HISTORY OF MATHEMATICS

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Mathematics is a unique aspect of human thought, and its history differs in essence from all other histories.

RECOMMENDED BOOKS:

(i) A History of Mathematics by Uta C. Merzbach and Carl B. Boyer

(ii) A History of Mathematics by Florian Cajori

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INTRODUCTION

As time goes on, nearly every field of human endeavor is marked by changes which can be considered as correction or extension. Thus, the changes in the evolving history of political and military events are always chaotic; there is no way to predict the rise of a Genghis Khan, for example, or the consequences of the short-lived Mongol Empire.

Other changes are a matter of fashion and subjective opinion. The cave-paintings of 25,000 years ago are generally considered great art, and while art has continuously—even chaotically—changed in the subsequent millennia, there are elements of greatness in all the fashions. Similarly, each society considers its own ways natural and rational, and finds the ways of other societies to be odd, laughable, or repulsive. But only among the sciences is there true progress; only there is the record one of continuous advance toward ever greater heights. And yet, among most branches of science, the process of progress is one of both correction and extension.

Aristotle, one of the greatest minds ever to contemplate physical laws, was quite wrong in his views on falling bodies and had to be corrected by Galileo in the 1590s.

Galen, the greatest of ancient physicians, was not allowed to study human cadavers and was quite wrong in his anatomical and physiological conclusions. He had to be corrected by Vesalius in 1543 and Harvey in 1628.

Even **Newton,** the greatest of all scientists, was wrong in his view of the nature of light, of the achromaticity of lenses, and missed the existence of spectral lines. His masterpiece, the laws of motion and the theory of universal gravitation, had to be modified by Einstein in 1916.

Now we can see what makes mathematics unique.

Only in mathematics is there no significant correction—only extension. Once the Greeks had developed the deductive method, they were correct in what they did, correct for all time.

Euclid was incomplete and his work has been extended enormously, but it has not had to be corrected. His theorems are, every one of them, valid to this day.

Ptolemy may have developed an erroneous picture of the planetary system, but the system of trigonometry he worked out to help him with his calculations remains correct forever.

Each great mathematician adds to what came previously, but nothing needs to be uprooted. The contemplation of the various steps by which mankind has come into possession of the vast stock of mathematical knowledge can hardly fail to interest the mathematician. He takes pride in the fact that his science, more than any other, is an exact science, and that hardly anything ever done in mathematics has proved to be useless. The chemist smiles at the childish efforts of alchemists, but the mathematician finds the geometry of the Greeks and the arithmetic of the Hindoos as useful and admirable as any research of to-day. He is pleased to notice that though, in course of its development, mathematics has had periods of slow growth, yet in the main it has been pre-eminently a progressive science.

The history of mathematics may be instructive as well as agreeable; it may not only remind us of what we have, but may also teach us how to increase our store. Says De Morgan, “The early history of the mind of men with regard to mathematics leads us to point out our own errors; and in this respect it is well to pay attention to the history of mathematics." It warns us against hasty conclusions; it points out the importance of a good notation upon the progress of the science; it discourages excessive specialization on the part of investigators, by showing how apparently distinct branches have been found to possess unexpected connecting links; it saves the student from wasting time and energy upon problems which were, perhaps, solved long since; it discourages him from attacking an unsolved problem by the same method which has led other mathematicians to failure; it teaches that fortifications can be taken in other ways than by direct attack, that when repulsed from a direct assault it is well to reconnoiter and occupy the surrounding ground and to discover the secret paths by which the apparently unconquerable position can be taken.

The importance of this strategic rule may be emphasized by citing a case in which it has been violated. An untold amount of intellectual energy has been expended on the quadrature of the circle, yet no conquest has been made by direct assault. The circle-squarers have existed in crowds ever since the period of Archimedes. After innumerable failures to solve the problem at a time, even, when investigators possessed that most powerful tool, the differential calculus, persons versed in mathematics dropped the subject, while those who still persisted were completely ignorant of its history and generally misunderstood the conditions of the problem. Our problem," says De Morgan, is to square the circle with the old allowance of means: Euclid's postulates and nothing more. We cannot remember an instance in which a question to be solved by a definite method was tried by the best heads, and answered at last, by that method, after thousands of complete failures." But progress was made on this problem by approaching it from a different direction and by newly discovered paths. Lambert proved in 1761 that the ratio of the circumference of a circle to its diameter is incommensurable. Some years ago, Lindemann demonstrated that this ratio is also transcendental and that the quadrature of the circle, by means of the ruler and compass only, is impossible. He thus showed by actual proof that which keen-minded mathematicians had long suspected; namely, that the great army of circle-squarers have, for two thousand years, been assaulting a fortification which is as indestructible as the firmament of heaven.

Another reason for the desirability of historical study is the value of historical knowledge to the teacher of mathematics. The interest which pupils take in their studies may be greatly increased if the solution of problems and the cold logic of geometrical demonstrations are interspersed with historical remarks and anecdotes. A class in arithmetic will be pleased to hear about the Hindoos and their invention of the “Arabic notation"; they will marvel at the thousands of years which elapsed before people had even thought of introducing into the numeral notation that Columbus-egg _ the zero; they will find it astounding that it should have taken so long to invent a notation which they themselves can now learn in a month. After the pupils have learned how to bisect a given angle, surprise them by telling of the many futile attempts which have been made to solve, by elementary geometry, the apparently very simple problem of the trisection of an angle. When they know how to construct a square whose area is double the area of a given square, tell them about the duplication of the cube _ how the wrath of Apollo could be appeased only by the construction of a cubical altar double the given altar, and how mathematicians long wrestled with this problem. After the class have exhausted their energies on the theorem of the right triangle, tell them the legend about its discoverer _ how Pythagoras, jubilant over his great accomplishment, sacrificed a hecatomb to the Muses who inspired him. When the value of mathematical training is called in question, quote the inscription over the entrance into the academy of Plato, the philosopher:

”Let no one who is unacquainted with geometry enter here."

Students in analytical geometry should know something of Descartes, and, after taking up the differential and integral calculus, they should become familiar with the parts that Newton, Leibniz, and Lagrange played in creating that science. In his historical talk it is possible for the teacher to make it plain to the student that mathematics is not a dead science, but a living one in which steady progress is made.

The history of mathematics is important also as a valuable contribution to the history of civilization. Human progress is closely identified with scientific thought. Mathematical and physical researches are a reliable record of intellectual progress. The history of mathematics is one of the large windows through which the philosophic eye looks into past ages and traces the line of intellectual development.

ANCIENT EGYPT

Sesostris . . . made a division of the soil of Egypt among the inhabitants. . . . If the river carried away any portion of a man's lot, . . .the king sent persons to examine, and determine by measurement the exact extent of the loss. . . . From this practice, I think, geometry first came to be known in Egypt, whence it passed into Greece.

Herodotus

THE ERA AND THE SOURCES

About 450 BCE, Herodotus, the inveterate Greek traveler and narrative historian, visited Egypt. He viewed ancient monuments, interviewed priests, and observed the majesty of the Nile and the achievements of those working along its banks. His resulting account would become a cornerstone for the narrative of Egypt's ancient history. When it came to mathematics, he held that geometry had originated in Egypt, for he believed that the subject had arisen there from the practical need for resurveying after the annual flooding of the river valley. A century later, the philosopher Aristotle speculated on the same subject and attributed the Egyptians' pursuit of geometry to the existence of a priestly leisure class.

The debate, extending well beyond the confines of Egypt, about whether to credit progress in mathematics to the practical men (the surveyors, or **"rope-stretchers"** or to the contemplative elements of society **"the priests and the philosophers"** has continued to our times. As we shall see, the history of mathematics displays a constant interplay between these two types of contributors.

In attempting to piece together the history of mathematics in ancient Egypt, scholars until the nineteenth century encountered two major obstacles. <u>The first was the inability to read the source materials that existed. The second was the scarcity of such materials</u>.

For more than thirty-five centuries, inscriptions used hieroglyphic writing, with variations from purely ideographic to the smoother hieratic and eventually the still more flowing demotic forms. After the third century CE, when they were replaced by Coptic and eventually supplanted by Arabic, knowledge of hieroglyphs faded.

The breakthrough that enabled modern scholars to decipher the ancient texts came early in the nineteenth century when the French scholar Jean-Francois Champollion, working with multilingual tablets, was able to slowly translate a number of hieroglyphs. These studies were supplemented by those of other scholars, including the British physicist Thomas Young, who were intrigued by <u>the **Rosetta Stone**, a tri-lingual basalt slab with inscriptions in hieroglyphic,</u>

demotic, and Greek writings that had been found by members of Napoleon's Egyptian expedition in 1799.

By 1822, Champollion was able to announce a substantive portion of his translations in a famous letter sent to the Academy of Sciences in Paris, and by the time of his death in 1832, he had published a grammar textbook and the beginning of a dictionary.

Although these early studies of hieroglyphic texts shed some light on Egyptian numeration, they still produced no purely mathematical materials. This situation changed in the second half of the nineteenth century.

We abstain from introducing additional Greek opinion regarding Egyptian mathematics, or from indulging in wild conjectures. We rest our account on documentary evidence. A hieratic papyrus, included in the Rhind collection of the British Museum, was deciphered by Eisenlohr in 1877, and found to be a mathematical manual containing problems in arithmetic and geometry. It was written by Ahmes sometime before 1700 b.c., and was founded on an older work believed by Birch to date back as far as 3400 b.c.! This curious papyrus _ the most ancient mathematical handbook known to us_ puts us at once in contact with the mathematical thought in Egypt of three or five thousand years ago. It is entitled **"Directions for obtaining the Knowledge of all Dark Things."** We see from it that the Egyptians cared but little for theoretical results. Theorems are not found in it at all. It contains" hardly any general rules of procedure, but chiefly mere statements of results intended possibly to be explained by a teacher to his pupils." In 1858, the Scottish antiquary Henry **Rhind** purchased a papyrus roll in Luxor that is about one foot high and some eighteen feet long. Except for a few fragments in the Brooklyn Museum, this papyrus is now in the British Museum. It is known as **the Rhind or the Ahmes Papyrus**, in honor of the scribe by whose hand it had been copied in about 1650 BCE. The scribe tells us that the material is derived from a prototype from the Middle Kingdom of about 2000 to 1800 BCE. Written in the hieratic script, it became the major source of our knowledge of ancient Egyptian mathematics.

Another important papyrus, known as the **Golenishchev or Moscow Papyrus**, was purchased in 1893 and is now in the Pushkin Museum of Fine Arts in Moscow. It, too, is about eighteen feet long but is only one-fourth as wide as the Ahmes Papyrus. It was written less carefully than the work of Ahmes was, by an unknown scribe of circa. 1890 BCE. It contains twenty-five examples, mostly from practical life and not differing greatly from those of Ahmes, except for two that will be discussed further on.

Yet another twelfth-dynasty papyrus, **the Kahun**, is now in London; a **Berlin papyrus** is of the same period. Other, somewhat earlier, materials are two wooden tablets from **Akhmim** of about 2000 BCE and a leather roll containing a list of fractions. Most of this material was deciphered with in a hundred years of Champollion's death. There is a striking degree of coincidence between certain aspects of the earliest known inscriptions and the few mathematical texts of the Middle Kingdom that constitute our known source material.

POSSIBLE SHORT QUESTIONS:

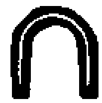
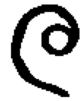
i. Write an ancient Egypt saying about origination of Geometry in Egypt.
ii. How does geometry come into picture in Egypt?
iii. Write the role of mathematics between two type of contributors i.e. Surveyor /Rope stretcher and the Priest/ the Philosophers.
iv. Write two major obstacles/problems to collect material in Egyptian math's history.
v. Write the role of Jean Prancois Champollin regarding Egyptian math's history.
vi. What was Rosetta Stone. Who found it?
vii. Who was the Rhind (the Ahmes)?
viii. Note on the Rhind Papyrus (the Ahmes Papyrus)?
ix. Write the title of Ahmes's Papyrus.
x. Note on the Moscow Papyrus (the Goleneshchev)?
xi. Name few documets of Egyptian mathematics history.

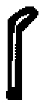
LONG QUESTIONS:

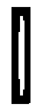
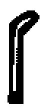
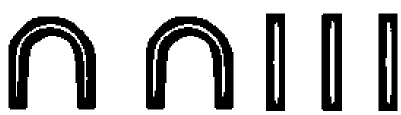
1) Briefly describe about the era and the sources of ancient Egypt.
2) Write a brief note on The Ahmes (the Rhind) and The Ahmes Papyrus.

NUMBERS AND FRACTIONS

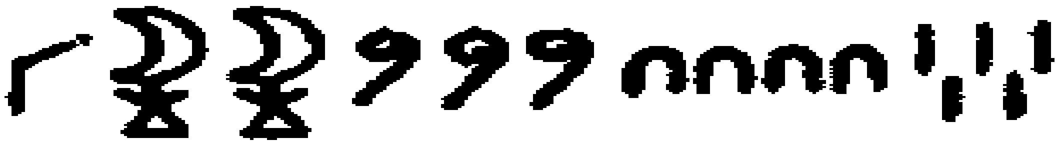
An insight into Egyptian methods of numeration was obtained through the ingenious deciphering of the hieroglyphics by Champollion, Young, and their successors.

The symbols used were the following:  for 1,  for 10,  for 100,

 for 1000,  for 10; 000,  for 100; 000,  for 1; 000; 000,

 for 10; 000; 000. The symbol for 1 represents a vertical staff  for 10; 000 a pointing finger  for 100; 000 a burbot  for 1; 000; 000, a man in astonishment . The significance of the remaining symbols is very doubtful. The writing of numbers with these hieroglyphics was very cumbrous. The unit symbol of each order was repeated as many times as there were units in that order. The principle employed was the *additive*. Thus, 23 was written .

A single vertical stroke represented a unit, an inverted wicket was used for 10, a snare somewhat resembling a capital C stood for 100, a lotus flower for 1,000, a bent finger for 10,000, a tadpole for 100,000, and a kneeling figure, apparently Heh, the god of the Unending, for 1,000,000. Through repetition of these symbols, the number 12,345, for example, would appear as



Sometimes the smaller digits were placed on the left, and other times the digits were arranged vertically. The symbols themselves were occasionally reversed in orientation, so that the snare might be convex toward either the right or the left.

Besides the hieroglyphics, Egypt possesses the hieratic and demotic writings, but for want of space we pass them by.

The more cursive hieratic script used by Ahmes was suitably adapted to the use of pen and ink on prepared papyrus leaves. Numeration remained decimal, but the tedious repetitive principle of hieroglyphic numeration was replaced by the introduction of ciphers or special signs to represent digits and multiples of powers of 10. The number 4, for example, usually was no longer represented by four vertical strokes but by a horizontal bar, and 7 is not written as seven strokes but as a single cipher  resembling a sickle. The hieroglyphic form for the number 28 was  the hieratic form was simply . Note that the cipher  for the smaller digit 8 (or two 4s) appears on the left, rather than on the right.

Herodotus makes an important statement concerning the mode of computing among the Egyptians. He says that they;

"calculate with pebbles by moving the hand from right to left, while the Hellenes move it from left to right."

Herein we recognize again that **instrumental method of figuring** so extensively used by peoples of antiquity.

The Egyptians used the decimal scale. Since, in figuring, they moved their hands horizontally, it seems probable that they used ciphering- boards with vertical columns. In each column there must have been not more than nine pebbles, for ten pebbles would be equal to one pebble in the column next to the left.

The **Ahmes papyrus** contains interesting information on the way in which the Egyptians employed fractions. Their methods of operation were, of course, radically different from ours. Fractions were a subject of very great difficulty with the ancients. Simultaneous changes in both numerator and denominator were usually avoided. In manipulating fractions the Babylonians kept the denominators (60) constant. The Romans likewise kept them constant, but equal to 12. The Egyptians and Greeks, on the other hand, kept the numerators constant, and dealt with variable denominators. Ahmes used the term "fraction" in a restricted sense, for he applied it only to unit-fractions, or fractions having unity for the numerator. It was designated by writing the denominator and then placing over it a dot. Fractional values which could not be expressed by any one unit-fraction were expressed as the sum of two or more of them. Thus, he wrote $\frac{1}{3}\frac{1}{15}$ in place of $\frac{2}{5}$. The first important problem naturally arising was,

HOW TO REPRESENT ANY FRACTIONAL VALUE AS THE SUM OF UNIT-FRACTIONS.

This was solved by aid of a table, given in the papyrus, in which all fractions of the form $\frac{2}{2n+1}$ (where n designates successively all the numbers up to 49) are reduced to the sum of unit-fractions. Thus, $\frac{2}{7} = \frac{1}{4}\ \frac{1}{28}$ and $\frac{2}{99} = \frac{1}{66}\ \frac{1}{198}$. When, by whom, and how this table was calculated, we do not know. Probably it was compiled empirically at different times, by different persons. It will be seen that by repeated application of this table, a fraction whose numerator exceeds two can be expressed in the desired form, provided that there is a fraction in the table having the same denominator that it has. Take, for example, the problem, to divide 5 by 21. In the first place, 5 = 1 + 2 + 2. From the table we get $\frac{2}{21} = \frac{1}{14}\ \frac{1}{42}$ then

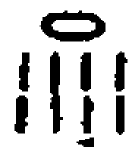
$$\frac{5}{21} = \frac{1}{21} + \left(\frac{1}{14}\ \frac{1}{42}\right) + \left(\frac{1}{14}\ \frac{1}{42}\right) = \frac{1}{21} + \left(\frac{2}{14}\ \frac{2}{42}\right) = \frac{1}{21}\ \frac{1}{7}\ \frac{1}{21} = \frac{1}{7}\ \frac{2}{21} = \frac{1}{7}\ \frac{1}{14}\ \frac{1}{42}$$

The papyrus contains problems in which it is required that fractions be raised by addition or multiplication to given whole numbers or to other fractions. For example, it is required to increase $\frac{1}{4}\ \frac{1}{8}\ \frac{1}{10}\ \frac{1}{30}\ \frac{1}{45}$ to 1 The common denominator taken appears to be 45, for the numbers are stated as $11\frac{1}{4}$, $5\frac{1}{2}\ \frac{1}{8}$, $4\frac{1}{2}$, $1\frac{1}{2}$, 1 The sum of these is $23\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}$ forty-fifths. Add to this $\frac{1}{9}\ \frac{1}{40}$ and the sum is $\frac{2}{3}$ Add $\frac{1}{3}$ and we have 1. Hence the quantity to be added to the given fraction is $\frac{1}{3}\ \frac{1}{9}\ \frac{1}{40}$

Having finished the subject of fractions, Ahmes proceeds to the solution of equations of one unknown quantity. **<u>The unknown quantity is called `hau' or heap</u>**. Thus the problem, "heap, its $\frac{1}{7}$, its whole , its make 19" i.e. $\frac{x}{7} + x = 19$ in this case , the solution is as follows:

$\frac{8x}{7} = 19$; $\frac{x}{7} = 2\frac{1}{4}\frac{1}{8}$; $x = 16\frac{1}{2}\frac{1}{8}$

Egyptian inscriptions indicate familiarity with large numbers at an early date. A museum at Oxford has a royal mace more than 5,000 years old, on which a record of 120,000 prisoners and 1,422,000 captive goats appears. These figures may have been exaggerated, but from other considerations it is clear that the Egyptians were commendably accurate in counting and measuring. The construction of the Egyptian solar calendar is an outstanding early example of observation, measurement, and counting. The pyramids are another famous instance; they exhibit such a high degree of precision in construction and orientation that ill-founded legends have grown up around them.

The principle of cipherization, introduced by the Egyptians some 4,000 years ago and used in the Ahmes Papyrus, represented an important contribution to numeration, and it is one of the factors that makes our own system in use today the effective instrument that it is. Egyptian hieroglyphic inscriptions have a special notation for unit fractions—that is, fractions with unit numerators. The reciprocal of any integer was indicated simply by placing over the notation for the integer an elongated oval sign. The fraction $\frac{1}{8}$ thus appeared as  and $\frac{1}{20}$ was written as . In the hieratic notation, appearing in papyri, the elongated oval is replaced by a dot, which is placed over the cipher for the corresponding integer (or over the right-hand cipher in the case of the reciprocal of a multi digit number). In the Ahmes Papyrus, for example, the fraction $\frac{1}{8}$ appears as , and $\frac{1}{20}$ is written as . Such unit fractions were freely handled in Ahmes's day, but the general fraction seems to have been an enigma to the Egyptians. They felt comfortable with the fraction $\frac{2}{3}$ For which they had a special hieratic sign ; occasionally, they used special signs for fractions of the form $\frac{n}{n+1}$, the complements of the unit fractions.

To the fraction $\frac{2}{3}$, the Egyptians assigned a special role in arithmetic processes, so that in finding one-third of a number, they first found two-thirds of it and subsequently took half of the result! They knew and used the fact that two-thirds of the unit fraction $\frac{1}{p}$ is the sum of the two unit fractions $\frac{1}{2p}$and $\frac{1}{6p}$; they were also aware that double the unit fraction $\frac{1}{2p}$ is the unit fraction $\frac{1}{p}$.

Yet it looks as though, apart from the fraction $\frac{2}{3}$, the Egyptians regarded the general proper rational fraction of the form $\frac{m}{n}$ not as an elementary "thing" but as part of an uncompleted process.

Where today we think of $\frac{3}{5}$ as a single irreducible fraction, Egyptian scribes thought of it as reducible to the sum of three unit fractions, $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{15}$

To facilitate the reduction of "mixed" proper fractions to the sum of unit fractions, the Ahmes Papyrus opens with a table expressing $\frac{2}{n}$ as a sum of unit fractions for all odd values of n from 5 to 101.

The equivalent of $\frac{2}{5}$ is given as $\frac{1}{3}$, $\frac{1}{15}$ and $\frac{2}{11}$ is written as $\frac{1}{6}$ and $\frac{1}{66}$ and $\frac{2}{15}$ is expressed as $\frac{1}{10}$ and $\frac{1}{30}$. The last item in the table decomposes $\frac{2}{101}$ into $\frac{1}{101}$ and $\frac{1}{202}$ and $\frac{1}{303}$ and $\frac{1}{606}$.

It is not clear why one form of decomposition was preferred to another of the indefinitely many that are possible. This last entry certainly exemplifies the Egyptian prepossession for halving and taking a third;

it is not at all clear to us why the decomposition $\frac{2}{n} = \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{2.3.n}$ is better than $\frac{1}{n} + \frac{1}{n}$. Perhaps one of the objects of the $\frac{2}{n}$ decomposition was to arrive at unit fractions smaller than $\frac{1}{n}$. Certain passages indicate that the Egyptians had some appreciation of general rules and methods above and beyond the specific case at hand, and this represents an important step in the development of mathematics.

POSSIBLE SHORT QUESTIONS:

i. What type of symbols were used in Ancient Egypt for 1,10,100,1000,10,000,100000,100000,10000000 etc?
ii. Besides the hieroglyphics, Egypt possesses the other writings, name them.
iii. Write the difference between hieroglyphics, the hieratic and demotic languages.
iv. Shortly note on instrumental method of fingering.
v. Shortly note on Ahmes's Papyrus.
vi. Shortly note on Ahmes's view about fraction.
vii. Who adapted the use of pen and ink on prepared papyrus leaves?
viii. Shortly note on Egyptian's view about fraction and compare with other civilizations.
ix. How to represent any fractional value as the sum of unit-fractions?
x. The papyrus contains problems in which it is required that fractions be raised by addition or multiplication to given whole numbers or to other fractions. Explain with example.
xi. Define heap.
xii. Write the number 12,345 in Egyptians symbols.
xiii. The Egyptians were commendably accurate in counting and measurement. Would you give an example?
xiv. What is meant by Cipher?
xv. Write the principle of Cipherization.
xvi. Exemplify the Egyptian prepossession for halving and taking a third with example.
xvii. Write the importance of fraction $\frac{2}{3}$ regarding reduction of fractions.

LONG QUESTIONS:

1) Briefly describe about the numbers and fractions in ancient Egypt.
2) How to represent any fractional value as the sum of unit-fractions in ancient Egypt?
3) Briefly describe about Egyptian Hieroglyphics system of symbols with examples.

ARITHMETIC OPERATIONS

The Ahmes papyrus doubtless represents the most advanced attainments of the Egyptians in arithmetic and geometry. It is remarkable that they should have reached so great prociency in mathematics at so remote a period of antiquity. But strange, indeed, is the fact that, during the next two thousand years, they should have made no progress whatsoever in it. The conclusion forces itself upon us, that they resemble the Chinese in the stationary character, not only of their government, but also of their learning. All the knowledge of geometry which they possessed when Greek scholars visited them, six centuries B.C., was doubtless known to them two thousand years earlier, when they built those stupendous and gigantic structures the pyramids. An explanation for this stagnation of learning has been sought in the fact that their early discoveries in mathematics and medicine had the misfortune of being entered upon their sacred books and that, in after ages, it was considered heretical to augment or modify anything therein. Thus the books themselves closed the gates to progress.

The **principal defect** of Egyptian arithmetic was the lack of a simple, comprehensive symbolism a defect which not even the Greeks were able to remove.

The 2/n table in the Ahmes Papyrus is followed by a short n/10 table for n from 1 to 9, the fractions again being expressed in terms of the favorites—unit fractions and the fraction $\frac{2}{3}$

for example, the fraction $\frac{9}{10}$ is broken into $\frac{1}{30}$ and $\frac{1}{5}$ and $\frac{2}{3}$.

Ahmes had begun his work with the assurance that it would provide a "complete and thorough study of all things . . . and the knowledge of all secrets," and therefore the main portion of the material, following **the $\frac{2}{n}$ and $\frac{n}{10}$ tables, consists of eighty-four widely assorted problems**. The first six of these require the division of one or two or six or seven or eight or nine loaves of bread among ten men, and the scribe makes use of the $\frac{n}{10}$ table that he has just given.
In the first problem, the scribe goes to considerable trouble to show that it is correct to give to each of the ten men one tenth of a loaf!

If one man receives $\frac{1}{10}$ of a loaf, two men will receive $\frac{2}{10}$ or $\frac{1}{5}$ and four men will receive $\frac{2}{5}$ of a loaf or $\frac{1}{3}+\frac{1}{15}$ of a loaf. Hence, eight men will receive $\frac{2}{3}+\frac{2}{15}$ of a loaf or $\frac{2}{3}+\frac{1}{10}+\frac{1}{30}$ of a loaf, and eight men plus two men will receive $\frac{2}{3}+\frac{1}{5}+\frac{1}{10}+\frac{1}{30}$ or a whole loaf.

Ahmes seems to have had a kind of equivalent to our least common multiple, which enabled him to complete the proof. In the division of seven loaves among ten men, the scribe might have chosen $\frac{1}{2}+\frac{1}{5}$ of a loaf for each, but the predilection for $\frac{2}{3}$ led him instead to $\frac{2}{3}$ and $\frac{1}{30}$ of a loaf for each.

The fundamental arithmetic operation in Egypt was addition, and our operations of multiplication and division were performed in Ahmes's day through successive doubling, or "duplation." Our own word "multiplication," or manifold, is, in fact, suggestive of the Egyptian process.

A multiplication of, say, 69 by 19 would be performed by adding 69 to itself to obtain 138, then adding this to itself to reach 276, applying duplation again to get 552, and once more to obtain 1104, which is, ofcourse, 16 times 69. Inasmuch as 19 = 16 + 2 + 1, the result of multiplying 69 by 19 is 1104 + 138 + 69, that is, 1311.

Occasionally, a multiplication by 10 was also used, for this was a natural concomitant of the decimal hieroglyphic notation. Multiplication of combinations of unit fractions was also a part of Egyptian arithmetic.

Problem 13 in the Ahmes Papyrus, for example, asks for the product of $\frac{1}{16}+\frac{1}{112}$ and $1+\frac{1}{2}+\frac{1}{4}$ the result is correctly found to be $\frac{1}{8}$

For division, the duplation process is reversed, and the divisor, instead of the multiplicand, is successively doubled. That the Egyptians had developed a high degree of artistry in applying the duplation process and the unit fraction concept is apparent from the calculations in the problems of Ahmes.

Problem 70 calls for the quotient when 100 is divided by $7+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
the result, $12+\frac{2}{3}+\frac{1}{42}+\frac{1}{126}$, is obtained as follows. Doubling the divisor successively, we first obtain $15+\frac{1}{2}+\frac{1}{4}$ then $31+\frac{1}{2}$ and finally 63, which is 8 times the divisor.
Moreover, $\frac{2}{3}$ of the divisor is known to be $5+\frac{1}{4}$. Hence, the divisor when multiplied by 8+ 4 + $\frac{2}{3}$ will total 9934, which is $\frac{1}{4}$ short of the product 100 that is desired.

Here a clever adjustment was made. In as much as 8 times the divisor is 63, it follows that the divisor when multiplied by $\frac{2}{63}$ will produce $\frac{1}{4}$.
From the $\frac{2}{n}$ table, one knows that $\frac{2}{63}=\frac{1}{42}+\frac{1}{26}$ hence, the desired quotient is $12+\frac{2}{3}+\frac{1}{42}+\frac{1}{126}$.
Incidentally, this procedure makes use of a commutative principle in multiplication, with which the Egyptians evidently were familiar.

Many of Ahmes's problems show knowledge of manipulations of proportions equivalent to the "rule of three."

Problem 72 calls for the number of loaves of bread of "strength" 45, which are equivalent to 100 loaves of "strength" 10, and the solution is given as $\frac{100}{10} \times 45$, or 450 loaves.

In bread and beer problems, the "strength," or pesu, is the reciprocal of the grain density, being the quotient of the number of loaves or units of volume divided by the amount of grain.

Bread and beer problems are numerous in the Ahmes Papyrus. **Problem 63**, for example, requires the division of 700 loaves of bread among four recipients if the amounts they are to receive are in the continued proportion 2/3 :1/2 :1/3 :1/4. The solution is found by taking the ratio of 700 to the sum of the fractions in the proportion.

In this case, the quotient of 700 divided by $1\frac{3}{4}$ is found by multiplying 700 by the reciprocal of the divisor, which is $\frac{1}{2} + \frac{1}{14}$. The result is 400; by taking 2/3 and 1/2 and 1/3 and 1/4 of this, the required shares of bread are found.

POSSIBLE SHORT QUESTIONS:

- **i.** Write The principal defect of Egyptian arithmetic.
- ii. Shortly note on the $\frac{2}{n}$ and $\frac{n}{10}$ tables.
- iii. Name few fundamental arithmetic operation in Egypt.
- iv. How did multiplication and division were perform in Ahmes's day's?
- v. Note on Problem 13.
- vi. Note on Problem 70.
- vii. Note on Problem 72.
- viii. Note on Problem 63.
- ix. Note on bread and beer problems

LONG QUESTIONS:

1) Briefly describe about arithmetic operations in ancient Egypt.
2) Note on bread and beer problems

''HEAP'' PROBLEMS

The Egyptian problems so far described are best classified as arithmetic, but there are others that fall into a class to which the term "algebraic" is appropriately applied. These do not concern specific concrete objects, such as bread and beer, nor do they call for operations on known numbers. Instead, they require the equivalent of solutions of linear equations of the form x + ax + b or x + ax + bx =c, where a and b and c are known and x is unknown. The unknown is referred to as **"aha," 'hue' or "heap"**.

Problem 24, for instance, calls for the value of heap if heap and 1 7 of heap is 19.

The solution given by Ahmes is not that of modern textbooks but is characteristic of a procedure now known as the "method of false position," or the "rule of false." A specific value, most likely a false one, is assumed for heap, and the operations indicated on the left-hand side of the equality sign are performed on this assumed number. The result of these operations is then compared with the result desired, and by the use of proportions the correct answer is found. In problem 24, the tentative value of the unknown is taken as 7, so that x + $\frac{1}{7}x$ is 8, instead of the desired answer, which was 19. Inasmuch as $8(2+\frac{1}{4}+\frac{1}{8}) = 19$, one must multiply 7 by $2+\frac{1}{4}+\frac{1}{8}$ to obtain the correct heap;

Ahmes found the answer to be $16+\frac{1}{2}+\frac{1}{8}$. Ahmes then "checked" his result by showing that if to $16+\frac{1}{2}+\frac{1}{8}$ one adds $\frac{1}{7}$ of this (which is $2+\frac{1}{4}+\frac{1}{8}$), one does indeed obtain 19.

Here we see another significant step in the development of mathematics, for the check is a simple instance of a proof. Although the method of false position was generally used by Ahmes, there is one problem **(Problem 30)** in which $x+\frac{2}{3}x+\frac{1}{2}x+\frac{1}{7}x = 37$ is solved by factoring the left-hand side of the equation and dividing 37 by $1+\frac{2}{3}+\frac{1}{2}+\frac{1}{7}$ the result being $16+\frac{1}{56}+\frac{1}{679}+\frac{1}{776}$

Many of the **"aha"** calculations in the Rhind (Ahmes) Papyrus appear to be practice exercises for young students. Although a large proportion of them are of a practical nature, in some places the scribe seemed to have had puzzles or mathematical recreations in mind.

Thus, **Problem 79** cites only

"seven houses, 49 cats, 343 mice, 2401 ears of spelt, 16807 hekats."

It is presumed that the scribe was dealing with a problem, perhaps quite well known, where in each of seven houses there are seven cats, each of which eats seven mice, each of which would have eaten seven ears of grain, each of which would have produced seven measures of grain. The problem evidently called not for the practical answer, which would be the number of measures of grain that were saved, but for the impractical sum of the numbers of houses, cats, mice, ears of spelt, and measures of grain. This bit of fun in the Ahmes Papyrus seems to be a forerunner of our familiar nursery rhyme:

As I was going to St. Ives,

I met a man with seven wives;

Every wife had seven sacks,

Every sack had seven cats,

Every cat had seven kits,

Kits, cats, sacks, and wives,

How many were going to St. Ives?

POSSIBLE SHORT QUESTIONS:

- **i.** Define “Heap”. Also write other names for the term ‘heap’.
- ii. Note on problem 24.
- iii. Note on problem 30.
- iv. In the Rhind (Ahmes) Papyrus what type of heap problems appeared?
- v. Note on problem 79.
- vi. Would you write a rhyme of modern age which relate with problem 79?

LONG QUESTIONS:

1) Briefly describe about Heap Problems in ancient Egypt.

GEOMETRIC PROBLEMS: The word “Geometry” is derived by ancient Greek, (Geo mean Earth, Metron mean measurement) is a branch of mathematics concerned with knowledge dealing with spatial relationships, for example, geometrical shapes, relative position of geometrical figures and the properties of space. A mathematician who work in the field of geometry is called a geometer.

BEGINNING: In the beginning geometry was a collection of rules for computing lengths, areas and volumes. Many were crude approximation derived by trial and error. This body of knowledge developed and used in construction, navigation and surveying by the Babylonian and Egyptians was passed to the Greeks. The Greek historian Herodotus (5^{th} century BC) credits the Egyptians with having originated the subject, but there is much evidence that the Babylonian, the Hindu civilization and the Chinese knew much of what was passed along the Egyptians.

Though there is great difference of opinion regarding the antiquity of Egyptian civilization, yet all authorities agree in the statement that, however far back they go, they find no uncivilized state of society. **Menes,** the first king, changes the course of the Nile, makes a great reservoir, and builds the temple of Phthah at Memphis." The Egyptians built the pyramids at a very early period. Surely a people engaging in enterprises of such magnitude must have known something of mathematics at least of practical mathematics.

All Greek writers are unanimous in ascribing, without tenvy, to Egypt the priority of invention in the mathematical sciences. Plato in Phadrus says:

“At the Egyptian city of Naucratis there was a famous old god whose name was Theuth; the birdwhich is called the Ibis was sacred to him, and he was the inventor of many arts, such as arithmetic and calculation and geometry and astronomy and draughts and dice, but his great discovery was the use of letters."

Aristotle says that mathematics had its birth in Egypt, because there the priestly class had the leisure needful for the study of it. Geometry, in particular, is said by Herodotus, Diodorus, Diogenes Laertius, Iamblichus, and other ancient writers to have originated in Egypt. In Herodotus we find this (II. c. 109): They said also that

“this king [Sesostris] divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levied yearly. But everyone from whose part the river tore away anything, had to go to him and notify what had happened; he then sent the overseers, who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax imposed. In this way, it appears to me, geometry originated, which passed thence to Hellas."

Some problems in this papyrus seem to imply a rudimentary knowledge of proportion.

The base-lines of the pyramids run north and south, and east and west, but probably only the lines running north and south were determined by astronomical observations. This, coupled with the fact that the word harpedonapte, applied to Egyptian geometers, means " rope-stretchers," would point to the conclusion that the Egyptian, like the Indian and Chinese geometers, constructed a right triangle upon a given line, by stretching around three pegs a rope consisting of three parts in the ratios 3 : 4 : 5, and thus forming a right triangle.

If this explanation is correct, then the Egyptians were familiar, 2000 years B.C., with the well-known property of the right triangle, for the special case at least when the sides are in the ratio 3 : 4 : 5.

On the walls of the celebrated temple of Horus at Edfu have been found hieroglyphics, written about 100 b.c., which enumerate the pieces of land owned by the priesthood, and give their areas. The area of any quadrilateral, however irregular, is there found by the formula $\frac{a+b}{2}.\frac{c+d}{2}$ Thus, for a quadrangle whose opposite sides are 5 and 8, 20 and 15, is given the area $113\frac{1}{2}\frac{1}{4}$

The incorrect formula of Ahmes of 3000 years B.C. yield generally closer approximations than those of the Edfu inscriptions, written 200 years after Euclid!

The fact that the geometry of the Egyptians consists chiefly of constructions, goes far to explain certain of its great defects. **The Egyptians failed** in two essential points without which a science of geometry, in the true sense of the word, cannot exist. **In the first place**, they failed to construct a rigorously logical system of geometry, resting upon a few axioms and postulates. A great many of their rules, especially those in solid geometry, had probably not been proved at all, but were known to be true merely from observation or as matters of fact. **The second** great defect was their inability to bring the numerous special cases under a more general view, and thereby to arrive at broader and more fundamental theorems. Some of the simplest geometrical truths were divided into numberless special cases of which each was supposed to require separate treatment.

In geometry the forte of the Egyptians lay in making constructions and determining areas. The area of an **isosceles triangle**, of which the sides measure 10 ruths and the base 4 ruths, was erroneously given as 20 square ruths, or half the product of the base by one side.

The area of an **isosceles trapezoid** is found, similarly, by multiplying half the sum of the parallel sides by one of the non-parallel sides.

The area of a **circle** is found by deducting from the diameter $\frac{1}{9}$ of its length and squaring the remainder. Here is taken $\pi = (\frac{16}{9})^2 = 3{:}1604$ … … … … a very fair approximation.

The papyrus explains also such problems as these, "To mark out in the field a right triangle whose sides are 10 and 4 units; or a trapezoid whose parallel sides are 6 and 4, and the non-parallel sides each 20 units.

It is often said that the ancient Egyptians were familiar with the Pythagorean theorem, but there is no hint of this in the papyri that have come down to us. There are nevertheless some geometric problems in the Ahmes Papyrus.

Problem 51 of Ahmes shows that the area of an isosceles triangle was found by taking half of what we would call the base and multiplying this by the altitude. Ahmes justified his method of finding the area by suggesting that the isosceles triangle can be thought of as two right triangles, one of which can be shifted in position, so that together the two triangles form a rectangle.

The isosceles trapezoid is similarly handled in **Problem 52**, in which the larger base of a trapezoid is 6, the smaller base is 4, and the distance between them is 20. Taking 12 of the sum of the bases, "so as to make a rectangle," Ahmes multiplied this by 20 to find the area.

In transformations such as these, in which isosceles triangles and trapezoids are converted into rectangles, we may see the beginnings of a theory of congruence and the idea of proof in geometry, but there is no evidence that the Egyptians carried such work further. Instead, their geometry lacks a clear-cut distinction between relationships that are exact and those that are only approximations. A surviving deed from Edfu, dating from a period some 1,500 years after Ahmes, gives examples of triangles, trapezoids, rectangles, and more general quadrilaterals. The rule for finding the area of the general quadrilateral is to take the product of the arithmetic means of the opposite sides. Inaccurate though the rule is, the author of the deed deduced from it a corollary—that the area of a triangle is half of the sum of two sides multiplied by half of the third side. This is a striking instance of the search for relationships among geometric figures, as well as an early use of the zero concept as a replacement for a magnitude in

The Egyptian rule for finding the area of a circle has long been regarded as one of the outstanding achievements of the time. **In Problem 50**, the scribe Ahmes assumed that the area of a circular field with a diameter of 9 units is the same as the area of a square with a side of 8 units.

If we compare this assumption with the modern formula $A = \pi r^2$, we find the Egyptian rule to be equivalent to giving π a value of about $3\frac{1}{6}$ a commendably close approximation, but here again we miss any hint that Ahmes was aware that the areas of his circle and square were not exactly equal.

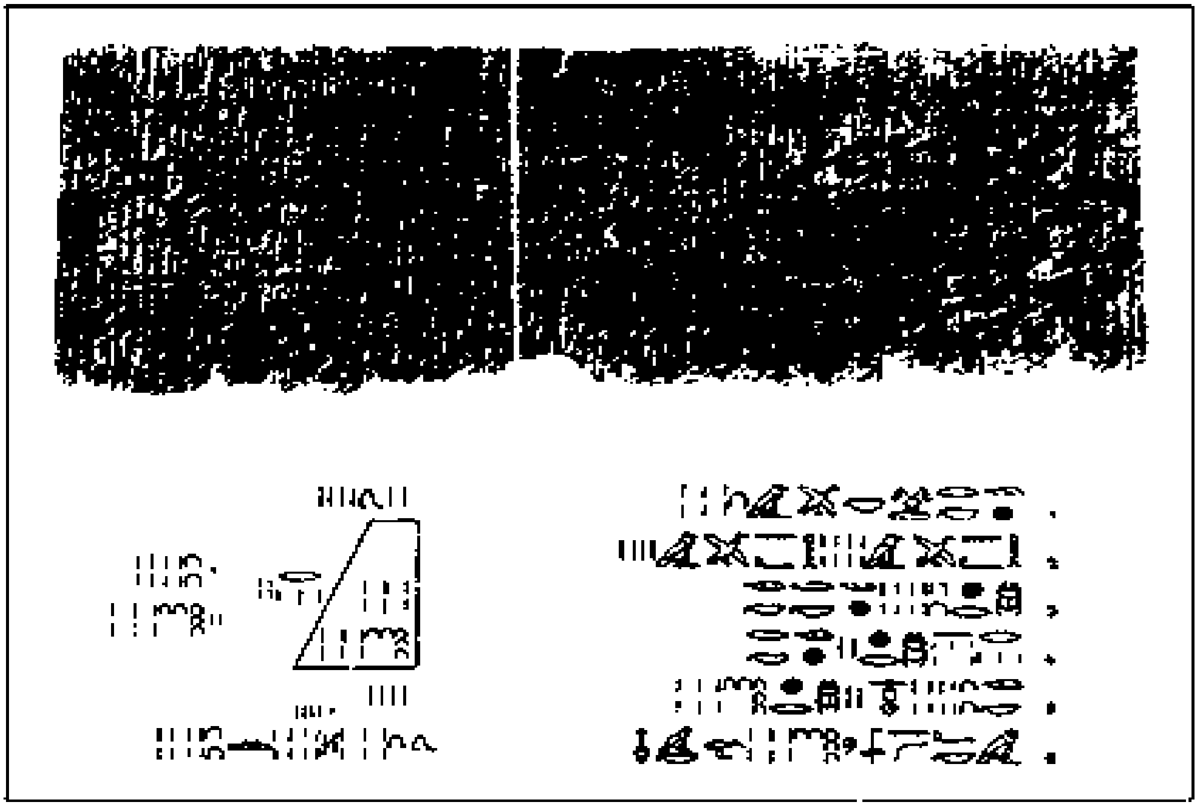
It is possible that **Problem 48** gives a hint to the way in which the Egyptians were led to their area of the circle. In this problem, the scribe formed an octagon from a square having sides of 9 units by trisecting the sides and cutting off the four corner isosceles triangles, each having an area of 41 2 units. The area of the octagon, which does not differ greatly from that of a circle inscribed within the square, is 63 units, which is not far removed from the area of a square with 8 units on a side.

That the number $4(\frac{8}{9})^2$ did indeed play a role comparable to our constant π seems to be confirmed by the Egyptian rule for the circumference of a circle, according to which the ratio of the area of a circle to the circumference is the same as the ratio of the area of the circumscribed square to its perimeter. This observation represents a geometric relationship of far greater precision and mathematical significance than the relatively good approximation for π.

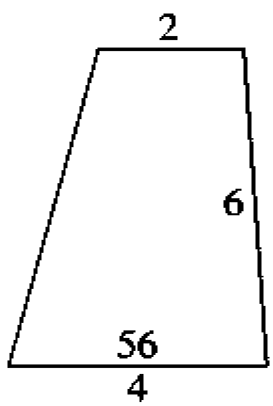
Degree of accuracy in approximation is not a good measure of either mathematical or architectural achievement, and we should not over emphasize this aspect of Egyptian work. Recognition by the Egyptians of interrelationships among geometric figures, on the other hand, has too often been overlooked, and yet it is here that they came closest in attitude to their successors, the Greeks. No theorem or formal proof is known in Egyptian mathematics, but some of the geometric comparisons made in the Nile Valley, such as those on the perimeters and the areas of circles and squares, are among the first exact statements in history concerning curvilinear figures.

The value of $\frac{22}{7}$ is often used today for π; but we must recall that Ahmes's value for π is about $3\frac{1}{6}$ not $3\frac{1}{3}$. That Ahmes's value was also used by other Egyptians is confirmed in a papyrus roll from the twelfth dynasty (the Kahun Papyrus), in which the volume of a cylinder is found by multiplying the height by the area of the base, the base being determined according to Ahmes's rule.

Associated with **Problem 14** in the Moscow Papyrus is a figure that looks like an isosceles trapezoid (see Fig),

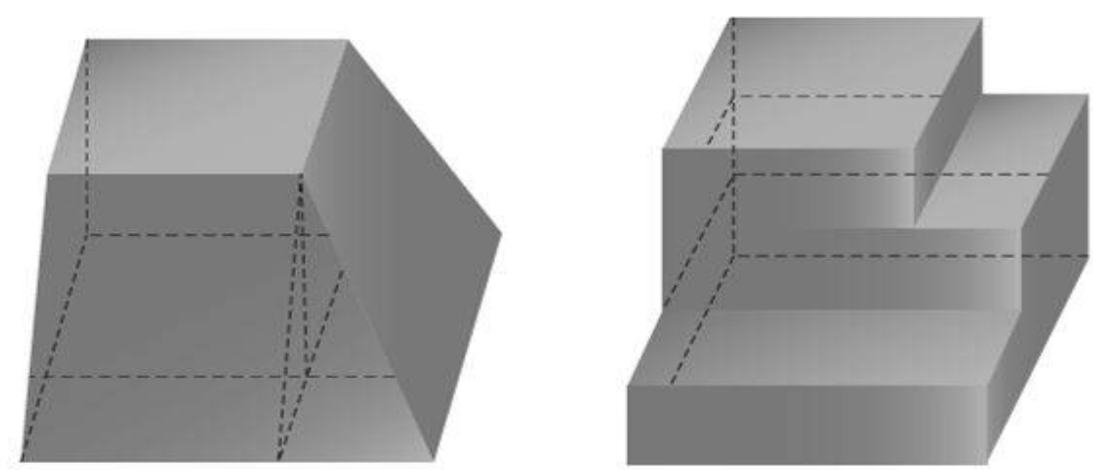


Reproduction (top) of a portion of the Moscow Papyrus, showing the problem of the volume of a frustum of a square pyramid, together with hieroglyphic transcription (below)



but the calculations associated with it indicate that a frustum of a square pyramid is intended. Above and below the figure are signs for 2 and 4, respectively, and within the figure are the hieratic symbols for 6 and 56. The directions alongside make it clear that the problem calls for the volume of a frustum of a square pyramid 6 units high if the edges of the upper and lower bases are 2 and 4 units, respectively. The scribe directs one to square the numbers 2 and 4 and to add to the sum of these squares the product of 2 and 4, the result being 28. This result is then multiplied by a third of 6, and the scribe concludes with the words “See, it is 56; you have found it correctly.” That is, the volume of the frustum has been calculated in accordance with the modern formula $V = h(a^2 + ab + b^2)/3$, where h is the altitude and a and b are the sides of the square bases. Nowhere is this formula written out, but in substance it evidently was known to the Egyptians. If, as in the deed from Edfu, one takes b = 0, the formula reduces to the familiar formula, one-third the base times the altitude, for the volume of a pyramid.

How these results were arrived at by the Egyptians is not known. An empirical origin for the rule on the volume of a pyramid seems to be a possibility, but not for the volume of the frustum. For the latter, a theoretical basis seems more likely, and it has been suggested that the Egyptians may have proceeded here as they did in the cases of the isosceles triangle and the isosceles trapezoid—they may mentally have broken the frustum into parallelepipeds, prisms, and pyramids. On replacing the pyramids and the prisms by equal rectangular blocks, a plausible grouping of the blocks leads to the Egyptian formula. One could, for example, have begun with a pyramid having a square base and with the vertex directly over one of the base vertices. An obvious decomposition of the frustum would be to break it into four parts as in Figure (below) a rectangular parallelepiped having a volume b2h, two triangular prisms, each with a volume of $b(a - b)h / 2$, and a pyramid of volume $(a - b)^2h / 3$. The prisms can be combined into a rectangular parallelepiped with dimensions b and a - b and h; and the pyramid can be thought of as a rectangular parallelepiped with dimensions a - b and a - b and h / 3. On cutting up the tallest parallelepipeds so that all altitudes are h / 3, one can easily arrange the slabs so as to form three layers, each of altitude h / 3, and having cross-sectional areas of a^2 and ab and b^2, respectively.



Problem 10 in the Moscow Papyrus presents a more difficult question of interpretation than does Problem 14. Here the scribe asks for the surface area of what looks like a basket with a diameter of $4\frac{1}{2}$. He proceeds as though he were using the equivalent of a formula $S = \left(1 - \frac{1}{9}\right)^2 (2x)x$ where x is $4\frac{1}{2}$, obtaining an answer of 32 units. Inasmuch as $\left(1 - \frac{1}{9}\right)^2$ is the Egyptian approximation of π/4, the answer 32 would correspond to the surface of a hemisphere of diameter$4\frac{1}{2}$, and this was the interpretation given to the problem in 1930. Such a result, antedating the oldest known calculation of a hemispherical surface by some 1,500 years, would have been amazing, and it seems, in fact, to have been too good to be true. Later analysis indicates that the "basket" may have been a roof—somewhat like that of a Quonset hut in the shape of a half-cylinder of diameter $4\frac{1}{2}$ and length $4\frac{1}{2}$. The calculation in this case calls for nothing beyond knowledge of the length of a semicircle, and the obscurity of the text makes it admissible to offer still more primitive interpretations, including the possibility that the calculation is only a rough estimate of the area of a domelike barn roof. In any case, we seem to have here an early estimation of a curvilinear surface area.

POSSIBLE SHORT QUESTIONS:

i. Shortly note on the term ‘Geometry’.
ii. Shortly note on the beginning of ‘Geometry’.
iii. Ancient Egyptians must have known something of mathematics at least of practical mathematics. Could you explain it with some example?
iv. Aristotle says that mathematics had its birth in Egypt, on what bases?
v. Name few historians who said that geometry originated in Egypt.
vi. Herodotus, said the Geometry originated in Egypt. On what bases? Quote its saying.
vii. The Egyptians were familiar, 2000 years B.C., with the well-known property of the right triangle. Is it true?
viii. The geometry of the Egyptians consists certain defects. Note on them.
ix. Write the views of Egyptians about area.
x. How did Egyptians find the area of isosceles triangle?
xi. How did Egyptians find the area of isosceles trapezoid?
xii. How did Egyptians find the area of circle?
xiii. It is often said that the ancient Egyptians were familiar with the Pythagorean theorem. Is it true?
xiv. Note on Problem 51. Or write Ahmes method to find the area of an isosceles triangle.
xv. Note on Problem 52. Or write Ahmes method to find the area of an isosceles trapezoid.
xvi. Note on Problem 50. Or write Ahmes method to find the area of an circle.
xvii. Write edfu corollary and its importance.
xviii. About relationships, what was the concept of Egytians?
xix. Write the rule for finding the area of the general quadrilateral in Egypt era.
xx. Note on Problem 48.
xxi. Write the role of $4(\frac{8}{9})^2$in Egyptian mathematics.
xxii. What were the concept of Egyptians about Degree of accuracy or about Angles?
xxiii. Write the Ahmes’s value for π.
xxiv. Note on Problem 14.
xxv. Note on problem 10.

LONG QUESTIONS:

1) Briefly describe about Geometrical concepts in ancient Egypt.

SLOPE PROBLEMS

In the construction of the pyramids, it had been essential to maintain a uniform slope for the faces, and it may have been this concern that led the Egyptians to introduce a concept equivalent to the cotangent of an angle. In modern technology, it is customary to measure the steepness of a straight line through the ratio of the “rise” to the “run.” In Egypt, it was customary to use the reciprocal of this ratio. There, the word **“seqt”** meant the horizontal departure of an oblique line from the vertical axis for every unit change in the height. The seqt thus corresponded, except for the units of measurement, to the batter used today by architects to describe the inward slope of a masonry wall or pier. The vertical unit of length was the cubit, but in measuring the horizontal distance, the unit used was the “hand,” of which there were seven in a cubit. Hence, the seqt of the face of a pyramid was the ratio of run to rise, the former measured in hands, the latter in cubits.

In **Problem 56** of the Ahmes Papyrus, one is asked to find the seqt of a pyramid that is 250 ells or cubits high and has a square base 360 ells on a side. The scribe first divided 360 by 2 and then divided the result by 250, obtaining $\frac{1}{2}+\frac{1}{5}+\frac{1}{50}$ in ells. Multiplying the result by 7, he gave the seqt as $5\,\frac{1}{25}$ in hands per ell. In other pyramid problems in the Ahmes Papyrus, the seqt turns out to be $5\frac{1}{4}$, agreeing somewhat better with that of the great Cheops Pyramid, 440 ells wide and 280 high, the seqt being $5\frac{1}{2}$ hands per ell.

POSSIBLE SHORT QUESTIONS:

i. How did Egyptian introduce about cotangent of an angle or Slope?
ii. In modern age we measure the steepness of a straight line through the ratio of the “rise” to the “run.” How did Egyptian measure?
iii. What is meant by the word seqt?
iv. Note on Problem 56.

LONG QUESTIONS:

1) Briefly describe about Slope concepts in ancient Egypt.

END WORDS ABOUT EGYPTIAN HISTORY

The knowledge indicated in extant Egyptian papyri is mostly of a practical nature, and calculation was the chief element in the questions. Where some theoretical elements appear to enter, the purpose may have been to facilitate technique. Even the once-vaunted Egyptian geometry turns out to have been mainly a branch of applied arithmetic. Where elementary congruence relations enter, the motive seems to be to provide mensurational devices. The rules of calculation concern specific concrete cases only. The Ahmes and Moscow papyri, our two chief sources of information, may have been only manuals intended for students, but they nevertheless indicate the direction and tendencies in Egyptian mathematical instruction. Further evidence provided by inscriptions on monuments, fragments of other mathematical papyri, and documents from related scientific fields serves to confirm the general impression. It is true that our two chief mathematical papyri are from a relatively early period, a thousand years before the rise of Greek mathematics, but Egyptian mathematics seems to have remained remarkably uniform throughout its long history. It was at all stages built around the operation of addition, a disadvantage that gave to Egyptian computation a peculiar primitivity, combined with occasionally astonishing complexity.

The fertile Nile Valley has been described as the world's largest oasis in the world's largest desert. Watered by one of the most gentlemanly of rivers and geographically shielded to a great extent from foreign invasion, it was a haven for peace-loving people who pursued, to a large extent, a calm and unchallenged way of life. Love of the beneficent gods, respect for tradition, and preoccupation with death and the needs of the dead all encouraged a high degree of stagnation. Geometry may have been a gift of the Nile, as Herodotus believed, but the available evidence suggests that Egyptians used the gift but did little to expand it. The mathematics of Ahmes was that of his ancestors and of his descendants. For more progressive mathematical achievements, one must look to the more turbulent river valley known as Mesopotamia.

BABYLONIAN (MESOPOTAMIAN)

How much is one god beyond the other god?

An Old Babylonian astronomical text

THE ERA AND THE SOURCES

The fourth millennium before our era was a period of remarkable cultural development, bringing with it the use of writing, the wheel, and metals. As in Egypt during the first dynasty, which began toward the end of this extra-ordinary millennium, so also in the Mesopotamian Valley there was at the time a high order of civilization. There the Sumerians had built homes and temples decorated with artistic pottery and mosaics in geometric patterns. Powerful rulers united the local principalities into an empire that completed vast public works, such as a system of canals to irrigate the land and control flooding between the Tigris and Euphrates rivers, where the overflow of the rivers was not predictable, as was the inundation of the Nile Valley. The cuneiform pattern of writing that the Sumerians had developed during the fourth millennium probably antedates the Egyptian hieroglyphic system.

The Mesopotamian civilizations of antiquity are often referred to as Babylonian, although such a designation is not strictly correct. The city of Babylon was not at first, nor was it always at later periods, the center of the culture associated with the two rivers, but convention has sanctioned the informal use of the name "Babylonian" for the region during the interval From about 2000 to roughly 600 BCE. When in 538 BCE Babylon fell to Cyrus of Persia, the city was spared, but the Babylonian Empire had come to an end.
"Babylonian" mathematics, however, continued through the Seleucid period in Syria almost to the dawn of Christianity.

Then, as today, the Land of the Two Rivers was open to invasions from many directions, making the Fertile Crescent a battlefield with frequently changing hegemony. One of the most significant of the invasions was that by **the Semitic Akkadians** under Sargon I (ca. 2276 2221 BCE), or Sargon the Great. He established an empire that extended from the Persian Gulf in the south to the Black Sea in the north, and from the steppes of Persia in the east to the Mediterranean Sea in the west. Under Sargon, the invaders began a gradual absorption of the indigenous Sumerian culture, including the cuneiform script. Later invasions and revolts brought various racial strains— **Ammorites, Kassites, Elamites, Hittites, Assyrians, Medes, Persians,** and others—to political power at one time or another in the valley, but there remained in the area a sufficiently high degree of cultural unity to justify referring to the civilization simply as Mesopotamian. In particular, the use of cuneiform script formed a strong bond.

Laws, tax accounts, stories, school lessons, personal letters—these and many other records were impressed on soft clay tablets with styluses, and the tablets were then baked in the hot sun or in ovens. Such written documents were far less vulnerable to the ravages of time than were Egyptian papyri; hence, as much larger body of evidence about Mesopotamian mathematics is available today than exists about the Nilotic system. From one locality aLONG, the site of ancient Nippur, we have some 50,000 tablets. The university libraries at Columbia, Pennsylvania, and Yale, among others, have large collections of ancient tablets from Mesopotamia, some of them mathematical. Despite the availability of documents, however, it was the Egyptian hieroglyphic, rather than the Babylonian cuneiform, that was first deciphered in modern times. The German philologist F.W.Grotefend had made some progress in the reading of Babylonian script early in the nineteenth century, but only during the second quarter of the twentieth century did substantial accounts of Mesopotamian mathematics begin to appear in histories of antiquity.

POSSIBLE SHORT QUESTIONS:

i. Write the time period of Babylonian Era.
ii. How did Babylonian save their records?
iii. In which institutes mostly material (mathematical) of Babylonian era saved?
iv. In 9th century who tried to read Babylonian Script?

LONG QUESTIONS:

1) Briefly describe about era and the sources of Babylonian histry of mathematics.

CUNEIFORM WRITING

The fertile valley of the Euphrates and Tigris was one of the primeval seats of human society. Authentic history of the peoples inhabiting this region begins only with the foundation, in Chaldaea and Babylonia, of a united kingdom out of the previously disunited tribes. Much light has been thrown on their history by the discovery of the art of reading the **cuneiform or wedge-shaped system of writing**.

In the study of Babylonian mathematics we begin with the notation of numbers. A vertical wedge  stood for 1, while the characters  and  signified 10 and 100 respectively. Grotefend believes the character for 10 originally to have been the picture of two hands, as held in prayer, the palms being pressed together, the fingers close to each other, but the thumbs thrust out. In the Babylonian notation two principals were employed _ the additive and multiplicative. Numbers below 100 were expressed by symbols whose respective values had to be added. Thus,  stood for 2,  for 3,  for 4,  for 23,  for 30. Here the symbols of higher order appear always to the left of those of lower order. In writing the hundreds, on the other hand, a smaller symbol was placed to the left of the 100, and was, in that case, to be multiplied by 100. Thus,  signified 10 times 100, or 1000. But this symbol for 1000 was itself taken for a new unit, which could take smaller coefficients to its left. Thus,  denoted, not 20 times 100, but 10 times 1000. Of the largest numbers written in cuneiform symbols, which have hitherto been found, none go as high as a million.

The early use of writing in Mesopotamia is attested to by hundreds of clay tablets found in Uruk and dating from about 5,000 years ago. By this time, picture writing had reached the point where conventionalized stylized forms were used for many things:  for water,  for eye, and combinations of these to indicate weeping. Gradually, the number of signs became smaller, so that of some 2,000 Sumerian signs originally used, only a third remained by the time of the Akkadian conquest. Primitive drawings gave way to combinations of wedges: water became  and eye . At first, the scribe wrote from top to bottom in columns from right to left; later, for convenience, the table was rotated counterclockwise through 90°, and the scribe wrote from left to right in horizontal rows from top to bottom. The stylus, which formerly had been a triangular prism, was replaced by a right circular cylinder—or, rather, two cylinders of unequal radius. During the earlier days of the Sumerian civilization, the end of the stylus was pressed into the clay vertically to represent 10 units and obliquely to represent a unit, using the smaller stylus; similarly, an oblique impression with the larger stylus indicated 60 units and a vertical impression indicated 3,600 units. Combinations of these were used to represent intermediate numbers.

POSSIBLE SHORT QUESTIONS:

i. What is Cuneiform Writing?
ii. Babylonian used special symbol for 10. Write it. Write Grotefend belief about it.
iii. In the Babylonian notation two principals were employed. Name them.
iv. Write the Babylonian notation for 1,2,3,4,23 and 30 etc.
v. How did Babylonian represent intermediate numbers?

LONG QUESTIONS:

1) Briefly describe about the Cuneiform Writing of Babylonian history of mathematics.

SEXAGESIMALS

As the Akkadians adopted the Sumerian form of writing, lexicons were compiled giving equivalents in the two tongues, and forms of words and numerals became less varied. Thousands of tablets from about the time of the Hammurabi dynasty (ca. 1800 1600 BCE) illustrate a number system that had become well established. The decimal system, common to most civilizations, both ancient and modern, had been submerged in Mesopotamia under a notation that made fundamental the base 60. Much has been written about the motives behind this change; it has been suggested that astronomical considerations may have been instrumental or that the sexagesimal scheme might have been the natural combination of two earlier schemes, one decimal and the other using the base 6. It appears more likely, however, that the base 60 was consciously adopted and legalized in the interests of metrology, for a magnitude of 60 units can be sub- divided easily into halves, thirds, fourths, fifths, sixths, tenths, twelfths, fifteenths, twentieths, and thirtieths, thus affording ten possible sub-divisions. Whatever the origin, the sexagesimal system of numeration has enjoyed a remarkably long life, for remnants survive, unfortunately for consistency, even to this day in units of time and angle measure, despite the fundamentally decimal form of mathematics in our society.

What led to the invention of the sexagesimal system? Why was it that 60 parts were selected? To this we have no positive answer. Ten was chosen, in the decimal system, because it represents the number of fingers. But nothing of the human body could have suggested 60. **Cantor** offers the following theory: At first the Babylonians reckoned the year at 360 days. This led to the division of the circle into 360 degrees, each degree representing the daily amount of the supposed yearly revolution of the sun around the earth. Now they were, very probably, familiar with the fact that the radius can be applied to its circumference as a chord 6 times, and that each of these chords subtends an arc measuring exactly 60 degrees. Fixing their attention upon these degrees, the division into 60 parts may have suggested itself to them. Thus, when greater precision necessitated a subdivision of the degree, it was partitioned into 60 minutes. In this way the sexagesimal notation may have originated. The division of the day into 24 hours, and of the hour into minutes and seconds on the scale of 60, is due to the Babylonians.

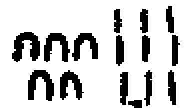
POSSIBLE SHORT QUESTIONS:

i. What is Sexagesimal system?
ii. What led to the invention of the sexagesimal system? This question have no exact answer but regarding this Cantor offered a theory. Write that.

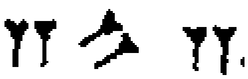
LONG QUESTIONS:

1) Briefly describe about the Sexagesimal system of Babylonian history of mathematics.

POSITIONAL NUMERATION

Babylonian cuneiform numeration, for smaller whole numbers, proceeded along the same lines as did the Egyptian hieroglyphic, with repetitions of the symbols for units and tens. Where the Egyptian architect, carving on stone, might write 59 as , the Mesopotamian scribe could similarly represent the same number on a clay tablet through fourteen wedge-shaped marks—five broad sideways wedges or “angle-brackets,” each representing 10 units, and nine thin vertical wedges, each standing for a unit, all juxtaposed in a neat group as . Beyond the number 59, however, the Egyptian and Babylonian systems differed markedly. Perhaps it was the inflexibility of the Mesopotamian writing materials, possibly it was a flash of imaginative insight that made the Babylonians realize that their two symbols for units and tens sufficed for the representation of any integer, however large, without excessive repetitiveness. This was made possible through their invention, some 4,000 years ago, of the positional notation—the same principle that accounts for the effectiveness of our present numeral forms. That is, the ancient Babylonians saw that their symbols could be assigned values that depend on their relative positions in the representation of a number. Our number 222 makes use of the same cipher three times, but with a different meaning each time. Once it represents two units, the second time it means two 10s, and finally it stands for two 100s (that is, twice the square of the base 10). In a precisely analogous way, the Babylonians made multiple use of such a symbol as . When they wrote  clearly separating the three groups of two wedges each, they understood the right-hand group to mean two units, the next group to mean twice their base, 60, and the left-hand group to signify twice the square of their base. This numeral, therefore, denoted $2(60)^2 + 2(60) + 2$ (or 7,322 in our notation).

A wealth of primary material exists concerning Mesopotamian mathematics, but, oddly enough, most of it comes from two periods widely separated in time. There is an abundance of tablets from the first few hundred years of the second millennium BCE (the Old Babylonian age), and many tablets have also been found dating from the last few centuries of the first millennium BCE (the Seleucid period). Most of the important contributions to mathematics will be found to go back to the earlier period, but one contribution is not in evidence until almost 300 BCE. The Babylonians seem at first to have had no clear way in which to indicate an "empty" position—that is, they did not have a zero symbol, although they sometimes left a space where a zero was intended. This meant that their forms for the numbers 122 and 7,202 looked very much alike, for  might mean either $2(60) + 2$ or $2(60)^2 + 2$. Context in many cases could be relied on to relieve some of the ambiguity, but the lack of a zero symbol, such as enables us to distinguish at a glance between 22 and 202, must have been quite inconvenient.

By about the time of the conquest by Alexander the Great, however, a special sign, consisting of two small wedges placed obliquely, was invented to serve as a placeholder where a numeral was missing. From that time on, as long as cuneiform was used, the number  ,or $2(60)^2 + 0(60) + 2$, was readily distinguishable from ,  or $2(60) + 2$. The Babylonian zero symbol apparently did not end all ambiguity, for the sign seems to have been used for intermediate empty positions only. There are no extant tablets in which the zero sign appears in a terminal position. This means that the Babylonians in antiquity never achieved an absolute positional system. Position was only relative; hence, the symbol  could represent $2(60) + 2$ or $2(60)^2 + 2(60)$ or $2(60)^3 + 2(60)^2$ or any one of indefinitely many other numbers in which two successive positions are involved.

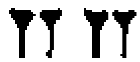
POSSIBLE SHORT QUESTIONS:

i. Write the symbolic representation of 59 in Egyptian mathematics as well as Babylonian times.
ii. Write special symbol for 59 in Babylonian mathematics history.
iii. Is the zero symbol presented in Babylonian mathematics?
iv. Since Babyloinians had no symbiol for zero then what did they use for value having zero magnitude?
v. Write $2(60)^2 + 0(60) + 2$ in Babylonian symbols.
vi. Shortly note on Posintional system of Babylonians. As they have no absolute positional system.

LONG QUESTIONS:

1) Briefly describe about the Positional numeration of Babylonian history of mathematics.

SEXAGESIMAL FRACTIONS

Had Mesopotamian mathematics, like that of the Nile Valley, been based on the addition of integers and unit fractions, the invention of the positional notation would not have been greatly significant at the time. It is not much more difficult to write 98,765 in hieroglyphic notation than in cuneiform, and the latter is definitely more difficult to write than the same number in hieratic script. The secret of the superiority of Babylonian mathematics over that of the Egyptians lies in the fact that those who lived "between the two rivers" took the most felicitous step of extending the principle of position to cover fractions as well as whole numbers. That is, the notation  ' was used not only for 2(60) + 2, but also for 2+ 2(60) or for $2(60) + 2(60)^2$ or for other fractional forms involving two successive positions. This meant that the Babylonians had at their command the computational power that the modern decimal fractional notation affords us today. For the Babylonian scholar, as for the modern engineer, the addition or the multiplication of 23.45 and 9.876 was essentially no more difficult than was the addition or the multiplication of the whole numbers 2,345 and 9,876, and the Mesopotamians were quick to exploit this important discovery.

POSSIBLE SHORT QUESTIONS:

i. Write the superiority of Babylonian mathematics over Egyptian mathematics.
ii. On what mathematical concept, Babylonian had at their command?

LONG QUESTIONS:

1) Briefly describe about the Positional numeration of Babylonian history of mathematics.

APPROXIMATIONS

An Old Babylonian tablet from the Yale Collection (No. 7289) includes the calculation of the square root of 2 to three sexagesimal places, the answer being written . In modern characters, this number can be appropriately written as 1;24,51,10, where a semicolon is used to separate the integral and fractional parts, and a comma is used as a separatrix for the sexagesimal positions. This form will generally be used throughout this chapter to designate numbers in sexagesimal notation. Translating this notation into decimal form, we have $1+24(60)+51(60)^2+10(60)^3$. This Babylonian value for $\sqrt{2}$ is equal to approximately 1.414222, differing by about 0.000008 from the true value. Accuracy in approximations was relatively easy for the Babylonians to achieve with their fractional notation, which was rarely equaled until the time of the Renaissance.

The effectiveness of Babylonian computation did not result from their system of numeration aLONG. Mesopotamian mathematicians were skillful in developing algorithmic procedures, among which was a square-root process often ascribed to later men. It is sometimes attributed to the Greek scholar Archytas (428 365 BCE) or to Heron of Alexandria (ca. 100 CE); occasionally, one finds it called Newton's algorithm. This Babylonian procedure is as simple as it is effective. Let x = $\sqrt{a}$ be the root desired, and let a_1 be a first approximation to this root; let a second approximation be found from the equation $b_1 = a/a_1$.If a_1 is too small, then b_1 is too large, and vice versa. Hence, the arithmetic mean $a_2 = \frac{1}{2}(a_1 + b_1)$ is a plusible next approximation. Inasmuch as a_2 is always too large, the next approximation, $b_2 = a/a_2$, will be too small, and one takes the arithmetic mean $a_3 = \frac{1}{2}(a_2 + b_2)$ to obtain a still better result; the procedure can be continued indefinitely. The value of $\sqrt{2}$ on Yale Tablet 7289 will be found to be that of a_3,where a_1 = 1;30. In the Babylonian square-root algorithm, one finds an iterative procedure that could have put the mathematicians of the time in touch with infinite processes, but scholars of that era did not pursue the implications of such problems.

The algorithm just described is equivalent to a two-term approximation to the binomial series, a case with which the Babylonians were familiar. If $\sqrt{a^2+b}$ is desired, the approximation a_1 = a leads to $b_1 = (a_2+b)/a$ and $a_2 = (a_1+b_1)/2 = a + b/(2a)$, which is in agree ment with the first two terms in the expansion of $(a^2 + b)^{1/2}$ and provides an approximation found in Old Babylonian texts.

POSSIBLE SHORT QUESTIONS:

i. Write the symbolic representation of $\sqrt{2}$ in Babylonian symbols.
ii. Write the sexagesimal representation of $\sqrt{2}$ in Babylonian symbols.
iii. In modern mathematics Which symbol is used to separate the integral and fractional parts? Show with example.
iv. In modern mathematics Which symbol is used as a separatrix for the sexagesimal positions? Show with example.
v. Explain Newton's algorithm with example in Babylonian mathematics history.
vi. Write the Babylonian Procedure to find Square root. Or write Babylonian Square root method.

LONG QUESTIONS:

1) Briefly describe about the Approximation concept of Babylonian history of mathematics.

TABLES

A substantial proportion of the cuneiform tablets that have been unearthed are "table texts," including multiplication tables, tables of reciprocals, and tables of squares and cubes and of square and cube roots written, of course, in cuneiform sexagesimals. One of these, for example, carries the equivalents of the entries shown in the following table:

2	30
3	20
4	15
5	12
6	10
8	7, 30
9	6, 40
10	6
12	5

The product of elements in the same line is in all cases 60, the Babylonian number base, and the table apparently was thought of as a table of reciprocals. The sixth line, for example, denotes that the reciprocal of 8 is $7/60+ 30/ (60)^2$. It will be noted that the reciprocals of 7 and 11 are missing from the table, because the reciprocals of such "irregular" numbers are nonterminating sexagesimals, just as in our decimal system the reciprocals of 3, 6, 7, and 9 are infinite when expanded decimally. Again, the Babylonians were faced with the problem of infinity, but they did not consider it systematically. At one point, however, a Mesopotamian scribe seems to give upper and lower bounds for the reciprocal of the irregular number 7, placing it between 0;8,34,16,59 and 0;8,34,18.

It is clear that the fundamental arithmetic operations were handled by the Babylonians in a manner not unlike that which would be employed today, and with comparable facility. Division was not carried out by the clumsy duplication method of the Egyptians, but through an easy multiplication of the dividend by the reciprocal of the divisor, using the appropriate items in the table texts. Just as today the quotient of 34 divided by 5 is easily found by multiplying 34 by 2 and shifting the decimal point, so in antiquity the same division problem was carried out by finding the product of 34 by 12 and shifting one sexagesimal place to obtain $6\frac{48}{60}$. Tables of reciprocals in general furnished reciprocals only of "regular" integers—that is, those that can be written as products of twos, threes, and fives—although there are a few exceptions. One table text includes the approximations $\frac{1}{59}$ = ;1,1,1 and $\frac{1}{61}$ = ;0,59,0,59. Here we have sexagesimal analogues of our decimal expressions $\frac{1}{9} = .11\overline{1}$ and $\frac{1}{11} =.09\overline{09}$, unit fractions in which the denominator is one more or one less than the base, but it appears again that the Babylonians did not notice, or at least did not regard as significant, the infinite periodic expansions in this connection.

One finds among the Old Babylonian tablets some table texts containing successive powers of a given number, analogous to our modern tables of logarithms or, more properly speaking, of antilogarithms.Exponential (or logarithmic) tables have been found in which the first ten powers are listed for the bases 9 and 16 and 1,40 and 3,45 (all perfect squares). The question raised in a problem text asking to what power a certain number must be raised in order to yield a given number is equivalent to our question **"What is the logarithm of the given number in a system with a certain number as base?"** The chief differences between the ancient tables and our own, apart from matters of language and notation, are that no single number was systematically used as a base in various connections and that the gaps between entries in the ancient tables are far larger than in our tables. Then, too, their "logarithm tables" were not used for general purposes of calculation, but rather to solve certain very specific questions.

Despite the large gaps in their exponential tables, Babylonian mathematicians did not hesitate to interpolate by proportional parts to approximate intermediate values. Linear interpolation seems to have been a commonplace procedure in ancient Mesopotamia, and the positional notation lent itself conveniently to the rule of three. A clear instance of the practical use of interpolation within exponential tables is seen in a problem text that asks how long it will take money to double at 20 percent annually; the answer given is 3;47,13,20. It seems to be quite clear that the scribe used linear interpolation between the values for $(1;12)^3$ and $(1;12)^4$, following the compound interest formula $a= P(1+r)^n$, where r is 20 percent, or $\frac{12}{60}$, and reading values from an exponential table with powers of 1;12.

POSSIBLE SHORT QUESTIONS:

i. Shortly describe Babylonia table concept by using an example.
ii. How did the Babylonians form there tables?
iii. Division method of Babylonia was quite different the Egyptians. Explain it using an example.
iv. Write the chief differences between the ancient tables and modern tables.

LONG QUESTIONS:

1) Briefly describe about the Table concept of Babylonian history of mathematics.

EQUATIONS

One table for which the Babylonians found considerable use is a tabulation of the values of $n^3 + n^2$ for integral values of n, a table essential in Babylonian algebra; this subject reached a considerably higher level in Mesopotamia than in Egypt. Many problem texts from the Old Babylonian period show that the solution of the complete three-term quadratic equation afforded the Babylonians no serious difficulty, for flexible algebraic operations had been developed. They could transpose terms in an equation by adding equals to equals, and they could multiply both sides by like quantities to remove fractions or to eliminate factors. By adding 4ab to $(a-b)^2$ they could obtain $(a+b)^2$,forthey were familiar with many simple forms of factoring. They did not use letters for unknown quantities, for the alphabet had not yet been invented, but words such as “length,” “breadth,” “area,” and “volume” served effectively in this capacity. That these words may well have been used in a very abstract sense is suggested by the fact that the Babylonians had no qualms about adding a “length” to an “area” or an “area” to a “volume.” Egyptian algebra had been much concerned with linear equations, but the Babylonians evidently found these too elementary for much attention. In one problem, the weight x of a stone is called for if $(x+\frac{x}{7})+\frac{1}{11}(x+\frac{x}{7})$ is one mina; the answer is simply given as 48;7,30 gin, where 60 gin make a mina. In another problem in an Old Babylonian text, we find two simultaneous linear equations in two unknown quantities, called respectively the “first silver ring” and the “second silver ring.” If we call these x and y in our notation, the equations are $x/7 + y/11 = 1$ and $6x/7 = 10y/11$. The answer is expressed laconically in terms of the rule $\frac{x}{7} = \frac{11}{7+11} + \frac{1}{72}$ and $\frac{y}{11} = \frac{7}{7+11} - \frac{1}{72}$

In another pair of equations, part of the method of solution is included in the text. Here $\frac{1}{4}$ width + length = 7 hands,and length + width = 10 hands. The solution is first found by replacing each “hand” with 5 “fingers” and then noticing that a width of 20 fingers and a length of 30 fingers will satisfy both equations. Following this, however, the solution is found by an alternative method equivalent to an elimination through combination. Expressing all dimensions in terms of hands, and letting the length and the width be x and y, respectively, the equations become y + 4x = 28 and x + y = 10. Subtracting the second equation from the first, one has the result 3x = 18; hence, x = 6 hands, or 30 fingers, and y = 20 fingers.

POSSIBLE SHORT QUESTIONS:

i. Did Babylonian use letters for unknown quantities?

LONG QUESTIONS:

1) Briefly describe about the equation concept of Babylonian history of mathematics.

QUADRATIC EQUATIONS

The solution of a three-term quadratic equation seems to have exceeded by far the algebraic capabilities of the Egyptians, but Otto Neugebauer in 1930 disclosed that such equations had been handled effectively by the Babylonians in some of the oldest problem texts. For instance, one problem calls for the side of a square if the area less the side is 14,30. The solution of this problem, equivalent to solving $x^2 - x = 870$, is expressed as follows:

Take half of 1, which is 0;30, and multiply 0;30 by 0;30, which is 0;15; add this to 14,30 to get l4,30;15. This is the square of 29;30. Now add 0;30 to 29;30, and the result is 30, the side of the square.

The Babylonian solution is, of course, exactly equivalent to the formula $x = \sqrt{\left(\frac{p}{2}\right)^2 + q} + \frac{p}{2}$ for a root of the equation $x^2 - px = q$, which is the quadratic formula that is familiar to high school students of today. In another text, the equation $1x^2 + 7x = 6;15$ was reduced by the Babylonians to the standard type $x^2 + px = q$ by first multiplying through by 11 to obtain $(11x)^2 + 7(11x) = 1,8;45$. This is a quadratic in normal form in the unknown quantity $y = 11x$, and the solution for y is easily obtained by the familiar rule $y = \sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2}$, from which the value of x is then determined. This solution is remarkable as an instance of the use of algebraic transformations.

Until modern times, there was no thought of solving a quadratic equation of the form $x^2 + px + q = 0$, where p and q are positive, for the equation has no positive root. Consequently, quadratic equations in ancient and medieval times and even in the early modern period were classified under three types:

1. $x^2 + px + q = 0$
2. $x^2 = px + q$
3. $x^2 + q = px$

All three types are found in Old Babylonian texts of some 4,000 years ago. The first two types are illustrated by the problems given previously; the third type appears frequently in problem texts, where it is treated as equivalent to the simultaneous system $x + y = p$, $xy = q$. So numerous are problems in which one is asked to find two numbers when given their product and either their sum or their difference that these seem to have constituted for the ancients, both Babylonian and Greek, a sort of "normal form" to which quadratics were reduced. Then, by transforming the simultaneous equations $xy = a$ and $x \pm y = b$ into the pair of linear equations $x \pm y = b$ and $x \mp y = \sqrt{b^2} \mp 4a$; the values of x and y are found through an addition and a subtraction. A Yale cuneiform tablet, for example, asks for the solution of the system $x + y = 6;30$ and $xy = 7;30$. The instructions of the scribe are essentially as follows. First find

From the last two equations, it is obvious that x

$$\frac{x+y}{2} = 3;15$$

and then find

$$\left(\frac{x+y}{2}\right)^2 = 10;33,45.$$

Then,

$$\left(\frac{x+y}{2}\right)^2 - xy = 3;3,45$$

and

$$\sqrt{\left(\frac{x+y}{2}\right)^2 - xy} = 1;45.$$

Hence,

$$\left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right) = 3;15 + 1;45$$

and

$$\left(\frac{x+y}{2}\right) - \left(\frac{x-y}{2}\right) = 3;15 - 1;45.$$

and $y = 1\frac{1}{2}$. Because the quantities x and y enter symmetrically in the given conditional equations, it is possible to interpret the values of x and y as the two roots of the quadratic equation $x^2 + 7;30 = 6;30x$. Another Babylonian text calls for a number that when added to its reciprocal becomes 2;0,0,33,20. This leads to a quadratic of type 3, and again we have two solutions, 1;0,45 and 0;59,15,33,20.

POSSIBLE SHORT QUESTIONS:

i. solve the side of a square if the area less the side is 14,30.Or equivalently solve $x^2 - x = 870$ in Babylonian style.
ii. Write Babylonian formula to find the solution of quadratic equation.
iii. In ancient and medieval times and even in the early modern period quadratic equations were classified under three types. Write the types.

LONG QUESTIONS:

1) Briefly describe about quadratic equation concept of Babylonian history of mathematics.

CUBIC EQUATIONS

The Babylonian reduction of a quadratic equation of the form $ax^2 + bx = c$ to the normal form $y^2 + by = ac$ through the substitution $y = ax$ shows the extraordinary degree of flexibility in Mesopotamian algebra. There is no record in Egypt of the solution of a cubic equation, but among the Babylonians there are many instances of this.

Pure cubics, such as $x^3 = 0;7,30$, were solved by direct reference to tables of cubes and cube roots, where the solution $x = 0;30$ was read off. Linear interpolation within the tables was used to find approximations for values not listed in the tables. Mixed cubics in the standard form $x^3 + x^2 = a$ were solved similarly by reference to the available tables, which listed values of the combination $n^3 + n^2$ for integral values of n from 1 to 30. With the help of these tables, they easily read off that the solution, for example, of $x^3 + x^2 = 4,12$ is equal to 6. For still more general cases of equations of the third degree, such as $144x^3 + 12x^2 = 21$, the Babylonians used their method of substitution. Multiplying both sides by 12 and using $y = 12x$, the equation becomes $y^3 + y^2 = 4,12$, from which y is found to be equal to 6, hence x is just $\frac{1}{2}$ or 0;30. Cubics of the form $ax^3 + bx^2 = c$ are reducible to the Babylonian normal form by multiplying through by a^2 / b^3 to obtain $(ax / b)^3\ 1(ax / b)^2 = ca^2/ b^3$, a cubic of standard type in the unknown quantity ax / b. Reading off from the tables the value of this unknown quantity, the value of x is determined. Whether the Babylonians were able to reduce the general four- term cubic, $ax^3 + bx^2 + ex = d$, to their normal form is not known. It is not too unlikely that they could reduce it, as is indicated by the fact that a solution of a quadratic suffices to carry the four-term equation to the three-term form $px^3 + qx^2 = r$, from which, as we have seen, the normal form is readily obtained.

There is, however, no evidence now available to suggest that the Mesopotamian mathematicians actually carried out such a reduction of the general cubic equation. With modern symbolism, it is a simple matter to see that $(ax)^3 + (ax)^2 = b$ is essentially the same type of equation as $y^3 + y^2 = b$, but to recognize this without our notation is an achievement of far greater significance for the development of mathematics than even the vaunted positional principle in arithmetic that we owe to the same civilization. Babylonian algebra had reached such an extraordinary level of abstraction that the equations $ax^4 + bx^2 = c$ and $ax^8 + bx^4 = c$ were recognized as nothing worse than quadratic equations in disguise.i.e., quadratics in x^2 and x^4

POSSIBLE SHORT QUESTIONS:

i. Did the Egyptian have an idea about Cubic equation?
ii. Shortly note on reduction of cubic equations in Baylonian times with examples.

LONG QUESTIONS:

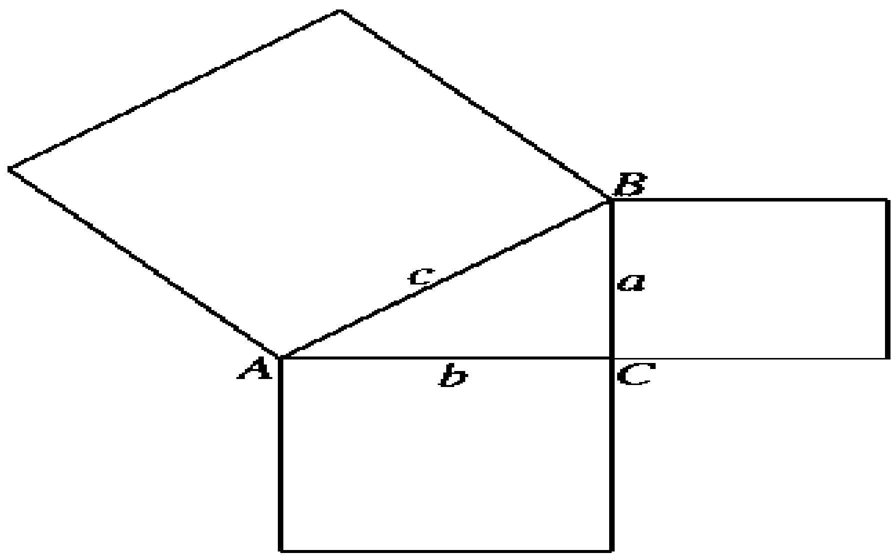
1) Briefly describe about cubic equation concept of Babylonian history of mathematics.

MEASUREMENTS: PYTHAGOREAN TRIADS

The algebraic achievements of the Babylonians are admirable, but the motives behind this work are not easy to understand. It has commonly been supposed that virtually all pre-Hellenic science and mathematics were purely utilitarian, but what sort of real-life situation in ancient Babylon could possibly lead to problems involving the sum of a number and its reciprocal or a difference between an area and a length? If utility was the motive, then the cult of immediacy was less strong than it is now, for direct connections between purpose and practice in Babylonian mathematics are far from apparent. That there may well have been toleration for, if not encouragement of,mathematics for its own sake is suggested by a tablet (No.322) in the Plimpton Collection at Columbia University. The tablet dates from the Old Babylonian period (ca. 1900 1600 BCE), and the tabulations it contains could easily be interpreted as a record of business accounts. Analysis, however, shows that it has deep mathematical significance in the theory of numbers and that it was perhaps related to a kind of proto-trigonometry. Plimpton 322 was part of a larger tablet, as is illustrated by the break along the left-hand edge, and the remaining portion contains four columns of numbers arranged in fifteen horizontal rows. The right-hand column contains the digits from 1 to 15, and, evidently, its purpose was simply to identify in order the items in the other three columns, arranged as follows:

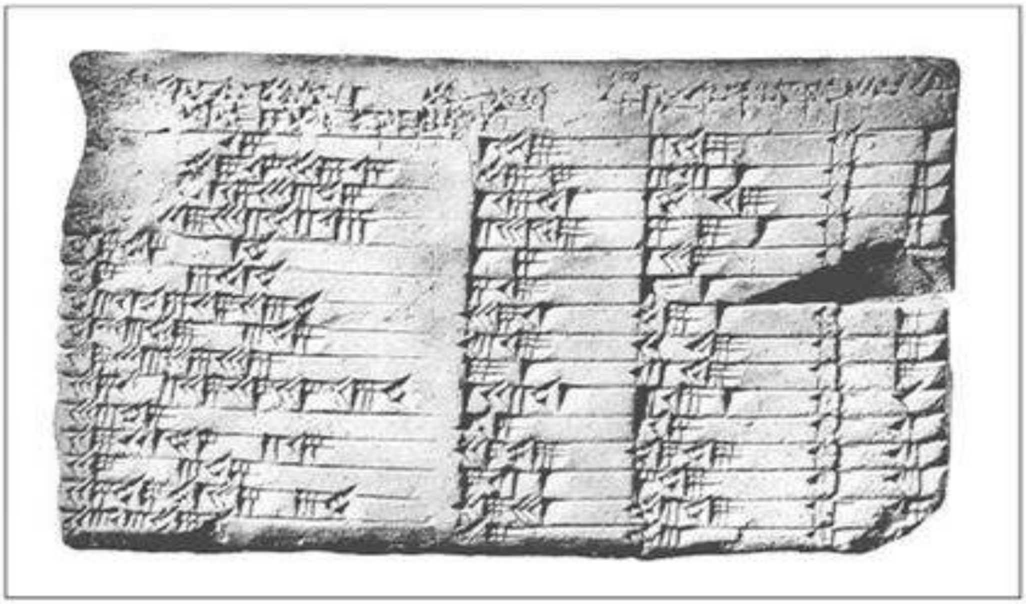
1,59,0,15	1,59	2,49	1
1,56,56,58,14,50,6,15	56,7	1,20,25	2
1,55,7,41,15,33,45	1,16,41	1,50,49	3
1,53,10,29,32,52,16	3,31,49	5,9,1	4
1,48,54,1,40	1,5	1,37	5
1,47,6,41,40	5,19	8,1	6
1,43,11,56,28,26,40	38,11	59,1	7
1,41,33,59,3,45	13,19	20,49	8
1,38,33,36,36	8,1	12,49	9
1,35,10,2,28,27,24,26,40	1,22,41	2,16,1	10
1,33,45	45,0	1,15,0	11
1,29,21,54,2,15	27,59	48,49	12
1,27,0,3,45	2,41	4,49	13
1,25,48,51,35,6,40	29,31	53,49	14
1,23,13,46,40	56	1,46	15

The tablet is not in such excellent condition that all of the numbers can still be read, but the clearly discernible pattern of construction in the table made it possible to determine from the context the few items that were missing because of small fractures. To understand what the entries in the table probably meant to the Babylonians, consider the right triangle ABC (Fig)



If the numbers in the second and third columns (from left to right) are thought of as the sides a and c, respectively, of the right triangle, then the first, or left-hand, column contains in each case the square of the ratio of c to b. The left-hand column, therefore, is a short table of values of $\sec^2 A$, but we must not assume that the Babylonians were familiar with our secant concept. Neither the Egyptians nor the Babylonians introduced a measure of angles in the modern sense. Nevertheless, the rows of numbers in Plimpton 322 are not arranged in haphazard fashion, as a superficial glance might imply. If the first comma in column one (on the left) is replaced by a semicolon, it is obvious that the numbers in this column decrease steadily from top to bottom. Moreover, the first number is quite close to $\sec^2 45^0$, and the last number in the column is approximately $\sec^2 31^0$, with the intervening numbers close to the values of $\sec^2 A$ as A decreases by degrees from 45^0 to 31^0. This arrangement obviously is not the result of chance aLONG. Not only was the arrangement carefully thought out, but the dimensions of the triangle were also derived according to a rule. Those who constructed the table evidently began with two regular sexagesimal integers, which we shall call p and q, with $p > q$, and then formed the triple of numbers $p^2- q^2$ and 2pq and $p^2 + q^2$. The three integers thus obtained are easily seen to form a Pythagorean triple, in which the square of the largest is equal to the sum of the squares of the other two. Hence, these numbers can be used as the dimensions of the right triangle ABC, with $a= p^2 - q^2$ and $b = 2pq$ and $c = p^2 + q^2$. Restricting themselves to values of p less than 60 and to corresponding values of q such that $1< p/q<1 + \sqrt{2}$, that is, to right triangles for which $a < b$, the Babylonians presumably found that there were just 38 possible pairs of values of p and q satisfying the conditions, and for these they apparently formed the 38 corresponding Pythagorean triples. Only the first 15, arranged in descending order for the ratio $(p^2 + q^2)/2pq$, are included in the table on the tablet, but it is likely that the scribe had intended to continue the table on the other side of the tablet. It has also been suggested that the portion of Plimpton 322 that has broken off from the left side contained four additional columns, in which were tabulated the values of p and q and 2pq and what we should now call $\tan^2 A$.

The Plimpton Tablet 322 might give the impression that it is an exercise in the theory of numbers, but it is likely that this aspect of the subject was merely ancillary to the problem of measuring the areas of squares on the sides of a right triangle.



Plimpton 322

The Babylonians disliked working with the reciprocals of irregular numbers, for these could not be expressed exactly in finite sexagesimal fractions. Hence, they were interested in values of p and q that should give them regular integers for the sides of right triangles of varying shapes, from the isosceles right triangle down to one with a small value for the ratio a / b.
For example, the numbers in the first row are found by starting with p = 12 and q = 5, with the corresponding values a = 119 and b = 120 and c = 169. The values of a and c are precisely those in the second and third positions from the left in the first row on the Plimpton tablet; the ratio c^2 / b^2 = 28561 / 14400 is the number 1;59,0,15 that appears in the first position in this row. The same relationship is found in the other fourteen rows; the Babylonians carried out the work so accurately that the ratio c^2 / b^2 in the tenth row is expressed as a fraction with eight sexagesimal places, equivalent to about fourteen decimal places in our notation.

So much of Babylonian mathematics is bound up with tables of reciprocals that it is not surprising to find that the items in Plimpton 322 are related to reciprocal relationships. If a= 1, then 1 = (c + b)(c - b), so that c + b and c - b are reciprocals. If one starts with c+ b = n, where n is any regular sexagesimal, then c – b = 1/n; hence, a = 1 and $b = \frac{1}{2}(n - \frac{1}{n})$ and $c = \frac{1}{2}(n + \frac{1}{n})$ are a

Pythagorean fraction triple, which can easily be converted to a Pythagorean integer triple by multiplying each of the three by 2n. All triples in the Plimpton tablet are easily calculated by this device.

The account of Babylonian algebra that we have given is representative of their work, but it is not intended to be exhaustive. There are in the Babylonian tablets many other things, although none so striking as those the Plimpton Tablet 322; as in this case, many are still open to multiple interpretations. For instance, in one tablet the geometric progression $1+2+2^2+.......+2^9$ is summed, and in another the sum of the series of squares $1^2+2^2+3^2+........+10^2$ is found. One wonders whether the Babylonians knew the general formulas for the sum of a geometric progression and the sum of the first n perfect squares. It is quite possible that they did, and it has been conjectured that they were aware that the sum of the first n perfect cubes is equal to the square of the sum of the first n integers. Nevertheless, it must be borne in mind that tablets from Mesopotamia resemble Egyptian papyri in that only specific cases are given, with no general formulations.

POSSIBLE SHORT QUESTIONS:

i. Note on Pathgorian Triads.
ii. Note on Plimpton 322. Write also its time.
iii. To understand what the entries in the table Plimpton 322 probably meant to the Babylonians, could you give an example?
iv. The Babylonians disliked working with the reciprocals of irregular numbers, is it true? What was there there area of interest? Examplify.
v. How did triples in the Plimpton tablet calculated?
vi. Did the Babylonians aware about geometric progression? If yes then write any Geometric Progression.
vii. Did the Babylonians aware about the sum of the first n perfect cubes? And how did they tackle about it?

LONG QUESTIONS:

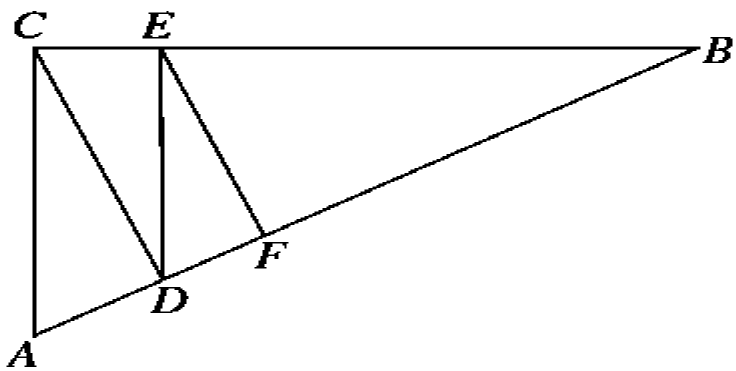
1) Briefly describe about measurement concept of Babylonian history of mathematics.
2) Briefly note on Plimpton 322 (The Babylonian Tablet)

POLYGONAL AREAS

It used to be held that the Babylonians were better in algebra than were the Egyptians, but that they had contributed less to geometry.The first half of this statement is clearly substantiated by what we have learned in previous paragraphs; attempts to bolster the second half of the comparison generally are limited to the measure of the circle or to the volume of the frustum of a pyramid. In the Mesopotamian Valley, the area of a circle was generally found by taking three times the square of the radius, and in accuracy this falls considerably below the Egyptian measure. Yet the counting of decimal places in the approximations for π is scarcely an appropriate measure of the geometric stature of a civilization, and a twentieth-century discovery has effectively nullified even this weak argument.

In 1936, a group of mathematical tablets was unearthed at Susa, a couple of hundred miles from Babylon, and these include significant geometric results. True to the Mesopotamian penchant for making tables and lists, one tablet in the Susa group compares the areas and the squares of the sides of the regular polygons of three, four, five, six, and seven sides. The ratio of the area of the pentagon, for example, to the square on the side of the pentagon is given as 1;40, a value that is correct to two significant figures. For the hexagon and the heptagon, the ratios are expressed as 2;37,30 and 3;41, respectively. In the same tablet, the scribe gives 0;57,36 as the ratio of the perimeter of the regular hexagon to the circumference of the circumscribed circle, and from this, we can readily conclude that the Babylonian scribe had adopted 3;7,30, or $3\frac{1}{8}$,as an approximation for π. This is at least as good as the value adopted in Egypt. Moreover, we see it in a more sophisticated context than in Egypt, for the tablet from Susa is a good example of the systematic comparison of geometric figures. One is almost tempted to see in it the genuine origin of geometry, but it is important to note that it was not so much the geometric context that interested the Babylonians as the numerical approximations that they used in mensuration. Geometry for them was not a mathematical discipline in our sense, but a sort of applied algebra or arithmetic in which numbers are attached to figures.

There is some disagreement as to whether the Babylonians were familiar with the concept of similar figures, although this appears to be likely. The similarity of all circles seems to have been taken for granted in Mesopotamia, as it had been in Egypt, and the many problems on triangle measure in cuneiform tablets seem to imply a concept of similarity. A tablet in the Baghdad Museum has a right triangle ABC (Fig) with sides a = 60 and b = 45 and c = 75, and it is subdivided into four smaller right triangles, ACD, CDE, DEF,and EFB.



The areas of these four triangles are then given as 8,6 and 5,11;2,24 and 3,19;3,56,9,36 and 5,53;53,39,50,24, respectively. From these values, the scribe computed the length of AD as 27, apparently using a sort of "similarity formula" equivalent to our theorem that areas of similar figures are to each other as squares on corresponding sides. The lengths of CD and BD are found to be 36 and 48, respectively, and through an application of the "similarity formula" to triangles BCD and DCE, the length of CE is found to be 21;36. The text breaks off in the middle of the calculation of DE.

POSSIBLE SHORT QUESTIONS:

i. The Babylonians were better in algebra than were the Egyptians, but that they had contributed less to geometry. Would you explain it?
ii. How did Mesopotamian (Babylonian) measure the area of circle?
iii. Note on Susa Tablet by Baylonian. Why this table is named by Susa? When did it discovered?
iv. Write Babylonian approximation for π.
v. How did they think about Geometry (Babylonians)?
vi. Whether the Babylonians were familiar with the concept of similar figures or not?

LONG QUESTIONS:

1) Briefly describe about polygonal area concept of Babylonian history of mathematics.

GEOMETRY AS APPLIED ARITHMETIC

In geometry the Babylonians accomplished almost nothing. Besides the division of the circumference into 6 parts by its radius, and into 360 degrees, they had some knowledge of geometrical figures, such as the triangle and quadrangle,which they used in their auguries. Like the Hebrews (1 Kin. 7:23), they took π = 3. Of geometrical demonstrations there is, of course, no trace. "As a rule, in the Oriental mind the intuitive powers eclipse the severely rational and logical."

Measurement was the keynote of algebraic geometry in the Mesopotamian Valley, but a major flaw, as in Egyptian geometry, was that the distinction between exact and approximate measures was not made clear. The area of a quadrilateral was found by taking the product of the arithmetic means of the pairs of opposite sides, with no warning that this is in most cases only a crude approximation. Again, the volume of a frustum of a cone or a pyramid was sometimes found by taking the arithmetic mean of the upper and lower bases and multiplying by the height; sometimes, for a frustum of a square pyramid with areas a^2 and b^2 for the lower and upper bases, the formula

$$V = \left(\frac{a+b}{2}\right)^2 h$$

was applied. For the latter, however, the Babylonians also used a rule equivalent to

$$V = \left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right] h$$

a formula that is correct and reduces to the one used by the Egyptians. It is not known whether Egyptian and Babylonian results were always independently discovered, but in any case, the latter were definitely more extensive than the former, in both geometry and algebra. The Pythagorean theorem, for example, does not appear in any form in surviving documents from Egypt, but tablets even from the Old Babylonian period show that in Mesopotamia the theorem was widely used. A cuneiform text from the Yale Collection, for example, contains a diagram of a square and its diagonals in which the number 30 is written along one side and the numbers 42;25,35 and 1;24,51,10 appear along a diagonal. The last number obviously is the ratio of the lengths of the diagonal and a side, and this is so accurately expressed that it agrees with $\sqrt{2}$ to within about a millionth. The accuracy of the result was made possible by knowledge of the Pythagorean theorem. Sometimes, in less precise computations, the Babylonians used 1;25 as a rough-and-ready approximation to this ratio. Of more significance than the precision of the values, however, is the implication that the diagonal of any square could be found by multiplying the side by $\sqrt{2}$. Thus, there seems to have been some awareness of general principles, despite the fact that these are exclusively expressed in special cases.

Babylonian recognition of the Pythagorean theorem was by no means limited to the case of a right isosceles triangle. In one Old Babylonian problem text, a ladder or a beam of length 0;30 stands against a wall; the question is, how far will the lower end move out from the wall if the upper end slips down a distance of 0;6 units? The answer is correctly found by use of the Pythagorean theorem. Fifteen hundred years later, similar problems, some with new twists, were still being solved in the Mesopotamian Valley. A Seleucid tablet, for example, proposes the following problem. A reed stands against a wall. If the top slides down 3 units when the lower end slides away 9 units, how long is the reed? The answer is given correctly as 15 units.

Ancient cuneiform problem texts provide a wealth of exercises in what we might call geometry, but which the Babylonians probably thought of as applied arithmetic. A typical inheritance problem calls for the partition of a right-triangular property among six brothers. The area is given as 11,22,30 and one of the sides is 6,30; the dividing lines are to be equidistant and parallel to the other side of the triangle. One is asked to find the difference in the allotments. Another text gives the bases of an isosceles trapezoid as 50 and 40 units and the length of the sides as 30; the altitude and the area are required

(van der Waerden 1963, pp. 76 77).

The ancient Babylonians were aware of other important geometricrelationships.Like the Egyptians, they knew that the altitude in an isosceles triangle bisects the base. Hence, given the length of a chord in a circle of known radius, they were able to find the apothem. Unlike the Egyptians, they were familiar with the fact that an angle inscribed in a semicircle is a right angle, a proposition generally known as the Theorem of Thales, despite the fact that Thales lived more than a millennium after the Babylonians had begun to use it. This misnaming of a well-known theorem in geometry is symptomatic of the difficulty in assessing the influence of pre-Hellenic mathematics on later cultures. Cuneiform tablets had a permanence that could not be matched by documents from other civilizations, for papyrus and parchment do not so easily survive the ravages of time. Moreover, cuneiform texts continued to be recorded down to the dawn of the Christian era, but were they read by neighboring civilizations, especially the Greeks? The center of mathematical development was shifting from the Mesopotamian Valley to the Greek world half a dozen centuries before the beginning of our era, but reconstructions of early Greek mathematics are rendered hazardous by the fact that there are virtually no extant mathematical documents from the pre-Hellenistic period. It is important, therefore, to keep in mind the general characteristics of Egyptian and Babylonian mathematics so as to be able to make at least plausible conjectures concerning analogies that may be apparent between pre-Hellenic contributions and the activities and attitudes of later peoples.

There is a lack of explicit statements of rules and of clear-cut distinctions between exact and approximate results. The omission in the tables of cases involving irregular sexagesimals seems to imply some recognition of such distinctions, but neither the Egyptians nor the Babylonians appear to have raised the question of when the area of a quadrilateral (or of a circle) is found exactly and when only approximately. Questions about the solvability or unsolvability of a problem do not seem to have been raised, nor was there any investigation into the nature of proof. The word "proof" means various things at different levels and ages; hence, it is hazardous to assert categorically that pre-Hellenic peoples had no concept of proof, nor any feeling of the need for proof. There are hints that these people were occasionally aware that certain area and volume methods could be justified through a reduction to simpler area and volume problems. Moreover, pre-Hellenic scribes not infrequently checked or "proved" their divisions by multiplication; occasionally, they verified the procedure in a problem through a substitution that verified the correctness of the answer. Nevertheless, there are no explicit statements from the pre-Hellenic period that would indicate a felt need for proofs or a concern for questions of logical principles. In Mesopotamian problems, the words "length" and "width" should perhaps be interpreted much as we interpret the letters x and y,for the writers of cuneiform tablets may well have moved on from specific instances to general abstractions. How else does one explain the addition of a length to an area? In Egypt also, the use of the word for quantity is not incompatible with an abstract interpretation such as we read into it today. In addition, there were in Egypt and Babylonia problems that have the earmarks of recreational mathematics. If a problem calls for a sum of cats and measures of grain, or of a length and an area, one cannot deny to the perpetrator either a modicum of levity or a feeling for abstraction. Of course, much of pre-Hellenic mathematics was practical, but surely not all of it. In the practice of computation, which stretched over a couple of millennia, the schools of scribes used plenty of exercise material, often, perhaps, simply as good clean fun.

POSSIBLE SHORT QUESTIONS:

i. How did they think about Geometry (Babylonians)?
ii. Write the major flaw (Problem) in Babylonian measurement system.
iii. How did the Babylonian find the area of a quadrilateral?
iv. How did the Babylonian find the area of a frustum of a cone or a pyramid?
v. The Pythagorean theorem did not used by Egypt, but did Babylonian use it?
vi. Both Babylonian and Egyptians were lack of explicit statements of rules and of clear-cut distinctions between exact and approximate results. Is it true?

LONG QUESTIONS:

1) Briefly describe about Geometrical concept of Babylonian history of mathematics.

ANCIENT AND MEDIEVAL INDIA (THE HINDOOS)

A mixture of pearl shells and sour dates . . . or of costly crystal and common pebbles.

Al Biruni's India

EARLY MATHEMATICS IN INDIA (HISTORICAL BACKGROUND)

Despite developppping quite independently of chinese (and probably also of Babylonian mathematics) some mathematical discoveries were made at a very early time in india. Before Perso – Arabic mathematicians, work on mathematics was started in india. Brahmi numerals are the basis of the system predate the common era. Brahmi and Karosthi numerals were used in Mauriya Empire period, both appearing on 3rd century BC edicts of Ashok Budhist used the symbol 1,4,6 around 300 BC. They were also familiar with 2,4,6,7 and 9. The Brahmi numerals were the ancestor of Hindu – Arabic Glyphics 1 to 9. 10,20,30 numerals were also in their counting.The actual numeral system, including positional notation and use of zero, is in principle independent of the glyphs used and significantly younger than the Brahmi numerals.

The development of positional decimal system takes its origin in Hindu mathematics during the Gupta Period. Around 500 BC Aryabhatya mark zero and Brahmasphuta Siddhanta explained mathematical role of zero. The Sansikrat translation of Prakrit preserve positional use of zero. These indian developments were take up in Islamic mathematics in 8th century as recoreded in Al – Qifti's Chronolgy of the Scholars (early 13th century)

Archaeological excavations at Mohenjo Daro and Harappa give evidenceof an old and highly cultured civilization in the Indus Valley during the era of the Egyptian pyramid builders (ca. 2650 BCE), but we have no Indian mathematical documents from that age. There is evidence of structured systems of weights and measures, and samples of decimal-based numeration have been found. During this period and succeeding centuries, however, major movements and conquests of people occurred on the Indian subcontinent. Many of the languages and the dialects that evolved as a result have not been deciphered. It is therefore difficult at this stage to plot a time-space chart of mathematical activities for this vast area. The linguistic challenges are compounded by the fact that the earliest known Indian language samples were part of an oral tradition, rather than a written one. Nevertheless, Vedic Sanskrit, the language in 186 question, presents us with the earliest concrete information about ancientIndian mathematical concepts. The Vedas, groups of ancient, essentially religious texts, include references to large numbers and decimal systems. Especially interesting are dimensions, shapes, and proportions given for bricks used in the construction of ritual fire altars. India, like Egypt, had its "rope-stretchers," and the sparse geometric lore acquired in connection with the laying out of temples and the measurement and construction of altars took the form of a body of knowledge known as the Sulbasutras, or “rules of the cord.” Sulba(or sulva) refers to cords used for measurements, and sutra means a book of rules or aphorisms relating to a ritual or a science. The stretching of ropes is strikingly reminiscent of the origin of Egyptian geometry, and its association with temple functions reminds one of the possible ritual origin of mathematics. Yet, the difficulty of dating the rules is also matched by doubt concerning the influence the Egyptians had on later Hindu mathematicians. Evenmore so than in the case of China, there is a striking lack of continuity of tradition in the mathematics of India.

POSSIBLE SHORT QUESTIONS:

i. What type of numerals were the ancestor of Hindu – Arabic Glyphics?
ii. In whose time period indian positional mathematical system developed?
iii. Who introduced zero. And who explained the mathematical role of zero?
iv. When did indian mathematical knowledge entered in Muslim mathematic world? Or when did Muslims know about indian working about mathematics?
v. Shortly note on SalbaSatras (SalvsaSatras)

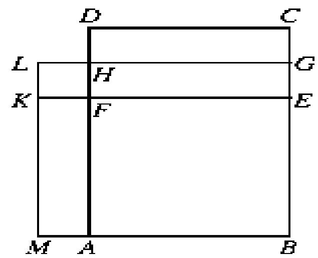
LONG QUESTIONS:

1) Briefly describe about Early mathematics in india or Historical Background of Indian mathematics.

THE SULBASUTRAS:

As early as 8th century BCE, long before Pythagoras, a text known as SulbaSatras (SulvaSatras) listed simple Pythagorian Triples(Triads), Pythagorian theorem is also stated for square and rectangle in this text. Sulvastras also contain Geometrical calculations of linear equations. It also gives square root of 2 obtained by adding $1+\frac{1}{3}+\frac{1}{3\times4}-\frac{1}{3\times4\times34}$ which the value of 1.4142156 correct to 5 decimal places.

There are a number of Sulbasutras; the major extant ones, all in verse, are associated with the names of Baudhayama, Manava, Katyayana, and the best-known, Apastamba. They may date from the first half of the first millennium BCE, although earlier and later dates have been suggested as well. We find rules for the construction of right angles by means of triples of cords the lengths of which form “Pythagorean” triads, such as 3, 4, and 5; or 5, 12, and 13; or 8, 15, and 17; or 12, 35, and 37. Although Mesopotamian influence in the Sulbasutras is not unlikely, we know of no conclusive evidence for or against this. Apastamba knew that the square on the diagonal of a rectangle is equal to the sum of the squares on the two adjacent sides. Less easily explained is another rule given by Apastamba—one that strongly resembles some of the geometric algebra in Book II of Euclid's Elements. To construct a square equal in



area to the rectangle ABCD **(Fig)**,

lay off the shorter sides on the longer, so that AF = AB = BE = CD, and draw HG bisecting segments CE and DF;extend EF to K, GH to L,and AB to M so that FK = HL = FH = AM, and draw LKM. Now construct a rectangle with a diagonal equal to LG and with a shorter side HFE. Then, the longer side of this rectangle is the side of the square desired. There are also rules for transforming rectilinear into curvilinear shapes and vice versa. So conjectural are the origin and the period of the Sulbasutras that we cannot tell whether the rules are related to early Egyptian surveying or to the later Greek problem of altar doubling.

POSSIBLE SHORT QUESTIONS:

i. Shortly note on SalvaSatras text.
ii. What type of concepts were include (discussed) in SalbaStras?

LONG QUESTIONS:

1) Briefly describe about The Salbastras of indian mathematics.

THE SIDDHANTAS

There are references to arithmetic and geometric series in Vedic literature that purport to go back to 2000 BCE, but no contemporary documents from India are available to confirm this. It has also been claimed that the first recognition of incommensurables is to be found in India during the Sulbasutra period, but such claims are not well substantiated. The period of the Sulbasutras was followed by the age of **the Siddhantas, or systems (of astronomy).** Five different versions of the Siddhantas are known by the names: **Paulisha Siddhanta, Surya Siddhanta, Vasisishta Siddhanta, Paitamaha Siddhanta, and Romanka Siddhanta**. Of these, the Surya Siddhanta (System of the Sun), written about 400 CE, is the only one that seems to be completely extant. According to the text, written in epic stanzas, it is the work of Surya, the Sun God. The main astronomical doctrines are evidently Greek, but with the retention of considerable old Hindu folklore. The Paulisha Siddhanta, which dates from about 380 CE, was summarized by the Hindu mathematician **Varahamihira (fl. 505 CE),** who also listed the other four Siddhantas. It was referred to frequently by the Arabic scholar al-Biruni, who suggested a

Greek origin or influence. Later writers report that the Siddhantas were in substantial agreement on substance, only the phraseology varied; hence, we can assume that the others, such as the Surya Siddhanta, were compendia of astronomy comprising cryptic rules in Sanskrit verse, with little explanation and without proof.

It is generally agreed that the Siddhantas stem from the late fourth or the early fifth century, but there is sharp disagreement about the origin of the knowledge that they contain. Indian scholars insist on the originality and independence of the authors, whereas Western writers are inclined to see definite signs of Greek influence. It is not unlikely, for example, that the Paulisha Siddhanta was derived in considerable measure from the work of the astrologer Paul, who lived in Alexandria shortly before the presumed date of composition of the Siddhantas. (Al-Biruni, in fact, explicitly attributes this Siddhanta to Paul of Alexandria.) This would account in a simple manner for the obvious similarities between portions of the Siddhantas and the trigonometry and the astronomy of Ptolemy. **The Paulisha Siddhanta, for example, uses the value $3\,\frac{177}{1250}$ for π, which is in essential agreement with the Ptolemaic sexagesimal value 3;8,30.**

Even if Indian authors did acquire their knowledge of trigonometry from the cosmopolitan Hellenism at Alexandria, the material in their hands took on a significantly new form. Whereas the trigonometry of Ptolemy had been based on the functional relationship between the chords of a circle and the central angles they subtend, the writers of the Siddhantas converted this to a study of the correspondence between half of a chord of a circle and half of the angle subtended at the center by the whole chord. Thus was born, **apparently in India, the predecessor of the modern trigonometric function known as the sine of an angle, and the introduction of the sine function represents the chief contribution of the Siddhantas to the history of mathematics. It was through the Indians, and not the Greeks, that our use of the half chord has been derived, and our word "sine," through misadventure in translation (see further on), has descended from the Sanskrit name jiva.**

POSSIBLE SHORT QUESTIONS:

i. Shortly note on The Siddhantas.
ii. What is mean by The Siddhantas?
iii. Name five different versions of the Siddhantas.
iv. Shortly write about the concept of Sine function in indian text.
v. Write The Paulisha Siddhanta's value for 'Pi'.

LONG QUESTIONS:

1) Briefly describe about The Siddhantas of indian mathematics.

ARYABHATA

During the sixth century, shortly after the composition of the Siddhantas, there lived two Indian mathematicians who are known to have written books on the same type of material. The older and more important of the two was Aryabhata, whose best-known work, written around 499 CE and titled Aryabhatiya, is a slim volume, written in verse, covering astronomy and mathematics. The names of several Hindu mathematicians before this time are known, but nothing of their work has been preserved beyond a few fragments. In this respect, then, the position of the Aryabhatiya of Aryabhata in India is somewhat akin to that of the Elements of Euclid in Greece some eight centuries earlier. Both are summaries of earlier developments, compiled by a single author. There are, however, more striking differences than similarities between the two works. The Elements is a well-ordered synthesis of pure mathematics with a high degree of abstraction, a clear logical structure, and an obvious pedagogical inclination; the Aryabhatiya is a brief descriptive work, in 123 metrical stanzas, intended to supplement rules of calculation used in astronomy and mensurational mathematics, with no appearance of deductive methodology. About a third of the work is on ganitapada, or mathematics. This section opens with the names of the powers of 10 up to the tenth place and then proceeds to give instructions for square and cube roots of integers. Rules of mensuration follow, about half of which are erroneous. The area of a triangle is correctly given as half the product of the base and altitude, but the volume of a pyramid is also taken to be half of the product of the base and the altitude. The area of a circle is found correctly as the product of the circumference and half of the diameter, but the volume of a sphere is incorrectly stated to be the product of the area of a great circle and the square root of this area. Again, in the calculation of areas of quadrilaterals, correct and incorrect rules appear side by side. The area of a trapezoid is expressed as half of the sum of the parallel sides multiplied by the perpendicular between them, but then follows the incomprehensible assertion that the area of any plane figure is found by determining two sides and multiplying them. One statement in the Aryabhatiya to which Indian scholars have pointed with pride is as follows:

Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000. (Clark 1930, p. 28)

Here we see the equivalent of 3.1416 for π, but it should be recalled that this is essentially the value Ptolemy had used. The likelihood that Aryabhata here was influenced by Greek predecessors is strengthened by his adoption of the myriad, 10,000, as the number of units in the radius. A typical portion of the Aryabhatiya is that involving arithmetic progressions, which contains arbitrary rules for finding the sum of the terms in a progression and for determining the number of terms in a progression when given the first term, the common difference, and the sum of the terms. The first rule had long been known by earlier writers. The second is a curiously complicated bit of exposition:

Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term, and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two. The result will be the number of terms.

Here, as elsewhere in the Aryabhatiya, no motivation or justification is given for the rule. It was probably arrived at through a solution of a quadratic equation, knowledge of which might have come from Mesopotamia or Greece. Following some complicated problems on compound interest (that is, geometric progressions), the author turns, in flowery language, to the very elementary problem of finding the fourth term in a simple proportion:

In the rule of three multiply the fruit by the desire and divide by the measure. The result will be the fruit of the desire.

This, of course, is the familiar rule that if $a/b = c/x$, then $x = bc / a$, where a is the "measure," b the "fruit," c the "desire," and x the "fruit of the desire." The work of Aryabhata is indeed a potpourri of the simple and the complex, the correct and the incorrect. The Arabic scholar al- Biruni, half a millennium later, characterized Indian mathematics as a mixture of common pebbles and costly crystals, a description quite appropriate to Aryabhatiya.

POSSIBLE SHORT QUESTIONS:

i. Shortly note on Aryabhatiya and who wrote it?
ii. Write the statement quoted in the Aryabhatiya about circumference of a circle.
iii. Write the statement for finding the sum of the terms in a progression and for determining the number of terms in a progression discussed in Aryabhatiya.
iv. In Aryabhatiya the writer use the flowery language to present geometric progressions. Could you write it?

LONG QUESTIONS:

1) Briefly describe about Aryabhatiya of indian mathematics.

NUMERALS

Despite developpping quite independently of chinese (and probably also of Babylonian mathematics) some mathematical discoveries were made at a very early time in india. Before Perso – Arabic mathematicians, work on mathematics was started in india. Brahmi numerals are the basis of the system predate the common era. Brahmi and Karosthi numerals were used in Mauriya Empire period, both appearing on 3rd century BC edicts of Ashok Budhist used the symbol 1,4,6 around 300 BC. They were also familiar with 2,4,6,7 and 9. The Brahmi numerals were the ancestor of Hindu – Arabic Glyphics 1 to 9. 10,20,30 numerals were also in their counting.The actual numeral system, including positional notation and use of zero, is in principle independent of the glyphs used and significantly younger than the Brahmi numerals.

The development of positional decimal system takes its origin in Hindu mathematics during the Gupta Period. Around 500 BC Aryabhatya mark zero and Brahmasphuta Siddhanta explained mathematical role of zero. The Sansikrat translation of Prakrit preserve positional use of zero. These indian developments were take up in Islamic mathematics in 8th century as recoreded in Al – Qifti's Chronolgy of the Scholars (early 13th century)

The second half of the Aryabhatiya is on the reckoning of time and on spherical trigonometry; here we note an element that would leave a permanent impression on the mathematics of later generations—the decimal place-value numeration. It is not known just how Aryabhata carried out his calculations, but his phrase "from place to place each is ten times the preceding" is an indication that the application of the principle of position was in his mind. "Local value" had been an essential part of Babylonian numeration, and perhaps the Hindus were becoming aware of its applicability to the decimal notation for integers in use in India. The development of numerical notations in India seems to have followed about the same pattern found in Greece. Inscriptions from the earliest period at Mohenjo Daro show at first simple vertical strokes, arranged into groups, but by the time of Asoka (third century BCE) a system resembling the Herodianic was in use. In the newer scheme the repetitive principle was continued, but new symbols of higher order were adopted for 4, 10, 20, and 100. This so-called Karosthi script then gradually gave way to another notation, known as the Brahmi characters, which resembled the alphabetic cipherization in the Greek Ionian system; one wonders whether it was only a coincidence that the change in India took place shortly after the period when in Greece the Herodianic numerals were displaced by the Ionian.

From the Brahmi ciphered numerals to our present-day notation for integers, two short steps are needed. The first is a recognition that through the use of the positional principle, the ciphers for the first nine units can also serve as the ciphers for the corresponding multiples of 10 or equally well as ciphers for the corresponding multiples of any power of 10. This recognition would make superfluous all of the Brahmi ciphers beyond the first nine. It is not known when the reduction to nine ciphers occurred, and it is likely that the transition to the more economical notation was made only gradually. It appears from extant evidence that the change took place in India, but the source of the inspiration for the change is uncertain. Possibly, the so-called Hindu numerals were the result of internal development aLONG; perhaps they developed first along the western interface between India and Persia, where remembrance of the Babylonian positional notation may have led to modification of the Brahmi system. It is possible that the newer system arose along the eastern interface with China, where the pseu-dopositional rod numerals may have suggested the reduction to nine ciphers. There is also a theory that this reduction may first have been made at Alexandria within the Greek alphabetic system and that subsequently the idea spread to India. During the later Alexandrian period, the earlier Greek habit of writing common fractions with the numerator beneath the denominator was reversed, and it is this form that was adopted by the Hindus, without the bar between the two. Unfortunately, the Hindus did not apply the new numeration for integers to the realm of decimal fractions; hence, the chief potential advantage of the change from Ionian notation was lost.

The earliest specific reference to the Hindu numerals is found in 662 in the writings of Severus Sebokt, a Syrian bishop. After Justinian closed the Athenian philosophical schools, some of the scholars moved to Syria, where they established centers of Greek learning. Sebokt evidently felt piqued by the disdain for non-Greek learning expressed by some associates; hence, he found it expedient to remind those who spoke Greek that “there are also others who know something.” To illustrate his point, he called attention to the Hindus and their “subtle discoveries in astronomy,” especially “their valuable methods of calculation, and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs” (Smith 1958, Vol. I, p. 167). That the numerals had been in use for some time is indicated by the fact that they occur on an Indian plate of the year 595 CE, where the date 346 is written in decimal place value notation.

POSSIBLE SHORT QUESTIONS:

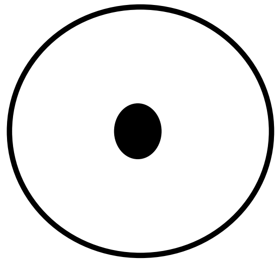
i. Shortly note on Numeral concept of indian mathematics.

LONG QUESTIONS:

1) Briefly describe about Numeral concept of indian mathematics.

THE SYMBOL FOR ZERO

The ancient Hindu Symbol of a circle with a dot in the middle, known as bindu or bindhu, symbolizing the void and the negation of the self, was probably instrumental in the use of a circle as a representation of the concept of zero.



Around 500 BC Aryabhatya mark zero and Brahmasphuta Siddhanta explained mathematical role of zero. The Sansikrat translation of Prakrit preserve positional use of zero.Brahmagupta established the basic mathematical rules dealing with zero. e.g. 1 + 0 = 1, 1 – 0 = 1, 1× 0 = 0
It should be remarked that the reference to nine symbols, rather than ten, implies that the Hindus evidently had not yet taken the second step in the transition to the modern system of numeration—the introduction of a notation for a missing position—that is, a zero symbol. The history of mathematics holds many anomalies, and not the least of these is the fact that "the earliest undoubted occurrence of a zero in India is in an inscription of 876" (Smith 1958, Vol. II, p. 69)—that is, more than two centuries after the first reference to the other nine numerals. It is not even established that the number zero (as distinct from a symbol for an empty position) arose in conjunction with the other nine Hindu numerals. It is quite possible that zero originated in the Greek world, perhaps at Alexandria, and that it was transmitted to India after the decimal posi- tional system had been established there.

The history of the zero placeholder in positional notation is further complicated by the fact that the concept appeared independently, well before the days of Columbus, in the western as well as the eastern hemisphere.

With the introduction, in the Hindu notation, of the tenth numeral, a round goose egg for zero, the modern system of numeration for integers was completed. Although the medieval Hindu forms of the ten numerals differ considerably from those in use today, the principles of the system were established. The new numeration, which we generally call the Hindu system, is merely a new combination of three basic principles, all of ancient origin: (1) a decimal base; (2) a positional notation; and (3) a ciphered form for each of the ten numerals. Not one of these three was originally devised by the Hindus, but it presumably is due to them that the three were first linked to form the modern system of numeration.

It may be well to say a word about the form of the Hindu symbol for zero—which is also ours. It was once assumed that the round form originally stemmed from the Greek letter omicron, the initial letter in the word "ouden," or "empty," but recent investigations seem to believe such an origin. Although the symbol for an empty position in some of the extant versions of Ptolemy's tables of chords does seem to resemble an omicron, the early zero symbols in Greek sexagesimal fractions are round forms variously embellished and differing markedly from a simple goose egg. Moreover, when in the fifteenth century in the Byzantine Empire a decimal positional system was fashioned out of the old alphabetic numerals by dropping the last eighteen letters and adding a zero symbol to the first nine letters, the zero sign took forms quite unlike an omicron. Sometimes it resembled an inverted form of our small letter h; other times, it appeared as a dot.

POSSIBLE SHORT QUESTIONS:

i. Who introduced zero. And who explained the mathematical role of zero?
ii. Write the indian symbol for zero.
iii. Write three basic principles of modern numeral system
(also called Hindu system).
iv. Greek also use zero symbol. Write about it.
v. Write a short comparison about the concept regarding zero in Indian history and Greek history.

LONG QUESTIONS:

1) Briefly describe about Zero concept of indian mathematics.

TRIGONOMETRY

The development of our system of notation for integers was one of the two most influential contributions of India to the history of mathematics. The other was the introduction of an equivalent of the sine function in trigonometry to replace the Greek tables of chords. The earliest tables of the sine relationship that have survived are those in the Siddhantas and the Aryabhatiya. Here the sines of angles up to 90^0 are given for twenty-four equal intervals of $3\frac{3}{4}^{\circ}$ each. In order to express arc length and sine length in terms of the same unit, the radius was taken as 3,438 and the circumference as $360.60 = 21{,}600$. This implies a value of π agreeing to four significant figures with that of Ptolemy. In another connection, Aryabhata used the value $\sqrt{10}$ for π, which appeared so frequently in India that it is sometimes known as the Hindu value.

For the sine of $3\frac{3}{4}^{\circ}$, the Siddhantas and the Aryabhatiya took the number of units in the arc—that is, $60 \times 3\frac{3}{4}$ or 225. In modern language, the sine of a small angle is very nearly equal to the radian measure of the angle (which is virtually what the Hindus were using). For further items in the sine table, the Hindus used a recursion formula that may be expressed as follows. If the n^{th} sine in the sequence from n = 1to n = 24 is designated as S_n and if the sum of the first n sines is S_n ,then $S_{n+1} = S_n + S_1 - S_n / S_1$. From this rule, one easily deduces that $\text{Sin}7\frac{1}{2} = 449$, $\text{Sin } 11\frac{1}{4}^{\circ} = 671$, $\text{Sin}15^0 = 890$, and so on,upto $\text{Sin}90^0 = 3{,}438$—the values listed in the table in the Siddhantas and the Aryabhatiya. Moreover, the table also includes values for what we call the versed sine of the angle (that is, $1 - cos\theta$ in modern trigonometry or 3,438 $[1 - cos\ \theta]$ in Hindu trigonometry) from vers $3\frac{3}{4}^{\circ} = 7$ to vers $90^0 = 3{,}438$. If we divide the items in the table by 3,438, the results are found to be in close agreement with the corresponding values in modern trigonometric tables (Smith 1958, Vol. II).

POSSIBLE SHORT QUESTIONS:

- i. Define trigonometry.
- ii. Write Ariyabhatiya value for 'Pi'.

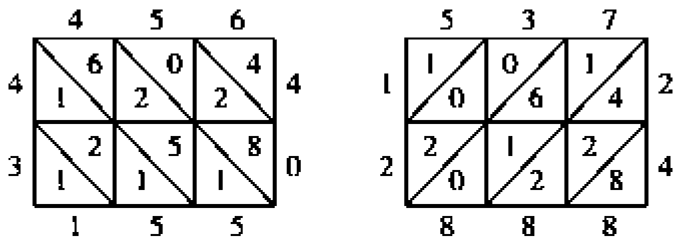
LONG QUESTIONS:

1) Briefly describe about Trigonometry concept of indian mathematics.

MULTIPLICATION

Trigonometry was evidently a useful and accurate tool in astronomy. How the Indians arrived at results such as the recursion formula is uncertain, but it has been suggested that an intuitive approach to difference equations and interpolation may have prompted such rules. Indian mathematics is frequently described as "intuitive," in contrast to the stern rationalism of Greek geometry. Although in Indian trigonometry there is evidence of Greek influence, the Indians seem to have had no occasion to borrow Greek geometry, concerned as they were with simple mensurational rules. Of the classical geometric problems or the study of curves other than the circle, there is little evidence in India, and even the conic sections seem to have been overlooked by the Indians, as by the Chinese. Hindu mathematicians were instead fascinated by work with numbers, whether it involved the ordinary arithmetic operations or the solution of determinate or indeterminate equations. Addition and multiplication were carried out in India much as they are by us today, except that the Indians seem at first to have preferred to write numbers with the smaller units on the left, hence to work from left to right, using small blackboards with white removable paint or a board covered with sand or flour. Among the devices used for multiplication was one that is known under various names:

lattice multiplication, gelosia multiplication, or cell or grating or quadrilateral multiplication. The scheme behind this is readily recognized in two examples. In the **first** example **(Fig.),**



the number 456 is multiplied by 34. The multiplicand has been written above the lattice and the multiplier appears to the left, with the partial products occupying the square cells. Digits in the diagonal rows are added, and the product 15,504 is read off at the bottom and the right. To indicate that other arrangements are possible, a second example is given in **second Fig**, in which the multiplicand 537 is placed at the top, the multiplier 24 is on the right, and the product 12,888 appears to the left and along the bottom. Still other modifications are easily devised. In fundamental principle, gelosia multiplication is, of course, the same as our own, the cell arrangement being merely a convenient device for relieving the mental concentration called for in "carrying over" from place to place the 10s arising in the partial products. The only "carrying" required in lattice multiplication is in the final additions along the diagonals.

POSSIBLE SHORT QUESTIONS:

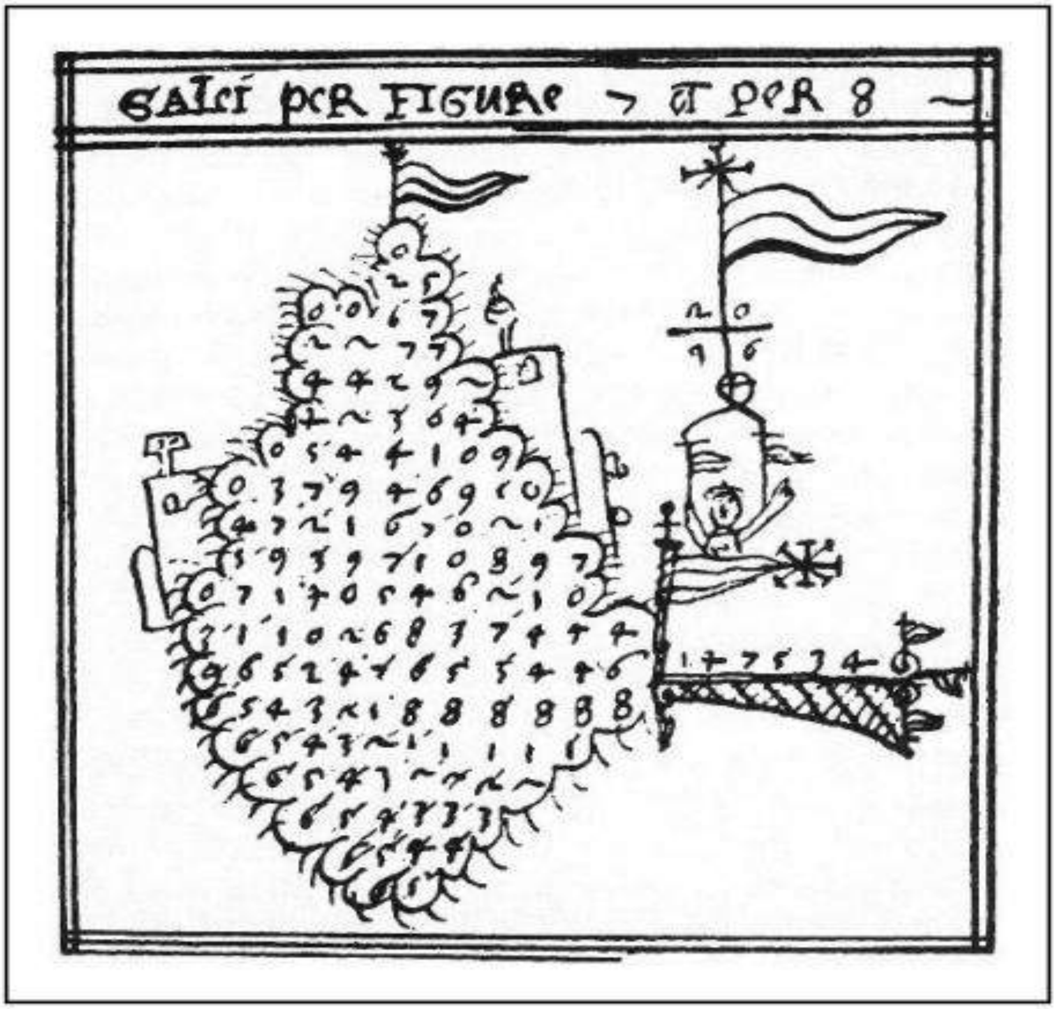
i. Write the writing pattern regarding numbers in indian mathematics.
ii. Discuss few multiplication method in indian mathematics with example.
iii. Which indian multiplication method is same as our own method?

LONG QUESTIONS:

1) Briefly describe about Multiplication concept of indian mathematics.

LONG DIVISION

It is not known when or where gelosia multiplication arose, but India seems to be the most likely source. It was used there at least by the twelfth century, and from India, it seems to have been carried to China and Arabia. From the Arabs, it passed over to Italy in the fourteenth and fifteenth centuries, where the name gelosia was attached to it because of the resemblance to gratings placed on windows in Venice and elsewhere. (The current word "jalousie" seems to stem from the Italian gelosia and is used for Venetian blinds in France, Germany, Holland, and Russia.) The Arabs (and, through them, the later Europeans) appear to have adopted most of their arithmetic devices from the Hindus, so it is likely that the pattern of long division known as the “scratch method” or the “galley method” (from its resemblance to a boat) also came from India. (See the following illustration.)

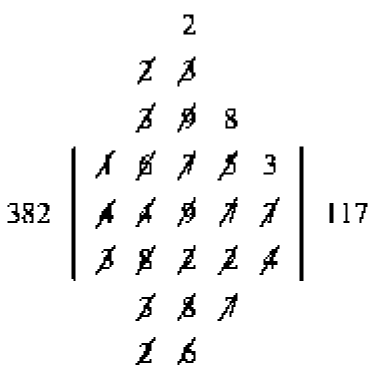


Galley division, sixteenth century. From an unpublished manuscript of a Venetian monk. The title of the work is “Opus Arithmetica D. Honorati veneti monachj coenobij S. Lauretig.” From Mr. Plimpton’s library.

To illustrate the method, let it be required to divide 44,977 by 382. In Fig.

```
          117
382)44977
    382
     677
     382
     2957
     2674
      283
```

we give the modern method, in Fig the galley method. The latter closely parallels the former, except that the dividend appears in the middle, for subtractions are performed by canceling digits and placing differences above, rather than below, the minuends. Hence, the remainder, 283, appears above and to the right, rather than below. The process in Fig. is easy to follow if we note that the digits in a given subtrahend, such as 2,674, or in a given difference, such as 2,957, are not necessarily all in the same row and that subtrahends are written below the middle and differences above the middle. Position in a column is significant, but not position in a row. The determination of roots of numbers probably followed a somewhat similar “galley” pattern, cou pled in the later years with the binomial theorem in “Pascal triangle” form, but Indian writers did not provide explanations for their calculations or proofs for their statements. It is possible that Babylonian and



Chinese influences played a role in the problem of evolution or root extraction. It is often said that the “proof by nines,” or the “casting out of nines,” is a Hindu invention, but it appears that the Greeks knew earlier of this property, without using it extensively, and that the method came into common use only with the Arabs of the eleventh century.

POSSIBLE SHORT QUESTIONS:

i. Note on gelosia multiplication method.
ii. Note on the “scratch method” or the “galley method”.
iii. Divide 44,977 by 382 using Galley method.

LONG QUESTIONS:

1) Briefly describe about Long division concept of indian mathematics.

BRAHMAGUPTA

The last few paragraphs may leave the unwarranted impression that there was a uniformity in Hindu mathematics, for we have frequently localized developments as merely "of Indian origin," without specifying the period. The trouble is that there is a high degree of uncertainty in Hindu chronology. Material in the important Bakshali manuscript, containing an anonymous arithmetic, is supposed by some to date from the third or fourth century, by others from the sixth century, and by others from the eighth or ninth century or later, and there is a suggestion that it may not even be of Hindu origin. We have placed the work of Aryabhata around the year 500 CE, but there were two mathematicians named Aryabhata, and we cannot with certainty ascribe results to our Aryabhata, the elder. Hindu mathematics presents more historical problems than does Greek mathematics, for Indian authors referred to predecessors infrequently, and they exhibited surprising independence in mathematical approach. Thus, it is that Brahmagupta (fl. 628 CE), who lived in Central India somewhat more than a century after Aryabhata, has little in common with his predecessor, who had lived in eastern India. Brahmagupta mentions two values of π—the "practical value" 3 and the "neat value" $\sqrt{10}$ —but not the more accurate value of Aryabhata; in the trigonometry of his best-known work, the Brahmasphuta Siddhanta, he adopted a radius of 3,270, instead of Aryabhata's 3,438. In one respect, he does resemble his predecessor—in the juxtaposition of good and bad results. He found the "gross" area of an isosceles triangle by multiplying half of the base by one of the equal sides; for the scalene triangle with base fourteen and sides thirteen and fifteen, he found the "gross area" by multiplying half of the base by the arithmetic mean of the other sides. In finding the "exact" area, he used the Archimedean-Heronian formula. For the radius of the circle circumscribed about a triangle, he gave the equivalent of the correct trigonometric result $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, but this, of course, is only a reformulation of a result known to Ptolemy in the language of chords. Perhaps the most beautiful result in Brahmagupta's work is the generalization of "Heron's" formula in finding the area of a quadrilateral. This formula,

$$K = \sqrt{(S-a)(S-b)(S-c)(S-d)}$$

where a, b, c, d are the sides and s is the semiperimeter, still bears his name, but the glory of his achievement is dimmed by failure to remark that the formula is correct only in the case of a cyclic quadrilateral. The correct formula for an arbitrary quadrilateral is

$$K = \sqrt{(S-a)(S-b)(S-c)(S-d)} - abcdCos^2 \propto$$

where α is half of the sum of two opposite angles. As a rule for the "gross" area of a quadrilateral, Brahmagupta gave the pre-Hellenic formula, the product of the arithmetic means of the opposite sides. For the quadrilateral with sides a = 25, b = 25, c = 25, d = 39, for example, he found a "gross" area of 800.

POSSIBLE SHORT QUESTIONS:

i. Note on Brahmagupta.
ii. Write few working of Brahmagupta.
iii. Brahmagupta mentions two values of π write that .
iv. Write Brahmagupta area of isosceles triangle as well as scalene triangle.
v. Write Hero ("Heron's") formula.
vi. Write Brahmagupta formula to find area of quadrilateral.
vii. Write The correct formula for an arbitrary quadrilateral also mention the failure of Brahmagupta formula to find area of quadrilateral.

LONG QUESTIONS: Briefly describe about Brahmagupta of indian mathematics.

BRAHMAGUPTA'S FORMULA: Brahmagupta's contributions to algebra are of a higher order than are his rules of mensuration, for here we find general solutions of quadratic equations, including two roots even in cases in which one of them is negative.The systematized arithmetic of negative numbers and zero is, in fact, first found in his work. The equivalents of rules on negative magnitudes were known through the Greek geometric theorems on subtraction, such as $(a - b)(c - d) = ac + bd - ad - bc$, but the Indians converted these into numerical rules on positive and negative numbers. Moreover, although the Greeks had a concept of nothingness, they never interpreted this as a number, as did the Indians. Yet here again Brahmagupta spoiled matters somewhat by asserting that $0 \div 0 = 0$, and on the touchy matter of $a \div 0$, for a ≠0, he did not commit himself:

Positive divided by positive, or negative by negative, is affirmative. Cipher divided by cipher is naught. Positive divided by negative is negative. Negative divided by affirmative is negative. Positive or negative divided by cipher is a fraction with that for denominator.
(Colebrook 1817, Vol. I)

It should also be mentioned that the Hindus, unlike the Greeks, regarded irrational roots of numbers as numbers. This was of enormous help in algebra, and Indian mathematicians have been much praised for taking this step. We have seen the lack of nice distinction on the part of Hindu mathematicians between exact and inexact results, and it was only natural that they should not have taken seriously the difference between commensurable and incommensurable magnitudes. For them, there was no impediment to the acceptance of irrational numbers, and later generations uncritically followed their lead until in the nineteenth-century mathematicians established the real number system on a sound basis.

Indian mathematics was, as we have said, a mixture of good and bad. But some of the good was superlatively good, and here Brahmagupta deserves high praise. Hindu algebra is especially noteworthy in its development of indeterminate analysis, to which Brahmagupta made several contributions. For one thing, in his work we find a rule for the formation of Pythagorean triads expressed in the form ,$\frac{1}{2}\left(\frac{m^2}{n-n}\right)$,$\frac{1}{2}\left(\frac{m^2}{n+n}\right)$, but this is only a modified form of the old Babylonian rule, with which he may have become familiar. Brahmagupta's area formula for a quadrilateral, mentioned previously, was used by him in conjunction with the formulas

$\sqrt{\frac{(ab+cd)(ac+bd)}{(ad+bc)}}$ and $\sqrt{\frac{(ab+cd)(ad+bc)}{(ac+bd)}}$ for the diagonals to find quadrilaterals whose sides, diagonals, and areas are all rational. Among them was the quadrilateral with sides a = 52, b = 25, c = 39, d = 60, and diagonals 63 and 56. Brahmagupta gave the "gross" area as 1; 933$\frac{3}{4}$ despite the fact that his formula provides the exact area, 1,764 in this case.

POSSIBLE SHORT QUESTIONS:

i. Write Brahmagupta's contributions to algebra.
ii. Write Brahmagupta's comment about division.
iii. did indian was a concept of distinction between exact and inexact results, between commensurable and incommensurable magnitudes, between rational and irrational numbers?
iv. Write Brahmagupta's area formula for a quadrilateral.

LONG QUESTIONS: Briefly describe about Brahmagupta's formulae of indian mathematics.

INDETERMINATE EQUATIONS Like many of his countrymen, Brahmagupta evidently loved mathematics for its own sake, for no practical-minded engineer would raise questions such as those Brahmagupta asked about quadrilaterals. One admires his mathematical attitude evenmorewhen one finds that he was apparently the first one to give a general solution of the linear Diophantine equation $ax + by = c$, where a, b, and c are integers. For this equation to have integral solutions, the greatest common divisor of a and b must divide c, and Brahmagupta knew that if a and b are relatively prime, all solutions of the equation are given by $x = p + mb, y = q - ma$, where m is an arbitrary integer.He also suggested theDiophantine quadratic equation $x^2 = 1 + py^2$, which was mistakenly named for John Pell (1611 1685) but first appeared in the Archimedean cattle problem. The Pell equation was solved for some cases by Brahmagupta's countryman Bhaskara (1114 ca. 1185). It is greatly to the credit of Brahmagupta that he gave all integral solutions of the Linear Diophantine equation,whereas Diophantus himself had been satisfied to give one particular solution of an indeterminate equation. In as much as Brahmagupta used some of the same examples as Diophantus, we see again the likelihood of Greek influence in India—or the possibility that they both made use of a common source, possibly from Babylonia. It is also interesting to note that the algebra of Brahmagupta, like that of Diophantus, was syncopated. Addition was indicated by juxtaposition, subtraction by placing a dot over the subtrahend, and division by placing the divisor below the dividend, as in our fractional notation but without the bar. The operations of multiplication and evolution (the taking of roots), as well as unknown quantities, were represented by abbreviations of appropriate words.

POSSIBLE SHORT QUESTIONS:

i. Who was the first one to give a general solution of the linear Diophantine equation?
ii. Write the Pell equation. And who tried to solve it?
iii. Write Brahmagupta's algebraic operations.

LONG QUESTIONS: Briefly note on intermediate equations in indian mathematics.

MANTRAS WORKING (POWER OF TEN) Mantras from early Vedic period (before 1000 BCE) invoke power of ten from a hundred all the way upto a trillion and provide use of arithmetic operations i.e. addition, subtraction, multiplication etc. a 4^{th} century CE Sansikrit text report Buddha enumerating numbers upto 10^{53} over and above these values upto number 10^{421}. They also estimated that there are 10^{80} atoms in univers.

BHASKARA

Bhaskara II, who lived in the 12^{th} century, was one of the most accomplished of all Indians. He explained the operation of division by zero. He noticed that dividing 1 into two pieces yiedls a half, so $1 \div \frac{1}{2} = 2$ etc. so dividing 1 by smaller and smaller numbers (fractions) yield a large number. Ultimately dividing 1 into pieces of zero size yields infinite many pieces, indicating that $1 \div 0 = \infty$. Bhaskara also contributed to solution of quadratic, cubic equations. He also work on analysis of spherical trigonometry.

India produced a number of later medieval mathematicians, but we shall describe the work of only one of these—Bhaskara, the leading mathematician of the twelfth century. It was he who filled some of the gaps in Brahmagupta's work, as by giving a general solution of the Pell equation and by considering the problem of division by zero. Aristotle had once remarked that there is no ratio by which a number such as 4 exceeds the number zero, but the arithmetic of zero had not been part of Greek mathematics, and Brahmagupta had been noncommittal on the division of a number other than zero by the number zero. It is therefore in Bhaskara's Vija-Ganita that we find the first statement that such a quotient is infinite.

Statement: *Dividend 3. Divisor 0. Quotient the fraction 3/0. This fraction of which the denominator is cipher, is termed an infinite quantity. In this quantity consisting of that which has cipher for a divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God.*

This statement sounds promising, but a lack of clear understanding of the situation is suggested by Bhaskara's further assertion that $\frac{a}{0} .0 = a$.

Bhaskara was one of the last significant medieval mathematicians from India, and his work represents the culmination of earlier Hindu contributions. In his best-known treatise, the Lilavati, he compiled problems from Brahmagupta and others, adding new observations of his own. The very title of this book may be taken to indicate the uneven quality of Indian thought, for the name in the title is that of Bhaskara's daughter, who, according to legend, lost the opportunity to marry because of her father's confidence in his astrological predictions. Bhaskara had calculated that his daughter might propitiously marry only at one particular hour on a given day. On what was to have been her wedding day, the eager girl was bending over the water clock, as the hour for the marriage approached, when a pearl from her headdress fell, quite unnoticed, and stopped the outflow of water. Before the mishap was noted, the propitious hour had passed. To console the unhappy girl, the father gave her name to the book we are describing.

POSSIBLE SHORT QUESTIONS:

i. Note on Bhaskara and his workings.
ii. Write Bhaskara's contribution in indian mathematics.
iii. Write the time period of Bhaskara in indian mathematics.
iv. Write the concept of indians (Bhaskara's) regarding divisor as zero in division.
v. Who was the Lilavatti?
vi. Write the story described in legend relating to Lilavatti.
vii. Write the name of Bhaskara's book.

LONG QUESTIONS:

Briefly describe about Bhaskara of indian mathematics.

THE LILAVATI

The Lilavati, like the Vija-Ganita, contains numerous problems dealing with favorite Hindu topics: linear and quadratic equations, both determinate and indeterminate; simple mensuration; arithmetic and geometric progressions; surds; Pythagorean triads; and others. The "broken bamboo" problem, popular in China (and also included by Brahmagupta), appears in the following form: if a bamboo 32 cubits high is broken by the wind so that the tip meets the ground 16 cubits from the base, at what height above the ground was it broken? Also making use of the Pythagorean theorem is the following problem: A peacock is perched a top a pillar at the base of which is a snake's hole. Seeing the snake at a distance from the pillar, which is three times the height of the pillar, the peacock pounces on the snake in a straight line before it can reach its hole. If the peacock and the snake have gone equal distances, how many cubits from the hole do they meet?

These two problems well illustrate the heterogeneous nature of the Lilavati, for despite their apparent similarity and the fact that only a single answer is required, one of the problems is determinate and the other is indeterminate. In the treating of the circle and the sphere, the Lilavati also fails to distinguish between exact and approximate statements. The area of the circle is correctly given as one-quarter the circumference multiplied by the diameter and the volume of the sphere as one-sixth the product of the surface area and the diameter, but for the ratio of circumference to diameter in a circle, Bhaskara suggests either 3,927 to 1,250 or the "gross" value $\frac{22}{7}$. The former is equivalent to the ratio mentioned, but not used, by Aryabhata. There is no hint in Bhaskara or other Hindu writers that they were aware that all ratios that had been proposed were only approximations. Yet, Bhaskara severely condemns his predecessors for using the formulas of Brahmagupta for the area and the diagonals of a general quadrilateral, because he saw that a quadrilateral is not uniquely determined by its sides. Evidently, he did not realize that the formulas are indeed exact for all cyclic quadrilaterals.

Many of Bhaskara's problems in the Lilavati and the Vija-Ganita were evidently derived from earlier Hindu sources; hence, it is no surprise to note that the author is at his best in dealing with indeterminate analysis. In connection with the Pell equation, $x^2 = 1 + py^2$, proposed earlier by Brahmagupta, Bhaskara gave particular solutions for the five cases p = 8, 11, 32, 61, and 67. For $x^2 = 1 + 61y^2$, for example, he gave the solution x = 1,776,319,049 and $y = 22{,}615{,}390$. This is an impressive feat in calculation, and its verification aLONG will tax the efforts of the reader.
Bhaskara's books are replete with other instances of Diophantine problems.

POSSIBLE SHORT QUESTIONS:

i. Write the topics discussed in Lilavatti.
ii. Describe The "broken bamboo" problem.
iii. Write the defect of Lilavatti.
iv. Name two books of old indian mathematics.

LONG QUESTIONS:

1) Briefly note on Lilavatti of indian mathematics.

MADHAVA AND THE KERALESE SCHOOL

Beginning in the late fourteenth century, a group of mathematicians emerged along the southwestern coast of India and came to be known as members of the "Keralese School," named after their geographic location of Kerala. The group appears to have started under the leadership of **Madhava**, who is best known for his expansion of the power series for sines and cosines that is usually named after Newton and the series for $\frac{\pi}{4}$ credited to Leibniz. Among his other contributions are a computation of π that is accurate to eleven decimal places, computation of the circumference of a circle using polygons, and expansion of the arctangent series usually attributed to James Gregory, as well as various other series expansions and astronomical applications.

Few of Madhava's original verses have been documented; most of his work has come down to us through descriptions and references by his students and other later members of the Keralese school. The Keralese school, with its astonishing achievements in series expansions and geometric, arithmetic, and trigonometric procedures, as well as astronomical observations, has inspired considerable speculation concerning transmission and influence. Until now, there is inadequate documentation to support any of the related major conjectures. There is, however, a great deal to be learned from recent translations of these and prior texts. (We have given only a few examples of results usually associated with the seventeenth-century giants of western Europe. For samples of translations providing a closer appreciation of the nature of the mathematical issues found in the ancient and medieval Sanskrit texts, the reader is referred to Plofker 2009.)

POSSIBLE SHORT QUESTIONS:

i. Note on Keralese School. Also shortly describe about Madhava.

LONG QUESTIONS: Briefly describe about Madhava And The Keralese School.

SUMMARY ABOUT THE HINDOOS. (FROM A HISTORY OF MATHEMATICS BY FLORIAN CAJORI)

The first people who distinguished themselves in mathematical research, after the time of the ancient Greeks, belonged, like them, to the Aryan race. It was, however, not a European, but an Asiatic nation, and had its seat in far-off India.

Unlike the Greek, Indian society was fixed into castes. The only castes enjoying the privilege and leisure for advanced study and thinking were the Brahmins, whose prime business was religion and philosophy, and the **Kshatriyas**, who attended to war and government.

Very striking was the difference in the bent of mind of the Hindoo and Greek; for, while the Greek mind was pre-eminently geometrical, the Indian was first of all arithmetical. The Hindoo dealt with number, the Greek with form. Numerical symbolism, the science of numbers, and algebra attained in India far greater perfection than they had previously reached in Greece. On the other hand, we believe that there was little or no geometry in India of which the source may not be traced back to Greece. Hindoo trigonometry might possibly be mentioned as an exception, but it rested on arithmetic more than on geometry.

This is shown plainly by the Greek origin of some of the technical terms used by the Hindoos. Hindoo astronomy was influenced by Greek astronomy. Most of the geometrical knowledge which they possessed is traceable to Alexandria and to the writings of Heron in particular. In algebra there was, probably, a mutual giving and receiving. We suspect that Diophantus got the first glimpses of algebraic knowledge from India. On the other hand, evidences have been found of Greek algebra among the Brahmins. The earliest knowledge of algebra in India may possibly have been of Babylonian origin. When we consider that Hindoo scientists looked upon arithmetic and algebra merely as tools useful in astronomical research, there appears deep irony in the fact that these secondary branches were after all the only ones in which they won real distinction, while in their pet science of astronomy they displayed an inaptitude to observe, to collect facts, and to make inductive investigations.

We shall now proceed to enumerate **the names of the leading Hindoo mathematicians**, and then to review briefly Indian mathematics. We shall consider the science only in its complete state, for our data are not sufficient to trace the history of the development of methods. Of the great Indian mathematicians, or rather, astronomers, for India had no mathematicians proper, **Aryabhatta** is the earliest. He was born 476 A.D., at **Pataliputra**, on the upper Ganges. His celebrity rests on a work entitled Aryabhattiyam, of which the third chapter is devoted to mathematics. About one hundred years later, mathematics in India reached the highest mark. At that time flourished **Brahmagupta** (born 598). In 628 he wrote his **Brahma-sphuta-siddhanta ("**The Revised System of Brahma"), of which the twelfth and eighteenth chapters belong to mathematics. To the fourth or fifth century belongs an anonymous astronomical work, called **Surya-siddhanta** (“Knowledge from the Sun"), which by native authorities was ranked second only to the Brahma-siddhanta, but is of interest to us merely as furnishing evidence that Greek science influenced Indian science even before the time of Aryabhatta. The following centuries produced only two names of importance; namely, **Cridhara**,who wrote a **Ganita-sara** (“Quintessence of Calculation"),and **Padmanabha**, the author of an algebra. The science seems to have made but little progress at this time; for a work entitled **Siddhantaciromani** (“Diadem of an Astronomical System"), written by **Bhaskara Acarya** in 1150, stands little higher than that of Brahmagupta, written over 500 years earlier. The two most important mathematical chapters in this work are the **Lilavati** (“the beautiful," i.e. the noble science) and **Viga-ganita** (“root-extraction"), devoted to arithmetic and algebra. From now on, the Hindoos in the Brahmin schools seemed to content themselves with studying the masterpieces of their predecessors. Scientific intelligence decreases continually, and in modern times a very deficient Arabic work of the sixteenth century has been held in great authority.

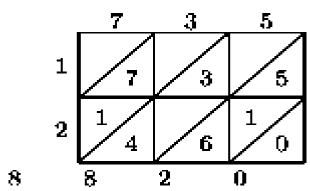
The mathematical chapters of the Brahma-siddhanta and Siddhantaciromani were translated into English by H. T. Colebrooke, London, 1817. The Surya-siddhanta was translated by E. Burgess, and annotated by W. D. Whitney, New Haven, Conn., 1860.

The grandest achievement of the Hindoos and the one which, of all mathematical inventions, has contributed most to the general progress of intelligence, is the invention of the principle of position in writing numbers. Generally we speak of our notation as the”Arabic" notation, but it should be called the “Hindoo" notation, for the Arabs borrowed it from the Hindoos. That the invention of this notation was not so easy as we might suppose at first thought, may be inferred from the fact that, of other nations, not even the keen-minded Greeks possessed one like it. We inquire, who invented this ideal symbolism, and when? But we know neither the inventor nor the time of invention. That our system of notation is of Indian origin is the only point of which we are certain.The nine figures for writing the units are supposed to have been introduced earliest, and the sign of zero and the principle of position to be of later origin. This view receives support from the fact that on the island of Ceylon a notation resembling the Hindoo, but without the zero has been preserved. We know that Buddhism and Indian culture were transplanted to Ceylon about the third century after Christ, and that this culture remained stationary there, while it made progress on the continent. It seems highly probable, then, that the numerals of Ceylon are the old, imperfect numerals of India. In Ceylon, nine figures were used for the units, nine others for the tens, one for 100, and also one for 1000. These 20 characters enabled them to write all the numbers up to 9999. Thus, 8725 would have been written with six signs, representing the following numbers: 8, 1000, 7, 100, 20, 5. These **Singhalesian** signs, like the old Hindoo numerals, are supposed originally to have been the initial letters of the corresponding numeral adjectives. There is a marked resemblance between the notation of Ceylon and the one used by Aryabhatta in the first chapter of his work, and there only. Although the zero and the principle of position were unknown to the scholars of Ceylon, they were probably known to Aryabhatta; for, in the second chapter, he gives directions for extracting the square and cube roots, which seem to indicate a knowledge of them. It would appear that the zero and the accompanying principle of position were introduced about the time of Aryabhatta. These are the inventions which give the Hindoo system its great superiority, its admirable perfection.

There appear to have been several notations in use in different parts of India, which differed, not in principle, but merely in the forms of the signs employed. Of interest is also a symbolical system of position, in which the figures generally were not expressed by numerical adjectives, but by objects suggesting the particular numbers in question. Thus, for 1 were used the words moon, Brahma, Creator, or form; for 4, the words **Veda**, (because it is divided into four parts) or ocean, etc. The following example, taken from the Suryasiddhanta, illustrates the idea. The number 1; 577; 917; 828 is expressed from right to left as follows: **Vasu** (a class of 8 gods) + two + eight + mountains (the 7 mountain-chains) + form + digits (the 9 digits) + seven + mountains + lunar days (half of which equal 15). The use of such notations made it possible to represent a number in several different ways. This greatly facilitated the framing of verses containing arithmetical rules or scientific constants, which could thus be more easily remembered.

At an early period the Hindoos exhibited great skill in calculating, even with large numbers. Thus, they tell us of an examination to which Buddha, the reformer of the Indian religion, had to submit, when a youth, in order to win the maiden he loved. In arithmetic, after having astonished his examiners by naming all the periods of numbers up to the 53d, he was asked whether he could determine the number of primary atoms which, when placed one against the other, would form a line one mile in length. Buddha found the required answer in this way: 7 primary atoms make a very minute grain of dust, 7 of these make a minute grain of dust, 7 of these a grain of dust whirled up by the wind, and so on. Thus he proceeded, step by step, until he finally reached the length of a mile. The multiplication of all the factors gave for the multitude of primary atoms in a mile a number consisting of 15 digits. This problem reminds one of the `Sand-Counter' of Archimedes.

After the numerical symbolism had been perfected, figuring was made much easier. Many of the Indian modes of operation differ from ours. The Hindoos were generally inclined to follow the motion from left to right, as in writing. Thus, they **added** the left-hand columns first, and made the necessary corrections as they proceeded. For instance, they would have added 254 and 663 thus: 2 + 6 = 8, 5 + 6 = 11, which changes 8 into 9, 4 + 3 = 7. Hence the sum 917. In **subtraction** they had two methods. Thus in 821   348 they would say, 8 from 11 = 3, 4 from 11 = 7, 3 from 7 = 4. Or they would say, 8 from 11 = 3, 5 from 12 = 7, 4 from 8 = 4. In multiplication of a number by another of only one digit, say 569 by 5, they generally said, 55 = 25, 56 = 30, which changes 25 into 28, 59 = 45, hence the 0 must be increased by 4. The product is 2845. In the multiplication with each other of many-figured numbers, they first multiplied, in the manner just indicated, with the left-hand digit of the multiplier, which was written above the multiplicand, and placed the product above the multiplier. On multiplying with the next digit of the multiplier, the product was not placed in a new row, as with us, but the first product obtained was corrected, as the process continued, by erasing, whenever necessary, the old digits, and replacing them by new ones, until finally the whole product was obtained. We who possess the modern luxuries of pencil and paper, would not be likely to fall in love with this Hindoo method. But the Indians wrote “with a cane-pen upon a small blackboard with a white, thinly liquid paint which made marks that could be easily erased, or upon a white tablet, less than a foot square, strewn with red our, on which they wrote the figures with a small stick, so that the figures appeared white on a red ground." Since the digits had to be quite large to be distinctly legible, and since the boards were small, it was desirable to have a method which would not require much space. Such a one was the above method of multiplication. Figures could be easily erased and replaced by others without sacrificing neatness. But the Hindoos had also other ways of multiplying, of which we mention the following: The tablet was divided into squares like a chess-board. Diagonals were also drawn, as



seen in the figure.

The multiplication of 12 × 735 = 8820 is exhibited in the adjoining diagram. The manuscripts extant give no information of how divisions were executed. The correctness of their additions, subtractions, and multiplications was tested “by excess of 9's." In writing fractions, the numerator was placed above the denominator, but no line was drawn between them.

We shall now proceed to the consideration of **some arithmetical problems** and the Indian modes of solution. A favorite method was that of **inversion**. With laconic brevity, Aryabhatta describes it thus: “Multiplication becomes division, division becomes multiplication; what was gain becomes loss, what loss, gain; inversion." Quite different from this quotation in style is the following problem from Aryabhatta, which illustrates the method: “Beautiful maiden with beaming eyes, tell me, as thou understandst the right method of inversion, which is the number which multiplied by 3, then increased by $\frac{3}{4}$ of the product, divided by 7, diminished by $\frac{1}{3}$ of the quotient, multiplied by itself, diminished by 52, the square root extracted, addition of 8, and division by 10, gives the number 2?" The process consists in beginning with 2 and working backwards. Thus, $(2.10 - 8)^2 + 52 = 196$, $\sqrt{196} = 14$, and $14.\frac{3}{2}\ 7.\frac{4}{7}\ \div 3 = 28$, the answer.

Here is another example taken from Lilavati, a chapter in Bhaskara's great work:"The square root of half the number of bees in a swarm has own out upon a jessamine-bush, $\frac{8}{9}$ of the whole swarm has remained behind; one female bee flies about a male that is buzzing within a lotus-ower into which he was allured in the night by its sweet odour, but is now imprisoned in it. Tell me the number of bees." Answer, 72.

The pleasing poetic garb in which all arithmetical problems are clothed is due to the Indian practice of writing all school- books in verse, and especially to the fact that these problems, propounded as puzzles, were a favourite social amusement. Says Brahmagupta: "These problems are proposed simply for pleasure; the wise man can invent a thousand others, or he can solve the problems of others by the rules given here. As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems, and still more if he solves them."

The Hindoos solved problems in interest, discount, partnership, alligation, summation of arithmetical and geometric series, devised rules for determining the numbers of combinations and permutations, and invented magic squares. It may here be added that chess, the profoundest of all games, had its origin in India.

The Hindoos made frequent use of the "rule of three," and also of the method of "falsa positio," which is almost identical with that of the "tentative assumption" of Diophantus. These and other rules were applied to a large number of problems.

Passing now to **algebra**, we shall first take up the **symbols of operation.** Addition was indicated simply by juxtaposition as in Diophantine algebra; subtraction, by placing a dot over the subtrahend; multiplication, by putting after the factors, bha, the abbreviation of the word bhavita, "the product"; division, by placing the divisor beneath the dividend; squareroot, by writing ka, from the word karana (irrational), before the quantity. The unknown quantity was called by Brahmagupta yavattavat (quantum tantum). When several unknown quantities occurred, he gave, unlike Diophantus, to each a distinct name and symbol. The first unknown was designated by the general term "unknown quantity." The rest were distinguished by names of colours, as the black, blue, yellow, red, or green unknown. The initial syllable of each word constituted the symbol for the respective unknown quantity.
Thus y$\hat{a}$ meant x; k$\hat{a}$ (from kalaka = black)meant y; y$\hat{a}$k$\hat{a}$bha, "x times y"; ka 15 ka 10,
: $\sqrt{15} - \sqrt{10}$."

<u>The Indians were the first to recognise the existence of absolutely negative quantities</u>. They brought out the difference between positive and negative quantities by attaching to the one the idea of `possession,' to the other that of `debts.' The conception also of opposite directions on a line, as an interpretation of + and   quantities, was not foreign to them. They advanced beyond Diophantus in observing that a quadratic has always two roots. Thus Bhaskara gives $x = 50$ and $x =\ 5$ for the roots of $x^2 - 45x = 250$. “But," says he, “the second value is in this case not to be taken, for it is inadequate; people do not approve of negative roots." Commentators speak of this as if negative roots were seen, but not admitted.

Another important generalisation, says Hankel, was this, that the Hindoos never connected their arithmetical operations to rational numbers. For instance, Bhaskara showed

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

how, by the formula

the square root of the sum of rational and irrational numbers could be found. The Hindoos never discerned the dividing line between numbers and magnitudes, set up by the Greeks, which, though the product of a scientific spirit, greatly retarded the progress of mathematics. They passed from magnitudes to numbers and from numbers to magnitudes without anticipating that gap which to a sharply discriminating mind exists between the continuous and discontinuous. Yet by doing so the Indians greatly aided the general progress of mathematics. “Indeed, if one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space magnitudes, then the learned Brahmins of Hindostan are the real inventors of algebra."

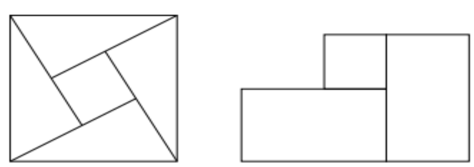
Let us now examine more closely the Indian algebra. In extracting the square and cube roots they used the formulas $(a+b)^2 = a^2 + 2ab + b^2$ and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. In this connection Aryabhatta speaks of dividing a number into periods of two and three digits. From this we infer that the principle of position and the zero in the numeral notation were already known to him. In figuring with zeros, a statement of Bhaskara is interesting. A fraction whose denominator is zero, says he, admits of no alteration, though much be added or subtracted. Indeed, in the same way, no change takes place in the in finite and immutable Deity when worlds are destroyed or created, even though numerous orders of beings be taken up or brought forth. Though in this he apparently evinces clear mathematical notions, yet in other places he makes a complete failure in figuring with fractions of zero denominator.

In the Hindoo solutions of determinate equations, Cantor thinks he can see traces of Diophantine methods. Some technical terms betray their Greek origin. Even if it be true that the Indians borrowed from the Greeks, they deserve great credit for improving and generalising the solutions of linear and quadratic equations. Bhaskara advances far beyond the Greeks and even beyond Brahmagupta when he says that "the square of a positive, as also of a negative number, is positive; that the square root of a positive number is twofold, positive and negative. There is no square root of a negative number, for it is not a square." Of equations of higher degrees, the Indians succeeded in solving only some special cases in which both sides of the equation could be made perfect powers by the addition of certain terms to each.

Incomparably greater progress than in the solution of determinate equations was made by the Hindoos in the treatment of indeterminate equations. Indeterminate analysis was a subject to which the Hindoo mind showed a happy adaptation. We have seen that this very subject was a favourite with Diophantus, and that his ingenuity was almost inexhaustible in devising solutions for particular cases. But the glory of having invented general methods in this most subtle branch of mathematics belongs to the Indians. The Hindoo indeterminate analysis differs from the Greek not only in method, but also in aim. The object of the former was to find all possible integral solutions. Greek analysis, on the other hand, demanded not necessarily integral, but simply rational answers. Diophantus was content with a single solution; the Hindoos endeavoured to find all solutions possible. Aryabhatta gives solutions in integers to linear equations of the form $ax + by = c$, where a, b, c are integers. The rule employed is called the pulveriser. For this, as for most other rules, the Indians give no proof. Their solution is essentially the same as the one of Euler. Euler's process of reducing $\frac{a}{b}$ to a continued fraction amounts to the same as the Hindoo process of finding the greatest common divisor of a and b by division. This is frequently called the Diophantine method. Hankel protests against this name, on the ground that Diophantus not only never knew the method, but did not even aim at solutions purely integral. These equations probably grew out of problems in astronomy. They were applied, for instance, to determine the time when a certain constellation of the planets would occur in the heavens. Passing by the subject of linear equations with more than two unknown quantities, we come to indeterminate quadratic equations. In the solution of $xy = ax+by+c$, they applied the method re-invented later by Euler, of decomposing $(ab + c)$ into the product of two integers mn and of placing $x = m + b$ and $y = n + a$.

Remarkable is the Hindoo solution of the quadratic equation $cy^2 = ax^2+b$. With great keenness of intellect they recognized in the special case $y^2 = ax^2 + 1$ a fundamental problem in indeterminate quadratics. They solved it by the cyclic method. "It consists," says De Morgan, "in a rule for finding an indefinite number of solutions of $y^2 = ax^2 + 1$ (a being an integer which is not a square), by means of one solution given or found, and of feeling for one solution by making a solution of $y^2 = ax^2 + b$ give a solution of $y^2 = ax^2 + b^2$. It amounts to the following theorem: If p and q be one set of values of x and y in $y^2 = ax_2 + b$ and p' and q' the same or another set, then qp + pq' and pp' + qq' are values of x and y in $y^2 = ax^2 + b^2$. From this it is obvious that one solution of $y^2 = ax^2 + 1$ may be made to give any number, and that if, taking b at pleasure, $y^2 = ax^2 + b^2$ can be solved so that x and y are divisible by b, then one preliminary solution of $y^2 = ax^2 + 1$ can be found. Another mode of trying for solutions is a combination of the preceding with the **cuttaca (pulveriser)**." These calculations were used in astronomy.

Hindoo geometry is far inferior to the Greek. In it are found no definitions, no postulates, no axioms, no logical chain of reasoning or rigid form of demonstration, as with Euclid. Each theorem stands by itself as an independent truth. Like the early Egyptian, it is empirical. Thus, in the proof of the theorem of the right triangle, Bhaskara draws the right triangle four times in the square of the hypotenuse, so that in the middle there remains a square whose side equals the difference between the two sides of the right triangl



Arranging this square and the four triangles in a different way, they are seen, together, to make up the sum of the square of the two sides. "Behold!" says Bhaskara, without adding another word of explanation. Bretschneider conjectures that the Pythagorean proof was substantially the same as this. In another place, Bhaskara gives a second demonstration of this theorem by drawing from the vertex of the right angle a perpendicular to the hypotenuse, and comparing the two triangles thus obtained with the given triangle to which they are similar. This proof was unknown in Europe till Wallis rediscovered it. The Brahmins never inquired into the properties of figures. They considered only metrical relations applicable in practical life. In the Greek sense, the Brahmins never had a science of geometry. Of interest is the formula given by Brahmagupta for the area of a triangle in terms of its sides. In the great work attributed to Heron the Elder this formula is first found. Whether the Indians themselves invented it, or whether they borrowed it from Heron, is a disputed question. Several theorems are given by Brahmagupta on quadrilaterals which are true only of those which can be inscribed on a circle, a limitation which he omits to state. Among these is the proposition of Ptolemus, that the product of the diagonals is equal to the sum of the products of the opposite sides. The Hindoos were familiar with the calculation of the areas of circles and their segments, of the length of chords and perimeters of regular inscribed polygons. An old Indian tradition makes $\pi = 3$, also $= \sqrt{10}$; but Aryabhatta gives the value $\frac{31416}{10000}$. **Bhaskara gives two values, the "accurate" $\frac{3927}{1250}$, and the "inaccurate" Archimedean value, $\frac{22}{7}$.**

Greater taste than for geometry was shown by the Hindoos for **trigonometry**. Like the Babylonians and Greeks, they divided the circle into quadrants, each quadrant into 90 degrees and 5400 minutes. The whole circle was therefore made up of 21; 600 equal parts. From Bhaskara's `accurate' value for π it was found that the radius contained 3438 of these circular parts. This last step was not Grecian. The Greeks might have had scruples about taking a part of a curve as the measure of a straight line. Each quadrant was divided into 24 equal parts, so that each part embraced 225 units of the whole circumference, and corresponds to $3\frac{3}{4}$ degrees. Notable is the fact that the Indians never reckoned, like the Greeks, with the whole chord of double the arc, but always with the sine (joa) and versed sine. Their mode of calculating tables was theoretically very simple. The sine of 90^0 was equal to the radius, or 3438; the sine of 30^0 was evidently half that, or 1719. Applying the formula $\sin^2 a + \cos^2 a = r^2$, they obtained $\sin 45^0 = \sqrt{\frac{r^2}{2}}$ = 2431. Substituting for cos a its equal sin(90 - a), and making a = 60 , they obtained $\sin 60^0 = \frac{\sqrt{3r^2}}{2}$ = 2978. With the sines of 90, 60, 45, and 30 as starting-points, they reckoned the sines of half the angles by the formula ver sin2a = 2 $\sin^2$ a, thus obtaining the sines of $22^0\,30'$, $11^0 15'$, $7^0\,30'$, $3^0 45'$. They now figured out the sines of the complements of these angles, namely, the sines of $86^0\,15'$, $82^0\,30'$, $78^0\,45'$, 75^0 , $67^0\,30'$; then they calculated the sines of half these angles; then of their complements; then, again, of half their complements; and so on. By this very simple process they got the sines of angles at intervals of $3^0\,45'$. In this table they discovered the unique law that if a, b, c be three successive arcs such that a - b = b - c = $3^0\,45'$, then sin a - sin b = (sin b - sin c) - $\frac{Sinb}{225}$. This formula was afterwards used whenever a re-calculation of tables had to be made. No Indian trigonometrical treatise on the triangle is extant. In astronomy they solved plane and spherical right triangles.

It is remarkable to what extent Indian mathematics enters into the science of our time. Both the form and the spirit of the arithmetic and algebra of modern times are essentially Indian and not Grecian. Think of that most perfect of mathematical symbolisms _ the Hindoo notation, think of the Indian arithmetical operations nearly as perfect as our own, think of their elegant algebraical methods, and then judge whether the Brahmins on the banks of the Ganges are not entitled to some credit. Unfortunately, some of the most brilliant of Hindoo discoveries in indeterminate analysis reached Europe too late to exert the influence they would have exerted, had they come two or three centuries earlier.

THE ISLAMIC HEGEMONY

Ah, but my Computations, People say, Have squared the Year to human Compass, eh?
If so, by striking from the Calendar Unborn To-morrow, and dead Yesterday.
Omar Khayyam (Rubaiyat in the FitzGerald version)

ARABIC CONQUESTS

One of the most transformative developments affecting mathematics in the Middle Ages was the remarkable spread of Islam.Within one century from 622 CE, the year of the Prophet Muhammad's (S.A.W) Hegira, Islam had expanded from Arabia to Persia, to North Africa, and to Spain.At the time that Brahmagupta was writing, the Sabean Empire of Arabia Felix had fallen, and the peninsula was in a severe crisis. It was inhabited largely by desert nomads, known as Bedouins, who could neither read nor write. Among them was the prophet Muhammad (S.A.W), born in Macca in about 570. During his journeys, Muhammed (S.A.W) came in contact with Jews and Christians, and the amalgam of religious feelings that were raised in his mind led to the belief that he was the apostle of God sent to lead his people. For some ten years, he preached at Macca but in 622, faced by a plot on his life, he accepted an invitation to Madina. This "flight," known as the Hegira, marked the beginning of the Muhammadan era—one that was to exert a strong influence on the development of mathematics. Muhammad (S.A.W) now became a military, as well as a religious, leader. Ten years later, he had established a Muhammadan state, with its center at Macca, within which Jews and Christians, being also mono-theistic, were afforded protection and freedom of worship.Being a Muslim state and Civilization, mathematics was need of the day. In muslim state there was a need to measure the Ushra, Zakat, Ghanaem, dayt, legacy,ownership etc so in this way mathematics was a religios need for the muslims. This is also a need of the present day for the muslims. In 632, while planning to move against the Byzantine Empire, Muhammad (S.A.W) died in Madina. His sudden death in no way impeded the expansion of the Islamic state, for his followers overran neighboring territories with astonishing rapidity. Within a few years, Damascus and Jerusalem and much of the Mesopotamian Valley fell to the conquerors; by 641, Alexandria, which for many years had been the mathematical center of the world, was captured. As happens so often in these conquests, the books in the library were burned. The extent of the damage done at that time is unclear; it has been assumed that following depredations by earlier military and religious fanatics and long ages of sheer neglect, there may have been relatively few books left to fuel the flames in the library that had once been the greatest in the world.

For more than a century, the Arab conquerors fought among themselves and with their enemies, until by about 750 the warlike spirit subsided. By this time, a schism had arisen between the western Arabs in Morocco and the eastern Arabs, who, under the caliph al-Mansur, had established a new capital at Baghdad, a city that was shortly to become the new center for mathematics. Yet the caliph at Baghdad could not even command the allegiance of all Moslems in the eastern half of his empire, although his name appeared on coins of the realm and was included in the prayers of his "subjects." The unity of the Arab world, in other words, was more economic and religious than it was political. Arabic was not necessarily the common language, although it was a kind of lingua franca for intellectuals. Hence, it may be more appropriate to speak of the culture as Islamic, rather than Arabic, although we shall use the terms more or less interchangeably.

During the first century of the Arabic conquests, there had been political and intellectual confusion, and possibly this accounts for the difficulty in localizing the origin of the modern system of numeration. The Arabs were at first without known intellectual interest, and they had little culture, beyond a language, to impose on the peoples they conquered. In this respect, we see a repetition of the situation when Rome conquered Greece, of which it was said that in a cultural sense, captive Greece took captive the captor Rome. By about 750 CE,the Arabs were ready to have history repeat itself, for the conquerors became eager to absorb the learning of the civilizations they had overrun. We learn that by the 770s, an astronomical-mathematical work known to the Arabs as **the Sindhind** was brought to Baghdad. A few years later, perhaps about 775, this Siddhanta was translated into Arabic, and it was not long afterward (ca. 780) that Ptolemy's astrological Tetrabiblos was translated into Arabic from the Greek. Alchemy and astrology were among the first studies to appeal to the dawning intellectual interests of the conquerors. The "Arabic miracle" lies not so much in the rapidity with which the political empire rose, as in the alacrity with which, their tastes once aroused, the Arabs absorbed the learning of their neighbors.

POSSIBLE SHORT QUESTIONS:

i. What were the need of mathematics for Muslims on religious basis?

THE HOUSE OF WISDOM (BAIT AL-HIKMA)

The first century of the Muslim Empire had been devoid of scientific achievement. To Baghdad at that time were called scholars from Syria, Iran, and Mesopotamia, including Jews and Nestorian Christians; under three great Abbasid patrons of learning—al-Mansur, Haroun al-Rasheed, and al-Mamun—the city became a new Alexandria. During the reign of the second of these caliphs, familiar to us today through the Arabian Nights, part of Euclid was translated. It was during the caliphate of al-Mamun (809 - 833), however, that the Arabs fully indulged their passion for translation. The caliph is said to have had a dream in which Aristotle appeared, and as a consequence al-Mamun determined to have Arabic versions made of all of the Greek works he could lay his hands on, including Ptolemy's Almagest and a complete version of Euclid's Elements. From the Byzantine Empire, with which the Arabs maintained an uneasy peace, Greek manuscripts were obtained through treaties. Al-Mamun established in Baghdad a "House of Wisdom" (Bait al-hikma) comparable to the ancient Museum in Alexandria. Major emphasis from its beginning was placed on translations, initially from Persian to Arabic, later from Sanskrit and Greek.Gradually, the House of Wisdom included a collection of ancient manuscripts, obtained largely from Byzantine sources. Finally, an observatory was added to the institutional holdings. Among the mathematicians and astronomers there, we note Mohammed ibn Musa al-Khwarizmi, whose name, like that of Euclid, was later to become a household word in Western Europe. Others active in the ninth century of translation were the brothers Banu Musa, al Kindi, and Thabit ibn Qurra. By the thirteenth century, during the Mongol invasion of Baghdad, the library of the House of Wisdomwas destroyed; this time, we are told, books were not burned but thrown into the river, which was equally effective because water quickly washed out the ink.

POSSIBLE SHORT QUESTIONS:

i. Note on The House of Wisdom. Who established it.
ii. What was the main purpose of The House of Wisdom?
iii. Who destroyed The House of Wisdom?
iv. Write few ancient Muslim translators.

LONG QUESTIONS:

Briefly describe about The House of Wisdom of Muslim Civilization.

ABU ABDULLAH MUHAMMAD IBN MUSA AL-KHWARIZMI

The father of Algebra Abu Abdullah Muhammad ibn Musa al-Khwarizmi (780 - 850) was born in the middle east. Muhammad ibn Musa al-Khwarizmi (ca. 780 ca. 850) wrote more than half a dozen astronomical and mathematical works, of which the earliest were probably based on the Sindhind. Besides astronomical tables and treatises on the astrolabe and the sundial, al-Khwarizmi wrote two books on arithmetic and algebra that played very important roles in the history of mathematics. One of these survives only in a unique copy of a Latin translation with the title De numero indorum, the original Arabic version having since been lost. In this work, based presumably on an Arabic translation of Brahmagupta, al-Khwarizmi gave so full an account of the Hindu numerals that he is probably responsible for the widespread but false impression that our system of numeration is Arabic in origin. Al-Khwarizmi made no claim to originality in connection with the system, the Hindu source of which he assumed as a matter of course, but when Latin translations of his work subsequently appeared in Europe, cursory readers began to attribute not only the book but also the numeration to the author. The new notation came to be known as that of al-Khwarizmi or, more carelessly, algorismi; ultimately, the scheme of numeration that made use of the Hindu numerals came to be called simply “algorism” or “algorithm,” a word that, originally derived from the name al-Khwarizmi, now means, more generally, any peculiar rule of procedure or operation—such as the Euclidean method for finding the greatest common divisor.

AL-KHWARIZMI WORKS

Al-khwarizmi’s working in many field is given as follows;

i. Astronomy
ii. Calendar
iii. Arithmetic
iv. Algebra

Al-khwarizmi introduced many of the invensions of mathematics to the people of that age. Most of the Al-khwarizmi’s working in the field of astronomy. He wrote about a hundred astronomical tables. One of these **“Zij Al Sindhind(زیج السندہند)”.** He also wrote geogerophy text **“Kitab Surat al Ard(کتاب صورت الارض) ”.** Al-khwarizmi calculated the interval between Jewish era and Selucdid era with began oct 1, 312 BC.Al-khwarizmi’s arithmetic treatises was possibly entitled **“Kitab-al-Iam-Wal-Tafriq be – Hisab-al-Hind.** (کتاب الایام والتفریق الہند)”

POSSIBLE SHORT QUESTIONS:

i. Note on Alkhawaizmi. Name + date of birth and death+ place.etc.
ii. What is Algorism, Algorithm?
iii. Name few books of Alkhawarizm and their description.

AL-JABR

One of the earliest Islamic Algebra texts written about 825 CE by Al Khawarizmi was entitled by **“Al kitab Al Mukhtasar al Jabar wal Muqabalah (الکتاب المختصر الجبر والمقابلہ)** The book of restring and balancing. The term Al Jabar can be translated as “restoring and refers to operation of Transposing a subtracted quantity on one sied of an equation to the other side where it became an added quantity. e.g.

$3x + 2 = 4 - 2x \Rightarrow 3x + 2 + 2x = 4 - 2x + 2x \Rightarrow 3x + 2x + 2 = 4 \Rightarrow 5x + 2 = 4$

The term Al Muqabalah can be translated as “comparing and refers to the reduction of positive terms by subtracting equal amount from both sides of equations. e.g.

$5x + 2 = 4 \Rightarrow 5x + 2 - 2 = 4 - 2 \Rightarrow 5x = 2$

Through his arithmetic, al-Khwarizmi’s name has become a common English word; through the title of his most important book, **Hisob al-jabr wa’l muqabalah**, he has supplied us with an even more popular household term. From this title has come the word “algebra,” for it is from this book that Europe later learned the branch of mathematics bearing this name. Neither al-Khwarizmi nor other Arabic scholars made use of syncopation or of negative numbers.

The Al-jabr has come down to us in two versions, Latin and Arabic, but in the Latin translation, Liber algebrae et al mucabala, a considerable portion of the Arabic draft is missing. The Latin, for example, has no preface, perhaps because the author’s preface in Arabic gave fulsome praise to Muhammad (S.A.W), the prophet, and to al-Mamun, “the Commander of the Faithful.” Al-Khwarizmi wrote that the latter had encouraged him to

compose a short work on Calculating by (the rules of) Completion and Reduction, confining it to what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partitions, lawsuits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned (Karpinski 1915, p. 96).

It is not certain just what the terms al-jabr and muqabalah mean, but the usual interpretation is similar to that implied in the previous translation. The word “al-jabr” presumably meant something like “restoration” or “completion” and seems to refer to the transposition of subtracted terms to the other side of an equation; the word “muqabalah” is said to refer to “reduction” or “balancing”—that is, the cancellation of like terms on opposite sides of the equation. Arabic influence in Spain long after the time of al-Khwarizmi is found in Don Quixote, where the word “algebrista” is used for a bone-setter, that is, a “restorer.”

POSSIBLE SHORT QUESTIONS:

i. What is meant by Al Jabar and Al Muqabalah?

LONG QUESTIONS:

1) Briefly describe about The working of Al Khawarizmi.
2) Note on Al kitab Al Mukhtasar al Jabar wal Muqabalah.

QUADRATIC EQUATIONS

The Latin translation of al-Khwarizmi's Algebra opens with a brief introductory statement of the positional principle for numbers and then proceeds to the solution, in six short chapters, of the six types of equations made up of the three kinds of quantities: roots, squares, and numbers (i.e., x, x^2 , and numbers). Chapter I, in three short paragraphs, covers the case of squares equal to roots, expressed in modern notation as $x^2 = 5x$, $x^2/3 = 4x$,and $5x^2 = 10x$, giving the answers x = 5, x = 12, and x = 2, respectively. (The root x = 0 was not recognized.) Chapter II covers the case of squares equal to numbers, and Chapter III solves the case of roots equal to numbers, again with three illustrations per chapter to cover the cases in which the coefficient of the variable term is equal to, more than, or less than 1. Chapters IV, V, and VI are more interesting, for they cover in turn the three classical cases of three-term quadratic equations:

(1) squares and roots equal to numbers,

(2) squares and numbers equal to roots, and

(3) roots and numbers equal to squares.

The solutions are "cookbook" rules for "completing the square" applied to specific instances. Chapter IV, for example, includes the three illustrations $x^2 + 10x = 39$, $2x^2 + 10x = 48$, and $\frac{1}{2}x^2 + 5x = 28$. In each case, only the positive answer is given. In Chapter V, only a single example, $x^2 + 21 = 10x$,is used,but both roots,3 and 7,are given, corresponding to the rule $x = 5 \mp \sqrt{25 - 21}$. Al-Khwarizmi here calls attention to the fact that what we designate as the discriminant must be positive:

You ought to understand also that when you take the half of the roots in this form of equation and then multiply the half by itself; if that which proceeds or results from the multiplication is less than the units above-mentioned as accompanying the square, you have an equation.

In Chapter VI, the author again uses only a single example, $3x + 4 = x^2$, for whenever the coefficient of x^2 is not unity, the author reminds us to divide first by this coefficient (as in Chapter IV). Once more, the steps in completing the square are meticulously indicated, without justification, the procedure being equivalent to the solution $= 1\frac{1}{2} + \sqrt{\left(1\frac{1}{2}\right)^2 + 4}$. Again, only one root is given, for the other is negative.

The six cases of equations given previously exhaust all possibilities for linear and quadratic equations having a positive root. The arbitrariness of the rules and the strictly numerical form of the six chapters remind us of ancient Babylonian andmedieval Indianmathematics. The exclusion of indeterminate analysis, a favorite Hindu topic, and the avoidance of any syncopation, such as is found in Brahmagupta, might suggest Mesopota- mia as more likely a source than India. As we read beyond the sixth chapter, however, an entirely new light is thrown on the question. Al-Khwarizmi continues:

> We have said enough so far as numbers are concerned, about the six types of equations. Now, however, it is necessary that we should demonstrate geometrically the truth of the same problemswhichwe have explained in numbers.

POSSIBLE SHORT QUESTIONS:

i. In Al Khawarizmi's book (Latin version) how many chapters included?
ii. Describe about chapter division for topics.

LONG QUESTIONS:

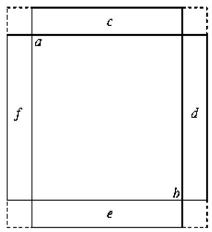
Briefly describe about The working of Al Khawarizmi on Quadratic equations.

GEOMETRIC FOUNDATION

To find Circumference al-Khwarizmi provides three rules with the diameter 'd' and Periphery 'P' (outer limit or edge of an area) and the approximate valued of $\pi = \frac{P}{d}$

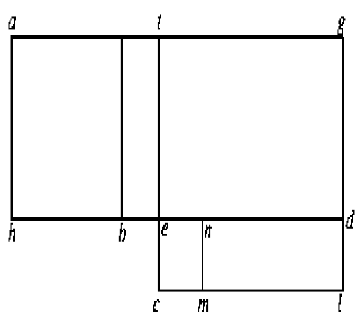
Thus $\pi = \frac{P}{d} = \frac{P = 3\frac{1}{7}d}{d} = 3.1439$ or $\pi = \frac{P}{d} = \frac{P = \sqrt{10d^2}}{d} = 3.1623$ or $\pi = \frac{P}{d} = \frac{P = \frac{62832d}{20000}}{d} = 3.1416$

The Algebra of al-Khwarizmi betrays unmistakable Hellenic elements, but the first geometric demonstrations have little in common with classical Greek mathematics. For the equation $x^2 + 10x = 39$, al-Khwarizmi drew a square, ab, to represent x^2 , and on the four sides of this square he placed rectangles c, d, e, and f, each $2\frac{1}{2}$ units wide. To complete the larger square, one must add the four small corner squares (dotted in Fig.)



each of which has an area of $6\frac{1}{4}$ units. Hence, to "complete the square" we add 4 times 61 4 units, or 25 units, thus obtaining a square of total area 39 + 25 = 64 units (as is clear from the right-hand side of the given equation). The side of the large square must therefore be 8 units, from which we subtract 2 times $2\frac{1}{2}$, or 5, units to find that x = 3, thus proving that the answer found in Chapter IV is correct.

The geometric proofs for Chapters V and VI are somewhat more involved. For the equation $x^2 + 21 = 10x$, the author draws the square "ab" to represent x^2 and the rectangle bg to represent 21 units. Then the large rectangle, comprising the square and the rectangle bg, must have an area equal to 10x, so that the side ag or hd must be 10 units. If, then, one bisects hd at e, draws et perpendicular to hd, extends te to c so that tc = tg, and completes the squares tclg and cmne (Fig.),



the area tb is equal to the area md. But the square tl is 25, and the gnomon tenmlg is 21 (because the gnomon is equal to the rectangle bg). Hence, the square nc is 4, and its side ec is 2. Inasmuch as ec = be, and because he = 5, we see that x = hb = 5 - 2 or 3, which proves that the arithmetic solution given in Chapter V is correct. A modified diagram is given for the root x = 5 + 2 = 7, and an analogous type of figure is used to justify geometrically the result found algebraically in Chapter VI.

POSSIBLE SHORT QUESTIONS:

i. Write Al Khawarizmi's method to find circumference.
ii. Write geometrical description for quadratic equations according to Al Khawarizmi.

ALGEBRAIC PROBLEMS (Just Read)

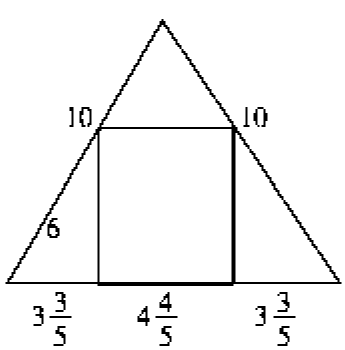
The Algebra of al-Khwarizmi contains more than the solution of equations, material that occupies about the first half. There are, for example, rules for operations on binomial expressions, including products such as (10 + 2)(10 - 1) and (10 + x)(10 - x). Although the Arabs rejected negative roots and absolute negative magnitudes, they were familiar with the rules governing what are now known as signed numbers. There are also alternative geometric proofs of some of the author's six cases of equations. Finally, the Algebra includes a wide variety of problems illustrating the six chapters or cases. As an illustration of the fifth , for example, al-Khwarizmi asks for the division of 10 into two parts in such a way that "the sum of the products obtained by multiplying each part by itself is equal to fifty eight." The extant Arabic version, unlike the Latin, also includes an extended discussion of inheritance problems, such as the following:

A man dies, leaving two sons behind him, and bequeathing one-third of his capital to a stranger. He leaves ten dirhems of property and a claim of ten dirhems upon one of the sons.

The answer is not what one would expect, for the stranger gets only 5 dirhems. According to Arabic law, a son who owes to the estate of his father an amount greater than the son's portion of the estate retains the whole sum that he owes, one part being regarded as his share of the estate and the remainder as a gift from his father. To some extent, it seems to have been the complicated nature of laws governing inheritance that encouraged the study of algebra in Arabia.

A PROBLEM FROM HERON (Just Read)

A few of al-Khwarizmi's problems give rather clear evidence of Arabic dependence on the Babylonian-Heronian stream of mathematics. One of them presumably was taken directly from Heron, for the figure and the dimensions are the same. Within an isosceles triangle having sides of 10 yards and a base of 12 yards (Fig.),



A square is to be inscribed, and the side of this square is called for. The author of the Algebra first finds through the Pythagorean theorem that the altitude of the triangle is 8 yards, so that the area of the triangle is 48 square yards. Calling the side of the square the "thing," he notes that the square of the "thing" will be found by taking from the area of the large triangle the areas of the three small triangles lying outside the square but inside the large triangle. The sum of the areas of the two lower small triangles he knows to be the product of the "thing" by 6 less half of the "thing," and the area of the upper small triangle is the product of 8 less the "thing" by half of the "thing."Hence, he is led to the obvious conclusion that the "thing" is $4\frac{4}{5}$ yards—the side of the square. The chief difference between the form of this problem in Heron and that of al- Khwarizmi is that Heron had expressed the answer in terms of unit fractions as $4\frac{1}{2}\frac{1}{5}\frac{1}{10}$. The similarities are so much more pronounced than the differences that we may take this case as confirmation of the general axiom that continuity in the history of mathematics is the rule, rather than the exception.Where a discontinuity seems to arise, we should first consider the possibility that the apparent saltus may be explained by the loss of intervening documents.

'ABD AL-HAMID IBN-TURK (JUST READ)

The Algebra of al-Khwarizmi is usually regarded as the first work on the subject, but a publication in Turkey raises some question about this. A manuscript of a work by 'Abd-al-Hamid ibn-Turk, titled "Logical Necessities in Mixed Equations," was part of a book on Al-jabr wa'l muqabalah, which was evidently very much the same as that by al-Khwarizmi and was published at about the same time—possibly even earlier. The surviving chapters on "Logical Necessities" give precisely the same type of geometric demonstration as al- Khwarizmi's Algebra and in one case the same illustrative example, $x^2 + 21 = 10x$. In one respect, 'Abd al-Hamid's exposition is more thorough than that of al-Khwarizmi for he gives geometric figures to prove that if the discriminant is negative, a quadratic equation has no solution. Similarities in the work of the two men and the systematic organization found in them seem to indicate that algebra in their day was not so recent a development as has usually been assumed. When textbooks with a conventional and well-ordered exposition appear simultaneously, a subject is likely to be considerably beyond the formative stage. Successors of al-Khwarizmi were able to say, once a problem had been reduced to the form of an equation, "Operate according to the rules of algebra and almucabala." In any case, the survival of al-Khwarizmi's Algebra can be taken to indicate that it was one of the better textbooks typical of Arabic algebra of the time. It was to algebra what Euclid's Elements was to geometry—the best elementary exposition available until modern times—but al-Khwarizmi's work had a serious deficiency that had to be removed before it could effectively serve its purpose in the modern world: a symbolic notation had to be developed to replace the rhetorical form. This step the Arabs never took, except for the replacement of number words by number signs.

THABIT IBN-QURRA (ثابت ابن قرٰی)

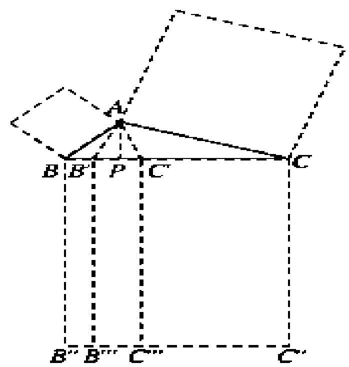
The ninth century was a glorious one in mathematical transmission and discovery. It produced not only al-Khwarizmi in the first half of the century, but also Thabit ibn-Qurra (826 - 901) some auther write (836 - 901) in the second half. Thabit, a Sabean, was born in Harran, the ancient Mesopotamian city that is located in present-day southeastern Turkey and once lay along one of the notable trade routes of the region. He died in Baghdad. Thabit, trilingual since his youth, came to the attention of one of the Musa brothers, who encouraged him to come to Baghdad to study with his brothers in the House of Wisdom. Thabit became proficient in medicine, as well as in mathematics and astronomy, and, when appointed court astronomer by the caliph of Baghdad, established a tradition of translations, especially from Greek and Syriac. To him we owe an immense debt for translations into Arabic of works by Euclid, Archimedes, Apollonius, Ptolemy, and Eutocius. Had it not been for Thabit's efforts, the number of Greek mathematical works extant today would be smaller. For example, we should have only the first four, rather than the first seven, books of Apollonius's Conics.

CONTRIBUTIONS AND ACHEIVEMENTS:

He is created with dozen of treatises, covering a wide range of fields and topics. He developed a theory about trepidation and oscillation of the equinoctial (equal of day and night) points, of which many scholars debated in the Middle Ages.

According to Copernicus, Thabit determined the length of Sidereal Year as 365 days, 6 hours, 9 minutes and 12 seconds (an error of 2 seconds) . He observe the condition of equilibrium of bodies, beams and leavers. He also wrote on Philosophical and Cosmological topics, questioning some of the fundamentals of the Aristotelian Cosmos (Outer Space). Thabit and his grandson Ibrahim ibn Sinan studied the curves which are needed for making sundials (device to measure time) that is commendable and is a great source of inspiration for the learners.

Moreover, Thabit had so thoroughly mastered the content of the classics he translated that he suggested modifications and generalizations. To him is due a remarkable formula for amicable numbers: if p, q,and r are prime numbers, and if they are of the form $p = 3.2^n - 1$, $q = 2^{n-1} - 1$, and $r = 2^{n-2} - 1$, then $2^n pq$ and $2^n r$ are amicable numbers, for each is equal to the sum of the proper divisors of the other. Like Pappus, Thabit also gave a generalization of the Pythagorean theorem that is applicable to all triangles, whether right or scalene. If from vertex A of any triangle ABC one draws lines inter- secting BC in points B' and C; such that angles AB'B and AC'C are each equal to angle A (Fig. 11.4)



Then $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}(\overline{BB'} + \overline{CC'})$:

Thabit gave no proof of the theorem, but this is easily supplied through theorems on similar triangles. Alternative proofs of the Pythagorean theorem, works on parabolic and paraboloidal segments, a discussion of magic squares, angle trisections, and new astronomical theories are among Thabit's further contributions to scholarship. Thabit boldly added a ninth sphere to the eight previously assumed in simplified versions of Aristotelian-Ptolemaic astronomy, and instead of the Hipparchan precession of the equinoxes in one direction or sense only, Thabit proposed a "trepidation of the equinoxes" in a reciprocating type of motion.

POSSIBLE SHORT QUESTIONS:

i. Shortly note on Thabit ibn-Qurra .
ii. Shortly note on working of Thabit ibn-Qurra.
iii. Write the length of Sidereal Year according to Thabit ibn-Qurra.

LONG QUESTIONS: Briefly describe about The working of Thabit ibn-Qurra.

NUMERALS

The Abbasides at Bagdad encouraged the introduction of the sciences by inviting able specialists to their court, irrespective of nationality or religious belief. Medicine and astronomy were their favourite sciences. Thus Haroun-al-Raschid, the most distinguished Saracen ruler, drew Indian physicians to Bagdad. In the year 772 there came to the court of Caliph Almansur a Hindoo astronomer with astronomical tables which were ordered to be translated into Arabic. These tables, known by the Arabs as the Sindhind, and probably taken from the Brahma sphuta siddhanta of Brahmagupta, stood in great authority. They contained the important Hindoo table of sines. Doubtless at this time, and along with these astronomical tables, the Hindoo numerals, with the zero and the principle of position, were introduced among the Saracens. Before the time of Muhammad (S.A.W) the Arabs had no numerals. Numbers were written out in words.Gradually it became the practice to employ the 28 Arabic letters of the alphabet for numerals, in analogy to the Greek system. This notation was in turn superseded by the Hindoo notation, which quite early was adopted by merchants, and also by writers on arithmetic.

As regards the form of the so-called Arabic numerals, the statement of the Arabic writer Albiruni (died 1039), who spent many years in India, is of interest. He says that the shape of the numerals, as also of the letters in India, differed in different localities, and that the Arabs selected from the various forms the most suitable. An Arabian astronomer says there was among people much difference in the use of symbols, especially of those for 5, 6, 7, and 8. The symbols used by the Arabs can be traced back to the tenth century. We find material differences between those used by the Saracens in the East and those used in the West. But most surprising is the fact that the symbols of both the East and of the West Arabs deviate so extraordinarily from the Hindoo Devanagari numerals (= divine numerals) of to-day, and that they resemble much more closely the apices of the Roman writer Boethius. This strange similarity on the one hand, and dissimilarity on the other, is difficult to explain. The most plausible theory is the one of Woepcke:

(1) that about the second century after Christ, before the zero had been invented, the Indian numerals were brought to Alexandria, whence they spread to Rome and also to West Africa;

(2) that in the eighth century, after the notation in India had been already much modified and perfected by the invention of the zero, the Arabs at Bagdad got it from the Hindoos;

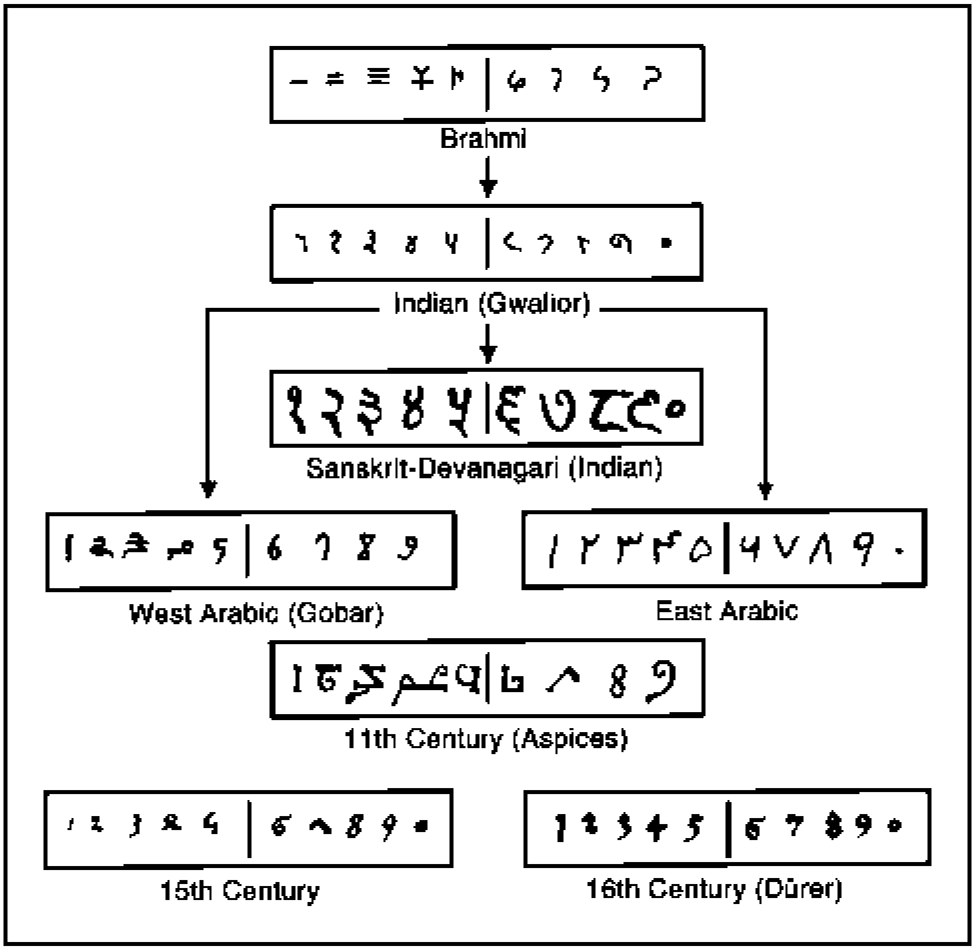
(3) that the Arabs of the West borrowed the Columbus-egg, the zero, from those in the East, but retained the old forms of the nine numerals, if for no other reason, simply to be contrary to their political enemies of the East;

(4) that the old forms were remembered by the West-Arabs to be of Indian origin, and were hence called Gubar-numerals (= dust-numerals, in memory of the Brahmin practice of reckoning on tablets strewn with dust or sand;

(5) that, since the eighth century, the numerals in India underwent further changes, and assumed the greatly modified forms of the modern Devanagari-numerals.

Within the confines of the Arabic empire lived peoples of very varied ethnic backgrounds: Syrian, Greek, Egyptian, Persian, Turkish, and many others. Most of them shared a common faith, Islam, although Christians and Jews were tolerated; very many shared a common language, Arabic, although Greek and Hebrew were sometimes used.There was considerable factionalism at all times, and it sometimes erupted into conflict. Thabit himself had grown up in a pro-Greek community, which opposed him for his pro-Arabic sympathies. Cultural differences occasionally became quite apparent, as in the works of the tenth- and eleventh-century scholars Abu'l-Wefa (940 998) and al-Karkhi (or al-Karagi, ca. 1029). In some of their works, they used the Hindu numerals, which had reached Arabia through the astronomical Sindhind; at other times, they adopted the Greek alphabetic pattern of numeration (with, of course, Arabic equivalents for the Greek letters). Ultimately, the superior

Hindu numerals won out, but even within the circle of those who used the Indian numeration, the forms of the numerals differed considerably. Variations had obviously been prevalent in India, but in Arabia variants were so striking that there are



Genealogy of our digits. Following Karl Menninger, Zahlwort und Ziffer (Go ¨ ttingen: Vanderhoeck & Ruprecht 1957 1958, 2 vols.), Vol. II, p. 233

theories suggesting entirely different origins for forms used in the eastern and western halves of the Arabic world. Perhaps the numerals of the Saracens in the east came directly from India, while the numerals of the Moors in the west were derived from Greek or Roman forms.More likely, the variants were the result of gradual changes taking place in space and time, for the Arabic numerals of today are strikingly different from the modern Devanagari (or "divine") numerals still in use in India. After all, it is the principles within the system of numeration that are important, rather than the specific forms of the numerals. Our numerals are often known as Arabic, despite the fact that they bear little resemblance to those now in use in Egypt, Iraq, Syria, Arabia, Iran, and other lands within the Islamic culture—that is, the forms

١٢٣٤٥٦٧٨٩٠ We call our numerals Arabic because the principles in the two systems are the same and because our forms may have been derived from the Arabic. Yet the principles behind the Arabic numerals presumably were derived from India; hence, it is better to call ours the Hindu or the Hindu-Arabic system (see the illustration above).

As in numeration, there was competition between systems of Greek and Indian origin, so also in astronomical calculations there were at first in Arabia two types of trigonometry—the Greek geometry of chords, as found in the Almagest, and the Hindu tables of sines, as derived through the Sindhind. Here, too, the conflict resulted in triumph for the Hindu aspect, and most Arabic trigonometry was ultimately built on the sine function. It was, in fact, again through the Arabs, rather than directly from the Hindus, that this trigonometry of the sine reached Europe.

Sometimes attempts are made to attribute the functions tangent, cotangent, secant, and cosecant to specific times and even to specific individuals, but this cannot be done with any assurance. In India and Arabia, there had been a general theory of shadow lengths, as related to a unit of length or gnomon, for varying solar altitudes. There was no one standard unit of length for the staff or the gnomon used, although a hand span or a man's height was frequently adopted. The horizontal shadow, for a vertical gnomon of given length, was what we call the cotangent of the angle of elevation of the sun. The "reverse shadow"—that is, the shadow cast on a vertical wall by a stick or a gnomon projecting horizontally from the wall—was what we know as the tangent of the solar elevation. The "hypotenuse of the shadow"—that is, the distance from the tip of the gnomon to the tip of the shadow—was the equivalent of the cosecant function, and the "hypotenuse of the reverse shadow" played the role of our secant. This shadow tradition seems to have been well established in Asia by the time of Thabit ibn-Qurra, but values of the hypotenuse (secant or cosecant) were seldom tabulated.

POSSIBLE SHORT QUESTIONS:

i. Define numeral.
ii. Shortly note on Arab's numerals.
iii. Before establishling the Arabic letters (28) how did Arabs wrote numerals?
iv. write about numeral changings from time to time regarding muslim history.
v. Why we call our number **"the Hindoo or Hindoo – Arabic system"**?
vi. Write short camparison of Arabic geometry for different civilizations.

LONG QUESTIONS:

Briefly note on Arab's numerals.

TENTH- AND ELEVENTH-CENTURY HIGHLIGHTS (JUST READ)

With **Abu'l-Wefa**, trigonometry assumes amore systematic forminwhich such theorems as double and half-angle formulas are proved.Although the Hindu sine function had displaced the Greek chord, it was nevertheless the Almagest of Ptolemy that motivated the logical arrangement of trigono- metric results. The law of sines had been known to Ptolemy in essence and is implied in the work of Brahmagupta, but it is frequently attributed to Abu'l-Wefa and his contemporary Abu Nasr Mensur because of their clear-cut formulation of the law for spherical triangles. Abu'l-Wefa also made up a new sine table for angles differing by $\frac{1}{4}°$, using the equivalent of eight decimal places. In addition, he contributed a table of tangents and made use of all six of the common trigonometric functions, together with relations among them, but his use of the new functions seems not to have been widely followed in the medieval period.

Abu'l-Wefa was a capable algebraist, as well as a trigonometer. He commented on al-Khwarizmi's Algebra and translated from the Greek one of the last great classics—the Arithmetica of Diophantus. His successor al-Karkhi evidently used this translation to become an Arabic disciple of Diophantus—but without Diophantine analysis! That is, al-Karkhi was concerned with the algebra of al-Khwarizmi, rather than with the indeterminate analysis of the Hindus, but like Diophantus (and unlike al-Khwarizmi), he did not limit himself to quadratic equations—despite the fact that he followed the Arabic custom of giving geometric proofs for quadratics. In particular, to al-Karkhi is attributed the first numerical solution of equations of the form $ax^{2n} + bx^{n} = c$ (only equations with positive roots were considered), where the Diophantine restriction to rational numbers was abandoned. It was in just this direction, toward the algebraic solution (in terms of radicals) of equations of more than the second degree, that the early developments in mathematics in the Renaissance were destined to take place. The time of al-Karkhi—the early eleventh century—was a brilliant era in the history of Arabic learning, and a number of his contemporaries deserve brief mention—brief not because they were less capable, but because they were not primarily mathematicians.

Ibn-Sina (980 1037), better known to the West as Avicenna, was the foremost scholar and scientist in Islam, but in his encyclopedic interests, mathematics played a smaller role than medicine and philosophy. He made a translation of Euclid and explained the casting-out of nines (which consequently is sometimes unwarrantedly attributed to him), but he is better remembered for his application of mathematics to astronomy and physics.

As Avicenna reconciled Greek learning with Muslim thought, so his contemporary al-Biruni (973 1048)made the Arabs—hence, us—familiar with Hindu mathematics and culture through his well-known book titled India. An indefatigable traveler and a critical thinker, he gave a sympathetic but candid account, including full descriptions of the Siddhantas and the positional principle of numeration. It is he who told us that Archimedes was familiar with Heron's formula and gave a proof of this and of Brahmagupta's formula, correctly insisting that the latter applies only to a cyclic quadrilateral. In inscribing a nonagon in a circle, al-Biruni reduced the problem, through the trigonometric formula for cos 3θ, to solving the equation $x^3 = 1 + 3x$, and for this, he gave the approximate solution in sexagesimal fractions as 1;52,15,17,13—equivalent to more than six-place accuracy. Al-Biruni also gave us, in a chapter on gnomon lengths, an account of the Hindu shadow reckoning. The boldness of his thought is illustrated by his discussion of whether the earth rotates on its axis, a question to which he did not give an answer. (Earlier, Aryabhata seems to have suggested a rotating earth at the center of space.)

Al-Biruni also contributed to physics, especially through studies in specific gravity and the causes of artesian wells, but as a physicist and a mathematician he was excelled by ibn-al-Haitham (ca. 965 1039), known to the West as Alhazen. The most important treatise written by Alhazen was the Treasury of Optics, a book that was inspired by work of Ptolemy on reflection and refraction and that in turn inspired scientists of medieval and early modern Europe. Among the questions that Alhazen considered were the structure of the eye, the apparent increase in the size of the moon when near the horizon, and an estimate, from the observation that twilight lasts until the sun is 19^0 below the horizon, of the height of the atmosphere. The problem of finding the point on a spherical mirror at which light from a source will be reflected to the eye of an observer is known to this day as "Alhazen's problem." It is a "solid problem" in the old Greek sense, solvable by conic sections, a subject with which Alhazen was quite familiar. He extended Archimedes' results on conoids by finding the volume generated by revolving about the tangent at the vertex the area bounded by a parabolic arc and the axis and an ordinate of the parabola.

OMAR KHAYYAM (JUST READ)

Arabic mathematics can with some propriety be divided into four parts:

(1) an arithmetic presumably derived from India and based on the principle of position;

(2) an algebra that, although from Greek, Hindu, and Babylonian sources, nevertheless in Muslim hands assumed a characteristically new and systematic form;

(3) a trigonometry the substance of which came chiefly from Greece but to which the Arabs applied the Hindu form and added new functions and formulas;

(4) a geometry that came from Greece but to which the Arabs contributed generalizations here and there. There was a significant contribution about a century after Alhazen by a man who in the East is known as a scientist but whom the West recalls as one of the greatest Persian poets.

Omar Khayyam (ca. 1050 1123), the “tent-maker,” wrote an Algebra that went beyond that of al-Khwarizmi to include equations of the third degree. Like his Arabic predecessors, Omar Khayyam provided both arithmetic and geometric solutions for quadratic equations; for general cubic equations, he believed (mistakenly, as the sixteenth century later showed), arithmetic solutions were impossible; hence, he gave only geometric solutions. The scheme of using intersecting conics to solve cubics had been used earlier by Menaechmus, Archimedes, and Alhazen, but Omar Khayyamtook the praiseworthy step of generalizing themethod to cover all third-degree equations (having positive roots). When in an earlier work he came across a cubic equation, he specifically remarked, “This cannot be solved by plane geometry [i.e., using straightedge and compasses only] since it has a cube in it. For the solution we need conic sections” (Amir-Moez 1963, p. 328).For equations of a higher degree than three, Omar Khayyam evidently did not envision similar geometric methods, for space does not contain more than three dimensions, “what is called square-square by algebraists in continuous magnitude is a theoretical fact. It does not exist in reality in any way.” The procedure that Omar Khayyam so tortuously—and so proudly applied to cubic equations can be stated with far greater succinctness in modern notation and concepts as follows. Let the cubic be $x^3+ax^2+b^2x+c^3=0$. Then, if for x^2 in this equation we substitute 2py, we obtain (recalling that $x^3 =x^2 . x$)the result $2pxy+2apy+b^2x+c^3=0$. Because the resulting equation represents a hyperbola, and the equality $x^2=2py$ used in the substitution represents a parabola, it is clear that if the hyperbola and the parabola are sketched on the same set of coordinate axes, then the abscissas of the points of intersection of the two curves will be the roots of the cubic equation. Obviously, many other pairs of conic sections can be used in a similar way to solve the cubic.

Our exposition of Omar Khayyam’s work does not do justice to his genius, for, lacking the concept of negative coefficients, he had to break the problem into many separate cases according as the parameters a, b, c are positive, negative, or zero.Moreover, he had to specifically identify his conic sections for each case, for the concept of a general parameterwas not at hand in his day.Not all roots of a given cubic equationwere given, for he did not accept the appropriateness of negative roots and did not note all intersections of the conic sections. It should also be mentioned that in the earlier Greek geometric solutions of cubic equations, the coefficients had been line segments, whereas in the work of Omar Khayyam they were specific numbers. One of the most fruitful contributions of Arabic eclecticism was the tendency to close the gap between numerical and geometric algebra.The decisive step in this direction came much later with Descartes, but Omar Khayyam was moving in this direction when he wrote, “Who-ever thinks algebra is a trick in obtaining unknowns has thought it in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras are geometric facts which are proved.” In replacing Euclid’s theory of proportions with a numerical approach, he came close to a definition of the irrational and struggledwith the concept of real number in general.

In his Algebra, Omar Khayyam wrote that elsewhere he had set forth a rule that he had discovered for finding fourth, fifth, sixth, and higher powers of a binomial, but such a work is not extant. It is presumed that he was referring to the Pascal triangle arrangement, one that seems to have appeared in China at about the same time. Such a coincidence is not easy to explain, but until further evidence is available, independence of discovery is to be assumed. Intercommunication between Arabia and China was not extensive at that time, but there was a silk route connecting China with Persia, and information might have trickled along it.

THE PARALLEL POSTULATE (Just read)

Islamic mathematicians were clearly more attracted to algebra and trigonometry than to geometry, but one aspect of geometry held a special fascination for them—the proof of Euclid's fifth postulate. Even among the Greeks, the attempt to prove the postulate had become virtually a "fourth famous problem of geometry," and several Muslim mathematicians continued the effort. Alhazen had begun with a trirectangular quadrilateral (sometimes known as "Lambert's quadrangle" in recognition of efforts in the eighteenth century) and thought that he had proved that the fourth anglemust also be a right angle. From this "theorem" on the quadrilateral, the fifth postulate can easily be shown to follow. In his "proof,"Alhazen had assumed that the locus of a point that moves so as to remain equidistant from a given line is necessarily a line parallel to the given line—an assumption shown in modern times to be equivalent to Euclid's postulate. Omar Khayyam criticized Alhazen's proof on the ground that Aristotle had condemned the use of motion in geometry. Omar Khayyam then began with a quadrilateral the two sides of which are equal and are both perpendicular to the base (usually known as a "Saccheri quadrilateral," again in recognition of eighteenth-century efforts), and he asked about the other (upper) angles of the quadrilateral, which necessarily are equal to each other. There are, of course, three possibilities. The angles may be (1) acute, (2) right, or (3) obtuse. The first and third possibilities Omar Khayyam ruled out on the basis of a principle, which he attributed to Aristotle, that two converging lines must intersect—again, an assumption equivalent to Euclid's parallel postulate.

NASIR AL- DIN AL – TUSI (نصیرالدین طوسی)

When Omar Khayyam died in 1123, Islamic science was in a state of decline, but Muslim contributions did not come to a sudden stop with his death. Both in the thirteenth century and again in the fifteenth century, we find an Islamic mathematician of note. At Maragha, for example, Nasir al-Din (Eddin) al-Tusi (1201 1274), an astronomer to Hulagu Khan, a grandson of the conqueror Genghis Khan and a brother of Kublai Khan, continued efforts to prove the parallel postulate, starting from the usual three hypotheses on a Saccheri quadrilateral. His "proof" depends on the following hypothesis, again equivalent to Euclid's:

If a line u is perpendicular to a line w at A, and if line v is oblique to w at 3, then the perpendiculars drawn from u upon v are less than AB on the side on which v makes an acute angle with w and greater on the side on which v makes an obtuse angle with w.

The views of al -Tusi, the last in the sequence of three Arabic precursors of non-Euclidean geometry, were translated and published by John Wallis in the seventeenth century. It appears that this work was the starting point for the developments by Saccheri in the first third of the eighteenth century.

Continuing the work of Abu'l-Wefa, al-Tusi was responsible for the first systematic treatise on plane and spherical trigonometry, treating the material as an independent subject in its own right and not simply as the handmaid of astronomy, as had been the case in Greece and India. The six usual trigonometric functions are used, and rules for solving the various cases of plane and spherical triangles are given. Unfortunately, the work of al-Tusi had limited influence, inasmuch as it did not become well known in Europe. In astronomy, however, al-Tusi made a contribution that may have come to the attention of Copernicus. The Arabs had adopted theories of both Aristotle and Ptolemy for the heavens; noticing elements of conflict between the cosmologies, they sought to reconcile and refine them. In this connection, al-Tusi observed that a combination of two uniform circular motions in the usual epicyclic construction can produce a reciprocating rectilinear motion. That is, if a point moves with uniform circular motion clockwise around the epicycle, while the center of the epicycle moves counterclockwise with half of this speed along an equal deferent circle, the point will describe a straight-line segment. (In other words, if a circle rolls without slipping along the inside of a circle whose diameter is twice as great, the locus of a point on the circumference of the smaller circle will be a diameter of the larger circle.) This "theorem of Nasir Eddin" became known to, or was rediscovered by, Nicholas Copernicus and Jerome Cardan in the sixteenth century.

AL-KASHI (JUST READ)

The mathematics of Islam continued to decline after al-Tusi, but our account of the Muslim contribution would not be adequate without reference to the work of a figure in the early fifteenth century. Jamshid al-Kashi (ca. 1380 1429) found a patron in the prince Ulugh Beg, who was a grandson of the Mongol conqueror Tamerlane. In Samarkand, where he held his court, Ulugh Beg had built an observatory and established a center of learning, and al-Kashi joined the group of scientists gathered there. In numerous works, written in Persian and Arabic, al-Kashi contributed to mathematics and astronomy. He also produced a major textbook for the use of students in Samarkand, which provided an introduction to arithmetic, algebra, and their applications to architecture, surveying, commerce, and other interest areas. His computational skills appear to have been unequalled. Noteworthy is the accuracy of his computations, especially in connection with the solution of equations by a special case of Horner's method, derived perhaps from the Chinese. From China, too, al-Kashi may have taken the practice of using decimal fractions. Al-Kashi is an important figure in the history of decimal fractions, and he realized the significance of his contribution in this respect, regarding himself as the inventor of decimal fractions. Although to some extent he had had precursors, he was perhaps the first user of sexagesimal fractions to suggest that decimals are just as convenient for problems requiring many-place accuracy. Nevertheless, in his systematic computations of roots, he continued to make use of sexagesimals. In illustrating his method for finding the nth root of a number, he took the sixth root of the sexagesimal $34,59,1,7,14,54,23,3,47,37; 40$. This was a prodigious feat of computation, using the steps that we follow in Horner's method—locating the root, diminishing the roots, and stretching or multiplying the roots—and using a pattern similar to our synthetic division. Al-Kashi evidently delighted in long calculations, and he was justifiably proud of his approximation of π, which was more accurate than any of the values given by his predecessors. He expressed his value of 2π in both sexagesimal and decimal forms. The former—6;16,59,28,34,51,46,15,50— is more reminiscent of the past, and the latter, 6.2831853071795865, in a sense presaged the future use of decimal fractions. No mathematician approached the accuracy in this tour de force of computation until the late sixteenth century. His computational skills appear to have been at the basis of the table of sines produced at the Samarkand observatory. In al-Kashi, the binomial theorem in "Pascal triangle" form again appears, just about a century after its publication in China and about a century before it was printed in European books.

The number of significant Islamic contributors to mathematics before al-Kashi was considerably larger than our exposition would suggest, for we have concentrated only on major figures, but after al-Kashi the number is negligible. It was very fortunate indeed that when Arabic learning began to decline, scholarship in Europe was on the upgrade and was prepared to accept the intellectual legacy bequeathed by earlier ages.

WHAT IS MATHEMATICS - AN OVERVIEW

Mathematics is based on deductive reasoning though man's first experience with mathematics was of an inductive nature. This means that the foundation of mathematics is the study of some logical and philosophical notions. We elaborate in simple terms that the deductive system involves four things:

(1) A set of primitive undefined terms;

(2) Definitions evolved from the undefined terms;

(3) Axioms or postulates; (4) Theorems and their proofs.

We also include some historical remarks on the nature of mathematics.

KEYWORDS : Mathematics Education, Deductive Reasoning, Inductive Reasoning, Primitive Undefined Terms, Axioms, Theorem, Direct Proof, Indirect Proof, Platonism, Formalism

INTRODUCTION

Mathematics is not only concerned with everyday problems, but also with using imagination, intuition and reasoning to find new ideas and to solve puzzling problems. One method used by mathematicians in discovering new ideas is to perform experiments. This is called the "experimental method" or "inductive reasoning". When a scientist takes a large number of careful observations and from them infers some probable results or when he repeats an experiment many times and from these data arrives at some probable conclusion, he is using inductive reasoning. That is to say, from a large number of specific cases he obtains a single general inference. The other method is based on reasoning rather than on experiments or observations. This is called "deductive reasoning". When a mathematician begins with a set of acceptable conditions, called the hypothesis and by a series of logical implications reaches a valid conclusion, he employs deductive reasoning. The major difference in the two methods is implied in the two words: “probable” with respect to inductive reasoning and `valid' relative to deductive reasoning. For example, if we perform an experiment successfully say a thousand times, then another twenty successful trails would lend credence to the result, but we have no assurance whatever that the experiment will not fall on the very next trail. On the other hand in a deductive system, once we accept the hypothesis, the validity of our conclusion is inevitable provided each implication in the reasoning process is a logical consequence of what which proceeds it. Here "consistency" of a logical system means that no theorem of the system contradicts another and "validity" means that the system's rules of proof never allow a false inference from true premises.

DEDUCTIVE REASONING SYSTEM: As mentioned above, mathematics is based on deductive reasoning though man's first experience with mathematics was of an inductive nature. The ancient Egyptians and Babylonians developed many mathematical ideas through observation and experimentation and made use of this mathematics in their daily life. Then the Greeks became interested in philosophy and logic and placed a great emphasis on reasoning. For example, in Geometry, the axiomatic development was first developed by them from 500 to 300 BC, and was described in detail by Euclid around 300 BC. They accepted a few most basic mathematical assumptions and used them to prove deductively most of the geometric facts we know today. Our high school geometry is an excellent example of a deductive system. Recall that in the study of geometry, we began with a set of undefined terms, such as point, line etc. We then made some definitions, for example, those of angle, parallel lines, perpendicular lines, triangle etc. Next, we listed a number of statements concerning these undefined and defined terms which we accepted to be true without proof; these assumptions we called, axioms or postulates. Finally, we were able to prove a considerable number of propositions or theorems by deductive reasoning. In summary, we observe that the study of foundations of mathematics involves an abstract deductive system consisting of:

1. A set of primitive undefined terms; 2. Definitions evolved from the undefined terms;

3. Axioms or postulates; 4. Theorems and their proofs.

We now discuss each of them as follow.

UNDEFINED TERMS

To build a mathematical system based on logic, the mathematician begins by using some words to express their ideas, such as `number' or a `point'. These words are undefined and are sometimes called `primitive terms'. These words usually have some meaning because of experience we have had with them. It may seem strange that in mathematics, a field with which precision and accuracy are commonly associated, we do not (and cannot) `start from scratch' but find it necessary to begin with a set of undefined terms. Why do we not start with precise definitions? An attempt to define any of the fundamental undefined terms, such as point, set, number or element demonstrate that we are soon led to what is referred to as 'circular reasoning'. For example, let us try to define `point'. What is a `point'? A point is a position at which something exists. But what is meant by position? The location of an object, naturally. But what does location of an object imply: a point. So we are back where we started. In a like manner, an attempt to define any of the other undefined terms of mathematics would also result in circular reasoning. Hence, it should now be clear that the use of primitive terms is indispensable because they serve as the foundation upon which the system rests. For obvious reasons, the primitive terms, a mathematician chooses must be simple in form and as small in number as possible. They usually appeal to the intuition, more or less, but it is important to distinguish the intuitive ideas behind them and the part they play in the theory. It is not completely true to say that a primitive term has no formal meaning. It may have content because of the logical position we put in it.

DEFINITIONS: : A definition, Bertrand Russell says, is a declaration that a certain newly introduced term or combination of terms is to mean the same as a certain other combination of terms, of which the meaning is already known. It assigns a meaning to a term by means of primitive terms and terms already defined. It is to be observed that although we employ definitions, yet “definitions" does not appear among our primitive ideas because, strictly speaking, the definitions are no part of our subject. Practically, of course, if we introduce no definitions, our formulae would very soon become so lengthy, as to be unimaginable. In spite of the fact that definitions are theoretically superfluous, it is nevertheless true, that they often convey more important information than is contained in the proposition in which they are used. Definitions clarify and simplify expressions. We need to define our terms so that we can use short names for complex ideas. Also definitions contain an analysis of a common idea and can, therefore, classify, that we wish to single out quadrilaterals with opposite parallel sides. We may do this by means of a definition: "a parallelogram is a quadrilateral whose opposite sides are parallel". If we assume in this definition that `quadrilateral', `opposite sides' and `parallel' have been defined previously, then what we have done is to define the class of parallelograms.

AXIOMS AND POSTULATES: At the start of every mathematical theory (such as Real Numbers System, Group Theory, Topology, Quantum Mechanics), some kinds of foundations are needed. For this purpose, a set of independent fundamental statements is asserted. These assertions are called axioms and postulates. Both the axioms and the postulates have their roots in antiquity. To quote Aristotle, "Every demonstrative science must start from indemonstrable principles. Otherwise, the steps of demonstration would be endless". Both the axioms and the postulates presumably are principles, so clearly true that we accept them without a corresponding proof. In Euclid's time (300 BC), axioms referred specifically to an assumption in geometry. Today the distinction is disregarded and both terms are used interchangeably. The axioms of a mathematical theory are usually stipulated at the beginning of the theory, immediately after announcing the primitive terms. These terms are the bricks with which we build up these axioms. The axioms may contain such statements as: "Things equal to the same thing are equal to one another". "Every line is a set of points". They are necessary if we are to avoid an infinite regression which would certainly result if we only accepted what we could prove. Once the axioms have been chosen, we become more severe about the subsequent propositions.

THEOREMS AND THEIR PROOFS:

A `theorem' is a statement whose truth is established by formal proof. The bulk of any branch of mathematics consists of the collection of theorems that pertain to that particular area. Much of the beauty of mathematics lies in the sequential development of the subject through the proofs of its theorems. A `proof' is a chain of reasoning that succeeded in establishing a conclusion by showing that it follows logically from premises that already are known to be true. In proving a theorem, we may use our undefined and defined terms, and our axioms and of course any theorem we prove, the more knowledge we have at our disposal to prove additional theorems. In any mathematical theory, to prove the first theorem, A (say), the only arguments that can be used are the axioms. And to prove a second theorem, B (say), we may use the axioms and Theorem A and similarly for the subsequent theorems. Hence we state the principle: "a proof demonstrates the validity of a proposition using an argument based entirely on the axioms and the previously established theorems".

KINDS OF PROOFS:

There are two kinds of proof: direct proof and indirect proof. Most theorems have the form "a statement p implies another statement q". To demonstrate such a statement we proceed with an assumption usually called the hypothesis in the following ways:

Assert p (i.e. suppose p is given). From this we construct a demonstration that ends with the statement q.

This program makes up what we called a "direct proof". The `indirect proof', also called "proof by contradiction" (reductio ad absurdum, in Latin), depends essentially on the notion of negation. This idea can be stated in the following form:

"To prove the Theorem A indirectly, we affirm its negation. From this we construct an argument that concludes with the negation of a result already known to be true. This is a legitimate proof of Theorem A".

TRUTH OF ASSERTIONS:

We have spoken of the truth of certain assertions. What does the word `truth' mean in this context? A proposition is true if it can be proved by means of the axioms and theorems proved previously. Notice that a theorem may be true in one theory but false in another; it all depends on the initial axioms. In "plane geometry", the statement that the sum of the angles of a triangle is two right angles is true, but it is no longer true in "Riemannian geometry". In "classical mechanics" mass is indestructible, but in "quantum theory", a mass can be destroyed and replaced by energy.

SOME REMARKS AND QUOTES

A. The above described Deductive System is also called Formalism. In fact, there are two dominant schools of thought about the nature of Mathematics: one is the Platonist or Realist (deriving from Plato) and the other is Formalist. The Platonists believe that mathematical objects exist independent of us and inhabit a world of their own. They are not invented by us but rather discovered. Formalists on the other hand believe that there are no such things as mathematical objects. Mathematics consists of definitions, axioms and theorems invented by mathematicians and have no meaning in themselves except that which we ascribe to them. This school of thought was introduced by David Hilbert in 1921. During the 1920's shock waves had run through the science of physics, because of Heisenberg's Uncertainty Principle (introduced first by the German physicist Werner Heisenberg in 1927). This principle states that you can never simultaneously know the exact position and the exact speed of an object. In 1931, a 25 year old Austrian mathematician Kurt Gödel shocked the worlds of mathematics and philosophy by showing that there are mathematical truths which simply cannot be proved. Kurt Gödel (1906-1978) was regarded as a brilliant mathematician and perhaps the greatest logician since Aristotle. His famous “incompleteness theorem” was a fundamental result about axiomatic systems, showing that in any axiomatic mathematical system, there are propositions that cannot be proved or disproved within the axioms of the system. In particular the consistency of the axioms cannot be proved. This ended a hundred years of attempts to establish axioms which would put the whole of mathematics on an axiomatic basis. These included some major attempts by several logicians and mathematicians of that time (such as Germans' Richard Dedekind (1831-1916), Georg Cantor (1845-1918), Friedrich Frege (1848-1925), David Hilbert (1862-1943), Ernst Zermelo (1871-1953), Italian's Giuseppe Peano (1858-1932),), Dutch L.E.J. Brouwer (1881-1966), British Bertrand Russell (1872-1970)).

B. Gödel's results did not destroy the fundamental idea of formalism, but it did demonstrate that any system would have to be more comprehensive than that envisaged by International Journal of Mathematics and Computational Science Vol. 1, No. 3, 2015, pp. 98-101 101 Hilbert and others. In fact, these results were a landmark in 20th-century mathematics, showing that mathematics is not a finished object, as had been believed. It also implies that a computer can never be programmed to answer all mathematical questions. Among physicists, Gödel is known as the man who proved that time travel to the past was possible under Einstein's equations.

C. <u>Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true</u> – (Bertrand Russell, Mysticism and Logic (1917)

D. "Obvious" is the most dangerous word in mathematics (E.T. Bell, 1883-1960).

E. To arrive at the simplest truth, as Newton knew and practiced, requires years of contemplation. Not activity. Not reasoning. Not calculating. Not busy behaviour of any kind. Not reading. Not talking. Not making an effort. Not thinking. Simply bearing in mind what it is one needs to know. (George Spencer Brown, 1923)

F. Pure mathematics is on the whole distinctly more useful than applied. For what is useful above all is technique, and mathematical technique is taught mainly through pure mathematics. (G.H. Hardy, 1877-1947)

G. As one ancient stated, teaching is not a matter of pouring knowledge from one mind into another as one pours water from one glass into another. It is more like one candle igniting another. Each candle burns with its own fuel.

H. For more than two thousand years some familiarity with mathematics has been regarded as an indispensable part of the intellectual equipment of every cultured person. Today the traditional place of mathematics in education is in grave danger. (Richard Courant and Herbert Robbins)

GEOMETRY

The word “Geometry” is derived by ancient Greek, (Geo mean Earth, Metron mean measurement) is a branch of mathematics concerned with knowledge dealing with spatial relationships, for example, geometrical shapes, relative position of geometrical figures and the properties of space. A mathematician who work in the field of geometry is called a geometer.

BEGINNING: In the beginning geometry was a collection of rules for computing lengths, areas and volumes. Many were crude approximation derived by trial and error. This body of knowledge developed and used in construction, navigation and surveying by the Babylonian and Egyptians was passed to the Greeks. The Greek historian Herodotus (5th century BC) credits the Egyptians with having originated the subject, but there is much evidence that the Babylonian, the Hindu civilization and the Chinese knew much of what was passed along the Egyptians.

CLASSICAL GREEK GEOMETRY: Classical geometry was focused in compass and strategic construction. Geometry was revolutionized by Euclid who introduced mathematical rigor and axiomatic methods still in used today. His book ‘The Element’ is a best known source for his work.

CONTRIBUTION OF DIFFERENT CIVILIZATIONS

BABYLONIANS GEOMETRY

Babylonians were able to compute areas of rectangles, right and isosceles triangles, trapozides and circles. They computed the area of a circle as the square of the circumference divided by twelve. The Babylonians were also responsible for dividing the circumference of a circle in to 360 equal parts. They also used the Pythagorian Theorem.

EGYPTIANS GEOMETRY

The Egyptinas were not nearly as inventive as the Babylonians but they were extensive user of mathematics, especially geometry. They were extremely accurate in their constructions, making the right angles in the Great Pyramid of Giza to maintain high level accuracy. They computed the area of a circle to be the square of 8/9 of the diameter.

EARLY GREEK GEOMETRY

The ancient knowledge of geometry was passed on to the Greeks.
Thales of Miletus (624 – 547 BC) developed the first logical geometry. Orderly development of theorems by proof was the distinctive characteristic of Greek mathematics was new. This mathematics of Thales was continued over the next two centuries by Pythagoras of Samos (569 – 475 BC) and his disciples.

EUCLIDEAN GEOMETRY: Euclidean Geometry is a mathematical well known system attributed to the Greek mathematician Euclid of Alexandria. The study of plane and solid figures on the basis of axioms and theorems employed by Euclid. Euclid's text Elements was the first systematic discussion of geometry. It has been one of the most influential book in history, as much for its method as for its mathematical content. The method consists of assuming a small set of intuitively appealing axioms, and then proving many others prospositions from those axioms. Although many of Euclid's results had been stated by earlier Greek mathematics, Ecuclid was the first to show how these prospositions could be fitted together into a comprehensive deductive and logical system.

DIFFERENTIAL GEOMETRY

Differential Geometry has been of increasing importance to mathematical physics due to Einstein's general relativity postulation that the universe is curved. Contemporary differential geometry is intrinsic, meaning that the spaces it considers are smooth manifolds whose geometric structure is governed by a Riemannian metric, which determines how distances are measured near each point.

TOPOLOGY AND GEOMETRY

The field of Topology, which saw massive development in the 20th century, is in a technical sense a type of transformation geometry, in which transformations are homeomorphisms. This has often been expressed in the form of the dictum ‘ topology is rubber sheet geometry’.

ALGEBRAIC GEOMETRY

The field of Algebraic Geometry is the modern incarnation of the Cartesian geometry of coordinates. This led to the introduction of schemes and greater emphasis on topological methods, including various chorology theories. The study of low dimensional algebraic verieties, algebraic curve, algebraic surfaces and algebraic verieties of dimensions 3 **(algebraic three fold)**, has been far advanced. Gr$\ddot{o}$bner basis theory and real algebraic geometry are among more applied subfields of modern algebraic geometry. Arithmetic geometry is an active field combininig algebraic geometry and number theory. Other direction of research involve moduli spaces and complex geometry. Algebra – geometric methods are commonly applied in string and brain theory.

AREA

Area is the quantity that expresses the extent of two dimensional figure or shape, or planar lamina, in the plane. Surface area is its analog on the two dimensional surface of a three dimensional object. Area can be understood as the amount of material with a given thickness that would be necessary to fashion a model fo the shape, or the amount of paint necessary to cover the surface with a single coat. It is the two – dimensional analog of the length of a curve (a one dimensional concept) or the volume of a solid. The area of a shape can be measured by comparing the shape to squares of a fixed size. In the International System of Units (SI System) , the standard unit of area is the square meter (m^2), which is the area of a squre whose sides are one meter long. A shape with an area of three square meters would have the same area as three such squares. In mathematics, the unit square si defined to have area one, and the area of any other shape or surface is a dimensionless real number.

Area plays an important **role** in modern mathematics. In addition to its obvious importance in geometry and calculus, area is related to the definition of determinants in linear algebra, is a basic property of surface in differential geometry. In analysis, the area of a subset of a plane is defined using Lebesgue measure, though not every subset is measureable. In general, area in higher mathematics is seen as a special case of volume for two dimensional regions. It can be defined through the use of axioms, defining it as a function of a collection of certain plane figures to the set of real numbers. It can be proved that such a function exists.

AREA CONCEPT IN GREEK: Calculation of area dates back 5th century BCE, since area of a disk was studied by Ancient Greeks. However in modern times, area is computed using methods of integral calculus and real analysis. Hippocrates (بقراط) of Chios was the first to show that the area of a disk (the region enclosed by a circle) is proportional to the square of its diameter, as part of his quadrature (زاویے کا فاصلہ) of the lune of Hippocrates, but did not identify the constant of proportionality. Eudoxus of Cnidus, also in 5th century, also found that the area of a disk is proportional to its raduius squared. **Archimeds** used tools of Euclidean geometry to show that the area inside a circle is equal to that of a right triangle whose base has the length of the circle's circumference and whose height equals to the circle's radius, in his book **Measurement of a Circle**. Swiss scientist John Heinrich Lambort in 1761 proved that π, the ratio of circle's area to its squared radius, is irrational. Heron (Hero) of Alexandria found formula for the area of a triangle in terms of its sides, and a proof can be found in his book, **Metrica**, written about 60 CE. In the 7th century CE, Brahmagupta developed a formula, for the area of a cyclic quadrilateral in a circle in terms of its sides. In 1842 the German mathematicians Carl Anton and George Chritian independently found a formula, for the area of any quadrilateral.

CIRCLE AREA: In the 5th century BCE, Hippocrates (بقراط) of Chios was the first to show that the area of a disk (the region enclosed by a circle) is proportional to the square of its diameter, as part of his quadrature (زاویے کا فاصلہ) of the lune of Hippocrates, but did not identify the constant of proportionality. Eudoxus of Cnidus, also in 5th century, also found that the area of a disk is proportional to its raduius squared.

TRIANGLE AREA: Heron (Hero) of Alexandria found formula for the area of a triangle in terms of its sides, and a proof can be found in his book, **Metrica**, written about 60 CE. It has been suggested that Archimedes knew the formula over two centuries earlier. And since **Metrica is a collection of mathematical knowledge available in the ancient world**,it is possible that the formula predates the reference given in that work.

QUADRILITERAL AREA: in the 7th century CE, Brahmagupta developed a formula **(Brahmagupta's formula)**, for the area of a cyclic quadrilateral in a circle in terms of its sides. In 1842 the German mathematicians Carl Anton Brettschneider and Karl George Chritian von Staudt independently found a formula, known as **Brettschneider's formula** for the area of any quadrilateral.

GENERAL POLYGON AREA: The development of Cartesian Coordinates by Rene Descartes in 17th century allowed the development of the Surveyor's Formula for the area of any Polygon with know vertex location by Gauss in the 19th century.

ANALYSIS

Analysis is the branch of mathematics dealing with limits and related theories, such as differentiation, integration, measure, infinite series and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. It may be distinguished from geometry, however it can be applied to any space of mathematical objects that has a definition of nearness or specific distances between objects.

Mathematical analysis formally developed in the 17^{th} century during the scientific revolution, but many of its ideas can be traced back to earlier mathematicians. Early results in analysis were implicitly present in the early days of ancient Greek mathematics. The modern foundations of mathematical analysis were established in 17^{th} century in Europe. Descarts and Fermat independently developed analytic Geometry and a few decay later Newton and Leibnitz independently developed infinitesimal calculus, which grew with the stimulus applied work that continued through the 18^{th} century, into analysis topics such as the calculus of variations, ODE's and PDE's, Fourier analysis and generating functions. The last third of century saw the arithmetization of analysis by Weierstrass, who thought that geometric reasoning was inherently misleading and introduced the "epsilon – delta" definition of limit. Instead Cauchy formulated calculus in terms of geometric ideas and infinitesimal. He also introduced the concept of the Cauchy sequence and started the Formal Theory of complex analysis.

The historical origin of calculus (analysis) can be found in attempts to calculating the spatial quantities such as the length of the curved line or the area enclosed by a curve. The area inside a curve, for instance, is of direct interest in land measurement, but the same technique also determines the mass of a uniform sheet of material bounded by some choosen curve or the quantity of paint needed to cover an irregularly shaped surface. Similarly the mathematical technique for finding a tangent line to a curve at a given point can also be used to calculated the steepness of a curved hill or the angle through which a boat must turn to avoid a collision. Less directly, it is related to the extremely important question of the calculation of instataneous velocity of Other instantaneous rates of change, such as the cooling of a warm object in a cold room or the propagation of a disease organism through a human population. Analysis came into being because many aspects of the natural world can profitably be considered as being continuous at least to an excellent degree of approximation. This article beings with a brief introduction to the historical background of analysis and to basic concepts such as number system, functions, continuity, infinite series, and limits, all of which are necessary for the understanding of analysis.

THE CONCEPT OF LIMITS

The definition of the limits as we know it has a long and deep history. One could argue that the informal idea of a limits starts with the ancient Greeks in Zenu's paradox and the Achilles and the tortoise. This paradox cause uproar in the mathematics community at the time, and the concept of limits were left alone until Network and Leibnitz around. Before that in the late 17^{th} century both Network and Leibnitz had a conception of limiting process which is the core nation of a limit. Neither of them gave it any sort of formulization the way weierstrass did, it is definitely clear that the modern conception of limit depends fundamentally on the nation of function it can be said that the modern nation of function starts with Newton Leibnitz and is formulize a bit after weierstrass, probably with cantor.

According to Eli Maor's: "The facts on file: calculus handbook", the origins of concept of limits are attributed to Eudoxus of Cinidus (370 BCE), who formulated a principle known as the method of exhaustion:

"If from any magnitude there be subtracted a part nit less than its half. From the remainder another past not less than its half, and so on, there will at length remain a magnitude less than any pre-assigned magnitude of the same kids" (Eudoxus)

The doctrine of limits is sometimes claimed to have replaced that of infinitesimals when analysis was rigorized in the 19^{th} century: while it is true that cantor, Dedekind and weierstrass attempted to eliminate infinitesimal from analysis, the history of the limits concept is most complex. Newton had explicitly written that his ultimate ratios were not actually ratios but, rather, limits of prime ratios. Infect the sources of a rigorous notion of limits are considerably older than the 19^{th} Century. In the concept of Leibnitzian Mathematics the limits of $f\ (x)$ as $x \to x_o$ can be viewed as the " assignable part" of $f(x_0 + dx)$ where dx is an "inassignable" infinitesimal increment.

Historical roots of limit notion are not as ancient as historical roots of infinitesimal methods. T.Wallis (1616-1703), in his Arithmetic Infinitorum (1655) introduced an arithmetical concept of the limit f a function. The number whose difference from the function can be lower than any given quantity M. Kline underlines the "Its formulation is still vague but there is the correct idea". (Kline, 1972), as regard this Wallis vague formulation.

In Mathematics the concepts of limit formally expresses the notion of arbitrary closeness. That is a limits is a value that a variable quantity approaches as closely as one desires. The operations of differentiation and integration from calculus are both based on the theory of limits. The theory of limit is based on a particular property of real numbers , namely that between two real numbers, no matter how close together they are, there is always another one between any two real numbers, they are always infinity many more.

Nearness is key to understanding limits: only after nearness is defined does a limit acquire an exact meaning. Relevantly, a neighborhood of points near Neighborhoods are definitive components of infinite limits of a sequence.

Archimedes first developed the idea of limits to measure curved figures and the volume of sphere in the third century BC. By carving these figures into small pieces that can be approximated then increasing the numbers of pieces can be approximated, then increasing the numbers of prices, the limit of the sum of prices can give the desired quantity. Archimedes thesis the method was lost until 1906, when mathematicians discovered the Archimedes came close to discovering infinitesimal calculus. As Archimedes work was unknown until the 20^{th} century other developed the modern mathematical concept of limits.Englishman Sir Isaac Newton and German Gottfried Wilhelm von Leibnitz independently developed the general principles of calculus (of theory of limits) in the 17^{th} century.

HISTORY OF CALCULUS

Caculus is the mathematical study of contiunuos change in the same way that geometry is study of shapes and algebra is study of generalization of arithmetical operation.

History: Calculus known in its early history as infinitesimal calculus which is focused on limits, functions, derivatives, integrals and infinite series.

Ancient Calculus: the ancient period introduced some of ideas that led to integral calculus but these ideas were not in systematic way. In **Egyptian Moscow Papyrus (1820 BC)** calculations of volume and area can be found but they are in concerete form. Indians (8^{th} century BC) had a long history of trigonometry. They gave different methods of differentiation of some trigonometric functions. **Babylonians** may discovered the trapezoidal rule which doing astronomical observation of Jupiter. **Greek mathematics** found the concept of area and volume i.e. Method of Exhaustion used by Eudoxus (408 – 355 BC). In 4^{th} century AD in China Liu Hui reinvented this method to find area of circle. **Democrats** was the first person using the division of object into an infinite number of cross sections at the same time. **Zeno of Elea** discredited infinitesimal by his paradoxes. **Archimedes** invented heuristic method i.e. The Quadrature of Parabola. He was the first to find the tangent to a curve other than a circle. In the 17^{th} century European mathematicians Isaac Barrow, Rene Descartes, Blasé Pascat and Fermet discussed the idea of derivative. In the mid of 17^{th} century Newton and Leibnitz independently discovered calculus. Newton’s idea about calculus based on Fermet’s work.

NEWTON'S CONTRIBUTION IN CALCULUS

Full name: Sir Isaac Newton and born in England on January 4,1643. He is credited as one of th greatest minds of the 17th century scientific revolution with discoveries in optics, motion and mathematics. He lived with his grandmother, graduated from Cambridge's University. He started his training as the chosen heir of Isaac Barrow in Cambridge.

Work: He derived the three law of motion which form basic principles of modern physics. He discovered **Binomial Theorem** in 1664 which describe the algebraic expression of power of a binomial. He developed Approximation method (Newton's method) to find roots of function. He was the first person who used infinted power series. He gave the **method of Fluixion** during the Plague Year (1665 – 1666). His initially work was to find a slope at any point on a curve. He calculated the derivative in order to find slope. He called this metho "**method of Fluixion**" here is the word Fluxion means flow a latin word.

He then established the opposite of differentiation i.e. integration which is called **Method of Fluents**. Newton also gave the **1st Fundamental theorem of Calculus** which based on the concept of integration and stated as ;

If a function is integrated and then differentiated the original function can be obtained because differentiation and integration are inverse functions.

Symbol: For derivative he used the notation $\dot{Y}, \ddot{Y}$ and for integration he used $\bar{x}$ or $\boxed{x}$

In 1671 he wrote **The Principia** his famous book. This book published in 1687. In this book he defined **Fluxional Calculus**. He was best know master of Royal Mint (1699). He was the **President of Royal Society** (1703). He was the first Scientist that is ever to be knighted in 1705. He died in March 1727. For Newton the calculus was Geometrical while Leibniz look it towards Analysis.

LEIBNIZ CONTRIBUTION IN CALCULUS

Full name: Gotfried Wilhelm von Leibniz and born in Germany on July 1, 1646. He was a Polymith and interested in mataphysics, law, economics, politics, logic and mathematics. He died in November 14,1716. He discovered a method of arranging linear equations into array now called **Matrix**. He introduced 'Leibniz Wheel'. He also introduced notion of self – similarity and Principle of continuity which was called Topology, his best known quote was

"The Best of All Possible Worlds"

Work: He developed the binary system. In 1673 he introduced the calculating machine in which binary system was used. It can do $\times, \div, +, -$ or even calculate extract roots. He developed calculus in 1674. And published his views in 1684.

Symbol: For derivative he used the notation $\frac{dy}{dx}$ and for integration he used $\int$.
He also gave the **product rule of differentiation** which is $\frac{d}{dy}\left[\int_{x=a}^{b} f(x,y)dx\right] = \int_{x=a}^{b} \frac{\partial f(x,y)}{\partial y} dx$
He also contributed in method of solving differential equations for example as follows;

- ❖ Determination of rate of decay i.e. $N = N_0 e^{kt}$
- ❖ Projection of population growth i.e. $P(t) = Ae^{kt}$
- ❖ Calculation of interest i.e. $A(t) = Pe^{rt}$

His work with integration can be applied in determining the followings;

- ❖ Displacement
- ❖ Force
- ❖ Work.

His best contribution was discovery of the **Fundamental Principle of Infinitesimal Calculus**. His notations was easily to use so was adopted universly.

Contraversy: Newton's concepts based on limits and concerete reality while Leibniz focused on infinite and abstract of calculus.For Newton the calculus was Geometrical while Leibniz look it towards Analysis. Newton did not published his fidings until 1687 while he recorded his discoveries in 1675. In 1699 The Royal Academy gave full credit to Newton for his discoveries in Calculus while charged Leibniz for plagiarizing Newton's work. In 1715 Royal Society handed down its decision and gave credit to both Newton and Leibniz for their great development in Calculus. Leibniz was very conscious about the importance of good notation and put a lot of thoughts into the symbols he used. While the Newton gave no importance to this thing. Consequently, much of the notations that is used in calculus today is due to Leibniz.

In their development of calculus both Newton and Leibniz used infinitesimal quantities that are infinitely small and yet non – zero of course, such infinitesimal do not really exist but Newton and Leibniz found it convenient to use thes quantities in their computations and their derivation of results. **Lord Bishop Barkley** made serious criticism of the calculus referring to infinitesimal as **the Ghosts of departed quatities**. Barkley's Criticism were well found and important in that they focused the attention of mathematicians on a logical clearification of the calculus. It was to be over 100 years, however, before calculus was to be made rigorous. Ultimately Cauchy, Weierstrass and Reimann reformulated calculus in terms of limits rather than infinitesimal. The development of calculus can roughly be described in three periods, that are **Anticipation, Development and Rigorization.**

In the **Anticipation stage** techniques were being used by mathematicians that involved process to find areas under curves or maximize quantities.

In the **Development stage** Newton and Leibniz created the foundations of calculus and brought all of these techniques together under the umbrella of the derivatives and integrals. However, their methods were not always logical sound and it took mathematicans a long time during the **Rigorization stage** to justify them and calculus on a sound mathematical foundation.

GAUSS'S CONTRIBUTION IN CALCULUS

Carl Friedrich Gauss was arguably the greatest mathematician who ever lived. He was born April 30, 1777, in Brunswick, Germany. His father was a laborer. Gauss mother, though illiterate, was more perceptive.

Gauss's mathematical genius was first apparent when, at the age of three, he detected an error in his father's computation. This is especially impressive in light of the fact that he had never been taught arithmetic. At the age of ten, he among his teacher by mentally calculating the sum of all numbers from 1 to 100. Gauss explained that he had noticed a pattern that suggested a formula; in any event , he arrived almost instantly at the correct answer.
(Larson calculus , Calculus 10e)

Gauss entered Brunswick College carolinum in 1792. At the academy Gauss independently discovers Bode's law , the binomial theorem and the arithmetic geometric mean, as well as law of quadratic reciprocity and the prime number theorem. Gauss constructed a regular 17-gon by ruler and compasses and was published as Section VII of Gauss's famous work, Disquisitions' Arithmetical. He published this book in the summer of 1801 there were seven section, but last section devoted to number theory in 1809, a major two volume treatise on the motion of celertial bodies . In the first volume he discussed differential equations, conic section, elliptic orbits, while the second volume was about planet's orbit.

The later work was inspired by geodesic problems and was principally concerned with potential theory. In fact, Gauss found himself more and more interested in geodesy in the 1820's. From the early 1800's Gauss's had an interest in the question of the possible existence of a non - Euclidean geometry. In a book review in 1816 he discussed proofs which deduced the axiom of parallels from the other Euclidean axioms, suggesting that he believed in the existence of non-Euclidean geometry, although he was rather vague.

- Gauss work in analysis , numerical analysis , vector calculus and calculus of variations:
  - Gaussian quadrature
  - Gauss's -Hermit quadrature
  - Gauss's -Kroners quadrature formula
  - Gauss's -Jacobi quadrature
  - Gauss's -Newton algorithm
  - Gauss's criterion
  - Gauss's -Legendre algorithm
  - Gauss's complex multiplication algorithm
  - Gauss's theorem (divergence theorem)
  - Gauss's pseudospectral method
  - Complex analysis and Convex analysis
  - Gauss's -Lucas theorem
  - Gauss's continued fraction
  - Gauss's hypergeometric theorem
  - (http://www.history.mcs.st_andrews.ac.uk/Biographies/Gauss.html)

ARCHIMEDIAN'S GEOMETRY

Archimedes (287 BC – 212 BC) was born in the city of Syracuse in Italy. His father Phidias was an astronomer and good mathematician. He belonged to a noble family. He went to Egypt for study in Alexendria university. After completing study he became a good inventer and mathematician.

- He solved many problems about physics.
- He worked on the new crown of king to check it purely golden of having a mixing of silver metal.
- He invented Pully, Fulkrum and Lever. All three are still popular. He said "give me space to stand and a long lever and fulcrum, I will move the Earth"

BETTLE'S INVENTIONS

- When a war held among Rome and Carthage before 3rd Century BC, the Archimedes city became Hub or Bettle Field. The king of Syracuse felt danger and join one of them against other one to survive. After it, king asked Archimedes to invent weapons.
- Archimedes invent machines to throwing huge stones upto huge distances.
- He also invent Catapullit

but unfortunately, the city of Archimedes was occupied by Carthage and one day the general of Carthage demanded Archimedes by a Soldier but Archimedes deny him to go there because he was busy in some mathematical problem solving. The Soldier had killed Archimedes. The tomb shape of Archimedes grave was a Sphere that was circumscribed by a cylinder.

GEOMETRY SKILLS

Archimedes worked on mediums of different geometries for light passing

- He gave relationship between volume of surface area of sphere and their circumscribing cyliders.
- He found area of circle by putting one polygon inside and one outside the circle. He firsts enclosed the circle in a triangle, then in a square, then in a pentagon or hexagon etc. each time approximating the area became more closely. Method is called **Exhausion method**.
- He found approximating value of π as $\frac{31}{7} \approx 3.1429$ and $\frac{310}{71} \approx 3.1408$
- He calculated the volume of sphere by slicing it into a series of cylinders and adding up the volumes of all cylinders. He observed that thinner the slices exact results are produced.
- He formulated area of parabolic curved segment which is equals to $\frac{4}{3}(area\ of\ triangle\ inside\ the\ parabola)$
- He found area of circumscribing cylinder (including two bases) $= 6\pi r^2$
- He found volume of circumscribing cylinder $= 2\pi r^3$
- He fond area of sphere $= 4\pi r^2$
- He fond volume of sphere= $\frac{4}{3}\pi r^3$

HELLINISTIC MATHEMATICS ; EUCLID

The story of axiomatic geometry begins with Euclid, the most famous mathematician in history. He is often reffered to as ‘Father of Geometry”. We know essentially nothing about Euclid’s life, save that he was a Greek who lived and worked in Alexandria, Egypt, around 300 BCE. His best known work is the Elements , a thirteen-volume treatise that organized and systematized essentially all of the knowledge of geometry and number theory that had been developed in the Western world upto that time.

POSTULATES: It is in the postulates that the great genius of Euclid’s achievement becomes evident. Although mathematicians before Euclid had provided proofs of some isolated geometric facts (for example, the Pythagorean theorem was probably proved at least two hundred years before Euclid’s time), it was apparently Euclid who first conceived the idea of arranging all the proofs in a strict logical sequence. Euclid realized that not every geometric fact can be proved, because every proof must rely on some prior geometric knowledge; thus any attempt to prove everything is doomed to circularity. He knew, therefore, that it was necessary to begin by accepting some facts without proof. He chose to begin by postulating five simple geometric statements:

Euclid’s Postulate 1: It is possible to draw a straight line from any point to any point.

Euclid’s Postulate 2: It is possible To produce a finite straight line continuously in a straight line.

Euclid’s Postulate 3: It is possible To describe a circle with any center and distance.

Euclid’s Postulate 4: All right angles are equal to one another.

Euclid’s Postulate 5: If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS (Axioms): Following his five postulates, Euclid states five “common notions,” which are also meant to be self-evident facts that are to be accepted without proof:
Common Notion 1: Things which are equal to the same thing are also equal to one another.

Common Notion 2: If equals be added to equals, the wholes are equal.

Common Notion 3: If equals be subtracted from equals, the remainders are equal.

Common Notion 4: Things which coincide with one another are equal to one another.

Common Notion 5: The whole is greater than the part.

PROPOSITIONS: Euclid refers to every mathematical statement that he proves as a proposition. This is somewhat different from the usual practice in modern mathematical writing, where a result to be proved might be called **a theorem** (an important result, usually one that requires a relatively lengthy or difficult proof); **a proposition** (an interesting result that requires proof but is usually not important enough to be called a theorem); **a corollary** (an interesting result that follows from a previous theorem with little or no extra effort); or **a lemma** (a preliminary result that is not particularly interesting in its own right but is needed to prove another theorem or proposition). Even though Euclid's results are all called propositions, the first thing one notices when looking through them is that, like the postulates, they are of two distinct types. Some propositions describe constructions of certain geometric configurations. Other propositions (traditionally called theorems) assert that certain relationships always hold in geometric configurations of a given type.

Here are the statements of Euclid's first three propositions:
Euclid's Proposition I.1. On a given finite straight line to construct an equilateral triangle.

Euclid's Proposition I.2. To place a straight line equal to a given straight line with one end at a given point.

Euclid's Proposition I.3. Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

AL TUSSI WORKING ON EUCLID AXIOM

When Omar Khayyam died in 1123, Islamic science was in a state of decline, but Muslim contributions did not come to a sudden stop with his death. Both in the thirteenth century and again in the fifteenth century, we find an Islamic mathematician of note. At Maragha, for example, Nasir al-Din (Eddin) al-Tusi (1201 1274), an astronomer to Hulagu Khan, a grandson of the conqueror Genghis Khan and a brother of Kublai Khan, continued efforts to prove the parallel postulate, starting from the usual three hypotheses on a Saccheri quadrilateral. His "proof" depends on the following hypothesis, again equivalent to Euclid's:

If a line u is perpendicular to a line w at A, and if line v is oblique to w at 3, then the perpendiculars drawn from u upon v are less than AB on the side on which v makes an acute angle with w and greater on the side on which v makes an obtuse angle with w.

The views of al -Tusi, the last in the sequence of three Arabic precursors of non-Euclidean geometry, were translated and published by John Wallis in the seventeenth century. It appears that this work was the starting point for the developments by Saccheri in the first third of the eighteenth century.

For the first time in the world Nasir al-Din who introduced trigonometry as a separate science. He also wrote on binomial coefficients which are introduced by pascal. The paper in Mathematics, namely:

i. Al Mukhtasar bi jami al-Hisab Takht wal Turab (an overview of the entire calculation with table and earth)
ii. Kitab al jabr wal muqabala (treatise of algebra)
iii. Al-ul-Mudua proposal (treatise of Euclid's postulates)
iv. Qawaid al-Handasa (rules of geometry)
v. Kitab al tajrid fi ilm al-mantiq (overview of logic)
vi. Kitab shakl al-qatta (treatise of quadriteral)
vii. Kitab Shikl al-Qita (treatise of rectangular area)

THE RESEARCH RESULT : The fifth postulate of five postulates Euclid be a contradiction because the fifth postulate actually not a postulate but a theorem that requires proof in Euclidean geometry. Nasir al-Din took part there. He also proved the fifth postulate actually is a theorem in Euclidean geometry.

PREHISTORIC MATHEMATICS

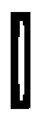
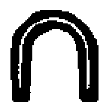
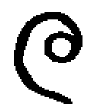
Our prehistoric ancestors would have had a general sensibility about amounts and would have instinctively know the difference between , say, one and two antilopes. But the intellectual leap from the concerete idea of two things to the invension of a symbol or word for the abstract idea of 'two' took many ages to come about. Even today there are isolated hunter gatherer tribes in Amizonia which only have words for 'one' 'two' and 'many'and other which only have words for number upto five. In the absence of settled agricultural and trade, there is a little need for a formal system of numbers.

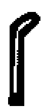
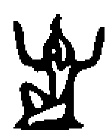
Early man kept track of regular occurences such as the phases of the moon and the seasons.some of the very ealiest evidence of mankind thinking about numbers is from notched bones in Africa dating back to 35,000 to 20,000 years ago. But this is really mere counting and tallying rather than mathematics as such. In the very beginning , human life was simple. An early ancient herdsman compared sheep (or cattle) of his herd with a pile of stones when the herd left for grazing and again on its return for missing animals. In the earliest system probably the vertical strokes or bars such as I,II,III etc. were used for the numbers 1,2,3….. the symbol 'IIIII' was used by many people including the ancient Egyptians for the number of finger of one hand.

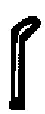
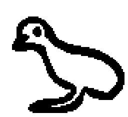
Historically, finger counting, or the practice of counting by fives and tens, seems to have come later than counter-casting by twos and threes, yet the quinary and decimal systems almost invariably displaced the binary and ternary schemes. A study of several hundred tribes among the American Indians, for example, showed that almost one-third used a decimal base, and about another third had adopted a quinary or a quinary-decimal system; fewer than a third had a binary scheme, and those using a ternary system constituted less than 1 percent of the group.The **vigesimal system**, with the number 20 as a base, occurred in about 10 percent of the tribes.

Mathematics proper initially developes largely as a response to Bureacratic needs when civilizations settled and developed agriculture – for the measurements of plots of lands, the taxation of the individuals etc, and this first occurred in **the Summariyand and Babylonian Civilzations of the Mesopotamia (roughly , modern Iraq) and in Ancient Egypt.**

ANCIENT EGYPT: An insight into Egyptian methods of numeration was obtained through the ingenious deciphering of the hieroglyphics by Champollion, Young, and their successors.

The symbols used were the following:  for 1,  for 10,  for 100,

 for 1000,  for 10; 000,  for 100; 000,  for 1; 000; 000,

 for 10; 000; 000. The symbol for 1 represents a vertical staff  for 10; 000 a pointing finger  for 100; 000 a burbot  for 1; 000; 000, a man in astonishment  . The significance of the remaining symbols is very doubtful. The writing of numbers with these hieroglyphics was very cumbrous. The unit symbol of each order was repeated as many times as there were units in that order. The principle employed was the *additive*. Thus, 23 was written

A single vertical stroke represented a unit, an inverted wicket was used for 10, a snare somewhat resembling a capital C stood for 100, a lotus flower for 1,000, a bent finger for 10,000, a tadpole for 100,000, and a kneeling figure, apparently Heh, the god of the Unending, for 1,000,000. Through repetition of these symbols, the number 12,345, for example, would appear as

Besides the hieroglyphics, Egypt possesses the hieratic and demotic writings, but for want of space we pass them by.

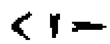
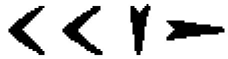
The more cursive hieratic script used by Ahmes was suitably adapted to the use of pen and ink on prepared papyrus leaves. Numeration remained decimal, but the tedious repetitive principle of hieroglyphic numeration was replaced by the introduction of ciphers or special signs to represent digits and multiples of powers of 10. The number 4, for example, usually was no longer represented by four vertical strokes but by a horizontal bar, and 7 is not written as seven strokes but as a single cipher  resembling a sickle. The hieroglyphic form for the number 28 was  the hieratic form was simply . Note that the cipher  for the smaller digit 8 (or two 4s) appears on the left, rather than on the right.

"HEAP" PROBLEMS: The Egyptian problems so far described are best classified as arithmetic, but there are others that fall into a class to which the term "algebraic" is appropriately applied. These do not concern specific concrete objects, such as bread and beer, nor do they call for operations on known numbers. Instead, they require the equivalent of solutions of linear equations of the form x + ax + b or x + ax + bx =c, where a and b and c are known and x is

BEGINNING OF GEOMETRY: In the beginning geometry was a collection of rules for computing lengths, areas and volumes. Many were crude approximation derived by trial and error. This body of knowledge developed and used in construction, navigation and surveying by the Babylonian and Egyptians was passed to the Greeks. The Greek historian Herodotus (5^{th} century BC) credits the Egyptians with having originated the subject, but there is much evidence that the Babylonian, the Hindu civilization and the Chinese knew much of what was passed along the Egyptians.

BABYLONIANS:

The fertile valley of the Euphrates and Tigris was one of the primeval seats of human society. Authentic history of the peoples inhabiting this region begins only with the foundation, in Chaldaea and Babylonia, of a united kingdom out of the previously disunited tribes. Much light has been thrown on their history by the discovery of the art of reading the **cuneiform or wedge-shaped system of writing**.

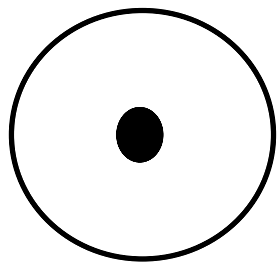
In the study of Babylonian mathematics we begin with the notation of numbers. A vertical wedge  stood for 1, while the characters  and  signified 10 and 100 respectively. Grotefend believes the character for 10 originally to have been the picture of two hands, as held in prayer, the palms being pressed together, the fingers close to each other, but the thumbs thrust out. In the Babylonian notation two principals were employed _ the additive and multiplicative. Numbers below 100 were expressed by symbols whose respective values had to be added. Thus,  stood for 2,  for 3,  for 4,  for 23,  for 30. Here the symbols of higher order appear always to the left of those of lower order. In writing the hundreds, on the other hand, a smaller symbol was placed to the left of the 100, and was, in that case, to be multiplied by 100. Thus,  signified 10 times 100, or 1000. But this symbol for 1000 was itself taken for a new unit, which could take smaller coefficients to its left. Thus,  denoted, not 20 times 100, but 10 times 1000. Of the largest numbers written in cuneiform symbols, which have hitherto been found, none go as high as a million.

They aslo introduced the concept of Sexagesimals positons.

HISTORICAL BACKGROUND OF INDIANS

Despite developpping quite independently of chinese (and probably also of Babylonian mathematics) some mathematical discoveries were made at a very early time in india. Before Perso – Arabic mathematicians, work on mathematics was started in india. Brahmi numerals are the basis of the system predate the common era. Brahmi and Karosthi numerals were used in Mauriya Empire period, both appearing on 3rd century BC edicts of Ashok Budhist used the symbol 1,4,6 around 300 BC. They were also familiar with 2,4,6,7 and 9. The Brahmi numerals were the ancestor of Hindu – Arabic Glyphics 1 to 9. 10,20,30 numerals were also in their counting.The actual numeral system, including positional notation and use of zero, is in principle independent of the glyphs used and significantly younger than the Brahmi numerals.

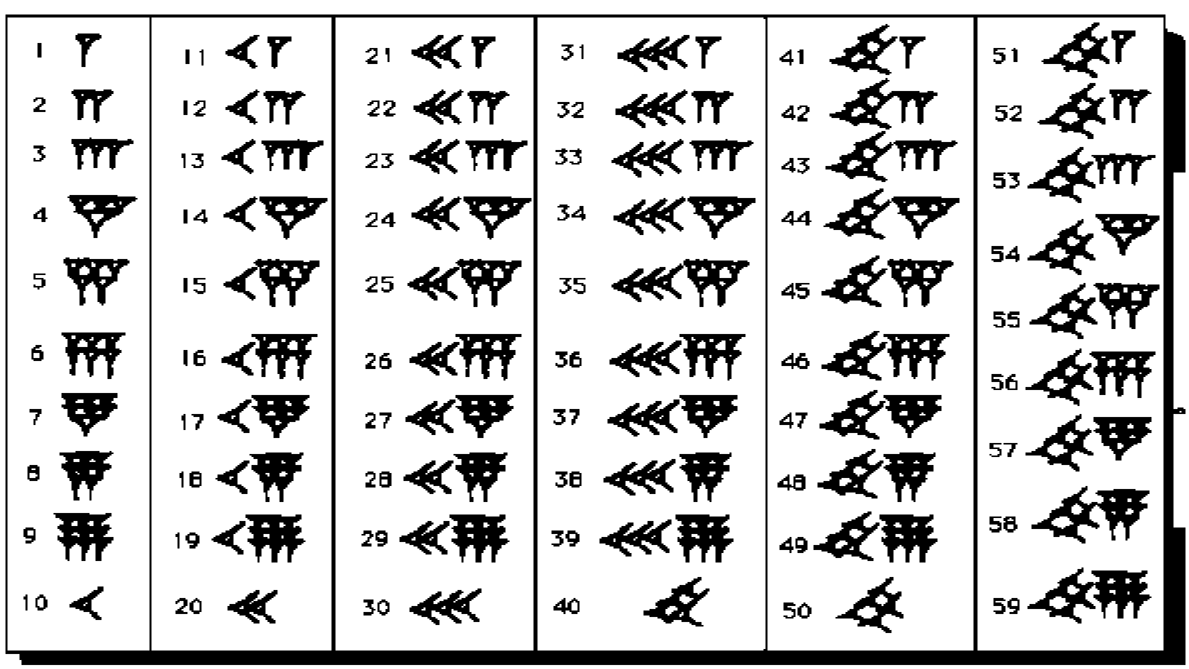
The development of positional decimal system takes its origin in Hindu mathematics during the Gupta Period. Around 500 BC Aryabhatya mark zero and Brahmasphuta Siddhanta explained mathematical role of zero. The Sansikrat translation of Prakrit preserve positional use of zero. These indian developments were take up in Islamic mathematics in 8^{th} century as recoreded in Al – Qifti's Chronolgy of the Scholars (early 13^{th} century). The ancient Hindu Symbol of a circle with a dot in the middle, known as bindu or bindhu, symbolizing the void and the negation of the self, was probably instrumental in the use of a circle as a representation of the concept of zero.



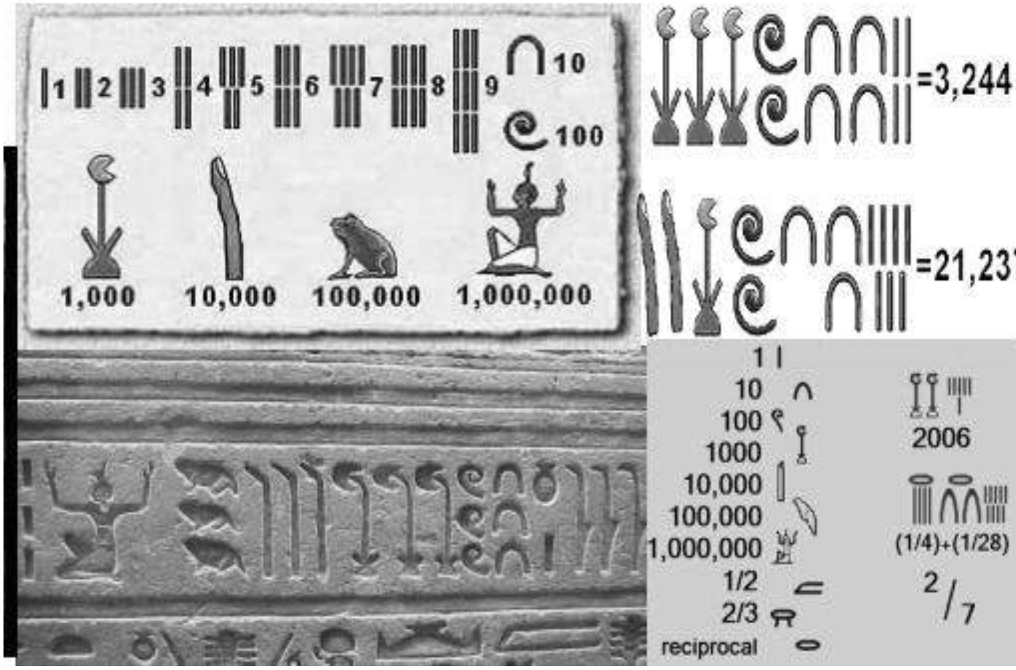
Around 500 BC Aryabhatya mark zero and Brahmasphuta Siddhanta explained mathematical role of zero. The Sansikrat translation of Prakrit preserve positional use of zero. And continuing the Historic Journey many other Civilizations play their role in the improvement of mathematical concepts some of them as follows;

- The Muslims
- The Chinese
- The Summariyans
- The Greeks

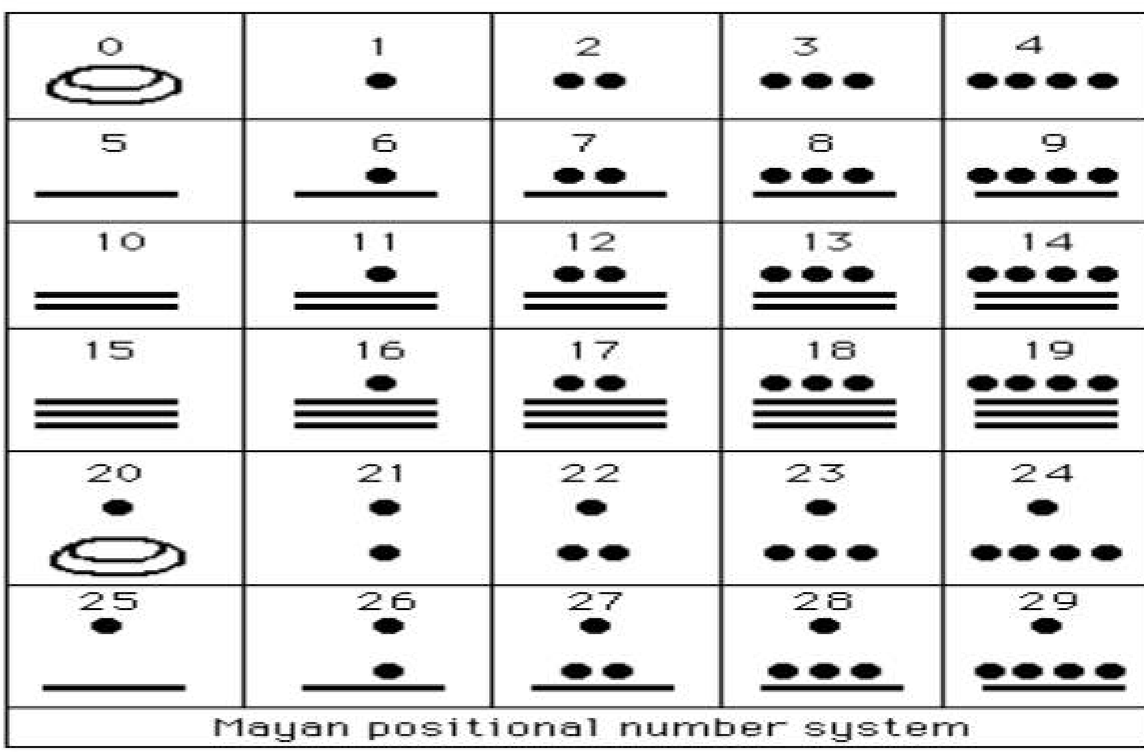
BABYLONIAN NUMERALS

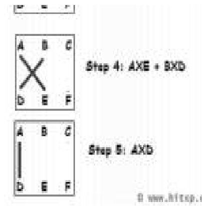


EGYPTIAN NUMERALS



MAYA NUMERALS

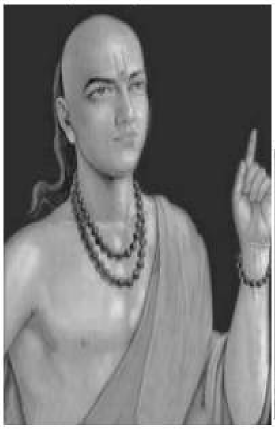




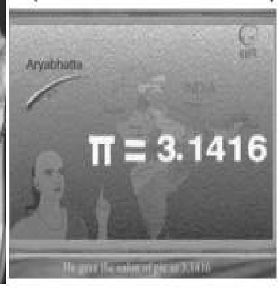
Indian Mathematics: originates in **Vedic mathematics** in Sanskrit sutras with multiplication rules and formulas (like areas of geometric figures, may be Pythagoras Th) hidden between the Vedic hymns.

Aryabhata 476-550 AD book **Aryabhatiya** (on Astronomy) in 121 verses

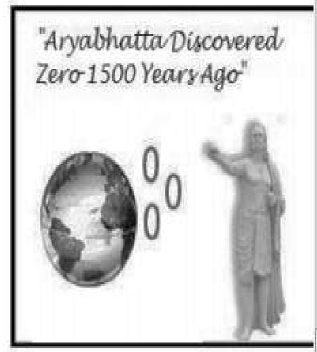
1) The place-value system; he did not use zero, but some argue that its knowledge was implicit



2) Approximation of π = 3.1416; he had possibly a guess that π is an irrational number
3) The oldest use of alphabet numerals in place of old-style word numerals



4) Used arithmetic and Geometric progressions
5) Used trigonometry for the computation of eclipses



- Egypt; 3000B.C.
  - Positional number system, base 10
  - Addition, multiplication, division. Fractions.
  - Complicated formalism; limited algebra.
  - Only perfect squares (no irrational numbers).
  - Area of circle; $(8D/9)^2 \rightarrow \Pi=3.1605$. Volume of pyramid.

- Babylon; 1700-300B.C.
  - Positional number system (base 60; sexagesimal)
  - Addition, multiplication, division. Fractions.
  - Solved systems of equations with many unknowns
  - No negative numbers. No geometry.
  - Squares, cubes, square roots, cube roots
  - Solve quadratic equations (but no quadratic formula)
  - Uses: Building, planning, selling, astronomy (later)

- Greece; 600B.C. – 600A.D. Papyrus created!
  - Pythagoras; mathematics as abstract concepts, properties of numbers, irrationality of √2, Pythagorean Theorem $a^2+b^2=c^2$, geometric areas
  - Zeno paradoxes; infinite sum of numbers is finite!
  - Constructions with ruler and compass; 'Squaring the circle', 'Doubling the cube', 'Trisecting the angle'
  - Plato; plane and solid geometry

Aristotle; mathematics and the physical world (astronomy, geography, mechanics), mathematical formalism (definitions, axioms, proofs via construction)

- Euclid; *Elements* – 13 books. Geometry, algebra, theory of numbers (prime and composite numbers, irrationals), method of exhaustion (calculus!), Euclid's Algorithm for finding greatest common divisor, proof that there are infinitely many prime numbers, **Fundamental Theorem of Arithmetic** (all integers can be written as a product of prime numbers)
- Apollonius; conic sections
- Archimedes; surface area and volume, centre of gravity, hydrostatics
- Hipparchus and Ptolemy; Trigonometry (circle has 360°, sin, cos, tan; $\sin^2 + \cos^2 = 1$), the *Almagest* (astronomy; spherical trigonometry).
- Diophantus; introduction of symbolism in algebra, solves polynomial equations

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The notion of **infinity** 20th Century

∞ or $\aleph_0$

Second symbol known as **"Aleph"**

Timeline of (some) Math Events till 13th Century

Date	Name	Nationality	Major Achievements
35000 BC		African	First notched tally bones
3100 BC		Sumerian	Earliest documented counting and measuring system
2700 BC		Egyptian	Earliest fully-developed base 10 number system in use
2600 BC		Sumerian	Multiplication tables, geometrical exercises and division problems
2000-1800 BC		Egyptian	Earliest papyri showing numeration system and basic arithmetic
1800-1600 BC		Babylonian	Clay tablets dealing with fractions, algebra and equations
1650 BC		Egyptian	Rhind Papyrus (instruction manual in arithmetic, geometry, unit fractions, etc)
1200 BC		Chinese	First decimal numeration system with place value concept
1200-900 BC		Indian	Early Vedic mantras invoke powers of ten from a hundred all the way up to a trillion
800-400 BC		Indian	"Sulba Sutra" lists several Pythagorean triples and simplified Pythagorean theorem for the sides of a square and a rectangle, quite accurate approximation to √2
650 BC		Chinese	Lo Shu order three (3 x 3) "magic square" in which each row, column and diagonal sums to 15
624-546 BC	Thales	Greek	Early developments in geometry, including work on similar and right triangles
570-495 BC	Pythagoras	Greek	Expansion of geometry, rigorous approach building from first principles, square and triangular numbers, Pythagoras' theorem
500 BC	Hippasus	Greek	Discovered potential existence of irrational numbers while trying to calculate the value of √2
490-430 BC	Zeno of Elea	Greek	Describes a series of paradoxes concerning infinity and infinitesimals
470-410 BC	Hippocrates of Chios	Greek	First systematic compilation of geometrical knowledge, Lune of Hippocrates
460-370 BC	Democritus	Greek	Developments in geometry and fractions, volume of a cone
428-348 BC	Plato	Greek	Platonic solids, statement of the Three Classical Problems, influential teacher and popularizer of mathematics, insistence on rigorous proof and logical methods
410-355 BC	Eudoxus of Cnidus	Greek	Method for rigorously proving statements about areas and volumes by successive approximations
384-322 BC	Aristotle	Greek	Development and standardization of logic (although not then considered part of mathematics) and deductive reasoning
300 BC	Euclid	Greek	Definitive statement of classical (Euclidean) geometry, use of axioms and postulates, many formulas, proofs and theorems including Euclid's Theorem on infinitude of primes
287-212 BC	Archimedes	Greek	Formulas for areas of regular shapes, "method of exhaustion" for approximating areas and value of π, comparison of infinities
276-195 BC	Eratosthenes	Greek	"Sieve of Eratosthenes" method for identifying prime numbers

262-190 BC	Apollonius of Perga	Greek	Work on geometry, especially on cones and conic sections (ellipse, parabola, hyperbola)
200 BC		Chinese	"Nine Chapters on the Mathematical Art", including guide to how to solve equations using sophisticated matrix-based methods
190-120 BC	Hipparchus	Greek	Develop first detailed trigonometry tables
36 BC		Mayan	Pre-classic Mayans developed the concept of zero by at least this time
10-70 AD	Heron (or Hero) of Alexandria	Greek	Heron's Formula for finding the area of a triangle from its side lengths, Heron's Method for iteratively computing a square root
90-168 AD	Ptolemy	Greek/Egyptian	Develop even more detailed trigonometry tables
200 AD	Sun Tzu	Chinese	First definitive statement of Chinese Remainder Theorem
200 AD		Indian	Refined and perfected decimal place value number system
200-284 AD	Diophantus	Greek	Diophantine Analysis of complex algebraic problems, to find rational solutions to equations with several unknowns
220-280 AD	Liu Hui	Chinese	Solved linear equations using a matrices (similar to Gaussian elimination), leaving roots unevaluated, calculated value of π correct to five decimal places, early forms of integral and differential calculus
400 AD		Indian	"Surya Siddhanta" contains roots of modern trigonometry, including first real use of sines, cosines, inverse sines, tangents and secants
476-550 AD	Aryabhata	Indian	Definitions of trigonometric functions, complete and accurate sine and versine tables, solutions to simultaneous quadratic equations, accurate approximation for π (and recognition that π is an irrational number)
598-668 AD	Brahmagupta	Indian	Basic mathematical rules for dealing with zero (+, - and x), negative numbers, negative roots of quadratic equations, solution of quadratic equations with two unknowns
600-680 AD	Bhaskara I	Indian	First to write numbers in Hindu-Arabic decimal system with a circle for zero, remarkably accurate approximation of the sine function
780-850 AD	Muhammad Al-Khwarizmi	Persian	Advocacy of the Hindu numerals 1 - 9 and 0 in Islamic world, foundations of modern algebra, including algebraic methods of "reduction" and "balancing", solution of polynomial equations up to second degree
908-946 AD	Ibrahim ibn Sinan	Arabic	Continued Archimedes' investigations of areas and volumes, tangents to a circle
953-1029 AD	Muhammad Al-Karaji	Persian	First use of proof by mathematical induction, including to prove the binomial theorem
966-1059 AD	Ibn al-Haytham (Alhazen)	Persian/Arabic	Derived a formula for the sum of fourth powers using a readily generalizable method, "Alhazen's problem", established beginnings of link between algebra and geometry
1048-1131	Omar Khayyam	Persian	Generalized Indian methods for extracting square and cube roots to include fourth, fifth and higher roots, noted existence of different sorts of cubic equations
1114-1185	Bhaskara II	Indian	Established that dividing by zero yields infinity, found solutions to quadratic, cubic and quartic equations (including negative and irrational solutions) and to second order Diophantine equations, introduced some preliminary concepts of calculus
1170-1250	Leonardo of Pisa (Fibonacci)	Italian	Fibonacci Sequence of numbers, advocacy of the use of the Hindu-Arabic numeral system in Europe, Fibonacci's identity (product of two sums of two squares is itself a sum of two squares)
1201-1274	Nasir al-Din al-Tusi	Persian	Developed field of spherical trigonometry, formulated law of sines for plane triangles
1202-1261	Qin Jiushao	Chinese	Solutions to quadratic, cubic and higher power equations using a method of repeated approximations
1238-1298	Yang Hui	Chinese	Culmination of Chinese "magic" squares, circles and triangles, Yang Hui's Triangle (earlier version of Pascal's Triangle of binomial co-efficients)

حرفِ آخر (17-09-2019)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (اٰمین)

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چک نمبر 105 شمالی (گودھے والا) سرگودھا

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