarrow S_1 = y_1 \mu^*(E_1) + \dots + y_{p-1} \mu^*(E_{p-1}) + u \mu^*(E_p^{(1)}) + y_p \mu^*(E_p^{(2)}) + \dots + y_n \mu^*(E_n) \quad \text{—— (2)}$$

where

$$E_p = \{x;\ y_{p-1} \le f(x) < y_p\}$$
$$E_p^{(1)} = \{x;\ y_{p-1} \le f(x) < u\}$$
$$E_p^{(2)} = \{x;\ u \le f(x) < y_p\}$$

Also $E_p = E_p^{(1)} \cup E_p^{(2)}$ and $E_p^{(1)} \cap E_p^{(2)} = \phi$

$$\Rightarrow \mu^*(E_p) = \mu^*(E_p^{(1)}) + \mu^*(E_p^{(2)})$$

Now from eq (1)

$$S = y_1 \mu^*(E_1) + \dots + y_p \mu^*(E_p^{(1)}) + y_p \mu^*(E_p^{(2)}) + \dots + y_n \mu^*(E_n) \quad (3)$$

eq (2) $\Rightarrow$ $S_1 = y_1 \mu^*(E_1) + \dots + y_{p-1} \mu^*(E_{p-1}) + u \mu^*(E_p^{(1)}) + y_p \mu^*(E_p^{(2)}) + \dots + y_n \mu^*(E_n)$

Subtract eq (2) and (3), we get

$$S_1 - S = u \mu^*(E_p^{(1)}) - y_p \mu^*(E_p^{(1)})$$

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$$S_1 - S = \mu^*(E_p^{(1)})\,(u - y_p) \quad \text{—— (A)}$$

From fig $(u - y_p) < 0$ and we know $\mu^*(E_p^{(1)}) \geq 0$.

$$\Rightarrow S_1 - S < 0$$

$$\Rightarrow \boxed{S_1 < S}$$

on the similar way since.

$$s = y_0\mu^*(E_1) + y_1\mu^*(E_2) + \cdots + y_{p-1}\mu^*(E_p) + \cdots + y_{n-1}\mu^*(E_n)$$

and

$$s_1 = y_0\mu^*(E_1) + \cdots + y_{p-1}\mu^*(E_p^{(1)}) + u\,\mu^*(E_p^{(2)}) + y_p\mu^*(E_{p+1}) + \cdots + y_{n-1}\mu^*(E_n)$$

Now from above

$$s - s_1 = y_{p-1}\mu^*(E_p) - \left[y_{p-1}\mu^*(E_p^{(1)}) + u\,\mu^*(E_p^{(2)})\right]$$

$$= y_{p-1}\mu^*(E_p^{(1)} \cup E_p^{(2)}) - y_{p-1}\mu^*(E_p^{(1)}) - u\,\mu^*(E_p^{(2)})$$

$$= \cancel{y_{p-1}\mu^*(E_p^{(1)})} + y_{p-1}\mu^*(E_p^{(2)}) - \cancel{y_{p-1}\mu^*(E_p^{(1)})} - u\,\mu^*(E_p^{(2)})$$

$$s - s_1 = \mu^*(E_p^{(2)})\,(y_{p-1} - u)$$

From fig $(y_{p-1} - u) < 0$ and we know $\mu^*(E_p^{(2)}) \geq 0$.

$$\Rightarrow s - s_1 < 0$$

$$\Rightarrow \boxed{s < s_1}$$

Theorem:-

Prove that an upper sum for any

partition is not less than a lower sum corresponding to the same or any other partition.

Proof:-

Case-1:-

let S and s be the upper and lower sum corresponding to the same partition.

Then $S \leq s$ is obvious.

Case:2:-

Let S_1 and s_1 be the upper and lower sum corresponding to a partition (say) P_1, and let S_2 and s_2 be the upper and lower sum corresponding to an another partition (say) P_2.

Then we have to show that

$$S_1 \geq s_2 \quad \text{and} \quad S_2 \geq s_1$$

If S_3 and s_3 are the upper and lower sum corresponding to the union of above partitions.

Then

$$S_3 \leq S_1 \quad \text{and} \quad S_3 \leq S_2$$

$$\text{and} \quad s_3 \geq s_1 \quad \text{and} \quad s_3 \geq s_2$$

$$\text{and} \quad S_3 \geq s_3 \quad \text{is obvious.}$$

Also we know that

$$S_1 \geq s_1 \quad \text{and} \quad S_2 \geq s_2$$

Now as

$$S_1 \geq S_3 \geq s_3 \geq s_1$$

$$\Rightarrow \boxed{S_1 \geq s_1}$$

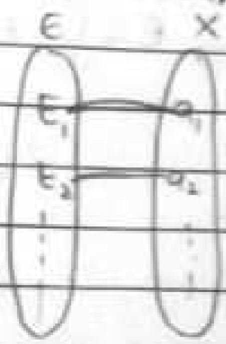
Also $S_2 \geq S_3 \geq s_3 \geq s_2$

$$\Rightarrow \boxed{S_2 \geq s_2}$$

Simple Function:-

let $f : E \to \mathbb{R}$ be a function then f is said to be simple function if it assumes only finite number of values which are finite.

If $f : E \to \mathbb{R}$ is simple function and taking distinct values $a_1, a_2, \dots, a_n$.



Then there exists disjoint sub-sets $E_1, E_2, \dots, E_n$ of E such that

$$f(E_i) = \{a_i\}$$

and $E = \bigcup_{i=1}^{n} E_i$

then we can write f as:

$$\boxed{f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)}$$

e.g: If $x \in E_3 \Rightarrow f(x) = a_3$ as

$$f(x) = a_1 \chi_{E_1}(x) + a_2 \chi_{E_2}(x) + a_3 \chi_{E_3}(x) + \dots + a_n \chi_{E_n}(x)$$

$$\Rightarrow f(x) = 0 + 0 + a_3 \cdot 1 + \dots + 0$$

$$\Rightarrow f(x) = a_3$$

Also it is well known that f is measurable iff $E_1, E_2, \ldots, E_n$ are measurable sets.

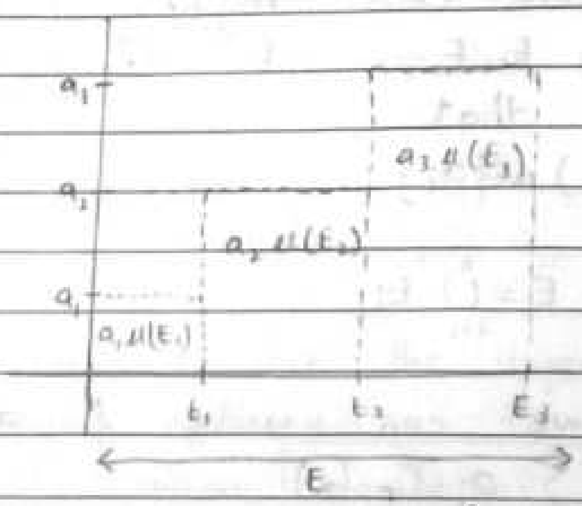
Lebesgue Integral of Simple Function:-

let $f: E \to \mathbb{R}$ be simple measurable function then we can write

$$f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

where $f(E_i) = a_i$, $E_i \cap E_j = \phi$

$E = \bigcup_{i=1}^{n} E_i$, E_i's are measurable.



the lebesgue integral of simple function f is defined by

$$\int_E f\,d = \int_E f(x)\,dx = \int_E f\,dx = \sum_{i=1}^{n} a_i\,\mu(E_i)$$

Q:- let $f: E \to \mathbb{R}$ by defined by $f(x) = 2$ $\forall\, x \in E$. Find the lebesgue integral of f.

Sol:-

Clearly f is simple function with $a_1 = 2$ and $E_1 = E$.

then $\int_E f(x)\,dx = \sum_{i=1}^{n} a_i\,\mu(E_i)$

$= a_1\,\mu(E_1)$

$= 2\,\mu(E)$

$\Rightarrow \int_E f(x)\,dx = 2\,\mu(E)$

If $E = [2, 10]$

i.e. $f : [2, 10] \to \mathbb{R}$

$\Rightarrow f(x) = 2 \quad \forall \; x \in [2, 10]$.

then

$\int_E f(x)\,dx = 2\,\mu[2, 10]$

$= 2(10 - 2)$

$= 2(8)$

$\int_E f(x)\,dx = 16$ ∴ Ans.

Note:- $\int_E dx = \mu(E)$.

Q:- let $f : [1, 10] \to \mathbb{R}$ be defined by

$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in } [1, 10]. \\ 0 & \text{if } x \text{ is irrational in } [1, 10]. \end{cases}$

i) Find lebesgue integral of "f" over $[1, 10]$

ii) Is f Riemann integral.

Solution:-

Clearly f is simple function.

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Let A be the sub-set of all rational number in $[1,10]$ and B be the sub-set of all irrational number in $[1,10]$ then $f(A)=\{1\}$ and $f(B)=\{0\}$

So $a_1 = 1$, $E_1 = A$
$a_2 = 0$, $E_2 = B$

then

$$\int_{[1,10]} f d = a_1 \mu(E_1) + a_2 \mu(E_2)$$

$$= 1\, \mu(A) + 0 . \mu(B)$$

$$= 1.(0) + 0\,(9)$$

$$= 0$$

So $\int_E f(x) dx = 0$

Now $A \cup B = [1,10]$

$$\mu(A \cup B) = \mu[1,10]$$

$$\mu(A) + \mu(B) = (10-1)$$

$$0 + \mu(B) = 9$$

$$\mu(B) = 9$$

Next we find Riemann integrable

Then

S = upper sum $= 1 = \sum_{i=1}^{n} M_i \Delta x_i$

s = lower sum $= 0 = \sum_{i=1}^{n} m_i \Delta x_i$.

$\Rightarrow S \neq s$

So
f is not Riemann integrable.

Q:- let $f:[2,20] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3 & \text{if } x \in [2,10] \\ 5 & \text{if } x \in (10,15) \\ 4 & \text{if } x \in [15,20) \end{cases}$$

Find lebesgue integral of "f" over $[2,20]$. Is f Riemann integral.

Solution:-
Clearly f is simple function
with $a_1 = 3$, $E_1 = [2,10]$
$a_2 = 5$, $E_2 = (10,15)$
$a_3 = 4$, $E_3 = [15,20)$

$\mu(E_1) = \mu[2,10] = 10-2 = 8$
$\mu(E_2) = \mu(10,15) = 15-10 = 5$
$\mu(E_3) = \mu[15,20) = 20-15 = 5$

then $\int_{[2,20]} f\,d = a_1\mu(E_1) + a_2\mu(E_2) + a_3\mu(E_3)$

$= 3(8) + 5(5) + 4(5)$
$= 24 + 25 + 20$
$= 69$

$\Rightarrow \int_{[2,20]} f\,d = 69$ Ans.

Next we find Riemann Integrable.
Then
S = upper sum $= 5 = \sum_{i=1}^{n} M_i \Delta x_i$
s = lower sum $= 3 = \sum_{i=1}^{n} m_i \Delta x_i$

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→ rational countable hota hai
→ any countable ka measurable (u) Hamesha zero hota hai

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$S \neq s$

So,
f is not Riemann integrable

Q:- let $f : [1, 10] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in } [1, 10] \\ 3 & \text{if } x \text{ is irrational in } [1, 10] \end{cases}$$

i) Find lebesgue integral of "f" over [1, 10]
ii) IS f Riemann integrable.

Solution:-
Clearly f is Simple function. let A be the sub-set of all rational number in [1, 10] and B be the sub-set of all irrational number in [1, 10] then $f(A) = \{1\}$ and $f(B) = \{3\}$

So
$$a_1 = 1 \quad , \quad E_1 = A$$
$$a_2 = 3 \quad , \quad E_2 = B$$

then
$$\int_{[1,10]} f d = a_1 \mu(E_1) + a_2 \mu(E_2)$$
$$= 1 . \mu(A) + 3 . \mu(B)$$
$$= 1 . 0 + 3(9)$$
$$= 0 + 27$$
$$\int_{[1,10]} f d = 27$$

So $\int_E f(x)\, dx = 27$ Ans.

Now $A \cup B = [1, 10]$
$$\mu(A \cup B) = \mu([1, 10])$$
$$\mu(A) + \mu(B) = 10 - 1$$

$0 + \mu(B) = 9$

$\mu(B) = 9$

Now we find Riemann integrable then

S = upper sum $= 3 = \sum_{i=1}^{n} M_i \Delta x_i$

s = lower sum $= 1 = \sum_{i=1}^{n} m_i \Delta x_i$

$$S \neq s$$

So,
f is not Riemann integrable.

Theorem:-
Prove that $f(x)$ is lebesgue integrable iff for any $\epsilon > 0$, there exist a partition with upper and lower sum, "S" and "s" such that $S - s < \epsilon$.

Proof:-
let us choose for any $\epsilon > 0$, $\exists$ S and s such that $S - s < \epsilon$.
we have to show that $f(x)$ is lebesgue integrable i.e. we prove that $I = J$.

As we know that $S \geq I \geq J \geq s$

$\Rightarrow J \geq s$

$\Rightarrow -J \leq -s$ —— (i)

and $S \geq I \Rightarrow I \leq S$ —— (ii)

Now adding eq (i) and (ii), we get

$S - s \geq I - J$

$\Rightarrow I - J \leq S - s < \epsilon$

$\Rightarrow I - J < \epsilon$

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$\Rightarrow I - J = 0$

$\Rightarrow I = J$

Thus f is lebesgue integrable.

Conversely,

$I = \underset{P}{\text{glb}}\, S$ and $J = \underset{P}{\text{lub}}\, s$

Then for any $\epsilon > 0$, there exists partition with upper sum and lower sum, such that

$I + \epsilon/2 > S$ and $J - \epsilon/2 < s$ (by def of lub and glb)

$\Rightarrow -J + \epsilon/2 > -s$

$\Rightarrow I - J + \frac{\epsilon}{2} + \frac{\epsilon}{2} > S - s$

$\Rightarrow I - J + \epsilon > S - s$

But $I = J \Rightarrow I - J = 0$

$\Rightarrow 0 + \epsilon > S - s$

$\Rightarrow S - s < \epsilon + 0$

$\Rightarrow S - s < \epsilon$

which is the required proof.

Theorem:-

If $f: E \to \mathbb{R}$ is bounded and measurable function on E, with finite measure of E, then prove that "f" is lebesgue integrable.

Proof:-
To show that f is lebesgue integrable it is enough to show that for any $\epsilon > 0$, there exist a partition with upper and lower sum's S and s respectively, such that
$$S - s < \epsilon.$$

Now since $f : E \to \mathbb{R}$ is bounded,
Therefore take a partition $\{y_0, y_1, \cdots, y_n\}$, such that
$$y_k - y_{k-1} < \epsilon \quad \forall \quad k = 1, 2, 3, \cdots, n$$

Consider the upper and lower sums, w.r.t the above partition

$$S = \sum_{k=1}^{n} y_k \, \mu^*(E_k) \quad , \quad s = \sum_{k=1}^{n} y_{k-1} \, \mu^*(E_k)$$

where $E_i \cap E_j = \phi \quad \forall \quad i \neq j$, and

$$E_k = \{x ; y_{k-1} \leq f(x) < y_k\}$$

and $\bigcup_{k=1}^{n} E_k = E$, E_k's are measurable

Now $S - s = \sum_{k=1}^{n} y_k \, \mu^*(E_k) - \sum_{k=1}^{n} y_{k-1} \, \mu^*(E_k)$

$$= \sum_{k=1}^{n} (y_k - y_{k-1}) \, \mu^*(E_k)$$

$$< \sum_{k=1}^{n} \epsilon \, \mu^*(E_k)$$

$$< \epsilon \sum_{k=1}^{n} \mu^*(E_k)$$

$$< \epsilon \, \mu^*(E) \qquad (\because \mu^*(E) \text{ is finite})$$

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$\Rightarrow \quad S - s < \epsilon_0$, where $\epsilon_0 = \epsilon\mu^*(E)$

So by necessary and sufficient condition $f(x)$ is lebesgue integrable on E.

Simple function wala hai

Problem:-

If $\phi: E \to \mathbb{R}$, be constant function defined by $\phi(x) = c \quad \forall \; x \in E$

Then $\int_E \phi(x)\, dx = \int_E c\, dx = c\,\mu^*(E)$.

Proof:-

Since we know that

If $\phi: E \to \mathbb{R}$ be a simple function then

$$\int_E \phi\, d = \sum_{i=1}^{n} a_i\, \mu(E_i)$$

where $\phi(E_i) = a_i$

Also since $\phi(E) = \{c\} =$ finite $\forall x \in E$

$\Rightarrow \phi$ is simple function, so

$$\int_E \phi\, d = c\,\mu^*(E) + 0 + 0 + \cdots$$

$$= c\,\mu^*(E)$$

$$\Rightarrow \int_E \phi\, d = c\,\mu^*(E)$$

Theorem:-

Let $f: E \to \mathbb{R}$ be bounded and measurable functions with finite measure of E, Then $\int_E (f+c)\,dx = \int_E f\,dx + c\,\mu^*(E)$.

Proof:-
Given that $f(x)$ is bounded and measurable on E, So $f(x)$ is lebesgue integrable on E.
Thus $\int_E f(x)\,dx$ exists.

Now due to boundness of $f(x)$ we can write $\alpha < f(x) < \beta$.
let us choose mode of partition of $[\alpha, \beta]$ as

$$\alpha = y_0 < y_1 < y_2 < \cdots < y_n = \beta$$

Then the upper sum of $f(x)$ for the above mode of partition is

$S = \sum_{k=1}^{n} y_k\, \mu^*(E_k)$, where $E_k = \{x;\ y_{k-1} \le f(x) < y_k\}$

Therefore the upper sum "S_1" for the function $f(x)+c$ for the above partition is

$$S_1 = \sum_{k=1}^{n} (y_k + c)\, \mu^*(E_k)$$

$$S_1 = \sum_{k=1}^{n} y_k\, \mu^*(E_k) + \sum_{k=1}^{n} c\, \mu^*(E_k)$$

$$S_1 = \sum_{k=1}^{n} y_k\, \mu^*(E_k) + c \sum_{k=1}^{n} \mu^*(E_k)$$

$$S_1 = \sum_{k=1}^{n} y_k\, \mu^*(E_k) + c\, \mu^*(E)$$

$$\Rightarrow S_1 = S + c\, \mu^*(E)$$

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Taking infimum, we get

$$\Rightarrow \inf\{S_1\} = \inf\{S\} + c\,\mu^*(E)$$

lebesgue integral of f

$$\Rightarrow \int_E (f(x)+c)\,dx = \int_E f(x)\,dx + c\,\mu^*(E)$$

$$\Rightarrow \int_E (f+c)\,dx = \int_E f\,dx + c\,\mu^*(E)$$

proved!

Theorem:-

If $f, g : E \to \mathbb{R}$ are bounded and measurable (lebesgue integrable) functions with finite measure of E, then $\int_E (f+g)\,d = \int_E f\,d + \int_E g\,d$

Proof:-

Let $\{y_0, y_1, y_2, \dots, y_n\}$ be partition choosen for $f(x)$.

and $E_k = \{x;\ y_{k-1} < f(x) < y_k\}$

where $E_i \cap E_j = \phi \quad \forall\ i \neq j$

and $\bigcup_{i=1}^{n} E_i = E$

and let "S" and "s" be the upper and lower sum defined for "$f(x)$".

Since we know that if $A \cap B = \phi$

then $\int_{A\cup B} f = \int_A f + \int_B f$

Now let $\int_E (f+g)\,dx = \int_{\bigcup_{k=1}^{n} E_k} (f+g)\,dx$

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$$\int_E (f+g)\,dx = \int_{E_1} (f+g)\,dx + \int_{E_2} (f+g)\,dx + \dots + \int_{E_n} (f+g)\,dx$$

$$\int_E (f+g)\,dx = \sum_{k=1}^{n} \int_{E_k} (f+g)\,dx \quad \text{—— (A)}$$

$$\because y_{k-1} \le f(x) < y_k$$

$$\ge \sum_{k=1}^{n} \int_{E_k} [y_{k-1} + g(x)]\,dx$$

$$\int_E [f(x)+g(x)]\,dx \ge \sum_{k=1}^{n} y_{k-1} \int_{E_k} dx + \sum_{k=1}^{n} \int_{E_k} g(x)\,dx$$

$$= \sum_{k=1}^{n} y_{k-1}\, \mu^*(E_k) + \int_E g(x)\,dx$$

$$\int_E [f(x)+g(x)]\,dx \ge s + \int_E g(x)\,dx$$

Taking supremum (glb), we get

$$\int_E (f+g)\,d \ge \sup\{s\} + \int_E g(x)\,dx$$

$$\int_E (f+g)\,d \ge \int_E f\,d + \int_E g\,d \quad \text{—— (i)}$$

Similarly,

$$(A) \Rightarrow \int_E (f+g)\,d = \sum_{k=1}^{n} \int_{E_k} (f+g)\,dx$$

$$\because y_{k-1} \le f(x) < y_k$$

$$\le \sum_{k=1}^{n} \int_{E_k} (y_k + g)\,dx$$

$$\le \sum_{k=1}^{n} y_k \int_{E_k} dx + \sum_{k=1}^{n} \int_{E_k} g(x)\,dx$$

$$\le \sum_{k=1}^{n} y_k\, \mu^*(E_k) + \int_E g\,d$$

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$$\int_E (f+g)d \le S + \int_E g(x)d$$

Taking infimum, we get

$$\int_E (f+g)d \le \inf S + \int_E g\, d$$

$$\int_E (f+g)d \le \int_E f\, d + \int_E g\, d \quad \text{—— (ii)}$$

From eq (i) and (ii), we get

$$\int_E (f+g)d = \int_E f\, d + \int_E g\, d$$

which is the required result.

Theorem:-

If $f: E \to \mathbb{R}$ is lebesgue integrable function, and A, B be two measurable sets such that $A \cap B = \phi$ and $A \cup B = E$, Show that

$$\int_{A\cup B} f d = \int_A f d + \int_B f d.$$

Proof:-

Since we know that

characteristic function of $A\cup B$

$$\chi_{A\cup B} = \chi_A + \chi_B$$

Now

$$\int_{A\cup B} f \chi_{A\cup B}\, d = \int_{A\cup B} f(\chi_A + \chi_B)\, d$$

$\therefore \chi_{A\cup B} = 1$ because all $x \in A\cup B = E$

$$\Rightarrow \int_{A\cup B} f\, d = \int_{A\cup B} (f\chi_A + f\chi_B)\, d$$

$$= \int_{A\cup B} f\chi_A\, d + \int_{A\cup B} f\chi_B\, d$$

$$= \int_A f\chi_A\, d + \int_B f\chi_A\, d + \int_A f\chi_B\, d + \int_B f\chi_B\, d$$

Since $\int_B f\chi_A\, d = 0$ and $\int_A f\chi_B\, d = 0$

So;

$$\int_{A\cup B} f\, d = \int_A f\chi_A\, d + 0 + 0 + \int_B f\chi_B\, d$$

$$\int_{A\cup B} f\, d = \int_A f\chi_A\, d + \int_B f\chi_B\, d$$

Since $\chi_A = 1$, $\chi_B = 1$; $\forall\ x \in A$ and B.

$$\Rightarrow \int_{A\cup B} f\, d = \int_A f.(1)\, d + \int_B f.(1)\, d$$

$$\Rightarrow \int_{A\cup B} f\, d = \int_A f\, d + \int_B f\, d.$$

which is the required proof.

Theorem:-

If $f: E \rightarrow \mathbb{R}$ is lebesgue integrable (bounded and measurable) function

with $\mu^*(E)=0$, then
$$\int_E f(x)\,dx = 0.$$

Proof:-
Given that the set "E" has measure zero.
Let us choose the points of partitions
$$\alpha = y_0 < y_1 < y_2 < \cdots < y_n = \beta$$

and $E_k = \{x \in E ;\ y_{k-1} < f(x) < y_k\}$

$$E_i \cap E_j = \phi \quad \text{and} \quad \bigcup_{i=1}^{n} E_i = E$$

Now all E_i's are disjoint

$$\Rightarrow \mu^*(E) = \mu^*\left(\bigcup_{i=1}^{n} E_i\right)$$

$$\Rightarrow 0 = \sum_{i=1}^{n} \mu^*(E_i)$$

$$\Rightarrow \sum_{i=1}^{n} \mu^*(E_i) = 0$$

$$\Rightarrow \mu^*(E_i) = 0 \quad \forall\ i = 1, 2, \cdots, n$$

Now,
$$S = \sum_{k=1}^{n} y_k\, \mu^*(E_k)$$

$$= \sum_{k=1}^{n} y_k\,(0)$$

$$S = 0$$

Taking infimum, we get
$$\inf\{S\} = \inf\{0\}$$

$I = 0$

Also

$$s = \sum_{k=1}^{n} y_{k-1}\, \mu^*(E_k)$$

$$s = \sum_{k=1}^{n} y_{k-1} (0)$$

$s = 0$

Taking supremum, we get

$\Rightarrow \operatorname{Sup}\{s\} = \operatorname{Sup}\{0\}$

$\Rightarrow J = 0$

$\Rightarrow I = J = 0$

$$\Rightarrow \int_E f(x)\, dx = 0$$

Theorem:-

If $f, g : E \to \mathbb{R}$ are lebesgue integrable functions, such that $f = g$; a.e then $\int_E f d = \int_E g d$ is converse true.

Proof:-

let $E_1 = \{x \in E;\ f(x) \neq g(x)\}$

and $E_2 = \{x \in E;\ f(x) = g(x)\}$

Since $f = g$; a.e, So

$\mu^*(E_1) = \mu^*\{x \in E;\ f(x) \neq g(x)\} = 0$

Also since E_1 is measurable (because

it is null).

Also E is measurable (because f is measurable so domain is measurable)

$\Rightarrow E \setminus E_1 = E_2$ is measurable.

Now E_1 and E_2 are measurable sets and $E_1 \cap E_2 = \phi$.

So by previous theorem

Here since "f" is lebesgue integrable, so

$$\int_E (f-g)\,d = \int_{E_1 \cup E_2} (f-g)\,d$$

$$= \int_{E_1} (f-g)\,d + \int_{E_2} (f-g)\,d$$

Since we know that $m^*(E_1) = 0$ (use theorem)

and $f(x) = g(x) \Rightarrow f-g=0 \ \forall\ x \in E$

$$\int_E (f-g)\,d = 0+0$$

$$\int_E (f-g)\,d = 0$$

$$\int_E f\,d - \int_E g\,d = 0$$

$$\Rightarrow \int_E f\,d = \int_E g\,d \quad \text{proved!}$$

Theorem:-

If $f, g : E \to \mathbb{R}$ are lebesgue integrable and $f(x) \leq g(x)$ on E, then

$$\int_E f(x)\,dx \leq \int_E g(x)\,dx.$$

The result is also true if $f(x) \leq g(x)$ a.e on E.

Proof:-

Since $g(x) \geq f(x)$ on E

$$\Rightarrow g(x) - f(x) \geq 0 \text{ on } E$$

$$\Rightarrow \int_E [g(x) - f(x)]\,dx \geq 0$$

$$\Rightarrow \int_E g(x)\,dx - \int_E f(x)\,dx \geq 0$$

$$\Rightarrow \int_E g(x)\,dx \geq \int_E f(x)\,dx$$

$$\Rightarrow \int_E f(x)\,dx \leq \int_E g(x)\,dx.$$

proved!

Theorem:-

If $f: E \to \mathbb{R}$ is lebesgue integrable and $A \leq f(x) \leq B$, then

$$A\,\mu^*(E) \leq \int_E f(x)\,dx \leq B\,\mu^*(E)$$

Proof:-

Given $A \leq f(x) \leq B$

$$\Rightarrow \int_E A\,dx \leq \int_E f(x)\,dx \leq \int_E B\,dx$$

$$\Rightarrow A\int_E dx \le \int_E f(x)\,dx \le B\int_E dx$$

$$\Rightarrow A\,\mu^*(E) \le \int_E f(x)\,dx \le B\,\mu^*(E)$$

proved!

↳ Results for non-negative measurable functions:

Definition:-

If $f : E \to [0,\infty]$ is measurable function, then the set

$$R(f,E) = \{(x,y) : 0 \le y \le f(x) \text{ , iff } f(x) < \infty \text{ and } 0 \le y < f(x) \text{ , iff } f(x) = \infty\}$$

is called the Region of the function defined on set E. Then the lebesgue measure of $R(f,E)$ is called lebesgue Integral of $f(x)$. i.e.

$$\mu^*[R(f,E)] = \int_E f\,d = \int_E f(x)\,d\mu(x) = \int_E f(x)\,dx$$

* Tchebeshev's Inequality:-

Statment:-

Let f be a non-negative measurable function, on the set E, if $\alpha > 0$, then

$$\mu^*(\{x \in E ; f(x) > \alpha\}) \le \frac{1}{\alpha}\int_E f\,d$$

Proof:-

let $E_1 = \{x \in E ; f(x) > \alpha\}$

then $E_1 \subseteq E$

$$\Rightarrow \int_{E_1} f(x)\,dx \le \int_E f(x)\,dx$$

$$\Rightarrow \int_E f(x)\,dx \geq \int_{E_1} f(x)\,dx > \int_{E_1} \alpha\,dx = \alpha\,\mu^*(E_1)$$

$$\Rightarrow \int_E f(x)\,dx \geq \alpha\,\mu^*(E_1)$$

$$\Rightarrow \alpha\,\mu^*(E_1) \le \int_E f(x)\,dx$$

$$\Rightarrow \mu^*(E_1) \le \frac{1}{\alpha}\int_E f(x)\,dx$$

$$\Rightarrow \mu^*(\{x \in E ; f(x) > \alpha\}) \le \frac{1}{\alpha}\int_E f(x)\,dx$$

$$\Rightarrow \mu^*(\{x \in E ; f(x) > \alpha\}) \le \frac{1}{\alpha}\int_E f\,d.$$

proved!

Theorem:-

Let f be non-negative measurable function on the set E, such that $\int_E f\,d < \infty$, then prove that $f(x) < \infty$ a.e. i.e. f is finite a.e.

Proof:-

Let $E_1 = \{x \in E ; f(x) = \infty\}$, then $E_1 \subseteq E$

Now there are two cases:

Case-1:-
If E is null set, then since each subset of nullset is null, so E_1 is also null and so $\mu^*(E_1)=0$, which shows that $f(x) < \infty$ a.e.

Case-2:-
Let E is not a null set
$\Rightarrow \mu^*(E) \neq 0$
So obviously $\mu^*(E) > 0$.
We will show that $\mu^*(E_1)=0$.
Let on contrary suppose that $\mu^*(E_1) \neq 0$, so obviously $\mu^*(E_1) > 0$
Now

$$E_1 \subseteq E$$

$$\Rightarrow \int_{E_1} f\,d \leq \int_E f\,d$$

$$\Rightarrow \int_E f\,d \geqslant \int_{E_1} f\,d = \infty > \int_{E_1} a\,d \text{; where } a \in E_1$$

$$= a\,\mu^*(E_1)$$

$$\Rightarrow \int_E f\,d > a\,\mu^*(E_1) \text{ ; if } a \text{ is sufficient large}$$

$$\Rightarrow \int_E f\,d = \infty \quad \text{i.e } a \to \infty$$

which is contradiction of statement given. So this contradiction arises due to our wrong supposition and hence $\mu^*(E_1)=0$

which shows that $f(x) < \infty$ a.e.

Theorem:-
Let f be non-negative and measurable on E. Then $\int_E f\,d = 0$ iff $f = 0$ a.e in E.

Proof:-
Let $f = 0$ a.e, then

$$\int_E f\,d = \int_E (0)\,d = 0$$

$$\Rightarrow \int_E f\,d = 0$$

Conversely:
Let $\int_E f\,d = 0$

we need to prove that $f = 0$ a.e.
For this we show that for any $\alpha > 0$ the $\{x \in E : f(x) > \alpha\}$ has lebesgue measure zero.

Let $E_1 = \{x \in E : f(x) > \alpha\}$
So $E_1 \subseteq E$

therefore $\int_{E_1} f\,d \leq \int_E f\,d$

But $f > \alpha \quad \forall \; x \in E_1$

So,

$$\int_{E_1} f\,d \geq \int_{E_1} \alpha\,d$$

$$\Rightarrow \int_{E_1} \alpha\,d \leq \int_{E_1} f\,d \leq \int_E f\,d = 0$$

$$\Rightarrow \alpha \int_{E_1} d \leq 0$$

$$\Rightarrow \alpha \, \mu(E_1) \leq 0$$

$$\Rightarrow \alpha \, \mu(\{x \in E : f(x) > \alpha\}) \leq 0$$

$$\Rightarrow \alpha \, \mu(\{x \in E : f(x) > \alpha\}) = 0$$

$$\Rightarrow \mu(\{x \in E : f(x) > \alpha\}) = 0, \text{ for any } \alpha > 0$$

$$\Rightarrow \mu(\{x \in E : f(x) > \tfrac{1}{k}\}) = 0, \text{ for } k = 1, 2, 3, \dots$$

But

$$\{x \in E : f(x) > 1\} \subseteq \{x \in E : f(x) > \tfrac{1}{2}\} \subseteq \cdots \cdots \{x \in E : f(x) > \tfrac{1}{k}\}$$

let $f_1 = \{x \in E : f(x) > 1\}$, $f_2 = \{x \in E : f(x) > \frac{1}{2}\}$

$\cdots\cdots$ $f_k = \{x \in E : f(x) > \frac{1}{k}\}$

$$\Rightarrow f_1 \subseteq f_2 \subseteq \cdots \subseteq f_k$$

$$\Rightarrow \bigcup_{k=1}^{\infty} f_k = \bigcup_{k=1}^{\infty} \{x \in E : f(x) > \tfrac{1}{k}\}$$

$$\bigcup_{k=1}^{\infty} f_k = \{x \in E : f(x) > 0\}$$

But $\mu(f_k) = 0 \quad \forall \; k = 1, 2, 3, \dots$

$$\Rightarrow \mu\left(\bigcup_{k=1}^{\infty} f_k\right) = \mu\{x \in E : f(x) > 0\} = 0$$

$$\Rightarrow f = 0 \quad \text{a.e.}$$

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Convergence Theorem:- (for non-negative functions)

Let $\{f_k\}$ be a sequence of function then is it true that

$$\lim_{k\to\infty}\int_E f_k(x)\,dx = \int_E \lim_{k\to\infty} f_k(x)\,dx.$$

Ans: Not it is not always true.
leqin agar sequence pe hum different conditions put kar le, tho different conditions put karne se pir ye equal asakta hai.

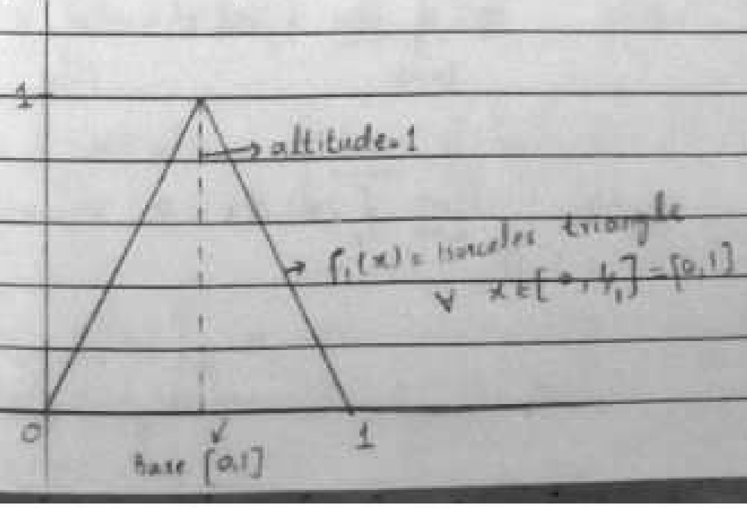
Example:-

Let f_n be a sequence of functions defined on $[0,1]$, as $f_k(x)$ is an isosceles triangle with base $[0,1/k]$ and altitude k, $\forall\ x\in[0,1/k]$, and $f_k(x)=0\ \forall\ x\in(1/k,1]$

i.e $f_n:[0,1]\longrightarrow \mathbb{R}$ by

$$f_k(x) = \begin{cases}\text{Isosceles triangle with base } [0,1/k] \text{ and altitude } k. & \text{iff } x\in[0,1/k]\\ 0 & \text{iff } x\in(1/k,1]\end{cases}$$

Then graph of $f_1(x)$ is



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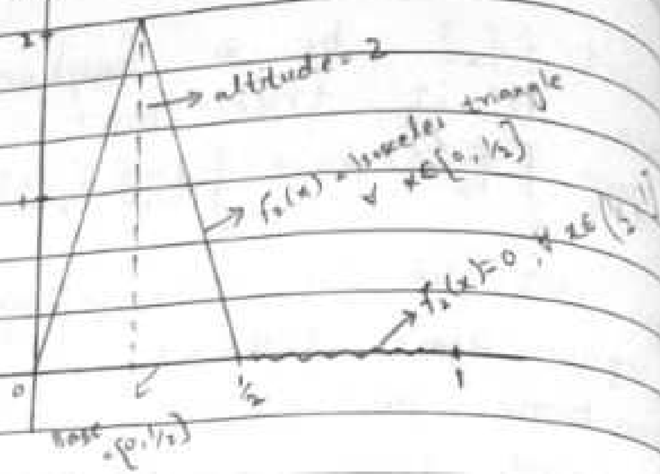
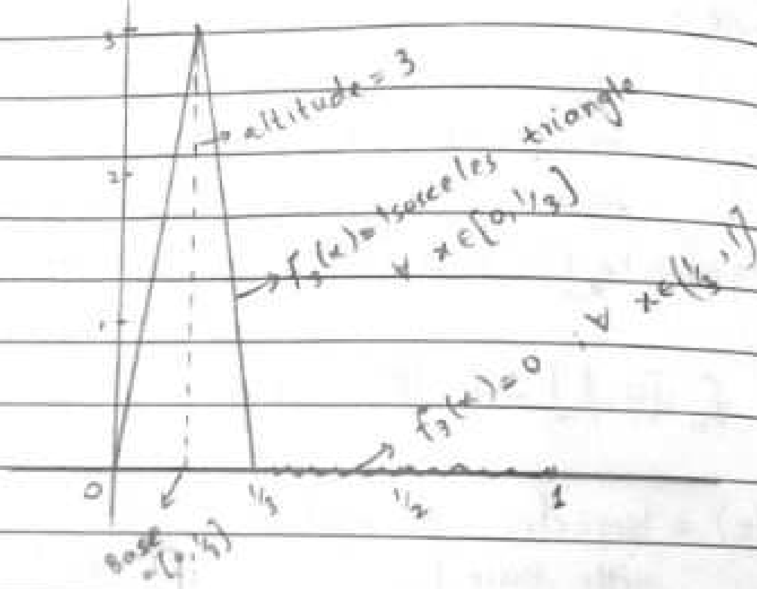
Graph of $f_2(x)$:-



Graph of $f_3(x)$:-



Here it is clear that as $k \to \infty$

$f_k(x) = 0$ i.e. $\lim_{k \to \infty} f_k(x) = 0$

$$\Rightarrow \int_{[0,1]} \lim_{k\to\infty} f_k(x)\, dx = \int_{[0,1]} 0\, dx$$

$$\Rightarrow \int_{[0,1]} \lim_{k\to\infty} f_k(x)\, dx = 0 \quad \text{——— (i)}$$

Now $\int_{[0,1]} f_k(x)\,dx = \int_{[0,1/k]} f_k(x)\,dx + \int_{(1/k,1]} f_k(x)\,dx$

$$= \frac{1}{2} \cdot k \cdot \frac{1}{k} + 0$$

$$\int_{[0,1]} f_k(x)\,dx = \frac{1}{2}$$

$$\lim_{k\to\infty} \int_{[0,1]} f_k(x)\,dx = \lim_{k\to\infty} \frac{1}{2}$$

$$\lim_{k\to\infty} \int_{[0,1]} f_k(x)\,dx = \frac{1}{2} \longleftarrow \text{(ii)}$$

From eq (i) and eq (ii), we get

$$\int_{[0,1]} \lim_{k\to\infty} f_k(x)\,dx \neq \lim_{k\to\infty} \int_{[0,1]} f_k(x)\,dx.$$

Monotone Convergence Theorem:-
(for non-negative functions)

Theorem:-
Let $\{f_n(x)\}$ be an increasing sequence of non-negative measurable functions, and if $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall\ x \in E$, then

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx.$$

Proof:-
Since $\{f_n(x)\}$ is an increasing sequence
i.e. $f_1 \le f_2 \le f_3 \le \cdots\cdots$

Therefore $R(f_1,E) \subseteq R(f_2,E) \subseteq \cdots$ and

$$\bigcup_{n=1}^{\infty} R(f_n,E) = R(f,E)$$

and $$\int_E f\,d = \mu^*(R(f,E))$$
$$= \lim_{n\to\infty} \mu^*(R(f_n,E))$$
$$= \lim_{n\to\infty} \int_E f_n\,d$$

$$\Rightarrow \int_E f\,d = \lim_{n\to\infty} \int_E f_n\,d$$

Since $\lim_{n\to\infty} f_n = f$

$$\Rightarrow \int_E \lim_{n\to\infty} f_n\,d = \lim_{n\to\infty} \int_E f_n\,d$$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx$$

proved!

Theorem:-

let $\{f_k\}$ be a sequence of non-negative measurable function, defined on E, then prove that

$$\int_E \sum_{k=1}^{\infty} f_k\,d = \sum_{k=1}^{\infty} \int_E f_k\,d$$

i.e. $$\int_E (f_1+f_2+f_3+\cdots)\,d = \int_E f_1\,d + \int_E f_2\,d + \cdots$$

i.e. linearity of lebesgue integration.

Proof:-

Let us define a function $F_n(x)$ as

$$F_n(x) = \sum_{k=1}^{n} f_k(x).$$

Since $\{f_k\}$ is a sequence of non-negative functions.

Therefore $\{F_n(x)\}$ is an increasing sequence. Also $F_n(x)$ is measurable $\forall$ n.

$\Rightarrow \{F_n(x)\}$ is an $\uparrow$ sequence of measurable functions. Therefore by using monotone-convergence theorem for non-negative functions we have

$$\int_E \lim_{n\to\infty} F_n(x)\,dx = \lim_{n\to\infty}\int_E F_n(x)\,dx$$

$$\Rightarrow \int_E \lim_{n\to\infty}\sum_{k=1}^{n} f_k(x)\,dx = \lim_{n\to\infty}\int_E \sum_{k=1}^{n} f_k(x)\,dx$$

$$\quad '' \qquad = \lim_{n\to\infty}\sum_{k=1}^{n}\int_E f_k(x)\,dx$$

$$\Rightarrow \int_E \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty}\int_E f_k(x)\,dx$$

proved!

Fatou's Lemma:- $(f_k \geq 0)$

If $\{f_k\}$ is a sequence of non-negative measurable functions, defined on E, then

$$\int_E (\liminf f_k)\,d \leq \liminf \int_E f_k\,d.$$

In particular if $\lim_{k\to\infty} f_k(x) = f(x)$, then

$$\int_E f(x)\,dx \leq \liminf \int_E f_k(x)\,dx.$$

Proof:-

As we know that a non-negative measurable function is lebesgue integrable, therefore $\int_E \liminf f_k \, dx < \infty$

i.e "$\liminf f_k$" is lebesgue integrable.

Now let $g_1 = \inf\{f_1, f_2, f_3, \dots\}$

$g_2 = \inf\{f_2, f_3, f_4, \dots\}$

$g_3 = \inf\{f_3, f_4, f_5, \dots\}$

$\vdots$

Then we know from definition of limit inf that

$\lim_{k\to\infty} g_k = \lim_{k\to\infty} \inf f_k$ and

also $0 \le g_k \le f_k$

Also $g_1 \le g_2 \le g_3 \le \dots$

Clearly $\{g_k\}$ is a monotonic increasing sequence of non-negative measurable functions. Therefore by using "monotone-convergence theorem for non-negative functions" we have

$$\lim_{k\to\infty} \int_E g_k(x)\,dx = \int_E \lim_{k\to\infty} g_k(x)\,dx$$

$$\Rightarrow \lim_{k\to\infty} \int_E g_k(x)\,dx = \int_E \liminf f_k(x)\,dx \quad \text{--- (1)}$$

But $g_k \le f_k$

$$\Rightarrow \int_E g_k \, d \leq \int_E f_k \, d$$

$$\Rightarrow \liminf \int_E g_k \, d \leq \liminf \int_E f_k \, d$$

here since $\lim_{k\to\infty} g_k(x)$ exist, so

$$\Rightarrow \lim_{k\to\infty} = \liminf = \limsup .$$

So,

$$\Rightarrow \lim_{k\to\infty} \int_E g_k(x)\, dx \leq \liminf \int_E f_k \, d \quad \text{—— (2)}$$

put eq (1) in eq (2), we get

$$\Rightarrow \int_E \liminf f_k(x)\, dx \leq \liminf \int_E f_k(x)\, dx .$$

$$\Rightarrow \int_E \liminf f_k \, d \leq \liminf \int_E f_k \, d .$$

proved!

Theorem:-

Let $\{f_k\}$ be a sequence of non-negative measurable functions, defined on E. If $\lim_{k\to\infty} f_k(x) = f(x)$, and if $\int_E f_k(x)\, dx \leq M \ \forall k$

Then prove that $\int_E f(x)\, dx \leq M$.

Proof:-

Since we know from "Fatous lemma" that $\int_E \liminf f_k \, d \leq \liminf \int_E f_k \, d$ —— (1)

But here $\lim_{k\to\infty} f_k(x) = f(x)$ —— (2)

Therefore

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Since $\lim_{k\to\infty} f_k(x) = f(x)$ exist

$\Rightarrow \liminf f_k = \limsup f_k = \lim f_k(x)$

$$\text{eq}\ (1) \Rightarrow \int_E f(x)\,dx = \int_E \lim_{k\to\infty} f_k(x)\,dx = \int \liminf f_k$$

$$\leq \liminf \int f_k(x)\,dx$$

$$\leq \liminf M \quad ; \forall k$$

$$= M$$

$$\Rightarrow \int_E f(x)\,dx \leq M$$

proved!

Lebesgue Dominated Convergence theorem for non-negative measurable functions

Theorem:-

let $\{f_k(x)\}$ be a sequence of non-negative measurable functions defined on E, if $\lim_{k\to\infty} f_k(x) = f(x)$, and if there exist a measurable function "$\phi(x)$" such that $f_k(x) \leq \phi(x)$ $\forall k$ and $\int_E \phi\, d$ is finite, then

$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$$

Proof:-

By "Fatous lemma" for the sequence $\{f_k\}$, we have $\int_E \liminf f_k\, d \leq \liminf \int_E f_k$

But since $\lim_{k \to \infty} f_k(x) = f(x)$

$$\Rightarrow \quad f(x) = \lim_{k \to \infty} f_k(x)$$

$$\Rightarrow \quad \int_E f(x)\,dx = \int_E \lim_{k \to \infty} f_k(x)\,dx$$

$$= \int_E \liminf f_k(x)\,dx \leq \liminf \int_E f_k(x)\,dx$$

$$\Rightarrow \quad \int_E f(x)\,dx \leq \liminf \int_E f_k(x)\,dx. \quad \text{——} \; (A)$$

To prove the required result it is enough to show that

$$\int_E f(x)\,dx \geq \liminf \int_E f_k(x)\,dx$$

For this, since it is given that

$$f_k(x) \leq \phi(x) \quad ; \; \forall \; x, k$$

$$\Rightarrow \quad \phi(x) - f_k(x) \geq 0 \quad ; \; \forall \; x, k$$

$\Rightarrow \{\phi(x) - f_k(x)\}$ is a sequence of non-negative measurable functions.

Therefore by using the "Fatous lemma" for the sequence $\{\phi - f_k\}$ we have

$$\int_E \liminf (\phi - f_k)\,d \leq \liminf \int_E (\phi - f_k)\,d \quad \text{——} \; (1)$$

But $\lim_{k \to \infty} f_k(x) = f(x) \Rightarrow -\lim_{k \to \infty} f_k = -f$

$$\Rightarrow \quad \lim_{k \to \infty} (\phi - f_k) = \phi - f$$

$\Rightarrow \{\phi - f_k\}$ converges to $\phi - f$.

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Therefore eq ① $\Rightarrow$

$$\int_E \liminf (\phi - f_k)\, d = \int_E \lim (\phi - f_k)\, d$$

$$= \int_E (\phi - f)\, d \leq \liminf \int_E (\phi - f_k)$$

$$\Rightarrow \int_E (\phi - f)\, d \leq \liminf \int_E (\phi - f_k)\, d$$

$$= \liminf \left[\int_E \phi\, d - \int_E f_k\, d\right]$$

$$= \liminf \int_E \phi\, d + \liminf \left(-\int_E f_k\, d\right)$$

Note: $\lim(-x_n) = \liminf(-x_n) = -\limsup(x_n)$

$$= \int_E \phi\, d - \limsup \int_E f_k\, d \quad \left[\because \int_E \phi\, d \text{ is constant}\right]$$

$$\Rightarrow \int_E (\phi - f)\, d \leq \int_E \phi\, d - \limsup \int_E f_k\, d$$

$$\Rightarrow \int_E \phi\, d - \int_E f\, d \leq \int_E \phi\, d - \limsup \int_E f_k\, d$$

$$\Rightarrow -\int_E f\, d \leq -\limsup \int_E f_k\, d$$

$$\Rightarrow \int_E f\, d \geq \limsup \int_E f_k\, d \quad \text{--- (B)}$$

From eq (A) and (B), we get

$$\Rightarrow \limsup \int_E f_k\, d \leq \int_E f\, d \leq \liminf \int_E f_k\, d$$

$$\leq \limsup \int_E f_k\, d$$

$$\Rightarrow \limsup \int_E f_k\, d = \int_E f\, d \quad \text{——— (a)}$$

Also from eq (A) and (B), we get

$$\liminf \int_E f_k\, d \le \limsup \int_E f_k\, d \le \int_E f\, d \le \liminf \int_E f_k\, d$$

$$\Rightarrow \liminf \int_E f_k\, d = \int_E f\, d \quad \text{———— (b)}$$

Using eq (a) and (b), we get

$$\int_E f\, d = \lim \sup \int_E f_k\, d = \lim \sup \int_E f_k\, d = \lim_{k\to\infty} \int_E f_k\, d$$

$$\Rightarrow \int_E f\, d = \lim_{k\to\infty} \int_E f_k\, d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k(x)\, dx = \int_E f(x)\, dx.$$

proved!

* $f^+ = \text{Max}\{f(x), 0\}$
  $f^- = -\text{Min}\{f(x), 0\}$

$$f = f^+ - f^-$$

$$\Rightarrow \int_E f\, d = \int_E f^+\, d - \int_E f^-\, d$$

$$\Rightarrow L(E) = L(\mu) = L'(\mu) = L_1(\mu) = L'(E) = \left\{ f : \int_E f(x)\, dx < \infty \right\}$$

Q: $|f| = f^+ + f^-$

Sol:-

Case-1:- If $f(x) \ge 0$

then $f^+(x) = f(x)$

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$f^-(x) = -\text{Min}\{f(x), 0\} = -0 = 0$

$\Rightarrow \quad f^+(x) + f^-(x) = f(x)$

Case - II :-

If $f(x) \leq 0$

then $f^+(x) = 0$

$f^-(x) = -\text{Min}\{f(x), 0\} = -f(x)$

$f^+(x) + f^-(x) = -f(x)$ —— ①

$\Rightarrow \quad |f(x)| = -f(x)$ —— ②

From eq ① and ②, we get

$\Rightarrow \quad |f(x)| = f^+(x) + f^-(x).$

Theorem:-

Let $f \in L(E)$ then prove that

$\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx.$

Proof:-

Since $\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$

$\left|\int_E f(x)\,dx\right| = \left|\int_E f^+(x)\,dx - \int_E f^-(x)\,dx\right|$

$\because |a-b| \leq |a| + |b|$

$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \left|\int_E f^+(x)\,dx\right| + \left|\int_E f^-(x)\,dx\right|$ —— (A)

as $f^+(x) \geq 0$ and $f^-(x) \geq 0$

So $\int_E f^+(x)\,dx \geq 0$ and $\int_E f^-(x)\,dx \geq 0$

Hence

$$\Rightarrow \left|\int_E f^+(x)\,dx\right| = \int_E f^+(x)\,dx$$

$$\Rightarrow \left|\int_E f^-(x)\,dx\right| = \int_E f^-(x)\,dx$$

eq Ⓐ $\Rightarrow$
$$\left|\int_E f(x)\,dx\right| \leq \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$$
$$\leq \int_E \left[f^+(x) + f^-(x)\right] dx$$
$$\leq \int_E |f(x)|\,dx$$

$$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$$

Theorem:-

let f be measurable function then f is lebesgue integrable iff $|f|$ is lebesgue integrable.

Proof:- OR $(f \in L(E) \Leftrightarrow |f| \in L(E))$

Suppose that

$|f| \in L(E)$ i.e. $|f|$ is integrable.

We prove that $f \in L(E)$.

Since

$$\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx \quad \text{——} ①$$

Also $|f| \in L(E) \Rightarrow \int_E |f(x)|\,dx < \infty$

eq ① $\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx < \infty$

$\Rightarrow \left| \int_E f(x)\,dx \right| < \infty$

$\Rightarrow \int_E f(x)\,dx < \infty$

$\Rightarrow f \in L(E)$

$\Rightarrow f$ is lebesgue integrable function.

Conversely:-

let $f \in L(E)$

we prove that $|f| \in L(E)$

Since $\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$

Also

$\int_E f(x)\,dx < \infty$

$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx < \infty$

$\Rightarrow \int_E f^+(x)\,dx < \infty$ and $\int_E f^-(x)\,dx < \infty$

$\Rightarrow \int_E f^+(x)\,dx + \int_E f^-(x)\,dx < \infty$

$\Rightarrow \int_E \left[f^+(x) + f^-(x) \right] dx < \infty$

$\Rightarrow \int_E |f(x)|\,dx < \infty$

$\Rightarrow |f| \in L(E)$

proved!

Theorem:-

If $f \in L(E)$ then f is finite a.e

Proof:-

Since we prove that "$f(x) \geq 0$

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and $\int_E f(x)\,dx < \infty$ then f is finite a.e"

Given that $f \in L(E)$

$\Rightarrow$ $|f| \in L(E)$ (proved)

i.e $|f(x)|$ is non-negative and integrable.
so by above result
$\Rightarrow$ $|f(x)|$ is finite a.e.
$\Rightarrow$ $f(x)$ is finite a.e.

which is the required result

Theorem:-
If $f \in L(E)$ and g are measurable such that $|g| \leq M$, then prove that $f.g \in L(E)$.

Proof:-
Since f is integrable (on set E) means that $\int_E |f|\,d\mu < \infty$

we need to show that $\int_E |fg|\,d\mu < \infty$.

But if $|g(x)| \leq M$ for all x, then the function $|fg|$ is bounded from above by $M|f|$, so we have

$$\int_E |fg|\,d\mu = \int_E |f|\,|g|\,d\mu \leq \int_E |f| M\,d\mu$$

$$\int_E |fg|\,d\mu \leq M\int_E |f|\,d\mu < \infty$$

$$\Rightarrow \int_E |fg|\,d\mu < \infty$$

$$\Rightarrow \int_E fg \, d\mu < \infty$$

$$\Rightarrow f \cdot g \in L(E)$$ proved!

* Convergence Theorem:-

→ Monotone Convergence Theorem:-

let $\{f_k\}$ be a sequence of measurable function defined on E

i) If $f_k \uparrow f$ a.e on E and there exists $\phi \in L(E)$ such that $f_k \geq \phi$ a.e on E $\forall$ k, then

$$\lim_{k\to\infty} \int_E f_k \, dx = \int_E f d$$

ii) If $f_k \downarrow f$ a.e on E and there exists $\phi \in L(E)$ such that $f_k \leq \phi$ a.e on E $\forall$ k, then

$$\lim_{k\to\infty} \int_E f_k(x) \, dx = \int_E f(x) \, dx$$

Proof:-

i) Since $f_k \geq \phi$

$$\Rightarrow f_k - \phi \geq 0$$

$\Rightarrow \{f_k - \phi\}$ is a sequence of non-negative measurable function as $f_1 \leq f_2 \leq f_3 \leq \cdots$

$$\Rightarrow f_1 - \phi \leq f_2 - \phi \leq f_3 - \phi \leq \cdots$$

$\Rightarrow \{f_k - \phi\}$ is increases sequence

Apply monotone convergence for non-negative function, then

$$\lim_{k\to\infty} \int_E (f_k - \phi) \, dx = \int_E \lim_{k\to\infty} (f_k - \phi) \, dx$$

$$\lim_{k\to\infty} \int_E (f_k - \phi)\, dx = \int_E (f - \phi)\, d$$

$$\lim_{k\to\infty} \int_E f_k\, dx - \int_E \phi\, d = \int_E f\, d - \int_E \phi\, d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k\, dx = \int_E f\, d$$

Since $f_k \leq \phi$

$\Rightarrow -f_k \geq -\phi$

$\Rightarrow -f_k + \phi \geq 0$

$\Rightarrow \{-f_k + \phi\}$ is a sequence of non-negative measurable function as $f_1 \leq f_2 \leq f_3 \leq \cdots$

$\Rightarrow -f_1 \geq -f_2 \geq -f_3 \geq \cdots$

$\Rightarrow -f_1 + \phi \geq -f_2 + \phi \geq -f_3 + \phi \geq \cdots$

$\Rightarrow \{-f_k + \phi\}$ is decreases sequence.

Apply monotone convergence for non-negative function, then

$$\lim_{k\to\infty} \int_E (-f_k + \phi)\, dx = \int_E \lim_{k\to\infty} (-f_k + \phi)\, dx$$

$$\lim_{k\to\infty} \int_E (-f_k)\, dx + \int_E \phi\, dx = \int_E (-f + \phi)\, dx$$

$$-\lim_{k\to\infty} \int_E f_k\, dx + \int \phi\, dx = -\int_E f\, dx + \int_E \phi\, dx$$

$$-\lim_{k\to\infty} \int_E f_k\, dx = -\int_E f\, dx$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k\, dx = \int_E f\, dx$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k(x)\, dx = \int_E f(x)\, dx.$$

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Fatou's Lemma :- (for arbitrary function).
Let $\{f_k\}$ be a sequence of measurable function and if there exists $\phi \in L(E)$ such that $\phi \in L(E)$, $f_k \geq \phi$ $\forall$ k, then prove that

$$\int_E \lim_{k\to\infty} \inf f_k \, d \leq \lim_{k\to\infty} \inf \int_E f_k \, d.$$

Proof:-

Since $f_k \geq \phi$

$\Rightarrow f_k - \phi \geq 0$

$\Rightarrow \{f_k - \phi\}$ is a sequence of non-negative measurable function.

Apply Fatou's lemma for non-negative function, then

$$\int_E \lim \inf (f_k - \phi) \, d \leq \lim \inf \int_E (f_k - \phi) \, d$$

$$\Rightarrow \int_E (\lim \inf f_k - \phi) \, d \leq \lim \inf \left[\int_E f_k \, d - \int_E \phi \, d \right]$$

$$\Rightarrow \int_E \lim \inf f_k \, d - \int_E \phi \, d \leq \lim \inf \int_E f_k \, d - \int_E \phi \, d$$

$$\Rightarrow \int_E \lim \inf f_k \, d \leq \lim \inf \int_E f_k \, d$$

proved

Corollary :-
Let $\{f_k\}$ be a sequence of measurable functions defined on E. If there exist a function $\phi(x) \in L(E)$ such that $f_k(x) \leq \phi(x)$ on E, $\forall$ k, then $\int_E \lim \sup f_k \, d \geq \lim \sup \int_E f_k \, d.$

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Proof:-

Since $f_k \le \phi$

$\Rightarrow -f_k \ge -\phi$

$\Rightarrow -f_k + \phi \ge 0$

$\Rightarrow \{-f_k+\phi\}$ is a sequence of non-negative measurable function.

Apply Fatou's lemma for non-negative function, then

$$\int_E \liminf (-f_k+\phi)\,d \le \liminf \int_E (-f_k+\phi)\,d$$

$$\int_E [\liminf (-f_k)+\phi]\,d \le \liminf \left[\int_E (-f_k)\,d + \int_E \phi\,d\right]$$

$$\int_E \liminf (-f_k)\,d + \cancel{\int \phi\,d} \le \liminf \int_E (-f_k)\,d + \cancel{\int_E \phi\,d}$$

$$\int_E \liminf (-f_k)\,d \le \liminf \int_E (-f_k)\,d \quad \text{--- (A)}$$

But since $\liminf(-a_n) = -\limsup(a_n)$.

So eq (1) becomes:

$$\Rightarrow \int_E [-\limsup(f_k)]\,d \le -\limsup \int_E f_k\,d$$

$$\Rightarrow \cancel{-}\int_E \limsup f_k\,d \le \cancel{-}\limsup \int_E f_k\,d$$

$$\Rightarrow \int_E \limsup f_k\,d \ge \limsup \int_E f_k\,d. \text{ proved!}$$

Lebesgue Dominated Convergence Theorem:-

Statement:- (for arbitrary sign)

Let $\{f_k\}$ be a sequence of measurable functions such that $f_k \to f$ a.e on E. If there exists $\phi \in L(E)$ such that

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$|f_k| \leq \phi \quad \forall\ k$, then

$$\lim_{k\to\infty} \int_E f_k\, d = \int_E f\, d.$$

Proof:-

Since $|f_k| \leq \phi \quad \forall\ k$

$\Rightarrow \quad -\phi \leq f_k \leq \phi$

$\Rightarrow \quad -\phi+\phi \leq f_k+\phi \leq \phi+\phi$

$\Rightarrow \quad 0 \leq f_k+\phi \leq 2\phi$

$\Rightarrow \{f_k+\phi\}$ is a sequence of non-negative functions such that $f_k+\phi \leq 2\phi$

So applying lebesgue dominated convergence theorem for non-negative function, we have

$$\lim_{k\to\infty} \int_E (f_k+\phi)\, d = \int_E \lim_{k\to\infty}(f_k+\phi)\, d$$

Since $\lim_{k\to\infty}(f_k+\phi) = f+\phi$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k\, d + \int_E \phi\, d = \int_E (f+\phi)\, d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k\, d + \int_E \phi\, d = \int_E f\, d + \int_E \phi\, d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k\, d = \int_E f\, d$$

proved!

↳ **Lebesgue bounded convergence theorem**

Statement:-

Let $\{f_k(x)\}$ be a sequence of measurable function defined on E, with the finite measure of E, such

that $\lim_{k\to\infty} f_k(x) = f(x)$ and if $|f_k| \le M$,
then
$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx.$$

Proof:-

Since we know from lebesgue dominated convergence theorem that

"If $\{f_k\}$ be a sequence of measurable functions defined on E, such that $\lim_{k\to\infty} f_k(x) = f(x)$ and if $|f_k| \le \phi$ on E,
then $\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$" — Ⓐ

Replacing "ϕ" by "M" in this statement.

Also $\int_E \phi(x)\,dx = \int_E M\,dx = M\int_E dx = M\mu(E) = \text{finite}$

$\Rightarrow$ $\phi = M$ is lebesgue integrable

i.e. $M \in L(E)$

So applying eq Ⓐ, we get
$|f_k| \le M$

Thus $\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$

proved!

Comparison between lebesgue and Riemann Integrals:-

Theorem:-

If the Riemann integrals $J = R\int_a^b f(x)\,dx$

exists, then $f(x)$ is lebesgue integrable on $[a,b]$ and $\int_{[a,b]} f(x)\,dx = J$.

where $\int_{[a,b]} f(x)\,dx$ shows lebesgue integrable

Proof:-
Consider the partition of $[a,b]$ into 2^n sub-interval by the point

$$x_k = a + \frac{k}{2^n}(b-a) \; ; \; k = 1,2,3,\cdots,2^n$$

Consider the Darboux sums.

$$\bar{S}_n = \sum_{k=1}^{2^n} \Delta x_k M_{nk} = \sum_{k=1}^{2^n} \frac{b-a}{2^n} M_{nk}$$

$$\bar{S}_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} M_{nk} \quad \text{——— (a)}$$

and

$$\underline{S}_n = \sum_{k=1}^{2^n} \Delta x_k m_{nk} = \sum_{k=1}^{2^n} \frac{b-a}{2^n} m_{nk}$$

$$\underline{S}_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} m_{nk} \quad \text{——— (b)}$$

where M_{nk} and m_{nk} are lub and glb of $f(x)$ in $[x_{k-1}, x_k]$.
So by definition:

$$\lim_{n\to\infty} \bar{S}_n = \lim_{n\to\infty} \underline{S}_n = J$$

Define sequences $\{\bar{f}_n\}$ and $\{\underline{f}_n\}$ as $\bar{f}_n(x) = M_{nk}$, $\underline{f}_n(x) = m_{nk}$.

clearly $\bar{f}_1 \geq \bar{f}_2 \geq \bar{f}_3 \geq \cdots$
and
Similarly $\underline{f}_1 \leq \underline{f}_2 \leq \underline{f}_3 \leq \cdots$

$\int_a^b \bar{f}_n(x)\,dx = \bar{S}_n$ and $\int_a^b \underline{f}_n(x)\,dx = \underline{S}_n$

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Now Since $\{\bar{f}_n\}$ is a non-increasing sequence and $\{\underline{f}_n\}$ is a non-decreasing sequence.

$$\left.\begin{array}{l}\lim_{n\to\infty} \bar{f}_n(x) = \bar{f}(x) \geq f(x) \\ \text{and} \\ \lim_{n\to\infty} \underline{f}_n(x) = \underline{f}(x) \leq f(x)\end{array}\right\} \text{———} \ (\star)$$

$$\therefore \lim_{n\to\infty} \bar{S}_n = \lim_{n\to\infty} \int_a^b \bar{f}_n(x)\,dx = \int_a^b \lim_{n\to\infty} \bar{f}_n(x)\,dx = \int_a^b \bar{f}(x)\,dx$$

$$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \int_{[a,b]} \lim_{n\to\infty} \bar{f}_n(x)\,dx = \lim_{n\to\infty} \int_{[a,b]} \bar{f}_n(x)\,dx$$

$$= \lim_{n\to\infty} \bar{S}_n = J = \lim_{n\to\infty} \underline{S}_n$$

$$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \lim_{n\to\infty} \underline{S}_n$$

$$= \lim_{n\to\infty} \int_{[a,b]} \underline{f}_n(x)\,dx$$

$$= \int_{[a,b]} \lim_{n\to\infty} \underline{f}_n(x)\,dx$$

$$= \int_{[a,b]} \underline{f}(x)\,dx$$

$$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \int_{[a,b]} \underline{f}(x)\,dx$$

Therefore

$$\int_{[a,b]} |\bar{f}(x) - \underline{f}(x)|\,dx = \int_{[a,b]} \left(\bar{f}(x) - \underline{f}(x)\right) dx \quad [\because \bar{f}(x) \geq f(x)]$$

$$= 0$$

$$\Rightarrow \int_{[a,b]} |\bar{f}(x) - \underline{f}(x)|\,dx = 0$$

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$\Rightarrow |\bar{f}(x) - \underline{f}(x)| = 0$ a.e.

$\Rightarrow \bar{f}(x) - \underline{f}(x) = 0$ a.e

$\Rightarrow \bar{f}(x) = \underline{f}(x)$ a.e.

So by eq (*)

$\Rightarrow \bar{f}(x) = \underline{f}(x) = f(x)$ a.e.

$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \int_{[a,b]} \underline{f}(x)\,dx = \int_{[a,b]} f(x)\,dx = J$

$\Rightarrow J = R\int_a^b f(x)\,dx$

$\Rightarrow J = \int_{[a,b]} f(x)\,dx$

which is the required result. proved!

LEBESGUE INTEGRAL:

Uniform Convergence Theorem:-

let $f_n \in L(E)$ and let $\{f_n\}$ converges uniformly to "f" on E and let $\mu^*(E) < \infty$. Then prove that $f \in L(E)$ and

$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E \lim_{k\to\infty} f_k(x)\,dx$$

$$= \int_E f(x)\,dx$$

Proof:-

Since $|f(x)| = |f(x) - f_k(x) + f_k(x)|$

$$|f(x)| \le |f(x) - f_k(x)| + |f_k(x)|$$

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Now since $\langle f_k \rangle$ converges to f, so by definition of convergence, for each $\epsilon > 0$ and in particular for $\epsilon = 1$, $\exists$ +ve integer n, such that

$$|f_k(x) - f(x)| < 1 \; ; \text{ for } k \geq n$$

so eq (1) $\Rightarrow$ $|f(x)| < 1 + |f_k(x)|$

$$\Rightarrow \int_E |f(x)|\,dx < \int_E 1\,dx + \int_E |f_k(x)|\,dx$$

$$= \mu^*(E) + \int_E |f_k(x)|\,dx$$

$$= \text{finite} \quad \text{as } f_k(x) \in L(E)$$

$\Rightarrow$ $|f(x)|$ is lebesgue integrable.

$\Rightarrow$ $f(x)$ is lebesgue integrable

$\Rightarrow$ $f(x) \in L(E)$.

Next we need to show that

$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$$

For this consider

$$\left|\int_E f(x)\,dx - \int_E f_k(x)\,dx\right| = \left|\int_E (f(x) - f_k(x))\,dx\right|$$

$$< \int_E |f(x) - f_k(x)|\,dx$$

$$\leq \int_E \sup_{x\in E} |f(x) - f_k(x)|\,dx$$

$$\leq \sup_{x\in E} |f(x) - f_k(x)| \int_E dx$$

$$\leq \sup_{x\in E} |f(x) - f_k(x)| \, \mu^*(E)$$

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Taking limit $k \to \infty$

$$\lim_{k\to\infty} \left| \int_E f(x)\,dx - \int_E f_k(x)\,dx \right| \leq \lim_{k\to\infty} \sup_{x\in E} \left| f(x) - f_k(x) \right| \mu^*(E)$$

$$= 0 \cdot \mu^*(E)$$

as $\lim_{k\to\infty} f_k(x) = f(x)$, $\lim_{k\to\infty} |f_k(x) - f(x)| = 0$

$$= 0$$

$$\Rightarrow \lim_{k\to\infty} \left| \int_E f(x)\,dx - \int_E f_k(x)\,dx \right| \leq 0$$

because $|x| \geq 0 \; ; \forall x$

$$\Rightarrow \lim_{k\to\infty} \left| \int_E f(x)\,dx - \int_E f_k(x)\,dx \right| = 0$$

$$\Rightarrow \lim_{k\to\infty} \left(\int_E f(x)\,dx - \int_E f_k(x)\,dx \right) = 0$$

$$\Rightarrow -\lim_{k\to\infty} \int_E f_k(x)\,dx + \int_E f(x)\,dx = 0$$

$$\Rightarrow \int_E f(x)\,dx = \lim_{k\to\infty} \int_E f_k(x)\,dx$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$$

which is the required result.

* Lebesgue Integral for unbounded function:-

* Let us consider a non-negative, an unbounded measurable function (say) "f" i.e.

$$f : E \to \mathbb{R} \quad , \quad E \subseteq \mathbb{R}^n .$$

$$f(x) \geq 0 \quad \forall\, x \in E$$

Let "p" is a natural number and

define an another function (say) "$[f(x)]_p$", such that

$$[f(x)]_p = \begin{cases} f(x) & ; \forall\, x \in E \text{ such that } f(x) \le p \\ p & ; \forall\, x \in E \text{ such that } f(x) \ge p. \end{cases}$$

Then for each p, the function $[f(x)]_p$ is bounded and measurable (prove it), and thus lebesgue integrable.

we define the lebesgue integral of $f(x)$ on E as

$$\int_E f(x)\,dx = \lim_{p\to\infty} \int_E [f(x)]_p\,dx \quad \text{———} \quad (*)$$

(prove it)

Q1: Show that $[f(x)]_p = \min\{f(x), p\}$.

Sol:-

Case-1:- when $f(x) \le p$,

then $[f(x)]_p = f(x)$ ——— ⓐ

Also since $f(x) \le p$

$\Rightarrow \min\{f(x), p\} = f(x)$ ——— ⓑ

eq ⓐ and ⓑ, we get

$[f(x)]_p = \min\{f(x), p\}$

Case-2:-

when $f(x) > p$,

then $[f(x)]_p = p$ ——— ⓒ

Also since $f(x) > p$

$\Rightarrow \min\{f(x), p\} = p$ ——— ⓓ

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From eq (c) and (d), we get

$$[f(x)]_p = \min\{f(x), p\}$$

proved!

Q2:- Show that $\lim_{p\to\infty} [f(x)]_p = f(x)$.

Sol:-

Since $[f(x)]_p = \min\{f(x), p\}$

$$\lim_{p\to\infty} [f(x)]_p = \lim_{p\to\infty} \min\{f(x), p\}$$

$$= \min\{f(x), \infty\}$$

$$= f(x)$$

$$\Rightarrow \lim_{p\to\infty} [f(x)]_p = f(x)$$

proved!

Q3:- Show that $\int_E f(x)dx = \lim_{p\to\infty} \int_E [f(x)]_p dx$

Sol:-

Using monotone convergence theorem

Since "$[f(x)]_p$" is an increasing ($\uparrow$) sequence of non-negative measurable function, such that

$$\lim_{p\to\infty} [f(x)]_p = f(x)$$

So,

$$\lim_{p\to\infty} \int_E [f(x)]_p dx = \int_E \lim_{p\to\infty} [f(x)]_p dx.$$

$$\lim_{p\to\infty} \int_E [f(x)]_p dx = \int_E f(x) dx$$

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$$\Rightarrow \int_E f(x)\,dx = \lim_{p\to\infty} \int_E [f(x)]_p\,dx$$

proved!

Lebesgue Intergral for arbitrary unbounded functions:-

let $f(x) \leq 0$

$\Rightarrow |f(x)| = -f(x)$

$\Rightarrow -|f(x)| = f(x)$

$\Rightarrow f(x) = -|f(x)|$

$$\Rightarrow \int_E f(x)\,dx = \int_E -|f(x)|\,dx$$

$$= -\int_E |f(x)|\,dx$$

So when $f(x) \leq 0$, then

$$\boxed{\int_E f(x)\,dx = -\int_E |f(x)|\,dx}$$

where the integral on the right is obtained as above since $|f(x)| \geq 0$.

→ In General case when $f(x)$ may have arbitrary sign (+ve, or -ve or zero), let us define

$$f^+(x) = \begin{cases} f(x) ; & \forall\ x \in E \text{ such that } f(x) \geq 0 \\ 0 ; & \forall\ x \in E \text{ such that } f(x) < 0 \end{cases}$$

and

$$f^-(x) = \begin{cases} 0 ; & \forall\ x \in E \text{ such that } f(x) \geq 0 \\ -f(x) ; & \forall\ x \in E \text{ such that } f(x) < 0 \end{cases}$$

where it is to be noted that both

$f^+(x)$ and $f^-(x)$ are non-negative, then it follows that

$$f(x) = f^+(x) - f^-(x) \quad \text{(prove it)} \text{ --- } (A)$$

$$\Rightarrow \int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$$

Also we can easily prove that

$$|f(x)| = f^+(x) + f^-(x) \text{ --- } (B)$$

$$\Rightarrow \int_E |f(x)|\,dx = \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$$

(proved it)

Proof:- eq (A) and eq (B)

eq (A) $\Rightarrow$ we have to prove that

$$f(x) = f^+(x) - f^-(x)$$

Case-1:- when $f(x) \geq 0$, then $f^+(x) = f(x)$ and $f^-(x) = 0$

$$\Rightarrow f^+(x) - f^-(x) = f(x) - 0$$

$$\Rightarrow f^+(x) - f^-(x) = f(x)$$

$$\Rightarrow \boxed{f(x) = f^+(x) - f^-(x)}$$

Case-2:- when $f(x) < 0$, then $f^+(x) = 0$ and $f^-(x) = -f(x)$

$$\Rightarrow f^+(x) - f^-(x) = 0 - (-f(x))$$

$$\Rightarrow f^+(x) - f^-(x) = f(x)$$

$$\Rightarrow \boxed{f(x) = f^+(x) - f^-(x)}$$

So. eq (A) proved!

eq (B) $\Rightarrow$ We have to prove that

$$|f(x)| = f^+(x) + f^-(x)$$

Case-1:- when $f(x) \geq 0 \Rightarrow |f(x)| = f(x)$ —— (a)

$f^+(x) = f(x)$ and $f^-(x) = 0$

$\Rightarrow f^+(x) + f^-(x) = f(x) + 0$

$\Rightarrow f^+(x) + f^-(x) = f(x)$ —— (b)

From (a) and (b), we get

$$\boxed{|f(x)| = f^+(x) - f^-(x)}$$

Case-2:- When $f(x) < 0$, then $|f(x)| = -f(x)$ —— (c)

$f^+(x) = 0$ and $f^-(x) = -f(x)$

$\Rightarrow f^+(x) - f^-(x) = 0 - [-f(x)]$

$\Rightarrow f^+(x) - f^-(x) = f(x)$ —— (d)

From eq (c) and (d), we get

$$\boxed{|f(x)| = f^+(x) - f^-(x)}$$

So eq (B) proved!

Theorem:-

Let $f(x)$ be an unbounded, non-negative, measurable function i-e $f(x) \geq 0$, then $\int_E f(x)dx$ exist iff $\int_E [f(x)]_p dx$ is uniformly bounded.

Proof:-

let $f(x)$ is an unbounded,

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non-negative and measurable function

i.e $f(x) \geq 0 \quad \forall \; x \in \text{Dom} f$.

let $\int_E f(x)\,dx$ exists

$\Rightarrow \int_E f(x)\,dx = M$

Since we know that

$[f(x)]_p = \min\{f(x), p\}$

$[f(x)]_p \leq f(x)$

$\int_E [f(x)]_p\,dx \leq \int_E f(x)\,dx = M$

$\int_E [f(x)]_p\,dx \leq M$

$\Rightarrow \langle \int_E [f(x)]_p\,dx \rangle$ is uniformly bounded sequence.

Conversely, let $\int_E [f(x)]_p\,dx$ is uniformly bounded.

we need to prove that $\int_E f(x)\,dx$ exists

Since $\left| \int_E [f(x)]_p\,dx \right| \leq M$

$\Rightarrow \int_E [f(x)]_p\,dx \leq M$

$\because \int_E [f(x)]_p\,dx \geq 0$

and $|x| = x$ iff $x \geq 0$

But $\int_E [f(x)]_p \, dx$ is monotonic increasing sequence.

So as it is also bounded.

$\Rightarrow \langle \int_E [f(x)]_p \, dx \rangle$ is convergent.

$\Rightarrow \lim_{p \to \infty} \int_E [f(x)]_p \, dx$ exist.

$\Rightarrow \int_E [f(x)]_p \, dx$ exist.

proved!

Theorem:-

If $|f(x)| \leq g(x)$, where $g(x)$ is integrable on E, then $f(x)$ is also integrable on E and $\int_E |f(x)| dx \leq \int_E g(x) \, dx$.

Prove this result for
(i) $f(x) \geq 0$ (ii) $f(x)$ having arbitrary sign.

Proof:-

Given that $|f(x)| \leq g(x)$
we need to prove that $f(x)$ is integrable.

Now
(i) when $f(x) \geq 0$

$\Rightarrow |f(x)| = f(x)$ and $|f(x)| \leq g(x)$

$\Rightarrow f(x) \leq g(x)$

$\Rightarrow [f(x)]_p \leq [g(x)]_p$

$\Rightarrow \int_E [f(x)]_p \, dx \leq \int_E [g(x)]_p \, dx$

$\Rightarrow \lim_{p \to \infty} \int_E [f(x)]_p \, dx \leq \lim_{p \to \infty} \int_E [g(x)]_p \, dx$

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$$\Rightarrow \int_E f(x)\,dx \le \int_E g(x)\,dx$$

Since $g(x)$ is integrable

So $\int_E g(x)\,dx$ exist

$$\Rightarrow \int_E f(x)\,dx \text{ exist}$$

$\Rightarrow f(x)$ is integrable on E.

$$\Rightarrow \int_E f(x)\,dx \le \int_E g(x)\,dx$$

$$\Rightarrow \int_E |f(x)|\,dx \le \int_E g(x)\,dx.$$

ii) Now when $f(x)$ has arbitrary sign, then since we know that

$$\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx \quad \text{—} (A)$$

Also $|f(x)| \le g(x)$

$$\Rightarrow f^+(x) - f^-(x) \le g(x)$$

$$\Rightarrow f^+(x) \le g(x) \text{ and } f^-(x) \le g(x)$$

So from above case (i) $f^+(x)$ and $f(x)$ are both integrable.

$$\Rightarrow \int_E f^+(x)\,dx \text{ and } \int_E f^-(x)\,dx \text{ exists}$$

$$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx \text{ exists.}$$

eq (A)

$$\Rightarrow \int_E f(x)\,dx \text{ exists.}$$

$\Rightarrow f(x)$ is integrable.

Also $\int_E f^+(x)\,dx + \int_E f^-(x)\,dx$ exist

$\Rightarrow \int_E (f^+(x) + f^-(x))\,dx$ exist.

$\Rightarrow \int_E |f(x)|\,dx$ exist

Now since $|f(x)| \leq g(x)$

$$\int_E |f(x)|\,dx \leq \int_E g(x)\,dx$$

which is the required result.

Theorem:-

A function $f(x)$ is integrable on E iff $|f(x)|$ is integrable on E and in such a case $\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$, where $f(x)$ is an unbounded function.

Proof:-

Let $f(x)$ is integrable.

Since $f(x) = f^+(x) - f^-(x)$, this shows that $f^+(x) - f^-(x)$ is integrable.

So,

$\Rightarrow \int_E (f^+(x) - f^-(x))\,dx$ exist

$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$ exist.

$\Rightarrow \int_E f^+(x)\,dx$ and $\int_E f^-(x)\,dx$ exist.

$\Rightarrow \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$ exist.

$\Rightarrow \int_E (f^+(x) + f^-(x))\,dx$ exist.

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$\Rightarrow \int_E |f(x)|\, dx$ exist.

$\Rightarrow |f(x)|$ is integrable.

Conversely;

Let $|f(x)|$ is integrable on E.

But $|f(x)| = f^+(x) + f^-(x)$

$\Rightarrow f^+(x) + f^-(x)$ is integrable.

$\Rightarrow \int_E (f^+(x) + f^-(x))\, dx$ exist.

$\Rightarrow \int_E f^+(x)\, dx + \int_E f^-(x)\, dx$ exist

$\Rightarrow \int_E f^+(x)\, dx$ and $\int_E f^-(x)\, dx$ exist

$\Rightarrow \int_E f^+(x)\, dx - \int_E f^-(x)\, dx$ exist

$\Rightarrow \int_E (f^+(x) - f^-(x))\, dx$ exist.

$\Rightarrow \int_E f(x)\, dx$ exist

$\Rightarrow f(x)$ is lebesgue integrable.

Also $\left|\int_E f(x)\, dx\right| = \left|\int_E (f^+(x) - f^-(x))\, dx\right|$

$= \left|\int_E f^+(x)\, dx - \int_E f^-(x)\, dx\right|$

$\because |a-b| \leq |a| + |b|$

$\leq \left|\int_E f^+(x)\, dx\right| + \left|\int_E f^-(x)\, dx\right|$

$$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$$

$$\leq \int_E \left(f^+(x) + f^-(x)\right)dx$$

$$\leq \int_E |f(x)|\,dx$$

$$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$$

proved!

Question:-

Prove that $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$ exists and find its value.

Solution:-

Since we know that for an unbounded, non-negative function $f(x)$.

$$\int_E f(x)\,dx = \lim_{p\to\infty} \int_E [f(x)]_p\,dx \quad \text{--- (1)}$$

We have to find $[f(x)]_p$

let us define

$$[f(x)]_p = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{for } \frac{1}{\sqrt[3]{x}} \leq p \text{ or } x \geq \frac{1}{p^3} \\ p & \text{for } \frac{1}{\sqrt[3]{x}} > p \text{ or } x < \frac{1}{p^3} \end{cases}$$

Then eq (1) $\Rightarrow \int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{p\to\infty} \int_0^8 [f(x)]_p\,dx$

$$\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{p\to\infty}\left[\int_0^{1/p^3} p\,dx + \int_{1/p^3}^8 \frac{1}{\sqrt[3]{x}}\,dx\right]$$

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$$\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{p\to\infty}\left[px\Big|_0^{1/p^3} + \int_{1/p}^{8} x^{-1/3}\,dx \right]$$

$$= \lim_{p\to\infty}\left[p\left(\frac{1}{p^3} - 0\right) + \frac{x^{-1/3+1}}{-1/3+1}\Big|_{1/p^3}^{8} \right]$$

$$= \lim_{p\to\infty}\left[p\left(1/p^3\right) + \frac{x^{2/3}}{2/3}\Big|_{1/p^3}^{8} \right]$$

$$= \lim_{p\to\infty}\left[\frac{1}{p^2} + \frac{3}{2}\, x^{2/3}\Big|_{1/p^3}^{8} \right]$$

$$= \lim_{p\to\infty}\left\{ \frac{1}{p^2} + \frac{3}{2}\left[8^{2/3} - \left(\frac{1}{p^3}\right)^{2/3}\right] \right\}$$

$$= \lim_{p\to\infty}\left[\frac{1}{p^2} + \frac{3}{2}\left(4 - \frac{1}{p^{2/3\times 3}}\right) \right]$$

$$= \lim_{p\to\infty}\left[\frac{1}{p^2} + \frac{3}{2}\left(4 - \frac{1}{p^2}\right) \right]$$

Applying limit

$$= \frac{1}{\infty^2} + \frac{3}{2}\left(4 - \frac{1}{\infty^2}\right)$$

$$= 0 + \frac{3}{2}(4-0)$$

$$= \frac{3}{\cancel{2}}(\cancel{4})^2$$

$$\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = 6 \quad : \text{Ans}$$

Thus the lebesgue integral exists and has the value 6.

Note:- The above Lebesgue integral can be evaluated as for Riemann integrals.

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Also that the Riemann integral $R\int_0^8 \frac{1}{\sqrt[3]{x}}dx$, exists as an improper integral defined as

$$\lim_{\epsilon\to 0} R\int_\epsilon^8 \frac{1}{\sqrt[3]{x}}dx = \lim_{\epsilon\to 0} R\int_\epsilon^8 x^{-1/3}dx$$

$$= \lim_{\epsilon\to 0} \frac{x^{-1/3+1}}{-1/3+1}\Bigg|_\epsilon^8$$

$$= \lim_{\epsilon\to 0} \frac{x^{2/3}}{2/3}\Bigg|_\epsilon^8$$

$$= \lim_{\epsilon\to 0} \frac{3}{2} x^{2/3}\Big|_\epsilon^8$$

$$= \lim_{\epsilon\to 0} \frac{3}{2}(8^{2/3} - \epsilon^{2/3})$$

$$= \frac{3}{2}\lim_{\epsilon\to 0}(4 - \epsilon^{2/3})$$

$$= \frac{3}{2}(4-0)$$

$$= \frac{3}{\cancel{2}_1}(\cancel{4})^2$$

$$\lim_{\epsilon\to 0} R\int_\epsilon^8 \frac{1}{\sqrt[3]{x}}dx = 6 \quad \text{Ans.}$$

Question:-

Prove that $\int_0^4 \frac{1}{\sqrt{x}}dx$ exists as a Lebesgue integral and find its value.

Solution:-

Since we know that for an unbounded, non-negative function $f(x)$.

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$$\int_{0}^{4} f(x)\,dx = \lim_{p\to\infty} \int_{0}^{4} [f(x)]_p \,dx \quad \text{--- (1)}$$

We have to find $[f(x)]_p$.

Let us define

$$[f(x)]_p = \begin{cases} \frac{1}{\sqrt{x}} & \text{for } \frac{1}{\sqrt{x}} < p \text{ or } x \geq \frac{1}{p^2} \\ p & \text{for } \frac{1}{\sqrt{x}} > p \text{ or } x < \frac{1}{p^2} \end{cases}$$

Then eq (1) $\Rightarrow \int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{p\to\infty} \int_0^4 [f(x)]_p \,dx$

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{p\to\infty} \left[\int_0^{1/p^2} p\,dx + \int_{1/p^2}^{4} \frac{1}{\sqrt{x}}\,dx \right]$$

$$= \lim_{p\to\infty} \left[px \Big|_0^{1/p^2} + \int_{1/p^2}^{4} x^{-1/2}\,dx \right]$$

$$= \lim_{p\to\infty} \left[p\left(\frac{1}{p^2} - 0\right) + \frac{x^{-1/2+1}}{-1/2+1} \Big|_{1/p^2}^{4} \right]$$

$$= \lim_{p\to\infty} \left[p\left(1/p^2\right) + \frac{x^{1/2}}{1/2} \Big|_{1/p^2}^{4} \right]$$

$$= \lim_{p\to\infty} \left[\frac{1}{p} + 2\,x^{1/2} \Big|_{1/p^2}^{4} \right]$$

$$= \lim_{p\to\infty} \left[\frac{1}{p} + 2\left\{ 4^{1/2} - \left(1/p^2\right)^{1/2} \right\} \right]$$

$$= \lim_{p\to\infty} \left[\frac{1}{p} + 2\left(2 - \frac{1}{p}\right) \right]$$

Applying limit

$$= \frac{1}{\infty} + 2\left(2 - \frac{1}{\infty}\right)$$

$$= 0 + 4 - 0$$

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = 4 \quad \text{Ans}$$

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Thus the Lebesgue integral exists and has the value 4.

Question:—

Prove that $\int_0^2 \frac{1}{x^2} dx$ does not exist.

Solution:—

Since we know that for an unbounded, non-negative function $f(x)$.

$$\int_E f(x)\, dx = \lim_{p\to\infty} \int_E [f(x)]_p\, dx$$

We have to find $[f(x)]_p$.

Let us define.

$$[f(x)]_p = \begin{cases} 1/x^2 & \text{for } 1/x^2 \le p \text{ or } x \ge \frac{1}{\sqrt{p}} \\ p & \text{for } 1/x^2 > p \text{ or } x < \frac{1}{\sqrt{p}} \end{cases}$$

Then eq ① $\Rightarrow \int_0^2 \frac{1}{\sqrt{x}} dx = \lim_{p\to\infty} \int_0^2 [f(x)]_p\, dx$

$$\int_0^2 \frac{1}{\sqrt{x}} dx = \lim_{p\to\infty} \left[\int_0^{1/\sqrt{p}} p\, dx + \int_{1/\sqrt{p}}^2 \frac{1}{x^2} dx \right]$$

$$= \lim_{p\to\infty} \left[p\, x \Big|_0^{1/\sqrt{p}} + \int_{1/\sqrt{p}}^2 x^{-2} dx \right]$$

$$= \lim_{p\to\infty} \left[p\left(\frac{1}{\sqrt{p}}\right) + \frac{x^{-3}}{-3} \Big|_{1/\sqrt{p}}^2 \right]$$

$$= \lim_{p\to\infty} \left[p(p^{-1/2}) - \frac{1}{3} \frac{1}{x^3} \Big|_{1/\sqrt{p}}^2 \right]$$

$$= \lim_{p\to\infty} \left[p^{1-1/2} - \frac{1}{3}\left(\frac{1}{2^3} - \frac{1}{(1/\sqrt{p})^3} \right) \right]$$

$$= \lim_{p\to\infty} \left[p^{1/2} - \frac{1}{3}\left(\frac{1}{8} - (\sqrt{p})^3 \right) \right]$$

Applying limit

Thus $\int_0^2 \frac{1}{x^2}\,dx = \infty$

$\Rightarrow \int_0^2 \frac{1}{x^2}\,dx$ does not exists.

Theorem:-

Show that $[f(x)]_p$ is bounded and measurable, and thus lebesgue integrable, for each p if $f(x)$ is measurable.

Proof:-

Since $[f(x)]_p = \min\{f(x), p\}$ —— (1)

$\Rightarrow [f(x)]_p \leq p \quad \forall\ p \in N$

$\Rightarrow [f(x)]_p$ is bounded for each p

Also since we know a result which states "If $f_1(x)$ and $f_2(x)$ are measurable on E, then
(a) $\max\{f_1(x), f_2(x)\}$ (b) $\min\{f_1(x), f_2(x)\}$ are measurable on E". —— (*)

Using this result (*), Here since $f(x)$ is given to be measurable and p being a constant function is measurable, so by eq (*).
$\min\{f(x), p\}$ is also measurable

$\Rightarrow [f(x)]_p$ is measurable

Note:- The result can be shown directly from the definition of the function $[f(x)]_p$.

Note:- What is the difference b/w Riemann integral and Lebesgue integral.

Ans:-

Both integral are used to calaulate the area under curve and the difference is

Riemann Integral	Lebesgue Integral
(i) Function should be continuous.	(i) Continuous / Discontinuous
(ii) Bounded	(ii) Bounded / unbounded
(iii) Countable (in case of sets).	(iii) Countable / uncountable (In case of sets).

Theorem:-

If $\int_E f(x)\,dx$ exists then prove that $f(x)$ is finite almost everywhere on E, where $f(x) \geq 0$ and unbounded.

Proof:-

let $f(x)$ be an unbounded, measurable function such that $f(x) \geq 0$.

let $\int_E f(x)\,dx$ exists.

We have to show that $f(x) < \infty$ a.e.

For this let $A = \{x \in E ; f(x) = \infty\}$

we will show that $\mu^*(A) = 0$

Since $A \subseteq E$

So,

$$\int_E [f(x)]_p\,dx \geq \int_A [f(x)]_p\,dx$$

$$\Rightarrow \int_E [f(x)]_p\, dx \geq \int_A [f(x)]_p\, dx \geq \int_A p\, dx = p\mu(A)$$

$$\Rightarrow \int_E [f(x)]_p\, dx \geq p\mu(A)$$

$$\Rightarrow \lim_{p\to\infty} \int_E [f(x)]_p\, dx \geq \lim_{p\to\infty} p\mu(A)$$

$$\Rightarrow \int_E f(x)\, dx \geq \lim_{p\to\infty} p\mu(A) \quad \text{--- (1)}$$

Now if $\mu(A) > 0$

$$\Rightarrow \lim_{p\to\infty} p\mu(A) = \infty$$

So eq (1) $\Rightarrow \int_E f(x)\, dx = \infty$

which is contradiction to the given that $\int_E f(x)\, dx$ exists.

So $\mu(A) \not> 0$ and $\mu(A) \not< 0$

$$\Rightarrow \mu(A) = 0$$

$$\Rightarrow \mu(\{x : f(x) = \infty\}) = 0$$

$\Rightarrow f(x)$ is finite almost everywhere.

Theorem:-

Let $f(x)$ be an unbounded, measurable function, such that $\int_E f(x)dx$ exists, if A is a measurable subset of E, then $\int_A f(x)dx$ also exists and in such a case

$$\int_A |f(x)|\, dx \leq \int_E |f(x)|\, dx.$$

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Proof:-
Given that $\int_E f(x)\,dx$ exist and $A \subseteq E$, where A is measurable. We are to prove that $\int_A f(x)\,dx$ exist and that $\int_A |f(x)|\,dx \le \int_E |f(x)|\,dx$.

Consider $f(x) \ge 0$, since $A \subseteq E$ and $[f(x)]_p$ is bounded.

So, $\int_A [f(x)]_p\,dx \le \int_E [f(x)]_p\,dx$

$$\lim_{p\to\infty} \int_A [f(x)]_p\,dx \le \lim_{p\to\infty} \int_E [f(x)]_p\,dx$$

$$\Rightarrow \int_A f(x)\,dx \le \int_E f(x)\,dx \quad \text{——— (1)}$$

But $\int_E f(x)\,dx$ exists, so $\int_A f(x)\,dx$ also exist.

Since $f(x) \ge 0 \Rightarrow |f(x)| = f(x)$

So eq(1) $\Rightarrow \int_A |f(x)|\,dx \le \int_E |f(x)|\,dx$.

So in this case the required answer proved!.

Now when $f(x)$ has arbitrary sign in such a case $f(x) = f^+(x) - f^-(x)$

$$\int_A f(x)\,dx = \int_A f^+(x)\,dx - \int_A f^-(x)\,dx$$

Also $\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$

Since $f^+(x) \geq 0$ and $f^-(x) \geq 0$
So from above case (i)

$$eq \text{ ①} \Rightarrow \int_A f^+(x)\,dx + \int_A f^-(x)\,dx \leq \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$$

$$\Rightarrow \int_A \left(f^+(x) + f^-(x)\right)dx \leq \int_E \left(f^+(x) + f^-(x)\right)dx$$

$$\Rightarrow \int_A |f(x)|\,dx \leq \int_E |f(x)|\,dx .$$

proved!

Theorem:-

If $E = E_1 \cup E_2$ where E_1 and E_2 are disjoint, then if $f(x)$ is an unbounded function on E, then

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx .$$

In general prove that if $E = \bigcup_{k=1}^{n} E_k$, where all E_k are disjoint then

$$\int_E f(x)\,dx = \sum_{k=1}^{n} \int_{E_k} f(x)\,dx$$

Proof:-

We prove the above result in general and that deduce from the particular for two sets.

Also we first consider $f(x) \geq 0$.

Since $[f(x)]_p$ is always a bounded and measurable function and we know from lebesgue integration theory of bounded function that " if $f(x)$

is bounded and measurable on E where $E = E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n$ where all E_i's are disjoint then

$$\int_E g(x)\,dx = \int_{E_1} g(x)\,dx + \int_{E_2} g(x)\,dx + \cdots + \int_{E_n} g(x)\,dx \text{ ''} \quad (*)$$

Using this result eq (*)

$$\int_E [f(x)]_p\,dx = \int_{E_1}[f(x)]_p\,dx + \int_{E_2}[f(x)]\,dx + \cdots + \int_{E_n}[f(x)]_n\,dx$$

$$\lim_{p\to\infty}\int_E [f(x)]_p\,dx = \lim_{p\to\infty}\int_{E_1}[f(x)]_p\,dx + \lim_{p\to\infty}\int_{E_2}[f(x)]_p\,dx + \cdots + \lim_{p\to\infty}\int_{E_n}[f(x)]_p\,dx$$

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx + \cdots + \int_{E_n} f(x)\,dx$$

or

$$\boxed{\int_E f(x)\,dx = \sum_{k=1}^{n}\int_{E_k} f(x)\,dx} \quad (A)$$

proved for $f(x) \geq 0$.

Now we consider the case when $f(x)$ has arbitrary sign and in such a case

$$f(x) = f^+(x) - f^-(x) \quad \text{and} \quad |f(x)| = f^+(x) + f^-(x)$$

Since $f^+(x) \geq 0$ and $f^-(x) \geq 0$

So using the above result eq (A) proved for $f(x) \geq 0$ we have

$$\int_E f^+(x)\,dx = \sum_{k=1}^{n}\int_{E_k} f^+(x)\,dx$$

and also $\displaystyle\int_E f^-(x)\,dx = \sum_{k=1}^{n}\int_{E_k} f^-(x)\,dx$

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$$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx = \sum_{k=1}^{n} \int_{E_k} f^+(x)\,dx - \sum_{k=1}^{n} \int_{E_k} f^-(x)\,dx$$

$$\Rightarrow \int_E \left(f^+(x) - f^-(x)\right) dx = \sum_{k=1}^{n} \int_{E_k} \left(f^+(x) - f^-(x)\right) dx$$

$$\Rightarrow \boxed{\int_E f(x)\,dx = \sum_{k=1}^{n} \int_{E_k} f(x)\,dx}$$

In particular when $E = E_1 \cup E_2$, then

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx$$

which is the required result.

Theorem:-

Let $f(x)$ be an unbounded and measurable function on E, such that $\mu(E) = 0$, then $\int_E f(x)\,dx = 0$

Proof:-

Let $f(x) \geq 0$, then since

$$\lim_{p \to \infty} \int_E [f(x)]_p\,dx = \int_E f(x)\,dx.$$

Now we have a result which states "If E has measure zero and $f(x)$ is bounded then $\int_E f(x)\,dx = 0$".

By this result, since here E has measure zero. and $[f(x)]_p$ is bounded, so

$$\int_E [f(x)]_p \, dx = 0$$

$$\lim_{p\to\infty} \int_E [f(x)]_p \, dx = 0$$

$$\Rightarrow \int_E f(x)\, dx = 0 \quad ; \text{ For } f(x) \geq 0$$

Now if $f(x)$ has arbitrary sign, then
$f(x) = f^+(x) - f^-(x)$

$$\Rightarrow \int_E f(x)\, dx = \int_E f^+(x)\, dx - \int_E f^-(x)\, dx \quad \text{———} \quad (1)$$

Since $f^+(x)$ and $f^-(x)$ are non-negative, and E has measure zero, so from above

$$\int_E f^+(x)\, dx = 0 \quad \text{and} \quad \int_E f^-(x)\, dx = 0$$

$$\Rightarrow \int_E f^+(x)\, dx - \int_E f^-(x)\, dx = 0$$

eq (1)

$$\Rightarrow \boxed{\int_E f(x)\, dx = 0}$$

which is the required result.

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CHAPTER: 5

L_p - SPACES (L^p)

Let (X, S, μ) be a Lebesgue measure space and consider the set
$$L_p = \{f : X \to \mathbb{R} : \int_X |f|^p dx < \infty\} \quad 0 < p < \infty.$$

i.e. L_p contains those functions "f" such that $|f|^p$ is integrable. (Also known as p integrable).

Question:-

Show that L_p is a linear space.

Proof:-

Let $f, g \in L_p = \{f : X \to \mathbb{R} ; \int_X |f|^p dx < \infty\}$

i) $\Rightarrow \int_X |f|^p dx < \infty$ and $\int_X |g|^p dx < \infty$

We will show that $f+g \in L_p$.
i.e. we will show that $\int_X |f+g|^p dx < \infty$. ($\because$ by def. of L_p)

Now $|f+g| \leq |f| + |g|$
If $|f| \leq |g|$

$$\Rightarrow |f(x) + g(x)| \leq |f(x)| + |f(x)|$$
$$\leq 2|f(x)| = 2\max\{|f(x)|, |g(x)|\}$$

Hence
$$|f(x) + g(x)| \leq 2\max\{|f(x)|, |g(x)|\}$$
$$|f(x) + g(x)|^p \leq 2^p \max\{|f(x)|^p, |g(x)|^p\}$$

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$$|f(x)+g(x)|^p \le 2^p\{|f(x)|^p+|g(x)|^p\}$$

$$\Rightarrow \int_X |f(x)+g(x)|^p\,dx \le 2^p\left\{\int_X |f(x)|^p dx+\int_X |g(x)|^p dx\right\}$$

$$\Rightarrow \int_X |f(x)+g(x)|^p\,dx < \infty$$

$\Rightarrow$ $f+g \in L_p$ if $p>0$

$\Rightarrow$ L_p is closed under addition.

Next we show that L_p is closed under scalar multiplication.
i.e. for any scalar (say) α and $f \in L_p$. we will show that $\alpha f \in L_p$.

For this we will show that $\int_X |\alpha f|^p dx < \infty$.

Now since $f \in L_p \Rightarrow \int_X |f|^p dx < \infty$

Consider

$$\int_X |\alpha f|^p dx = \int_X |\alpha|^p |f|^p dx$$

$$= |\alpha|^p \int_X |f|^p dx < \infty$$

$$\Rightarrow \int_X |\alpha f|^p dx < \infty$$

$\Rightarrow$ $\alpha f \in L_p$.

$\Rightarrow$ $L_p = \{f : X \to \mathbb{R} ; \int_X |f|^p dx < \infty\}$ is a vector space for $p>0$.

Lp as a normed linear space:-

Definition:-

Let us define a function $\|\cdot\| : L_p \to \mathbb{R}$ by

$$\|f\|_p = \left(\int_E |f(x)|^p \, d\mu\right)^{1/p}$$

i) $\|f\|_p \geq 0$ (ii) $\|\alpha f\|_p = |\alpha| \, \|f\|_p$.

$$\Rightarrow \|\alpha f\|_p = \left(\int_E |\alpha f|^p \, d\mu\right)^{1/p}$$

$$= \left(\int_E |\alpha|^p . |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha|^{p \times 1/p} \left(\int_E |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha| \left(\int_E |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha| \, \|f\|_p .$$

Thus

$$\|\alpha f\|_p = |\alpha| \, \|f\|_p .$$

Note:- $\|x\| = 0 \iff x = 0$

i) If $f(x) = 0 \quad \forall x$

$$\Rightarrow |f(x)| = 0$$

$$\Rightarrow |f(x)|^p = 0$$

$$\Rightarrow \int_E |f(x)|^p \, dx = 0$$

$$\Rightarrow \left(\int_E |f(x)|^p \, dx\right)^{1/p} = 0$$

$$\Rightarrow \|f\|_p = 0$$

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Conversely; let $\|f\|_p = 0$

$$\Rightarrow \left(\int_E |f|^p dx\right)^{1/p} = 0$$

$$\Rightarrow \int_E |f(x)|^p dx = 0$$

$$\Rightarrow |f(x)|^p = 0 \quad a.e.$$

$$\Rightarrow |f(x)| = 0 \quad a.e.$$

$$\Rightarrow f(x) = 0 \quad a.e.$$

$$\Rightarrow f = 0$$

we have define the relation "$\approx$" in L by $f = g$ if $f(x) = g(x)$ a.e.

Thus $\|f\|_p = 0$

$$\Rightarrow f = 0.$$

iii) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

$$\left(\int_E |f(x)+g(x)|^p d\mu\right)^{1/p} \leq \left(\int_E |f|^p d\mu\right)^{1/p} + \left(\int_E |g|^p d\mu\right)^{1/p}.$$

State and prove that lebesgue integral version of Holder Inequality.

Holder Inequality:-

Statement:-

Let $p, q > 1$ such that $1/p + 1/q = 1$. If $f \in L_p$ and $g \in L_q$, then $fg \in L_1$ and $\int_E |f \cdot g| d\mu \leq \left(\int_E |f|^p d\mu\right)^{1/p} \cdot \left(\int_E |g|^q d\mu\right)^{1/q}$ —— (1)

$$\Rightarrow \|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q.$$

Proof:-

If $f = 0$ a.e and $g = 0$ a.e, then clearly eq (1) holds.

Let us assume that $f \neq 0$, $g \neq 0$ a.e, then clearly $\|f\|_p > 0$ and $\|g\|_q > 0$.

Since we know that

$a^{\lambda} \cdot b^{1-\lambda} \leq \lambda a + (1-\lambda) b$ for $\lambda \in (0,1)$, $a, b > 0$

put $\lambda = 1/p$, $1-\lambda = 1/q$, $a = \left(\frac{|f(x)|^p}{\|f\|_p} \right)^{p}$,

$b = \left(\frac{|g(x)|^q}{\|g\|_q} \right)^{1/q}$

$$\Rightarrow \frac{|f(x)|}{\|f\|_p} \cdot \frac{|g(x)|}{\|g\|_q} \leq \frac{1}{p} \left(\frac{|f(x)|}{\|f\|_p} \right)^p + \frac{1}{q} \left(\frac{|g(x)|}{\|g\|_q} \right)^q$$

$$\Rightarrow \frac{|f(x) \cdot g(x)|}{\|f\|_p \cdot \|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$

Taking integral

$$\frac{1}{\|f\|_p \cdot \|g\|_q} \int_E |f(x) \cdot g(x)| \, du \leq \frac{1}{p \|f\|_p^p} \int_E |f(x)|^p \, du + \frac{1}{q \|g\|_q^q} \int_E |g(x)|^q \, du$$

$$= \frac{1}{p \|f\|_p^p} \|f\|_p^p + \frac{1}{q \|g\|_q^q} \|g\|_q^q$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{\|f\|_p \cdot \|g\|_q} \int_E |f(x) \cdot g(x)| \, du \leq 1$$

$$\Rightarrow \int_E |f(x) \cdot g(x)| \, du \leq \|f\|_p \cdot \|g\|_q$$

$$\Rightarrow \|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$$. proved!

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Minkowski's Inequality:-

Lebesgue integral version of Minkowski's inequality.

Let $1 \leq p < \infty$ and $f, g \in L_p$.

Then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

i.e.

$$\left(\int_E |f+g|^p d\mu\right)^{1/p} \leq \left(\int_E |f|^p d\mu\right)^{1/p} + \left(\int_E |g|^p d\mu\right)^{1/p}.$$

Proof:-

Since $f, g \in L^p$

$$\Rightarrow \int_E |f|^p d\mu < \infty \quad , \quad \int_E |g|^p d\mu < \infty$$

Now as $|f+g| \leq |f|+|g|$

$$\Rightarrow |f+g| \leq 2 \max\{|f|, |g|\}$$

$$\Rightarrow |f+g|^p \leq 2^p \max\{|f|^p, |g|^p\}$$

$$\Rightarrow |f+g|^p \leq 2^p\{|f|^p + |g|^p\}$$

$$\Rightarrow \int_E |f+g|^p d\mu \leq 2^p\left\{\int_E |f|^p d\mu + \int_E |g|^p d\mu\right\} < \infty$$

as $f, g \in L_p$.

$$\Rightarrow f+g \in L_p \quad (\because \text{by def: of } L_p)$$

Now consider

$$\|f+g\|_p^p = \int_E |f+g|^p d\mu = \int_E |f+g|^{p-1} \cdot |f+g| \, d\mu$$

$$\leq \int_E |f+g|^{p-1}(|f|+|g|) \, d\mu.$$

$$= \int_E |f| \, |f+g|^{p-1} d\mu + \int_E |g| \, |f+g|^{p-1} d\mu$$

$$\|f+g\|_p^p \leq \int_E |f| \, |f+g|^{p-1} d\mu + \int_E |g| \, |f+g|^{p-1} d\mu$$

Applying Holder inequality.

$$\|f+g\|_p^p \le \left(\int_E |f|^p d\mu\right)^{1/p} \cdot \left(\int_E |f+g|^{q(p-1)} d\mu\right)^{1/q} + \left(\int_E |g|^p d\mu\right)^{1/p} \cdot \left(\int_E |f+g|^{q(p-1)} d\mu\right)^{1/q}$$

$\frac{1}{p}+\frac{1}{q}=1$

$\frac{q+p}{pq}=1$

$\frac{q}{q}+\frac{p}{q}=q$

$1+\frac{p}{q}=q$

$\frac{p}{q}=q-1$

$p=(p-1)q$

where $\frac{1}{p}+\frac{1}{q}=1$

$$\Rightarrow \|f+g\|_p^p \le \|f\|_p \cdot \left(\int_E |f+g|^p d\mu\right)^{1/q} + \|g\|_p \left(\int_E |f+g|^p d\mu\right)^{1/q}$$

$$\Rightarrow \|f+g\|_p^p \le \left(\int_E |f+g|^p d\mu\right)^{1/q} \left[\|f\|_p + \|g\|_p\right]$$

$$\le \left(\int_E |f+g|^p d\mu\right)^{\frac{1}{p}\times\frac{p}{q}} \left[\|f\|_p + \|g\|_p\right]$$

$$\le \left(\|f+g\|_p\right)^{p/q} \left[\|f\|_p + \|g\|_p\right]$$

$$\Rightarrow \|f+g\|_p^p \le \|f+g\|_p^{p/q} \left[\|f\|_p + \|g\|_p\right]$$

$$\frac{\|f+g\|_p^p}{\|f+g\|_p^{p/q}} \le \|f\|_p + \|g\|_p$$

$$\|f+g\|_p^{p-p/q} \le \|f\|_p + \|g\|_p$$

$p-p/q = p(1-\frac{1}{q})$

$= p(\frac{1}{p})$

$= 1$

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

which is the required proof.

Note:- $L_p = \{f : \int_E |f|^p d\mu < \infty\}$

L_p is vector space if $p>0$

L_p is normed linear space if $p \ge 1$.

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Essential Bounded:-

Let f be a function defined on E, then a number $M \in \mathbb{R}$ is said to be essential bound if $|f(x)| \leq M$ a.e. If f has essential bound then f is said to be essential bounded function.

The essential supremum of a function f is denoted by $\|f\|_\infty$ and is defined by $\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e}\}$.

The collection of all measurable and essential bounded function is denoted by L_∞ or $L_\infty(\mu)$ or $L_\infty(E)$.

And L_∞ is normed linear space under the normed defined

$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e}\}$.

Note:-

$\Rightarrow L_p = \{f : \int_E |f|^p d\mu < \infty\}$ for $p \geq 1$,

L_p is normed linear space.

$$\|f\|_p = \left(\int_E |f|^p d\mu\right)^{1/p}$$

$\Rightarrow L_\infty = \{f : f \text{ is essential bounded}\}$

$\|f\|_\infty = \inf\{M : |f(x)| < M \text{ a.e}\}$

Theorem:-

If $\mu(E) < \infty$, then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

Proof:-

Since $\|f\|_\infty = \inf\{\alpha : \mu\{x \in E : |f(x)| > \alpha\} = 0\}$

let us suppose $\|f\|_\infty = M$

$$\mu\{x \in E : |f(x)| > M\} = 0$$

i.e $|f(x)| \leq M$ a.e.

we need to prove that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$

Clearly $M - 1/p < M$ for any $p > 0$

By definition of essential supremum that M is the infimum value such that

$$\mu\{x ; |f(x)| > M\} = 0$$

But since $M - 1/p < M$, so

$$\Rightarrow \mu\{x ; |f(x)| > M - 1/p\} > 0$$

(because $\mu(A) \not< 0$ for any set A)

let $A = \{x \in E : |f(x)| > M - 1/p\}$

Now $\|f\|_p = \left(\int_E |f|^p d\mu\right)^{1/p} \geq \left(\int_A |f|^p d\mu\right)^{1/p}$ ($\because A \subseteq E$)

$$> \left(\int_A \left(M - \frac{1}{p}\right)^p d\mu\right)^{1/p}$$

$$= \left(M - 1/p\right)^{p \times 1/p} \left(\int_A d\mu\right)^{1/p}$$

$$= \left(M - \frac{1}{p}\right)(\mu(A))^{1/p}$$

$$\Rightarrow \|f\|_p > \left(M - \frac{1}{p}\right)(\mu(A))^{1/p}$$

Taking limit inf on b.s

$$\liminf \|f\|_p \geq \liminf\left(M - 1/p\right) \cdot \liminf (\mu(A))^{1/p}$$

$$\geq (M - 0) \cdot 1 = M$$

$$\liminf \|f\|_p \geq M \quad \text{——} \quad (1)$$

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Also $\|f\|_p = \left(\int_E |f|^p d\mu\right)^{1/p}$

$$\leq \left(\int_E M^p d\mu\right)^{1/p}$$

$$\leq M^{p \times 1/p} \left(\int_E d\mu\right)^{1/p}$$

$$\leq M(\mu(E))^{1/p}$$

$\Rightarrow \|f\|_p \leq M(\mu(E))^{1/p}$

Taking limit Sup on both sides

$$\lim \sup \|f\|_p \leq \lim \sup \left[M(\mu(E))^{1/p}\right]$$

$$\leq M \lim \sup (\mu(E))^{1/p}$$

$$= M(\mu(E))^{\lim_{p\to\infty}(1/p)}$$

$$= M(\mu(E))^{1/\infty}$$

$$= M(\mu(E))^0$$

$$= M \cdot 1$$

$$= M$$

$\Rightarrow \lim \sup \|f\|_p \leq M$ —— ②

But $\lim \inf \|f\|_p \leq \lim \sup \|f\|_p$ —— ③

From eq ①, ② and ③, we get

$\lim \sup \|f\|_p \leq M \leq \lim \inf \|f\|_p \leq \lim \sup \|f\|_p$

$\Rightarrow \lim \sup \|f\|_p \leq M \leq \lim \sup \|f\|_p$.

$\Rightarrow \lim \sup \|f\|_p = M$

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$$\lim_{p\to\infty} \|f\|_p = M = \|f\|_\infty$$

Thus

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$$

proved!

Example:-

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p \quad ; \text{ if } \mu(E) < \infty$$

If $\mu(E) < \infty$, then the above result fails.

Sol:-

Let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = c$; $\forall\ x \in (0,\infty)$ and $c \neq 0$

where $E = (0,\infty)$

$\mu(E) = \infty$

Then

$$\text{ess Sup } f = \|f\|_\infty = \inf\{M ; |f(x)| \leq M \text{ a.e}\}$$
$$= c$$

$$\Rightarrow \text{ ess Sup } f = \|f\|_\infty = c \quad \text{---} \ (1)$$

Also since

$$\|f\|_p = \left(\int_E |f(x)|^p \, d\mu\right)^{1/p}$$

$$\lim_{p\to\infty} \|f\|_p = \left(\int_0^\infty |c|^p \, d\mu\right)^{1/p}$$

$$= |c|^{p\cdot 1/p} \cdot \left(\int_0^\infty d\mu\right)^{1/p}$$

$$= |c| \cdot \left(\mu(E)\Big|_0^\infty\right)^{1/p}$$

$$= \infty$$

$\lim\limits_{p \to \infty} \|f\|_p = \infty$ —— (2)

From eq (1) and (2), we get

$$\lim_{p \to \infty} \|f\|_p \neq \|f\|_\infty .$$

Holder Inequality for L^∞:-

If $f \in L^\infty$ and $g \in L^1$, then

$\|f \cdot g\| \leq \|f\|_\infty \cdot \|g\|_1$.

Proof:-

Let us suppose $\|f\|_\infty = M$

Since $\|f(x)\| \leq \|f\|_\infty = M$ a.e

$\Rightarrow \|f\| \leq M$ a.e.

Now

$|f \cdot g| \leq |f(x)| \cdot |g(x)|$ a.e.

$|fg| \leq M \; |g(x)|$ a.e.

$\int_E |f \cdot g| \, d\mu \leq M \int_E |g(x)| \, d\mu$

$\Rightarrow \left(\int_E |f(x) \; g(x)|^1 d\mu \right)^1 \leq M \left(\int_E |g(x)|^1 d\mu \right)^1$

$\Rightarrow \|f \cdot g\|_1 \leq M \|g\|_1$

$\Rightarrow$ But $M = \|f\|_\infty$

So $\|f \cdot g\|_1 \leq \|f\|_\infty \|g\|_1$.

Proved!!

OR Holder Inequality for L^p:-

$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q$

If $p \to \infty$ then $q = 1$

$\|f \cdot g\|_1 \leq \|f\|_\infty \|g\|_1$ proved

$\because \frac{1}{p} + \frac{1}{q} = 1$

$\frac{1}{\infty} + \frac{1}{q} = 1$

$0 + 1/q = 1 \Rightarrow 1/q = 1$

$\Rightarrow \boxed{q = 1}$

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Minkowski's Inequality for L^∞:-

If $f, g \in L_p$; $p \geq 1$, then $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Proof:-

Since $|f(x)| \leq \|f\|_\infty$ a.e.

and $|g(x)| \leq \|g\|_\infty$ a.e.

Now

$$|f(x)+g(x)| \leq |f(x)| + |g(x)|$$

$$\leq \|f\|_\infty + \|g\|_\infty$$

$$\Rightarrow |f(x)+g(x)| \leq \|f\|_\infty + \|g\|_\infty \quad \text{a.e} \quad \text{--- (1)}$$

But $|f(x)+g(x)| \leq \|f+g\|_\infty$ a.e --- (2)

From eq (1) and (2), we get

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

proved!

Note:-

Holder inequality for L_p.

$$\int_E |f.g|\, du \leq \left(\int_E |f|^p du\right)^{1/p} \cdot \left(\int_E |g|^q du\right)^{1/q}$$

$$\|f.g\|_1 \leq \|f\|_p \cdot \|g\|_q$$

put $p = q = 2$

$$\int_E |f.g|\, du \leq \left(\int_E |f|^2 du\right)^{1/2} \cdot \left(\int_E |g|^2 du\right)^{1/2}$$

$$\|f.g\|_1 \leq \|f\|_2 . \|g\|_2$$

It is known as Cauchy Schwarz inequality

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Theorem:-

Let (X, S, μ) be finite Lebesgue measure space i.e. $\mu(X) < \infty$, then for $1 \le p < q < \infty$, we have
$$L_q(\mu) \subseteq L_p(\mu).$$

Proof:-

Since $p < q \Rightarrow q > p$

$\Rightarrow \frac{q}{p} > 1$

let $r = q/p$.

let us choose "s" such that
$$\frac{1}{r} + \frac{1}{s} = 1$$

we need to prove that $L_q(\mu) \subseteq L_p(\mu)$.
let us suppose that $f \in L_q(\mu)$
For this we need to show that $f \in L_p(\mu)$.

If $f \in L_q(\mu)$.

$\Rightarrow \int_X |f|^q d\mu < \infty$ $\quad \because r = q/p$, $rp = q$

$\Rightarrow \int_X |f|^{rp} d\mu < \infty$

$\Rightarrow \int_X (|f|^p)^r d\mu < \infty$

$\Rightarrow |f|^p \in L_r$

and $1 \in L_s(X)$ as $\int_X 1\, d\mu = \mu(X) < \infty$

$\Rightarrow 1 \in L_s$ and $|f|^p \in L_r$ and $\frac{1}{r} + \frac{1}{s} = 1$.

$f \in L_p,\ g \in L_q$
$|f \cdot g| \in L_1$

$\Rightarrow |f|^p \cdot 1 \in L_1$
$\Rightarrow |f|^p \in L_1$

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$\Rightarrow \int_X (|f|^r)^{'} d\mu < \infty$

$\Rightarrow \int_X |f|^p d\mu < \infty$

$\Rightarrow f \in L_p$

or

$\Rightarrow f \in L_p(\mu)$.

So $\quad L_q(\mu) \subseteq L_p(\mu)$.

proved!

Convergence in L_p (or Mean Convergence):-

Let $\{f_n\}$ be a sequence of functions in L_p then we say f_n converge to $f \in L_p$ or f_n converge in measure to $f(x)$, if

$$\lim_{n\to\infty} \int_E |f_n(x) - f(x)|^p d\mu = 0$$

or

$$f_n \to f \quad \text{in } L_p$$

If $\|f_n(x) - f(x)\|_p < \epsilon \quad \forall \quad n \geq N_0$.

We say that $\{f_n\}$ is cauchy sequence in L_p.

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_E |f_n(x) - f_m(x)|^p dx = 0 \quad ; \quad m, n \geq N.$$

Theorem:-

If $f_n \in L_p$ and $\{f_n\}$ converges in mean or L_p then $\lim_{n\to\infty} f_n(x)$ is unique.

Proof:-

let us suppose that

$\lim_{n\to\infty} f_n(x) = f(x)$ and $\lim_{n\to\infty} f_n(x) = g(x)$

where $f \neq g$.

Now $\left(\int_E |f-g|^p d\mu\right)^{1/p} = \left(\int_E |f-f_n+f_n-g|^p d\mu\right)^{1/p}$

$\leq \left(\int_E |f-f_n|^p d\mu\right)^{1/p} + \left(\int_E |f_n-g|^p d\mu\right)^{1/p}$

$\lim_{n\to\infty}\left(\int_E |f-g|^p d\mu\right)^{1/p} \leq \lim_{n\to\infty}\left(\int_E |f-f_n|^p d\mu\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |f_n-g|^p d\mu\right)^{1/p}$

$\left(\lim_{n\to\infty}\int_E |f-g|^p d\mu\right)^{1/p} \leq \left(\lim_{n\to\infty}\int_E |f_n-f|^p d\mu\right)^{1/p} + \left(\lim_{n\to\infty}\int_E |f_n-g|^p d\mu\right)^{1/p}$

$\Rightarrow \left(\int_E |f-g|^p d\mu\right)^{1/p} \leq (0)^{1/p} + (0)^{1/p} = 0$

$\Rightarrow \left(\int_E |f-g|^p d\mu\right)^{1/p} = 0$

$\Rightarrow \int_E |f-g|^p d\mu = 0$

$\Rightarrow |f-g|^p = 0$

$\Rightarrow |f-g| = 0$ a.e.

$\Rightarrow f-g = 0$ a.e.

$\Rightarrow f = g$ a.e.

$\Rightarrow f(x) = g(x)$ a.e.

Thus the $\lim_{n\to\infty} f_n(x)$ is unique.

Theorem:-

Show that L^p is a metric space.

Proof:-

Let us define D on $L^p \times L^p$ by

$D(f,g) = \|f-g\|_p$

We need to show that D is a metric space.

i) Clearly $D \geq 0$.

ii) $D(f,g) = 0 \iff \|f-g\|_p = 0$

$$\iff \left(\int_E |f-g|^p du\right)^{1/p} = 0$$

$$\iff \int_E |f-g|^p du = 0$$

$$\iff |f-g|^p = 0$$

$$\iff |f-g| = 0$$

$$\iff f-g = 0$$

$$\iff f = g$$

Thus $D(f,g) = 0 \iff f = g$

iii) $D(f,g) = \|f-g\|_p$

$$= \|g-f\|_p$$

$$= D(g,f)$$

Thus $D(f,g) = D(g,f)$

iv) If $f, g, h \in L_p$, then

$D(f,g) \leq D(f,h) + D(h,g)$

$$D(f,g) = \|f-g\|_p$$
$$= \|f-h+h-g\|_p$$
$$\leq \|f-h\|_p + \|h-g\|_p$$
$$\leq D(f,h) + D(h,g)$$

$\Rightarrow D(f,g) \leq D(f,h) + D(h,g)$.

Thus D satisfies all the conditions of metric space.

Hence D is a metric space.

Theorem:-

If $f_n \to f$ in L^p and $g_n \to g$ in L^p then $f_n + g_n \to f+g$ in L^p.

Proof:-

Given that

$\Rightarrow f_n \to f$ in L^p

i.e $\lim_{n\to\infty} \int_E |f_n(x) - f(x)|^p dx = 0$

and

$\Rightarrow g_n \to g$ in L^p

i.e $\lim_{n\to\infty} \int_E |g_n(x) - g(x)|^p dx = 0$

we need to prove that $f_n + g_n \to f+g$ in L^p

Now

$$\left(\int_E |f_n + g_n - (f+g)|^p dx\right)^{1/p} = \left(\int_E |f_n - f + g_n - g|^p dx\right)^{1/p}$$

$$\left(\int_E |f_n + g_n - (f+g)|^p dx\right)^{1/p} \leq \left(\int_E |f_n - f|^p dx\right)^{1/p} + \left(\int_E |g_n - g|^p dx\right)^{1/p}$$

Taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty}\left(\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n-f|^p dx\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |g_n-g|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n-f|^p dx\right)^{1/p} + \left(\lim_{n\to\infty}\int_E |g_n-g|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} \le (0)^{1/p} + (0)^{1/p} = 0$$

$$\left(\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} = 0$$

$$\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx = 0$$

Hence $f_n+g_n \longrightarrow f+g$ in L_p.

Theorem:-

If $\lim_{n\to\infty} f_n(x) = f(x)$ in L_p (or mean converges), then prove that $\lim_{n\to\infty}\int_E |f_n(x)|^p dx = \int_E |f(x)|^p dx$.

Equivalently $\lim_{n\to\infty} \|f_n\| = \|f\|$.

Proof:-

Given that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ in } L_p$$

i.e.

$$\lim_{n\to\infty}\int_E |f_n(x)-f(x)|^p dx = 0$$

We need to prove that

$$\lim_{n\to\infty}\int_E |f_n(x)|^p dx = \int_E |f(x)|^p dx.$$

Now

$$\left(\int_E |f_n(x)|^p dx\right)^{1/p} = \left(\int_E |f_n(x)-f(x)+f(x)|^p dx\right)^{1/p}$$

$$\le \left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \left(\int_E |f_n(x)|^p dx\right)^{1/p} \le \left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

Taking limit $n \to \infty$ on b.s

$$\lim_{n\to\infty}\left(\int_E |f_n(x)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p} \le (0)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p} \le \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\lim_{n\to\infty}\int_E |f_n(x)|^p dx \le \int_E |f(x)|^p dx \quad \text{——— (1)}$$

Now

$$\left(\int_E |f(x)|^p dx\right)^{1/p} = \left(\int_E |f(x)-f_n(x)+f_n(x)|^p dx\right)^{1/p}$$

$$\le \left(\int_E |f(x)-f_n(x)|^p dx\right)^{1/p} + \left(\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \left(\int_E |f(x)|^p dx\right)^{1/p} \le \left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\int_E |f_n(x)|^p dx\right)^{1/p}$$

Taking limit $n \to \infty$ on b.s

$$\lim_{n\to\infty}\left(\int_E |f(x)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\left(\int_E |f(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\left(\int_E |f(x)|^p dx\right)^{1/p} \le (0)^{1/p} + \left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\left(\int_E |f(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \int_E |f(x)|^p dx \le \lim_{n\to\infty}\int_E |f_n(x)|^p dx \quad \text{——— (2)}$$

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From eq ① and eq ②, we get

$$\lim_{n\to\infty} \int_E |f_n(x)|^p dx = \int_E |f(x)|^p dx$$

proved!

Theorem:-

If $\{f_n(x)\}$ converges to $f(x)$ in L^p (f_n in mean convergent). Prove that $\{f_n(x)\}$ is cauchy sequence in L^p.

Proof:-

Since $\{f_n(x)\}$ converges to $f(x)$ in L^p

i.e $f_n \to f$ in L^p

$$\Rightarrow \lim_{n\to\infty} \int_E |f_n - f|^p dx = 0$$

Let $f_m \to f$ in L^p.

$$\Rightarrow \lim_{m\to\infty} \int_E |f_m - f|^p dx = 0$$

We need to prove that $\{f_n(x)\}$ is cauchy sequence in L^p.

Consider

$$\left(\int_E |f_n(x) - f_m(x)|^p dx\right)^{1/p} = \left(\int_E |f_n(x) - f(x) + f(x) - f_m(x)|^p dx\right)^{1/p}$$

$$\leq \left(\int_E |f_n(x) - f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x) - f_m(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \left(\int_E |f_n(x) - f_m(x)|^p dx\right)^{1/p} \leq \left(\int_E |f_n(x) - f(x)|^p dx\right)^{1/p} + \left(\int_E |f_m(x) - f(x)|^p dx\right)^{1/p}$$

Taking limit $n\to\infty$, $m\to\infty$ on b.s

$$\lim_{\substack{n\to\infty\\ m\to\infty}}\left(\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \lim_{m\to\infty}\left(\int_E |f_m(x)-f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{\substack{n\to\infty\\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\lim_{m\to\infty}\int_E |f_m(x)-f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{\substack{n\to\infty\\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} \le (0)^{1/p} + (0)^{1/p} = 0$$

$$\left(\lim_{\substack{n\to\infty\\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} = 0$$

$$\Rightarrow \lim_{\substack{n\to\infty\\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx = 0$$

$$\Rightarrow \lim_{\substack{n\to\infty\\ m\to\infty}}\int_E |f_n(x)-f(x)|^p dx = 0$$

Hence $\{f_n(x)\}$ is cauchy sequence in L^p.

proved!

Convergent in Measure:-

Let $M = \{f : f \text{ is measurable function}\}$

A sequence $\{f_n(x)\}$ converges to $f \in M$ in measure, we write $f_n \xrightarrow{\mu} f$ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \ge \epsilon\}) = 0$$

Note:-

$\Rightarrow x \in \{x : f(x) + g(x) \ge \epsilon\}$

$\Rightarrow x \in \{x : f(x) \ge \epsilon\} \cup \{x : g(x) \ge \epsilon\}$

Theorem:-

If $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ (f_n converges to f in measure, g_n converges to g in measure), then prove that

$\alpha f_n(x) + \beta g_n(x) \xrightarrow{\mu} \alpha f(x) + \beta g(x)$

Proof:-

Consider

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let $A = \{x : |\alpha f_n(x) + \beta g_n(x) - (\alpha f(x) + \beta g(x))| \geq \epsilon\}$
where $\alpha, \beta \neq 0$

$A = \{x : |\alpha f_n(x) - \alpha f(x) + \beta g_n(x) - \beta g(x)| \geq \epsilon\}$

$A = \{x : |\alpha (f_n(x) - f(x)) + \beta (g_n(x) - g(x))| \geq \epsilon\}$

$A = \{x : |\alpha| |f_n(x) - f(x)| + |\beta| |g_n(x) - g(x)| \geq \epsilon\}$

$A \subseteq \{x : |\alpha| |f_n(x) - f(x)| \geq \epsilon/2\} \cup \{x : |\beta| |g_n(x) - g(x)| \geq \epsilon/2\}$

$A \subseteq \{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\} \cup \{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2|\beta|}\}$

Taking measure (μ) on b.s.

$\mu(A) \leq \mu\left(\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\}\right) + \mu\left(\{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2|\beta|}\}\right)$

Taking limit $n \to \infty$ on b.s

$\lim_{n\to\infty} \mu(A) \leq \lim_{n\to\infty} \mu\left(\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\}\right) + \lim_{n\to\infty} \mu\left(\{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2|\beta|}\}\right)$

$\lim_{n\to\infty} \mu(A) \leq 0 + 0 = 0$

$\lim_{n\to\infty} \mu(A) = 0$

$\lim_{n\to\infty} \mu\left(\{x : |\alpha f_n(x) + \beta g_n(x) - (\alpha f(x) + \beta g(x))| \geq \epsilon\}\right) = 0$

Thus $\alpha f_n(x) + \beta g_n(x) \xrightarrow{\mu} \alpha f(x) + \beta g(x)$ in L'

proved!

Note:-

If $f(x) \leq g(x)$

let $A = \{x : f(x) \geq \epsilon\}$, $B = \{x : g(x) \geq \epsilon\}$

If $x \in A \Rightarrow f(x) \geq \epsilon$

but $|g(x)| \geq f(x) \geq \epsilon$

So $g(x) \geq \epsilon$

$\Rightarrow x \in B$

Thus $A \subseteq B$.

Theorem:-

If $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, then $f = g$ a.e.

Proof:-

Given that

$\Rightarrow f_n \xrightarrow{\mu} f$ i.e. $\lim\limits_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$

and

$\Rightarrow g_n \xrightarrow{\mu} g$ i.e. $\lim\limits_{n\to\infty} \mu(\{x : |g_n(x) - g(x)| \geq \epsilon\}) = 0$

Consider

$|f(x) - g(x)| = |f(x) - f_n(x) + f_n(x) - g(x)|$

$|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$

$|f(x) - g(x)| \leq |f_n(x) - f(x)| + |f_n(x) - g(x)|$ —— (1)

Now

$\{x : |f(x) - g(x)| \geq 2\epsilon\} \subseteq \{x : |f_n(x) - f(x)| + |f_n(x) - g(x)| \geq 2\epsilon\}$ using eq (1)

$\{x : |f(x) - g(x)| \geq 2\epsilon\} \subseteq \{x : |f_n(x) - f(x)| \geq \epsilon\} \cup \{x : |f_n(x) - g(x)| \geq \epsilon\}$

Taking measure (μ) on b.s

$\mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) \leq \mu\{\{x : |f_n(x) - f(x)| \geq \epsilon\}) + \mu(\{x : |f_n(x) - g(x)| \geq \epsilon\})$

Taking limit $n \to \infty$.

$\lim\limits_{n\to\infty} \mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) \leq \lim\limits_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\})$

$+ \lim\limits_{n\to\infty} \mu(\{x : |f_n(x) - g(x)| \geq \epsilon\})$

$$\lim_{n\to\infty} \mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) < 0 + 0 = 0$$

$\because f_n \to f,\ f_n \to g$

$$\lim_{n\to\infty} \mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) = 0$$

$$\mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) = 0$$

$\Rightarrow \quad f(x) - g(x) = 0 \quad a.e$

$\Rightarrow \quad f(x) = g(x) \quad a.e.$

$\Rightarrow \quad f = g \quad a.e$

proved!

Theorem:-

let $\langle f_n(x) \rangle$ be a sequence of integrable function such that $f_n(x)$ converges to $f(x)$ in means, then show that $f_n \xrightarrow{\mu} f$.

Proof:-

Since $\{f_n\}$ converges to $f(x)$ in means

So $\lim_{n\to\infty} \int_E |f_n(x) - f(x)|^p dx = 0$

We show that $f_n \xrightarrow{\mu} f$.

i.e we show that $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \delta\}) = 0$

let $E_n = \{x : |f_n(x) - f(x)| \geq \delta\}$

Clearly $E_n \subseteq E$

$\Rightarrow \quad |f_n(x) - f(x)| \geq \delta \quad \forall \quad x \in E_n$

$\Rightarrow \quad |f_n(x) - f(x)|^p \geq \delta^p \quad \forall \quad x \in E_n$

$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p dx \geq \int_{E_n} \delta^p dx$

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$$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p \, dx \geq \delta^p \int_{E_n} dx = \delta^p \mu(E_n)$$

$$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p \, dx \geq \delta^p \mu(E_n) \quad \text{—— (1)}$$

Since $E_n \subseteq E$

$$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p \, dx \leq \int_{E} |f_n(x) - f(x)|^p \, dx$$

$$\Rightarrow \delta^p \mu(E_n) \leq \int_{E_n} |f_n(x) - f(x)|^p \, dx \leq \int_{E} |f_n(x) - f(x)|^p \, dx$$

using eq (1).

$$\Rightarrow \delta^p \mu(E_n) \leq \int_{E} |f_n(x) - f(x)|^p \, dx$$

Taking Limit $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \delta^p \mu(E_n) \leq \lim_{n\to\infty} \int_{E} |f_n(x) - f(x)|^p \, dx$$

$$\Rightarrow \lim_{n\to\infty} \delta^p \mu(E_n) \leq 0$$

$$\Rightarrow \delta^p \lim_{n\to\infty} \mu(E_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} \mu(E_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \delta\}) = 0$$

Hence $f_n \xrightarrow{\mu} f$.

proved!

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