MATHTMATICAL METHODS

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Dedicated

To

My Honorable Teacher

Sir Muhammad Awais Aun

&

My Parents

Lecture # 01

Differential equations:

Many physical phenomena are described by a function whose value at a given point depends on its value at neighboring points. The equation determining this function thus contains derivatives of the function such as first derivative to indicate the slope at a point, a second derivative to indicate the curvature and so on. Such an equation is called a differential equation. Familiar physical situations that are described by differential equations are the flow of a fluid, the vibrations of a drum head and the dissipation of heat in a material.

An equation containing / involving one or more derivatives of an unknown function is called a differential equation. There are two basic types of differential equations

(i) Ordinary differential equations (ODEs)
(ii) Partial differential equations (PDEs)

They are distinguished by the number of independent variables that enter / are involved in the equation. A differential equation for a function of a single independent variable contains only ordinary derivatives of the function and is called an ordinary differential equation. For a function of two or more independent variables, a partial differential equation expresses a relation among partial derivatives of that function.

The general form of an ODE for a function y of an independent variable can be written in terms of F of the arguments x and y together with the derivatives of y as:

$$F\left(x, y, y', y'', ...\right) = 0 \qquad (1)$$

Where prime stands for derivatives of y w.r.t x i.e.

$$y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, ..., y^n = \frac{d^n y}{dx^n}$$

Order: The order of a differential equation is the order of the highest derivative appearing as an argument of the equation. For example, the most general form of a first order ordinary differential equation is

$$F\left(x, y, y'\right) = 0 \qquad (2)$$

The general form of an nth-order ordinary differential equation is there given by the expression:

$$F\left(x, y, y', y'', ... y^{(n)}\right) = 0 \qquad (3)$$

If the function F in (3) is a polynomial in the highest order derivative of y appearing in the argument list, then the degree of the differential equation is the power to which the highest derivative raided i.e. the degree of the polynomial. An equation is said to be linear if F is first degree in y and in each of the derivatives appearing as an argument of F.

The general form of a linear nth order ordinary differential equation is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \qquad \_\_\_\_(4)$$

Where f(x) and the coefficients $a_0(x), a_1(x), ..., a_n(x)$ are known functions.

Examples: Consider the following ODEs:

$$\frac{d^2y}{dx^2} + y - \in a_n(x)\frac{dy}{dx}\left(1 - y^2\right) = 0 \qquad \_\_\_(i)$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \qquad \_\_\_(ii)$$

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} + y = 0 \qquad \_\_\_(iii)$$

In equation (i), $\in$ is a constant. This equation (Van der Pol equation) is of second order and of first degree. This is a non-linear equation because of the term: $\frac{dy}{dx}y^2$.

(An equation that not linear is called a non-linear equation) i.e.
$\frac{d^2y}{dx^2} + \frac{g}{L}\sin y = 0$ is a non-linear equation? Equation (ii) is a particular case of

Bessel's equation and is seen to be a linear second order ODE. Equation (iii) is of first order and second degree is there is a nonlinear equation.

Solution of a Differential Equation:

A function y = g(x) is called a solution of a given nth order ODE (4) on some interval (say) a < x < b if g(x) is defined and n times differentiable throughout that interval and is such that the equation becomes an identity when y and its derivatives $y', y'', \ldots. y^{(n)}$ are replaced by $g', g'', \ldots. g^{(n)}$ respectively.

Equation (4) is called homogeneous if f(x) = 0. Otherwise it is called non-homogeneous.

Linearly Independent Functions and the Wronskian:

Two functions $f_1(x)$ and $f_2(x)$ are said to linearly dependent on an interval I = [a,b] if there exist two constants c_1 and c_2 not both zero such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \qquad (1)$$

For all $x \in I$

Two functions $f_1(x)$ and $f_2(x)$ are said to linearly independent on an interval I = [a,b] if there exist no linear combination of the two functions that vanishes over the interval i.e. if the only choice of constants c_1 and c_2 that satisfies

$$c_1 f_1(x) + c_2 f_2(x) = 0 \qquad \text{over the interval } c_1 = 0 \text{ and } c_2 = 0$$

Criterion for Linear Independent Functions:

The idea of linearly independent functions can be extended to any number of functions. Consider a set of n functions $\{f_1, f_2, \ldots.. f_n\}$ over an interval. These functions satisfy the following relation.

$$c_1 f_1 + c_2 f_2 + \ldots.. + c_n f_n = 0 \qquad (1)$$

For the constants c_k , k = 1,2,3,….n

To find a way of identifying linearly independent of the above set of functions, suppose that these functions form a set of linearly dependent set over the interval. So there exist a set of constants c_k , k = 1,2,3,….n such that the relation (1) is satisfied. It is also supposed that each these functions can be differentiated n−1 times with respect to x. Equation (1) is satisfied over the entire in question and the functions have the required number of derivatives. So up to n−1 derivatives of the above equation may be taken to obtain the n−1 equation as:

$$c_1 \frac{d^k f_1}{dx^k} + c_2 \frac{d^k f_2}{dx^k} + \dots\dots + c_n \frac{d^k f_n}{dx^k} = 0 \qquad (2)-(n-1)$$

For k = 1,2,3,…n−1

Thus Equations (1)–(n) form a system of n linearly homogeneous equations for the constants c_k. For these equations to have a solution with not all of the c_k being equal to zero, the determinant of the matrix of coefficients must vanish:

$$\begin{vmatrix} f_1 & f_2 & . & . & . & f_n \\ f_1^{(1)} & f_2^{(1)} & . & . & . & f_n^{(1)} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ f_1^{(n-1)} & f_2^{(n-1)} & . & . & . & f_n^{(n-1)} \end{vmatrix} = 0$$

This determinant is called Wronskian of the functions. f_k and is denoted by $W(f_1, f_2, \dots\dots f_n)$ *i*.e.,

$$W(f_1, f_2, \dots\dots f_n) = \begin{vmatrix} f_1 & f_2 & . & . & . & f_n \\ f_1^{(1)} & f_2^{(1)} & . & . & . & f_n^{(1)} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ f_1^{(n-1)} & f_2^{(n-1)} & . & . & . & f_n^{(n-1)} \end{vmatrix}$$

Since the vanishing of $W(f_1, f_2, \dots\dots f_n)$ is a necessary condition for the functions f_k to be linearly dependent, then the sufficient condition for the functions to be linearly independent is that the Wronskian is non-vanishing.

E.g, the sufficient condition for linearly independence of two non-zero function f_1 & f_2 takes the following form: $W(f_1, f_2) = f_1 f_2^{'} - f_1^{'} f_2 \neq 0$.

If $W(f_1, f_2) = f_1 f_2^{'} - f_1^{'} f_2 = 0$ then it implies

$\frac{f_1^{'}}{f_1} = \frac{f_2^{'}}{f_2} \Rightarrow f_1 = cf_2$ This shows that f_1 is proportional to f_2 . f_1 and f_2 are linearly dependent. Consider the functions $f_1 = e^{-x}$, $f_2 = e^x$ *and* $f_3 = e^{2x}$. The Wronskian is

$$W(f_1, f_2, f_3) = \begin{vmatrix} e^{-x} & e^x & e^{2x} \\ -e^{-x} & e^x & 2e^{2x} \\ e^{-x} & e^x & 4e^{2x} \end{vmatrix} = 6e^{2x} \neq 0$$

This shows that function are linearly independent.

General Solution:

A solution of a differential equation of nth order (linear or not) is called a general solution if it contains n arbitrary independent constants. Here independence means that the solution cannot be reduced to a form containing less than n arbitrary constants. If definite values are assigned to the n constants, then the solution so obtained is called a particular solution of that equation. A set of n linearly independent solutions $f_1(x), f_2(x), \ldots . f_n(x)$ of the linear homogenous equation.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

(1)

On an interval is called a basis or a fundamental system of solutions of (1) on the interval. If $f_1, f_2, \ldots . f_n$ is such a basis then the expression $y(x) = c_1 f_1(x) + c_2 f_2(x) + \ldots . + c_n f_n(x)$ is a general solution of (1).

Consider the functions $f_1 = e^{2x}$, $f_2 = \cos x$, $f_3 = \sin x$ which are solutions of the equation $y''' - 2y'' + y' - 2y = 0$. Its general solution is $y(x) = Ae^{2x} + B\cos x + C\sin x$. Because $f_1 = e^{2x}$, $f_2 = \cos x$, $f_3 = \sin x$ are linearly independent as

$$W(f_1, f_2, f_3) = \begin{vmatrix} e^{2x} & \cos x & \sin x \\ 2e^{2x} & -\sin x & \cos x \\ 4e^{2x} & -\cos x & -\sin x \end{vmatrix} = 5e^{2x} \neq 0$$

Auxiliary Conditions:

Consider an nth order linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(y) \_\_\_\_\_\_ (1)$$

The functions $a_0(x), a_1(x), a_2(x), \ldots, a_n(x)$ are called the coefficient functions of the differential equation and are generally assumed to be continuous in the interval (a,b) and $a_n(x)$ is not identically zero therein. In most of application, the unknown function, y in (1) must satisfy certain restraints called auxiliary conditions. The number of these conditions is equal to the order of the differential equation. When the conditions are satisfied at a single point, they are called initial conditions whereas they are specified at different points they are called boundary conditions. The problem of solving the equation (1) subject to initial condition is called the initial value problem. The problem of solving the equation (1) under boundary conditions is termed as the boundary value problem.

A mathematical problem is properly posed (well posed) if it satisfies the following conditions:

1. Existance: There is at least one solution
2. Uniqueness: There is at most one solution
3. Stability: The solution depends continuously on the boundary data

Separated and Mixed Boundary Conditions:

Consider a second order ordinary differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \quad \_\_\_\_\_\_ (1)$$

With typical boundary conditions having the general form

$$a_{11}y(a) + a_{12}y'(a) = \alpha \quad \_\_\_\_\_\_ (2a)$$

$$a_{21}y(b) + a_{22}y'(b) = \beta \quad \_\_\_\_\_\_ (2b)$$

a_{ij}'s ,α,β are constants where the most boundary conditions arising in practice are special cases of (2). These conditions are unmixed or separated boundary conditions because the first conditions is specified only at the end point x = a and the second only at the end point x=b.

More general form of the boundary conditions is

$$a_{11}y(a)+a_{12}y'(a)+b_{11}y(b)+b_{12}y'(b)=\alpha \quad \_\_\_\_\_\_(3a)$$

$$a_{21}y(b)+a_{22}y'(b)+b_{21}y(a)+b_{22}y'(a)=\beta \quad \_\_\_\_\_\_(3b)$$

The equation (1) is said to be homogeneous if f(x)= 0. Otherwise it is called non-homogeneous. When α=β=0 , the boundary conditions are called homogeneous. All other specifications of the differential equations or boundary conditions are called non-homogeneous.

Boundary Value Problems:

A boundary value problem is written as:

The general second order differential equation

$$a_2(x)y''+a_1(x)y'+a_0(x)y=f(x) \quad \_\_\_\_\_\_(1)$$

With unmixed boundary conditions:

$$a_{11}y(a)+a_{12}y'(a)=\alpha \quad \_\_\_\_\_\_(2a)$$

$$a_{21}y(b)+a_{22}y'(b)=\beta \quad \_\_\_\_\_\_(2b)$$

Writing the boundary value problem in the differential operator form as:

$$M=a_2(x)D^2+a_1(x)D+a_0(x) \text{ where } D=\frac{d}{dx},\ D^2=\frac{d^2}{dx^2}$$

So, equation (1) is written as:

$$M[y]=a_2(x)y''+a_1(x)y'+a_0(x)y=f(x)$$

Similarly, boundary operators B_1 and B_2 are introduced by

$$B_1[y]=a_{11}y(a)+a_{12}y'(a)M$$

$$B_2[y] = a_{21}y(b) + a_{22}y'(b)$$

In terms of these operators, the above boundary value can be expressed as

$$M[y] = f(x), B_1[y] = \alpha,\ B_2[y] = \beta$$

Definition: An operator M is said to be linear if and only if

$$M\left[c_1 f(x) + c_2 g(x)\right] = c_1 M\left[f(x)\right] + c_2 M\left[g(x)\right]$$

Where f(x) and g(x) are given functions c_1 and c_2 are any constants.

Abel's Formula:

If y_1 and y_2 are linearly independent solutions of the second order ODE:

$$y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

on some interval, where $a_1(x)$ and $a_0(x)$ are continuous show that the Wronskian $W(y_1,y_2)$ satisfies for some constant , C, $W(y_1,y_2) = C\exp\left[-a\int a_1(x)dx\right]$

Proof:

Since y_1 and y_2 are solutions of (1) then

$$y_1'' + a_1(x)y_1' + a_0(x)y_1 = 0 \qquad (2)$$

$$y_2'' + a_1(x)y_2' + a_0(x)y_2 = 0 \qquad (3)$$

Multiply (2) by $y_2(x)$ and (3) by $y_1(x)$ and subtract we get

$$y_1''y_2 + a_1(x)y_1'y_2 + a_0(x)y_1y_2 = 0$$

$$\pm y_2''y_1 \pm a_1(x)y_2'y_1 \pm a_0(x)y_2y_1 = 0$$

$$y_2''y_1 - y_1''y_2 + a_1(x)\left(y_1y_2' - y_1'y_2\right) = 0 \qquad \_\_\_\_\_ (4)$$

Now $W(y_1, y_2) = y_1y_2' - y_1'y_2$ *and* $\dfrac{dW}{dx} = y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2$

Sp, (4) becomes $\frac{dW}{dx}+a_1(x)W=0$

Integration gives $\ln W+\int a_1(x)dx=\ln C$ *which implies* $\ln(W/C)=-\int a_1(x)dx$

$$W=Ce^{-\int a_1(x)dx} \quad or \quad W=C\exp\left[-\int a_1(x)dx\right]$$

Question: If $y_1(x)$ is a non-trivial solution of $y''+a_1(x)y'+a_0(x)y=0$ show that its second linearly independent solution $y_2(x)$ is given by

$$y_2=y_1\int\frac{\exp\left[-\int a_1(x)dx\right]}{y_1^2}dx$$

Solution: From Abel's Formula $W=\exp\left[-\int a_1(x)dx\right]$ *and* $W(y_1,y_2)=y_1y_2'-y_1'y_2$ which implies

$$y_1y_2'-y_1'y_2=\exp\left[-\int a_1(x)dx\right]$$

Divide this expression by y_1^2 *we get*

$$\frac{y_1y_2'-y_1'y_2}{y_1^2}=\frac{\exp\left[-\int a_1(x)dx\right]}{y_1^2}\Rightarrow\frac{d}{dx}\left(\frac{y_2}{y_1}\right)=\frac{\exp\left[-\int a_1(x)dx\right]}{y_1^2}$$

Integration gives $\Rightarrow\frac{y_2}{y_1}=\int\frac{\exp\left[-\int a_1(x)dx\right]}{y_1^2}dx\Rightarrow y_2=y_1\int\frac{\exp\left[-\int a_1(x)dx\right]}{y_1^2}dx$

Eigenvalue Problems:

(Eigenfunctions and eigenvalues)

A homogeneous problem consists of a homogeneous differential equation and homogeneous boundary conditions.

Definition: When a linear operator acts on a function, it transforms the function into another function that is some scalar multiple of the original function. The function is called an eigen or characteristic or proper function and the scalar is an eigen or characteristic or proper or latent value. For example, if

y = sin(mx) or y = cos(mx) and the differential operator: $\frac{d^2}{dx^2}$. *Then* $\frac{d^2y}{dx^2}=-m^2\sin(mx)$ *or* $\frac{d^2y}{dx^2}=-m^2\cos(mx)$. Hence sin(mx) or cos(mx) is an eigen function $-m^2$ is the corresponding eigenvalue. Finding the eigenfunctions and eigenvalues is called the eigenvalue problem. Frequently the solutions of a differential equation depend upon a parameter, λ which may assume various values during a given discussion. This parameter can appear in the coefficients of the differential equations, in the boundary conditions or in both. For a large class of problems of practical significance, the typical eigenvalue equation is written in the following form:

$$M[y]+\lambda y = 0 \qquad (1)$$

Where $$M = a_2(x)D^2 + a_1(x)D + a_0(x)$$

In the coming problems, λ appears in the differential equations but not in the boundary conditions. The general solution of (1) must depend upon both x and the parameter. λ : if y_1 and y_2 constitute linearly independent solutions of (1) the general solution is written as

$$y = C_1y_1(x,\lambda) + C_2y_2(x,\lambda) \qquad (2)$$

Subjecting this solution function to the homogeneous boundary conditions:

$$B_1[y] = 0 \quad , \quad B_2[y] = 0 \qquad (3)$$

Leads to a coefficient determinant: $\Delta = \begin{vmatrix} B_1[y_1] & B_1[y_2] \\ B_2[y_1] & B_2[y_2] \end{vmatrix}$

that must also depend upon λ in this situation. The basic problem is to determine all values of λ for which $\Delta(\lambda)=0$ i.e. determine all values of λ for which the homogeneous BVP (1) and (3) admits non-trivial solutions and then find the solutions corresponding to those values of λ. These special values of λ are called eigenvalues and the corresponding non-trivial solutions are called eigenfunctions.

The general problem described here is called an eigenfunctions or Sturm-Liouville problem. Some special eigenvalue equations are the followings:

(1) $y'' + \lambda y = 0 \quad (Helmholtz\ equation)$

(2) $(1-x^2)y'' - 2xy' + \lambda y = 0 \quad (Legendre\ equation)$

(3) $x^2 y'' + xy' + (\lambda x^2 - \nu^2)y = 0 \quad (Bessel\ equation)$

(4) $(1-x^2)y'' - (1-x)y' + \lambda y = 0 \quad (Laguerre\ equation)$

(5) $y'' - 2xy' + \lambda y = 0 \quad (Hermite\ equation)$

Adjoint Equation:

Consider a second order linear homogeneous differential equation of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \qquad (1)$$

Where P(x) has a continuous second order derivative, Q(x) has a continuous first order derivative, R(x) is a continuous and P(x) ≠ 0 on a ≤ x ≤ b. Then the adjoint equation to the equation (1) is

$$\frac{d^2}{dx^2}\left[P(x)y\right] - \frac{d}{dx}\left[Q(x)y\right] + R(x)y = 0$$

i.e. after taking the indicated derivatives,

$$P(x)\frac{d^2y}{dx^2} + \left[2P'(x) - Q(x)\right]\frac{dy}{dx} + \left[P''(x) - Q'(x) + R(x)\right]y = 0 \qquad (2)$$

Where the primes denote the differentiation w.r.t x. It is also assumed that in the adjoint equation (2) each coefficient function is continuous on a ≤ x ≤ b.

Example: Determine the adjoint equation to the following

(i) $x\frac{d^2y}{dx^2} - \cos x\frac{dy}{dx} + (\sin x)y = 0$

Here P(x) = x , Q(x) = –cosx , R(x) = sinx

So adjoint equation is $x\frac{d^2y}{dx^2}-\left[2(1)-(-\cos x)\right]\frac{dy}{dx}+\left[0-\sin x+\sin x\right]y=0$

$$\Rightarrow x\frac{d^2y}{dx^2}-(2+\cos x)\frac{dy}{dx}+0y=0$$

(ii) $x^2\frac{d^2y}{dx^2}-7x\frac{dy}{dx}+8y=0$

P(x) = x^2 , Q(x) = 7x , R(x) =8

So adjoint equation is $x\frac{d^2y}{dx^2}-\left[2(2x)-7x\right]\frac{dy}{dx}+\left[2-7+8\right]y=0$

$$\Rightarrow x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y=0$$

Theorem: For the second order linear differential equation

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$$

Prove that the adjoint of the adjoint equation is always the original equation.

Proof: The adjoint equation of DE: $P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$ (1) is

$$P(x)\frac{d^2y}{dx^2}+\left[2P'(x)-Q(x)\right]\frac{dy}{dx}+\left[P''(x)-Q'(x)+R(x)\right]y=0 \qquad (2)$$

Suppose that

$P_1(x)=P(x),\ Q_1(x)=2P'(x)-Q(x),\ R_1(x)=P''(x)-Q'(x)+R(x)$. So (2) is written as

$$P_1(x)\frac{d^2y}{dx^2}+Q_1(x)\frac{dy}{dx}+R_1(x)y=0 \qquad (3)$$

The adjoint equation to (3) is

$$P_1(x)\frac{d^2y}{dx^2}+\left[2P_1'(x)-Q_1(x)\right]\frac{dy}{dx}+\left[P_1''(x)-Q_1'(x)+R_1(x)\right]y=0 \qquad (4)$$

Now $P_1(x) = P(x)$

$2P_1'(x) - Q_1(x) = 2P'(x) - 2P'(x) + Q(x) = Q(x)$ and

$P_1''(x) - Q_1'(x) + R_1(x) = P''(x) - 2P''(x) + Q'(x) + P''(x) - Q'(x) + R(x) = R(x)$

Replace these expressions in (4) we obtain

$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$ which is the original equation.

Self-Adjoint Equation:

A second order linear differential equation $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$ is called self-adjoint if it is identical with its adjoint equation OR is said to be self-adjoint if its adjoint is the same as original.

Theorem: Consider a second order linear differential equation:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \qquad (1)$$

Where P(x) has a continuous second order derivative, Q(x) has a continuous first order derivative, R(x) is a continuous and P(x) ≠ 0 on a ≤ x ≤ b. A necessary and sufficient condition for the equation (1) to self-adjoint is $\frac{d}{dx}\left[P(x)\right] = Q(x)$ (2)

on a ≤ x ≤ b.

Proof: By definition the adjoint equation of the equation (1) is

$$P(x)\frac{d^2y}{dx^2} + \left[2P'(x) - Q(x)\right]\frac{dy}{dx} + \left[P''(x) - Q'(x) + R(x)\right]y = 0$$

(3)

If the condition (2) is satisfied the $P'(x) = Q(x)$, $P''(x) = Q'(x)$ and (3) becomes as

$$P(x)\frac{d^2y}{dx^2}+\left[2Q(x)-Q(x)\right]\frac{dy}{dx}+\left[Q'(x)-Q'(x)+R(x)\right]y=0$$

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$$

Which is identical with (1). So (3) is the self-adjoint equation.

Conversely, if equations (1) and (3) are identical then

$2P'(x)-Q(x)=Q(x)$, $P''(x)-Q'(x)+R(x)=R(x)$

OR $\quad P'(x)=Q(x)\,, P''(x)-Q'(x)=0$

$\quad P'(x)-Q(x)=0$

Integrating which c = 0

Hence $\frac{d}{dx}\left[P(x)\right]=Q(x)$ which is the condition (2).

Corollary: Suppose that the linear second order differential equation:

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0 \qquad (1)$$

is the self-adjoint. The DE (1) can be written in the form:

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+R(x)y=0 \qquad (2)$$

Proof: Since DE (1) is self-adjoint then $P'(x)=Q(x)$

$$P(x)\frac{d^2y}{dx^2}+P'(x)\frac{dy}{dx}+R(x)y=0$$

$$\Rightarrow \quad \frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+R(x)y=0 \qquad (2)$$

Self-Adjoint form of a Differential Equation:

A homogeneous second order linear differential equation is said to be in self adjoint form if and only if the form:

$$p(x)y'' + p'(x)y' + [q(x) + \lambda r(x)]y = 0 \quad or \quad \frac{d}{dx}[p(x)y'] + [q(x) + \lambda r(x)]y = 0$$

Where p(x) > 0 and r(x) > 0 in (a,b) and p'(x) , q(x) and r(x) are all continuous functions in the interval [a,b].

If the general eigenvalue equation is of the form:

$$A_2(x)y'' + A_1(x)y' + [A_0(x) + \lambda]y = 0 \qquad (1)$$

Where $A_0(x)$, $A_1(x)$ & $A_2(x)$ are continuous functions in the interval [a,b] and $A_2(x)>0$ on the interval [a,b]. This differential equation is not in self-adjoint form unless $A_1(x) = A'_2(x)$. The equation (1) can be converted into self-adjoint by multiplying throughout by multiplying by the function $\mu(x) = \frac{p(x)}{A_2(x)}$

obtaining $p(x)y'' + \mu(x)A_1(x)y' + \mu(x)[A_0(x) + \lambda]y = 0 \qquad (2)$

The equation (2) is now in self-adjoint form provided we pick p(x) such that

$$p'(x) = \mu(x)A_1(x) = \frac{p(x)A_1(x)}{A_2(x)} \Rightarrow \frac{p'(x)}{p(x)} = \frac{A_1(x)}{A_2(x)}$$

Solving this first order differential equation for p(x) we get

$$p(x) = \exp\left[\int \frac{A_1(x)}{A_2(x)}dx\right] \quad So, \ from \ (2)$$

$$p(x)y'' + p'(x)y' + [q(x) + \lambda r(x)]y = 0 \qquad where$$

$$q(x) = \mu(x)A_0(x) = \frac{p(x)A_0(x)}{A_2(x)} \quad and \quad r(x) = \mu(x)\frac{p(x)}{A_2(x)}$$

Self-Adjoint Operator:

The differential operator: $L = D\left[p(x)D\right] + q(x)$ is called self-operator where D = d/dx.

In terms of L, we can express a homogeneous self-adjoint differential equation in the compact form:

$$L[y] + \lambda r(x) y = 0$$

Example: Convert the DE: $x^2 y'' + xy' + \lambda y = 0$ into self-adjoint form.

Here $A_2(x) = x^2$, $A_1(x) = x$

$$p(x) = \exp\left[\int \frac{A_1(x)}{A_2(x)} dx\right] = \exp\left[\int \frac{x}{x^2} dx\right] = \exp\left[\int \frac{1}{x} dx\right] = \exp[\ln x] = x$$

$$r(x) = \frac{p(x)}{A_2(x)} = \frac{x}{x^2} = \frac{1}{x}, hence, we\ get$$

$$\left[p(x) y'\right]' + \lambda r(x) y = 0 \quad or \left[xy'\right]' + \frac{\lambda}{x} y = 0$$

Symmetric Operator:

A self-adjoint, L is said to be symmetric operator on the interval [a,b] if and only if

$\int_a^b \{uL[v] - vL[u]\} dx = 0$ for any functions, u and v having continuous second order derivatives on the interval and satisfying the prescribed boundary condition associated with L. Eigenvalue problems for which the operator, L is symmetric are also referred to as self-adjoint operator but not be a self-adjoint problem. The symmetry property to the operator is closely related to the kind of boundary conditions prescribed along with L.

Lagrange's Identity:

If L = D[p(x)D]+q(x) is self-adjoint and functions, u and v have continuous second order derivatives on an interval the relation

$uL[v]-vL[u]=\frac{d}{dx}\left[p(x)W(u,v)(x)\right]$ is called the Lagrange's identity where $W(u,v)=uv'-u'v$ is the Wronskian function of u and v.

Proof: Here

$$uL[v]-vL[u]=u\{D[p(x)Dv]+q(x)v\}-v\{D[p(x)Du]+q(x)u\}$$

$$=uD[p(x)Dv]+q(x)uv-vD[p(x)Du]-q(x)uv$$

$$=uD[p(x)Dv]-vD[p(x)Du]$$

$$=uD[p(x)Dv]+Du(p(x)Dv)-vD[p(x)Du]-Dv(p(x)Du)$$

$$=D[p(x)uDv]-D[p(x)vDu]$$

$$=D[p(x)(uDv-vDu)]$$

$$=D[p(x)(uv'-u'v)]$$

$$=D[p(x)W(u,v)(x)] \quad Where\ D=\frac{d}{dx}$$

Green's Identity or Green's formula:

This identity is based on Lagrange's identity integrating both sides of Lagrange's identity,

$$\int_a^b \{uL[v]-vL[u]\}dx=\int_a^b D[p(x)W(u,v)(x)]dx=p(x)W(\mathrm{u},v)(x)\Big|_a^b$$

Theorem:

A self-adjoint L = D[p(x)D]+q(x) is a symmetric operator on the interval [a,b] if and only if $p(x)W(\mathrm{u},v)(x)\Big|_a^b=0$ u and v satisfy the described boundary conditions associated L and have continuous second order derivative in the interval [a,b].

Proof: Suppose that L = D[p(x)D]+q(x) is a symmetric operator on the interval [a,b] then for any functions u and v

$$\int_a^b \{uL[v]-vL[u]\}dx=0$$

And by Green's formula

$$\int_a^b \{uL[v]-vL[u]\}\,dx=\int_a^b D\left[p(x)W(u,v)(x)\right]dx$$

$$=p(x)W(\mathrm{u},v)(x)\Big|_a^b$$

$$\Rightarrow p(x)W(\mathrm{u},v)(x)\Big|_a^b=0$$

Suppose that

$$p(x)W(\mathrm{u},v)(x)\Big|_a^b=0$$

$$\int_a^b \{uL[v]-vL[u]\}\,dx$$

$=p(x)W(\mathrm{u},v)(x)\Big|_a^b=0$ which implies L is symmetric.

Lecture # 02

Differential Equation (D.E):

An equation involving one dependent variable and its derivative w.r.t one or more independent variables is called Differential Equation (D.E)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

Order of D.E:

The highest derivative accruing in the D.E is called the order of D.E.

Degree of D.E:

The power of highest derivative accruing in the D.E is called degree of D.E.

Ordinary D.E (O.D.E):

The differential equation which involves one or more derivative of an unknown function of a single independent variable is called O.D.E

$$x^2dy + y^2dx = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{y^2}{x^2}$$

Partial D.E (P.D.E):

The D.E which involves derivative of an unknown function of two or more independent variable is called P.D.E.

$$\frac{\partial^3 u}{\partial y^3} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial z}\right)^2 + 4x^3 + 4y^2 + 4z = 0$$

Initial Condition:

To find the solution of a D.E subject to a certain condition if condition is relate to one value of independent variable $y = y_0$ at $= x_0$. Then the condition is called initial condition or one point boundary condition.

e.g. y(0) = 1 , $y'(0) = 2$, $y''(0) = 3$

or y(2) = 2 , $y'(2) = 1$, $y''(2) = 0$

Boundary Condition:

To find the solution of D.E subject to a certain condition if the condition relate to two different value of independent variable. Then these problems are called two point boundary value problem or simply boundary value problem and such condition is called Boundary condition.

e.g. y(1) = 2 , y(2) = 3 ,

or y(0) = 1 , $y'(1) = 3$, $y''(4) = 5$

Examples:

(i) $\frac{d^2y}{dx^2} + y = 0$, y(0) = 1 , $y\left(\frac{\pi}{2}\right) = 1$

General Solution

y = $c_1$sinx+$c_2$cosx

at y(0) = 1 ⇒ 1 = $c_1(0) + c_2$ ⇒ $c_2 = 1$

at $y\left(\frac{\pi}{2}\right) = 1$ ⇒ 1 = $c_1 + c_2\,(0)$ ⇒ $c_1 = 1$

⇒ y = sinx + cosx

(ii) $\frac{dy}{dx} + y = 2xe^{-x}$ y (-1) = e + 3

General solution y = $(x^2 + c)e^{-x}$

By putting

e + 3 = (1+c) e

$\frac{e+3}{e} = 1 + c$

⇒ $1 + \frac{3}{e} = 1 + c$ ⇒ $c = \frac{3}{e}$

⇒ $y = \left(x^2 + \frac{3}{e}\right)e^{-x}$

(iii) $\frac{dy}{dx} = -\frac{y}{x}$, y(3) = 4

General solution $x^2 + y^2 = c^2$

By putting

$9 + 16 = c^2$ ⇒ $c^2 = 25$

$x^2 + y^2 = 25$

(iv) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$, y(0) = –2 , $y'(0) = 6$

General solution y = Ae^{4x}+Be^{-3x}

At y(0) = –2 ⇒ –2 = A + B ...(i)

Now y' = 4Ae^{4x} –3Be^{-3x}

At $y'(0) = 6$ ⇒ 6 = 4A–3B ..(ii)

Multipying (i) by 3 and add in (ii)

–6 = 3A+3B

6 = 4A–3B

0 = 7A ⇒ A = 0

Put in (i) ⇒ B = –2

⇒ y = $-2e^{-3x}$

(v) $x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$

y (2) = 0 , $y'(2) = 2$, $y''(2) = 6$

General solution

y = c_1x+c_2x^2+c_3x^3

At y (2) = 0 ⇒ 0 = c_1+4c_2+8c_3 …(i)

Now y' = c_1+2c_2x + 3c_3x^2

At $y'(2) = 2$ ⇒ 2 = c_1+4c_2+12c_3 …(ii)

Now y'' = 2c_2+6c_3x

At $y''(2) = 6$ ⇒ 6 = 2c_2+12c_3 …(iii)

Divide (iii) by 2 ⇒ 3 = c_2+6c_3

⇒ $c_2 = 3 - 6c_3$ put in (ii)

2 = c_1+4(3 − 6c_3)+12c_3

2 = c_1 + 12–24c_3+12c_3

2–12 = $c_1 - 12c_3$

c_1 = 12c_3–10

Put the value of c_1 & c_2 in (i)

0 = (12c_3–10)+4(3–6c_3)+8c_3

0 = 24c_3–20+12–24c_3+8c_3

0 = -8 + 8c_3

8c_3 = 8 ⇒ c_3 = 1

c_2 = 3 – 6(1) ⇒ c_2 = –3

c_1 = 12(1)–10 ⇒ c_1 = 2

⇒ y = $2x - 3x^2 + x^3$

(vi) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ y(0) = 0 , y(1) = 1

General solution y = $c_1e^x + c_2e^{2x}$

At y(0) = 0 ⇒ $0 = c_1 + c_2$ ...(i)

At y(1) = 1 ⇒ $1 = c_1e + c_2e^2$ …(ii)

Multiplying (i) by e and subtract from (ii)

$$1 = c_1e + c_2e^2$$

$$\underline{-0 = c_1e \pm c_2e}$$

$$1 = c_2e^2 - c_2e$$

$$1 = c_2e(e-1)$$

$$\Rightarrow \quad c_2 = \frac{1}{e(e-1)}$$

Put in (1) $\Rightarrow \quad 0 = c_1 + \frac{1}{e(e-1)}$

$$\Rightarrow \quad c_1 = -\frac{1}{e(e-1)}$$

$$\Rightarrow \quad y = -\frac{1}{e(e-1)}e^x + \frac{1}{e(e-1)}e^{2x}$$

Lecture # 03

Fourier Transform:

Let f(x) be a real valued function s.t f(x) → 0 as x →±∞. Then its Fourier transform is defined as

$$\mathfrak{F}[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx = F(k)$$

$$\textit{If fourier transform}$$

$$\mathfrak{F}[f(x)] = F(k)$$

Then its inverse Fourier transform is defined as

$$\mathfrak{F}^{-1}[F(k)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} F(k)dk = F(x)$$

Question: Fourier transformation of $Ne^{-\alpha x^2}, \alpha > 0$

Solution:

We know $\mathfrak{F}[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx = F(k)$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} Ne^{-\alpha x^2} dx$$

$$=\frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha x^2+ikx} dx$$

$$=\frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\alpha x^2-ikx)} dx \qquad \_\_\_\_\_\_(1)$$

$$Let\ \alpha x^2 - ikx = \left(\sqrt{\alpha}x\right)^2 - 2x\sqrt{\alpha}\left(\frac{ik}{2\sqrt{\alpha}}\right) + \left(\frac{ik}{2\sqrt{\alpha}}\right)^2 - \left(\frac{ik}{2\sqrt{\alpha}}\right)^2$$

$$=\left(\sqrt{\alpha}x - \frac{ik}{2\sqrt{\alpha}}\right)^2 - \left(\frac{ik}{2\sqrt{\alpha}}\right)^2 \quad put\ in\ (1)$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\sqrt{\alpha}x - \frac{ik}{2\sqrt{\alpha}}\right)^2 - \left(\frac{ik}{2\sqrt{\alpha}}\right)^2\right]} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha}x - \frac{ik}{2\sqrt{\alpha}}\right)^2} e^{\left(\frac{ik}{2\sqrt{\alpha}}\right)^2} dx$$

$$= \frac{N}{\sqrt{2\pi}} e^{\left(\frac{ik}{2\sqrt{\alpha}}\right)^2} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha}x - \frac{ik}{2\sqrt{\alpha}}\right)^2} dx$$

$$Put \quad z = \sqrt{\alpha}x - \frac{ik}{2\sqrt{\alpha}}$$

$$dz = \sqrt{\alpha}dx \;\Rightarrow\; dx = \frac{1}{\sqrt{\alpha}}dz$$

$$= \frac{N}{\sqrt{2\pi}} e^{\frac{i^2k^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-z^2} . \frac{1}{\sqrt{\alpha}} dz$$

$$= \frac{N}{\sqrt{2\pi}} e^{\frac{-k^2}{4\alpha}} . \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \frac{N}{\sqrt{2\pi}} e^{\frac{-k^2}{4\alpha}} . \frac{1}{\sqrt{\alpha}} . \sqrt{\pi} \qquad \because \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

$$\mathcal{F}[f(x)] = \frac{N}{\sqrt{2\alpha}} e^{\frac{-k^2}{4\alpha}}$$

Question: Fourier transform

f(x) = 1 if $|x| \le$ a

f(x) = 0 if $|x| >$ a

Solution: We know

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{a} e^{ikx} f(x)dx+\int_{-a}^{a} e^{ikx} f(x)dx+\int_{a}^{\infty} e^{ikx} f(x)dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ikx} f(x)dx \qquad \because\; f(x)=0\;\; for\; -\infty\to -a\;\&\, a\to\infty$$

$$=\frac{1}{\sqrt{2\pi}}\left|\frac{e^{ikx}}{ik}\right|_{-a}^{a}$$

$$=\frac{2}{\sqrt{2\pi}\,K}\left(\frac{e^{ika}-e^{-ika}}{2i}\right)$$

$$=\frac{2}{\sqrt{2\pi}}\frac{\sin Ka}{K}$$

Attenuation Property:

Question:

If Fourier transform ꟻ[f(x)] = F(k) then ꟻ[e^{ax}f(x)] = F(k-ia)

Solution:

$$ꟻ[e^{ax}f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{ax} f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(ikx+ax)} f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(ikx-i^2ax)} f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k-ia)x} f(x)dx$$

= F(k-ia)

Shifting property:

Question: If Fourier transform $\mathcal{F}$[f(x)]= F(k) then $\mathcal{F}$[f(x-a)] = e^{ika}F(k)

Solution:

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

f(x) replacing by f(x-a)

$$\mathcal{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x-a)dx$$

Put x-a = z $\Rightarrow$ x = z+a

dx = dz in R.H.S

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(z+a)} f(z)dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(ikz+ika)} f(z)dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikz} e^{ika} f(z)dz$$

$$= \frac{e^{ika}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikz} f(z)dz$$

$$\mathcal{F}[f(x-a)] = e^{ika} F(k)$$

Question:

If Fourier transform $\mathcal{F}$[f(x)] = F(k) then $\mathcal{F}$[f(x+a)] = e^{-ika}F(k)

Solution:

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

f(x) replacing by f(x+a)

$$\mathcal{F}[f(x+a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x+a)dx$$

Put x+a = z $\Rightarrow$ x = z-a

dx = dz   in R.H.S

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(z-a)} f(z)dz$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(ikz-ika)} f(z)dz$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikz} e^{-ika} f(z)dz$$

$$=\frac{e^{-ika}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikz} f(z)dz$$

$$\mathcal{F}[f(x+a)]=e^{-ika} F(k)$$

Even, Odd function:

A real valued function f(x), a < x < b is said to be even if f(-x) = f(x). It is said to be odd if f(-x) = -f(x).

e.g.   Even function = x^2 , cosx

And   Odd function = x^3 , x^5 , sinx

Lecture # 04

Question: Find the Fourier transform of $g(x)=e^{-\alpha x^2}\cos\beta x$

Solution: We know

$$\mathcal{F}[g(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}g(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}e^{-\alpha x^2}\cos\beta dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}e^{-\alpha x^2}\left(\frac{e^{i\beta x}+e^{-i\beta x}}{2}\right)dx$$

$$=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{ikx}e^{-\alpha x^2}e^{i\beta x}+e^{ikx}e^{-\alpha x^2}e^{-i\beta x}\right)dx$$

$$=\frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}e^{ikx}e^{-\alpha x^2}e^{i\beta x}dx+\int_{-\infty}^{\infty}e^{ikx}e^{-\alpha x^2}e^{-i\beta x}dx\right]$$

$Put\ \ f(x)=e^{-\alpha x^2}$

$$=\frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}e^{ikx}f(x)e^{i\beta x}dx+\int_{-\infty}^{\infty}e^{ikx}f(x)e^{-i\beta x}dx\right]$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}e^{ikx}e^{i\beta x}f(x)dx+\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}e^{ikx}e^{-i\beta x}f(x)dx\right]$$

$$=\frac{1}{2}\left[\mathcal{F}\left(e^{i\beta x}f(x)\right)+\mathcal{F}\left(e^{-i\beta x}f(x)\right)\right]$$

$$=\frac{1}{2}\left[F\left(k-i(i\beta)\right)+F\left(k-i(-i\beta)\right)\right]\quad\because\mathcal{F}\left[e^{ax}f(x)\right]=F(k-ia)$$

$$=\frac{1}{2}\left[F\left(k+\beta\right)+F\left(k-i\beta\right)\right]\_\_\_\_\_\_\_\_(1)$$

Where $\quad F(k)=\mathcal{F}[f(x)]=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}e^{ikx}f(x)dx$

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{-\alpha x^2}dx$$

$$F(k)=\frac{1}{\sqrt{2\alpha}}e^{-\frac{k^2}{4\alpha}} \qquad \because \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-kx}e^{-\alpha x^2}dx=\frac{1}{\sqrt{2\alpha}}e^{-\frac{k^2}{4\alpha}}$$

$$F(k+\beta)=\frac{1}{\sqrt{2\alpha}}e^{-\frac{(k+\beta)^2}{4\alpha}}$$

$$=\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2+2k\beta)}{4\alpha}} = \frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2)}{4\alpha}}.e^{\frac{-k\beta}{2\alpha}}$$

$$\Rightarrow \quad F(k-\beta)=\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k-\beta)^2}{4\alpha}}$$

$$=\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2-2k\beta)}{4\alpha}} = \frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2)}{4\alpha}}.e^{\frac{k\beta}{2\alpha}}$$

Put these values in (1)

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2)}{4\alpha}}.e^{\frac{-k\beta}{2\alpha}}+\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2)}{4\alpha}}.e^{\frac{k\beta}{2\alpha}}\right]$$

$$=\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2)}{4\alpha}}\left[\frac{e^{\frac{-k\beta}{2\alpha}}+e^{\frac{k\beta}{2\alpha}}}{2}\right]$$

$$\mathfrak{F}[g(x)]=\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+\beta^2)}{4\alpha}}\cosh\left(\frac{k\beta}{2\alpha}\right)$$

Fourier Sine Transform:

Let f(x) is define for x > 0. The Fourier sine transform of f(x) is denoted by

$\mathfrak{F}_s$[f(x)] or F_s(k) and defined as

$$\mathfrak{F}_s[f(x)] = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(kx)dx \quad ; (k>0)$$

Inverse Fourier:

The function f(x) is called inverse Fourier sine transform of F_s(k) and it is also denoted by F_s^{-1}(k) and is defined as

$$F_s^{-1}(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin(kx)dx$$

Question: Find Fourier sine transform f(x) = $\frac{1}{x}$

Solution: By definition

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin(kx)dx$$

Put kx = y $\Rightarrow$ $\frac{1}{x} = \frac{k}{y}$

kdx = dy $\Rightarrow$ dx = $\frac{dy}{k}$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{k}{y}\sin y\frac{dy}{k}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin y}{y}dy$$

$$= \sqrt{\frac{2}{\pi}}.\frac{\pi}{2} = \sqrt{\frac{\pi}{2}} \qquad \because \int_0^\infty \frac{\sin y}{y}dy = \frac{\pi}{2}$$

Question: Find Fourier Sine Transform

f(x) = $e^{-3x} + e^{-4x}$

Solution: We know that

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(e^{-3x} + e^{-4x}\right)\sin(kx)dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty e^{-3x}\sin(kx)dx + \int_0^\infty e^{-4x}\sin(kx)dx\right]$$

$$\because \int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx))$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{\frac{e^{-3x}}{9+k^2}(-3\sin(kx) - k\cos(kx)\right\}_0^\infty + \left\{\frac{e^{-4x}}{4+k^2}(-4\sin(kx) - k\cos(kx)\right\}_0^\infty\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{0 - \frac{1}{9+k^2}(-k)\right\} + \left\{0 - \frac{1}{16+k^2}(-k)\right\}\right]$$

$$F_S(\mathrm{k}) = \sqrt{\frac{2}{\pi}}\left[\frac{k}{9+k^2} + \frac{k}{16+k^2}\right]$$

Question: Find Fourier sine transform

$$f(x) = e^{-|x|} \quad or \quad f(x) = \begin{cases} e^{-(-x)} & x < 0 \\ e^{-x} & x > 0 \end{cases}$$

Solution: We know

$$F_S(\mathrm{k}) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx \quad \Rightarrow \quad = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\sin(kx)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin(kx)dx \qquad because \;\; x<0 \; not \;\; found$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+k^2}\left(-\sin(kx) - k\cos(kx)\right)\Big|_0^\infty\right] by \;\; formula$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{1+k^2}(0-k)\right] \quad \Rightarrow \quad \sqrt{\frac{2}{\pi}}.\frac{k}{1+k^2}$$

Question: Find Fourier sine transform

$$f(x)=\begin{cases}\sin x & 0<x<a\\ 0 & x>0\end{cases}$$

Solution:

We know $F_s(k)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^a f(x)\sin(kx)dx+\int_a^\infty f(x)\sin(kx)dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^a \sin(kx)\sin x dx+0\right]$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^a 2\sin(kx)\sin x dx$$

$\because$ $2sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin(k-1)}{k-1}x\Big|_0^a+\frac{\sin(k+1)}{k+1}x\Big|_0^a\right]$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\left(\frac{\sin(k-1)}{k-1}-0\right)-\left(\frac{\sin(k+1)}{k+1}-0\right)\right]$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin(k-1)a}{k-1}-\frac{\sin\left(k+1\right)a}{k+1}\right]$$

Lecture # 05

Fourier Transformation Derivatives of a function:

Let f(x) be a function

$$\mathfrak{F}[\mathrm{f}(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{f}(x)dx$$

$$\mathfrak{F}[\mathrm{f}'(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{f}'(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[e^{ikx} f(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty} ik\, e^{ikx}\,\mathrm{f}(x)dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[0-ik\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{f}(x)dx\right]$$

$$=(-ik)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{f}(x)dx$$

$$\mathfrak{F}[f'(x)]=(-ik)\,\mathfrak{F}[f(x)]$$

Similarly, $\mathfrak{F}[f''(x)]=(-ik)^2\,\mathfrak{F}[f(x)]$

.

.

.

$$\mathfrak{F}[f^{n}(x)]=(-ik)^{n}\,\mathfrak{F}[f(x)]$$

Example:

Find Fourier $y''+y'=f(\mathrm{x})$

Solution:

$$\mathfrak{F}[y'']+\mathfrak{F}[y']=\mathfrak{F}[f(\mathrm{x})]$$

$$(ik)^2\,\mathcal{F}[y] + (-ik)\,\mathcal{F}[y] = F(\text{k})$$

$$\mathcal{F}[y][-k^2 + (-ik)] = F(\text{k})$$

$$\mathcal{F}[y] = \frac{F(k)}{-k^2 + (-ik)}$$

Taking inverse Fourier on both sides

$$y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{F(k)}{-k^2 + (-ik)} dk$$

Question:

$$\phi_{xx} = \phi_{tt}$$

$$\phi(x,0) = f(x)$$

$$\phi_t(x,0) = 0$$

$$\phi, \phi_x \to 0 \; As \; x \to \pm\infty$$

Solution:

$$\mathcal{F}[\phi_{xx}] = \mathcal{F}[\phi_{tt}]$$

$$\mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}(\text{x},\text{t})\right] = \mathcal{F}\left[\frac{\partial^2 \phi}{\partial t^2}(\text{x},\text{t})\right]$$

$$(-ik)^2\,\mathcal{F}\big[\phi(\text{x},\text{t})\big] = \frac{\partial^2}{\partial t^2}\big[\mathcal{F}\,\phi(\text{x},\text{t})\big]$$

Suppose $\mathcal{F}\big[\phi(\text{x},\text{t})\big] = \bar{\phi}(k,t)$ ....(i)

$$(-ik)^2\,\bar{\phi}(k,t) = \frac{\partial^2}{\partial t^2}\bar{\phi}(k,t)$$

$$-k^2\,\bar{\phi}(k,t) = \frac{\partial^2}{\partial t^2}\bar{\phi}(k,t)$$

$$\frac{\partial^2}{\partial t^2}\bar{\phi}(k,t) + k^2\,\bar{\phi}(k,t) = 0$$

$$\left(\frac{\partial^2}{\partial t^2}+k^2\right)\bar{\phi}(k,t)=0$$

$$\frac{\partial^2}{\partial t^2}+k^2=0 \quad \& \ \bar{\phi}(k,t)\neq 0$$

$$\frac{\partial^2}{\partial t^2}=-k^2=i^2k^2$$

$$\frac{\partial}{\partial t}=\pm ik$$

$$\bar{\phi}(k,t)=Axos(kt)+B\sin(kt) \qquad ...(ii)$$

(1) $\phi(x,0)=f(x)$

$$\mathcal{F}\left[\phi(x,0)\right]=\mathcal{F}\left[f(x)\right]$$

$$\bar{\phi}(k,0)=F(k) \qquad ...(iii)$$

(2) $\phi_t(x,0)=0$

$$\mathcal{F}\left[\phi_t(x,0)\right]=0$$

$$\mathcal{F}\left[\frac{\partial}{\partial t}\phi(x,0)\right]=0$$

$$\frac{\partial}{\partial t}\mathcal{F}\left[\phi(x,0)\right]=0$$

$$\frac{\partial}{\partial t}\bar{\phi}(x,0)=0 \qquad \ldots\text{(iv)}$$

Now put t = 0 in (ii) and compare with (iii)

$$\bar{\phi}(\text{k},0)=A$$

A = F(k) on comparing

Now diff. (ii) partially w.r.t ‘t’

$$\frac{\partial}{\partial t}\bar{\phi}(x,0)=-Ak\sin(kt)+Bk\cos(kt)$$

Put t = 0 and compare with (iv)

$$\frac{\partial}{\partial t}(\text{k},0) = Bk$$

$$\text{Bk} = 0$$

$$\text{B} = 0$$

Put the value of A and B in (i) we have

$$\bar{\phi}(\text{k},\text{t}) = F(k)\cos(kt)$$

$$\mathcal{F}[\phi(\text{x},\text{t})] = F(k)\cos(kt) \qquad \text{By (i)}$$

Taking inverse Fourier on both sides

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}F(k)\cos(kt)\,dk$$

Question: $\phi_{xx} = \phi_t$

$$\phi(x,0) = e^{-ax^2}\ ,\ a > 0$$

$$\phi, \phi_x \to 0\ As\ x \to \pm\infty$$

Solution:

$$\mathcal{F}[\phi_{xx}] = \mathcal{F}[\phi_t]$$

$$\mathcal{F}\left[\frac{\partial^2\phi}{\partial x^2}(\text{x},\text{t})\right] = \mathcal{F}\left[\frac{\partial\phi}{\partial t}(\text{x},\text{t})\right]$$

$$-(ik)^2\ \mathcal{F}[\phi(\text{x},\text{t})] = \frac{\partial}{\partial t}\mathcal{F}[\phi(\text{x},\text{t})]$$

$$-k^2\ \mathcal{F}[\phi(\text{x},\text{t})] = \frac{\partial}{\partial t}\ \mathcal{F}[\phi(\text{x},\text{t})]$$

Suppose $\mathcal{F}[\phi(\text{x},\text{t})] = \bar{\phi}(k,t)$ ....(i)

$$-k^2\bar{\phi}(k,t) = \frac{\partial}{\partial t}\bar{\phi}(k,t)$$

$$\frac{\partial}{\partial t}\bar{\phi}(k,t)+k^2\bar{\phi}(k,t)=0$$

$$\left(\frac{\partial}{\partial t}+k^2\right)\bar{\phi}(k,t)=0$$

$$\frac{\partial}{\partial t}+k^2=0 \quad \& \ \bar{\phi}(k,t)\neq 0$$

$$\frac{\partial}{\partial t}=-k^2$$

$$\bar{\phi}(k,t)=Ae^{-k^2t} \qquad ...(ii)$$

Given condition

$$\phi(x,0)=e^{-ax^2}$$

$$\mathcal{F}\left[\phi(x,0)\right]=\mathcal{F}\left[e^{-ax^2}\right]$$

$$\bar{\phi}(k,0)=\frac{e^{\frac{-K^2}{4a}}}{\sqrt{2a}} \qquad ...(iii)$$

$$\bar{\phi}(k,0)=A$$

$$A=\frac{e^{\frac{-K^2}{4a}}}{\sqrt{2a}} \ \textit{on comparing}$$

$$\text{(ii)} \Rightarrow \quad \bar{\phi}(k,t)=\frac{e^{\frac{-K^2}{4a}}}{\sqrt{2a}}\,e^{-k^2t}$$

$$\mathcal{F}[\phi(k,t)]=\frac{e^{-k^2t-\frac{K^2}{4a}}}{\sqrt{2a}} \qquad \text{By (1)}$$

Taking inverse Fourier on both sides

$$\phi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\,\frac{e^{-k^2t-\frac{K^2}{4a}}}{\sqrt{2a}}\,dk$$

$$\phi(x,t)=\frac{1}{2\sqrt{a\pi}}\int_{-\infty}^{\infty}e^{-ikx-k^2t-\frac{K^2}{4a}}\,dk$$

Assignment:

$$\phi_{xxxx} = \phi_{tt}$$

$$\phi(x,0) = f(x)$$

$$\phi_t(x,0) = g(x)$$

$$\phi, \phi_x, \phi_{xx} \to 0 \ As \ x \to \pm\infty$$

Solution:

$$\mathcal{F}[\phi_{xxxx}] = \mathcal{F}[\phi_{tt}]$$

$$\mathcal{F}\left[\frac{\partial^4 \phi}{\partial x^4}(\mathrm{x,t})\right] = \mathcal{F}\left[\frac{\partial^2 \phi}{\partial t^2}(\mathrm{x,t})\right]$$

$$-(ik)^4 \ \mathcal{F}[\phi(\mathrm{x,t})] = \frac{\partial^2}{\partial t^2}\mathcal{F}[\phi(\mathrm{x,t})]$$

$$k^4 \ \mathcal{F}[\phi(\mathrm{x,t})] = \frac{\partial^2}{\partial t^2}\mathcal{F}[\phi(\mathrm{x,t})]$$

Suppose $\mathcal{F}[\phi(\mathrm{x,t})] = \overline{\phi}(k,t)$ ....(i)

$$k^4 \overline{\phi}(k,t) = \frac{\partial^2}{\partial t^2}\overline{\phi}(k,t)$$

$$\frac{\partial^2}{\partial t^2}\overline{\phi}(k,t) - k^4 \overline{\phi}(k,t) = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - k^4\right)\overline{\phi}(k,t) = 0$$

$$\frac{\partial^2}{\partial t^2} - k^4 = 0 \quad \& \ \overline{\phi}(k,t) \neq 0$$

$$\frac{\partial^2}{\partial t^2} = k^4$$

$$\frac{\partial}{\partial t} = \pm k^2$$

$$\overline{\phi}(k,t) = Ae^{-k^2 t} + Be^{-k^2 t} \qquad ...(ii)$$

Given condition

$$\phi(x,0) = f(x)$$
$$\mathcal{F}[\phi(x,0)] = \mathcal{F}[f(x)]$$
$$\overline{\phi}(k,0) = F(k) \qquad ...(iii)$$

Put t = 0 in (ii) and compare with (iii)

$$\overline{\phi}(k,0) = A + B$$
$$A + B = F(k) \qquad ....(\text{iv}) \qquad \textit{on comparing}$$

(2) $\phi_t(x,0) = g(x)$

$$\mathcal{F}[\phi_t(x,0)] = \mathcal{F}[g(x)]$$
$$\mathcal{F}\left[\frac{\partial}{\partial t}\phi(x,0)\right] = G(k)$$
$$\frac{\partial}{\partial t}\mathcal{F}[\phi(x,0)] = G(k)$$
$$\frac{\partial}{\partial t}\overline{\phi}(x,0) = G(k) \qquad \ldots(\text{v})$$

Now diff. (ii) partially w.r.t 't'

$$\frac{\partial}{\partial t}\overline{\phi}(\text{k},\text{t}) = k^2\,\text{A}e^{k^2 t} - k^2\,\text{B}e^{-k^2 t}$$

Put t = 0 and compare with (v)

$$\frac{\partial}{\partial t}\overline{\phi}(\text{k},\text{t}) = k^2 A - k^2\,\text{B}$$

$$k^2 A - k^2\,\text{B} = \text{G(k)} \qquad \text{on comparing}$$

$$k^2(A - \text{B}) = \text{G(k)}$$

$$(A - \text{B}) = \frac{G(\text{k})}{k^2} \qquad ....(iv)$$

Adding (iv) and (iv)

$$2A=F(k)+\frac{G(\mathrm{k})}{k^2}$$

$$A=\frac{1}{2}\left[F(k)+\frac{G(\mathrm{k})}{k^2}\right]$$

Subtracting (iv) and (vi)

$$2B=F(k)-\frac{G(\mathrm{k})}{k^2}$$

$$B=\frac{1}{2}\left[F(k)-\frac{G(\mathrm{k})}{k^2}\right]$$

Put the value of A and B in (ii)

$$\bar{\phi}(k,t)=\frac{1}{2}\left[F(k)+\frac{G(\mathrm{k})}{k^2}\right]e^{k^2t}+\frac{1}{2}\left[F(k)-\frac{G(\mathrm{k})}{k^2}\right]e^{-k^2t}$$

$$\mathcal{F}\left[\phi(k,t)\right]=\frac{1}{2}\left[F(k)+\frac{G(\mathrm{k})}{k^2}\right]e^{k^2t}+\frac{1}{2}\left[F(k)-\frac{G(\mathrm{k})}{k^2}\right]e^{-k^2t}$$

Taking inverse Fourier on both sides

$$\left[\phi(k,t)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\left[\frac{1}{2}\left[F(k)+\frac{G(\mathrm{k})}{k^2}\right]e^{-k^2t}+\frac{1}{2}\left[F(k)-\frac{G(\mathrm{k})}{k^2}\right]e^{-k^2t}\right]dk$$

Lecture # 06

Fourier Cosine Transformation:

Let f(x) is define for all $x \geq 0$. The Fourier cosine transformation of f(x) is denoted by F_c and define as $F_c\{f(x)\}=\int_0^\infty f(x)\cos(sx)\,dx \qquad ;(s>0)$

Inverse Fourier Cosine Transformation:

It is denoted as $F_c^{-1}\{F_c(s)\}$ and define as $F_c^{-1}\{F_c(s)\}=\frac{1}{\sqrt{2\pi}}\int_0^\infty F_c(\mathrm{s})\cos(sx)\,ds$

Question: Find Fourier cosine transformation of $f(x)=\begin{cases}\cos x & ;\ 0\leq x<a\\ 0 & ;\ x>a\end{cases}$

Solution: By definition of Fourier cosine transformation

$$F_c\{f(x)\}=\int_0^\infty f(x)\cos(sx)\,dx$$

$$=\int_0^a f(x)\cos(sx)\,dx+\int_a^\infty f(x)\cos(sx)\,dx$$

$$=\int_0^a \cos x\,\cos(sx)\,dx$$

$$=\frac{1}{2}\int_0^a 2\cos x\,\cos(sx)\,dx$$

$$\because\ 2\cos A\cos B=\cos(A+B)+\cos(\mathrm{A}-\mathrm{B})$$

$$=\frac{1}{2}\int_0^a\left(\cos(x+sx)+\cos(x-sx)\right)dx$$

$$=\frac{1}{2}\int_0^a\left(\cos(sx+x)-\cos(sx-x)\right)dx$$

$$=\frac{1}{2}\left[\int_0^a \cos(s+1)x\,dx-\int_0^a \cos(s-1)x\,dx\right]$$

$$=\frac{1}{2}\left[\left.\frac{\sin(s+1)x}{s+1}\right|_0^a-\left.\frac{\sin(s-1)x}{s-1}\right|_0^a\right]$$

$$=\frac{1}{2}\left[\frac{\sin(s+1)\mathrm{a}}{s+1}-\frac{\sin(s-1)\mathrm{a}}{s-1}\right]$$

Question:

$$f(x)=\begin{cases} x & ;\ 0<x<1 \\ 2-1 & ;\ 1<x<2 \\ 0 & ;\ x>2 \end{cases}$$

Solution:

By definition of Fourier cosine transformation

$$F_c\{f(x)\}=\int_0^1 f(x)\cos(sx)\,dx+\int_1^2 f(x)\cos(sx)\,dx+\int_2^\infty f(x)\cos(sx)\,dx$$

$$F_c\{f(x)\}=\int_0^1 x\cos(sx)\,dx+\int_1^2 (2-x)\cos(sx)\,dx+0$$

$$=I_1+I_2 \qquad \_\_\_\_\_\_\_(1)$$

$$I_1=\int_0^1 x\cos(sx)\,dx=\left.\frac{x\sin(sx)}{s}\right|_0^1-\int_0^1\frac{\sin(sx)}{s}.1dx$$

$$I_1=\left.\frac{\sin(s)}{s}+\frac{\cos(sx)}{s^2}\right|_0^1$$

$$I_1=\frac{\sin(s)}{s}+\frac{\cos(s)}{s^2}$$

$$I_1=\frac{\mathrm{s}\ \sin(\mathrm{s})+\cos(s)}{s^2}$$

$$I_2 = \int_1^2 (2-x)\cos(sx)\,dx = \left.\frac{(2-x)\sin(sx)}{s}\right|_1^2 - \int_1^2 \frac{\sin(sx)}{s}.(-1)dx$$

$$I_2 = \frac{-\sin(s)}{s} + \left(\left. -\frac{\cos(sx)}{s^2} \right|_1^2 \right)$$

$$I_2 = \frac{-\sin(s)}{s} - \frac{\cos(2s)}{s^2} + \frac{\cos(s)}{s^2}$$

$$I_2 = \frac{-s\sin(s) - \cos(2s) + \cos(s)}{s^2}$$

By putting these values in (1)

$$F_c(s) = \frac{s\sin(s) + \cos(s)}{s^2} + \frac{-s\sin(s) - \cos(2s) + \cos(s)}{s^2}$$

$$F_c(s) = \frac{s\sin(s) + \cos(s) - s\sin(s) - \cos(2s) + \cos(s)}{s^2}$$

$$F_c(s) = \frac{2\cos(s) - \cos(2s)}{s^2}$$

Question: $f(x) = e^{-3x} + e^{-4x}$

Solution: By definition of Fourier cosine transformation

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos(sx)\,dx$$

$$F_c\{f(x)\} = \int_0^\infty e^{-3x}\cos(sx)\,dx + \int_0^\infty e^{-4x}\cos(sx)\,dx$$

$$\because \int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$F_c\{f(x)\} = \left.\frac{e^{-3x}}{9+s^2}(-3\cos(sx) + s\sin(sx))\right|_0^\infty + \left.\frac{e^{-4x}}{16+s^2}(-4\cos(sx) + s\sin(sx))\right|_0^\infty$$

$$F_c\{f(x)\} = \frac{e^{-3(\infty)}}{9+s^2}(-3\cos(s.\infty)+s\sin(s.\infty)) - \frac{e^{-3(0)}}{9+s^2}(-3\cos(s.0)+s\sin(s.0))$$

$$+\frac{e^{-4(\infty)}}{16+s^2}(-4\cos(s.\infty)+s\sin(s.\infty)) - \frac{e^{-4(0)}}{16+s^2}(-4\cos(s.0)+s\sin(s.0))$$

$$= 0 - \frac{1}{9+s^2}(-3+0) + 0 - \frac{1}{16+s^2}(-4+0)$$

$$= \frac{3}{9+s^2} + \frac{4}{16+s^2}$$

Lecture # 07

Laplace Transformation:

Let f(t) be a function define for t > 0. The Laplace transformation of f(t) is denoted by $\mathcal{L}$[f(t)] or $\hat{f}$(s) and defined as

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Question: Find Laplace transformation of f(t) = t^n, n > −1

Solution:

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st} t^n dt$$

$$Let \quad st = x \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{dx}{s}$$

$$\mathcal{L}[t^n] = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$\mathcal{L}[t^n] = \int_0^\infty e^{-x} \frac{x^n}{s^n}\frac{dx}{s}$$

$$\mathcal{L}[t^n] = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n dx$$

$$\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \qquad \because \Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

$$*\,\mathcal{L}[t^0] = \frac{\Gamma(1)}{s} = \frac{1}{s} \qquad \because \Gamma(1) = 1$$

$$*\,\mathcal{L}[t^{-1}] = does\,not\,exist\,as\,n > -1$$

$$* \mathcal{L}[t^{-1/2}] = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}} \qquad \because \Gamma(1/2) = \sqrt{\pi}$$

Question: $f(t) = e^{at}$

Solution:
$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-st+at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty$$

$$= 0 - \left[\frac{-1}{-(s-a)} \right]$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad , \quad s > a$$

$$* \mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

$$* \mathcal{L}[e^{4t}] = \frac{1}{s-4}$$

Question: $f(t) = \sin at$

Solution:
$$\mathcal{L}[\sin at] = \int_0^\infty e^{-st} \sin at \, dt$$

$$\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$=\frac{e^{-st}}{(-s)^2+a^2}\left[-s\sin at-a\cos at\right]_0^{\infty}$$

$$=0-\frac{1}{s^2+a^2}[0-a]$$

$$\mathcal{L}[\sin at]=\frac{a}{s^2+a^2}$$

$$*\ \mathcal{L}[\sin 4t]=\frac{4}{s^2+16}$$

Question: $f(t)=\cos at$

Solution: $$\mathcal{L}[\cos at]=\int_0^{\infty} e^{-st}\cos at\,dt$$

$$\because \int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\cos bx+b\sin bx\right]$$

$$=\frac{e^{-st}}{(-s)^2+a^2}\left[-s\cos at+a\sin at\right]_0^{\infty}$$

$$=0-\frac{1}{s^2+a^2}[-s+0]$$

$$\mathcal{L}[\cos at]=\frac{s}{s^2+a^2}$$

$$*\ \mathcal{L}[\sin 3t]=\frac{s}{s^2+9}$$

Question: $f(t)=\sinh at$

Solution: $$\mathcal{L}[\sinh at]=\int_0^{\infty} e^{-st}\sinh at\,dt$$

$$=\int_0^{\infty} e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$

$$=\frac{1}{2}\left[\int_0^\infty e^{-st}e^{at}dt-\int_0^\infty e^{-st}e^{-at}dt\right]$$

$$=\frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]$$

$$=\frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$=\frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$=\frac{a}{s^2-a^2}\quad , \ s>|a|$$

$$*\ \mathcal{L}[\sinh 3t]=\frac{3}{s^2-9}$$

Question: $f(t)=\cosh at$

Solution: $\mathcal{L}[\cosh at]=\int_0^\infty e^{-st}\cosh at\,dt$

$$=\int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt$$

$$=\frac{1}{2}\left[\int_0^\infty e^{-st}e^{at}dt+\int_0^\infty e^{-st}e^{-at}dt\right]$$

$$=\frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$=\frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$=\frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$$= \frac{s}{s^2 - a^2}$$

$$* \mathcal{L}[\cosh 4t] = \frac{s}{s^2 - 16}$$

Shifting Property:

Let $\mathcal{L}[\text{f}(t)] = F(s)$

then $\mathcal{L}[\text{e}^{at}\text{f}(t)] = F(s-a)$

$$\mathcal{L}[\text{e}^{at}\text{f(t)}] = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$\mathcal{L}[\text{e}^{at}\text{f(t)}] = \int_0^\infty e^{-st+at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$\mathcal{L}[\text{e}^{at}\text{f}(t)] = F(s-a) \qquad \because \int_0^\infty e^{-st} f(t)dt = F(s)$$

$$* \mathcal{L}[\text{e}^{-at}\text{f}(t)] = F(s+a)$$

Laplace transformation of derivative of function of f(t):

As $\mathcal{L}[\text{f(t)}] = \int_0^\infty e^{-st} f(t)dt$

$$\mathcal{L}\left[f'(t)\right] = \int_0^\infty e^{-st} f'(t)dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt$$

$$= \left[0 - f(0)\right] + s\int_0^\infty e^{-st} f(t)dt$$

$$\mathcal{L}\left[f'(t)\right] = s\mathcal{L}\left[f(t)\right] - f(0)$$

$$\mathcal{L}\left[f''(t)\right] = s\mathcal{L}\left[f'(t)\right] - f'(0)$$

$$\mathcal{L}\left[f''(t)\right] = s\left[s\mathcal{L}\left[f(t)\right] - f(0)\right] - f'(0)$$

$$\mathcal{L}\left[f''(t)\right] = s^2\mathcal{L}\left[f(t)\right] - sf(0) - f'(0)$$

$$\vdots \qquad \vdots$$

$$\mathcal{L}\left[f^n(t)\right] = s^n\mathcal{L}\left[f(t)\right] - s^{n-1}f(0) - sf^{n-2}(0) - f^{n-1}(0)$$

Derivative of Laplace transformation:

As $$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

$$\frac{dF}{ds} = \int_0^\infty \frac{d}{ds} e^{-st} f(t)dt$$

$$= \int_0^\infty (-t)e^{-st} f(t)dt$$

$$= (-1)\int_0^\infty e^{-st} t\, f(t)dt$$

$$= (-1)\mathcal{L}\left[f(t)\right]$$

$$\mathcal{L}\left[f(t)\right] = (-1)\frac{dF}{ds}$$

$$\frac{d^2F}{ds^2} = (-1)\int_0^\infty \frac{d}{ds} e^{-st} t dt$$

$$= (-1)^2 \int_0^\infty e^{-st} t^2 \, dt$$

$$= (-1)^2 \mathcal{L}\left[t^2 f(t)\right]$$

$$\mathcal{L}\left[t^2 f(t)\right] = (-1)^2 \frac{d^2F}{ds^2}$$

. .

. .

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^nF}{ds^n}$$

Question: $\mathcal{L}\left[t^2 \sin t\right] = ?$

Solution: Here f(t) = sint

$$F(s) = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

$$\mathcal{L}\left[t^2 \sin t\right] = (-1)^2 \frac{d^2F}{ds^2}$$

$$= (-1)^2 \frac{d^2F}{ds^2}\left[\frac{1}{s^2+1}\right]$$

$$= (-1)^2 \frac{d}{ds}\left[\frac{d}{ds}\left(s^2+1\right)^{-1}\right]$$

$$= (-1)^2 \frac{d}{ds}\left[(-1)\left(s^2+1\right)^{-1}(2s)\right]$$

$$= (-1)^2 (-2)\frac{d}{ds}\left[\frac{s}{\left(s^2+1\right)^2}\right]$$

$$=(-1)^2(-2)\left[\frac{\left(s^2+1\right)^2.1-s.2\left(s^2+1\right).2s}{\left(s^2+1\right)^4}\right]$$

$$=(-1)^2(-2)\left(s^2+1\right)\left[\frac{s^2+1-4s^2}{\left(s^2+1\right)^3}\right]$$

$$=(-1)^2(-2)\left[\frac{1-3s^2}{\left(s^2+1\right)^3}\right]$$

$$\mathcal{L}\left[t^2\sin t\right]=(-1)^2\left[\frac{2\left(3s^2-1\right)}{\left(s^2+1\right)^3}\right]$$

Unit function:

The unit function is defined as

H(t–a) = 0 ; t < a

H(t–a) = 1 ; t ≤ a

Proof:

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

$$F(s)=\int_0^a e^{-st}f(t)dt+\int_a^\infty e^{-st}f(t)dt$$

$$F(s)=\int_0^a e^{-st}H(t-a)dt+\int_a^\infty e^{-st}H(t-a)dt$$

$$F(s)=0+\int_a^\infty e^{-st}(1)dt$$

$$F(s)=\int_a^\infty e^{-st}dt \qquad \Rightarrow \qquad F(s)=\left.\frac{e^{-st}}{-s}\right|_a^\infty=0-\left(\frac{e^{-at}}{-s}\right)=\frac{e^{-at}}{s}$$

Lecture # 08

Theorem:

Let $\mathcal{L}\left[f(t)\right]=F(s)$ then $\mathcal{L}\left[H(t-a)f(t-a)\right]=e^{-as}F(s)$

Where H(t–a) = 0 ; t < a & H(t–a) = 1 ; t ≥ a

Solution:

$$\mathcal{L}\left[H(t-a)f(t-a)\right]=\int_0^\infty e^{-st}H(t-a)f(t-a)\,dt$$

$$=\int_0^a e^{-st}H(t-a)f(t-a)\,dt+\int_a^\infty e^{-st}H(t-a)f(t-a)\,dt$$

$$=0+\int_a^\infty e^{-st}f(t-a)\,dt$$

Put z = t – a ⇒ t = z + a

dz = dt when t = ∞ , z = ∞

$$=\int_a^\infty e^{-s(z+a)}f(z)\,dz$$

$$=\int_a^\infty e^{-sz}e^{-sa}f(z)\,dz$$

$$=e^{-as}\int_a^\infty e^{-sz}f(z)\,dz$$

$=e^{-as}F(s)$ proved

Question: $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y=\cos at$ $\quad\quad y(0)=1\,,\ y'(0)=0$

Solution: By applying Laplace transform on both side

$$\mathcal{L}\left[\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y\right]=\mathcal{L}\left[\cos at\right]$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right]-4\mathcal{L}\left[\frac{dy}{dt}\right]+3\mathcal{L}\left[y\right]=\frac{s}{s^2+a^2} \qquad \because\ \mathcal{L}\left[\cos at\right]=\frac{s}{s^2+a^2}$$

$$s^2\mathcal{L}\left[y\right]-sy(0)-y'(0)-4s\mathcal{L}\left[y\right]+4y(0)+3\mathcal{L}\left[y\right]=\frac{s}{s^2+a^2}$$

$$s^2\mathcal{L}\left[y\right]-s-0-4s\mathcal{L}\left[y\right]+4+3\mathcal{L}\left[y\right]=\frac{s}{s^2+a^2}$$

$$\left(s^2-4s+3\right)\mathcal{L}\left[y\right]=s-4+\frac{s}{s^2+a^2}$$

$$\mathcal{L}\left[y\right]=\frac{s-4}{\left(s^2-4s+3\right)}+\frac{s}{\left(s^2+a^2\right)\left(s^2-4s+3\right)} \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Now by partial fraction

$$\frac{s-4}{\left(s-1\right)\left(s-3\right)}=\frac{A}{s-1}+\frac{B}{s-3}$$

$$s-4=A\left(s-3\right)+B\left(s-1\right)$$

Put s–1 = 0 $\Rightarrow$ s = 1

$$1-4=A\left(1-3\right)+B\left(1-1\right)$$

$$-3=-2A$$

$$\Rightarrow \qquad A=\frac{3}{2}$$

Put s–3 = 0 $\Rightarrow$ s = 3

$$3-4=A(3-3)+B(3-1)$$

$$-1=2B$$

$$\Rightarrow \qquad B=-\frac{1}{2}$$

$$\frac{s-4}{(s-1)(s-3)}=\frac{3}{2(s-1)}-\frac{1}{2(s-3)}=\frac{\alpha_1}{(s-1)}-\frac{\alpha_2}{(s-3)} \quad where\ \alpha_1=\frac{3}{2},\alpha_2=-\frac{1}{2}$$

Now

$$\frac{s}{(s^2+a^2)(s^2-4s+3)}=\frac{C}{s-1}+\frac{D}{s-3}+\frac{Es+F}{s^2+a^2}$$

$$s=C(s^2+a^2)(s-3)+D(s^2+a^2)(s-1)+(Es+F)(s-1)(s-3)$$

Put s–1 = 0 $\Rightarrow$ s = 1

$$1=C(1^2+a^2)(1-3)$$

$$C=\frac{-1}{2(a^2+1)}$$

Put s–3 = 0 $\Rightarrow$ s = 3

$$3=d(a^2+3^2)(3-1)$$

$$D=\frac{3}{2(a^2+9)}$$

$$s=C(s^3+a^2s-3s^2-3a^2)+d(s^3+a^2s-s^2-a^2)+Es^3-4Es^2+3Es+Fs^2-4Fs+3F$$

By comparing coefficient of s^3

$$0 = C + D + E$$

$$E = -C - D$$

$$E=\frac{1}{2\left(a^2+1\right)}-\frac{3}{2\left(a^2+9\right)}$$

$$E=\frac{a^2+9-3a^2-3}{2\left(a^2+1\right)\left(a^2+9\right)}=\frac{-2a^2+6}{2\left(a^2+1\right)\left(a^2+9\right)}=\frac{3-a^2}{\left(a^2+1\right)\left(a^2+9\right)}$$

By comparing coefficient of s^2

$$0=-3C-D-4E+F$$

$$F=3C+D+4E$$

$$F=\frac{-3}{2\left(a^2+1\right)}+\frac{3}{2\left(a^2+9\right)}+\frac{4(3-a^2)}{\left(a^2+1\right)\left(a^2+9\right)}$$

$$=\frac{-3\left(a^2+9\right)+3\left(a^2+1\right)+8(3-a^2)}{2\left(a^2+1\right)\left(a^2+9\right)}$$

$$=\frac{-27-3a^2+3+3a^2+24-8a^2}{2\left(a^2+1\right)\left(a^2+9\right)}$$

$$=\frac{-8a^2}{2\left(a^2+1\right)\left(a^2+9\right)}$$

$$F=\frac{-4a^2}{\left(a^2+1\right)\left(a^2+9\right)}$$

$$\frac{Es+F}{s^2+a^2}=\frac{\frac{\left(3-a^2\right)}{\left(a^2+1\right)\left(a^2+9\right)}-\frac{-4a^2}{\left(a^2+1\right)\left(a^2+9\right)}}{s^2+a^2}$$

$$=\frac{\left(3-a^2\right)s}{\left(s^2+a^2\right)\left(a^2+1\right)\left(a^2+9\right)}-\frac{4a\,.\,a}{\left(s^2+a^2\right)\left(a^2+1\right)\left(a^2+9\right)}$$

$$=\alpha\frac{s}{\left(s^2+a^2\right)}-\beta\frac{a}{\left(s^2+a^2\right)}$$

Where $\alpha=\dfrac{\left(3-a^2\right)}{\left(a^2+1\right)\left(a^2+9\right)}$, $\beta=\dfrac{4a}{\left(a^2+1\right)\left(a^2+9\right)}$

$$\frac{s}{\left(s^2+a^2\right)\left(s^2-4s+3\right)}=\frac{-1}{2\left(a^2+1\right)(s-1)}+\frac{3}{2\left(a^2+9\right)(s-3)}+\alpha\frac{s}{\left(s^2+a^2\right)}-\beta\frac{a}{\left(s^2+a^2\right)}$$

$$\frac{s}{\left(s^2+a^2\right)\left(s^2-4s+3\right)}=\frac{\alpha_3}{(s-1)}+\frac{\alpha_4}{(s-3)}+\frac{\alpha s}{\left(s^2+a^2\right)}-\frac{\beta a}{\left(s^2+a^2\right)}$$

Where $\alpha_3=\dfrac{-1}{2\left(a^2+1\right)}$, $\alpha_4=\dfrac{3}{2\left(a^2+9\right)}$

Put these values in (i)

$$\mathcal{L}[y]=\frac{\alpha_1}{(s-1)}-\frac{\alpha_2}{(s-3)}+\frac{\alpha_3}{(s-1)}+\frac{\alpha_4}{(s-3)}+\frac{\alpha s}{\left(s^2+a^2\right)}-\frac{\beta a}{\left(s^2+a^2\right)}$$

$$y=\mathcal{L}^{-1}\left[\frac{\alpha_1}{(s-1)}\right]-\alpha_2\mathcal{L}^{-1}\left[\frac{1}{(s-3)}\right]+\alpha_3\mathcal{L}^{-1}\left[\frac{1}{(s-1)}\right]+\alpha_4\mathcal{L}^{-1}\left[\frac{1}{(s-3)}\right]+\alpha\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)}\right]-\beta\mathcal{L}^{-1}\left[\frac{a}{\left(s^2+a^2\right)}\right]$$

$$y=\alpha_1e^t-\alpha_2e^{3t}+\alpha_3e^t+\alpha_4e^{3t}+\alpha\cos at-\beta\sin at$$

Put the values of $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha,\beta$

$$y=\frac{3}{2}e^t-\frac{1}{2}e^{3t}-\frac{1}{2\left(a^2+1\right)}e^t+\frac{3}{2\left(a^2+9\right)}e^{3t}+\frac{\left(3-a^2\right)}{\left(a^2+1\right)\left(a^2+9\right)}\cos at-\frac{4a}{\left(a^2+1\right)\left(a^2+9\right)}\sin at$$

is required answer.

Question: $\dfrac{dw}{dt}+aw=H(t-1)$ $\qquad w(0)=1$

Solution: By applying Laplace on both sides

$$\mathcal{L}\left[\frac{dw}{dt}+aw\right]=\mathcal{L}\left[H(t-1)\right]$$

$$\mathcal{L}\left[\frac{dw}{dt}\right]+a\mathcal{L}[w]=\frac{e^{-s}}{s}$$

$$s\mathcal{L}[w]-w(0)+a\mathcal{L}[w]=\frac{e^{-s}}{s}$$

$$(s+a)\mathcal{L}[w]-1=\frac{e^{-s}}{s}$$

$$(s+a)\mathcal{L}[w]=1+\frac{e^{-s}}{s}$$

$$\mathcal{L}[w]=\frac{1}{(s+a)}+\frac{e^{-s}}{s(s+a)} \qquad \_\_\_\_\_\_\text{(i)}$$

$$\frac{1}{s(s+a)}=\frac{A}{s}+\frac{B}{s+a}$$

$$1=A(s+a)+B(s)$$

Put s = 0

$$1=A(a)$$

$$\Rightarrow \qquad A=\frac{1}{a}$$

Put s + a = 0 $\Rightarrow$ s = –a

$$1=B(-a)$$

$$\Rightarrow \qquad B=-\frac{1}{a}$$

$$\frac{1}{s(s+a)}=\frac{1}{as}-\frac{1}{a(s+a)} \qquad \_\_\_\text{(i)}$$

$$\mathcal{L}[w]=\frac{1}{s+a}+\frac{e^{-s}}{as}-\frac{e^{-s}}{a(s+a)}$$

$As\ \mathcal{L}\left[H(t-a)f(t-a)\right]=e^{-as}F(s)$

$\mathcal{L}^{-1}\frac{e^{-s}}{s}, a=1$

$f(t)=\frac{1}{s}$

$\mathcal{L}^{-1}\frac{e^{-s}}{s}=H(t-1)(1)$

$\mathcal{L}^{-1}\frac{e^{-s}}{s+a}, a=1$

$f(t)=\frac{1}{s+a}$

$\mathcal{L}^{-1}\frac{1}{s+a}=e^{-at}$

Here t = t–1

$\mathcal{L}^{-1}\frac{e^{-s}}{s+a}=H(t-1)e^{-a(t-1)}$

$$\mathcal{L}^{-1}\mathcal{L}[w]=\mathcal{L}^{-1}\left[\frac{1}{s+a}+\frac{e^{-s}}{as}-\frac{e^{-s}}{a(s+a)}\right]$$

$$w=\mathcal{L}^{-1}\left[\frac{1}{s+a}\right]+\frac{1}{a}\mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right]-\frac{1}{a}\mathcal{L}^{-1}\left[\frac{e^{-s}}{(s+a)}\right]$$

$$w=e^{at}+\frac{1}{a}H(t-1)-\frac{1}{a}H(t-1)e^{-a(t-1)}$$

is the required answer.

Question: $\frac{\partial w}{\partial x}+x\frac{\partial w}{\partial t}=0$; $w(x,0)=0$, $\mathrm{w}(0,\mathrm{t})=\mathrm{t}$

Solution: Apply Laplace transform on both sides

$$\mathcal{L}\left[\frac{\partial w}{\partial x}+x\frac{\partial w}{\partial t}\right]=\mathcal{L}[0]$$

$$\mathcal{L}\left[\frac{\partial w}{\partial x}\right]+x\mathcal{L}\left[\frac{\partial w}{\partial t}\right]=\mathcal{L}[0]$$

$$\frac{\partial}{\partial x}\mathcal{L}[w(x,t)]+x[s\mathcal{L}w(x,t)-w(x,0)]=0$$

$$\frac{\partial}{\partial x}\mathcal{L}[w(x,t)]+x[s\mathcal{L}w(x,t)]=0$$

Say $\mathcal{L}[w(x,t)]=\overline{w}(x,s)$

$$\frac{\partial}{\partial x}\overline{w}(x,s)+xs\,\overline{w}(x,s)=0$$

$$\frac{\partial}{\partial x}\overline{w}(x,s)=-xs\,\overline{w}(x,s)$$

$$\frac{\partial\overline{w}}{\overline{w}}=-xs\,\partial x$$

On integrating both sides

$$\int \frac{\partial \overline{w}}{\overline{w}} = -s\int x\partial x$$

$$\ln \overline{w} = -\frac{x^2}{2}s + A(s)$$

$$\overline{w} = e^{-\frac{x^2}{2}s + A(s)}$$

$$\overline{w} = e^{-\frac{x^2}{2}s} . e^{A(s)}$$

$$\overline{w} = e^{-\frac{x^2}{2}s} . B(s) \qquad \_\_(i)$$

$$\because B(s) = e^{A(s)} = constant$$

Given that

w(x,0) = 0 & w(0,t) = t

$$\mathcal{L}\left[w(x,0)\right] = \mathcal{L}[0] \quad \& \quad \mathcal{L}\left[w(0,t)\right] = \mathcal{L}[t]$$

$$\overline{w}(x,0) = 0 \quad \& \quad \overline{w}(0,s) = \frac{1}{s^2}$$

(i) $\Rightarrow$ $\overline{w}(x,s) = e^{-\frac{x^2}{2}s} . B(s)$

Put x = 0

$$\frac{1}{s^2} = B(s)$$

Put in (i)

$$\mathcal{L}[w(x,t)] = \frac{1}{s^2} e^{-\frac{x^2}{2}s} \quad \because \mathcal{L}[w(x,t)] = \overline{w}(x,s)$$

$$w(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2} e^{-\frac{x^2}{2}s}\right]$$

$$w = H\left(t - \frac{x^2}{2}\right) f\left(t - \frac{x^2}{2}\right)$$

is the required answer

$$As\ \mathcal{L}\left[H(t-a)f(t-a)\right] = e^{-as}F(s)$$

$$a = -\frac{x^2}{2}, f(s) = \frac{1}{s^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

Here t = t-a

Lecture # 09

Question: $\dfrac{\partial^2 w}{\partial x^2}=c^2\dfrac{\partial^2 w}{\partial t^2}$

Boundary condition w(0,t) = f(t)

$$\lim_{x\to\infty} w(x,t)=0$$

Initial condition $w(x,0)=0$, $\dfrac{\partial w}{\partial t}(x,0)=0$

Solution: Apply Laplace transform on both sides

$$\mathcal{L}\left[\frac{\partial^2 w}{\partial x^2}\right]=c^2\mathcal{L}\left[\frac{\partial^2 w}{\partial t^2}\right]$$

$$\frac{\partial^2}{\partial x^2}\mathcal{L}[w(x,t)]=c^2\left[s^2\mathcal{L}[w(x,t)]-w(x,0)-w'(x,0)\right]$$

By Initial condition

$$\frac{\partial^2}{\partial x^2}\mathcal{L}[w(x,t)]=c^2\left[s^2\mathcal{L}[w(x,t)]-0-0\right]$$

$$\frac{\partial^2}{\partial x^2}\mathcal{L}[w(x,t)]=c^2 s^2\mathcal{L}[w(x,t)]$$

Say $\mathcal{L}[w(x,t)]=\overline{w}(x,s)$ ______(i)

$$\frac{\partial^2}{\partial x^2}\overline{w}(x,s)=c^2 s^2\overline{w}(x,s)$$

$$\frac{\partial^2}{\partial x^2}\overline{w}(x,s)-c^2 s^2\overline{w}(x,s)=0$$

$$\left(\frac{\partial^2}{\partial x^2}-c^2 s^2\right)\overline{w}(x,s)=0$$

$$\frac{\partial^2}{\partial x^2}-c^2 s^2=0$$

$$\frac{\partial^2}{\partial x^2} = c^2\, s^2$$

$$\frac{\partial}{\partial x} = \pm c\, s$$

$$\overline{w}(x,s) = A(s)e^{csx} + B(s)e^{-csx} \qquad \_\_\_\text{(ii)}$$

Now by boundary condition

w(0,t) = f(t)

Applying Laplace transformation

$$\mathcal{L}\left[w(0,t)\right] = \mathcal{L}\left[f(t)\right]$$

$$\overline{w}(0,s) = F(s) \qquad \_\_\_\_\text{(iii)}$$

Put x = 0 in (ii) and compare with (iii)

$$\overline{w}(x,0) = A(s)e^{0} + B(s)e^{-0}$$

$$F(s) = A(s) + B(s) \qquad \_\_\_\_\_\_ \quad \text{(iv)}$$

$$\lim_{x\to\infty} w(x,t) = 0$$

Applying Laplace Transformation

$$\lim_{x\to\infty} \mathcal{L}\left[w(x,t)\right] = \mathcal{L}[0]$$

$$\lim_{x\to\infty} \overline{w}(x,\text{s}) = 0 \qquad \_\_\_\_(v)$$

Apply $\lim\limits_{x\to\infty}$ on eq (ii) and compare with (v)

$$\lim_{x\to\infty} \overline{w}(x,\text{s}) = \lim_{x\to\infty} A(s)e^{csx} + \lim_{x\to\infty} B(s)e^{-csx}$$

$$0 = A(s)e^{\infty} + B(s)e^{-\infty}$$

$$0 = A(s)e^{\infty} + B0$$

$$\Rightarrow \quad A(s) = 0$$

Put in (iv)

F(s) = 0 + B(s)

Put these values in (ii)

$$\overline{w}(x,s)=(0)e^{csx}+F(\mathrm{s})\mathrm{e}^{-csx}$$

$$\overline{w}(x,s)=F(\mathrm{s})\mathrm{e}^{-csx}$$

From eq (i)

$$\mathcal{L}\left[w(x,\mathrm{t})\right]=F(s)e^{-csx}$$

$$\mathcal{L}^{-1}\mathcal{L}\left[w(x,\mathrm{t})\right]=\mathcal{L}^{-1}F(s)e^{-csx}$$

$$w(x,t)=H(t-cx)f(t-cx)$$

$$\because \mathcal{L}\left[H(t-a)f(t-a)\right]=e^{-ax}F(\mathrm{s})$$

Lecture # 10

Question: Find Laplace transform of

$$\phi_{tt} = a^2\phi_{xx} - g$$

Boundary Condition $\phi(0,t) = 0$, $\lim_{x\to\infty}\phi_x(x,t) = 0$

Initial Condition $\phi(x,0) = 0$, $\phi_t(x,0) = 0$

Solution: Apply Laplace transform

$$\mathcal{L}[\phi_{tt}] = \mathcal{L}\left[a^2\phi_{xx}\right] - \mathcal{L}[g]$$

$$\mathcal{L}\left[\frac{\partial^2\phi(x,t)}{\partial t^2}\right] = a^2\mathcal{L}\left[\frac{\partial^2\phi(x,t)}{\partial x^2}\right] - g\mathcal{L}[1]$$

$$s^2\mathcal{L}\left[\phi(x,t)\right] - s\phi(x,0) - \phi_t(x,0) = a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - g.\frac{1}{s}$$

By initial condition

$$s^2\mathcal{L}\left[\phi(x,t)\right] - s(0) - 0 = a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - \frac{g}{s}$$

$$s^2\mathcal{L}\left[\phi(x,t)\right] = a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - \frac{g}{s}$$

$$a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - s^2\mathcal{L}\left[\phi(x,t)\right] = \frac{g}{s}$$

let $\mathcal{L}[\phi(\mathrm{x,t})] = \overline{\phi}(\mathrm{x,s})$

$$a^2\frac{\partial^2}{\partial x^2}\overline{\phi}(\mathrm{x,s}) - s^2\overline{\phi}(\mathrm{x,s}) = \frac{g}{s}$$

$$\left(a^2\frac{\partial^2}{\partial x^2} - s^2\right)\overline{\phi}(\mathrm{x,s}) = \frac{g}{s}$$

This is non-homogenous equation

For complementary solution we have

$$a^2\frac{\partial^2}{\partial x^2}-s^2=0$$

$$\frac{\partial^2}{\partial x^2}=\frac{s^2}{a^2}$$

$$\frac{\partial}{\partial x}=\pm\frac{s}{a}$$

$$\overline{\phi_c}(x,s)=Ae^{\frac{s}{a}x}+Be^{-\frac{s}{a}x}$$

For particular solution

$$\left(a^2\frac{\partial^2}{\partial x^2}-s^2\right)\overline{\phi}_p(\mathrm{x,t})=\frac{g}{s}$$

$$\left(a^2D^2-s^2\right)\overline{\phi}_p(\mathrm{x,t})=\frac{g}{s}$$

$$\overline{\phi}_p(\mathrm{x,t})=\frac{1}{\left(a^2D^2-s^2\right)}.\frac{g}{s}$$

$$\overline{\phi}_p(\mathrm{x,t})=\frac{1}{-s^2\left(1-\frac{a^2}{s^2}D^2\right)}.\frac{g}{s}$$

$$\overline{\phi}_p(\mathrm{x,t})=\frac{-1}{s^2}\left(1-\frac{a^2}{s^2}D^2\right)^{-1}.\frac{g}{s}$$

$$\overline{\phi}_p(\mathrm{x,t})=\frac{-1}{s^2}.\left(1+\frac{a^2}{s^2}D^2+...\right)\frac{g}{s}$$

$$\overline{\phi}_p(\mathrm{x,t})=\frac{-1}{s^2}.\left(\frac{g}{s}+0\right)$$

$$\overline{\phi}_p(\mathrm{x,t})=\frac{-g}{s^3}$$

$$\overline{\phi}(\mathrm{x},\mathrm{t})=\overline{\phi_c}(\mathrm{x},\mathrm{t})+\overline{\phi_p}(\mathrm{x},\mathrm{t})$$

$$\overline{\phi}(\mathrm{x},\mathrm{t})=Ae^{\frac{s}{a}x}+Be^{-\frac{s}{a}x}-\frac{g}{s^3} \quad \_\_\_\_\_(1)$$

Now by boundary condition

$$\phi(0,t)=0$$

$$\mathcal{L}\left[\phi(0,t)\right]=\mathcal{L}[0]$$

$$\overline{\phi}(0,\mathrm{s})=0 \quad \_\_\_\_\_(2)$$

Put x = 0 in (1) and compare with (2)

$$\overline{\phi}(0,\mathrm{s})=A+B-\frac{g}{s^3}$$

$$\Rightarrow \quad A+B-\frac{g}{s^3}=0 \quad \_\_\_\_\_(*)$$

Now $\lim\limits_{x\to\infty}\phi_x(x,t)=0$

$$\lim_{x\to\infty}\mathcal{L}\left[\phi_x(x,t)\right]=\mathcal{L}[0]$$

$$\lim_{x\to\infty}\overline{\phi_x}(x,\mathrm{s})=0 \quad \_\_\_\_(3)$$

Diff. eq (1) w.r.t x and applying $\lim\limits_{x\to\infty}$ and compare with (3)

$$\overline{\phi_x}(x,\mathrm{s})=\frac{s}{a}Ae^{\frac{s}{a}x}-\frac{s}{a}Be^{-\frac{s}{a}x}$$

$$\lim_{x\to\infty}\overline{\phi_x}(x,\mathrm{s})=\frac{s}{a}Ae^{\frac{s}{a}x}-0$$

$$0=\frac{s}{a}Ae^{\infty}-0 \quad \Rightarrow A=0$$

Put A = 0 in (*)

$$B-\frac{g}{s^3}=0 \quad \Rightarrow \quad B=\frac{g}{s^3}$$

Put the value of A and B in Eq (1)

$$\bar{\phi}(\mathrm{x,s})=\frac{g}{s^3}e^{-\frac{s}{a}x}-\frac{g}{s^3}$$

$$\mathcal{L}[\phi(\mathrm{x,t})]=\frac{g}{s^3}e^{-\frac{s}{a}x}-\frac{g}{s^3}$$

$$\mathcal{L}^{-1}\mathcal{L}[\phi(\mathrm{x,t})]=\mathcal{L}^{-1}\left[\frac{g}{s^3}e^{-\frac{s}{a}x}\right]-\mathcal{L}^{-1}\left[\frac{g}{s^3}\right]$$

$$[\phi(\mathrm{x,t})]=g\left[H\left(t-\frac{x}{a}\right)\frac{t^2}{2!}\right]-g.\frac{t^2}{2!} \qquad \because \mathcal{L}\left[H(t-a)f(t-a)\right]=F(s)e^{-as}\ \&\ \mathcal{L}\left[t^n\right]=\frac{n!}{s^{n+1}}$$

Question: Find Laplace transform of

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial\phi}{\partial t}$$

Boundary Condition $\qquad \phi(0,t)=1=\phi(1,t)$

Initial Condition $\qquad \phi(x,0)=1+\sin(\pi x)$

Solution: Apply Laplace transform

$$\mathcal{L}\left[\frac{\partial^2\phi}{\partial x^2}\right]=\mathcal{L}\left[\frac{\partial\phi}{\partial t}\right]$$

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right]=s\mathcal{L}\left[\phi(x,t)\right]-\phi(x,0)$$

By putting initial condition

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right]=s\mathcal{L}\left[\phi(x,t)\right]-1-\sin(\pi x)$$

let $\quad \mathcal{L}[\phi(\mathrm{x,t})]=\bar{\phi}(\mathrm{x,s})$

$$\frac{\partial^2}{\partial x^2}\overline{\phi}(\mathrm{x,s}) = s\overline{\phi}(\mathrm{x,s}) - 1 - \sin(\pi x)$$

$$\left(\frac{\partial^2}{\partial x^2} - s\right)\overline{\phi}(\mathrm{x,s}) = -1 - \sin(\pi x)$$

For complementary solution we have

$$\frac{\partial^2}{\partial x^2} - s = 0$$

$$\frac{\partial^2}{\partial x^2} = s$$

$$\frac{\partial}{\partial x} = \pm\sqrt{s}$$

$$\overline{\phi_c}(x,s) = Ae^{\sqrt{s}\,x} + Be^{-\sqrt{s}\,x}$$

For particular solution

$$\left(\frac{\partial^2}{\partial x^2} - s\right)\overline{\phi_p}(\mathrm{x,s}) = -1 - \sin(\pi x)$$

$$\left(D^2 - s\right)\overline{\phi}_p(\mathrm{x,s}) = -1 - \sin \pi x$$

$$\overline{\phi}_p(\mathrm{x,s}) = \frac{1}{\left(D^2 - s\right)}.(-1 - \sin \pi x)$$

$$\overline{\phi}_p(\mathrm{x,s}) = \frac{1}{\left(D^2 - s\right)}(-1) + \frac{1}{\left(D^2 - s\right)}(-\sin \pi x)$$

$$\overline{\phi}_p(\mathrm{x,s}) = \frac{-1}{-s\left(1 - \frac{D^2}{s}\right)} - \frac{1}{\left(D^2 - s\right)}(\sin \pi x)$$

$$\overline{\phi}_p(\mathrm{x,s}) = \frac{\left(1 - \frac{D^2}{s}\right)^{-1}}{s} - \frac{1}{\left(\pi^2 - s\right)}(\sin \pi x)$$

$$\bar{\phi}_p(\mathrm{x,s}) = \frac{1}{s}\left(1+\frac{D^2}{s}+..\right)(1)+\frac{1}{\pi^2+s}(\sin\pi x)$$

$$\bar{\phi}_p(\mathrm{x,s}) = \frac{1}{s}(1+0)+\frac{1}{\pi^2+s}(\sin\pi x)$$

$$\bar{\phi}_p(\mathrm{x,s}) = \frac{1}{s}+\frac{\sin\pi x}{\pi^2+s}$$

$$\bar{\phi}(\mathrm{x,s}) = \overline{\phi_c}(\mathrm{x,s})+\overline{\phi_p}(\mathrm{x,s})$$

$$\bar{\phi}(\mathrm{x,s}) = Ae^{\sqrt{s}\,x}+Be^{-\sqrt{s}\,x}+\frac{1}{s}+\frac{\sin(\pi x)}{\pi^2+s} \quad \_\_\_\_\_(i)$$

Now by boundary condition

$$\phi(0,t)=1$$

$$\mathcal{L}\left[\phi(0,t)\right]=\mathcal{L}[1]$$

$$\bar{\phi}(0,\mathrm{s})=\frac{1}{s} \qquad \_\_\_\_\_(\mathrm{ii})$$

Put x = 0 in (i) and compare with (ii)

$$\bar{\phi}(0,\mathrm{s}) = A+B+\frac{1}{s}$$

$$\Rightarrow \frac{1}{s} = A+B+\frac{1}{s}$$

$$\Rightarrow A+B=0$$

$$\Rightarrow A=-B \qquad \_\_\_\_(iii)$$

Now $\phi(1,t)=1$

$$\mathcal{L}\left[\phi(1,t)\right]=\mathcal{L}[1]$$

$$\bar{\phi}(1,\mathrm{s})=\frac{1}{s} \qquad \_\_\_\_(\mathrm{iv})$$

Put x = 1 in eq (i) and compare with (iv)

$$\bar{\phi}(1,s) = Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + \frac{1}{s}$$

$$\frac{1}{s} = Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + \frac{1}{s}$$

$$Ae^{\sqrt{s}} + Be^{-\sqrt{s}} = 0$$

Put A = – B

$$-Be^{\sqrt{s}} + Be^{-\sqrt{s}} = 0$$

$$\left(-e^{\sqrt{s}} + e^{-\sqrt{s}}\right)B = 0$$

$\Rightarrow$ B = 0

$\Rightarrow$ A = 0

Put in (i)

$$\bar{\phi}(x,s) = \frac{1}{s} + \frac{\sin(\pi x)}{\pi^2 + s}$$

$$\mathcal{L}[\phi(x,t)] = \frac{1}{s} + \frac{\sin(\pi x)}{\pi^2 + s} \qquad \because \bar{\phi}(x,s) = \mathcal{L}[\phi(x,t)]$$

$$\mathcal{L}^{-1}\mathcal{L}[\phi(x,t)] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{\sin(\pi x)}{\pi^2 + s}\right]$$

$$\mathcal{L}^{-1}\mathcal{L}[\phi(x,t)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{\sin(\pi x)}{\pi^2 + s}\right]$$

$$\phi(x,t) = 1 + \mathcal{L}^{-1}\left[\frac{\sin(\pi x)}{\pi^2 + s}\right]$$

Lecture # 11

Bessel Equation and Bessel Functions:

Bessel equation is an important equation in applied mathematics and is given by

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \qquad \_\_\_(i)$$

Where the perimeter ν (neu) is a given number and it is assumed to be real and non-negative from Frobenius method. Its solution is supposed to be in the power series from

$$y(x) = \sum_{m=0}^{\infty} C_m\, x^{m+r} \qquad \_\_\_(ii)$$

Substituting this expression and its derivatives into Bessel's equation we get the following

$$x^2\left(\sum_{m=0}^{\infty} C_m\, x^{m+r}\right)'' + x\left(\sum_{m=0}^{\infty} C_m\, x^{m+r}\right)' + \left(x^2 - \nu^2\right)\left(\sum_{m=0}^{\infty} C_m\, x^{m+r}\right) = 0$$

$$x^2\left(\sum_{m=0}^{\infty} C_m (m+r)(m+r-1)x^{m+r-2}\right) + x\left(\sum_{m=0}^{\infty} C_m (m+r)x^{m+r}\right) + x^2\left(\sum_{m=0}^{\infty} C_m x^{m+r}\right) - \nu^2\left(\sum_{m=0}^{\infty} C_m x^{m+r}\right) = 0$$

$$\sum_{m=0}^{\infty} C_m\left[(m+r)^2 - (m+r) + (m+r) - \nu^2\right]x^{m+r} + \sum_{m=0}^{\infty} C_m x^{m+r+2} = 0$$

$$\sum_{m=0}^{\infty} C_m\left[(m+r)^2 - \nu^2\right]x^{m+r} + \sum_{m=0}^{\infty} C_m x^{m+r+2} = 0 \qquad \_\_\_\_\_(iii)$$

From equation (ii) is an identity in x and true for every value of x. So, put the coefficient of term involving the smallest power of $x^r = 0$ so eq (iii) becomes

$$C_0\left[r^2 - \nu^2\right] = 0$$

$$C_0 \neq 0 \quad , \quad r^2 - \nu^2 = 0$$

$$r = \pm \nu$$

The roots are $r_1 = \nu \;\; and \;\; r_2 = -\nu$ a solution is determined from (iii)

$$\sum_{m=0}^{\infty} C_m \left[(m+r)^2 - \nu^2 \right] x^{m+r} + \sum_{m=0}^{\infty} C_m x^{m+r+2} = 0$$

Replace m by m – 2 in 2nd term

$$\sum_{m=0}^{\infty} C_m \left[(m+r)^2 - \nu^2 \right] x^{m+r} + \sum_{m=0}^{\infty} C_{m-2} x^{m+r} = 0 \qquad \_\_\_\_(iv)$$

$$Put \quad r = \nu \;\; in \; eq(iv)$$

$$\sum_{m=0}^{\infty} C_m \left[(m+\nu)^2 - \nu^2 \right] x^{m+\nu} + \sum_{m=0}^{\infty} C_{m-2} x^{m+\nu} = 0$$

$$\sum_{m=0}^{\infty} C_m \left[m^2 + \nu^2 - 2m\nu - \nu^2 \right] x^{m+\nu} + \sum_{m=0}^{\infty} C_{m-2} x^{m+\nu} = 0$$

$$C_m \left[m(m+2\nu) \right] + C_{m-2} = 0 \qquad \_\_\_\_(v)$$

Put m = 1

$$C_1 \left[1(1+2\nu) \right] + C_{1-2} = 0$$

$$C_1 (1+2\nu) + C_{-1} = 0$$

$$C_1 (1+2\nu) = 0 \qquad \because C_{-1} = 0$$

Put m = 3

$$C_3 \left[3(3+2\nu) \right] + C_{3-2} = 0$$

$$C_3 .3(3+2\nu) + C_1 = 0$$

$$C_3 .3(3+2\nu) = 0 \qquad \because C_1 = 0$$

$$C_3 = 0 \qquad and \;\; 3(3+2\nu) \neq 0$$

It follows that $C_1 = C_3 = C_5 = \ldots = C_{2m-1} = 0$

From eq (v) we have

$$C_m = \frac{(-1)C_{m-2}}{m(m+2\nu)}$$

Replace m by 2m

$$C_{2m} = \frac{(-1)C_{2m-2}}{2m(2m+2\nu)}$$

$$C_{2m} = \frac{(-1)C_{2m-2}}{2^2 m(m+\nu)} \qquad \_\_\_\_(vi)$$

This determine the coefficient $C_2, C_4, C_6 \ldots$ sufficiently C_0 is arbitrary suppose it is given as

$$C_0 = \frac{1}{2^\nu \Gamma(\nu+1)} \qquad \_\_\_\_(vii)$$

Where $\Gamma(\nu+1)$ is a gamma function and $\Gamma(\alpha)$ is defined by integral

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \;\; (\alpha > 0) \qquad \_\_\_(viii)$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$\Gamma(k+1) = k! \quad , \; k = 0,1,2,\ldots\ldots$$

From eq (vi)

$$C_{2m} = \frac{(-1)C_{2m-2}}{2^2 m(m+\nu)}$$

$$C_2 = \frac{(-1)C_0}{2^2.1!(+\nu)}$$

$$C_2 = \frac{(-1)}{2^2.1!(1+\nu)} \cdot \frac{1}{2^\nu \Gamma(\nu+1)}$$

$$C_2 = \frac{(-1)}{2^{2+\nu} 1! \Gamma(\nu+2)} \qquad \because \alpha\Gamma(\alpha) = \Gamma(\alpha+1)$$

$$C_4 = \frac{(-1)}{2^2.2!(2+\nu)} \cdot \frac{-1}{2^{2+\nu}1!\overline{|(\nu+2)}}$$

$$C_4 = \frac{(-1)^2}{2^{4+\nu}.2!\overline{|\nu+3}}$$

Its genera form is $$C_{2m} = \frac{(-1)^m}{2^{2m+\nu}.\text{m}!\overline{|(m+\nu+1)}}$$

Eq (ii) $$y(x) = \sum_{m=0}^{\infty} C_m \, x^{m+r}$$

$$y(x) = x^r \sum_{m=0}^{\infty} C_m \, x^m$$

$$Put \ r = \nu \quad \Rightarrow \quad y(x) = x^\nu \sum_{m=0}^{\infty} C_m \, x^m$$

$$y(x) = x^\nu \left[C_0 + C_1 x^1 + C_2 x^2 + C_3 x^3 + \ldots . \right]$$

$$y(x) = x^\nu \left[C_0 + C_2 x^2 + C_4 x^4 + C_6 x^6 + \ldots . \right]$$

$$y(x) = x^\nu \sum_{m=0}^{\infty} C_{2m} \, x^{2m}$$

$$y(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu} \ \text{m}!\overline{|(m+\nu+1)}} x^{2m}$$

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\text{m}!\overline{|(m+\nu+1)}} \left(\frac{x}{2}\right)^{2m+\nu} \qquad \_\_\_\_(ix)$$

This solution is known as the Bessel function of first kind with order ν and denoted by $J_\nu(x)$ i.e. replace ν by $-\nu$ in (ix)

$$J_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\text{m}!\overline{|(m-\nu+1)}} \left(\frac{x}{2}\right)^{2m-\nu}$$

If ν is not an integer a general solution of Bessel's equation for all $x \neq 0$ is given by

$$y(x) = a_1 J_\nu(x) + a_2 J_{-\nu}(x)$$

The integral value of ν is mostly denoted by n

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\mathrm{m}!\,\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n}$$

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\mathrm{m}!\,(m-n)!} \left(\frac{x}{2}\right)^{2m-n}$$

Basic Identities and Recurrence formula:

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$$

$$\text{L.H.S} = \frac{d}{dx}\left[x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{\mathrm{m}!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}\right]$$

$$= \frac{d}{dx}\left[\sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu}\,\mathrm{m}!\,\Gamma(m+\nu+1)} x^{2m+2\nu}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\, 2(m+\nu)\, x^{2m+2\nu-1}}{2^{2m+\nu}\,\mathrm{m}!\,(m+\nu)\,\Gamma(m+\nu)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{\mathrm{m}!\,\Gamma(m+\nu)} \left(\frac{x}{2}\right)^{2m+2\nu-1}$$

$$= x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{\mathrm{m}!\,\Gamma(m+\nu-1)} \left(\frac{x}{2}\right)^{2m+\nu-1}$$

$$= x^\nu J_{\nu-1}(x) = R.H.S$$

Lecture # 12

Prove that $$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = -x^{\nu}J_{\nu+1}(x)$$

Proof: As we know that $$J_\nu(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\, x^{2m+\nu}}{2^{2m+\nu}\, m!\,\Gamma(m+\nu+1)}$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \frac{d}{dx}\left[x^{-\nu}\sum_{m=0}^{\infty}\frac{(-1)^m\, x^{2m+\nu}}{2^{2m+\nu}\, m!\,\Gamma(m+\nu+1)}\right]$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \frac{d}{dx}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\, x^{2m}}{2^{2m+\nu}\, m!\,\Gamma(m+\nu+1)}\right]$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \sum_{m=0}^{\infty}\frac{(-1)^m\, 2m\, x^{2m-1}}{2^{2m+\nu}\, m!\,\Gamma(m+\nu+1)}$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \sum_{m=0}^{\infty}\frac{(-1)^m\, 2m\, x^{2m-1}}{2^{2m+\nu}\, m(m-1)!\,\Gamma(m+\nu+1)}$$

Replace m by m+1

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \sum_{m=-1}^{\infty}\frac{(-1)^{m+1}\, x^{2m+2-1}}{2^{2m+\nu-1}(m+1-1)!\,\Gamma(m+1+\nu+1)}$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \sum_{m=-1}^{\infty}\frac{(-1)^{m+1}\, x^{2m+1}}{2^{2m+\nu-1}\, m!\,\Gamma(m+1+\nu+1)}$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \frac{(-1)^{-1+1}\, x^{-2+1}}{2^{-2+\nu-1}(-1)!\,\Gamma(-1+\nu+1+1)} + \sum_{m=0}^{\infty}\frac{(-1)^{m+1}\, x^{2m+1}}{2^{2m+\nu+1}\, m!\,\Gamma(m+\nu+1+1)}$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = 0 + x^{-\nu}\sum_{m=0}^{\infty}\frac{(-1)^m(-1)^1\, x^{2m+\nu+1}}{2^{2m+\nu+1}\, m!\,\Gamma(m+\nu+1+1)} \quad \because (-1)! = \infty \text{ and } \frac{1}{\infty} = 0$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right]=-x^{-\nu}\sum_{m=0}^{\infty}\frac{(-1)^m\ x^{2m+\nu+1}}{2^{2m+\nu+1}\ \text{m!}\,\Gamma(m+\nu+1+1)}$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right]=-x^{-\nu}J_{\nu+1}(x)\quad \text{proved}$$

Question: Find the value of $J_\nu(x)$

Solution: As $\frac{d}{dx}\left[x^{\nu}J_\nu(x)\right]=x^{\nu}J_{\nu-1}(x)$ _____(i)

And $\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right]=-x^{-\nu}J_{\nu+1}(x)$ ____(ii)

From (i) $x^{\nu}J_\nu'(x)+\nu x^{\nu-1}J_\nu(x)=x^{\nu}J_{\nu-1}(x)$

$J_\nu'(x)+\nu x^{-1}J_\nu(x)=J_{\nu-1}(x)$ ___(iii)

From (ii) $x^{-\nu}J_\nu'(x)-\nu x^{-\nu-1}J_\nu(x)=-x^{-\nu}J_{\nu+1}(x)$

$J_\nu'(x)-\nu x^{-1}J_\nu(x)=-J_{\nu+1}(x)$ ___(iv)

Adding (iii) and (iv)

$$2J_\nu'(x)=J_{\nu-1}(x)-J_{\nu+1}(x)$$

$$J_\nu'(x)=\frac{1}{2}\left(J_{\nu-1}(x)-J_{\nu+1}(x)\right)$$

Subtracting (iv) and (iii)

$$2\nu x^{-1}J_\nu(x)=J_{\nu-1}(x)-J_{\nu+1}(x)$$

$$J_\nu(x)=\frac{x}{\nu}\frac{J_{\nu-1}(x)-J_{\nu+1}(x)}{2}$$

Parametric Bessel functions:

The differential equation is $x^2y''+xy'+\left(\lambda^2x^2-\nu^2\right)y=0$ ___(i)

is called the Parametric Bessel equation

Transforming the independent variable for Bessel Function:

Let $$z = \lambda x$$

$$\frac{dz}{dx} = \lambda$$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$$

$$\frac{dy}{dx} = \lambda \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda \frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\frac{dz}{dx}\left(\lambda \frac{dy}{dz}\right) = \frac{d}{dz}\lambda\left(\lambda \frac{dy}{dz}\right) = \lambda^2 \frac{d^2y}{dz^2} \quad put\ in(i)$$

$$x^2\lambda^2 \frac{d^2y}{dz^2} + x\lambda \frac{dy}{dz} + \left(z^2 - \nu^2\right)y = 0$$

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \nu^2\right)y = 0 \quad \_\_\_\_(ii)$$

This shows that $J_\nu(z)$ is the solution of (ii) which shows that $J_\nu(\lambda x)$ is the solution of (i).

Property No. 1

Orthogonality of Bessel functions:

If $\lambda\ and\ \mu$ be the roots of $J_\nu(x) = 0$ then show that $\int_0^1 x J_\nu(\lambda x) J_\nu(\mu x)dx = 0$

Proof: Consider the parametric equation

$$x^2y'' + xy' + \left(\lambda^2x^2 - \nu^2\right)y = 0 \quad \_\_\_(i)$$

$Let \quad y_1(x) = J_\nu(\lambda x) \quad , \quad y_2(x) = J_\nu(\mu x)$

$$x^2y_1'' + xy_1' + \left(\lambda^2x^2 - \nu^2\right)y_1 = 0 \quad \_\_\_(ii)$$

$$x^2 y_2'' + x y_2' + \left(\mu^2 x^2 - \nu^2\right) y_2 = 0 \qquad \_\_\_(iii)$$

Multiplying equation (ii) by y_2 and (iii) by y_1 and subtracting

$$x^2 y_1'' y_2 + x y_1' y_2 + \lambda^2 x^2 y_1 y_2 - \nu^2 y_1 y_2 = 0$$

$$-x^2 y_2'' y_1 \pm x y_2' y_1 \pm \mu^2 x^2 y_1 y_2 \mp \nu^2 y_1 y_2 = 0$$

$$x^2\left(y_1'' y_2 - y_1 y_2''\right) + x\left(y_1' y_2 - y_1 y_2'\right) + \left(\lambda^2 - \mu^2\right) x^2 y_1 y_2 = 0$$

$$x\left(y_1'' y_2 - y_1 y_2''\right) + \left(y_1' y_2 - y_1 y_2'\right) + \left(\lambda^2 - \mu^2\right) x y_1 y_2 = 0$$

$$\Rightarrow \left(\mu^2 - \lambda^2\right) x y_1 y_2 = x\left(y_1'' y_2 - y_1 y_2''\right) + \left(y_1' y_2 - y_1 y_2'\right)$$

$$\Rightarrow \left(\mu^2 - \lambda^2\right) x y_1 y_2 = \frac{d}{dx}\left[x\left(y_1' y_2 - y_1 y_2'\right)\right]$$

$$\Rightarrow \left(\mu^2 - \lambda^2\right) \int_0^1 x y_1 y_2 \, dx = \int_0^1 \frac{d}{dx}\left[x\left(y_1' y_2 - y_1 y_2'\right)\right] dx$$

$$\left(\mu^2 - \lambda^2\right) \int_0^1 x y_1 y_2 \, dx = x\left(y_1' y_2 - y_1 y_2'\right)\Big|_0^1$$

$$\int_0^1 x y_1 y_2 \, dx = \frac{y_1'(1) y_2(1) - y_1(1) y_2'(1)}{\left(\mu^2 - \lambda^2\right)} \qquad \_\_\_\_\_(iv)$$

$$As \quad y_1(x) = J_\nu(\lambda x) \Rightarrow y_1(1) = J_\nu(\lambda)$$

$$y_2(x) = J_\nu(\mu x) \Rightarrow y_2(1) = J_\nu(\mu)$$

$$y_1'(x) = \lambda J_\nu'(\lambda x) \Rightarrow y_1(1) = \lambda J_\nu'(\lambda)$$

$$y_2'(x) = \mu J_\nu'(\mu x) \Rightarrow y_2'(1) = \mu J_\nu'(\mu)$$

Put in (iv) $\Rightarrow$ $$\int_0^1 x y_1 y_2 \, dx = \frac{\lambda J_\nu'(\lambda) J_\nu(\mu) - \mu J_\nu'(\mu) J_\nu(\lambda)}{\left(\mu^2 - \lambda^2\right)} \qquad \_\_\_\_(v)$$

Let λ *and* μ be the roots of boundary equation $AJ_\nu(x) + BJ_\nu'(x) = 0$

Then $$AJ_\nu(\lambda) + BJ_\nu'(\lambda) = 0 \qquad \_\_\_\_\_(vi)$$

$$AJ_\nu(\mu)+BJ_\nu'(\mu)=0 \qquad \_\_\_\_\_(vii)$$

Multiplying (vi) by $J_\nu(\mu)$ and (vii) by $J_\nu(\lambda)$ and subtracting

$$AJ_\nu(\lambda)J_\nu(\mu)+BJ_\nu'(\lambda)J_\nu(\mu)=0$$

$$-AJ_\nu(\mu)J_\nu(\lambda)\pm BJ_\nu'(\mu)J_\nu(\lambda)=0$$

$$B\left(J_\nu'(\lambda)J_\nu(\mu)-J_\nu'(\mu)J_\nu(\lambda)\right)=0$$

$$J_\nu'(\lambda)J_\nu(\mu)-J_\nu'(\mu)J_\nu(\lambda)=0$$

Put in (v) $\Rightarrow$ $\int_0^1 xJ_\nu(\lambda x)J_\nu(\mu x)\,dx=\dfrac{0}{(\mu^2-\lambda^2)}=0$

This shows that $J_\nu(\lambda x)$ and $J_\nu(\mu x)$ are orthogonal

Property No. 2:

Prove that $\int_0^1 xJ_\nu^2(\lambda x)\,dx=\dfrac{1}{2}\left[J_\nu'^2(\lambda)+\left(1-\dfrac{\nu^2}{\lambda^2}\right)J_\nu^2(\lambda)\right]$

Proof: Consider the parametric Bessel equation

$$x^2y''+xy'+\left(\lambda^2x^2-\nu^2\right)y=0 \qquad \_\_\_(i)$$

Let $y_1(x)=J_\nu(\lambda x)$, $y_2(x)=J_\nu(\mu x)$ be the solution of differential equation (i). Then we have

$$\int_0^1 xJ_\nu(\lambda x)J_\nu(\mu x)\,dx=\frac{\lambda J_\nu'(\lambda)J_\nu(\mu)-\mu J_\nu'(\mu)J_\nu(\lambda)}{(\mu^2-\lambda^2)}$$

Put $\mu=\lambda \Rightarrow \int_0^1 xJ_\nu(\mu x)J_\nu(\lambda x)\,dx=\lim\limits_{\mu\to\lambda}\dfrac{\lambda J_\nu'(\lambda)J_\nu(\mu)-\mu J_\nu'(\mu)J_\nu(\lambda)}{(\mu^2-\lambda^2)}\quad\left(\dfrac{0}{0}\right)$

By L-Hospital rule

$$\int_0^1 xJ_\nu^2(\lambda x)\,dx=\lim_{\mu\to\lambda}\frac{\lambda J_\nu'(\lambda)J_\nu'(\mu)-\mu J_\nu(\lambda)J_\nu''(\mu)-J_\nu(\lambda)J_\nu'(\mu)}{2\mu}$$

$$\int_0^1 x J_\nu^2(\lambda x)\,dx = \frac{\lambda J_\nu'(\lambda) J_\nu'(\lambda) - \lambda J_\nu(\lambda) J_\nu''(\lambda) - J_\nu(\lambda) J_\nu'(\lambda)}{2\lambda}$$

$$\int_0^1 x J_\nu^2(\lambda x)\,dx = \frac{\lambda J_\nu'^2(\lambda) - \lambda J_\nu(\lambda) J_\nu''(\lambda) - J_\nu(\lambda) J_\nu'(\lambda)}{2\lambda} \qquad \_\_\_(ii)$$

From Bessel equation

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \qquad \_\_\_(iii)$$

$$Put \quad y = J_\nu(x) \quad in(iii)$$

$$x^2 J_\nu''(x) + x J_\nu'(x) + \left(x^2 - \nu^2\right) J_\nu(x) = 0 \qquad \text{Put } x = \lambda$$

$$\lambda^2 J_\nu''(\lambda) + \lambda J_\nu'(\lambda) + \left(\lambda^2 - \nu^2\right) J_\nu(\lambda) = 0$$

$$J_\nu''(\lambda) = \frac{-1}{\lambda^2}\left[\lambda J_\nu'(\lambda) + \left(\lambda^2 - \nu^2\right) J_\nu(\lambda)\right]$$

$$J_\nu''(\lambda) = -\frac{1}{\lambda} J_\nu'(\lambda) - \left(1 - \frac{\nu^2}{\lambda^2}\right) J_\nu(\lambda)$$

Put in (ii)

$$\int_0^1 x J_\nu^2(\lambda x)\,dx = \frac{1}{2\lambda}\left[\lambda J_\nu'(\lambda) - \lambda J_\nu(\lambda)\left\{-\frac{1}{\lambda} J_\nu'(\lambda) - \left(1 - \frac{\nu^2}{\lambda^2}\right) J_\nu(\lambda)\right\} - J_\nu(\lambda) J_\nu'(\lambda)\right]$$

$$\int_0^1 x J_\nu^2(\lambda x)\,dx = \frac{1}{2\lambda}\left[\lambda J_\nu'(\lambda) + J_\nu(\lambda) J_\nu'(\lambda) + \lambda\left(1 - \frac{\nu^2}{\lambda^2}\right) J_\nu(\lambda) - J_\nu(\lambda) J_\nu'(\lambda)\right]$$

$$\int_0^1 x J_\nu^2(\lambda x)\,dx = \frac{1}{2\lambda}\left[\lambda J_\nu'(\lambda) + \lambda\left(1 - \frac{\nu^2}{\lambda^2}\right) J_\nu(\lambda)\right]$$

$$\int_0^1 x J_\nu^2(\lambda x)\,dx = \frac{\lambda}{2\lambda}\left[J_\nu'(\lambda) + \left(1 - \frac{\nu^2}{\lambda^2}\right) J_\nu(\lambda)\right]$$

$$\int_0^1 x J_\nu^2(\lambda x)\,dx = \frac{1}{2}\left[J_\nu'(\lambda) + \left(1 - \frac{\nu^2}{\lambda^2}\right) J_\nu(\lambda)\right]$$

Lecture # 13

Some Results:

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$x^n J_n^{'}(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$$

$$\div x^n \qquad J_n^{'}(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \quad \_\_\_\_(A)$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$x^{-n} J_n^{'}(x) + nx^{-n-1} J_n(x) = -x^{-n} J_{n-1}(x)$$

$$\div x^{-n} \qquad J_n^{'}(x) + \frac{n}{x} J_n(x) = -J_{n+1}(x) \quad \_\_\_\_(B)$$

Adding (A) and (B)

$$2J_n^{'}(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$J_n^{'}(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \_\_\_\_(C)$$

Subtracting (B) from (A)

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \quad \_\_\_\_(D)$$

Hankel Transformation:

The infinite Hankel transform of function f(x) $(0 \le x < \infty)$ is defined as

$$H\{f(x)\} = \int_0^\infty f(x)\, x\, J_n(sx)\,dx$$

is denoted by $\overline{f}(s)$ i.e. $H\{f(x)\} = \overline{f}(s)$ where $J_n(sx)$ is a Bessel function of first kind with order n.

If $\overline{f}(s)$ is the Hankel transformation of the order 'n' of the function f(x) then

$\left[\overline{f}(s)\right]$ is denoted by $\overline{f_n}(s)$

i.e. $$\int_0^\infty f(x)\, x\, J_n(sx)dx = H\{f(x),n\} = \overline{f_n}(s)$$

Properties of Hankel transformation:

Linearity property:

Let f(x) and g(x) be two function and a,b are scalar's then

$$H\{af(x)+bg(x),n\} = \int_0^\infty \big(af(x)+bg(x)\big)x\, J_n(sx)dx$$

$$H\{af(x)+bg(x),n\} = \int_0^\infty \big(af(x)xJ_n(sx)+bg(x)xJ_n(sx)\big)\, dx$$

$$H\{af(x)+bg(x),n\} = a\int_0^\infty f(x)xJ_n(sx)\, dx + b\int_0^\infty g(x)xJ_n(sx)dx$$

$$H\{af(x)+bg(x),n\} = aH\{f(x),n\} + bH\{g(x),n\}$$

Scaling Property:

$$H\{f(ax)\} = \frac{1}{a^2}\overline{f}\left(\frac{s}{a}\right)$$

Since $$H\{f(ax)\} = \int_0^\infty f(ax)\, x\, J_n(sx)dx$$

Put ax = z

x = z/a

dx = 1/a dz

$$H\{f(ax)\} = \int_0^\infty f(z)\,\frac{z}{a}\, J_n\left(\frac{sz}{a}\right)\frac{1}{a}dz$$

$$H\{f(ax)\} = \frac{1}{a^2}\int_0^\infty f(z)\, z\, J_n\left(\frac{sz}{a}\right)dz$$

Replacing z by x

$$H\{f(ax)\}=\frac{1}{a^2}\int_0^\infty f(x)\,x\,J_n\left(\frac{sx}{a}\right)dx$$

$$H\{f(ax)\}=\frac{1}{a^2}\overline{f}\left(\frac{s}{a}\right) \qquad \because \overline{f_n}(s)=\int_0^\infty f(x)\,x\,J_n(sx)dx$$

Hanker transformation of the derivation of the function f(x).

Theorem: Let $\overline{f_n}(s)$ be the Hankel transformation of {f(x) , n} with order 'n' then show that

$$\overline{f_n'}(s)=\frac{s}{2n}\left\{(n-1)\overline{f}_{n+1}(s)-(n+1)\overline{f}_{n-1}(s)\right\}$$

Proof: By definition of Hankel transformation we have

$$\overline{f_n}(s)=\int_0^\infty f(x)\,x\,J_n(sx)dx$$

$$\overline{f_n'}(s)=f(x)xJ_n(sx)\Big|_0^\infty-\int_0^\infty f(x)\left[J_n(sx)+xJ_n'(sx)s\right]dx$$

$$\overline{f_n'}(s)=0-\int_0^\infty f(x)\left[J_n(sx)+(sx)J_n'(sx)\right]dx \quad \_\_\_\_(1)$$

We know that

$$J_n'(x)+\frac{n}{x}J_n(x)=J_{n-1}(x)$$

Multiply by x $\qquad xJ_n'(x)+nJ_n(x)=xJ_{n-1}(x)$

$$xJ_n'(x)=xJ_{n-1}(x)-nJ_n(x)$$

Replace x by (sx)

$$(sx)J_n'(sx)=(sx)J_{n-1}(sx)-nJ_n(sx)$$

Adding $J_n(sx)$ on both sides

$$J_n(sx)+(sx)J_n'(sx)+=(sx)J_{n-1}(sx)-nJ_n(sx)+J_n(sx)$$

$$J_n(sx)+(sx)J_n'(sx)=(sx)J_{n-1}(sx)+(1-n)J_n(sx) \quad \_\_\_(2)$$

By putting the value of (2) in (1) we have

$$\overline{f_n'}(s) = -\int_0^\infty f(x)\left[(sx)J_{n-1}(sx) + (1-n)J_n(sx)\right]dx \quad \_\_\_\_(3)$$

We know that $$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Replace x by sx

$$J_n(sx) = \frac{sx}{2n}\left[J_{n-1}(sx) + J_{n+1}(sx)\right] \quad \_\_\_(4)$$

By putting the value of (4) in (3) we have

$$\overline{f_n'}(s) = -\int_0^\infty f(x)\left[(sx)J_{n-1}(sx) + (1-n)\left\{\frac{sx}{2n}\left(J_{n-1}(sx) + J_{n+1}(sx)\right)\right\}\right]dx$$

$$\overline{f_n'}(s) = -\int_0^\infty (sx)f(x)\left[J_{n-1}(sx) + \frac{(1-n)}{2n}J_{n-1}(sx) + \frac{(1-n)}{2n}J_{n+1}(sx)\right]dx$$

$$\overline{f_n'}(s) = -\int_0^\infty (sx)f(x)\left[\left(\frac{2n+1-n}{2n}\right)J_{n-1}(sx) + \frac{(1-n)}{2n}J_{n+1}(sx)\right]dx$$

$$\overline{f_n'}(s) = -\int_0^\infty (sx)f(x)\left[\left(\frac{n+1}{2n}\right)J_{n-1}(sx) + \frac{(1-n)}{2n}J_{n+1}(sx)\right]dx$$

$$\overline{f_n'}(s) = -\frac{s}{2n}\left[\int_0^\infty (n+1)f(x)xJ_{n-1}(sx)dx + \int_0^\infty (1-n)f(x)xJ_{n-1}(sx)dx\right]$$

$$\overline{f_n'}(s) = \frac{s}{2n}\left[-(n+1)\int_0^\infty f(x)xJ_{n-1}(sx)dx - (1-n)\int_0^\infty f(x)xJ_{n-1}(sx)dx\right]$$

$$\overline{f_n'}(s) = \frac{s}{2n}\left[(n-1)\overline{f}_{n+1}(s) - (n+1)\overline{f}_{n-1}(s)\right]$$

Theorem: Show that

$$\overline{f_n''}(s)=\frac{s^2}{2n}\left[\frac{n+1}{n-1}\overline{f}_{n-2}(s)-\frac{2(n^2-3)}{n^2-1}\overline{f}_n(s)-\frac{n-1}{n+1}\overline{f}_{n+2}(s)\right]$$

Proof: We know that

$$\overline{f_n'}(s)=\frac{s}{2n}\left[(n-1)\overline{f}_{n+1}(s)-(n+1)\overline{f}_{n-1}(s)\right] \quad \text{\_\_\_\_(A)}$$

Similarly,

$$\overline{f_n''}(s)=\frac{s}{2n}\left[(n-1)\overline{f'}_{n+1}(s)-(n+1)\overline{f'}_{n-1}(s)\right] \quad \text{\_\_\_\_(B)}$$

Replace n by n+1 in (A) we have

$$\overline{f_{n+1}'}(s)=\frac{s}{2(n+1)}\left[n\overline{f}_{n+2}(s)-(n+2)\overline{f}_n(s)\right]$$

$$\overline{f_{n+1}'}(s)=\frac{s}{2}\left[\frac{n}{(n+1)}\overline{f}_{n+2}(s)-\frac{(n+2)}{(n+1)}\overline{f}_n(s)\right]$$

Replace n by n–1 in (A) we have

$$\overline{f_{n-1}'}(s)=\frac{s}{2(n-1)}\left[(n-2)\overline{f}_n(s)-n\overline{f}_{n-2}(s)\right]$$

$$\overline{f_{n-1}'}(s)=\frac{s}{2}\left[\frac{n-2}{n-1}\overline{f}_n(s)-\frac{n}{n-1}\overline{f}_{n-2}(s)\right]$$

Put these values in (B) we have

$$\overline{f_n''}(s)=\frac{s}{2n}\left[(n-1)\frac{s}{2}\left\{\frac{n}{(n+1)}\overline{f}_{n+2}(s)-\frac{(n+2)}{(n+1)}\overline{f}_n(s)\right\}-(n+1)\frac{s}{2}\left\{\frac{n-2}{n-1}\overline{f}_n(s)-\frac{n}{n-1}\overline{f}_{n-2}(s)\right\}\right]$$

$$\overline{f_n''}(s)=\frac{s^2}{4n}\left[\frac{n(n-1)}{(n+1)}\overline{f}_{n+2}(s)-\frac{(n-1)(n+2)}{n+1}\overline{f}_n(s)-\frac{(n+1)(n-2)}{n-1}\overline{f}_n(s)+\frac{n(n+1)}{n-1}\overline{f}_{n-2}(s)\right]$$

$$\overline{f''_n}(s)=\frac{s^2}{4n}\left[\frac{n(n-1)}{(n+1)}\overline{f}_{n+2}(s)-\left\{\frac{(n-1)(n+2)}{n+1}+\frac{(n+1)(n-2)}{n-1}\right\}\overline{f}_n(s)+\frac{n(n+1)}{n-1}\overline{f}_{n-2}(s)\right]\_\_(C)$$

$$\frac{(n-1)(n+2)}{n+1}+\frac{(n+1)(n-2)}{n-1}=\frac{(n-1)^2(n+2)+(n+1)^2(n-2)}{(n+1)(n-1)}$$

$$=\frac{(n^2-2n+1)(n+2)+(n^2+2n+1)(n-2)}{n^2-1}$$

$$=\frac{n^3+2n^2-2n^2-4n+n+2+n^3-2n^2+2n^2-4n+n-2}{n^2-1}=\frac{2n^3-6n}{n^2-1}$$

$$=\frac{2n(n^2-3)}{n^2-1}$$

Put in (C)

$$\overline{f''_n}(s)=\frac{s^2}{4n}\left[\frac{n(n-1)}{(n+1)}\overline{f}_{n+2}(s)-\frac{2n(n^2-3)}{n^2-1}\overline{f}_n(s)+\frac{n(n+1)}{n-1}\overline{f}_{n-2}(s)\right]$$

$$\overline{f''_n}(s)=\frac{s^2}{4n}.n\left[\frac{(n-1)}{(n+1)}\overline{f}_{n+2}(s)-\frac{2(n^2-3)}{n^2-1}\overline{f}_n(s)+\frac{n+1}{n-1}\overline{f}_{n-2}(s)\right]$$

$$\overline{f''_n}(s)=\frac{s^2}{2n}\left[\frac{n+1}{n-1}\overline{f}_{n-2}(s)-\frac{2(n^2-3)}{n^2-1}\overline{f}_n(s)-\frac{n-1}{n+1}\overline{f}_{n+2}(s)\right]$$

Question: Find the Hankel transformation of $f'(s)$ when $f(x)=\frac{e^{-ax}}{x}$; $n=1$

Solution: We know that

$$\overline{f'_n}(s)=\frac{s}{2n}\left[(n-1)\overline{f}_{n+1}(s)-(n+1)\overline{f}_{n-1}(s)\right]$$

n = 1 $\qquad \overline{f'_n}(s)=\frac{s}{2}\left[0-2\overline{f}_{n-1}(s)\right]$

$$\overline{f_n'}(s) = -s\overline{f}_0(s)$$

$$\overline{f_n'}(s) = -s\int_0^\infty f(x).x J_0(sx)\,dx$$

$$\overline{f_n'}(s) = -s\int_0^\infty \frac{e^{-ax}}{x}.x J_0(sx)\,dx$$

$$\overline{f_n'}(s) = -s\int_0^\infty e^{-ax} J_0(sx)\,dx$$

$$\overline{f_n'}(s) = -\frac{s}{\sqrt{s^2+a^2}} \qquad \because \int_0^\infty e^{-ax} J_0(sx)\,dx = \frac{1}{\sqrt{s^2+a^2}}$$

Theorem: Show that $\int_0^\infty e^{-ax} J_0(sx)\,dx = \frac{1}{\sqrt{s^2+a^2}}$

Proof: $\int_0^\infty e^{-ax} J_0(sx)\,dx = \int_0^\infty e^{-ax} \sum_{m=0}^\infty \frac{(-1)^m}{m!m!}\left(\frac{sx}{2}\right)^{2m} dx$

$$= \sum_{m=0}^\infty \frac{(-1)^m}{m!m!}\left(\frac{s}{2}\right)^{2m} \int_0^\infty e^{-ax} x^{2m}\,dx$$

Put $ax = z \Rightarrow dx = dz/a$

$$= \sum_{m=0}^\infty \frac{(-1)^m}{m!m!}\left(\frac{s}{2}\right)^{2m} \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{2m} \frac{dz}{a}$$

$$= \sum_{m=0}^\infty \frac{(-1)^m}{m!m!.a^{2m+1}}\left(\frac{s}{2}\right)^{2m} \int_0^\infty e^{-z} z^{2m}\,dz$$

$$= \sum_{m=0}^\infty \frac{(-1)^m}{m!m!.a^{2m+1}}\left(\frac{s}{2}\right)^{2m} \Gamma(2m+1) \qquad \because \int_0^\infty e^{-z} z^{2m}\,dz = \Gamma(2m+1)$$

$$= \sum_{m=0}^\infty \frac{(-1)^m s^{2m}}{m!m!2^{2m} a^{2m+1}}.2m!$$

$$= \frac{1}{a}\left[\sum_{m=0}^\infty \frac{(-1)^m 2m!}{m!m!2^{2m}}\left(\frac{s}{a}\right)^{2m}\right]$$

$$= \frac{1}{a}\left[1 + \frac{(-1)^1 2!}{1!1!\,2^2}\left(\frac{s}{a}\right)^2 + \frac{(-1)^2 4!}{2!2!\,2^4}\left(\frac{s}{a}\right)^4 + \ldots\right]$$

$$= \frac{1}{a}\left[1 + \frac{(-1).2}{4}\left(\frac{s}{a}\right)^2 + \frac{(-1)^2 24}{2!2!16}\left(\frac{s}{a}\right)^4 + \ldots\right]$$

$$= \frac{1}{a}\left[1 - \frac{1}{2}\left(\frac{s}{a}\right)^2 + \frac{3}{2!.4}\left(\frac{s}{a}\right)^4 + \ldots\right] \_\_\_\_(1)$$

$$\left(1+\left(\frac{s}{a}\right)^2\right)^{-1/2}=1+\left(-\frac{1}{2}\right)\left(\frac{s}{a}\right)^1+\frac{\left(\frac{-1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\left(\frac{s}{a}\right)^2\right)^2+\ldots.$$

$$\left(1+\left(\frac{s}{a}\right)^2\right)^{-1/2}=1-\frac{1}{2}\left(\frac{s}{a}\right)^1+\frac{1}{2!}\frac{3}{4}\left(\frac{s}{a}\right)^4+\ldots.$$

Eq (1) $\Rightarrow$

$$\int_0^\infty e^{-ax}J_0(sx)\,dx=\frac{1}{a}\left(1+\left(\frac{s}{a}\right)^2\right)^{-1/2}$$

$$\int_0^\infty e^{-ax}J_0(sx)\,dx=\frac{1}{a}\left(1+\frac{s^2}{a^2}\right)^{-1/2}$$

$$\int_0^\infty e^{-ax}J_0(sx)\,dx=\frac{1}{a}\left(\frac{a^2+s^2}{a^2}\right)^{-1/2}$$

$$\int_0^\infty e^{-ax}J_0(sx)\,dx=\frac{1}{a}\left(\frac{a^2}{a^2+s^2}\right)^{1/2}$$

$$\int_0^\infty e^{-ax}J_0(sx)\,dx=\frac{1}{a}\frac{a}{\sqrt{a^2+s^2}}=\frac{1}{\sqrt{a^2+s^2}}$$

Question: Find the Hankel transformation of

$$H\left\{f''(x)+\frac{1}{x}f'(x)-\frac{n^2}{x^2}f(x)\right\}=?$$

Solution: $H\{f(x)\}=\int_0^\infty f(x)\,x\,J_n(sx)\,dx$

$$H\{f''(x)\}=\int_0^\infty f''(x)\,x\,J_n(sx)\,dx$$

$$H\{f''(x)\}=f'(x)\,x\,J_n(sx)\Big|_0^\infty-\int_0^\infty f'(x)\frac{d}{dx}\big(x\,J_n(sx)\big)\,dx$$

$$H\left\{f''(x)\right\} = 0 - \int_0^\infty f'(x).\left(J_n(sx) + xJ_n'(sx)s\right)dx$$

$$H\left\{f''(x)\right\} = -\int_0^\infty \left[f'(x)J_n(sx) + sf'(x)xJ_n'(sx)\right]dx$$

$$H\left\{f''(x)\right\} = -\int_0^\infty f'(x).\frac{x}{x}J_n(sx)dx - s\int_0^\infty f'(x)xJ_n'(sx)dx$$

$$H\left\{f''(x)\right\} = -H\left\{\frac{f'(x)}{x}\right\} - s\int_0^\infty f'(x)xJ_n'(sx)dx$$

$$H\left\{f''(x)\right\} + H\left\{\frac{f'(x)}{x}\right\} = -s\int_0^\infty f'(x)xJ_n'(sx)dx$$

$$H\left\{f''(x) + \frac{f'(x)}{x}\right\} = -s\left[f(x)xJ_n'(sx)\Big|_0^\infty - \int_0^\infty f(x)\frac{d}{dx}\left(xJ_n'(sx)dx\right)\right]$$

$$H\left\{f''(x) + \frac{f'(x)}{x}\right\} = -s\left[0 - \int_0^\infty f(x)\frac{d}{dx}\left(xJ_n'(sx)dx\right)\right]$$

$$H\left\{f''(x) + \frac{f'(x)}{x}\right\} = s\int_0^\infty f(x)\frac{d}{dx}\left(xJ_n'(sx)\right)dx \quad \_\_\_(i)$$

We know that

$J_n(x)$ satisfies the Bessel equation

$$x^2y'' + xy' + \left(x^2 - n^2\right)y = 0$$

$$xy'' + y' + \left(\frac{x^2 - n^2}{x}\right)y = 0$$

$$\frac{d}{dx}\left[xy'\right] + \left(x - \frac{n^2}{x}\right)y = 0$$

$$\frac{d}{dx}\left[xJ_n'(x)\right] + \left(x - \frac{n^2}{x}\right)J_n(x) = 0 \qquad \text{Replace y by } J_n(x)$$

$$\frac{d}{dsx}\left[sxJ_n'(sx)\right]+\left(sx-\frac{n^2}{sx}\right)J_n(sx)=0 \qquad \text{Replace } x \text{ by } sx$$

$$\frac{sd}{sdx}\left[xJ_n'(sx)\right]+\left(sx-\frac{n^2}{sx}\right)J_n(sx)=0$$

$$\frac{d}{dx}\left[xJ_n'(sx)\right]=-\left(sx-\frac{n^2}{sx}\right)J_n(sx) \qquad \_\_\_(ii)$$

By putting the value of (ii) in (i)

$$H\left\{f''(x)+\frac{f'(x)}{x}\right\}=s\int_0^\infty f(x)\left(-sx+\frac{n^2}{sx}\right)J_n(sx)\,dx$$

$$=-s^2\int_0^\infty f(x)J_n(sx)dx+n^2\int_0^\infty f(x)\frac{1}{x}J_n(sx)dx$$

$$=-s^2H\{f(x)\}+n^2\int_0^\infty f(x)\frac{x}{x^2}J_n(sx)dx$$

$$=-s^2H\{f(x)\}+n^2H\left\{\frac{f(x)}{x^2}\right\}$$

$$H\left\{f''(x)+\frac{f'(x)}{x}\right\}-n^2H\left\{\frac{f(x)}{x^2}\right\}=-s^2H\{f(x)\}$$

$$H\left\{f''(x)+\frac{1}{x}f'(x)-\frac{n^2}{x^2}f(x)\right\}=-s^2H\{f(x)\}$$

Parseval Theorem:

If $\overline{f}(s)\, and\, \overline{g}(s)$ are two Hankel transformation of f(x) and g(x) then

$$\int_0^\infty xf(x)g(x)dx=\int_0^\infty s\overline{f}(s)\,\overline{g}(s)ds$$

Proof: Let $\int_0^\infty f(x)\,x\,J_n(sx)dx=H\{f(x)\}=\overline{f}(s)$

$$\int_0^{\infty} g(x)\, x\, J_n(sx)\,dx = H\{g(x)\} = \overline{g}(s)$$

$$\int_0^{\infty} \overline{f}(s)\overline{g}(s)\,ds = \int_0^{\infty}\left(s\overline{f}(s).\int_0^{\infty} g(x)\, x\, J_n(sx)\,dx\right)ds$$

$$\int_0^{\infty} \overline{f}(s)\overline{g}(s)\,ds = \int_0^{\infty} g(x)\, x\left[\int_0^{\infty} s\overline{f}(s)\,\mathrm{J}_n(sx)\,ds\right]dx$$

$$\because\; f(x) = H^{-1}\{\overline{f}(s)\} = \int_0^{\infty} s\overline{f}(s)\,\mathrm{J}_n(sx)\,ds$$

$$\int_0^{\infty} \overline{f}(s)\overline{g}(s)\,ds = \int_0^{\infty} g(x)x.f(x)\,dx$$

$$\int_0^{\infty} \overline{f}(s)\overline{g}(s)\,ds = \int_0^{\infty} x\, f(x)\, g(x)\,dx$$

Question: Find the Hankel Transformation of

$$f(x)\begin{cases}1 & if\;\; 0<x<a \\ 0 & if\;\;\; x>a\end{cases}$$

H{f(x); n=0} = ?

Solution: $H\{f(x);n=0\} = \int_0^{\infty} f(x)xJ_0(sx)\,dx$

$$= \int_0^{a} f(x)xJ_0(sx)\,dx + \int_a^{\infty} f(x)xJ_0(sx)\,dx$$

$$= \int_0^{a} 1.xJ_0(sx)\,dx + 0$$

$$= \int_0^{a} xJ_0(sx)\,dx \quad \_\_\_(i)$$

We know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

n = 1 $$\frac{d}{dx}\left[xJ_1(x)\right]=xJ_0(x)$$

Replace x by sx

$$\frac{d}{dsx}\left[sxJ_1(sx)\right]=sxJ_0(sx)$$

$$\frac{sd}{sdx}\left[xJ_1(sx)\right]=sxJ_0(sx)$$

$$\frac{d}{dx}\left[xJ_1(sx)\right]=sxJ_0(sx)$$

$$xJ_0(sx)=\frac{1}{s}\frac{d}{dx}\left[xJ_1(sx)\right]$$

Put in (i) $\Rightarrow$ $$H\{f(x);n=0\}=\frac{1}{s}\int_0^a\frac{d}{dx}\left[xJ_1(sx)\right]dx$$

$$H\{f(x);n=0\}=\frac{1}{s}\left[xJ_1(sx)\right]_0^a$$

$$H\{f(x);n=0\}=\frac{1}{s}\left[aJ_1(sa)-0\right]$$

$$H\{f(x);n=0\}=\frac{a}{s}J_1(sa)$$

Question: Show that $\int\limits_0^{\infty}e^{-ax}\,xJ_0(sx)\,dx=\frac{a}{\left(a^2+s^2\right)^{3/2}}$

Proof: $$\int\limits_0^{\infty}e^{-ax}\,xJ_0(sx)\,dx=\int\limits_0^{\infty}e^{-ax}\,x\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!}\left(\frac{sx}{2}\right)^{2m}dx$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!}\left(\frac{s}{2}\right)^{2m}\int\limits_0^{\infty}e^{-ax}x\,x^{2m}\,dx$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!}\left(\frac{s}{2}\right)^{2m}\int\limits_0^{\infty}e^{-ax}\,x^{2m+1}\,dx$$

Put $\quad ax = z \quad \Rightarrow dx = dz/a$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!}\left(\frac{s}{2}\right)^{2m}\int_0^{\infty}e^{-z}\left(\frac{z}{a}\right)^{2m+1}\frac{dz}{a}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!.\mathrm{a}^{2m+2}}\left(\frac{s}{2}\right)^{2m}\int_0^{\infty}e^{-z}z^{2m+1}\,dz$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!.\mathrm{a}^{2m+2}}\left(\frac{s}{2}\right)^{2m}\Gamma(2m+2)$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m s^{2m}}{m!m!2^{2m}\,\mathrm{a}^{2m+2}}\;(2m+1)!$$

$$=\frac{1}{a^2}\left[\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)!}{m!m!2^{2m}}\left(\frac{s}{a}\right)^{2m}\right]$$

$$=\frac{1}{a^2}\left[1+\frac{(-1)^1 3!}{1!1!\,2^2}\left(\frac{s}{a}\right)^2+\frac{(-1)^2 5!}{2!2!\,2^4}\left(\frac{s}{a}\right)^4+\ldots\right]$$

$$=\frac{1}{a^2}\left[1-\frac{6}{4}\left(\frac{s}{a}\right)^2+\frac{1}{21}\frac{120}{2016}\left(\frac{s}{a}\right)^4+\ldots\right]$$

$$=\frac{1}{a^2}\left[1-\frac{3}{2}\left(\frac{s}{a}\right)^2+\frac{1}{2!}\frac{15}{4}\left(\frac{s}{a}\right)^4+\ldots\right] \_\_\_\_(i)$$

$$\left(1+\left(\frac{s}{a}\right)^2\right)^{-3/2}=1+\left(-\frac{3}{2}\right)\left(\left(\frac{s}{a}\right)^2\right)^1+\frac{\left(\frac{-3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}\left(\left(\frac{s}{a}\right)^2\right)^2+\ldots.$$

$$\left(1+\left(\frac{s}{a}\right)^2\right)^{-3/2}=1-\frac{3}{2}\left(\frac{s}{a}\right)^2+\frac{1}{2!}\frac{15}{4}\left(\frac{s}{a}\right)^4+\ldots.$$

Eq (1) $\Rightarrow$

$$\int_0^\infty e^{-ax}xJ_0(sx)dx=\frac{1}{a^2}\left(1+\left(\frac{s}{a}\right)^2\right)^{-3/2}$$

$$\int_0^\infty e^{-ax}xJ_0(sx)dx=\frac{1}{a^2}\left(1+\frac{s^2}{a^2}\right)^{-3/2}$$

$$\int_0^\infty e^{-ax}xJ_0(sx)dx=\frac{1}{a^2}\left(\frac{a^2+s^2}{a^2}\right)^{-3/2}$$

$$\int_0^\infty e^{-ax}xJ_0(sx)dx=\frac{1}{a^2}\left(\frac{a^2}{a^2+s^2}\right)^{3/2}$$

$$\int_0^\infty e^{-ax}xJ_0(sx)dx=\frac{1}{a^2}\frac{a^3}{\sqrt{a^2+s^2}}=\frac{a}{\left(a^2+s^2\right)^{3/2}}$$

Question: Show that $\int_0^\infty xe^{-ax}J_1(sx)dx=\frac{s}{\left(s^2+a^2\right)^{3/2}}$

Solution: $\int_0^\infty xe^{-ax}J_1(sx)dx$

$$\because J_1(sx)=-J_0'(sx)$$

$$\int_0^\infty xe^{-ax}J_1(sx)dx=-\int_0^\infty \underset{I}{xe^{-ax}}\,\underset{II}{J_0'(sx)}dx$$

$$\int_0^\infty xe^{-ax}J_1(sx)dx=-\left[xe^{-ax}J_0(sx)\Big|_0^\infty-\int_0^\infty\frac{J_0(sx)}{s}\frac{d}{dx}\left(xe^{-ax}\right)dx\right]$$

$$\int_0^\infty xe^{-ax}J_1(sx)dx=-\left[0-\frac{1}{s}\int_0^\infty J_0(sx)\left(e^{-ax}-xe^{-ax}a\right)dx\right]$$

$$\int_0^\infty xe^{-ax}J_1(sx)dx=\frac{1}{s}\left[\int_0^\infty e^{-ax}J_0(sx)dx-a\int_0^\infty xe^{-ax}J_0(sx)dx\right]\_\_\_\_(i)$$

As we prove

$$\int_0^\infty e^{-ax} J_0(sx)\,dx = \frac{1}{\sqrt{s^2+a^2}}$$

$$\int_0^\infty e^{-ax} xJ_0(sx)\,dx = \frac{a}{\left(a^2+s^2\right)^{3/2}}$$

Put in (i)

$$\int_0^\infty xe^{-ax} J_1(sx)\,dx = \frac{1}{s}\left[\frac{1}{\left(s^2+a^2\right)^{1/2}} - \frac{a^2}{\left(s^2+a^2\right)^{3/2}}\right]$$

$$\int_0^\infty xe^{-ax} J_1(sx)\,dx = \frac{1}{s}\left[\frac{s^2+a^2-a^2}{\left(s^2+a^2\right)^{3/2}}\right]$$

$$\int_0^\infty xe^{-ax} J_1(sx)\,dx = \frac{s}{\left(s^2+a^2\right)^{3/2}}$$

Differential formula of Hankel transformation:

$$H_n\left\{x^{n-1}\left(x^{1-n}f(x)\right)'\right\} = ?$$

Solution: $H_n\left\{x^{n-1}\left(x^{1-n}f(x)\right)'\right\} = \int_0^\infty x^{n-1}\left(x^{1-n}f(x)\right)' x J_n(sx)\,dx$

$$H_n\left\{x^{n-1}\left(x^{1-n}f(x)\right)'\right\} = \int_0^\infty \underset{II}{\left(x^{1-n}f(x)\right)'}\,\underset{I}{x^n J_n(sx)}\,dx$$

$$= x^n J_n(sx).x^{1-n}f(x)\Big|_0^\infty - \int_0^\infty x^{1-n}f(x)\frac{d}{dx}\left[x^n J_n(sx)\right]dx$$

$$= 0 - \int_0^\infty x^{1-n}f(x)\frac{d}{dx}\left[x^n J_n(sx)\right]dx$$

$$\because \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

Replace x by sx

$$\frac{d}{dsx}\left[(sx)^n J_n(sx)\right]=(sx)^n J_{n-1}(sx)$$

$$\frac{s^n}{s}\frac{d}{dx}\left[(x)^n J_n(sx)\right]=s^n x^n J_{n-1}(sx)$$

$$\frac{d}{dx}\left[(x)^n J_n(sx)\right]=sx^n J_{n-1}(sx)$$

By putting value

$$=-\int_0^\infty x^{1-n} f(x) sx^n J_{n-1}(sx)dx$$

$$=-s\int_0^\infty f(x)x J_{n-1}(sx)dx$$

$$H_n\left\{x^{n-1}\left(x^{1-n}f(x)\right)'\right\}=-sH_{n-1}\left\{f(x)\right\}$$

Question: $H_n\left\{x^{-n-1}\left(x^{1+n}f(x)\right)'\right\}=?$

Solution: $H_n\left\{x^{-n-1}\left(x^{1+n}f(x)\right)'\right\}=\int_0^\infty x^{-n-1}\left(x^{1+n}f(x)\right)' x J_n(sx)dx$

$$H_n\left\{x^{-n-1}\left(x^{1+n}f(x)\right)'\right\}=\int_0^\infty \underset{II}{\left(x^{1+n}f(x)\right)'}\,\underset{I}{x^{-1}J_n(sx)}dx$$

$$=x^{-n}J_n(sx).x^{1+n}f(x)\Big|_0^\infty-\int_0^\infty x^{1+n}f(x)\frac{d}{dx}\left[x^{-n}J_n(sx)\right]dx$$

$$=0-\int_0^\infty x^{1+n}f(x)\frac{d}{dx}\left[x^{-n}J_n(sx)\right]dx$$

$$\because \frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$$

Replace x by sx

$$\frac{d}{dsx}\left[(sx)^{-n}J_n(sx)\right]=-(sx)^{-n}J_{n+1}(sx)$$

$$\frac{s^{-n}}{s}\frac{d}{dx}\left[(x)^{-n}J_n(sx)\right]=-s^{-n}x^{-n}J_{n+1}(sx)$$

$$\frac{d}{dx}\left[(x)^{-n}J_n(sx)\right]=-sx^{-n}J_{n+1}(sx)$$

By putting value

$$=-\int_0^\infty x^{1+n}f(x)\left(-sx^{-n}J_{n+1}(sx)\right)dx$$

$$=s\int_0^\infty f(x)\,x\,J_{n+1}(sx)\,dx$$

$$H_n\left\{x^{n-1}\left(x^{1-n}f(x)\right)'\right\}=sH_{n+1}\left\{f(x)\right\}$$

Lecture # 15

Question: Find Hankel transformation of

$$f(x)=\begin{cases} x^n & 0<x<a \quad ;\mathrm{n}\geq 0 \\ 0 & x>a \end{cases}$$

Solution: We know that

$$H\{f(x)\}=\int_0^\infty f(x)\,x\,J_n(sx)\,dx$$

$$H\{f(x)\}=\int_0^a f(x)\,x\,J_n(sx)\,dx+\int_a^\infty f(x)\,x\,J_n(sx)\,dx$$

$$H\{f(x)\}=\int_0^a x^n x\,J_n(sx)\,dx+0$$

$$H\{f(x)\}=\int_0^a x^{n+1}J_n(sx)\,dx \qquad \_\_\_(i)$$

$$\because \frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$$

Replace n by n+1

$$\frac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right]=x^{n+1}J_n(x)$$

Replace x by sx

$$\frac{d}{dsx}\left[(sx)^{n+1}J_{n+1}(sx)\right]=(sx)^{n+1}J_n(sx)$$

$$\frac{s^{n+1}}{s}\frac{d}{dx}\left[(x)^{n+1}J_{n+1}(sx)\right]=s^{n+1}x^{n+1}J_n(sx)$$

$$\frac{1}{s}\frac{d}{dx}\left[x^{n+1}J_{n+1}(sx)\right]=x^{n+1}J_n(sx)$$

Put in (i)

$$H\{f(x)\}=\frac{1}{s}\int_0^a\frac{d}{dx}\left[x^{n+1}J_{n+1}(sx)\right]dx$$

$$H\{f(x)\} = \frac{1}{s}\left[x^{n+1} J_{n+1}(sx)\right]_0^a$$

$$H\{f(x)\} = \frac{1}{s}\left[a^{n+1} J_{n+1}(sa) - 0\right]$$

$$H\{f(x)\} = \frac{a^{n+1}}{s} J_{n+1}(sa)$$

Green's Identity or Green's formula:

This identity is based on Lagrange's identity integrating both sides of Lagrange's identity,

$$\int_a^b \{uL[v] - vL[u]\}\, dx = \int_a^b D\left[p(x) W(u,v)(x)\right] dx = p(x) W(\mathrm{u},v)(x)\Big|_a^b$$

Theorem:

A self-adjoint L = D[p(x)D]+q(x) is a symmetric operator on the interval [a,b] if and only if $p(x)W(\mathrm{u},v)(x)\Big|_a^b = 0$ u and v satisfy the described boundary conditions associated L and have continuous second order derivative in the interval [a,b].

Proof: Suppose that L = D[p(x)D]+q(x) is a symmetric operator on the interval [a,b] then for any functions u and v

$$\int_a^b \{uL[v] - vL[u]\} dx = 0$$

And by Green's formula

$$\int_a^b \{uL[v] - vL[u]\}\, dx = \int_a^b D\left[p(x) W(u,v)(x)\right] dx$$

$$= p(x) W(\mathrm{u},v)(x)\Big|_a^b$$

$$\Rightarrow p(x) W(\mathrm{u},v)(x)\Big|_a^b = 0$$

Consider that

$$p(x)W(\mathrm{u},v)(x)\Big|_a^b = 0$$

$$\int_a^b \{uL[v]-vL[u]\}\,dx$$

$$= p(x)W(\mathrm{u},v)(x)\Big|_a^b = 0$$ which implies L is symmetric.

Green's functions and their applications:

The sudden excitation to a system denoted by $d_a(t)$, has a non-zero value over the short interval of time: a − ε < I < a + ε but is otherwise zero. The total impulse (force times duration) imparted to the system is thus defined by

$$I = \int_{-\infty}^{\infty} d_a(t)\,dt = \int_{a-\varepsilon}^{a+\varepsilon} d_a(t)\,dt, (\varepsilon > 0)$$

Let us idealize the function, $d_a(t)$ by requiring it to act over shorter and shorter interval of time allowing ε→0. Although the interval about t = a is shrinking to zero, we still want I = 1.

i.e. $$\frac{Lim}{\varepsilon \to 0} I = \frac{Lim}{\varepsilon \to 0} \int_{-\infty}^{\infty} d_a(t)\,dt = 1$$

We can use the result to this limit process to define an "idealized" unit impulse function, $\delta(t-a)$ which has the property of imparting a unit impulse to the system at time t = a but being zero for other values of t. The defining properties of this function are, therefore,

$$\delta(t-a) = 0,\ t \neq a$$

$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$$

Green's function for boundary value problems:

The general boundary value problem is characterized by

$$M[y] = F(x) \, , \, a < x < b,$$

$$B_1[y] = a_{11}y(a) + a_{12}y^{'}(a) = \alpha \qquad (1)$$

$$B_2[y] = a_{21}y(a) + a_{22}y^{'}(a) = \beta$$

Where the differential operator M is defined by

$$M[y] = A_2(x)\frac{d^2}{dx^2} + A_1(x)\frac{d}{dx} + A_0(x)$$

For developing a Green's function for this problem, it is preferred to the differential equation in self-adjoint form. To do so, multiply the differential equation by the function: $\mu(x) = p(x)/A_2(x)$ where

$$p(x) = \exp\int \frac{A_1(x)}{A_2(x)}dx \qquad (2)$$

So, the problem (1) assumes the form:

$$L[y] = f(x) \, , \, a < x < b, \qquad (3)$$

$$B_1[y] = \alpha \, , \, B_2 = [y] = \beta$$

Where $\quad f(x) = p(x)F(x)/A_2 . \text{q}(x) = p(x)A_0(x)/A_2(x)$

$$L[y] = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) \qquad (4)$$

Suppose that the solution of (3) has the general form:

$$y = y_H + y_p$$

Where y_H satisfies the boundary value problem:

$$L[y] = 0 \qquad (5)$$

$$B_1[y] = \alpha \, , \, B_2 = [y] = \beta$$

And y_p is a solution of

$$L[y] = f(x) \qquad (6)$$

$$B_1[y] = 0\ ,\ B_2 = [y] = 0$$

The general solution of (5) is

$$y_H = c_1 y_1(x) + c_2 y_2(x) \qquad (7)$$

Where $y_1(x)$ & $y_2(x)$ are linearly independent solutions of the homogeneous differential equation. The constants c_1 & c_2 are determined by imposing the non-homogeneous boundary conditions. Consider the problem (6). Suppose that the solution of (6) can be expressed in the integral form as:

$$y_p = -\int_a^b g(x,s) f(s) ds \qquad (8)$$

Where $g(x,s)$ is the Green's function to be defined. The minus sign in (8) is chosen such that g(x,s) will have the proper physical interpretation. Applying the differential operator, L to both sides if (8), assuming commutativity of L with integration, we find that

$$L\left[y_p\right] = L\left[-\int_a^b g(x,s) f(s) ds\right] = -\int_a^b L\left[g(x,s)\right] f(s) ds$$

Now it is argued that y_p is indeed a solution of the differential equation in (6), then the right-hand side of this last expression must equal f(x). This will happen provided

$$L[g] = -\delta(x-s) \qquad a < x < b \qquad (9)$$

Where $\delta(x-s)$ is the Dirac delta function.

In order to determine the unique solution, g(x,s), it is also found conditions other than (9) that contribute to its definition. Let us impose the homogenous boundary conditions in (6) on the solution (8), which leads to

$$B_1\left[y_p\right] = -\int_a^b B_1[g] f(s) ds = 0$$

$$B_2\left[y_p\right] = -\int_a^b B_2\left[g\right] f\left(s\right) ds = 0$$

Because f(x) can be almost any function, these relations are possible if

$$B_1\left[g\right] = 0\ ,\ B_2\left[g\right] = 0 \qquad (10)$$

Hence the Green's function, we are seeking a solution of the boundary value problem:

$$L\left[g\right] = -\delta\left(x - s\right) \qquad a < x < b$$

$$B_1\left[g\right] = 0\ ,\ B_2\left[g\right] = 0 \qquad (11)$$

Where s is fixed and a < x < b.

This problem is similar to that described by (6). Only the forcing function in (11) is a delta function rather than an arbitrary function, f(x). This means that solving the problem for g(x,s) is simpler than solving the corresponding problem of y and once the Green's functions has been found for a particular operator, L and set of boundary conditions, it may be used for solving (6) any number of times where only the function, f(x) changes from problem to problem. It is the feature of Green's function that makes it most useful in applications.

The presence of delta function in (9) shows the peculiar behavior of g(x,s) in the vicinity of x–s.

To investigate this behavior, form (8), we have

$$y_p\left(s^+\right) - y_p\left(s^-\right) = -\int_a^b \left[g\left(s^+, s\right) - g\left(s^-, s\right)\right] f\left(s\right) ds$$

Where

$$y_p\left(s^+\right) = \frac{Lim}{x \to 0^+} y_p\left(s + \varepsilon\right)\ ,\ y_p\left(s^-\right) = \frac{Lim}{x \to 0^-} y_p\left(s - \varepsilon\right)$$

Because the solution of a differential equation must be a continuous function, the left-hand side of the above expression vanishes and since f(x) is arbitrary, we deduce that

$$g\left(s^{+},s\right)=g\left(s^{-},s\right) \qquad (12)$$

Which implies that g(x,s) is continuous at x = s.

Next the behavior of the derivative of g(x,s) at x = s is investigated. Integrate both sides (9) with respect to x from $x=s^{-}\ to\, x=s^{+}$ i.e.

$$\int_{s^-}^{s^+} L[g]=-\int_{s^-}^{s^+}\delta(x-s)\,dx$$

$$\Rightarrow\int_{s^-}^{s^+}\left\{\frac{\partial}{\partial x}\left[p(x)\frac{\partial g(x,s)}{\partial x}\right]+q(x)g(x,s)\right\}dx=-\int_{s^-}^{s^+}\delta(x-s)\,dx$$

$$\Rightarrow\left[p(x)\frac{\partial g(x,s)}{\partial x}\right]_{x=s^-}^{x=s^+}\int_{s^-}^{s^+}q(x)g(x,s)\,dx=-\int_{s^-}^{s^+}\delta(x-s)\,dx$$

From the continuity of both q(x) and g(x,s) at x = s, it follows that the integral on the left hand side of this expression is zero. Also, using the integral property of the delta function and the fact that p(x) is continuous and non-zero on [a,b] the last expression reduces to

$$\frac{Lim}{x\to s^+}p(x)\frac{\partial g(x,s)}{\partial x}-\frac{Lim}{x\to s^-}p(x)\frac{\partial g(x,s)}{\partial x}=-1$$

$$\Rightarrow\frac{Lim}{x\to s^+}p(x)\times\frac{Lim}{x\to s^+}\frac{\partial g(x,s)}{\partial x}-\frac{Lim}{x\to s^-}p(x)\times\frac{Lim}{x\to s^-}\frac{\partial g(x,s)}{\partial x}=-1$$

$$\Rightarrow p\left(s^{+}\right)\times\frac{Lim}{x\to s^+}\frac{\partial g(x,s)}{\partial x}-p\left(s^{-}\right)\times\frac{Lim}{x\to s^-}\frac{\partial g(x,s)}{\partial x}=-1$$

$$\Rightarrow p(s)\times\frac{Lim}{x\to s^+}\frac{\partial g(x,s)}{\partial x}-p(s)\times\frac{Lim}{x\to s^-}\frac{\partial g(x,s)}{\partial x}=-1 \qquad (13)$$

$$\Rightarrow p(s)\left[\frac{Lim}{x\to s^+}\frac{\partial g(x,s)}{\partial x}-\frac{Lim}{x\to s^-}\frac{\partial g(x,s)}{\partial x}\right]=-1$$

$$\Rightarrow\left.\frac{\partial g(x,s)}{\partial x}\right|_{s^-}^{s^+}=\frac{-1}{p(s)}$$

Where the continuity of q(x) has been used at x = s.

This results (13) suggest that x =s, the derivative of g(x,s) has a jump discontinuity of magnitude, 1/p(x).

Def. The Green's function, g(x,s) associated with the boundary value problem:

$$L[y] = f(x) \ , a < x < b,$$

$$B_1[y] = \alpha \ , B_2 = [y] = \beta$$

Where $$L[y] = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$$

is a function satisfying the following conditions: (a < x < b)

(a) $L[g] = -\delta(x-s) \qquad (a < x < b), \qquad (s \ fixed)$

(b) $B_1[g] = 0 \ , \ B_2[g] = 0$

(c) $g(s^+, s) = g(s^-, s)$

(d) $\left.\dfrac{\partial g(x,s)}{\partial x}\right|_{s^-}^{s^+} = \dfrac{-1}{p(s)}$

Based on this definition of Green's function, an explicit formula for the Greens' functions, can be constructed. It is observed from (a) that if either x < s or x > s then L[g] = 0 from the definition of the delta function. Next if z_1 and z_2 are solutions of the homogeneous differential equation, L[g] = 0 such a way that

$$B_1[g] = 0 \ , \ B_2[g] = 0 \qquad (14)$$

From conditions (a) and (b) it follows that Green's functions has the form:

$$g(x,s) = \begin{cases} u(s)z_1(x) \ , \ x < s \\ v(s)z_2(x) \ , \ x > s \end{cases} \qquad (15)$$

Where u and v are functions to be determined. Imposing conditions (c) and (d), the unknown functions u & v must be chosen such that

$$v(s)z_2(s) - u(s)z_1(s) = 0$$

$$v(s)z_2'(s)-u(s)z_1'(s)=\frac{-1}{p(s)} \qquad (16)$$

The simultaneous solution of (16) yields

$$u(s)=-\frac{z_2(s)}{p(s)W(z_1,z_2)}$$
$$v(s)=-\frac{z_1(s)}{p(s)W(z_1,z_2)} \qquad (17)$$

Where $W(z_1,z_2)=z_1z_2'-z_1'z_2$ is the Wronskian function.

Result: It is a curious fact that if y_1 , y_2 are any solutions of the same homogeneous differential equation, the $p(x)W(y_1,y_2)$ is a constant number.

Proof: Let y_1 & y_2 be two solutions of

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=0 \qquad (1)$$

Then
$$\frac{d}{dx}\left[p(x)\frac{dy_1}{dx}\right]+q(x)y_1=0 \qquad (2)$$

$$\frac{d}{dx}\left[p(x)\frac{dy_2}{dx}\right]+q(x)y_2=0 \qquad (3)$$

Multiplying (2) by y_2 and (3) by y_1

$$y_2\frac{d}{dx}\left[p(x)\frac{dy_1}{dx}\right]+q(x)y_1y_2=0$$

$$y_1\frac{d}{dx}\left[p(x)\frac{dy_2}{dx}\right]+q(x)y_1y_2=0$$

$$-\qquad\qquad\qquad -$$

$$y_2\frac{d}{dx}\left[p(x)\frac{dy_1}{dx}\right]-y_1\frac{d}{dx}\left[p(x)\frac{dy_2}{dx}\right]=0$$

$$\Rightarrow\frac{d}{dx}\left[p(x)\frac{dy_1}{dx}y_2\right]-\frac{d}{dx}\left[p(x)\frac{dy_2}{dx}y_1\right]=0$$

$$\Rightarrow \frac{d}{dx}\left[p(x)\left(\frac{dy_2}{dx}y_1 - \frac{dy_1}{dx}y_2\right)\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[p(x)\left(\frac{dy_2}{dx}y_1 - \frac{dy_1}{dx}y_2\right)\right] = 0$$

Integrating

$$\frac{d}{dx}\left[p(x)\left(\frac{dy_2}{dx}y_1 - \frac{dy_1}{dx}y_2\right)\right] = c$$

$$\Rightarrow \frac{d}{dx}\left[p(x)W(y_1, y_2)(x)\right] = c$$

Thus since z_1 & z_2 are two such solutions, we can write

$$p(s)W(z_1, z_2)(s) = p(x)W(z_1, z_2)(x) = c$$

And the Green's function in (15) takes the form:

$$g(x,s) = \begin{cases} \dfrac{-z_1(x)z_2(s)}{c}, & a < x < s \\ \dfrac{-z_1(s)z_2(x)}{c}, & s < x < b \end{cases} \qquad (18)$$

Or equivalently

$$g(x,s) = \begin{cases} \dfrac{-z_1(s)z_2(x)}{c}, & a < s < x \\ \dfrac{-z_1(x)z_2(s)}{c}, & x < s < b \end{cases} \qquad (19)$$

Because C is constant, it follows from inspection of (19) that Green's function is symmetric in x & s i.e.,

$$g(x,s) = g(s,x)$$

Question: Construct the Green's function associated with boundary value problem; $y'' + k^2 y = F(x)$; $0 < x < 1$, $y(0) = \alpha$, $y(1) = \beta$

Solution: $$y'' + k^2 y = F(x)$$

$$\frac{d^2y}{dx^2}+k^2y=F(x)$$

$$\left(\frac{d^2}{dx^2}+k^2\right)y=F(x)$$

$$\left(\frac{d^2}{dx^2}+k^2\right)y=0 \quad \because f(x)=0 \text{ any constant function}$$

The auxiliary equation

$$\frac{d^2}{dx^2}+k^2=0$$

$$\frac{d^2}{dx^2}=-k^2$$

$$\frac{d}{dx}=\pm ik$$

$$y_c=c_1\cos kx+c_2\sin kx$$

Let $\quad y_1=\cos kx \quad , \; y_2=\sin kx$

In order to construct solution z_1 and z_2 they are assumed to be some linear combination of y_1 and y_2 , written as

$$z_1=c_1\cos kx+c_2\sin kx \qquad \_\_\_\_\_\_(i)$$

By applying first boundary condition y(0) = 0

$$0=c_1+0 \Rightarrow c_1=0$$

Put in (i)

$$z_1=c_2\sin kx$$

Take $c_1=1$

$$z_1=\sin kx$$

Again $\qquad z_2=d_1\cos kx+d_2\sin kx \qquad \_\_\_\_\_\_(ii)$

By applying second boundary condition y(1) = 0

$$\Rightarrow \quad 0 = d_1 \cos k + d_2 \sin k$$

$$\Rightarrow \quad d_2 = \frac{-d_1 \cos k}{\sin k}$$

Put in (ii) $\quad z_2 = d_1 \cos kx + \left(\frac{-d_1 \cos k}{\sin k}\right) \sin kx$

$$z_2 = \frac{d_1 \,\text{sink} \cos kx - d_1 \cos k \sin kx}{\sin k}$$

$$z_2 = \frac{d_1}{\sin k}\left[\text{sink} \cos kx - \cos k \sin kx\right]$$

$$z_2 = \frac{d_1}{\sin k} \sin k(1-x)$$

$$Take \frac{d_1}{\sin k} = 1$$

$$z_2 = \sin k(1-x)$$

Now c determined as

$$c = P(x)w(z_1, z_2)(x)$$

$$c = (1)\begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} \quad \because P(x) = 1$$

$$c = (1)\begin{vmatrix} \sin kx & \sin k(1-x) \\ k\cos kx & -k\cos k(1-x) \end{vmatrix}$$

$$c = -k \sin kx \cos k(1-x) - k \cos kx \sin k(1-x)$$

$$c = -k\left[\sin kx \cos k(1-x) + \cos kx \sin k(1-x)\right]$$

$$c = -k\left[\sin k(x+1-x)\right]$$

$$c = -k\sin k$$

Now green function

$$g(x,s) = \begin{cases} -\dfrac{z_1(x)z_2(s)}{c} & ;\ a < x < s \\ -\dfrac{z_1(s)z_2(x)}{c} & ;\ s \le x < b \end{cases}$$

$$g(x,s) = \begin{cases} -\dfrac{\sin kx.\sin k(1-s)}{-k\sin k} & ;\ 0 < x < s \\ -\dfrac{\sin kx.\sin k(1-x)}{-k\sin k} & ;\ s \le x < 1 \end{cases}$$

$$g(x,s) = \begin{cases} \dfrac{\sin kx.\sin k(1-s)}{k\sin k} & ;\ 0 < x < s \\ \dfrac{\sin kx.\sin k(1-x)}{k\sin k} & ;\ s \le x < 1 \end{cases}$$

Question: Use the method of Green's function to solve. Construct the Green's function associated with boundary value problem;

$$y'' + y = \sin x\ ;\ 0 < x < \frac{\pi}{2}\ ,\ y(0) = 1\ ,\ y\left(\frac{\pi}{2}\right) = -1$$

Solution:
$$y'' + y = \sin x$$

$$\frac{d^2y}{dx^2} + y = \sin x$$

For particular solution

$$\left(\frac{d^2}{dx^2} + 1\right)y = 0$$

The auxiliary equation

$$\frac{d^2}{dx^2} + 1 = 0$$

$$\frac{d^2}{dx^2} = -1$$

$$\frac{d}{dx} = \pm i$$

$$y_H = c_1 \cos x + c_2 \sin x \qquad \because y_H = particular\, solution$$

Put y(0) = 1

$$1 = c_1 + 0 \Rightarrow c_1 = 1$$

Put y(π/2) = –1

$$-1 = 0 + c_2 \Rightarrow c_2 = -1$$

$$\Rightarrow\ y_H = \cos x - \sin x$$

Let $y_1 = \cos x \quad , \ y_2 = \sin x$

In order to construct solution z_1 and z_2 they are assumed to be some linear combination of y_1 and y_2 , written as

$$z_1 = c_1 \cos x + c_2 \sin x \qquad \_\_\_\_\_\_(i)$$

By applying first boundary condition y(0) = 0

$$0 = c_1 + 0 \quad \Rightarrow \quad c_1 = 0$$

Put in (i)

$$z_1 = c_2 \sin x$$

Take $c_2 = 1$

Similarly, $z_2 = d_1 \cos x + d_2 \sin x \qquad \_\_\_\_\_\_(ii)$

By applying second boundary condition y(π/2) = 0

$$\Rightarrow \quad 0 = 0 + d_2 \quad \Rightarrow d_2 = 0$$

Put in (ii) $z_2 = d_1 \cos x$

Take $d_1 = 1$

$$z_2 = \cos x$$

Now c determined as

$$c = P(x)\,w(z_1, z_2)(x)$$

$$c = (1)\begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} \quad \because P(x) = 1$$

$$c = (1)\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$c = -\sin^2 x - \cos^2 x \quad = -\left(\sin^2 x + \cos^2 x\right) = -1$$

Now green function

$$g(x,s) = \begin{cases} -\dfrac{z_1(x)z_2(s)}{c} & ;\ a < x < s \\ -\dfrac{z_1(s)z_2(x)}{c} & ;\ s \le x < b \end{cases}$$

$$g(x,s) = \begin{cases} -\dfrac{\sin x.\cos(s)}{-1} & ;\ 0 < x < s \\ -\dfrac{\text{sins}.\cos(x)}{-1} & ;\ s \le x < \dfrac{\pi}{2} \end{cases}$$

$$g(x,s) = \begin{cases} \sin(x).\cos(s) & ;\ 0 < x < s \\ \sin(s).\cos(x) & ;\ s \le x < \dfrac{\pi}{2} \end{cases}$$

Now for y_p we have $\quad y_p = -\int_a^b g(x,s)\,f(s)\,ds$

$$y_p = -\left[\int_0^x g(x,s)\,f(s)\,ds + \int_x^{\frac{\pi}{2}} g(x,s)\,f(s)\,ds\right]$$

$$y_p = -\left[\int_0^x \sin(x)\cos(s)\sin(s)\,ds + \int_x^{\frac{\pi}{2}} \sin(s)\cos(x)\sin(s)\,ds\right]$$

$$y_p = -\left[\frac{\sin(x)}{2}\int_0^x 2\sin(s)\cos(s)\,ds + \cos(x)\int_x^{\frac{\pi}{2}} \sin^2(s)\,ds\right]$$

$$y_p = -\left[\frac{\sin(x)}{2}\int_0^x \sin(2s)\,ds + \cos(x)\int_x^{\frac{\pi}{2}}\left(\frac{1-\cos(2s)}{2}\right)ds\right]$$

$$y_p = -\left[\frac{\sin(x)}{2}\left(-\frac{\cos 2s}{2}\right)_0^x + \frac{\cos x}{2}\left(s - \frac{\sin(2s)}{2}\right)_x^{\frac{\pi}{2}}\right]$$

$$y_p = -\left[\frac{\sin(x)}{2}\left(-\frac{\cos(2x)}{2} + \frac{1}{2}\right) + \frac{\cos x}{2}\left(\left\{\frac{\pi}{2} - \frac{0}{2}\right\} - \left\{x - \frac{\sin(2x)}{2}\right\}\right)\right]$$

$$y_p = -\left[\frac{\sin(x)}{2}\left(\frac{1-\cos 2\text{x}}{2}\right) + \frac{\cos x}{2}\left(\frac{\pi}{2} - x + \frac{\sin(2x)}{2}\right)\right]$$

$$y_p = -\left[\frac{\sin(x)}{2}\left(\sin^2 x\right) + \frac{\pi}{4}\cos x - \frac{x}{2}\cos x + \frac{\cos\text{x}\sin(2x)}{2}\right]$$