Lecture Notes on

General Topology

by

Azhar Hussain

hafiziqbal30@yahoo.com

Department of Mathematics
University of Sargodha
Sargodha, Pakistan

Chapter-01

1 Introduction

Topology is the generalization of the Metric Space. The word Topology is composed of two words.

- **Top** means twisting instruments.
- **Logy** a Latin word means Analysis.

So, Topology means Twisting Analysis.
Topology is the combination of two main branches of Mathematics,one is Set theory and the other is Geometry (rubber sheet geometry). We call Set theory is the language of Topology. The course which we will study is basically known as **Point Set Topology** or **General topology**.

To define **Topology** in an other way is the qualitative geometry. The basic idea is that if one geometric object can be continuously transformed into another, then the two objects are considered as topologically same.
e.g. a circle and a square are topologically equivalent.
Physically, a rubber band can be stretched into the form of either a circle or a square. Similarly, many other shapes can also be viewed as topologically same.
e.g. If we take a piece of rubber and draw a circle on it then stretched it, in usual geometry there is no change but topologically there happened a change and the circle deform into an ellipse (or some other shape depending upon the force of stretchness). This is what the rubber sheet geometry means.
The term used to describe two geometric objects that are topologically equivalent is homeomorphic. So, in above example the circle and square, circle and ellipse are homeomorphic.

Definition 1. Suppose that X be a non-empty set and τ be the collection of subsets of X, then τ is called a topology on X if the following axioms are satisfied.

1. ϕ and X are in τ.
2. The union of the elements of any sub collection of τ is in τ.
3. The intersection of the elements of any finite sub collection of τ is in τ.

We call the set X together with topology τ is a topological space and denote it (X, τ). The subset A of X is an open subset of X if $A \in \tau$, so we can say that a topological space together with its subsets are all open, such that X and ϕ are both open and also the infinite union and finite intersection of open sets is also open.

Example 1. *Let $X = \{a, b, c\}$, and consider the collection*

$$\tau = \{X, \phi, \{a\}, \{b, c\}\}$$

- *X and ϕ belongs to τ.*
- *The union of any sub collection of τ belongs to τ.*
- *The intersection of finite sub collection of τ belongs to τ.*

All the three axios are satisfied, hence τ is a topology on X.

Exercise 1. Let $X = \{a, b, c, d\}$, make all possible topologies on X.

Example 2. *Consider $\mathbb{R}$, the set of real numbers, with*

$$\tau = \{S \subseteq \mathbb{R} : \forall x \in \ S \ \ \exists \ \epsilon > 0 \ \textit{such that} \ (x - \epsilon, x + \epsilon) \subseteq S\}$$

Now $(\mathbb{R}, \tau)$ is a topological space as,

- *$\phi, \mathbb{R} \in \tau$ trivially.*
- *Consider the class $\{A_\alpha\}$, where $\alpha \in I$ such that $\forall \ \alpha \in I$ we have $A_\alpha \in \tau$.*
  We show that $\bigcup_{\alpha \in I} A_\alpha \in \tau$.
  For this, let
  $$W = \bigcup_{\alpha \in I} A_\alpha$$
  Then for all $x \in W \ \exists \ \alpha \in I$ such that $x \in A_\alpha$. So by hypothesis $\exists \ \epsilon > 0$ such that
  $$(x - \epsilon, x + \epsilon) \subseteq A_\alpha \subseteq W \subseteq \tau$$
- *Let $A, B \in \tau$, we show that $A \bigcap B \in \tau$.*
  For this, let $x \in A \bigcap B$, then $\exists \ \epsilon_\alpha > 0$ and $\epsilon_\beta > 0$ such that
  $$(x - \epsilon_\alpha, x + \epsilon_\alpha) \subseteq A \quad \textit{and} \quad (x - \epsilon_\beta, x + \epsilon_\beta) \subseteq B$$
  Take
  $$\epsilon = min(\epsilon_\alpha, \epsilon_\beta)$$
  $$(x - \epsilon, x + \epsilon) \subseteq A \quad \textit{and} \quad (x - \epsilon, x + \epsilon) \subseteq B$$
  $$\Rightarrow \quad (x - \epsilon, x + \epsilon) \subseteq A \bigcap B$$
  $$\Rightarrow \quad A \bigcap B \in \tau$$
  Hence τ is a topology on $\mathbb{R}$ and it is called usual topology on $\mathbb{R}$.

Example 3. *Let X be any set, and $P(X)$ called the power set of X consisting of all subsets of X is a topology on X. It is called discrete topology.*
The collection consisting of the set X and empty set only is also a topology on X, it is called indiscrete topology or trivial topology.

Example 4. *If τ_1 and τ_2 are two topologies on X then show that $\tau_1 \bigcap \tau_2$ is also a topology on X. Also give an example.*
Solution. *Let τ_1 and τ_2 are two topologies on X, we have to show that $\tau_1 \bigcap \tau_2$ is also a topology on X.*
For this,

1. *Since $\phi, X \in \tau_1$ and $\phi, X \in \tau_2$*
$$\Rightarrow \phi, X \in \tau_1 \bigcap \tau_2.$$

2. *Let $G_i \in \tau_1 \bigcap \tau_2$*
$\Rightarrow G_i \in \tau_1$ *and* $G_i \in \tau_2$ *respectively.*
$\Rightarrow \bigcup G_i \in \tau_1$ *and* $\bigcup G_i \in \tau_2$ *($\because$ τ_1 and τ_2 are topologies).*
$\Rightarrow \bigcup G_i \in \tau_1 \bigcap \tau_2$.

3. *Let $G_1, G_2 \in \tau_1 \bigcap \tau_2$*
$\Rightarrow G_1, G_2 \in \tau_1$ *and* $G_1, G_2 \in \tau_2$
$\Rightarrow G_1 \bigcap G_2 \in \tau_1$ *and* $G_1, G_2 \in \tau_2$, *($\because$ τ_1 and τ_2 are topologies)*
$\Rightarrow G_1 \bigcap G_2 \in \tau_1 \bigcap \tau_2$. *All the axioms are satisfied, hence intersection of two topologies on X is also topology on X.*

Exercise 2. Give an example of the above example.

Exercise 3. Show by an example that the union of two topologies on a same set X is not a topology on X.

Definition 2. Let X be non-empty set and the class τ of all those subsets of X whose compliment is finite together with ϕ, is a topology on X called the **Co-finite** topology on X.

Remark 1. *Co-finite topology of a finite set is a discrete topology.*

Definition 3. Let (X, τ) be a topological space and N be an open set of X, let $p \in X$ then N is called an open **Neighborhood** of p if $p \in N$. And if p does not belongs to N then the set $N \backslash p$ is called **Deleted** open neighborhood of p.

Example 5. *Let $X\{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$.*
Since $b \in X \Rightarrow X$ is an open neighborhood of b.
Also $a \in X \Rightarrow X, \{a\}$ are open neighborhood of a.

Definition 4. Let (X, τ) be a topological space and $A \subset X$. Let $p \in X$, then p is called **Limit or Accumulation Point** of A iff for every open set G such that $p \in G$, contains a point of A different from p.
In other words
$$(G - \{p\}) \bigcap A \neq \phi$$

Example 6. *Let $X = \{a, b, c, d, e\}$, and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ be a topology on X. Consider the set $A = \{a, b, c\}$, now*

1. *$a \in X$, but there is no open set G(say) in X for which we have $(G - \{a\}) \bigcap A \neq \phi$. Hence a is not a limit point of A.*

2. *$b \in X$, and since the open set containing b are $\{b, c, d, e\}$ and X, and each contained a point of A different from b.*

3. *$c \in X$, and since the open set containing c are $\{c, d\}, \{a, c, d\}, \{b, c, d, e\}$ and X, but the open set $\{c, d\}$ does not contained a point of A different from c.*

4. *$d \in X$, and since the open set containing d are $\{c, d\}, \{a, c, d\}, \{b, c, d, e\}$ and X and each contained a point of A different from d.*

5. *$e \in X$, and since the open set containing e are $\{b, c, d, e\}$ and X, and each contained a point of A different from e.*

Definition 5. The set of all limit points of $A \subset X$ is called the **Derived Set** of A and is denoted by A'.

Example 7. *In example 6. the derived set is $\{b, d, e\}$.*

Exercise 4. Let $X = \{a, b, c\}$, the collection $\tau = \{\phi, X, \{a\}\}$ is a topology on X and $A = \{b, c\} \subset X$, find the derived set of A.

Definition 6. Let (X, τ) be a topological space and $A \subset X$, then A is called **Closed set** of X if A^c is open.

Remark 2. *The empty set ϕ and the set X are both open and closed.*

Example 8. *Let $X = \{a, b, c, d, e\}$, and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ be a topology on X, then the closed subset of X are*
$\phi, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}$.

Example 9. *If (X, τ) is a discrete topology then every subset of X is closed because compliment of all such subsets is open.*

Theorem 1. *Let X be a topological space. Then the class of closed subsets of X possesses the following properties.*

1. *ϕ and X are closed sets.*

2. *The intersection of any number of closed sets is closed.*

3. *The union of any two closed sets is closed.*

Proof. 1. Since $\phi, X \in \tau$ and ϕ and X are both open as well as closed.

2. Let $\{A_i\}$ be the collection of members of τ then $\bigcup_i A_i \in \tau$.
i.e. $\bigcup_i A_i$ is open. (union of open sets is open)
Taking complement and using DeMorgan's Law we get
$(\bigcup_i A_i)^c = \bigcap_i A_i^c$ is closed.
$\Rightarrow A_i$ is open $\forall\ i$.
$\Rightarrow A_i^c$ is closed $\forall\ i$.
Hence intersection of any number of closed sets is closed.

3. Let $A, B \in \tau$ implies $A \bigcap B \in \tau$. And since $A, B \in \tau$ then both A and B are open. Also since intersection of open sets is also open, so $A \bigcap B$ is open.
   $\Rightarrow A^c, B^c$ *and* $(A \bigcap B)^c$ are closed.
$$(A \bigcap B)^c = A^c \bigcup B^c$$
   $\Rightarrow$ Union of any two closed sets is closed in (X, τ).

Hence the class of closed subsets of X possesses all the three properties describe in statement. □

Theorem 2. *A subset A of a topological space X is closed iff A contains each of its limit point.*

Proof. Assume that A is closed, then we are to show that $A' \subset A$.
Let p be a limit point of A such that $p \notin A$ then $p \in A^c$.
But A^c is open since A is closed.
Hence $p \notin A'$ for A^c is open set, such that
$$p \in A^c \text{ and } A^c \bigcap A = \phi$$
$$\Rightarrow (A^c - \{p\}) \bigcap A = \phi$$
Which is a contradiction to the fact that $p \notin A$.
Thus $A' \subset A$ if A is closed.
Conversely.
Now assume that $A' \subset A$, then we are to show that A is closed.
For this, we show that A^c is open.
Let $p \in A^c$ then $p \notin A$, so $\exists$ an open set G such that $p \in G$ and
$$(G - \{p\}) \bigcap A = \phi$$
But $p \notin A$, hence
$$G \bigcap A = \phi$$
So $G \subset A^c$.
Thus p is an interior point of A^c and so A^c is open.
$\Rightarrow A$ is closed. □

Definition 7. Let A be a subset of a topological space X. The closure of A is the intersection of all closed supersets of A.
i.e. Closure of A is the smallest closed superset of A. Closure of a set A is denoted by $\bar{A}$.

Example 10. *Let $X = \{a, b, c, d, e\}$, and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ be a topology on X, find $\overline{\{b\}}, \overline{\{a, c\}}, \overline{\{b, d\}}$.*

Solution 1. Let $X = \{a, b, c, d, e\}$, and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ be a topology on X, then the closed subset of X are
$\phi, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}$.

1. For $\overline{\{b\}}$, the closed supersets of $\{b\}$ are $X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}$, now the intersection of all these closed supersets of $\{b\}$ is
$$\{b, e\}$$

2. For $\overline{\{a,c\}}$, the closed supersets of $\{a,c\}$ is the only X, now the intersection of all these closed supersets of $\{a,c\}$ is
$$X$$

3. For $\overline{\{b,d\}}$, the closed supersets of $\{b,d\}$ are $X, \{b,c,d,e\}$, now the intersection of all these closed supersets of $\{b,d\}$ is
$$\{b,c,d,e\}$$

Theorem 3. *Let A be a subset of a topological space X. Then the closure of A is the union of A and its derived set. i.e.*
$$\bar{A} = A \bigcup A'$$

Proof. Let $x \in \bar{A}$ then x belongs to each closed superset of A, say F. Also we know that
$$A \subset \bar{A}$$

If $x \in A$ then $x \in A \bigcup A'$. If $x \notin A$ then we are to show that $x \in A'$.
For this suppose that $x \notin A'$. Then $\exists$ an open set G containing x such that
$$(G - \{x\}) \bigcap A = \phi$$

i.e. $G \bigcap A = \phi$
$\Rightarrow A \subset G^c$. Then F is a closed superset of A and then $x \notin F$.
This is a contradiction to the fact that $x \in \bar{A}$. Therefore $x \in A'$.
Hence $x \in A \bigcup A'$. So,
$$\bar{A} \subset A \bigcup A' \tag{1}$$
Now let $x \in A \bigcup A'$, then $x \in A$ or $x \in A'$.
If $x \in A$ then clearly $x \in \bar{A}$. $\because A \subset \bar{A}$
If $x \notin A$ then $x \in A'$, so we are to show that $x \in \bar{A}$.
i.e. x belongs to each closed superset of A.
Assume that there is a closed superset F of A such that $x \notin F$.
Then $x \in F^c = G$ (*an open set*).
Since $x \in A'$, i.e. x is a limit point of A.
So $G \bigcap A \neq \phi \;\Rightarrow F^c \bigcap A \neq \phi$ $\because G = F^c$
Which is a contradiction to the fact that $A \subset F$.
Accordingly x belongs to each closed superset of A.
So $x \in \bar{A}$, hence
$$A \bigcup A' \subset \bar{A} \tag{2}$$
From (1) and (2) we have
$$\bar{A} = A \bigcup A'$$
□

Exercise 5. Let A and B be the subsets of topological space (X, τ), then $(A \bigcup B)' = A' \bigcup B'$.

Exercise 6. If $A \subset B$ then every limit point of A is also a limit point of B.

Definition 8. Let $A \subset X$ and $p \in X$, then p is called **Closure Point** of A iff $p \in \bar{A}$.

Remark 3. *From Theorem (3) a point $p \in X$ is closure point of $A \subset X$ iff $p \in A$ or $p \in A'$.*

Definition 9. A subset A of a topological space X is said to be **Dense** in X iff $\bar{A} = X$.

Example 11. *Consider the set Q of rational numbers. We know that*

$$\mathbb{R} = Q \bigcup Q' \tag{3}$$

Where Q' is the set of irrational numbers. And since in the usual topology of $\mathbb{R}$ every real number $a \in \mathbb{R}$ is a limit point of Q, i.e. $a \in Q'$.
So

$$\bar{Q} = Q \bigcup Q' \tag{4}$$

Where Q' is the set of irrational numbers and also the derived set of Q. Hence from (1) and9 (2)

$$\bar{Q} = \mathbb{R}$$

Definition 10. Let A be a subset of a topological space X. A point $p \in A$ is called **Interior Point** A if there exist an open set G such that $p \in G$ which contained in A. i.e.

$$p \in G \subset A \ \ \textit{where } G \textit{ is open.}$$

The set of interior points of A is called the interior of A and is denoted by

$$int(A) \ \textit{or} \ A^\circ.$$

Example 12. *Let $X = \{a, b, c, d, e\}$, and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}$, $\{b, c, d, e\}\}$ be a topology on X and $A = \{b, c, d\}$ is a subset of X, find A°.*

Solution 2. To find the interior of A, we have to check b, c, d as interior points.

1. Since $b \in A$, and the open subsets containing b are X, *and* $\{b, c, d, e\}$, but $b \in X \not\subset A$ and also $b \in \{b, c, d, e\} \not\subset A$, so $b \notin A^\circ$.

2. Since $c \in A$, and the open subsets containing c are $X, \{c, d\}, \{a, c, d\}$, *and* $\{b, c, d, e\}$, and since $c \in \{c, d\} \subset A$ so $c \in A^\circ$.

3. Since $d \in A$, and the open subsets containing d are $X, \{c, d\}, \{a, c, d\}$, *and* $\{b, c, d, e\}$, and since $d \in \{c, d\} \subset A$ so $d \in A^\circ$.

Theorem 4. *Prove that for any set A and B we have*

1. *If $A \subset B$ then $A^\circ \subset B^\circ$.*

2. $(A \bigcap B)^\circ = A^\circ \bigcap B^\circ$.

3. $A^\circ \bigcup B^\circ \subset (A \bigcup B)^\circ$.

Proof. 1. Consider the topology τ on X and let $A, B \subset X$ such that $A \subset B$.
We know that $A^\circ \subset A$ and $B^\circ \subset B$, $\because$ interior is the largest open set.
$\Rightarrow A^\circ \subset A \subset B \ \Rightarrow A^\circ \subset B$
$\Rightarrow A^\circ$ is an open subset of B. But B° is the largest open subset of B
$\Rightarrow A^\circ \subset B^\circ$.

2. We know that $A\bigcap B\subset A$ and also $A\bigcap B\subset B$.
   And we know that if $A\subset B$ then $A^\circ\subset B^\circ$,
   $\Rightarrow (A\bigcap B)^\circ\subset A^\circ$ and $(A\bigcap B)^\circ\subset B^\circ$
   $\Rightarrow (A\bigcap B)^\circ\subset A^\circ\bigcap B^\circ$ (1) Now, $A^\circ\subset A$ and $B^\circ\subset B$
   $\Rightarrow A^\circ\bigcap B^\circ\subset A\bigcap B$.
   $\Rightarrow A^\circ\bigcap B^\circ$ is an open subset of $A\bigcap B$, but $(A\bigcap B)^\circ$ is the union of all open subsets of $A\bigcap B$.
   $\Rightarrow A^\circ\bigcap B^\circ\subset (A\bigcap B)^\circ$ (2) From (1) and (2)
   $$A^\circ\bigcap B^\circ=(A\bigcap B)^\circ$$
3. We know that
   $A\subset A\bigcup B$ and $B\subset A\bigcup B$
   And we know that if $A\subset B$ then $A^\circ\subset B^\circ$.
   $\Rightarrow A^\circ\subset (A\bigcup B)^\circ$ and $B^\circ\subset (A\bigcup B)^\circ$
   $\Rightarrow A^\circ\bigcup B^\circ\subset (A\bigcup B)^\circ$.

$\square$

Proposition 1. *Show that the interior of A is the union of all open subsets of A. Furthermore*

1. *A° is open.*
2. *A° is the largest open subset of A, i.e. if G is an open subset of A then $G\subset A^\circ\subset A$.*
3. *A is open iff $A=A^\circ$.*

Proof. Let $\{G_i\}$ be collection of all open subsets of A. If $x\in A^\circ$ then by definition X belongs to an open subset set of A, i.e. there exist i_0 such that $x\in G_{i_0}$.
Since $x\in G_{i_0}\Rightarrow x\in\bigcup_i G_i$, so
$$A^\circ\subset\bigcup_i G_i \tag{5}$$
Now let $y\in\bigcup_i G_i$, then there exist some i_0 such that $y\in G_{i_0}$, where $G_{i_0}\subset A$
i.e. $y\in G_{i_0}\subset A$ (i.e. y is an interior point of A)
$\Rightarrow y\in A^\circ$
$$\Rightarrow\bigcup_i G_i\subset A^\circ \tag{6}$$
From (5) and (6)
$$A^\circ=\bigcup_i G_i$$
i.e. the interior of A is the union of all open subsets of A.

1. Since $A^\circ=\bigcup_i G_i\Rightarrow A^\circ$ is the union of open sets.
   $\Rightarrow A^\circ$ is open since it is the union of open sets.
2. If G is an open subset of A then $G\in\{G_i\}$
   $\Rightarrow G\subset\bigcup_i G_i$
   $\Rightarrow G\subset A^\circ\ \because A^\circ=\bigcup_i G_i$
   But $A^\circ\subset A$, so
   $G\subset A^\circ\subset A$.

3. Let A is an open set then we show that $A = A^\circ$
   Now since A is an open subset of of itself and A° is the largest open subset of A
$$A \subset A^\circ \tag{7}$$
   But
$$A^\circ \subset A \tag{8}$$
   So from (7) and (8)
$$A = A^\circ$$
   Now let $A = A^\circ$ then we show that A is open.
   And since A° is open since it is the union of open subsets and also $A = A^\circ$
   $\Rightarrow A$ is open.

□

Definition 11. The interior of the compliment of A is called the **Exterior** of A and is denoted by $ext(A)$
i.e. $ext(A) = int(A^c)$.

Definition 12. The set of point which do not belongs to $int(A)$ as well as $ext(A)$ called the **Boundary** of a set A and is denoted by $b(A)$.

Example 13. *Let $X = \{a, b, c, d, e\}$, and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}$, $\{b, c, d, e\}\}$ be a topology on X and $A = \{b, c, d\}$ is a subset of X, find $ext(A)$ and $b(A)$.*

Solution 3. Since $ext(A) = int(A^c)$, so $A^c = \{a, e\}$. Now we will check a and e as interior points.

1. Since $a \in A^c$ and the open set containing a are $X, \{a\},$ *and* $\{a, c, d\}$, and $a \in \{a\} \subset A^c$
   $\Rightarrow a \in int(A^c)$.

2. Since $e \in A^c$ and the open set containing e are $X,$ *and* $\{b, c, d, e\}$, but $e \in X \not\subset A^c$ and also $e \in \{b, c, d, e\} \not\subset A^c$.
   $\Rightarrow e \in int(A^c)$

So a is the only interior point of A^c.
$\Rightarrow int(A^c) = \{a\} = ext(A)$.
Now, since the boundary of A consists of all such points which or neither in $ext(A)$ nor in $int(A)$, so
$b(A) = \{b, e\}$.

Example 14. *Consider the set Q of rational numbers, find $int(Q), ext(Q)$ and $b(Q)$.*

Solution 4. For int(Q): Let $a \in Q$ and consider an open interval (b, c) where b and c are real numbers such that $a \in (b, c)$.
Now, $a \in (b, c) \not\subset Q$
$\Rightarrow a \notin int(Q)$
And since a was arbitrary element.
$\Rightarrow Q$ has no interior point.
$\Rightarrow int(Q) = \phi$.
For ext(Q): Since $ext(Q) = int(Q^c)$ (1) where Q^c is the set of irrational number. Now, let $a \in Q^c$ be an arbitray element and consider the open

interval (b,c) where b and c are real numbers such that $a \in (b,c)$.
Now $a \in (b,c) \not\subset Q^c$
$\Rightarrow a \notin int(Q^c)$
And since a was arbitrary $\Rightarrow Q^c$ has no interior point.

$$\Rightarrow int(Q^c) = \phi = ext(Q) \qquad using(1)$$

For b(Q): Since exterior and interior both are empty so all real numbers belongs to the boundary of Q. i.e.

$$b(Q) = \mathbb{R}$$

Theorem 5. *Show that*

1. $\bar{A} = A \bigcup b(A)$.
2. $A^\circ = A - b(A)$.
3. *A is closed iff* $b(A) \subset A$.
4. *A is both open and closed iff* $b(A) = \phi$.

Proof. 1. To prove $\bar{A} = A \bigcup b(A)$.
R.H.S.
$A \bigcup b(A)$, and by definition of $b(A)$, i.e. $b(A) = \bar{A} \bigcap \bar{A^C}$.
$\Rightarrow A \bigcup b(A) = A \bigcup (\bar{(A)} \bigcap \bar{A}^c)$.

$$\begin{aligned}
\Rightarrow A \bigcup b(A) &= (A \bigcup \bar{A}) \bigcap (A \bigcup \bar{A}^c), && \text{by distributive law.}\\
&= \bar{A} \bigcup (A \bigcup \bar{A}c) && \because \bar{A} = A \bigcup \bar{A}\\
&= \bar{A} \bigcup X && \because X = A \bigcup \bar{A}^c\\
&= \bar{A}\\
&= \text{L.H.S.}
\end{aligned}$$

2. To prove that $A^\circ = A - b(A)$.
R.H.S.

$$\begin{aligned}
A - b(A) &= A - (\bar{A} \bigcap \bar{A}^c) && \because b(A) = (\bar{A} \bigcap \bar{A}^c)\\
&= A \bigcap (\bar{A} \bigcap \bar{A}^c)^c && \because A - B = A \bigcap B^c\\
&= A \bigcap (\bar{A}^c \bigcup (\bar{A^c})^c) && \text{By DeMorgen's Law.}\\
A - b(A) &= (A \bigcap \bar{A}^c) \bigcup (A \bigcap (\bar{A}^c)^c) && (1)
\end{aligned}$$

Now

$$A \bigcap \bar{A}^c = \phi \quad \because A \subset \bar{a} \quad and \quad A \bigcap (\bar{A}^c)^c = A^\circ$$

So, (1) becomes

$$A - b(A) = \phi \bigcup A^\circ = A^\circ = L.H.S$$

$$\Rightarrow A - b(A) = A^\circ$$

3. Suppose that A is closed then we show that $b(A) \subset A$.
Since A is closed then $A = \bar{A}$ and also $\bar{A} = A \bigcup b(A)$.
$\bar{A} = A \bigcup b(A)$
$A = A \bigcup b(A) \Rightarrow b(A) \subset A$.
Now, if $b(A) \subset A$ then we show that A is closed.
Since $\bar{A} = A \bigcup b(A)$, and $b(A) \subset A$ then

$$\bar{A} = A$$

$\Rightarrow$ A is closed.

4. **Case-I**: A is open iff $b(A) = \phi$
   Suppose that A is open then we show that $b(A) = \phi$.
   Since $A^\circ = A - b(A)$ and $A = A^\circ$
   $\Rightarrow A^\circ = A^\circ - b(A)$
   $\Rightarrow b(A) = \phi$
   Now, if $b(A) = \phi$ then we show that A is open.
   Since $A^\circ = A - b(A) = A - \phi$
   $\Rightarrow A^\circ = A$
   $\Rightarrow A$ is open.
   Case-II: A is closed iff $b(A) = \phi$
   Suppose that A is closed the we show that $b(A)\phi$.
   Since $A = \bar{A}$ $\because$ A is closed, and also $\bar{A} = A \bigcup b(A)$
   $\Rightarrow \bar{A} = \bar{A} \bigcup b(A)$
   $\Rightarrow b(A) = \phi$
   Now, if $b(A) = \phi$ then we show that A is closed.
   Since $\bar{A} = A \bigcup b(A) \Rightarrow \bar{A} = A \bigcup \phi$
   $\Rightarrow \bar{A} = A$
   $\Rightarrow A$ is closed.

$\square$

Note 1. A subset A of a topological space X is said to be **nowhere dense** in X if the interior of closure of A is empty. i.e. $int(\bar{A}) = \phi$.

Example 15. *Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ is a subset of $\mathbb{R}$. Prove that A is nowhere dense in $\mathbb{R}$.*
***Solution**:*
Let $A \subset \mathbb{R}$, where $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$, then clearly 0 is the only limit point of A.
$\Rightarrow A' = \{0\}$.
Now, since $\bar{A} = A \bigcup A' = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\} \bigcup \{0\}$
$\Rightarrow \bar{A} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$.
Now if we take take any element of $\bar{A}$, e.g. $0 \in \bar{A}$, then by definition $\exists$ an open interval $(-r, r)$ such that $0 \in (-r, r) \not\subset \bar{A}$. Similarly, we can check all elements of $\bar{A}$ and we conclude that $int(\bar{A}) = \phi$
So, A is nowhere dense in $\mathbb{R}$

Example 16. *Let $A = \{x : x \in \mathbb{Q} \wedge 0 < x < 1\}$, then show that A is nowhere dense in $\mathbb{R}$.*
***Solution**:*
Let $A = \{x : x \in \mathbb{Q} \wedge 0 < x < 1\}$, then clearly $int(A) = \phi$
Now,

$$\bar{A} = A \bigcup A'$$

$$\Rightarrow \bar{A} = (0, 1) \bigcup \{0, 1\}$$

$$\Rightarrow \bar{A} = [0, 1]$$

Now, $int(\bar{A}) = int([0, 1]) = (0, 1) \neq \phi$
$\Rightarrow$ *A is not nowhere dense in $\mathbb{R}$.*

Definition 13. Let X be a topological space and $o \in X$. A subset N of X is a **Neighborhood** of p iff $\exists$ an open set G such that

$$p \in G \subset N$$

The class of neighborhood of $p \in X$ is denoted by N_p, and is called **Neighborhood System of** p.

Remark 4. *The relation " N is a Neighborhood of a point p " is the inverse of the relation " p is an interior point of p*

Example 17. *State the neighborhood of a point a, where $a \in \mathbb{R}$. And then for a complex number p.*
Solution:
Let a be any real number, i.e. $a \in \mathbb{R}$. Now for each $a \in \mathbb{R}$ there exist an open interval $(a-\delta, a+\delta)$ such that

$$a \in (a-\delta, a+\delta) \subset [a-\delta, a+\delta]$$

So each close interval with center $[a-\delta, a+\delta]$ with center a is a neighborhood of a.
And hence the intervals $[a-2\delta, a+2\delta], [a-3\delta, a+3\delta], \ldots\ldots.$ are neighborhood of a.
Similarly, if we take a point p on a complex plane $\mathbb{R}^{\nvDash}$, then every closed disc

$$\{q \in \mathbb{R}^{\nvDash} : d(p,q) < \delta \neq 0\}$$

with center p is a neighborhood of p, since it contains the open disc with center p.

Proposition 2.

1. *$\mathbb{N}_{\shortmid}$ is not empty and p belongs to each member of $\mathbb{N}_{\shortmid}$.*
2. *The intersection of any two members of $\mathbb{N}_{\shortmid}$ belongs to $l\mathbb{N}_{\shortmid}$.*
3. *Every superset of a member of $\mathbb{N}_{\shortmid}$ belongs to $\mathbb{N}_{\shortmid}$.*
4. *Each member $N \in \mathbb{N}_{\shortmid}$ is a superset of a member $G \in \mathbb{N}_{\shortmid}$ where G is a neighborhood of if each of its points, i.e. $G \in \mathbb{N}_{\mathfrak{g}}$ for every $g \in G$.*

Proof.

1. since $\mathbb{N}_{\shortmid}$ is the class of all neighborhood of $p \in X$. So it cannot be empty and $p \in N$ $\forall N \in \mathbb{N}_{\shortmid}$ is obvious by definition.
2. Let
$$\mathbb{N}_{\shortmid} = \{G_p : p \in X \wedge G_P \ is \ n.hood \ of \ p\}$$
Let $p \in G_p$ and $p \in H_p$ then $p \in G_p \bigcap H_p \subset \mathbb{N}_{\shortmid}$.
So it is clear that $G_p \bigcap H_p$ is open or close neighborhood of p according as G_p and H_p is open or closed respectively.
3. Every superset of each member of $\mathbb{N}_{\shortmid}$ belongs to $\mathbb{N}_{\shortmid}$.
As $G_p \subset H_p \in \mathbb{N}_{\shortmid}$ and $H_p \subset K_p \in \mathbb{N}_{\shortmid}$ and the largest superset is superset of itself.
i.e. for K_p being largest superset
$$K_p \subset K_p \in \mathbb{N}_{\shortmid}$$
4. $N \in \mathbb{N}_{\shortmid}$ is a superset of $G \in \mathbb{N}_{\shortmid}$ is clear. As
$$p \in G \subset N$$
and G is a neighborhood of each of its point is clear as
$$p \in G \subset G \ \forall \ p \in G$$
$\Rightarrow G$ is a neighborhood of p $\forall p \in G$.

$\square$

Definition 14. Let X be a topological space and $(a_1, a_2, a_3, ...)$ be a sequence of points of X, now this sequence $< a_n >$ is said to be converges to a point $b \in X$, or in other words b is a limit point of $< a_n >$, i.e.

$$\lim_{n \to \infty} a_n = b$$

iff for each open set G containing b $\exists$ a positive integer $n_0 \in N$ such that $n > n_0$

$$\Rightarrow a_n \in G$$

, i.e. if G contains almost all, i.e. all except a finite number of the terms of the sequence.

Example 18. *Let $(a_1, a_2, a_3, ...)$ be a sequence of points in indiscrete topological space (X, τ). Find out the point b in X such that sequence converges to b.*
Solution:
Let $(a_1, a_2, a_3, ...)$ be a sequence of points in indiscrete topological space (X, τ).i.e. $\tau = \{X, \phi\}$. $\Rightarrow X$ is the only set containing any point $b \in X$. Also X contains every term of the sequence (a_n). Hence the sequence $(a_1, a_2, a_3, ...)$ converges to every point $b \in X$.

Definition 15. Let τ_1 and τ_2 be two topologies on a non-empty set X. Now, suppose that each member of τ_1 is also a member of τ_2, i.e.

$$\tau_1 \subset \tau_2$$

Then we say τ_1 is **Coarser**,smaller or weaker than τ_2 or τ_2 is **Finer**, longer or smaller than τ_1.

Example 19. *Let $X = \{a, b\}$, and let $\mathfrak{D} = \{X, \phi, \{a\}, \{b\}\}$ be discrete topology, $\mathfrak{I} = \{X, \phi\}$ be indiscrete topology and $\tau = \{X, \phi, \{a\}\}$ is any other topology then observe that the topology τ is coarser than $\mathfrak{D}$ and finer than $\mathfrak{I}$. i.e.*

$$\mathfrak{D} \preceq \tau \preceq \mathfrak{I}$$

Definition 16. Let (X, τ) be a topological space and A be any non-empty subset of X. Now,the collection τ_A obtained by taking the intersection of A with the members of τ defines a topology on A, we call this **Relative** topology on A, and the topological space (A, τ_A) is called the subspace of (X, τ).
In other words a subset H of A is τ_A-open set, i.e.open relative to A, iff $\exists$ a τ-open subset G of X such that

$$H = G \bigcap A$$

Example 20. *Let $X = \{a, b, c, d, e\}$, and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}$, $\{b, c, d, e\}\}$ be a topology on X and $A = \{a, d, e\}$ is a subset of X, then find the topology relative to A.*
Solution:
Since $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ and $A = \{a, d, e\}$, now taking the intersection of A with the members of τ we get
$X \bigcap A = A$, $\phi \bigcap A = \phi$, $\{a\} \bigcap = \{a\}$, $\{c, d\} \bigcap A = \{d\}$, $\{a, c, d\} \bigcap A = \{a, d\}$, $\{b, c, d, e\} \bigcap A = \{d, e\}$
So, we write the resultant in a collection

$$\tau_A = \{A, \phi, \{a\}, \{d\}, \{a, d\}, \{d, e\}, \}$$

and clearly this collection forms a topology on A.

2 Base for a Topology

2.1 Definition

We can define a **Base** for a topology in two ways as

1. A **Base** for a topology τ on X is the class $\mathfrak{B}_1$ of open subsets of X such that every element of τ is the union of members of $\mathfrak{B}_1$.

2. Let X be any set, a basis for a topology τ on X is the class $\mathfrak{B}_2$ of open subsets of X such that

   (a) For every element $x \in X$ there is at least one element in $\mathfrak{B}_2$ say B_1 for which $x \in B_1$.

   (b) If x belongs to B_1 and B_2 such that $B_1, B_2 \in \mathfrak{B}_2$ then $\exists$ $B_3 \in \mathfrak{B}_2$ containing x such that $B_3 \subset B_1 \bigcap B_2$.

If the collection $\mathfrak{B}_2$ satisfies the above two conditions then we can define a topology τ on X generated by $\mathfrak{B}_2$ as follows;
A subset G of X is said to be open in X (i.e. an element of τ) if for every element $g \in G$ we have an element B of $\mathfrak{B}_2$ such that such that $g \in B$ and $B \subset G$.
Note that each basis element is itself an element of τ.
Now we show that the above two definitions for basis of a topology are equivalent.
i.e. We show that the basis define above generates the same topology.
i.e. $\mathfrak{B}_1 = \mathfrak{B}_2$ For this, we will show that $\mathfrak{B}_1$ is a subset of $\mathfrak{B}_2$ and $\mathfrak{B}_2$ is a subset of $\mathfrak{B}_1$.
Case-I:
First, we show that $\mathfrak{B}_1 \subset \mathfrak{B}_2$. For this consider an open set G of (X, τ). By 1^{st} definition we have some elements A_i of $\mathfrak{B}_1$ such that there union is the entire G. i.e.

$$\bigcup_i A_i = G$$

Hence each A_i is a subset of G.
Now, every point $g \in G$ is contained at least one of the A_i's and so each of them is a subset of G.
As for $g \in G$ we have $g \in \{g\} \subset G$.
Finally, since every element of X belongs to at least one element of τ (call it G) (this because of $X \in \tau$) and since G is the union of members of $\mathfrak{B}_1$, so by above argument there exists an element of $\mathfrak{B}_1$ which contained that point and is a subset of G. That is, for some $A_i \in \mathfrak{B}_1$ such that

$$g \in A_i \subset G$$

Hence every such $A_i \in \mathfrak{B}_1$ is an element of $\mathfrak{B}_2$, and since G was arbitrary so this all holds for all elements of τ to access all the elements of $\mathfrak{B}_1$ and so

$$\mathfrak{B}_1 \subset \mathfrak{B}_2 \tag{9}$$

Case-II:
Now we prove $\mathfrak{B}_2 \subset \mathfrak{B}_1$. For this, suppose that G is an open set and let $g \in G$ then by 2^{nd} definition there exists an element B_i (for some i) of $\mathfrak{B}_2$ such that $g \in B_i \subset G$ (for some i). And since g was arbitrary so this all holds for each $g \in G$.
Now, taking union of all such elements of $\mathfrak{B}_2$ we get

$$\bigcup_i B_i = G \; for \; each \; i$$

Hence all such B_i's satisfy the condition for being in $\mathfrak{B}_1$ w.r.t the open set G.
And since G was arbitrary, all above holds for every $G \in \tau$, and hence we can prove for all elements of $\mathfrak{B}_2$. And so every element of $\mathfrak{B}_2$ is also an element of $\mathfrak{B}_1$, so

$$\mathfrak{B}_2 \subset \mathfrak{B}_1 \tag{10}$$

So from (9) and (10) we get

$$\mathfrak{B}_1 = \mathfrak{B}_2$$

Example 21. *Let $X = \{a, b, c, d\}$ be a topological space and consider the class*

$$\mathfrak{B} = \{X, \phi, \{b\}, \{d\}, \{b, c\}, \{a, b\}\}$$

of subsets of X, show that $\mathfrak{B}$ form a base for a topology.
Solution:
Since $\mathfrak{B} = \{X, \phi, \{b\}, \{d\}, \{b, c\}, \{a, b\}\}$ be the class of subsets of X. Now taking the union of members of $\mathfrak{B}$ we get

$$\tau = \{X, \phi, \{b\}, \{d\}, \{b, d\}, \{b, c\}, \{a, b\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c\}\}$$

which is clearly defines a topology on X generated by the class $\mathfrak{B}$ and it is called base for τ.

Exercise 7. Apply the 2^{nd} definition on the above example and find the topology.

Remark 5. *:*

1. *The class of all open intervals on the real line $\mathbb{R}$ form a base for the usual topology on the real line.*
2. *The class of all open discs on the plane $\mathbb{R}^2$ form a base for the usual topology on the plane $\mathbb{R}^2$.*
3. *The class of all singletons of a set form a base for the discrete topology.*

Theorem 6. *Let $\mathfrak{B}$ be a class of subsets of a non-empty set X. Then $\mathfrak{B}$ is a base for some topology on X iff it possesses the following properties.*

1. *$X = \bigcup\{B : B \in \mathfrak{B}\}$*
2. *For any $B_1, B_2 \in \mathfrak{B}$, $B_1 \bigcap B_2$ is the union of members of $\mathfrak{B}$ or if $x \in B_1 \bigcap B_2$ then $\exists\ B_3 \in \mathfrak{B}$ such that $x \in B_3 \subset B_1 \bigcap B_2$.*

Proof. Suppose that $\mathfrak{B}$ is the base for some topology τ on X. Since X and ϕ are open then X is the union of members of $\mathfrak{B}$.
i.e. $X = \bigcup\{B : B \in \mathfrak{B}\}$, so (1) is satisfied.
Now let $B_1, B_2 \in \mathfrak{B}$, then B_1, B_2 are open therefore $B_1 \bigcap B_2$ is also open.
Now, $x \in B_1 \bigcap B_2$, then by definition of base there is $B_3 \in \mathfrak{B}$ such that

$$x \in B_3 \subset B_1 \bigcap B_2$$

Conversely:
Suppose for a class $\mathfrak{B}$ of subsets of X condition (1) and (2) holds, then we are to show that $\mathfrak{B}$ is the base for some topology on X.
Let τ be the collection of subsets of X obtained by taking union of members of $\mathfrak{B}$, we are to show that τ is a topology.

1. From (1), $X = \bigcup\{B : B \in \mathfrak{B}\}$, and by definition ϕ is the empty sub-collection of members of $\mathfrak{B}$
i.e. $\phi = \bigcup\{B \in \mathfrak{B} : B \in \phi \subset \mathfrak{B}\}$, hence $\phi \in \tau$.

2. Let G_i be the collection of members of τ then each G_i is the union of members of $\mathfrak{B}$. Then $\bigcup_i G_i$ is also the union of members of $\mathfrak{B}$.
i.e. $\bigcup_i G_i \in \tau$.

3. Let $G, H \in \tau$, then we show that $G \bigcap H \in \tau$.
For this, let $\{G_i : i \in I\}$ and $\{H_j : j \in I\}$ be two families of members of $\mathfrak{B}$ such that
$$G = \bigcup_i G_i \text{ and } H = \bigcup_j H_j$$
Then
$$\begin{aligned} G \bigcap H &= (\bigcup_i G_i) \bigcup (\bigcup_j H_j) \\ &= \bigcup\{G_i \bigcap H_j : i \in I \wedge j \in I\} \end{aligned}$$
By (2), $G_i \bigcap H_j$ is the union of members of $\mathfrak{B}$ $\forall$ i, j.
Then $G \bigcap H = \{G_i \bigcap H_j : i \in I \wedge j \in I\}$ is also the union of members of $\mathfrak{B}$ and so belongs to τ.
Hence τ is a topology on X w.r.t the base $\mathfrak{B}$.

□

Definition 17. Let (X, τ) be a topological space, a class S of open subsets of X, (i.e. $S \subset \tau$) is a subbase for τ on X iff finite intersection of members of S form a base for τ. Then S is called **Subbase** for τ.

Example 22. *Every open interval (a, b) in the real line $\mathbb{R}$ is the intersection of two infinite open intervals (a, ∞) and $(-\infty, b)$*
$$(a, b) = (a, \infty) \bigcap (-\infty, b)$$
But the open intervals form a base for the usual topology on $\mathbb{R}$, hence class S of all infinite open intervals form a subbase.

Example 23. *Let $X = \{a, b, c, d\}$ and $S = \{\{a, b\}\{b, c\}\{d\}\}$ be the class of subsets of X. Now, by taking the finite intersection of members of S we have*
$$\mathfrak{B} = \{\{a, b\}, \{b, c\}, \{d\}, \{b\}, \phi, X\}, here X = \bigcap\{B : B \in \phi \subset \mathfrak{B}\}$$
Now we show that $\mathfrak{B}$ is a base. For this, taking union of members of $\mathfrak{B}$ we get
$$\tau = \{\{a, b\}, \{b, c\}, \{d\}, \{b\}, \phi, X, \{a, b, d\}, \{b, c, d\}, \{b, d\}, \{a, b, c\}\}$$
which is a topology on X.
So, $\mathfrak{B}$ is a base for τ and S is the subbase for τ on X.

Theorem 7. *Any class $\mathcal{A}$ of subsets of a non-empty set X is the subbase for a unique topology τ on X. i.e. Finite intersection of members of $\mathcal{A}$ form a base for a topology τ on X.*

Proof. We show that the finite intersection of members of $\mathcal{A}$ satisfied the two conditions.
i.e. to be a base for a topology on X.

1. $X = \bigcup\{B : B \in \mathfrak{B}\}$

2. $B_1 \bigcap B_2$ is the union of members of $\mathfrak{B}$ for $B_1, B_2 \in \mathfrak{B}$

X will be the empty intersection of $\mathcal{A}$ and so $X \in \mathfrak{B}$ and so

$$X = \bigcup\{B : B \in \mathfrak{B}\}$$

If $B_1, B_2 \in \mathfrak{B}$ then B_1 and B_2 are finite intersection of members of $\mathcal{A}$. So $B_1 \bigcap B_2$ is also a finite intersection of members of $\mathcal{A}$ and there fore belongs to $\mathfrak{B}$.
Accordingly, $\mathfrak{B}$ is a base for topology τ on X for which $\mathcal{A}$ is a subbase. □

Theorem 8. *Let $\mathcal{A}$ be the class of non-empty subsets of X. Then the topology τ on X generated by $\mathcal{A}$ is the intersection of all topologies on X which contains $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be collection of subsets of X and $\{\tau_i\}$ be the class of topologies on X which contained $\mathcal{A}$, and let

$$\tau^{'} = \bigcap_i \tau_i \text{ and } \mathcal{A} \subset \tau^{'}$$

we wish to prove that $\tau = \tau^{'}$.
Since τ is a topology containing $\mathcal{A}$, and $\tau^{'}$ is the intersection of all such topologies so we have

$$\tau^{'} \subset \tau \tag{11}$$

Now suppose

$$G \in \tau \tag{12}$$

then by definition of topology we have

$$G = \bigcup\{A_{i_1} \bigcap A_{i_2} \bigcap ....\bigcap A_{i_n} : A_{i_k} \in \mathcal{A}\}$$

But $\mathcal{A} \subset \tau^{'}$
$\Rightarrow A_{i_k} \in \mathcal{A} \subset \tau^{'}$
$\Rightarrow A_{i_k} \in \tau^{'}$
$\Rightarrow \{A_{i_1} \bigcap A_{i_2} \bigcap ....\bigcap A_{i_n}\} \in \tau^{'}$
$\Rightarrow G = \bigcup\{A_{i_1} \bigcap A_{i_2} \bigcap ....\bigcap A_{i_n}\} \in \tau^{'}$
From (12) and (13)

$$\tau \subset \tau^{'} \tag{13}$$

From (11) and (14)

$$\tau = \tau^{'}$$

□

Definition 18. Let X be a topological space and let $p \in X$. A class $\mathfrak{B}_\mathfrak{p}$ of open subsets of X containing p is called a local base at p iff for each open set G containing p, $\exists G_p \in \mathfrak{B}_\mathfrak{p}$ such that $p \in G_p \subset G$.

Example 24. *Consider the usual topology* **??**12) *on the plane $\mathbb{R}^2$, and let $p \in \mathbb{R}^2$. Then the collection of all open discs with centered p is a local base at p. Similarly, the class of open intervals $(a - \delta, a + \delta)$ in the real line with centered $a \in \mathbb{R}$ is a local base at he point a.*

Proposition 3. *A point p in a topological space X is a limit point of $A \subset X$ iff each member of some local base $\mathfrak{B}_\mathfrak{p}$ at p contains a point of A different from p.*

Proof. Suppose that p is a limit point of A, i.e.

$$(G-\{p\})\bigcap A\neq\phi$$

where $p\in G\in\tau\ \forall G$.
But $\mathfrak{B}_\mathfrak{p}\subset\tau$, so in particular

$$(B-\{p\})\bigcap A\neq\phi$$

for all $B\in\mathfrak{B}_\mathfrak{p}$.
Conversely:
Suppose that there is some local base $\mathfrak{B}_\mathfrak{p}$ at p such that each member of $\mathfrak{B}_\mathfrak{p}$ contains a point of A different from p. We are to show that p is a limit point.
For this, let G be an open subset of X that contains p. Then $\exists B_0\in\mathfrak{B}_\mathfrak{p}$ for which $p\in B_0\subset G$. But then

$$(G-\{p\})\bigcap A\supset(B_0-\{p\})\bigcap A\neq\phi$$

So $(G-\{p\})\bigcap A\neq\phi$, which implies that p is a limit point of A. □

Theorem 9. *Let $\mathcal{S}$ be a subbase for a topological space Y. Then a function $f:X\to Y$ is continuous iff the inverse of each member of $\mathcal{S}$ is an open subset of X.*

Proof. Let $\mathcal{S}$ be a subbase for any topology τ^* on Y and let for every $S\in\mathcal{S}$ we have $f^{-1}[S]\in\tau$. We are to show that f is continuous. i.e. for $G\in\tau^*\Rightarrow f^{-1}[G]\in\tau$. Let $G\in\tau^*$ then by definition of subbase

$$G=\bigcup_i(S_{i_1}\cap S_{i_2}\cap.......\cap S_{i_n})$$

where $S_{i_k}\in\mathcal{S}$.
Hence $f^{-1}[g]=f^{-1}[\cup_i(S_{i_1}\cap S_{i_2}\cap.......\cap S_{i_n})]$
$=\cup_i f^{-1}[S_{i_1}\cap S_{i_2}\cap.......\cap S_{i_n}]$
$=\cup_i(f^{-1}[S_{i_1}]\cap f^{-1}[S_{i_2}]\cap........\cap f^{-1}[S_{i_n}])$
But $S_{i_k}\in\mathcal{S}\Rightarrow f^{-1}[S_{i_k}]\in\tau$.
Hence $f^{-1}[G]\in\tau$.
Since it is the union of finite intersection of open sets. Accordingly f is continuous.
Conversely
Suppose that f is continuous then the inverse of all open sets including the members of $\mathcal{S}$ is open. □

Theorem 10. *A function $f:X\to Y$ is continuous iff the inverse image of every closed subset of Y is closed in X.*

Proof. Suppose $f:X\to Y$ is continuous. Let $F\subset Y$ be closed. Then $F^c=Y-F$ is open in Y.
Since f is continuous so $f^{-1}[F^c]=f^{-1}[Y-F]=f^{-1}[Y]-f^{-1}[F]$
But $f^{-1}[Y]=X$, so $f^{-1}[F^c]=X-f^{-1}[F]$ is open in X, which implies that $f^{-1}[F]$ is closed.
Conversely
Assume that the inverse image of every closed subset of Y is closed in X. We prove that f is continuous.
Let $G\subset Y$ be open then $G^c=Y-G$ is closed and by assumption

$$f^{-1}[Y-G]=f^{-1}[Y]-f^{-1}[G]=X-f^{-1}[G]$$

i.e. $X-f^{-1}[G]$ closed in X.
$\Rightarrow f^{-1}[G]$ is closed in X.
Hence f is continuous. □

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