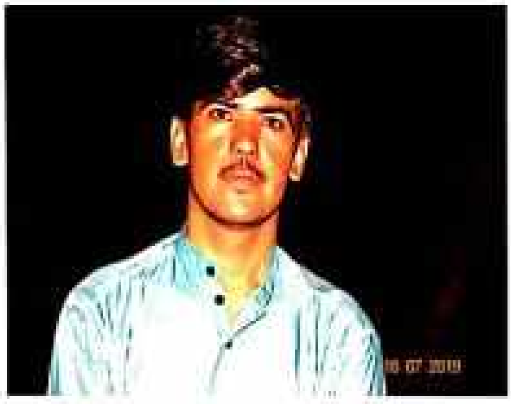

Written by :

UMAR SAEED Bunarai

Msc:Mathematics

Notes:

Fuzzy sets theory

Followed book:

Mathematics of fuzzy sets and fuzzy logic

For Students:

For MSc and BS

Dedicated to:

All my MSc friends

Spacial thanks to:

Fazal Ullah Fazal (Msc mathematics)

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Fuzzy set :- Let X is non empty set ($X \neq \phi$) then a fuzzy set is defined as follows;

$$A = \{(x, u_A(x)) \mid x \in X\}$$

Where x is a particular element of X and $u_A : X \longrightarrow [0, 1]$ is called membership function and $u_A(x)$ is called degree of membership of x.

Example :- $X = \{1, 2, 3, 4, 5\}$
then fuzzy set of X as;

$A = \{(1, 0.9), (2, 0.5), (3, 0.4), (4, 0.6), (5, 0.2)\}$
or also we written as.

$u_A(1) = 0.9$, $u_A(2) = 0.5$
$u_A(3) = 0.4$, $u_A(4) = 0.6$, $u_A(5) = 0.2$.

Lotfi Zadeh :-

* He was born in 1921 in Azerbaijan (His family from Iran).
* He was mathematician, electricle engineer and artificial inteligence reacher.
* professor of computer science at the universty of california, Berkeley (US) (Amerika).
* He published fuzzy set theory in 1965.
* In 1973 he proposed his theory of fuzzy logic.
* Died in 6 september 2017.

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Intersection of fuzzy sets:-

Let μ_A and μ_B be two fuzzy sets of X. Then intersection of μ_A and μ_B is denoted by $\mu_A \cap \mu_B$ and defined as

$$(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example :-

$$X = \{1, 2, 3\}$$

$\mu_A = \{(1, 0.8)(2, 0.3)(3, 0.7)\}$

$\mu_B = \{(1, 0.2)(2, 0.7)(3, 0.3)\}$.

$$(\mu_A \cap \mu_B) = \{(1, \min(0.8, 0.2)), (2, \min(0.3, 0.7)) (3, \min(0.7, 0.3))\}.$$

$$= \{(1, 0.2)(2, 0.3)(3, 0.3)\}$$

Example :-

$$X = \{x_1, x_2\}.$$

$\mu_A = \{(x_1, 0.3)(x_2, 0.8)\}$

$\mu_B = \{(x_1, 0.7)(x_2, 0.2)\}$

$$(\mu_A \cap \mu_B) = \{(x_1, 0.3)(x_2, 0.2)\}.$$

Union of fuzzy sets:-

Let μ_A and μ_B be two fuzzy sets of X. Then Union of μ_A and μ_B is denoted by $\mu_A \cup \mu_B$ and defined as.

$$(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$$

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Example :- $X = \{1, 2, 3\}$

$U_A = \{(1, 0.8) (2, 0.3) (3, 0.7)\}$
$U_B = \{(1, 0.2) (2, 0.7) (3, 0.3)\}$.

$((U_A \cup U_B)(x) = \{(1, max(0.8, 0.2)) (2, max(0.3, 0.7))$
$(3, max(0.7, 0.3))\}$.

$= \{(1, 0.8) (2, 0.7) (3, 0.7)\}$.

Example :-
$X = \{x_1, x_2\}$.

$U_A = \{(x_1, 0.3) (x_2, 0.8)\}$
$U_B = \{(x_1, 0.7) (x_2, 0.2)\}$.

$(U_A \cup U_B)(x) = \{(x_1, 0.7), (x_2, 0.8)\}$.

Universal set :- A set whose degree of membership is 1 called universal set.

empty set :- A set whose degree of membership is 0 called empty set.

Compliment :- Compliment of fuzzy set is denoted by $U_{\bar{A}}(x)$ and defined as:

$U_{A^c}(x) = 1 - U_A(x)$.

Example :- $X = \{1, 2, 3\}$
$A = \{(1, 0.6), (2, 0.3) (3, 0.7)\}$
then
$A^c(x) = \{(1, 1-0.6), (2, 1-0.3), (3, 1-0.7)\}$
$= \{(1, 0.4), (2, 0.7), (3, 0.3)\}$

Example :- $X = \{x_1, x_2, x_3\}$

$A = \{(x_1, 0.5)(x_2, 0.6)(x_3, 0.4)\}$.

$A^c = \{(x_1, 0.5)(x_2, 0.4)(x_3, 0.6)\}$

Cross product (cartesian product):-

Let $U_1, U_2, U_3, \dots, U_n$ be n- fuzzy sets of $X_1, X_2, X_3, \dots, X_n$ respectivly,

$U_1 \times U_2 \times U_3 \times \dots \times U_n(x_1, x_2, x_3, \dots, x_n)$

$= \min\{U_1(x_1), U_2(x_2), U_3(x_3), \dots U_n(x_n)\}$

Example :-

$X_1 = \{1, 2, 3\}$

$X_2 = \{4, 5, 6\}$

$A = \{(1, 0.2)(2, 0.3)(3, 0.5)\}$

$B(x) = \{(4, 0.8)(5, 0.6)(6, 0.3)\}$

$U_1 \times U_2(x_1, x_2) = \min(A(x_1), B(x_2))$

$$A \times B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \end{bmatrix} \end{array}$$

$= \{[(1,4), 0.2], [(1,5), 0.2], [(1,6), 0.2],$
$[(2,4), 0.3], [(2,5), 0.3], [(2,6), 0.3],$
$[(3,4), 0.5], [(3,5), 0.5], [(3,6), 0.3]\}$

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Example $X_1 = \{1, 2, 3, 4\}$
$X_2 = \{5, 6, 7, 8\}$.

$A (X_1) = \{(1, 0.3)\ (2, 0.5)\ (3, 0.7)\ (4, 0.9)\}$

$B (X_2) = \{(5, 0.2)\ (6, 0.4)\ (7, 0.6)\ (8, 0.8)\}$

$\mu_{A \times B_{1,2}} (x_1, x_2) = \min \{ \mu_1(x_1), \mu_2(x_2)\}$

$A \times B = \{ [(1,5), \min(0.3, 0.2)]\ [(1,6), \min(0.3, 0.4)]$
$[(1,7), \min(0.3, 0.6)]\ [(1,8), \min(0.3, 0.8)]$
$[(2,5), \min(0.5, 0.2)]\ [(2,6), \min(0.5, 0.6)]\ [(2,7), \min(0.5, 0.6)]$
$[(2,8), \min(0.5, 0.8)]\ [(3,5), \min(0.7, 0.2)]\ [(3,6), \min(0.7, 0.4)]$
$[(3,7), \min(0.7, 0.6)]\ [(3,8), \min(0.7, 0.8)]\ [(4,5), \min(0.9, 0.2)]$
$[(4,6), \min(0.9, 0.4)]\ [(4,7), \min(0.9, 0.6)]\ [(4,8), \min(0.9, 0.8)] \}$

$= \{ [(1,5), 0.2], [(1,6), 0.3]\ [(1,7), 0.3]\ [(1,8), 0.3]$
$[(2,5), 0.2]\ [(2,6), 0.4]\ [(2,7), 0.5]\ [(2,8), 0.5]$
$[(3,5), 0.2]\ [(3,6), 0.4]\ [(3,7), 0.6]\ [(3,8), 0.7]$
$[(4,5), 0.2]\ [(4,6), 0.4]\ [(4,7), 0.6]\ [(4,8), 0.8] \}$.

Level set (α-cutset) :-

Let X be a non empty set and μ_A be a fuzzy set of X than level set or α-cut set is defined as

$A_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha \}$ for $\alpha \in (0, 1]$

example :-

$X = \{1, 2, 3\}$

$\mu_A = \{(1, 0.3)\ (2, 0.7)\ (3, 0.5)\}$

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$\alpha = 0.3$

then $A_\alpha = \{ (x \in X \mid x \geq \alpha \}$

$A_{0.3} = \{1, 2, 3\}$

if $\alpha = 0.6$

then $A_{0.6} = \{2\}$

if $\alpha = 0.4$

then $A_{0.4} = \{2, 3\}$

Core set :- Let X be non-empty and U_A be a fuzzy of X then core set is defined as.

$$A_1 = \{ x \in X \mid U_A(x) \geq 1 \}.$$

Example :-

$$X = \{x_1, x_2, x_3\}$$

$$U_A = \{ (x_1, 1) (x_2, 1) (x_3, 0.9) \}$$

$$A_1 = \{x_1, x_2\}$$

Support set :- Let X be non-empty set and U_A be a fuzzy of X then support set be defined as.

$$\text{Supp } A = \{ x \in X \mid U_A(x) > 0 \}$$

Example :-

$$X = \{x_1, x_2, x_3\}$$

$$U_A = \{ (x_1, 0) (x_2, 0.3) (x_3, 0.1) \}$$

then

$$\text{Supp } A = \{x_2, x_3\}$$

Subset of fuzzy sets :-

Let U_A and U_B be two

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fuzzy sets of X then we said that u_A is subset of u_B if $u_A(x) \leq u_B(x)$ $\forall x \in X$

Example :-

$$X = \{x_1, x_2\}$$

$$u_A = \{(x_1, 0.2)\ (x_2, 0.3)\}$$

$$u_B = \{(x_1, 0.4)\ (x_2, 0.5)\}$$

$$u_C = \{(x_1, 0.4)\ \{x_2, 0.1)\}$$

Thus $u_A \leq u_B$ because $u_A(x) \leq u_B(x)$ $\forall x \in X$.

And

$u_A \nleq u_C$ because $u_A(x_2) \nleq u_B(x_2)$

Also $u_C \nleq u_A$ because $u_C(x_1) \nleq u_A(x_1)$.

Properties of Fuzzy sets :-

Note we denote Union by $\vee$ and intersection by $\wedge$

(1) Associative property :-

$$u_A \wedge (u_B \wedge u_C) = (u_A \wedge u_B) \wedge u_C$$

LHS

$$(u_A \wedge (u_B \wedge u_C))x = \min\{u_A(x), (u_B \wedge u_C)(x)\}$$

$$= \min\{u_A(x), \min(u_B(x), u_C(x))\}$$

$$= \min\{u_A(x), u_B(x), u_C(x)\}$$

RHS.

$$((u_A \wedge u_B) \wedge u_C)x = \min\{(u_A \wedge u_B)x, u_C(x)\}$$

$$= \min\{u_A(x), u_B(x), u_C(x)\}$$

RHS = LHS

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Similarly $\mu_A \vee (\mu_B \vee \mu_C) = (\mu_A \vee \mu_B) \vee \mu_C$

(2) Commutative property :-

$$\mu_A \wedge \mu_B = \mu_B \wedge \mu_A$$

$$\mu_A \vee \mu_B = \mu_B \vee \mu_A$$

(3) Identity property :-

$$A \wedge X = A$$

$$A \vee \phi = A$$

(4) Absorbtion by ϕ and X

$$A \wedge \phi = \phi$$

$$A \vee X = X$$

(5) Idempotence :-

$$A \vee A = A$$

$$A \wedge A = A.$$

proof :-

$$A \wedge A = A$$

$$(A \wedge A)x = \min(A(x), A(x))$$

$$= A(x)$$

$$A \wedge A = A$$

similarly all (4) and (5) properties...

(6) De Morgen Laws :-

$$\overline{A \wedge B} = \bar{A} \vee \bar{B}$$

LHS

$$(\overline{A \wedge B})x = 1 - (A \wedge B)x$$

$$= 1 - \min(A(x), B(x))$$

$$= \max\{1 - A(x), 1 - B(x)\}$$

$$= \max\{\bar{A}(x), \bar{B}(x)\}.$$

$$= \bar{A}(x) \vee \bar{B}(x)$$

$$\overline{A \wedge B} = \bar{A} \vee \bar{B}$$

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Similarly $\overline{A \vee B} = \bar{A} \wedge \bar{B}$.

(7) Distributive :-

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C).$$
$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

(8) Involution :-

$$\bar{\bar{A}} = A$$

$$\bar{A}(x) = 1 - A(x)$$
$$\bar{\bar{A}}(x) = 1 - \bar{A}(x) = 1 - (1 - A(x))$$
$$= A(x)$$
$$\bar{\bar{A}} = A$$

(9) Absorption :-

$$A \wedge (A \vee B) = A.$$

$$(A \wedge (A \vee B))x = \min(A(x), (A \vee B)(x)).$$
$$= \min(A(x), \max(A(x), B(x))).$$
$$= \min(A(x), B(x) \text{ or } A(x)).$$
$$= A(x).$$
$$A \wedge (A \vee B) = A$$

Similarly $A \vee (A \wedge B) = A$.

Assignment :- If A is a non-classical fuzzy set $A : X \rightarrow [0,1]$ (i.e. ~~then~~ there exist $x \in X$ with $A(x) \notin \{0,1\}$ then

$$A \wedge \bar{A} \neq \phi$$
$$A \vee \bar{A} \neq X$$

Proof :-

if $x \in X$ is such that $0 < A(x) < 1$ then $0 < \bar{A}(x) < 1$ and then

$0 < A \wedge \bar{A}(x) < 1$ and $0 < A \vee \bar{A} < 1$

For example :-

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Ph 09

$X = \{1, 2\}$

$A = \{(1, 0.4)\ (2, 0.3)\}$ and

$\bar{A} = \{(1, 0.6)\ (2, 0.7)\}$

then

$A \wedge \bar{A} = \{(1, \min(0.4, 0.6))\ ((2, \min(0.3, 0.7)))\}$

$A \wedge \bar{A} = \{(1, 0.4)\ (2, 0.3)\} \neq \phi$

$$\boxed{A \wedge \bar{A} \neq \phi}$$

Also

$A \vee \bar{A} = \{(1, \max(0.4, 0.6))\ (2, \max(0.3, 0.7))\}$

$A \vee \bar{A} = \{(1, 0.6)\ (2, 0.7)\} \neq X$

$$\Rightarrow \boxed{A \wedge \bar{A} \neq X}$$

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Ch # 2

Negation :- A Function $N:[0,1] \to [0,1]$ is called Negation if

* $N(0) = 1$ and $N(1) = 0$ and N is non-increasing ($x \le y \Rightarrow N(x) \ge N(y)$).

Strict negation :-

A Negation is called strict negation if it is stricktly decreasing

$x < y \Rightarrow N(x) > N(y)$

Strong negation :-

A strict negation is said to be strong negation if it is also invalutive i-e $N(N(x)) = x$.

Example :-

$$N(x) = 1 - x$$

* $N(0) = 1$, $N(1) = 0$.

and $x \le y$

$-x \ge -y \Rightarrow 1-x \ge 1-y$

$\Rightarrow N(x) \ge N(y)$

It is negation.

* $x < y \Rightarrow -x > -y$.

$\Rightarrow 1-x > 1-y \Rightarrow N(x) > N(y)$

It is strict negation.

* $N(x) = 1 - x$

$N(N(x)) = 1 - N(x)$

$= 1 - (1 - x)$

$= 1 - 1 + x$

$N(N(x)) = x$

It is strong negation.

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Example :- $N_\lambda(x) = \dfrac{1-x}{1+\lambda x}, \quad \lambda > -1$

* $N_\lambda(0) = 1$, $N_\lambda(1) = 0$ and.

$$x \le y$$

$$-x \ge -y \Rightarrow 1-x \ge 1-y$$

$$\Rightarrow \frac{1-x}{1+\lambda x} \ge \frac{1-y}{1+\lambda y}$$

$$\Rightarrow N_\lambda(x) \ge N_\lambda(y)$$

It is Negation.

* $x < y$

$$\frac{1-x}{1+\lambda x} > \frac{1-y}{1+\lambda y} \Rightarrow N_\lambda(x) > N_\lambda(y)$$

It is strict negation.

*

$$N_\lambda(x) = \frac{1-x}{1+\lambda x}$$

$$N(N_\lambda(x)) = \frac{1-N(x)}{1+\lambda N(x)}$$

$$= \frac{1-\frac{1-x}{1+\lambda x}}{1+\lambda\left(\frac{1-x}{1+\lambda x}\right)} = \frac{\frac{1+\lambda x-(1-x)}{\cancel{1+\lambda x}}}{\frac{1+\lambda x+\lambda(1-x)}{\cancel{1+\lambda x}}}$$

$$= \frac{1+\lambda x-1+x}{1+\lambda x+\lambda-\cancel{\lambda x}} = \frac{\lambda x+x}{1+\lambda} = \frac{\cancel{(\lambda+1)}x}{\cancel{1+\lambda}}$$

$$N(N_\lambda(x)) = x$$

It is strong Negation

$\Rightarrow N_\lambda(x) = \dfrac{1-x}{1+\lambda x}$ strong Negation called λ- Compement.

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Triangular Norms and Conorms :-

T-Norm (t-norm) :-

t-norm is function $T : [0,1] \times [0,1] = [0,1]^2 \rightarrow [0,1]$ that satisfies the following properties:

$T_1 : T(x,1) = x$ (identity)

$T_2 : T(x,y) = T(y,x)$ (commutative)

$T_3 : T(x, T(y,z)) = T(T(x,y), z)$ (associativity

T_4 : if $x \le u$ and $y \le v$ then
$T(x,y) \le T(u,v)$.

S-Norm (t-conorm) :-

s-norm is a function $S : [0,1]^2 \rightarrow [0,1]$ that satisfies following properties:

$S_1 : S(x,0) = x$ (identity)

$S_2 : S(x,y) = S(y,x)$ (Commutative)

$S_3 : S(x, S(y,z)) = S(S(x,y), z)$
associative

S_4 : if $x \le u$, $y \le v$ then
$S(x,y) \le S(u,v)$.

Example :- $T(x,y) = \min\{x,y\}$ $x,y \in \{0,1\}$

T_1: $T(x,1) = \min\{x,1\} = x$ satisfied

S_1: $T(x,0) = \min\{x,0\} = 0 \neq x$ thus s-norm condition not satisfied.

T_2 : $T(x,y) = \min\{x,y\} = \min\{y,x\} = T\{y,x\}$
satisfied.

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T_3: $T\{x, T(y,z)\} =$

$= \min\{x, \min(y,z)\}$

$= \min\{x,y,z\}$

$= \min\{\min(x,y), z\}$

$= \min\{T\{x,y\}, z\}$

$= T\{T(x,y), z\}$ satisfy

T_4: $x \le u$ and $y \le v$.

then $\min\{x,y\} \le \min\{u,v\}$

$\therefore T\{x,y\} \le T\{u,v\}$.

Thus

$T(x,y) = \min\{x,y\}$ is t-norm.

Example :-

$T(x,y) = x \cdot y$ is t-norm ?

T_1 : $T(x,1) = x$. satisfy

T_2 : $T(x,y) = xy = yx = T(y,x)$ satisfy

T_3 : $T(x, T(y,z)) = T(x, y\cdot z) = xyz$

$= (xy)\cdot z = T(xy, z)$

$= T(T(x,y), z)$ satisfy

T_4 : $x \le u$ and $y \le v$

$\Rightarrow xy \le uv$

$\Rightarrow T\{x,y\} \le T(u,v)$

$T(x,y) = xy$ is t-norm proved.

Example :-

$S(x,y) = x + y - xy$

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S_1: $S(x,0) = x$ satisfy

S_2: $S(x,y) = x+y-xy$

$= y+x-yx$

$= S(y,x)$. satisfy.

S_3: $S(x, S(y,z)) = S(x, y+z-yz)$

LHS→

$= x + y+z-yz - x(y+z-yz)$

$= x+y+z-yz-xy-xz+xyz$.

RHS $S(S(x,y),z)$

$= S(x+y-xy, z)$

$= x+y-xy+z - (x+y-xy)(z)$

$= x+y-xy+z-xz-yz+xyz$

$= x+y+z-yz-xy-xz+xyz$

$\Rightarrow$ LHS = RHS satisfy.

S_4: $x \le u$ and $y \le v$ $\because x \in [0,1]$

$(x+y-xy) \le (u+v-uv)$

$S(x,y) \le S(u,v)$.

example

$0.2 < 0.3$, $0.4 < 0.5$

$0.2 + 0.4 - (0.2)(0.4) \le 0.3 + 0.5 - (0.3)(0.5)$

$\boxed{0.52 < 0.65}$

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Definition :-

Let $F(x)$ be collections of all fuzzy sets of X. and $T, S : [0,1]^2 \to [0,1]$ at t-norm and t-conorm. then operation for any μ_A, μ_B can be defined as following: $(\mu_A T \mu_B)x = \mu_A(x) T \mu_B(x)$

$$(\mu_A S \mu_B)x = \mu_A(x) S \mu_B(x).$$

proposition :-(2.11)

given any t-norm T and t-conorm S we have $T(x,0) = 0$ and $S(x,1) = 1 \ \forall x \in [0,1]$

proof :- From T_1 we have

$T(0,1) = 0$

$T(0,x) \le T(0,1) = 0$ ∴ From T_4 as $0 \le 0, x \le 1$ $\Rightarrow T(0,x) \le T(0,1)$.

$\Rightarrow T(0,x) = 0$

$\boxed{T(x,0) = 0}$ ∴ From T_2, $T(x,y) = T(y,x)$

Now From S_1 we have

$S(1,0) = 1$.

$1 = S(1,0) \le S(1,x)$ ∴ From S_4 as $1 \le 1, 0 \le x$ $\Rightarrow S(1,0) \le S(1,x)$

$1 = S(1,x)$

$\Rightarrow S(1,x) = 1$

$\boxed{S(x,1) = 1}$ ∴ From S_2. $S(x,y) = S(y,x)$.

Proposition :- (2.12)

Let T and S be t-norm and t-conorm then,

(i) $T(x,y) \le x \wedge y$ and (ii) $S(x,y) \ge x \vee y$ for any $x, y \in [0,1]$

Proof :- since from T_{1} we have

$T(x,1) = x$

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$T(x,y) \le T(x,1) = x$ —① ∵ From T_4 as $x \le x, y \le 1$

$\Rightarrow T(x,y) \le T(x,1)$

Also From T_2 we have

$T(x,y) = T(y,x) \le T(y,1) = y$ —— ②

∵ From T_4, $y \le y$, $x \le 1$

$T(y,x) \le T(y,1)$

then From T_1.

$T(y,1) = y$

From ① and ②

$T(x,y) \wedge T(x,y) \le x \wedge y$

$\Rightarrow \boxed{T(x,y) \le x \wedge y}$

(ii)

$x = S(x,0) \le S(x,y)$ —① ∵ From S_4 and S_1

Also

$y = S(y,0) \le S(y,x) = S(x,y)$ —— ②

∵ From S_2, S_4 and S_1

From ① and ②

$S(x,y) \vee S(x,y) \ge x \vee y$

$\Rightarrow \boxed{S(x,y) \ge x \vee y}$

De-Morgen Triplet :-

A triplet (S,T,N) is called De-Morgen triplet if T is t-norm and S is s-norm and N is a strong Negation. Also Full fill De-Morgen law;

$$S(x,y) = N(T(N(x), N(y)))$$

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2.15

Example (1) :- $T(x,y) = x \wedge y$

$S(x,y) = x \vee y$

$N(x) = 1 - x$

Form a de-Morgen triplet ?

(1) $T(x,y) = x \wedge y$ it is a t-norm

see in ph 12 (example)

(2) $S(x,y) = x \vee y$

(i) $S_1: S(x,0) = \max(x,y) = \max(x,0)$

$= x$ satisfy.

(ii) $S(x,y) = \max(x,y) = \max(y,x) = S(y,x)$

(iii) $S(x, S(y,z)) = \max(x, y \vee z)$

$= \max(x, \max(y,z))$

$= \max(x,y,z)$

$S(S(x,y),z) = \max(x \vee y, z)$

$= \max(\max(x,y), z)$

$= \max(x,y,z)$

LHS = RHS

(iv) if $x \le u$, $y \le v$.

then $\max(x,y) \le \max(u,v)$

$S(x,y) \le S(u,v)$.

Thus it is s-norm.

(3) $N(x) = 1 - x$ it is a strong Negation see ph 10. example

(4) $S(x,y) = N(T(N(x), N(y)))$

$= N[N(x) \wedge N(y)]$ $\because$ T-Norm

$= 1 - [N(x) \wedge N(y)]$ $\because$ def Negation

$= (1 - N(x)) \vee (1 - N(y))$

$= (1 - (1-x)) \vee (1 - (1-y))$

$= x \vee y = S(x,y)$ $\because$ def S-Norm

It form a de-morgen triplet.

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example (2) :- $T(x,y) = x \cdot y$
$S(x,y) = x+y-xy$
$N(x) = 1-x$ form a de-morgen triplet

Solution :-

① $T(x,y) = x \cdot y$
It is t-Norm (see ph 13)

② $S(x,y) = x+y-xy$
It is S-Norm. (see ph 13)

③ $N(x) = 1-x$
It is strong negation (see ph 10)

if given example satisfy de-morgen law then it form a de-morgen triplet.

$S(x,y) = N[T(N(x), N(y))]$

$= N[N(x) \cdot N(y)]$ ∵ def T-Norm

$= 1-(N(x))(N(y))$ ∵ because of negation.

$= 1-[(1-x)(1-y)]$

$= 1-[1-y-x+xy]$

$= 1-1+y+x-xy$

$= x+y-xy$

$S(x,y) = S(x,y)$

Thus it form de-Morgen triplet.

example (3) (2.17) :-

$T(x,y) = (x+y-1) \vee 0$

$S(x,y) = (x+y) \wedge 1$

$N(x) = 1-x$

Form a demorgen triplet.

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$T(x,y) = \max[x+y-1, 0]$ pg 20

① (i) $T(x,1) = \max[x+y-1, 0] = \max[x, 0]$
$= x$.

(ii) $T(x,y) = \max[x+y-1, 0] = \max[0, x+y-1]$
$= T(y,x)$

(iii) $T(x, T(y,z)) = T(x, \max[y+z-1, 0])$
$= (x+(y+z-1)-1) \vee 0$
$= (x+y+z-2) \vee 0$.

$T(T(x,y), z) = T[(x+y-1) \vee 0, z]$
$= (x+y-1+z-1) \vee 0$
$= (x+y+z-2) \vee 0$ (satisfy).

(iv) $x \le u$, $y \le v$
$(x+y-1) \vee 0 \le (u+v-1) \vee 0$,
$T(x,y) \le T(u,v)$.

$T(x,y) = (x+y-1) \vee 0$ is t-norm.

(i) $S(x,0) = [(x+y) \wedge 1] = (x \wedge 1)$
$= x$.

(ii) $S(x,y) = (x+y) \wedge 1 = (y+x) \wedge 1 = S(y,x)$

(iii) $S(x, S(y,z)) =$
$= S(x, (y+z) \wedge 1) = (x+y+z) \wedge 1$.

$S(S(x,y), z) = S((x+y) \wedge 1, z)$
$= (x+y+z) \wedge 1$

(iv) $x \le u$, $y \le v$
$(x+y) \wedge 1 \le (u+v) \wedge 1$.
$S(x,y) \le S(u,v)$.

$S(x,y) = (x+y) \wedge 1$ is s-norm.

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$N(x) = 1-x$ is a strong Negation

(4) $S(x,y) = N[T(N(x), N(y))]$

$= N[(N(x) + N(y) - 1) \vee 0]$

$= N[(1-x+1-y-1) \vee 0]$

$= N[(1-x-y) \vee 0]$

$= 1 - [(1-x-y) \vee 0]$

$= 1 - \max[1-x-y, 0]$

$= \min[1-(1-x-y), 1-0]$

$= \min[x+y, 1] = (x+y) \wedge 1 = S(x,y)$

It is form a de-Morgen triplet.

Example (2.18):-

$$T(x,y) = \begin{cases} x \wedge y & \text{if } x+y > 1 \\ 0 & \text{if } x+y \le 1 \end{cases}$$

(1) $T(x,1) = \begin{cases} x \wedge 1 & \text{if } x+1 > 1 \\ 0 & \text{if } x+1 \le 1 \end{cases}$

since $x+1 > 1$

$= \min(x, 1)$

$= x$ satisfy.

(2) $T(x,y) = \begin{cases} x \wedge y & \text{if } x+y > 1 \\ 0 & \text{if } x+y \le 1 \end{cases}$

$= \begin{cases} y \wedge x & \text{if } y+x > 1 \\ 0 & \text{if } y+x \le 1 \end{cases}$

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$= T(y,x).$

(3) $T(x, T(y,z)) = \begin{cases} T(x, y\wedge z) & \text{if } y+z>1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} x\wedge y\wedge z & \text{if } x+y\wedge z>1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} x\wedge y\wedge z & \text{if } x+y+z>1 \\ 0 & \text{otherwise} \end{cases}$

Also

$T(T(x,y),z) = \begin{cases} T(x\wedge y, z) & \text{if } x+y>1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} x\wedge y\wedge z & \text{if } x\wedge y+z>1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} x\wedge y\wedge z & \text{if } x+y+z>1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow T(x, T(y,z)) = T(T(x,y),z)$

(4) if $x \le u$, $y \le v$.

Case (I) $x+y>1$, $u+v>1$

$x\wedge y \le u\wedge v$

$T(x,y) \le T(u,v)$

Case (II) $x+y\le 1$, $u+v>1$

$0 \le u\wedge y$

$T(x,y) \le T(u,v)$

Case (III) $x+y\le 1$, $u+v\le 1$

$0 \le 0.$

$T(x,y) \le T(u,v)$

$\Rightarrow T$ is t-norm.

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$$S(x,y) = \begin{cases} x \vee y & \text{if } x+y<1 \\ 1 & \text{otherwise} \end{cases}$$

(i) $$S(x,0) = \begin{cases} x \vee 0 & \text{if } x+0<1 \\ 1 & \text{otherwise} \end{cases}$$

$$\boxed{S(x,0) = x}$$

(ii) $$S(x,y) = \begin{cases} x \vee y & \text{if } x+y<1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} y \vee x & \text{if } y+x<1 \\ 1 & \text{otherwise} \end{cases}$$

$$S(x,y) = S(y,x)$$

(iii) $$S(x, S(y,z)) = \begin{cases} S(x, y \vee z) & \text{if } y+z<1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} x \vee y \vee z & \text{if } x+y \vee z<1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} x \vee y \vee z & \text{if } x+y+z<1 \\ 1 & \text{otherwise} \end{cases}$$

$$S(S(x,y),z) = \begin{cases} S(x \vee y, z) & \text{if } x+y<1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} x \vee y \vee z & \text{if } x+y+z<1 \\ 1 & \text{otherwise} \end{cases}$$

$\Rightarrow S(x, S(y,z)) = S(S(x,y),z)$

(iv) $x \le u$, $y \le v$

$x+y<1$ & $u+v<1$

$x \vee y \le u \vee v \Rightarrow S(x,y) \le S(u,v)$

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case II

$x+y \le 1$ and $u+v \ge 1 \Rightarrow x \vee y \le 1$

$\max(u,v) \le 1 \quad \Rightarrow s(x,y) \le s(u,v)$

$S(x,y) \le S(u,v)$.

case III $\quad x+y \ge 1$ and $u+v \ge 1$.

$1 \le 1$

$S(x,y) = S(u,v)$.

$\Rightarrow S(x,y)$ is t-conorm.

Now For de-Morgan triplit.

$S(x,y) = N[T(N(x), N(y))]$

$$= \begin{cases} N(N(x) \wedge N(y)) & \text{if } N(x)+N(y) > 1 \\ N(0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1-\min(N(x), N(y)) & \text{if } N(x)+N(y) > 1. \\ 1-(0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \max(1-N(x), 1-N(y)) & \text{if } (1-N(x))+(1-N(y)) < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \max(1-(1-x), 1-(1-y)) & \text{if } (1-(1-x)+(1-(1-y)) < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \max(x,y) & \text{if } x+y < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} x \vee y & \text{if } x+y < 1. \\ 1 & \text{otherwise} \end{cases}$$

$S(x,y) = S(x,y)$

Thus triplet (S,T,N) Form a demorgan triplet.

Fuzzy implication :- Let $I : [0,1]^2 \to [0,1]$ be a function if the following conditions are fulfilled:

(i) if $x \le y$, then $I(x,z) \ge I(y,z)$ i.e I is decreasing in it first variable.

(ii) if $y \le z$ then $I(x,y) \le I(x,z)$ i.e I is increasing in it second variable.

(iii) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$

then I is called fuzzy implication.

Example :- (2.35)

$$I_3(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y/x & \text{if } x > y \end{cases}$$

are fuzzy implication.

proof :-

(i) if $x \le y$, then $I(x,z) \ge I(y,z)$, i.e I is decreasing in it first variable.

$\Rightarrow$ if $x \le y$. there are three cases.

(a) $x \le y \le z$

$\Rightarrow x \le z \Rightarrow I(x,z) = 1$

$\Rightarrow y \le z \Rightarrow I(y,z) = 1$.

Thus $\boxed{I(y,z) = I(x,z)}$

(b) $x \le z < y$

$\Rightarrow x \le z \Rightarrow I(x,z) = 1$

$\Rightarrow y > z \Rightarrow I(y,z) = z/y$.

Thus $\boxed{I(x,z) > I(y,z)}$

(c) $z < x < y$

$\Rightarrow x > z \Rightarrow z/x$ since $x \le y$, $\frac{1}{x} \ge \frac{1}{y}$

$\Rightarrow y > z \Rightarrow z/y$

$\frac{z}{x} \ge \frac{z}{y}$

$\boxed{I(x,z) \ge I(y,z)}$

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Thus in all three cases I is decreasing in its 1st variable.

(ii) if $y \le z$ then $I(x,y) \le I(x,z)$
i.e I is increasing in 2nd varible
if $y \le z$ then there are three cases
(a) $x \le y \le z$
$\Rightarrow x \le y \Rightarrow I(x,y) = 1$
$\Rightarrow x \le z \Rightarrow I(x,z) = 1$.
Thus $\boxed{I(x,y) = I(x,z)}$

(b) $y < x \le z$.
$\Rightarrow x \le z \Rightarrow I(x,z) = 1$
$\Rightarrow y < x \Rightarrow I(x,y) = y/x$
Thus $\boxed{I(x,y) < I(x,z)}$.
(c) $y < z < x$
$\Rightarrow x > y \Rightarrow I(x,y) = y/x$
$\Rightarrow x > z \Rightarrow I(x,z) = z/x$
since $y \le z$
$$\frac{y}{x} \le \frac{z}{x}$$
$$\Rightarrow \boxed{I(x,y) \le I(x,z)}$$

Thus all in three cases I is increasing in 2nd variable.

(iii) $I(1,0) = 0$
since $x > y \Rightarrow I(1,0) = y/x = \frac{0}{1} = 0$
$\boxed{I(1,0) = 0}$
$I(0,0) = I(1,1)$
since $x = y$ in both cases then

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$I(0,0) = I(1,1) = 1$

Thus from (i), (ii) and (iii) I are fuzzy implication.

Example :- $I(x,y) = \max\{1-x, y\}$ is fuzzy implication.

proof :-

(i) if $x \le y$ then $I(x,z) \ge I(y,z)$.

$x \le y \Rightarrow I(x,z) \ge I(y,z)$

$\max\{1-x, z\} \ge \max\{1-y, z\}$.

example.

$x = 0.3,\ y = 0.4,\ z = 0.5$

$\max\{1-x, z\} \ge \max\{1-y, z\}$

$\max\{0.7, 0.5\} \ge \max\{0.6, 0.5\}$.

$\boxed{0.7 \ge 0.6}$

$\Rightarrow$ since $x \le y$.

$-x \ge -y$.

$1-x \ge 1-y$

$\max(1-x, z) \ge \max\{1-y, z\}$.

$\boxed{I(x,z) \ge I(y,z)}$.

(ii) $y \le z$.

$\Rightarrow \max(1-x, y) \le \max\{1-x, z\}$

$\boxed{I(x,y) \le I(x,z)}$

example $0.4 \le 0.6$

$\max\{0.3, 0.4\} \le \max\{0.3, 0.6\}$

$\boxed{0.4 \le 0.6}$.

(iii) $I(1,0) = \max\{1-1, 0\}$

$= \max\{0, 0\} = 0$

$\Rightarrow \boxed{I(1,0) = 0}$

$I(0,0) = \max\{1-0, 0\} = \max\{1, 0\} = \boxed{1}$

$\boxed{I(0,0) = 1}$

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$I(1,1) = \max\{1-1, 1\} = \max\{0,1\}$

$\Rightarrow \boxed{I(1,1) = 1}$

Thus from (i) (ii) and (iii) $I(x,y) = (1-x) \vee y$ is fuzzy implication.

Example: $I(x,y) = \min\{1-x+y, 1\}$ is fuzzy implication.

proof :-

(i) if $x \le y$

$x \le y$

$= -x \ge -y$

$= (1-x+z) \ge (1-y+z)$

$\min\{1-x+z, 1\} \ge \min\{1-y+z, 1\}$

$\boxed{I(x,z) \ge I(y,z)}$

(ii) if

$y \le z$

$= -x+y \le -x+z$

$= 1-x+y \le 1-x+z$

$\min\{1-x+y, 1\} \le \{1-x+z, 1\} \min$

$\boxed{I(x,y) \le I(x,z)}$

(iii) $I(1,0) = \min\{1-1+0, 1\} = \min\{0,1\}$

$\Rightarrow \boxed{I(1,0) = 0}$

$I(1,1) = \min\{1-1+1, 1\} = \min\{1,1\}$

$\boxed{I(1,1) = 1}$

$I(0,0) = \min\{1-0+0, 1\} = \min\{1,1\}$

$\Rightarrow \boxed{I(0,0) = 1}$

Thus from (i) (ii) and (iii) I is fuzzy implication.

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Theorem 2.36:- If I is a fuzzy implication and if N is a Negation then
$I'(x,y) = I(N(y), N(x))$ is a fuzzy implication.

Proof :-

(i) if $x \le y \Rightarrow N(y) \le N(x)$ ∵ by definition of Negation.

Since I is implication then by using 2nd condition
if $y \le z$ then $I(x,y) \le I(x,z)$.
since $N(y) \le N(x)$
then $I(N(z), N(y)) \le I(N(z), N(x))$
$I'(y,z) \le I'(x,z)$
$\Rightarrow$ if $x \le y$ then $\boxed{I'(y,z) \le I'(x,z)}$

(ii) if $y \le z$
$= N(z) \le N(y)$ ∵ by definition of Negation.

Since I is fuzzy implication then by using 1st condition.
if $x \le y \Rightarrow I(x,z) \ge I(y,z)$.
since
$N(z) \le N(y)$ or $N(y) \ge N(z)$.
$I(N(y), N(x)) \le I(N(z), N(x))$
$\boxed{I'(x,y) \le I(x,z)}$

(iii)
$I'(1,0) = I(N(0), N(1)) = I(1,0) = 0$
$I'(1,1) = I(N(1), N(1)) = I(0,0) = 1$
$I'(0,0) = I(N(0), N(0)) = I(1,1) = 1$

Thus $I'(x,y) = I(N(y), N(x))$ is a fuzzy implication.

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(i) $x \le y \Rightarrow N(x) \ge N(y)$

$S(N(x), z) \ge S(N(y), z)$

$I(x,z) \ge I(y,z)$

Ph 32 I is decreasing in 1st variable.

2.36

proposition :- Let I be fuzzy implication then

(i) $I(0,x) = 1 \quad \forall\, x \in [0,1]$

(ii) $I(x,1) = 1 \quad \forall\, x \in [0,1]$

proof. (i)

Since $I(0,0) = 1$

$1 = I(0,0) \le I(0,x)$

$\Rightarrow 1 \le I(0,x)$ —— (i)

Also $I(0,x) \le 1$ —— (ii)

$\Rightarrow 1 \le I(0,x) \le 1$

$\Rightarrow I(0,x) = 1. \quad \forall\, x \in [0,1]$

(ii) Since $I(1,1) = 1$

$1 = I(1,1) \le I(x,1)$

$I(x,1) \ge I(1,1) = 1$

$I(x,1) \ge 1$ —— (i)

Also

$I(x,1) \le 1$ —— (ii)

$1 \le I(x,1) \le 1$

$\Rightarrow \boxed{I(x,1) = 1} \quad \forall\, x \in [0,1]$

S-implication :-

Let S be t-conorm and N be a strong negation. Then $I(x,y) = S(N(x), y)$ is called S-implication.

proposition 2.39 :- $I(x,y) = S(N(x), y)$ is an fuzzy implication.

proof :- (i) Let $x \le z$ then $N(x) \ge N(z)$

and $S(N(x), y) \ge S(N(z), y)$

$I(x,y) \ge I(z,y)$

So I is decreasing in 1st variable

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$y \le z$

$S(N(x),y) \le S(N(x),z)$.

$= I(x,y) \le I(x,z)$

(ii) Similarly I is increasing in its 2nd varible.

(iii) $I(1,0) = S(N(1),0) = S(0,0) = 0$

$I(1,1) = S(N(1),1) = S(0,1) = 1$

$I(0,0) = S(N(0),0) = S(1,0) = 1$.

Thus $I(x,y) = S(N(x),y)$ is implication.

Example (2.41):-

Let $S(x,y) = x+y-xy$

$N(x) = 1-x$.

then $I(x,y) = 1-x+xy$ is the S-implication.

$S(x,y) = x+y-xy$ is t-conorm and $N(x) = 1-x$ is strong negation then S-implication as;

$I(x,y) = S(N(x),y)$

$= S(1-x,y)$

$= 1-x+y-(1-x)y$

$= 1-x+y-y+xy$

$I(x,y) = 1-x+xy$ is the S-implication.

Example :- (2.42)

$S(x,y) = \min(x+y,1)$

$N(x) = 1-x$

$I(x,y) = \min(1-x+y,1)$ is S-implication

proof :- S-implication is;

$I(x,y) = S(N(x),y)$

$= S(1-x,y)$

$= \min(1-x+y,1) = I(x,y)$ proof.

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Residual-Implication (R-implication) :-

Let T be a t-norm.

$$I_T(x,y) = \operatorname{Sup}\{z \mid T(x,z) \le y\}$$

is called R-implication.

proposition (2.45) :-

R-implication is implication

Proof :-

(i) $x_1 \le x_2 \Rightarrow T(x_1,z) \le T(x_2,z), \ \forall z \in [0,1]$

if $z_0 \in \{z \mid T(x_2,z) \le y\}$

$\Rightarrow z_0 \in \{z \mid T(x_1,z) \le y\}$

$$\{z \mid T(x_2,z) \le y\} \subseteq \{z \mid T(x_1,z) \le y\}$$

$$I_T(x_2,y) = \operatorname{Sup}\{z \mid T(x_2,z) \le y\} \le \operatorname{Sup}\{z \mid T(x_1,z) \le y\}$$

$$= I_T(x_1,y)$$

$$I_T(x_2,y) \le I_T(x_1,y)$$

ii)

$$I_T(1,0) = \operatorname{Sup}\{z \mid T(1,0) \le y\}$$

$$= \operatorname{Sup}\{z \mid 0 \le y\} =$$

$$= 0$$

$$I_T(1,1) = \operatorname{Sup}\{z \mid T(1,1) \le y\}$$

$$= \operatorname{Sup}\{z \mid 1 \le y\} = \operatorname{Sup}[0,1]$$

$$= 1$$

$$I_T(0,0) = \operatorname{Sup}\{z \mid T(0,0) \le y\} =$$

$$=$$

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Example 2.47 (i) :- $T(x,y) = \min(x,y)$ gives the R-implication.

$T(x,y) = x \wedge y$ then.

$I_T(x,y) = \sup\{z \mid x \wedge z \leq y\}$

There are two cases if $x \leq y$ then possible value of z is that at which we take minimum with x and the result is less or equal to y.

$\Rightarrow I_T(x,y) = \sup[0,1] = 1$.

if $x > y$

$I_T(x,y) = \sup\{z \mid x \wedge z \leq y\}$ then possible value of z will be less or equal to y.

$I_T(x,y) = \sup[0,y] = y$.

Thus $I_T = \begin{cases} 1 & x \leq y \\ y & x > y \end{cases}$

example $x = 0.2,\ y = 0.6$

if $x < y$.

$x \wedge z \leq 0.6$

$0.2 \wedge [0,1] \leq 0.6$

if $x = 0.6,\ y = 0.2$

$x > y$

$x \wedge z \leq y \Rightarrow 0.6 \wedge [0,y] \leq 0.2$

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example 2.47 (ii) :- $T(x,y) = xy$ gives the R-implication.

$T(x,y) = xy$

$I_T(x,y) = \text{Sup}\{z \mid xz \le y\}$

There are two cases

if $x \le y$

Then posisible value of $z \in [0,1]$

$I_T(x,y) = \text{Sup}\{[0,1] \mid x[0,1] \le y\}$

$= \text{Sup}\{[0,1]\} = 1$

if $x > y$

$I_T(x,y) = \text{Sup}\{[0, y/x] \mid xz \le y\}$

$= \text{Sup}\{[0, y/x] \mid xz \ge \frac{y}{x}\}$

$= \text{Sup}\{[0, y/x]\} = y/x$

$$I_T(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y/x & \text{if } x > y \end{cases}$$

example if $x \le y$

$x = 0.2,\ y = 0.6$

$\Rightarrow 0.2z \le 0.6$ if $z = 1$

$0.2 \le 0.6$ ✓ $z \in [0,1]$

if $x > y$, $x = 0.6,\ y = 0.2$

$0.6[0, \frac{0.2}{0.6}] < 0.2$

$0 \le 0.2$ ✓ $\boxed{z = 0}$

$0.2 < 0.2$ ✓ $\boxed{z = \frac{0.2}{0.6}}$

2.46 $T(x,1) = x$

$I_T = \text{Sup}\{z / T(x,1) \le y\}$

$= \text{Sup}\{z \mid x \le y\} \Rightarrow I_T(x,y) = \text{Sup}\{[0,1]\} = \boxed{1}$

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Fuzzy Equivalence :- (2.48)

A function $E : [0,1]^2 \longrightarrow [0,1]$ is a fuzzy equivalence if it satisfies the following conditions :

E_1 : $E(x,y) = E(y,x)$

E_2 : $E(0,1) = E(1,0) = 0$

E_3 : $E(x,x) = 1 \quad \forall\ x \in [0,1]$

E_4 : if $x \le x' \le y' \le y$ then $E(x,y) \le E(x',y')$

denoted be $x \Longleftrightarrow y$

Theorem 2.49 :-

The following statment are equivalent:

(a) E is fuzzy equivalence

(b) There exist a fuzzy implication I with the property $I(x,x) = 1 \ \forall\ x \in [0,1]$ such that

$$E(x,y) = \min[I(x,y), I(y,x)]$$

(b) There exist a fuzzy implication I with the property

$$E(x,y) = I(\max\{x,y\}, \min\{x,y\})$$

Proof :- $b \longrightarrow a$

Let $E(x,y) = \min[I(x,y), I(y,x)]$. Then we can easily check the ~~Condition~~ properties $E_1 \longrightarrow E_3$.

E_1 : $E(x,y) = \min[I(x,y), I(y,x)]$

$= \min[I(y,x), I(x,y)]$

$= E(y,x)$

$\Rightarrow E(x,y) = E(y,x)$ ✓

E_2 : $E(0,1) = \min[I(0,1), I(1,0)]$

$= \min[1, 0] = 0$

$E(1,0) = \min[I(1,0), I(0,1)] = 0$ ✓

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E_3: $E(x,x) = \min[I(x,x), I(x,x)]$

$= \min[1, 1]$ $\because I(x,x) = 1$

$E(x,x) = 1$ ✓

E_4: If $x \le x' \le y' \le y$ then

$E(x,y) \le E(x',y')$

$x \le y$

$I(x,x) \ge I(y,x)$

$x \le y$

$I(x,x) \le I(x,y)$

$\Rightarrow I(y,x) \le I(x,x) \le I(x,y)$

$\boxed{\text{means } E(x,y) = I(y,x)}$

$\Rightarrow E(x,y) = I(y,x)$

Similarly $E(x',y') = I(y',x')$

Now we show $E(x,y) \le E(x',y')$

or $I(y,x) \le I(y',x')$

$x \le x'$

$I(y',x) \le I(y',x')$

$y \le y'$

$I(y',x) \ge I(y,x)$

$\Rightarrow I(y,x) \le I(y',x) \le I(y',x')$

$\Rightarrow I(y,x) \le I(y',x')$

$\Rightarrow E(x,y) \le E(x',y')$

Hence E is fuzzy equivalence.

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Example $I(x,y) = \min\{1-x+y, 1\}$
Find $E(x,y)$ and then verify it is fuzzy equivalence

$$E(x,y) = \min[I(x,y), I(y,x)]$$

$$= \min[\min\{1-x+y, 1\}, \min\{1-y+x, 1\}]$$

$$E(x,y) = \min[1-x+y, 1-y+x]$$

(i) $E(x,y) = E(y,x)$

$$E(x,y) = \min[1-x+y, 1-y+x]$$

$$= \min[1-y+x, 1-x+y]$$

$E(x,y) = E(y,x)$ ✓

(ii) $E(0,1) = E(1,0) = 0$

$$E(0,1) = \min[1-0+1, 1-1+0] = \min[2, 0]$$

$$E(0,1) = 0$$

$E(1,0) = \min[1-1+0, 1-0+1] = 0$ ✓

(iii) $E(x,x) = 1 \quad \forall\ x \in [0,1]$

$E(x,x) = \min[1-x+x, 1-x+x] = 1$ ✓

(iv) if $x \le x' \le y' \le y \Rightarrow E(x,y) \le E(x',y')$

Since

$$I(x,y) = \min\{1-x+y, 1\}$$

if $x \le y$

$$\Rightarrow I(x,x) \ge I(y,x)$$

because $1 \ge I(y,x)$

if $x \le y$

$$\Rightarrow I(x,x) \le I(x,y)$$

$$\Rightarrow I(y,x) \le I(x,x) \le I(x,y)$$

$$\Rightarrow E(x,y) = I(y,x)$$

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Similarly $E(x',y') = I(y',x')$

Now we show $I(y,x) \leq I(y',x')$

$x \leq x'$

$$I(y',x) \leq I(y',x')$$

$y' \leq y$

$$I(y',x) \geq I(y,x)$$

$$\Rightarrow I(y,x) \leq I(y',x) \leq I(y',x')$$

$$\Rightarrow I(y,x) \leq I(y',x')$$

Thus $E(x,y) \leq E(x',y')$

Hence $E(x,y)$ is fuzzy Equivalence.

Example (Assignment):-

$$I(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{if } x > y \end{cases}$$

Find Fuzzy equivalence and then verify.

As $E(x,y) = \min\left[I(x,y), I(y,x) \right]$

There are three cases

(1) if $x = y$

$$E(x,y) = \min\left[1, 1\right] = 1$$

as $I(y,x) = \begin{cases} 1 & \text{if } y \leq x \\ x/y & \text{if } y > x \end{cases}$

(2) if $x \leq y$

$$E(x,y) \overset{\min[I(x,y), I(y,x)]}{=} \min\left[1, x/y\right] = x/y$$

(3) if $x > y$

$$E(y,x) = \min\left[y/x, 1\right] = y/x$$

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Thus $E(x,y) = \begin{cases} 1 & \text{if } x=y \\ x/y & \text{if } x<y \\ y/x & \text{if } y<x \end{cases}$

Verification:-

(a) : $E(x,y) = E(y,x)$

$$E(x,y) = \begin{cases} 1 & \text{if } x=y \\ x/y & \text{if } x<y \\ y/x & \text{if } y<x \end{cases}$$

$$= \begin{cases} 1 & \text{if } y=x \\ y/x & \text{if } y<x \\ x/y & \text{if } x<y \end{cases}$$

$E(x,y) = E(y,x)$

(b) $E(0,1) = E(1,0) = 0$

$$E(0,1) = \left\{ \frac{x}{y} \quad x<y \right.$$

$$= \left\{ \frac{0}{1} \quad 0<1 \Rightarrow 0 \right.$$

$$E(1,0) = \{ y/x \quad \text{if } y<x$$

$$= \left\{ \frac{0}{1} \quad 0<1 \Rightarrow 0 \right.$$

$E(0,1) = E(1,0) = 0$

(c) $E(x,x) = 1 \ \forall \ x \in [0,1]$

$$E(x,x) = \begin{cases} 1 & \text{if } y=x \\ x/x & \text{if } x<x \\ x/x & \text{if } x<x \end{cases}$$

$$= \{ 1 \quad \text{if } x=x = \boxed{1}$$

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(d) if $x \le x' \le y' \le y \Rightarrow E(x,y) \le E(x',y')$

$$E(x,y) = \min[I(x,y), I(y,x)]$$

$x \le y$

$$\Rightarrow I(x,x) \ge I(y,x)$$

or $$1 \ge I(y,x)$$

$x \le y$

$$\Rightarrow I(x,x) \le I(x,y)$$

$$I(y,x) \le I(x,x) \le I(x,y)$$

$$\Rightarrow E(x,y) = I(y,x)$$

Also $E(x',y') = I(y',x')$

Now we prove $E(x,y) \le E(x',y')$

$x \le x'$

$$I(y',x) \le I(y',x')$$

$y' \le y$

$$I(y',x) \ge I(y,x)$$

$$I(y,x) \le I(y',x) \le I(y',x')$$

$$E(x,y) \le E(x',y')$$

Thus $E(x,y)$ is fuzzy equivalence

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Chapter 3 Fuzzy Relation.

Classical Relation:- A subset $R \subseteq X \times Y$ where X and Y are classical sets is a classical relation.

It can be characteraized by a function $R : X \times Y \longrightarrow \{0,1\}$

$$R(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

Fuzzy Relation:-

Let X and Y be two classical sets. A mapping $R : X \times Y \longrightarrow [0,1]$ is called a fuzzy relation. OR

A fuzzy relation on $A \times B$ is denoted by R or $R(x,y)$ is defined as the set

$$R(x,y) = \{((x,y), \mu_R(x,y)) \mid (x,y) \in A \times B, \mu_R(x,y) \in [0,1]\}$$

* $\mu_R(x,y) \in [0,1]$ is a degree of relationship between x and y.
* We denote by $F(X \times Y)$ the family of all fuzzy relations b/w elements of X and Y.
* A fuzzy relation between elements in two finite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ can be represented as a matrix.

$$R = \begin{pmatrix} R(x_1,y_1) & R(x_1,y_2) & \cdots & R(x_1,y_n) \\ R(x_2,y_1) & R(x_2,y_2) & \cdots & R(x_2,y_n) \\ \vdots & & & \\ R(x_m,y_1) & R(x_m,y_2) & \cdots & R(x_m,y_n) \end{pmatrix}$$

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* Since fuzzy relations are themselves fuzzy sets, it is possible to perform fuzzy sets operations on them

(i) $N(R(x,y)) = 1 - R(x,y)$

(ii) $(R \vee S)(x,y) = R(x,y) \vee S(x,y)$

(iii) $(R \wedge S)(x,y) = R(x,y) \wedge S(x,y)$

(iv) $R^{-1}(x,y) = R(y,x)$

we also consider t-norm and s-norm operations

(v) $T(R,P)(x,y) = T(R(x,y), P(x,y))$

(vi) $S(R,P)(x,y) = S(R(x,y), P(x,y))$

* Example :- show how Fuzzy relation are represented

Let $V = \{1,2,3\}$ and $W = \{1,2,3\}$

A fuzzy relation R is a function defined in the space $V \times W$ which takes values from the interval $[0,1]$, expressed as $R: V \times W \to [0,1]$

$R = [\{\{1,1\}, 0.2\}, \{\{1,2\}, 0.6\}, \{\{1,3\}, 0.9\}, \{\{2,1\}, 1\}, \{\{2,2\}, 0.3\}, \{\{2,3\}, 0.5\}, \{\{3,1\}, 0\} \{\{3,2\}, 0.1\}, \{\{3,3\}, 0.7\}]$

In matrix form

$$R \triangleq \begin{array}{c|ccc} V\times W & 1 & 2 & 3 \\ \hline 1 & 0.2 & 0.6 & 0.9 \\ 2 & 1 & 0.3 & 0.5 \\ 3 & 0 & 0.1 & 0.7 \end{array}$$

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* **Composition :-** The operation "composition" combine the fuzzy relation in different variables say (x,y) and (y,z) ; $x \in A$, $y \in B$, $z \in C$.

Max-Min Composition :- Let $R \in F(x \times y)$ and $S \in F(y \times z)$ be fuzzy relations then $R \circ S \in F(X \times Z)$, defined as.

$$R \circ S(x,z) = \bigvee_{y \in Y} \left(R(x,y) \wedge S(y,z) \right)$$ is called the Max-Min Composition of R and S. denoted by $R \circ S$.

* if $R \in F(X \times Y)$ then we can defined $R^2 = R \circ R$ generally $R^n = R \circ R^{n-1}$ for $n \geq 2$

* Let $X = \{x_1, x_2, \dots x_n\}$, $Y = \{y_1, y_2, \dots, y_m\}$
$Z = \{z_1, z_2, \dots z_p\}$ be finite sets.
if $R = (r_{ij})_{i=1,\dots n, j=1,\dots m} \in F(X \times Y)$ and
$S = (S_{jk})_{j=1,\dots m, k=1,\dots p} \in F(Y \times Z)$ are fuzzy relations then the Composition

$T = (t_{ik})_{i=1,\dots n, k=1,\dots p} = R \circ S \in F(X \times Z)$ is

$$t_{ik} = \bigvee_{j=1}^{m} \left(r_{ij} \wedge S_{jk} \right)$$

Example (3.7) page 35 :-

if $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$

$S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$ then find $R \circ S$

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Solution: Consider

$$R \triangleq \begin{array}{c|ccc} X\times Y & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.7 & 0.2 \\ x_2 & 1 & 0 & 0.9 \end{array}, \quad S \triangleq \begin{array}{c|cc} Y\times Z & z_1 & z_2 \\ \hline y_1 & 0.8 & 0.3 \\ y_2 & 0.1 & 0 \\ y_3 & 0.5 & 0.6 \end{array}$$

$$R \circ S(x,z) = \vee\left(R(x,y) \wedge S(y,z)\right)$$

step(1) Compute minimum operation

$\min(R(x_1,y_1), S(y_1,z_1)) = \min(0.3, 0.8) = 0.3$
$\min(R(x_1,y_2), S(y_2,z_1)) = \min(0.7, 0.1) = 0.1$
$\min(R(x_1,y_3), S(y_3,z_1)) = \min(0.2, 0.5) = 0.2$

$\min(R(x_1,y_1), S(y_1,z_2)) = \min(0.3, 0.3) = 0.3$
$\min(R(x_1,y_2), S(y_2,z_2)) = \min(0.7, 0) = 0$
$\min(R(x_1,y_3), S(y_3,z_2)) = \min(0.2, 0.6) = 0.2$

$\min(R(x_2,y_1), S(y_1,z_1)) = \min(1, 0.8) = 0.8$
$\min(R(x_2,y_2), S(y_2,z_1)) = \min(0, 0.1) = 0$
$\min(R(x_2,y_3), S(y_3,z_1)) = \min(0.9, 0.5) = 0.5$

$\min(R(x_2,y_1), S(y_1,z_2)) = \min(1, 0.3) = 0.3$
$\min(R(x_2,y_2), S(y_2,z_2)) = \min(0, 0) = 0$
$\min(R(x_2,y_3), S(y_3,z_2)) = \min(0.9, 0.6) = 0.6$

Step (2) Compute maximum operation:

$\max(R(x_1,z_1)) = \max(0.3, 0.1, 0.2) = 0.3$
$\max(R(x_1,z_2)) = \max(0.3, 0, 0.2) = 0.3$
$\max(R(x_2,z_1)) = \max(0.8, 0, 0.5) = 0.8$
$\max(R(x_2,z_2)) = \max(0.3, 0, 0.6) = 0.6$

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Thus $RoS(x,z) =$

$X \times Z$	z_1	z_2
x_1	0.3	0.3
x_2	0.8	0.6

is the required Max-Min Composition.

Assignment No.1

Umar Saeed

Roll No 29 Msc 4th

proposition 3.8 :-

(i) The max-min Composition is associative

i.e $(R \circ S) \circ Q = R \circ (S \circ Q)$

where $R \in F(X \times Y)$, $S \in F(Y \times Z)$, $Q \in F(Z \times U)$.

proof :-

LHS $(R \circ S) \circ Q\ (x,u) = \max_z \{\min\{R \circ S(x,z), Q(z,u)\}\}$

$= \max_z \{\min\{\max_y\{\min\{R(x,y), S(y,z)\}\}, Q(z,u)\}\}$

now for some $y_k \in Y$, $k \in \mathbb{N}$

$= \max_z\{\min\{\min\{R(x,y_k), S(y_k,z)\}, Q(z,u)\}\}$

now for $z_m \in Z$, $m \in \mathbb{N}$

$= \min\{\min\{R(x,y_k), S(y_k,z_m)\}, Q(z_m,u)\}$

$= \min\{R(x,y_k), S(y_k,z_m), Q(z_m,u)\}$ —— ①

RHS

$R \circ (S \circ Q)(x,u) = \max_y\{\min\{R(x,y), S \circ Q(y,u)\}\}$

$= \max_y\{\min\{R(x,y), \max_z\{\min\{S(y,z), Q(z,u)\}\}\}\}$

$= \max_y\{\min\{R(x,y), \min\{S(y,z_m), Q(z_m,u)\}\}\}$

for $z_m \in Z$, $m \in \mathbb{N}$

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$= \min\{R(x,y_k), \min\{S(y_k, z_m), Q(z_m, u)\}\}$

for $y_k \in Y$, $k \in \mathbb{N}$

$= \min\{R(x,y_k), S(y_k, z_m), Q(z_m, u)\}$ — ②

From ① and ② we have

LHS = RHS

$(R \circ S) \circ Q = R \circ (S \circ Q)$

(ii) Let $R_1, R_2 \in F(X \times Y)$ and $Q \in F(Y \times Z)$.
If $R_1 \leq R_2$ then $R_1 \circ Q \leq R_2 \circ Q$

Proof :-

$R_1 \circ Q(x,z) = \bigvee_y \{R_1(x,y) \wedge Q(y,z)\}$

$R_2 \circ Q(x,z) = \bigvee_y \{R_2(x,y) \wedge Q(y,z)\}$

$\bigvee_y \{R_1(x,y) \wedge Q(y,z)\} \leq \bigvee_y \{R_2(x,y) \wedge Q(y,z)\}$

$\Rightarrow R_1 \circ Q(x,z) \leq R_2 \circ Q(x,z)$

Proposition 3.9 :- For any $R, S \in F(X \times Y)$ and $Q \in F(Y \times Z)$ we have

(i) $(R \vee S) \circ Q = (R \circ Q) \vee (S \circ Q)$

proof :-

$(R \vee S) \circ Q = \bigvee_{y \in Y} \{(R \vee S)(x,y) \wedge Q(y,z)\}$

$= \bigvee_{y \in Y} (R(x,y) \wedge Q(y,z)) \vee (S(x,y) \wedge Q(y,z))$

$\leq \bigvee_{y \in Y} R(x,y) \wedge Q(y,z) \vee \bigvee_{y \in Y} S(x,y) \wedge Q(y,z)$

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$(R \vee S) \circ Q \leq (R \circ Q) \vee (S \circ Q)$ — ①

on the other have $R \circ Q \leq (R \vee S) \circ Q$

and $S \circ Q \leq (R \vee S) \circ Q$

$\Rightarrow (R \circ Q) \vee (S \circ Q) \leq (R \vee S) \circ Q$ — ②

From ① and ② we have

$(R \vee S) \circ Q = (R \circ Q) \vee (S \circ Q)$

(ii) $(R \wedge S) \circ Q \leq (R \circ Q) \wedge (S \circ Q)$

$$(R \wedge S) \circ Q(x,z) = \bigvee_{y \in Y} (R \wedge S)(x,y) \wedge Q(y,z)$$

$$= \bigvee_{y \in Y} \big(R(x,y) \wedge Q(y,z)\big) \wedge \big(S(x,y) \wedge Q(y,z)\big)$$

$$\leq \bigvee_{y \in Y} \big(R(x,y) \wedge Q(y,z)\big) \wedge \bigvee_{y \in Y} \big(S(x,y) \wedge Q(y,z)\big)$$

$(R \wedge S) \circ Q(x,z) \leq (R \circ Q) \wedge (S \circ Q)$

Min-Max Composition:- Let $R \in F(X \times Y)$ and $S \in F(Y \times Z)$, then $R \bullet S \in F(X \times Z)$, defined as

$$R \bullet S(x,z) = \bigwedge_{y \in Y} R(x,y) \vee S(y,z)$$ is called min-max composition of the fuzzy relation R and S.

Example (3.12) if $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$, $S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$

Find $R \bullet S$.

Solution:-

(i) Compute max operation:-

$\max[R(x_1,y_1), S(y_1,z_1)] = \max[0.3, 0.8] = 0.8$

$\max[R(x_1,y_2), S(y_2,z_1)] = \max[0.7, 0.1] = 0.7$

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$\max[R(x_1, y_3), S(y_3, z_1)] = \max[0.2, 0.5] = 0.5$

$= \max[0.3, 0.3] = 0.3$

$= \max[0.7, 0] = 0.7$

$= \max[0.2, 0.6] = 0.6$

$= \max[1, 0.8] = 1$

$= \max[0, 0.1] = 0.1$

$= \max[0.9, 0.5] = 0.9$

$= \max[1, 0.3] = 01$

$= \max[0, 0] = 0$

$= \max[0.9, 0.6] = 0.9$

(ii) Compute ~~mimum~~ minimum operation :-

$R \circ S(x, z) = \min[R(x_1, z_1)] = \min[0.8, 0.7, 0.5]$

$= 0.5$

$= \min[0.3, 0.7, 0.6] = 0.3$

$= \min[1, 0.1, 0.9] = 0.1$

$= \min[1, 0, 0.9] = 0$

$R \circ S(x, z) =$

$X \times Z$	Z_1	Z_2
X_1	0.5	0.3
X_2	0.1	0

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Assignment (02) Umar Saeed Roll no. 29.

Proposition (3.13) :- (i) The Min-Max Composition is associative i.e. $(R \circ S) \circ T = R \circ (S \circ T)$

For $R \in F(X \times Y)$, $S \in F(Y \times Z)$, $T \in F(Z \times U)$

Proof :- LHS

$$(R \circ S) \circ T = \min_z \{\max\{R \circ S(x,z), T(z,u)\}\}$$

$$= \min_z \{\max\{\min_y \{\max\{R(x,y), S(y,z)\}, T(z,u)\}\}\}$$

$$= \min_z \{\max\{\max\{R(x,y_k), S(y_k,z)\} T(z,u)\}\}$$

for some $y_k \in Y$ and $k \in N$

$$= \max\{\max\{R(x,y_k), S(y_k,z_m)\}, T(z_m,u)\}$$

For some $z_m \in Z$ and $m \in N$

$$= \max\{R(x,y_k), S(y_k,z_m), T(z_m,u)\} \quad \text{--- (1)}$$

R.H.S

$$R \circ (S \circ T)(x,u) = \min_y \{\max\{R(x,y), S \circ T(y,u)\}\}$$

$$= \min_y \{\max\{R(x,y), \min_z \{\max\{S(y,z), T(z,u)\}\}\}\}$$

$$= \min_y \{\max\{R(x,y), \max\{S(y,z_m), T(z_m,u)\}\}\}$$

For some $z_m \in Z$, $m \in N$

$$= \max\{R(x,y_k), \max\{S(y_k,z_m), T(z_m,u)\}\}$$

For some $y_k \in Y$, $k \in N$

$$= \max\{R(x,y_k), S(y_k,z_m), T(z_m,u)\} \quad \text{--- (2)}$$

From (1) and (2) we have

$$(R \circ S) \circ T = R \circ (S \circ T)$$

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(ii) Consider $R_1, R_2 \in F(X \times Y)$ and $Q \in F(Y \times Z)$.
if $R_1 \leq R_2$ then
$$R_1 \circ Q \leq R_2 \circ Q$$

Proof :-
$$R_1 \circ Q = \bigwedge_{y \in Y} R_1(x,y) \vee Q(y,z)$$
$$\leq \bigwedge_{y \in Y} R_2(x,y) \vee Q(yz) = R_2 \circ Q$$
$$\Rightarrow R_1 \circ Q(x,z) \leq R_2 \circ Q(x,z) \quad \text{if } R_1 \leq R_2$$

Proposition 3.14 :- For any $R, S \in F(X \times Y)$ and $T \in (Y \times Z)$ we have
(i) $(R \wedge S) \circ T = (R \circ T) \wedge (S \circ T)$
(ii) $(R \vee S) \circ T \geq (R \circ T) \vee (S \circ T)$

Proof :-
(i) $$(R \wedge S) \circ T = \bigwedge_{y \in Y} ((R \wedge S) \vee T(y,z))$$
$$= \bigwedge_{y \in Y} [R(x,y) \vee T(y,z) \wedge S(x,y) \vee T(y,z)]$$
$$\geq [\bigwedge_{y \in Y} R(x,y) \vee T(y,z)] \wedge [\bigwedge_{y \in Y} S(x,y) \vee T(y,z)]$$
$$\geq R \circ T(x,z) \wedge (S \circ T)$$
$$\geq (R \circ T) \wedge (S \circ T) \quad -①$$
on the other hand
$$R \circ T \geq (R \wedge S) \circ T$$
$$S \circ T \geq (R \wedge S) \circ T$$
$$\Rightarrow (R \circ T) \wedge (S \circ T) \geq (R \wedge S) \circ T \quad -②$$
From ① and ②
$$(R \wedge S) \circ T = (R \circ T) \wedge (S \circ T)$$

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(ii) $(R \vee S) \bullet T = \bigwedge_{y \in Y} (R \vee S) \vee T(y,z)$

$= \bigwedge_{y \in Y} (R(x,y) \vee T(y,z)) \vee (S(x,y) \vee T(y,z))$

$\geq \bigwedge_{y \in Y} [R(x,y) \vee T(y,z)] \vee \bigwedge_{y \in Y} [S(x,y) \vee T(y,z)]$

$\geq (R \bullet T) \vee (S \bullet T)$

$(R \vee S) \bullet T \geq (R \bullet T) \vee (S \bullet T)$

Remark (3.10) :- $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow (R \wedge S) \circ Q = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

and $(R \circ Q) \wedge (S \circ Q) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Remark (3.15) :- $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow (R \vee S) \bullet T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

and $(R \bullet T) \vee (S \bullet T) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

Proposition (3.16) :- If we consider the standard negative we have (i) $\overline{R \circ S} = \bar{R} \bullet \bar{S}$, (ii) $\overline{R \bullet S} = \bar{R} \circ \bar{S}$

(i) $\overline{R \circ S}(x,z) = \overline{\bigvee_{y \in Y} R(x,y) \wedge S(y,z)}$

$= \bigwedge_{y \in Y} \overline{R(x,y) \wedge S(y,z)}$

$= \bigwedge_{y \in Y} \overline{R(x,y)} \vee \overline{S(y,z)} = \bar{R} \bullet \bar{S}$

(ii) similar way.

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Min $\longrightarrow$ Composition:

Let $\rightarrow$ be the standard Godel implication defined as

$$x \rightarrow y = \text{Sup}\{z \in [0,1] \; x \wedge z \leq y\} = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Proposition 3.18: For any $x, y, z \in [0,1]$ we have

(i) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

proof:

Case (1): if $x \vee y \leq z$ then $x \vee y \rightarrow z = 1$ — (A)

also we have $x \leq z$ and $y \leq z$

then $x \rightarrow z = 1$, $y \rightarrow z = 1$

$(x \rightarrow z) \wedge (y \rightarrow z) = 1 \wedge 1 = 1$ — (B)

For $A = B$

Case (2): if $x \vee y > z$ then $(x \vee y) \rightarrow z = z$ — (C)

also we have $x \rightarrow z = z$, $y \rightarrow z = z$

Thus $(x \rightarrow z) \wedge (y \rightarrow z) = z$ — (D).

Thus in both cases

$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

(ii) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$

Case (1): if $x \wedge y \leq z$ then

$(x \wedge y) \rightarrow z = 1$ — (A) Also $x \leq z$ and $y \leq z$.

then $x \rightarrow z = 1$, $y \rightarrow z = 1$.

so $(x \rightarrow z) \vee (y \rightarrow z) = 1$ — (B)

$A = B$

Case (2): if $x \wedge y > z$ the

$x \wedge y \rightarrow z = z$ — (C) also $x > z$, $y > z$

then $x \rightarrow z = z$, $y \rightarrow z = z$.

so $(x \rightarrow z) \vee (y \rightarrow z) = z$ — (D)

Thus in Both cases

$(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$

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(iii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$

Case (1) if $x \leq y \vee z$ then $x \to y \vee z = 1$

also $x \leq y$ ~~and~~ or $x \leq z$ then

$(x \to y) \vee (x \to z) = 1 \vee 1 = 1$.

Case (2): if $x \not\leq y \vee z$ then

$x \to y \vee z = y \vee z$

also $x > y \Rightarrow x \to y = y$

or $x > z \Rightarrow x \to z = z$

$(x \to y) \vee (x \vee z) = y \vee z$

Thus in Both cases

$x \to (y \vee z) = (x \to y) \vee (x \to z)$

(iv) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

Case (1) if $x \leq y \wedge z$ then

$x \to (y \wedge z) = 1$ also

$x \leq y$ or $x \leq z$ then $x \to y = 1$ or $x \to z = 1$.

$(x \to y) \wedge (x \to z) = 1 \wedge 1 = 1$

Case (2) if $x > y \wedge z$ then

$x \to y \wedge z = y \wedge z$ also

$x > y \Rightarrow x \to y = y$ ~~and~~ or

$x > z \Rightarrow x \to z = z$ thus

$(x \to y) \wedge (x \to z) = y \wedge z$

In Both cases

$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(V) $x \wedge (x \to y) \leq y$

Case (1) if $x \leq y$ then $x \wedge (x \to y) = x \wedge 1$

$= x \leq y$ (already)

Case (2) if $x > y$ the $x \wedge (x \to y)$

$= x \wedge y = y$

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In Both cases

$x \wedge (x \to y) \le y$.

(vi) $x \to (x \wedge y) \ge y$

Case (1) if $x \le y$ then $x \to x \wedge y = x \to x = 1 \ge y$

Case (2) if $x > y$

$x \to (x \wedge y) = \cancel{x \wedge y}\ x \to y = y$

Thus $x \to (x \wedge y) \ge y$

(vii)

$(x \to y) \to y \ge x$

Case (1) if $x \le y$ beacuse $1 \not\le y$

$(x \to y) \to y = 1 \to y = y \ge x$ (already)

Case (2) $x > y$

$(x \to y) \to y = y \to y = 1 \ge x$

$\Rightarrow (x \to y) \to y \ge x$

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In the present work we ~~simply~~ mainly use the subcomposition and we call it simply min → composition.

Proposition 3.21:- If $R, S \in F(X \times Y)$ and $Q \in F(Y \times Z)$ are such that $R \leq S$ then

$$R \triangleleft Q(x,z) \geq S \triangleleft Q(x,z).$$

Since → is decreasing in the first argument we have

$$R \triangleleft Q(x,z) = \bigwedge_{y \in Y} R(x,y) \longrightarrow Q(y,z)$$

$$\geq \bigwedge_{y \in Y} S(x,y) \longrightarrow Q(y,z)$$

$$= S \triangleleft Q(x,z).$$

Proposition 3.22:- For any $R, S \in F(X \times Y)$ and $Q \in F(Y \times Z)$ we have

(i) $(R \vee S) \triangleleft Q = (R \triangleleft Q) \wedge (S \triangleleft Q)$

From previous preposition

$(R \vee S) \triangleleft Q \leq R \triangleleft Q$ and

$(R \vee S) \triangleleft Q \leq S \triangleleft Q$

thus

$$(R \vee S) \triangleleft Q \leq (R \triangleleft Q) \wedge (S \triangleleft Q) \quad \text{——} (a)$$

Also

$$(R \vee S) \triangleleft Q(x,z) = \bigwedge_{y \in Y}\Big(\big(R(x,y) \vee S(x,y)\big) \longrightarrow Q(y,z)\Big)$$

$$= \bigwedge_{y \in Y}\big(R(x,y) \longrightarrow Q(y,z)\big) \wedge \big(S(x,y) \rightarrow Q(y,z)\big)$$

$$\geq \bigwedge_{y \in Y}\big(R(x,y) \longrightarrow Q(y,z)\big) \wedge \bigwedge_{y \in Y}\big(S(x,y) \rightarrow Q(y,z)\big)$$

$$= (R \triangleleft Q) \wedge (S \triangleleft Q) \quad \text{——} (b).$$

From (a) and (b).

$$(R \vee S) \triangleleft Q = (R \triangleleft Q) \wedge (S \triangleleft Q)$$

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(ii) $(R \wedge S) \triangleleft Q \geq (R \triangleleft Q) \vee (S \triangleleft Q)$

as

$$(R \wedge S) \triangleleft Q \geq (R \triangleleft Q)$$

and $$(R \wedge S) \triangleleft Q \geq (S \triangleleft Q)$$

thus

$$(R \wedge S) \triangleleft \geq (R \triangleleft Q) \vee (S \triangleleft Q)$$

Fuzzy Relational Eq's with Max-Min and Min→ Compositions :-

We consider the following two fuzzy relational eq

$$R \circ P = Q \quad \text{and}$$

$$R \triangleleft P = Q \quad \text{with}$$

$R \in F(X \times Y)$, $P \in F(Y \times Z)$ and

$Q \in F(X \times Z)$

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6)

Assignment : No. 02

Umar Saeed Roll no. 29

Theorem 3.23 :- The following inequalities hold true.

(i) $P \leq R^{-1} \triangleleft (R \circ P)$

Proof :- For every $y \in Y$ and $z \in Z$ we have

$$R^{-1} \triangleleft (R \circ P)(y,z) = \bigwedge_{x \in X} R^{-1}(y,x) \longrightarrow (R \circ P)(x,z)$$

$$= \bigwedge_{x \in X} R^{-1}(y,x) \longrightarrow \bigvee_{t \in Y} R(x,t) \wedge P(t,z)$$

$$\geq \bigwedge_{x \in X} R(x,y) \longrightarrow R(x,y) \wedge P(y,z)$$

$$\geq \bigwedge_{x \in X} P(y,z) = P(y,z)$$

$\because x \rightarrow y = 1$ if $x \leq y$

$x \rightarrow y = y$ if $x > y$

$$\Rightarrow P \leq R^{-1} \triangleleft (R \circ P)$$

(ii) $R \circ (R^{-1} \triangleleft Q) \leq Q$

Proof :- For every $x \in X$ and $z \in Z$ we have

$$R \circ (R^{-1} \triangleleft Q)(x,z) = \bigvee_{y \in Y} R(x,y) \wedge (R^{-1} \triangleleft Q)(y,z)$$

$$= \bigvee_{y \in Y} R(x,y) \wedge \Big(\bigwedge_{t \in X} R^{-1}(y,t) \rightarrow Q(t,z) \Big)$$

$$\leq \bigvee_{y \in Y} R(x,y) \wedge \big(R^{-1}(y,x) \rightarrow Q(x,z) \big)$$

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$$\leq \bigvee_{y\in Y} R(x,y) \wedge (R(x,y) \longrightarrow Q(x,z))$$

$$\leq \bigvee_{y\in Y} R(x,y) \wedge Q(x,z) \leq \bigvee_{y\in Y} Q(x,z) = Q(x,z)$$

$\because\ x \longrightarrow y = y$ if $x > y$

$\because\ x \rightarrow y = 1$ if $x \leq y$

Thus

$$R \circ (R^{-1} \lhd Q)(x,z) \leq Q(x,z)$$

(iii) $R \leq (P \lhd (R \circ P)^{-1})^{-1}$

Proof:- For every $x \in X$, $y \in Y$

$$(P \lhd (R\circ P)^{-1})^{-1}(x,y) = P \lhd (R \circ P)^{-1}(y,x)$$

$$= \bigwedge_{z\in Z} P(y,z) \longrightarrow (R\circ P)^{-1}(z,x)$$

$$= \bigwedge_{z\in Z} P(y,z) \longrightarrow (R\circ P)(x,z)$$

$$= \bigwedge_{z\in Z} P(y,z) \longrightarrow \bigvee_{t\in Y} R(x,t) \wedge P(t,z)$$

$$\geq \bigwedge_{z\in Z} P(y,z) \longrightarrow R(x,y) \wedge P(y,z)$$

$$\geq \bigwedge_{z\in Z} R(x,y) \wedge P(y,z) \geq \bigwedge_{z\in Z} R(x,y) = R(x,y)$$

(iv) $(P \lhd Q^{-1})^{-1} \circ P \leq Q$

Proof:- For every $x \in X$, $z \in Z$ we have

$$(P \lhd Q^{-1})^{-1} \circ P(x,z) = \bigvee_{y\in Y} (P \lhd Q^{-1})^{-1}(x,y) \wedge P(y,z)$$

$$= \bigvee_{y\in Y} (P \lhd Q^{-1})(y,x) \wedge P(y,z)$$

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$$= \bigvee_{y \in Y} \left(\bigwedge_{t \in Z} P(y,t) \longrightarrow Q^{-1}(t,x) \right) \wedge P(y,z)$$

$$\leq \bigvee_{y \in Y} \left(P(y,z) \longrightarrow Q^{-1}(z,x) \right) \wedge P(y,z)$$

$$\leq \bigvee_{y \in Y} \left(P(y,z) \rightarrow Q(x,z) \right) \wedge P(y,z)$$

$$\leq \bigvee_{y \in Y} Q(x,z) \wedge P(y,z) \leq \bigvee_{y \in Y} Q(x,z)$$

Thus

$$(P \triangleleft Q^{-1})^{-1} \circ P \leq Q$$

Proposition:- 3.26 :- Following inequility holds true:

(i) $(Q \triangleleft P^{-1}) \triangleleft P \geq Q$

Proof :- taking $\longrightarrow$ decreasing in its first arguments we have.

$$(Q \triangleleft P^{-1}) \triangleleft P(x,z) = \bigwedge_{y \in Y} \left(Q \triangleleft P^{-1}(x,y) \longrightarrow P(y,z) \right)$$

$$= \bigwedge_{y \in Y} \left(\bigwedge_{t \in Z} Q(x,t) \longrightarrow P^{-1}(x,t) \right) \longrightarrow P(y,z)$$

$$= \bigwedge_{y \in Y} \left(\bigwedge_{t \in Z} Q(x,t) \longrightarrow P(t,x) \right) \longrightarrow P(y,z)$$

$$\geq \bigwedge_{y \in Y} \left(Q(x,z) \longrightarrow P(y,z) \right) \longrightarrow P(y,z)$$

$$\geq \bigwedge_{y \in Y} \left(Q(x,z) \right) = Q(x,z)$$

Thus $((Q \triangleleft P^{-1}) \triangleleft P(x,z) \geq Q$

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(ii) $(R \lhd P) \lhd P^{-1} \geq R$

Proof :-

$$(R \lhd P) \lhd P^{-1}(x,y) = \bigwedge_{z \in Z} (R \lhd P)^{(x,z)} \longrightarrow P^{-1}(z,y)$$

$$= \bigwedge_{z \in Z} (R \lhd P)(x,z) \longrightarrow P^{-1}(z,y)$$

$$= \bigwedge_{z \in Z} \left(\bigwedge_{t \in Y} R(x,t) \rightarrow P(t,z) \right) \rightarrow P^{-1}(z,y)$$

$$\geq \bigwedge_{z \in Z} \left(R(x,y) \rightarrow P(y,z) \right) \rightarrow P(y,z)$$

$$\geq \bigwedge_{z \in Z} R(x,y)$$

Thus $(R \lhd P) \lhd P^{-1} \geq R$

(iii) $R^{-1} \circ (R \lhd P) \leq P$

Proof :-

$$R^{-1} \circ (R \lhd P)(y,z) = \bigvee_{x \in X} R^{-1}(y,x) \wedge (R \lhd P)(x,z)$$

$$= \bigvee_{x \in X} R(x,y) \wedge \left(\bigwedge_{t \in Y} R(x,t) \longrightarrow P(t,z) \right)$$

$$\leq \bigvee_{x \in X} R(x,y) \wedge \left(R(x,y) \longrightarrow P(y,z) \right)$$

From proposition 3.18

$x \wedge (x \longrightarrow y) \leq y$.

$$\leq \bigvee_{x \in X} P(y,z) = P(y,z)$$

Thus $R^{-1} \circ (R \lhd P)(y,z) \leq P(y,z)$.

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(iv) $R \triangleleft (R^{-1} \circ Q) \geq Q$

Proof:

$$R \triangleleft (R^{-1} \circ Q)(x,z) = \bigwedge_{y \in Y} R(x,y) \to (R^{-1} \circ Q)(y,z)$$

$$= \bigwedge_{y \in Y} R(x,y) \longrightarrow \left(\bigvee_{t \in X} \left(R^{-1}(y,t) \wedge Q(t,z) \right) \right)$$

$$\geq \bigwedge_{y \in Y} R(x,y) \to \left(R(x,y) \wedge Q(x,z) \right)$$

From proposition 3.18

$x \to (x \wedge y) \geq y$

$$\geq \bigwedge_{y \in Y} Q(x,z) = Q(x,z)$$

Thus

$$R \triangleleft (R^{-1} \circ Q) \geq Q.$$

Theorem 3.27:

(i) Consider the equation with $R \triangleleft P = Q$ with unknown R. The equation has a solution iff $Q \triangleleft P^{-1}$ is a solution. and in this case it is the greatest solution of this Eq.

(ii) Consider the eq. $R \triangleleft P = Q$ with unknown P. The equation has a solution iff $R^{-1} \circ Q$ is a solution. and in this case it is the least solution of this eq.

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