Functional Analysis: Handwritten Notes

by

Mr. Tahir Hussain Jaffery

Partial Contents

Chapter 01: Normed Linear Spaces

- Norm
- Examples of normed
- Normed linear space
- Bounded linear operator
- Continuous linear operator
- Norm of operator
- Isomorphism
- Isometric isomorphism
- Topological isomorphism
- Equivalent norm
- Compact space
- Holder inequality
- Minkwoski's inequality

Chapter 02: Banach Space

- Complete space
- Banach space
- Convergent series in spaces
- Absolutely convergent (or absolutely summable)
- Dual space
- Algebraic dual or conjugate space
- Quotient spaces
- Equivalent module M
- Equivalence class
- Norm of quotient space

Chapter 03: Fundamental Theorems of Functional Analysis

- Interior of a set
- Nowhere dense (or rare)
- Finite category (or meager)
- Baire's category theorem
- The principal of uniform boundedness or Banach-Steinhaus theorem
- Subadditive, positive homogeneous
- Sublinear functional
- Semi-norm
- Extensions, restriction
- Hahn-Banach theorem (real version)
- Hahn-Banach theorem (complex version)
- Hahn-Banach theorem for non-linear spaces
- The open mapping theorem
- norm linear spaces
- Graph of T, Set in the norm linear space
- The closed graph theorem

Chapter 04: Inner Product Spaces

- Inner product on complex linear space
- Consequences of the definition of inner product spaces
- Examples of inner product spaces
- Schwarz's inequality or Cauchy-Bunyakoski's inequality
- Polarization identity
- Parallelogram law for inner product spaces
- Continuous or jointly continuous
- Orthogonal and orthonormal sets
- Pythagorean thoerem
- Generalization of Pythagorean thoerem
- Orthonormal sets
- Complete orthonormal sets

Chapter 05: Hilbert Spaces

- Pre-Hilbert space
- Hilbert space
- Examples of Hilbert spaces
- Schwarz's inequality
- Parallelogram law
- Convex set
- Polarization identity
- Orthogonal in Hilbert space
- Pythagorean theorem
- Generalized Pythagorean theorem
- Orthogonal complement
- Sum of subspaces $M + N$
- Direct sum of the subspaces
- Projection theorem
- Orthonormal sets in Hilbert spaces
- Generalization of Bessel's inequality
- Zorn's lemma
- The Gram-Schmidt orthogonalization process
- The conjugate space of a Hilbert space
- Riesz representation theorem

"Functional Analysis" 1

CHAPTER No:1 {Normed Linear spaces}.

Def: (1.1): Norm: A norm on a linear space X is a real valued function $\|\cdot\|$ (ie $\|\cdot\| : X \to \mathbb{R}$) whose value at x, denoted by $\|x\|$, have the following properties.

(a) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$; $\forall\ x_1, x_2 \in X$.

(b) $\|\alpha x\| = |\alpha|\, \|x\|$; for any scaler α and $x \in X$.

(c) $\|x\| \geq 0$; $\forall\ x \in X$.

(d) $\|x\| = 0$ iff $x = 0$; $\forall\ x \in X$.

The pair $(X, \|\cdot\|)$ is called a normed linear space or normad vector space.

Remark (1.2): If x is a vector, its length is $\|x\|$, the length $\|x_1 - x_2\|$ of the vector difference $x_1 - x_2$ is the distance b/w the end points of the vectors x_1 and x_2.

Examples (1.3):

① The real linear space $\mathbb{R}$ is a normed linear space with norm $\|\cdot\| : \mathbb{R} \to \mathbb{R}$ defined by
$\|x\| = |x|$; $\forall\ x \in \mathbb{R}$.

Pf: (a) For any $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned} \|x_1 + x_2\| &= |x_1 + x_2| \quad \text{(by def:)} \\ &\leq |x_1| + |x_2| \\ &= \|x_1\| + \|x_2\| \end{aligned}$$

ie $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ $\forall\ x_1, x_2 \in \mathbb{R}$.

(b) For any scaler α and $x \in \mathbb{R}$, we have (2)

$\|\alpha x\| = |\alpha x| = |\alpha||x| = |\alpha|\,\|x\|$

(c) For any $x \in \mathbb{R}$, we have:

$\|x\| = |x| \geq 0 \Rightarrow \|x\| \geq 0.$

(d) For any $x \in \mathbb{R}$, we have:

$\|x\| = |x|$

Thus $\|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0.$

ie $\|x\| = 0 \Leftrightarrow x = 0.$

(2) The Complex linear space $\mathbb{C}$ is a normed linear space with the norm defined by:

$\|z\| = |z|$; $\forall\ z \in \mathbb{C}.$

Pf: (a) For any $z_1, z_2 \in \mathbb{C}$, we have:

$\|z_1 + z_2\| = |z_1 + z_2|$ (by definition)

$\leq |z_1| + |z_2|$ (property of Complex nos:)

$= \|z_1\| + \|z_2\|$ (by def:)

ie $\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|$; $\forall\ z_1, z_2 \in \mathbb{C}.$

(b) For any scaler α and $z \in \mathbb{C}$, we have:

$\|\alpha z\| = |\alpha z|$ (by def:)

$= |\alpha||z|$ (property of Complex nos:)

$= |\alpha|\,\|z\|$ (by def:)

(c) For any $z \in \mathbb{C}$, we have:

$\|z\| = |z| = 0$ iff $z = 0.$

(d) For any $z \in \mathbb{C}$, we have:

$\|z\| = |z| \geq 0.$ (property of Complex nos:)

ie $\|z\| \geq 0$; $\forall\ z \in \mathbb{C}.$

Hence the complex linear space $\mathbb{C}$ is a normed linear space with the norm defined above.

* * * * * * * * * * * * * * * * * * *

(3) The spaces $\mathbb{R}^n$ (n-dimensional Euclidean space) and $\mathbb{C}^n$ (n-dimensional unitary space) of all n-tuples of real and complex numbers are normal linear spaces with the norms defined by:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \; ; \qquad 1 \le p \le \infty$$

$$\therefore \|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p} \longrightarrow (1)$$

where $x_i = (x_1, x_2, \ldots, x_n)$

OR $\|x\| = \max\{|x_i| \; ; \; i = 1, 2, \ldots, n\}$.

$$= \max\{|x_1|, |x_2|, \ldots, |x_n|\} \longmapsto (2)$$

where $x_i = (x_1, x_2, \ldots, x_n)$, $1 \le i \le n$

<u>Note:</u> we can define more than one norm on a linear space.

Next, we introduce some special normed linear spaces.

(4) $l^p(x)$, when $\mathbb{C}^n$ or $\mathbb{R}^n$ is considered as normed linear spaces with the norm (1) of Example(3), we denote the space by $l^p(x)$.

Notice that we shall use $l^p(x)$ for both the

moreover, the question of whether the space under discussion is real or complex will either be clear from the context or we shall make a specific statement if necessary.

(5) $l^p = l_p$ = the space of all sequences $x = \{x_n\}$ with $\sum_{i=1}^{\infty} |x_i|^p < \infty$, $p \ge 1$; then this space l^p is a n.l.s with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \; ; \; \forall x \in l^p$$

(6) $l^\infty = l_\infty$ is the space of all bounded sequences $x = \{x_i\}$, Then l^∞ is a n.l.s with the norm:

$$\|x\|_\infty = \sup |x_i| \quad ; \quad 1 \le i \le \infty$$
$$= \sup \{ |x_1|, |x_2|, \ldots \}.$$

(7) $C[a,b]$ is the space of all continuous real valued functions defined on $[a,b]$ ie $f : [a,b] \longrightarrow \mathbb{R}$, which is continuous.

Then $C[a,b]$ is a n.l.s with norms:

(i) $\|f\| = \sup |f(x)|$; $\forall\, f \in C[a,b]$, $x \in [a,b]$.

(ii) $\|f\| = \int_a^b |f(x)|\, dx$; $\forall\, f \in C[a,b]$.

(8) C = This is the space of all convergent sequences in l^∞.

C_0 = This is also the space of all sequences in l^∞ converging to zero.

Then C and C_0 are normed linear spaces with norm as in l^∞.

Note that $C_0 \subset C \subset l^\infty$.

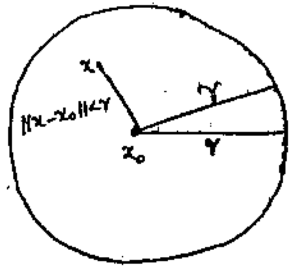
Definition (1.4) :- Let X be a normed linear space.

(a) An open sphere (or open ball) with centre x_0 and radius $r > 0$ is the set:

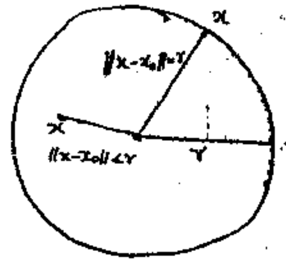
$$B(x_0; r) = \{ x \in X : \|x - x_0\| < r \}$$

A closed sphere (or ball) with centre x_0 and radius $r > 0$ is the set:

$$\bar{B}(x_0; r) = \{ x \in X : \|x - x_0\| \le r \}$$



$B(x_0; r)$

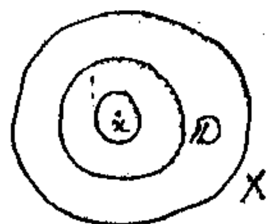


$\bar{B}(x_0; r)$

By the surface (or boundary) of this ball, we mean the set:

$$S(x_0; r) = \{x \in X : \|x - x_0\| = r\}$$

(b) A set D in X is said to be open if for every $x \in D$, there exists a ball with centre x ~~and~~ which is contained in D.



(c) A set D in X is said to be closed if for any sequence $\{x_n\}$ in D with $x_n \to x$ implies that $x \in D$.

(d) A set D is said to be bounded in X if there exists a constant M such that $\|x\| \leq M$; $\forall\, x \in D$.

(e) A set D is said to be compact if whenever $\{x_n\}$ is in D, there exists a cgt subsequence of $\{x_n\}$ whose limit is in D.

(f) A sequence $\{x_n\}$ is called bounded, if there exists a real constant $K > 0$ such that $\|x_n\| \leq K \;\; \forall\, n$.

Proposition (1.5):

(a) Every norm linear space X is a metric space w.r.t the metric $d(x,y) = \|x-y\|$; $\forall x,y \in X$.

(b) $\big| \|x\| - \|y\| \big| \leq \|x-y\|$; $\forall$ $x,y \in X$.

Proof: (a) let X be a norm linear space. Define a mapping $d : X \times X \longrightarrow \mathbb{R}$ by:

$$d(x,y) = \|x-y\| ; \forall x,y \in X.$$

we show that d is a metric on X.

Since (i) $d(x,y) = \|x-y\| \geqslant 0$ (by def:)

ie $d(x,y) \geqslant 0$.

(II) $d(x,y) = \|x-y\| = 0$ iff $x-y=0$ (by def:)
iff $x=y$

ie $d(x,y) = 0$ iff $x=y$.

(iii) $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

ie $d(x,y) = d(y,x)$.

(iv) $d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\|$
$= d(x,y) + d(y,z)$

so $d(x,z) \leq d(x,y) + d(y,z)$.

Hence d is a metric on norm linear space X, known as metric induced by a norm and hence X with d is a metric space.

(b) $\big| \|x\| - \|y\| \big| \leq \|x-y\|$; $\forall$ $x,y \in X$.

Pf: we can write: $x = x-y+y$

$\Rightarrow \|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|$. $\longrightarrow$ ①

similarly we can write: $y = y-x+x$.

$\Rightarrow \|y\| = \|y-x+x\| \leq \|y-x\| + \|x\|$

$\Rightarrow -\|y-x\| \leq \|x\| - \|y\|$

$\Rightarrow -\|x-y\| \leq \|x\| - \|y\| \longrightarrow (2)$

Combining (1) and (2), we have:

$$-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|.$$

$$\Rightarrow \quad \big|\|x\| - \|y\|\big| \leq \|x-y\|.$$

Definition (1.6):

let X be a normed linear space and let $\{x_n\}$ be a sequence in X. Then

(a) we say that the sequence $\{x_n\}$ of elements of X converges to the limit $x \in X$ if for every $\varepsilon > 0$, there exists a +ve integer N such that

$\|x_n - x\| < \varepsilon$ for $n \geq N$.

In other words, we say that $\{x_n\}$ is convergent to the limit $x \in X$ iff $\lim\limits_{n \to \infty} \|x_n - x\| = 0$.

Symbolically we write $x_n \to x$ as $n \to \infty$.

(b) we say that the sequence $\{x_n\}$ in X is a Cauchy sequence if for every $\varepsilon > 0$, there exists a +ve integer N such that:

$\|x_m - x_n\| < \varepsilon$ for $m, n \geq N$.

In other words, $\{x_n\}$ is a Cauchy sequence iff $\lim\limits_{\substack{m \to \infty \\ n \to \infty}} \|x_m - x_n\| = 0$.

Exercise (1.7): let X be a norm linear space.

① If The limit of a sequence $\{x_n\}$ in X exists Then it is unique.

② Every Convergent sequence in X is a Cauchy sequence, but The Converse is not true, in general.

③ A Cauchy sequence is Convergent iff it has a Convergent subsequence.

④ Every Cauchy sequence in X is bounded.

Proposition (1.8): let X be a norm linear space

ⓐ Norm is a Continuous function

ie if $x_n \to x$ Then $\|x_n\| \to \|x\|$

oR If $\{x_n\}$ is a Convergent sequence in X, Then $\|x_n\|$ is a Convergent sequence in $\mathbb{R}$.

ⓑ Addition and scaler multiplication are jointly Continuous in X ie If $x_n \to x$ and $y_n \to y$ Then $x_n + y_n \to x + y$.

and if $x_n \to x$ and $\alpha_n \to \alpha$, Then $\alpha_n x_n \to \alpha x$.

ⓒ If $x_n \to x$ and $y_n \to y$, Then $ax_n + by_n \to ax + by$ where a and b are Constants.

Proof: ⓐ Since $x_n \to x$. So by definition:

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{①}$$

Now $\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\|$ [using (1.5) (b)]

so $\lim\limits_{n\to\infty} \big| \|x_n\| - \|x\| \big| \leq \lim\limits_{n\to\infty} \|x_n - x\| = 0$ (by ①)

Thus $\|x_n\| \to \|x\|$ ie norm is a Continuous function.

(b) Since $x_n \to x$ and $y_n \to y$. So by definition ⑨

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - y\| = 0$$

Now
$$\|(x_n + y_n) - (x+y)\| = \|x_n - x + y_n - y\|$$
$$\leq \|x_n - x\| + \|y_n - y\|.$$

So
$$\lim_{n\to\infty} \|(x_n+y_n) - (x+y)\| \leq \lim_{n\to\infty} \|x_n - x\| + \lim_{n\to\infty} \|y_n - y\|$$
$$= 0 + 0 \quad (\text{from above})$$
$$= 0$$

Hence $x_n + y_n \longrightarrow x+y$

Next we show that $\alpha_n x_n \longrightarrow \alpha x$.

Since $\alpha_n \to \alpha$, so by definition; we have

$$\lim_{n\to\infty} |\alpha_n - \alpha| = 0$$

Now
$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$$
$$\leq \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\|.$$
$$= \|\alpha_n (x_n - x)\| + \|x(\alpha_n - \alpha)\|$$
$$= |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\|.$$

So
$$\lim_{n\to\infty} \|\alpha_n x_n - \alpha x\| \leq \lim_{n\to\infty} |\alpha_n| \|x_n - x\| + \lim_{n\to\infty} |\alpha_n - \alpha| \|x\|.$$
$$= |\alpha_n| \lim_{n\to\infty} \|x_n - x\| + \lim_{n\to\infty} |\alpha_n - \alpha| \|x\|.$$
$$= 0 + 0 \quad (\text{from above})$$
$$= 0$$

Hence $\alpha_n x_n \longrightarrow \alpha x$

ie scalar multiplication and addition are jointly continuous.

— * — * — * — * — * — * —

(c) Since $x_n \to x$ and $y_n \to y$, so we have:

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - y\| = 0.$$

Now $\|(ax_n + by_n) - (ax+by)\| = \|ax_n - ax + by_n - by\|$

$\leq \|ax_n - ax\| + \|by_n - by\|$

Now

$$\lim_{n\to\infty} \|(ax_n + by_n) - (ax+by)\| \leq \lim_{n\to\infty}\left(\|ax_n - ax\| + \|by_n - by\|\right)$$

$$= \lim_{n\to\infty} \|a(x_n - x)\| + \lim_{n\to\infty} \|b(y_n - y)\|$$

$$= |a| \lim_{n\to\infty} \|x_n - x\| + |b| \lim_{n\to\infty} \|y_n - y\|$$

$$= 0 + 0 \text{ (from above)}$$

$$= 0$$

Thus $ax_n + by_n \longrightarrow ax + by$

which completes the proof.

Bounded Linear operators:

Before defining a bounded linear operator, we recall ~~some~~ definitions and results from "Algebra".

Definition: let X and Y be linear spaces with the same scaler field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$).

let A be a function with $D(A)$ in X and range $R(A)$ in Y [ie $A: D(A) \subset X \longrightarrow R(A) \subset Y$]

Then A is called a linear operator if $D(A)$ is a subspace of X and if:

$$\left.\begin{aligned} &(a)\ A(x_1 + x_2) = Ax_1 + Ax_2 \ ; \forall\ x_1, x_2 \in D(A) \\ &(b)\ A(\alpha x) = \alpha A(x) \ ; \forall \alpha \in \mathbb{K} \text{ and } x \in D(A) \end{aligned}\right\} \to (1)$$

clearly condition (1) is equivalent to:

$$A(\alpha x_1 + \beta x_2) = \alpha Ax_1 + \beta Ax_2 \ ; \forall\ \alpha, \beta \in \mathbb{K} \text{ and } x_1, x_2 \in D(A).$$

If $D(A) = X$, we often say that A is a linear operator on (or from) X to Y.

Remark: ① It follows immediately by induction from ⓐ and ⓑ of above definition that

$$A(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) = \alpha_1 A x_1 + \alpha_2 A x_2 + \cdots + \alpha_n A x_n.$$

② If $\alpha = 0$ in above definition, then we have $A(0) = 0$.

③ An important subset of the domain of A is the null space of A denoted by $\mathcal{N}(A)$ and is defined by:

$$\mathcal{N}(A) = \{x \in D(A) : Ax = 0\}.$$

It is readialy verified that $\mathcal{N}(A)$ is a subspace.

[illegible] $\Rightarrow \alpha x_1 + \beta x_2 \in \mathcal{N}(A)$

Examples:-

① The identity operator $I : X \longrightarrow X$ defined by $I(x) = x \; ; \forall x \in X$ is clearly a linear operator from X into itself.

② Zero operator $T : X \longrightarrow Y$ defined by: $T(x) = 0 \; ; \forall x \in X$ is clearly linear operator.

Note that a zero operator is also called Null operator OR Trivial operator.

③ Consider the linear space P of all polynomials $p(x)$ with real coefficients, defined on $[0,1]$. Then the mapping D defined by:

$D(P) = \dfrac{dP}{dx}$, is a linear operator from P into itself.

④ The mapping T defined by: $T(f) = \int_0^1 f(x)\,dx$ is clearly seen to be a linear operator of $C[0,1]$, the space of continuous real functions defined on the closed unit interval $[0,1]$ into the real linear space of all real nos: ie $T : C[0,1] \longrightarrow \mathbb{R}$.

—— * —— * —— * —— * —— * ——

Definition: (a) A mapping $T: D(T) \subset X \longrightarrow Y$ is said to be injective or one-to-one if different points in the domain has different images. ie if for any $x_1, x_2 \in D(T)$, we have:

$$x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$$

or equivalently $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$.

(b) A mapping $T: D(T) \subset X \longrightarrow Y$ is said to be surjective or onto if $R(T) = Y$ ie if every element of Y is the image of at least one element in X.

(c) If T is both injective and surjective, Then it is called bijective.

Notations: If a linear operator A has an inverse, Then it is denoted by A^{-1}. The statement "A^{-1} exists" is the same as "A has an inverse". It is known that A^{-1} exists iff A is one-to-one ie $Ax_1 = Ax_2 \Longrightarrow x_1 = x_2$.

Thm (A): let A be a linear operator, Then A^{-1} exists iff $Ax = 0 \Rightarrow x = 0$.
when A^{-1} exists, Then A^{-1} is a linear operator.

Thm (B): If A is a linear operator from a linear space X into a linear space Y. Then A^{-1} exists iff A is one-to-one and onto.

Theorem: Let A be a linear operator, then A^{-1} exists iff $Ax=0 \Rightarrow x=0$. When A^{-1} exists, it is also a linear operator.

Proof: Before proving the above result, we remember the following [redacted] fact:

"The inverse of an operator A exists iff A is one-to-one ie if $Ax_1=Ax_2 \Rightarrow x_1=x_2$; $\forall x_1,x_2 \in D(A)$."

Now we prove the required result.

First let us suppose that A^{-1} exists. Suppose x is an arbitrary vector in $D(A)$ such that $Ax=0$.

But as A is a linear operator, so that $A(0)=0$

i.e. $Ax=A(0)$. But A^{-1} exists, so A is one-to-one

therefore $Ax=A(0) \Rightarrow x=0$.

Conversely, let us suppose that $Ax=0 \Rightarrow x=0$. We are to prove that A^{-1} exists and for this we will show that A is one-to-one.

For this let $Ax_1=Ax_2$, where $x_1,x_2 \in D(A)$

$\Rightarrow Ax_1-Ax_2=0 \Rightarrow A(x_1-x_2)=0$ (as A is linear)

$\Rightarrow x_1-x_2=0$ [by supposition]

$\Rightarrow x_1=x_2$

which shows that A is one-to-one.

Consequently A^{-1} exists. Hence proved.

Finally we show that when A^{-1} exists, then it is also a linear operator.

Now let $x_1,x_2 \in D(A)$, then we can find y_1,y_2 in $R(A)$ such that $Ax_1=y_1$ and $Ax_2=y_2$

Since A^{-1} exists, so that $x_1=A^{-1}y_1$ and $x_2=A^{-1}y_2$.

Now $y_1+y_2 = Ax_1+Ax_2 = A(x_1+x_2)$ [$\because A$ is Linear]

Since A^{-1} exists, so $A^{-1}(y_1+y_2)=x_1+x_2=A^{-1}(y_1)+A^{-1}(y_2)$

Again let $\alpha \in \mathbb{K}$ and consider αy_1.

Now $\alpha y_1 = \alpha(Ax_1) = A(\alpha x_1)$ [$\because A$ is linear]

$\Rightarrow A^{-1}(\alpha y_1) = \alpha x_1 = \alpha A^{-1}(y_1)$ [$\because A^{-1}$ exists]

$\Rightarrow A^{-1}(\alpha y_1) = \alpha A^{-1}(y_1)$

Hence A^{-1} is also a linear operator.

<u>Theorem</u>: A^{-1} exists iff $N(A) = \{0\}$, when A is linear operator.

<u>Proof</u>: First we recall that A^{-1} exists iff A is one-one. Now suppose that A^{-1} exists, we prove that $N(A) = \{0\}$. For this let $x \in N(A)$, so by def:, $Ax = 0$. But as A is a linear operator, so $A(0) = 0$ therefore $Ax = A(0)$. Since A^{-1} exists, so A is one-one. Hence $x = 0$. therefore $N(A) = \{0\}$.

Conversely suppose that $N(A) = \{0\}$ and we show that A^{-1} exists and to show that A^{-1} exists, we show that A is one-to-one.

For this let $Ax_1 = Ax_2$, where $x_1, x_2 \in D(A)$

$\Rightarrow Ax_1 - Ax_2 = 0$

$\Rightarrow A(x_1 - x_2) = 0$ [$\because A$ is linear]

$\Rightarrow x_1 - x_2 \in N(A) = \{0\}$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

Thus A is one-to-one, Consequently A^{-1} exists. This completes the required proof.

Definition (1.9): Let X and Y be two normed linear spaces over a field $\mathbb{K}$ and $T: X \to Y$ be a linear operator, Then

ⓐ we say that T is continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|Tx - Tx_0\| < \epsilon$ whenever $\|x - x_0\| < \delta$.

ⓑ we say that T is continuous on X if it is continuous for every point of X.

OR T is continuous on X iff for any sequence $\{x_n\}$ in X with $x_n \to x$ implies $Tx_n \to Tx$.

ⓒ T is continuous at the origin iff $x_n \to 0$ implies $Tx_n \to 0$.

ⓓ we say that T is uniformly continuous on X if for ~~every~~ any $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that:

$\|Tx_1 - Tx_2\| < \epsilon$ whenever $\|x_1 - x_2\| < \delta$.

Proposition (1.10): ⓐ A uniformly continuous function is continuous.

ⓑ A continuous function on a compact space is uniformly continuous.

Definition (1.11): An operator $T: X \to Y$ is said to be bounded if there exists a constant $M > 0$: such that $\|Tx\| \le M\|x\|$; $\forall x \in X$.

Theorem (1.12) Let $T: X \to Y$ be a linear operator from a n.l. space X into a n.l.s Y; Then

(a) If T is continuous at some point $x_0 \in X$, then T is uniformly continuous.

(b) T is (uniformly) continuous iff T is bounded.

Proof: (a) Let T be continuous at some point $x_0 \in X$, then by definition, for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$\|Tx - Tx_0\| < \varepsilon$ whenever $\|x - x_0\| < \delta \quad \to (1)$

Let y_1, y_2 be any two points in X.

Let $w = y_1 - y_2 + x_0$, Then $w \in X$ because X is a linear space (closed under addition).

Suppose $\|w - x_0\| < \delta$, Then by (1), we have:

$\|Tw - Tx_0\| < \varepsilon$

i.e $\|y_1 - y_2 + x_0 - x_0\| < \delta$ implies $\|T(y_1 - y_2 + x_0) - Tx_0\| < \varepsilon$

i.e $\|y_1 - y_2\| < \delta$ implies $\|Ty_1 - Ty_2 + Tx_0 - Tx_0\| < \varepsilon$. ($\because T$ is linear operator)

i.e $\|Ty_1 - Ty_2\| < \varepsilon$ whenever $\|y_1 - y_2\| < \delta$.

Thus T is uniformly continuous on X.

Note: The converse of this result is also true, because by proposition (1.10 (a)), we have: "Every unifi. continuous function is continuous".

$\boxed{b}$ Suppose that T is bounded. So by definition there exists a constant $M > 0$ such that

$\|Tx\| \le M\|x\|$, $\forall\, x \in X$.

Now consider any point $x_0 \in X$. Let $\epsilon > 0$ be given. Then for every $x \in X$ such that $\|x - x_0\| < \delta$ where $\delta = \epsilon/M$, we have:

$$
\begin{aligned}
\|Tx - Tx_0\| &= \|T(x - x_0)\| && (\because T \text{ is linear})\\
&\le M\|x - x_0\| && (\because T \text{ is bounded})\\
&< M.\delta\\
&= M.\epsilon/M\\
&= \epsilon
\end{aligned}
$$

ie $\|Tx - Tx_0\| < \epsilon$ whenever $\|x - x_0\| < \delta$

Since x_0 was an arbitrary point of X, This result shows that T is continuous on X.

$\Rightarrow$ T must be continuous at some point of X.

Therefor by part (a), it is uniformly continuous.

Conversely, if T is continuous at origin, then there exits $\delta > 0$ such that:

$$\|Tu\| \le 1 \text{ if } \|u\| \le \delta \quad (\because T0 = 0)$$

Given any $x \in X$, we may write:

$x = c\,u$, where $\|u\| = \delta$ and $c = \delta^{-1}\|x\| > 0$
$\boxed{\because x \text{ is } LS}$ ie c is constt.

Then $Tx = T(cu) \Rightarrow \|Tx\| = \|T(cu)\| = c\,\|Tu\|$

$$
\begin{aligned}
&\le c \quad (\because \|Tu\| \le 1)\\
&= \delta^{-1}\|x\|
\end{aligned}
$$

If we put $m = \delta^{-1}$, Then we have,

$\|Tx\| \le m\|x\|$ $\forall\, x \in X$, which shows that T is bounded.

Proof (b): Suppose that T is continuous on X, then the statement "T is continuous at some point of X" is obviously true.

Conversely, suppose that T is continuous at some point $x_0 \in X$, then by definition for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$\|Tx - Tx_0\| < \varepsilon$ whenever $\|x - x_0\| < \delta$, $\forall x \in X$. $\longrightarrow$ (i)

we show that T is continuous on X.

For this let y be any arbitrary point of X, then we can write: $x - y = (x - y + x_0) - x_0$

Clearly $x - y + x_0 \in X$ ($\because X$ is a linear space)

Now ~~$\|(x-y+x_0) - x_0\| < \delta$~~

Since the condition (i) is true $\forall x \in X$ and since $x - y + x_0 \in X$; so by (i), we can write:

$\|T(x-y+x_0) - Tx_0\| < \varepsilon$ W $\|(x-y+x_0) - x_0\| < \delta$

$\Rightarrow \|Tx - Ty + Tx_0 - Tx_0\| < \varepsilon$ W $\|x - y + x_0 - x_0\| < \delta$

$\Rightarrow \|Tx - Ty\| < \varepsilon$ W $\|x - y\| < \delta$

$\Rightarrow$ T is continous at y. But y was an arbitrary point of X, so T is continuous on every point of X, consequently T is continuous on X.

Pf: (c):- Suppose that T is bounded, then by definition there exists a +ve constant M such that:

$\|Tx\| \le M\|x\|$; $\forall x \in X \longrightarrow (*)$

We show that T is continuous on X.

For this let $\{x_n\}$ be a sequence in X such that $x_n \to x$. In order to show that T is continuous on X, we show that $Tx_n \to Tx$.

Now since $\{x_n\}$ is a sequence in X, so by condition (*), we have:

Theorem (1.13): let X and Y be norm linear spaces and $T: X \to Y$ be a linear operator, then

ⓐ T is continuous on X iff it is uniformly contin on X

ⓑ T is contin: on X iff it is continuous at some point of X.

ⓒ T is continuous on X iff it is bounded.

Proof: ⓐ suppose that T is continuous on X, then it is continuous at every point of X.

let $x_0 \in X$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \epsilon$ whenever $\|x - x_0\| < \delta$; (i)

we shall show that T is uniformly continuous on X.

For this let x_1, x_2 be any two points of X and let $w = x_1 - x_2 + x_0$, Then $w \in X$ ($\because X$ is a linear space)

So Replacing x by w in (i), we get:

$\|T(w) - T(x_0)\| < \epsilon$ whenever $\|w - x_0\| < \delta$

ie $\|T(x_1 - x_2 + x_0) - T(x_0)\| < \epsilon$ W $\|x_1 - x_2 + x_0 - x_0\| < \delta$

ie $\|Tx_1 - Tx_2 + Tx_0 - Tx_0\| < \epsilon$ W $\|x_1 - x_2\| < \delta$

ie $\|Tx_1 - Tx_2\| < \epsilon$ W $\|x_1 - x_2\| < \delta$

which shows that T is uniformly continuous on X.

conversely, suppose that T is uniformly continuous on X. then by definition, for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$\|Tx_1 - Tx_2\| < \epsilon$ W $\|x_1 - x_2\| < \delta$; $\forall x_1, x_2 \in X$

$\Rightarrow T$ is continuous at $x_2 \in X$. But $x_2 \in X$ is an arbitrary point of X, so T is continuous on X. this completes the proof.

$\|T(x_n - x)\| \leq M \|x_n - x\|$

$\Rightarrow \|Tx_n - Tx\| \leq M\|x_n - x\|$ ($\because T$ is linear)

$$\Rightarrow \lim_{n\to\infty} \|Tx_n - Tx\| \leq \lim_{n\to\infty} M\|x_n - x\|$$

$$= M \lim_{n\to\infty} \|x_n - x\|$$

$$= 0 \quad (\because x_n \to x)$$

$\Rightarrow \lim_{n\to\infty} \|Tx_n - Tx\| = 0$ ($\because$ norm is always greater or equal to zero)

$\Rightarrow Tx_n \longrightarrow Tx$.

Hence T is continuous on X.

Conversely, suppose that T is continuous on X, we shall show that T is bounded. On contrary let us suppose that T is unbounded, then we can find a sequence $\{x_n\}$ in X such that:

$$\|Tx_n\| > n\|x_n\| \quad \forall n. \Rightarrow \frac{\|Tx_n\|}{n\|x_n\|} > 1 \quad \forall n.$$

Let us choose $y_n = \dfrac{x_n}{n\|x_n\|}$, Then $y_n \in X$ as X is a linear space.

$$\Rightarrow T(y_n) = T\left(\frac{x_n}{n\|x_n\|}\right)$$

$$\Rightarrow \|Ty_n\| = \left\|T\left(\frac{x_n}{n\|x_n\|}\right)\right\| = \frac{\|Tx_n\|}{n\|x_n\|} > 1 \quad \text{(by above)}$$

i.e. $\|Ty_n\| > 1$.

Since $y_n = \dfrac{x_n}{n\|x_n\|} \Rightarrow \|y_n\| = \left\|\dfrac{x_n}{n\|x_n\|}\right\| = \dfrac{\|x_n\|}{n\|x_n\|} = \dfrac{1}{n}$

$\Rightarrow \|y_n\| = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

$\Rightarrow y_n \to 0$ as $n \longrightarrow \infty$

$\Rightarrow Ty_n \longrightarrow T(0) = 0$ ($\because T$ is continuous on X)

$\Rightarrow Ty_n \longrightarrow 0 \Rightarrow \|Ty_n\| \longrightarrow 0$

$\Rightarrow \|Tx_n\| \longrightarrow 0$ $\left(\because \|Ty_n\| = \dfrac{\|Tx_n\|}{n\|x_n\|}\right)$

So $\|Ty_n\| = \dfrac{\|Tx_n\|}{n\|x_n\|} = 0 < 1$, which is a contradiction to the fact that $\|Ty_n\| > 1$, so our supposition was wrong and hence T is bounded. #

Definition (1.14):
let X and Y be two normed linear spaces and let $T: X \longrightarrow Y$ be a bounded (Continuous) linear operator, then the norm of T is defined as:
$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$
The norm of T is also defined by the following formulae.
$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| \quad \text{and} \quad \|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Theorem (1.15): let $T: X \to Y$ be a Continuous (bounded) linear operator from a n.l. space X into a n.l. space Y, Then

ⓐ $\|T\| < \infty$ ⓑ $\|Tx\| \leq \|T\| \|x\|$; $\forall x \in X$.

Proof: ⓐ since T is a bounded linear operator, so by definition, There exists a constant say $M > 0$ such That $\|Tx\| \leq M \|x\|$; $\forall x \in X$

Then $\sup_{\|x\|=1} \|Tx\| \leq M \sup_{\|x\|=1} \|x\|$

$\Rightarrow \sup_{\|x\|=1} \|Tx\| \leq M \cdot 1 \Rightarrow \sup_{\|x\|=1} \|Tx\| \leq M.$

$\Rightarrow \|T\| \leq M < \infty$ (by def. of $\|T\|$)

$\Rightarrow \|T\| < \infty$.

ⓑ If $x = 0$, Then the inequality is obvious.

If $x \neq 0$, Then put $y = \frac{x}{\|x\|}$ so That $\|y\| = 1$

Thus $Ty = T\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|} \cdot Tx$. ($\because T$ is linear $\hookrightarrow T(\alpha x) = \alpha Tx$)

$\Rightarrow \|Ty\| = \frac{\|Tx\|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \|T\| \hookrightarrow (1).$

$\|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$

Also $\|Ty\| = \frac{\|Tx\|}{\|x\|}$ gives

$$\|Tx\| = \|Ty\|\,\|x\| \le \|T\|\,\|x\| \quad (\text{by } ①)$$

$$\Rightarrow \|Tx\| \le \|T\|\,\|x\| \;;\ \forall\, x \in X. \quad \#\text{proved}$$

<u>Proposition:</u> Let T be a bounded linear operator. Then $x_n \to x \Rightarrow Tx_n \to Tx$.

<u>Proof:</u> Since $x_n \to x$, so that $\|x_n - x\| \to 0$ as $n \to \infty$ or $\lim_{n\to\infty} \|x_n - x\| = 0$.

Now $\|Tx_n - Tx\| = \|T(x_n - x)\|$ ($\because T$ is linear)

$\le \|T\|\,\|x_n - x\|$ ($\because T$ is bounded)

i.e $\lim_{n\to\infty} \|Tx_n - Tx\| = 0$ $\to 0$ as $n \to \infty$

Hence $Tx_n \to Tx$ as $n \to \infty$.

<u>Theorem (1.16):</u> Suppose $T: X \to Y$ be a linear operator where X and Y are n.l.spaces. Then T^{-1} exists and is continuous on its domain of definition iff there exists a constant $m > 0$ such that:

$$m\|x\| \le \|Tx\| \;;\ \forall\, x \in X.$$

<u>Proof:-</u> Suppose there exists a constant $m > 0$ such that $m\|x\| \le \|Tx\| \;;\ \forall x \in X$ $\to ①$

In order to prove that T^{-1} exists, it is enough to show that $Tx = 0 \Rightarrow x = 0$. (Thm A).

Suppose that $Tx = 0$, then ① becomes:

$$m\|x\| \le \|0\| = 0 \Rightarrow m\|x\| = 0 \Rightarrow \|x\| = 0$$
$$\Rightarrow x = 0.$$

i.e $Tx = 0$ implies $x = 0$

Thus T^{-1} exists.

Now To prove The Continuity of T^{-1}, we define $Tx = y$, where $x \in X$ and $y \in Y$.

Since T^{-1} exists, so $T^{-1}y = x$.

Hence from ①, we have:

$m\|T^{-1}y\| \leq \|y\| \Rightarrow \|T^{-1}y\| \leq \frac{1}{m}\|y\|$

for all y in the range of T, which is the domain of T^{-1}.

~~So by Thm (1.12)~~

so that T^{-1} is bounded and by Thm (1.12) T^{-1} is continuous.

Conversely, If T^{-1} exists and is continuous, then by Thm (1.12), T^{-1} is bounded and so we have:

$\|T^{-1}y\| \leq \frac{1}{m}\|y\|$; $\forall$ y in the range of T.

i.e $m\|T^{-1}y\| \leq \|y\|$

But $Tx = y$ or $T^{-1}y = x$. So that

$m\|x\| \leq \|Tx\|$; $\forall x \in X$.

which Completes the required Proof.

Definition (1.17):

① let X and Y be two normed linear spaces. A mapping T on X into Y is called an Isomorphism if the following conditions are satisfied.

ⓐ $T(x+y) = Tx + Ty$ ⓑ $T(\alpha x) = \alpha(Tx)$ $\forall x, y \in X$ and any scalar α.

ⓒ T is one-to-one.

ie if T is linear and one-to-one.

The spaces X and Y are said to be Isomorphic if there exists an Isomorphism of X onto Y (ie $T(X) = Y$).

② An Isometric Isomorphism between two normed linear spaces X and Y is an Isomorphism $T: X \longrightarrow Y$ such that

$\|Tx\| = \|x\|$; $\forall x \in X$.

The spaces X and Y are said to be Isometrically Isomorphic or Congruent if there exists an Isometric Isomorphism of X onto Y.

③ A topological Isomorphism between two normed linear spaces X and Y is an Isomorphism $T: X \longrightarrow Y$ such that T and T^{-1} are continuous in their respective domain.

The spaces X and Y are said to be topologically Isomorphic if there is a homeomorphism T of X onto Y that is also linear operator. For this reason X and Y may be called linearly homeomorphic.

Remark (1.18):

① In order to show that X and Y are congruent, it is necessary and sufficient that there exists a linear operator T with domain X and range Y such that $\|Tx\| = \|x\|$; $\forall\, x \in X$.

② Two norm linear spaces may be Isomorphic but not necessarly congruent. (Find example).

③ Topological Isomorphism is an equivalence relation ie it is reflexive, symmetric and Transitive.

Theorem (1.19): If X and Y are norm linear spaces they are topologically Isomorphic iff there exists a linear operator T with domain X and range Y and +ve constants m, M such that:

$$m\|x\| \le \|Tx\| \le M\|x\| \;;\; \forall\, x \in X \longrightarrow ①$$

Proof: suppose that there exists a linear operator T with domain X and range Y and +ve constants m, M such that ① is satisfied.

we may write ① into two inequalities ie

$$m\|x\| \le \|Tx\| \;;\; \forall\, x \in X \hookrightarrow ②$$

and $\|Tx\| \le M\|x\|$; $\forall\, x \in X$ $\hookrightarrow ③$

Now by Thm (1.16) "T^{-1} exists and is continuous iff $m\|x\| \le \|Tx\|$; $\forall\, x \in X$ ie ② is satisfied".

Also by Thm (1.12), "T is continuous iff $\|Tx\| \le M\|x\|$ $\forall\, x \in X$ ie ③ is satisfied".

Hence combining the two results, we get:

T^{-1} exists and both T, T^{-1} are continuous if There exists Constants $m>0$, $M>0$ such That

$$m\|x\| \le \|Tx\| \le M\|x\| \; ; \; \forall x \in X.$$

which implies That X and Y are topologically Isomorphic iff There exists a linear operator T with domain X and range Y and positive constants m & M such That:

$$m\|x\| \le \|Tx\| \le M\|x\| \; ; \; \forall x \in X$$

which Completes The proof of The Theorem.

T^{-1} exists [illegible] Contin: T, T^{-1} [illegible]

Definition (1.20):

let X be a linear space (or vector space). A norm $\|\cdot\|_1$ on X is said to be equivalent to a norm $\|\cdot\|_2$ on X iff There exists constants m, M both positive such That:

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1 \; ; \; \forall x \in X.$$

Theorem (1.21): let X be a linear space and suppose two norms $\|x\|_1$ and $\|x\|_2$ are defined on X. These norms define the same topology on X iff There exists +ve Constants m, M such That

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1 \; ; \; \forall x \in X \quad (\text{ie They are equiv})$$

Proof: let X_1, X_2 be the normed linear spaces That becomes with The norms $\|x\|_1$ and $\|x\|_2$ respectively.

ie $X_1 = (X, \|x\|_1)$, $X_2 = (X, \|x\|_2)$.

let us define $Tx = x$ and consider T as an operator with domain X_1 and range X_2 (ie $T: X_1 \to X_2$ is linear with domain $D(T) = X_1$ & range $R(T) = X_2$).

Suppose that there exists +ve constants m, M such that $m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$; $\forall x \in X$.

Since $Tx = x$, so that

$$m\|x\|_1 \le \|Tx\|_2 \le M\|x\|_1 \;;\; \forall x \in X.$$

Hence by Thm (1.19):

$m\|x\|_1 \le \|Tx\|_2 \le M\|x\|_1$; $\forall x \in X$ iff

X_1 and X_2 are topologically Isomorphic iff

T^{-1} exists and both T and T^{-1} are Continuous iff

the open sets in X_1 are the same as the open sets in X_2 (by def: of Continuity of $\underline{X_1 \,\&\, X_2}$ $T \,\&\, T^{-1}$).

Thus proving that the two norms define the same topology on X; since elements (open sets) of both the topologies are same.

which completes the required proof.

Finite dimensional norm linear spaces:

Theorem (1.22): Any two norm linear spaces of same finite dimension with the same scaler field are topologically Isomorphic.

Proof: let X_1, X_2 be two norm linear spaces of the same finite dimension with the same scaler field. we need to show that X_1 is topologically Isomorphic to X_2.
the case when $n=0$ is trivial. so we may assume that $n \geq 1$. It will suffice to prove that "if X is an n-dimensional n.l-space, it is topologically Isomorphic to $l'(n)$.
In order to prove that $l'(n)$ and X are topologically Isomorphic, we need to show that there exists a linear operator T with domain $l'(n)$ and range X and +ve constants m, M such that:

$$m\|\eta\| \leq \|T\eta\| \leq M\|\eta\| \ ; \ \forall \eta \in l'(n)$$

(see Thm 1-20).

let $\{x_1, x_2, x_3, ---, x_n\}$ be a basis for X.
Define an operator $T: l'(n) \longrightarrow X$ by:

$$T(\eta) = \eta_1 x_1 + \eta_2 x_2 + \cdots + \eta_n x_n = \sum_{j=1}^{n} \eta_j x_j \quad (*)$$

for all $\eta = (\eta_1, \eta_2, ---, \eta_n) \in l'(n)$

then T is linear. we show that for all $\eta = (\eta_1, \eta_2, ---, \eta_n) \in l'(n)$, there exists $m>0$, $M>0$ such that

$$m\|\eta\| \leq \|T\eta\| \leq M\|\eta\|$$

that is

$$\|T\eta\| \leq M\|\eta\| \longrightarrow (1)$$

$$m\|\eta\| \leq \|T\eta\| \longrightarrow (2)$$

If $\eta = 0$, then (1) and (2) are obviously true.

If $\eta \neq 0$, then by (*),

$$\|T\eta\| = \left\|\sum_{j=1}^{n} \eta_j x_j\right\|$$

$$\leq \sum_{j=1}^{n} \|\eta_j x_j\|$$

$$= \sum_{j=1}^{n} |\eta_j| \, \|x_j\|$$

Let us take $M = \max\{\|x_1\|, \|x_2\|, \dots, \|x_n\|\}$

then $\|T\eta\| \leq M(|\eta_1| + |\eta_2| + \dots + |\eta_n|)$

$$= M\|\eta\| \quad [\because \eta = (\eta_1, \eta_2, \dots, \eta_n) \in l(n)]$$

which implies that (1) is true for $\eta \neq 0$.

From (2), note that:

$$m\|\eta\| \leq \|T\eta\| \quad \text{iff} \quad m \leq \frac{\|T\eta\|}{\|\eta\|}$$

$$\text{iff} \quad m \leq \frac{\|T(\eta_1, \eta_2, \dots, \eta_n)\|}{\|\eta\|}$$

$$\text{iff} \quad m \leq \|T(\beta)\|$$

where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, where

$$\beta_i = \frac{\eta_i}{\|\eta\|}, \quad \|\eta\| = |\eta_1| + |\eta_2| + \dots + |\eta_n|$$

then $\|\beta\| = 1$, because $\beta = (\beta_1, \beta_2, \dots, \beta_n)$

$$\Rightarrow \|\beta\| = \sum_{j=1}^{n} |\beta_j| = |\beta_1| + |\beta_2| + \dots + |\beta_n|$$

$$= \frac{|\eta_1|}{\|\eta\|} + \frac{|\eta_2|}{\|\eta\|} + \dots + \frac{|\eta_n|}{\|\eta\|}$$

$$= \frac{|\eta_1| + |\eta_2| + \cdots + |\eta_n|}{\|\eta\|}$$

$$= \frac{\|\eta\|}{\|\eta\|} = 1.$$

In order to prove ②, it is enough to show that there exists a constant $m > 0$ such that $m \leq \|T\beta\|$ for all $\beta \in l'(n)$ with $\|\beta\| = 1$.

We define a mapping $f : l'(n) \to \mathbb{R}$ by

$f(\eta) = \|T\eta\|$ for all $\eta \in l'(n)$ → (**)

then f is continuous function, because for any $\gamma \in l'(n)$, we have:

$$|f(\eta) - f(\gamma)| = \big|\, \|T\eta\| - \|T\gamma\| \,\big|$$
$$\leq \|T\eta - T\gamma\| \quad (\text{by Prop: } (1.5))$$
$$= \|T(\eta - \gamma)\| \quad (\because T \text{ is linear})$$
$$\leq c\,\|\eta - \gamma\| \quad , \text{where } c > 0$$

i.e. $|f(\eta) - f(\gamma)| \leq c\,\|\eta - \gamma\|$, $c > 0$.

Putting $\delta = \varepsilon / c$, we have:

$$\|\eta - \gamma\| < \delta \Rightarrow |f(\eta) - f(\gamma)| < \varepsilon$$

Thus f is continuous at $\gamma \in l'(n)$.

But γ is choosen arbitrary in $l'(n)$. Hence f is continuous on $l'(n)$.

Now we know (from Analysis) that "the surface of the unit sphere in $l'(n)$ is compact", that is

$$K = \{\eta \in l'(n) : \|\eta\| = 1\} \text{ is compact in } l'(n).$$

Hence the restriction of f to K namely g i.e. $f|_K = g$ is also continuous, because f is continuous. Also we know that "A real valued function on a compact set attains its maximum and minimum". Thus g attains its minimum, that is there exists $\eta \in K$ such that $g(\eta) \leq g(\beta)$; $\forall \beta \in K$, which yields $\|T\eta\| \leq \|T\beta\|$ (by (**)).

$$\Rightarrow 0 \leq m \leq \|T\beta\|, \text{ where } \|T\eta\| = m$$

If $m = 0$, then $\|T\eta\| = 0$ iff $T\eta = 0$ iff $\sum_{j=1}^{n} \eta_j x_j = 0$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$.

But each η_i cannot be zero, because

$\|\eta\| = 1$ (by def: of K).

So $\{x_1, x_2, \ldots, x_n\}$ is linearly dependent, which is a contradiction to the fact that $\{x_1, x_2, \ldots, x_n\}$ is linearly independent. Hence our supposition was wrong and so there exists a constant $m > 0$ such that $m \leq \|T\beta\|$; $\forall \beta \in l'(n)$ with $\|\beta\| = 1$.

This implies that there exists a constant $m > 0$ such that $m\|\eta\| \leq \|T\eta\|$, which is the inequality (II).

So we have proved that there exists a linear operator T with domain $l'(n)$ and range n-dimensional norm linear space X and constants $m>0$, $M>0$ such that :

$$m\|\eta\| \leq \|T\eta\| \leq M\|\eta\| \;;\forall\, \eta \in l'(n).$$

Hence by theorem (1.19), $l'(n)$ and X are topologically Isomorphic and Consequently X_1 and X_2 are topologically Isomorphic.

This Completes the required proof of the theorem.

<u>Remark:</u> Let X and Y are topologically Isomorphic norm linear spaces and if one of them is complete (as a metric space), then other is also complete.

<u>Theorem (1.23)</u>: A finite dimensional norm linear space is complete.

<u>Proof:</u> By above remark, If X and Y are two topologically Isomorphic norm linear spaces, and if one of them is complete, then does the other.

Note that the space $l'(n)$ is topologically Isomorphic to the space $l'(1)$ (ie the real or complex field) which is complete. Thus the finite dimensional norm linear space $l'(n)$ is complete.

More generally, if X is any finite dimensional norm linear space, then we know that every finite dimensional norm linear space X is topologically Isomorphic to $l'(n)$ and hence X is complete. (by above Remark)

Theorem (1.24): If X is a norm linear space, then every finite dimensional subspace of X is necessarily closed.

Proof: let X be a norm linear space and M be a finite dimensional subspace of X, Then by above result, it is Complete.
Then by a result stating that " Every Complete subspace of a metric space is closed", we have That M is closed.

Definition (1.25):
A metric space X is said to be Compact (or sequentially compact) if every sequence in X has a Convergent subsequence.
A subset M of X is said to be Compact if every sequence in M has a convergent subsequence whose limit is an element of M.

Theorem (1.26) Continuous mapping Theorem
let X and Y be metric spaces and $T: X \to Y$ be Continuous mapping, then the image of a Complete Subset M of X under T is Compact.

Theorem (1.27): If X is a finite dimensional normed linear space then each closed and bounded set in X is compact.

Proof: Let X be a finite dimensional norm linear space and M be a closed and bounded set in X. We show that M is compact in X.

We know that "Any two norm linear spaces of the same finite dimencion with the same scaler field are topologically Isomorphic", so there exists a topological Isomorphism $T: X \xrightarrow{\text{onto}} l'(n)$.

Since $M \subset X$, then ~~the map~~ $T(M) = K$, closed and bounded in $l'(n)$. [$\because T$ is a homeomorphism], and so K is compact. [using Hiene-Borel theorem], because in space $l'(n)$, we have from analysis that "each closed and bounded set in $l'(n)$ is always compact".
Since T^{-1} exits (ie $T^{-1}: l'(n) \to X$) and is continuous so using the fact that "Continuous image of compact set is compact", we can say that $T^{-1}(K) = M$ is compact in X. which completes the proof.

Theorem (1.28): If X is finite dimensional n.l. space, then each compact subset M of X is closed and bounded.

Proof: We know that "each compact set in a metric space is closed and bounded". Hence if M is a compact set in X, it must be closed and bounded. ($\because$ every n.l. space is a metric space)

—x—x—x—x—x—x—x—x—

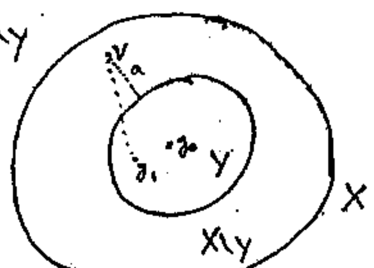
Remark: Combining Thm (1.27) and Thm (1.28), we have the following theorem.

Theorem (1.29): If X is a finite dimensional norm linear space, then each subset of X is compact iff it is closed and bounded.

(1.30) Lemma (F. Riesz's Lemma):

let Y be a subspace of a norm linear space X (of any dimension) such that Y is closed and a proper subset of X, then for every real number θ in the interval $(0,1)$, there exists a vector $x \in X$ such that $\|x\| = 1$ and $\|x-y\| \geq \theta \quad \forall y \in Y$.

Proof: we consider any vector $v \in X \setminus Y$ and denote its distance from Y by a, that is



$$a = \inf_{y \in Y} \|v-y\|$$

clearly $a > 0$ [because norm is always non-negative but $v \in X \setminus Y$]

Since Y is closed. we know $\theta \in (0,1)$. By definition of an infimum, there is a $y_0 \in Y$ such that

$$a \leq \|v-y_0\| \leq \frac{a}{\theta} \quad \left(\text{since } \theta \in (0,1)\text{, so } a < \frac{a}{\theta}\right)$$

let $x = c(v-y_0)$, where $c = \dfrac{1}{\|v-y_0\|}$

then $\|x\| = \|c(v-y_0)\| = c\|v-y_0\| = \dfrac{1}{\|v-y_0\|} \cdot \|v-y_0\|$

ie $\|x\| = 1$.

and we remain to show that $\|x-y\| \geq \theta \ \forall y \in Y$.

Now $\|x-y\| = \|c(v-y_0)-y\| = c\,\|(v-y_0)-c^{-1}y\|$

$$= c\,\|v-(y_0+c^{-1}y)\|$$

$$= c\,\|v-y_1\|, \text{ where } y_1 = y_0 + c^{-1}y.$$

The form of y_1 shows that $y_1 \in Y$ ($\because Y$ is a subspace)

Hence $\|v-y_1\| \geq a$ ($\because a = \inf_{y\in Y}\|v-y\|$, by graph).

Now $\|x-y\| = c\,\|v-y_1\|$

$$\geq c.a$$

$$= \frac{1}{\|v-y_0\|}\cdot a \quad \left(\because c = \frac{1}{\|v-y_0\|}\right)$$

$$\geq \frac{a}{a/\theta} \quad \left(\because a \leq \|v-y_0\| \leq \frac{a}{\theta}\right)$$

$$= \theta$$

So that $\|x-y\| \geq \theta$, where $\theta \in (0,1)$.

Since $y \in Y$ was choosen arbitrary; Therefore $\|x-y\| \geq \theta$; $\forall\, y \in Y$. This completes the proof.

Theorem (1.31) (Converse of 1.27)

let X be a norm linear space and suppose that the surface of the unit sphere $S = \{x \in X : \|x\|=1\}$ in X is Compact, then X is finite dimensional.

Proof: let X be a norm linear space. We need to show that X is finite dimensional.

we assume that $\dim X = \infty$, But S is Compact in X, and we show that this leads to a contradiction.

We choose any $x_1 \in S$. Define $X_1 = \langle x_1 \rangle$
ie x_1 generates a one dimensional space X_1 of X.
Then X_1 is closed (by Thm 1.1.24) and is a proper subspace of X, because $\dim X = \infty$.

Hence by Riez's Lemma, There is $x_2 \in S$ such that $\|x_2 - x_1\| \geq \theta = \frac{1}{2}$

Define $X_2 = \langle x_1, x_2 \rangle$, a two dimensional space generated by x_1, x_2 in S. So X_2 is a proper closed subspace of X. Again by Riez's Lemma, there is an $x_3 \in S$ such that for all $x \in X$, we have:

$$\|x_3 - x\| \geq \tfrac{1}{2}.$$

In particular, if $x = x_1$, then $\|x_3 - x_1\| \geq \frac{1}{2}$.

and if $x = x_2$, then $\|x_3 - x_2\| \geq \frac{1}{2}$.

Proceeding by induction, we obtain an infinite sequence $\{x_n\}$ in S such that

$$\|x_m - x_n\| \geq \tfrac{1}{2} \quad (m \neq n).$$

obviously $\{x_n\}$ Cannot have a convergent subsequence because $\{x_n\}$ itself ~~Cannot~~ is not a convergent sequence.
This facts Contradicts the compactness of S
($\because S$ is compact iff every sequence in S converges to a point in S)
Hence our supposition that $\dim X = \infty$ was false, and so $\dim X < \infty$.

—— * —— * —— * —— * —— * —— * —— * ——

Theorem (1.32): on a finite dimensional norm linear space, any two norms are equivalent.

Proof: Before proving this result, we state the following lemma.

"If $\{x_1, x_2, \dots, x_n\}$ be a linearly independent set in n.l. space, then there exists a constant $c > 0$ such that for each scalars $\alpha_1, \alpha_2, \dots, \alpha_n$

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i| \text{"}.$$

Now we prove the required result.

let $\{x_1, x_2, \dots, x_n\}$ be a basis for X.

If $x \in X$, so it can be uniquely expressed as

$$x = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n = \sum_{i=1}^{n} \xi_i x_i. \quad \longrightarrow (1)$$

where ξ_i are scalers.

let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms defined on X.

By above lemma, there exists a constant $c > 0$ such that:

$$\|x\|_1 = \left\| \sum_{i=1}^{n} \xi_i x_i \right\|_1 \geq c \sum_{i=1}^{n} |\xi_i| \quad \longrightarrow (2)$$

Since $\|\cdot\|_2$ is a norm on X, so

$$\|x\|_2 = \left\| \sum_{i=1}^{n} \xi_i x_i \right\|_2$$

$$\leq \sum_{i=1}^{n} \| \xi_i x_i \|_2 \qquad (\because \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|)$$

$$= \sum_{i=1}^{n} |\xi_i| \|x_i\|_2$$

$$\leq K \sum_{i=1}^{n} |\xi_i|$$

expand

where $K = \max \|x_i\|_2$, $1 \le i \le n$.

$c > 0$
$K = \max \|x_i\|_2$
so $K > 0$
$\therefore c/K > 0$.

so that $\frac{c}{K}\|x\|_2 \le \frac{c}{K}\cdot K \sum_{i=1}^{n} |\xi_i|$

$\Rightarrow m\|x\|_2 \le c \sum_{i=1}^{n} |\xi_i|$, where $m = c/K$.
$\therefore m > 0$

$\Rightarrow m\|x\|_2 \le \|x\|_1 \longrightarrow (3)$.

If we interchange the róle of $\|\cdot\|_1$ and $\|\cdot\|_2$, we get:

$\|x\|_1 \le M\|x\|_2$, where $M > 0$. $\longrightarrow (4)$

Combining (3) and (4), we get:

$m\|x\|_2 \le \|x\|_1 \le M\|x\|_2$ where $M > 0$, $m > 0$.

Hence $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proposition (1.33): If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, p and q are Holder Conjugate (or simply Conjugate) of each other, then for $a \ge 0$, $b \ge 0$, we have the following inequality $a^{1/p} \cdot b^{1/q} \le \frac{a}{p} + \frac{b}{q}$.

Proof: If $a = 0$ or $b = 0$, proposition is clearly satisfied.

we assume the case when both $a > 0$, $b > 0$.

Now if $k \in (0,1)$, define $f(t)$ for $t \ge 1$ by:

$$f(t) = k(t-1) - t^k + 1. \longrightarrow (1)$$

Note that $f(1) = 0$ and $f(t) \ge 0$ for all other values of t.

we have $0 \le f(t) = k(t-1) - t^k + 1$

$$\Rightarrow t^k \le kt + (1-k) \longrightarrow (2)$$

If $a \geq b$, then put $t = \frac{a}{b}$ and $k = \frac{1}{p}$.

so that ② becomes:

$$\left(\frac{a}{b}\right)^{1/p} \leq \frac{1}{p}\left(\frac{a}{b}\right) + \left(1 - \frac{1}{p}\right)$$

$$\Rightarrow a^{1/p} \cdot b^{-1/p} \leq \frac{1}{p} \cdot \frac{a}{b} + \frac{1}{q} \quad \left(\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p}\right)$$

$$\Rightarrow a^{1/p} \cdot b^{1-1/p} \leq \frac{a}{p} + \frac{b}{q} \quad (\text{mult: by } b)$$

$$\Rightarrow a^{1/p} \cdot b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

Now if $a < b$, then put $t = b/a$, $k = 1/q$

so that ② becomes:

$$\left(\frac{b}{a}\right)^{1/q} \leq \frac{1}{q} \cdot \frac{b}{a} + \left(1 - 1/q\right)$$

$$\Rightarrow a^{-1/q} \cdot b^{1/q} \leq \frac{1}{q} \cdot \frac{b}{a} + \frac{1}{p} \left(\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{p} = 1 - 1/q\right)$$

$$\Rightarrow a^{1-\frac{1}{q}} \cdot b^{1/q} \leq \frac{b}{q} + \frac{a}{p} \quad (\text{mult: by } a)$$

$$\Rightarrow a^{1/p} \cdot b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

which completes the proof.

(1.34) Hölder Inequality:

If $1 < p < \infty$ and $x = (x_1, x_2, \dots, x_n)$,

$y = (y_1, y_2, \dots, y_n)$; then

$$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \|y\|_q \quad \text{, where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof: we know that for $x = (x_1, x_2, \dots, x_n)$,

$y = (y_1, y_2, \dots, y_n)$.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \text{ and } \|y\|_q = \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}$$

If $x = 0$ or $y = 0$, then the inequality is obvious.

we assume that x, y are both non-zero, then

assume that:

$$a_i = \left(\frac{|x_i|}{\|x\|_p} \right)^p , \quad b_i = \left(\frac{|y_i|}{\|y\|_q} \right)^q \longrightarrow (1)$$

Then by proposition (1.33), we have:

$$a_i^{1/p} \cdot b_i^{1/q} \le \frac{a_i}{p} + \frac{b_i}{q} \longrightarrow (2)$$

Therefore
$$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} = \frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_q}$$

$$= a_i^{1/p} \cdot b_i^{1/q} \quad [\text{by } (1)]$$

$$\le \frac{a_i}{p} + \frac{b_i}{q} \quad [\text{by } (2)].$$

$$= \frac{\left(\frac{|x_i|}{\|x\|_p} \right)^p}{p} + \frac{\left(\frac{|y_i|}{\|y\|_q} \right)^q}{q} \quad [\text{by } (1)]$$

P.T.O

ie $$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} \le \frac{\left(\frac{|x_i|}{\|x\|_p}\right)^p}{p} + \frac{\left(\frac{|y_i|}{\|y\|_q}\right)^q}{q}.$$

Taking finite summation of both sides,

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} \le \sum_{i=1}^{n} \frac{\left(\frac{|x_i|}{\|x\|_p}\right)^p}{p} + \sum_{i=1}^{n} \frac{\left(\frac{|y_i|}{\|y\|_q}\right)^q}{q}$$

$$= \sum_{i=1}^{n} \frac{\frac{|x_i|^p}{\|x\|_p^p}}{p} + \sum_{i=1}^{n} \frac{\frac{|y_i|^q}{\|y\|_q^q}}{q} \quad \hookrightarrow ③$$

Thus by above recall and ③, we have:

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} \le \frac{\frac{\|x\|_p^p}{\|x\|_p^p}}{p} + \frac{\frac{\|y\|_q^q}{\|y\|_q^q}}{q}$$

$$= \frac{1}{p} + \frac{1}{q}.$$

$$= 1 \quad \text{(Given)}$$

$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \|y\|_q$ for $x = (x_1, x_2, \dots, x_n)$
$y = (y_1, y_2, \dots, y_n)$

which completes the proof.

Remark: when $p = q = 2$; we have:

$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_2 \|y\|_2$, which is called the Cauchy's inequality.

(1.35) **Minkowski's Inequality:**

If $1 \le p < \infty$ and $x = (x_1, x_2, \ldots, x_n)$,

$y = (y_1, y_2, \ldots, y_n)$, then $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

Proof: For $p=1$, the inequality is simply the triangle inequality. So we assume that $1 < p < \infty$.

we have $x+y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

thus $\|x+y\|_p = \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p}$

$$\text{So } \|x+y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p$$

$$= \sum_{i=1}^{n} |x_i + y_i| \, |x_i + y_i|^{p-1}$$

$$\le \sum_{i=1}^{n} \left(|x_i| + |y_i| \right) |x_i + y_i|^{p-1}$$

$$= \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$

$$\le \|x\|_p \, \|x+y\|_p^{p-1} + \|y\|_p \, \|x+y\|_p^{p-1}$$

(By applying Holder inequality)

$$= \left(\|x\|_p + \|y\|_p \right) \|x+y\|_p^{p-1}.$$

ie $\|x+y\|_p^p \le \left(\|x\|_p + \|y\|_p \right) \|x+y\|_p^{p-1}$.

$\Rightarrow \|x+y\|_p^{p-(p-1)} \le \|x\|_p + \|y\|_p$ (Dividing by $\|x+y\|_p^{p-1}$)

$\Rightarrow \|x+y\|_p \le \|x\|_p + \|y\|_p$

which is the desired Minkowski's inequality.

Remark: For $P = 2$, we have from above:

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

which is the famous Schwarz's inequality.

The end chapter # 1

Written by: TAHIR HUSSAIN JAFFERY

Peshawar University.

20.11.2003.

CHAPTER # 2 { BANACH SPACES }

Definition (2.1): (a) A normed linear space X is called complete if every Cauchy sequence in X converges to a limit point in it.

(b) A complete normed linear space is called a Banach space. OR

A normed linear space which is complete as a metric space is called a Banach space.

Remark (2.2): If a normed linear space is not complete, we may complete it by adjoining certain ideal elements to X so as to obtain a complete space $\hat{X}$.

ie X may be inlarged to form a Banach space $\hat{X}$ in which X is dense (ie $X \subset \hat{X}$, $\bar{X} = \hat{X}$).

The complete space $\hat{X}$ is called the "Banach space completion of X".

The space $\hat{X}$ is essentially unique, in the sense that any other Banach space containing X as a dense subspace must be isometrically isomorphic to $\hat{X}$.

<u>Theorem (2.3):</u> Let X and Y be normed linear spaces and let T be a continuous linear operator on X into Y. Then there exists a uniquely determined continuous linear operator $\hat{T}$ of $\hat{X}$ into $\hat{Y}$ such that $\hat{T}x = Tx$ if $x \in X$

and $\|\hat{T}\| = \|T\|$.

<u>Proof:</u> In order to obtain $\hat{T}$, we suppose that $\hat{x} \in \hat{X}$ and select a sequence $\{x_n\}$ in X such that $x_n \to \hat{x}$. Then $\{x_n\}$ is a Cauchy sequence ($\because$ every cgt seq: is a Cauchy sequence).

Now $\|Tx_n - Tx_m\| = \|T(x_n - x_m)\|$ ($\because T$ is linear)

$\leq \|T\| \|x_n - x_m\|$ ($\because T$ is bounded)

$\to 0$ as $m, n \to \infty$ ($\because \{x_n\}$ is a Cauchy sequence)

Therefore $\{Tx_n\}$ is a Cauchy sequence in Y and hence in $\hat{Y}$. So $\{Tx_n\}$ is convergent in $\hat{Y}$ because $\hat{Y}$ is complete.

let $Tx_n \longrightarrow \hat{y} \in \hat{Y}$.

Now we define $\lim\limits_{n \to \infty} Tx_n = \hat{T}\hat{x} \longrightarrow (1)$

Then $\hat{T}$ is a map on $\hat{X}$ into $\hat{Y}$

Now $\hat{T}$ is linear, for if $\hat{T}\hat{x}_1 = \lim\limits_{n \to \infty} Tx_n$ (by (1))

and $\hat{T}\hat{x}_2 = \lim\limits_{n \to \infty} Ty_n$, Then

$$\hat{T}(\hat{x_1}+\hat{x_2}) = \lim_{n\to\infty} T(x_n+y_n)$$

$$= \lim_{n\to\infty} Tx_n + \lim_{n\to\infty} Ty_n \quad (\because T \text{ is linear})$$

$$= \hat{T}\hat{x_1} + \hat{T}\hat{x_2}. \quad (\text{by } ①)$$

and $\hat{T}(\alpha\hat{x}) = \lim_{n\to\infty} T(\alpha x_n)$ [by ①].

$$= \alpha \lim_{n\to\infty} Tx_n \quad [\because T \text{ is linear}]$$

$$= \alpha \hat{T}\hat{x}.$$

Hence $\hat{T}$ is linear.

Next we show that $\hat{T}$ is Continuous.

If $x \in X$, then $x \in \hat{X}$ and so by ①, we have

$$\hat{T}x = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n \quad [\because T \text{ is } \overset{\text{Continuous}}{\text{linear}}]$$

$$= Tx \quad [\because x_n \to \hat{x}].$$

Hence $\hat{T}x = Tx$ if $x \in X \longrightarrow ②$

Also since $x_n \to \hat{x}$, so $Tx_n \to T\hat{x}$ [$\because T$ is Continuous].

ie $\lim_{n\to\infty} Tx_n = T\hat{x} \longrightarrow ③$

and hence $\|\hat{T}\hat{x}\| = \|\lim_{n\to\infty} Tx_n\|$ (by ①)

$$= \|T\hat{x}\| \quad (\text{by } ③).$$

$$\leq \|T\|\|\hat{x}\| \quad \left[\because \|Tx\| \leq \|T\|\|x\| \ \forall x \in X\right]$$

which shows that $\hat{T}$ is bounded and hence continuous.

Also $\|\hat{T}\| \leq \|T\|$ (by def: of norm of an operator). $\to$ (4)

we need to show that $\|T\| \leq \|\hat{T}\|$

we have by (2), $\|Tx\| = \|\hat{T}x\|$

$\leq \|\hat{T}\|\,\|x\|$ [$\because \hat{T}$ is linear]

So that

$\|T\| \leq \|\hat{T}\|$ (by def: of norm of an operator) $\to$ (5)

Form (4) & (5), we get:

$$\|\hat{T}\| = \|T\|.$$

Since X is considered to be dense in $\hat{X}$, so $\hat{T}$ thus obtained is unique.

This completes the proof.

Examples: (1) The spaces $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers are Banach spaces.

(2) The space $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces.

Sol: (1) The spaces $\mathbb{R}$ and $\mathbb{C}$ with norm defined by $\|x\| = |x|$ are normedlinear spaces (by chap 1).

we also know from "Analysis" that every Cauchy sequence of real and complex numbers converges ie $\mathbb{R}$ and $\mathbb{C}$ are complete. Thus $\mathbb{R}$ and $\mathbb{C}$ are complete normedlinear spaces and hence Banach spaces.

Recall: let X be a Complete metric space &

(a) Y be a Complete subspace of X, Then Y is closed.

(b) Y be a closed subspace of X, Then Y is Complete.

Theorem (2.4): A linear subspace X_0 of a Banach space X is itself a Banach space iff X_0 is closed.

Proof: let X_0 is a Complete n.l. subspace of The Banach space X. Since X is a Banach space, so it is Complete as a metric space ($\because$ every n.l. space is a metric space).

Since X_0 is a Complete subspace of The Complete space X, So by above recall (a), X_0 is closed.

Conversely, let X_0 be a closed subspace of The Complete space X; Then by reall (b), X_0 is complete. ie X_0 is a Banach space.

which completes the required proof.

Definition (2.5): let X be a normed linear space and let $x_1+x_2+\cdots+x_n+\cdots = \sum_{i=1}^{\infty} x_i$ be a formal series of elements of X and $S_n = x_1+x_2+\cdots+x_n = \sum_{i=1}^{n} x_i$ be the partial sum of elements of X, Then $\{S_n\}$ is a sequence in X.

(a) we say that the series $\sum_{i=1}^{\infty} x_i$ is Convergent (or summable) in X if there is an element $x \in X$ such that The sequence $\{S_n\}$ of its partial sum Converges to x and we write $x = \sum_{i=1}^{\infty} x_i$.

(b) we say that the series $\sum_{i=1}^{\infty} x_i$ is absolutely Convergent (or absolutely summable) if $\sum_{i=1}^{\infty} \|x_i\|$ is Convergent ie $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

<u>Theorem (2.6)</u>: A normed linear space X is a Banach space iff every absolutely Convergent series in X is Convergent.

<u>Proof</u>: Let X be a Banach space. let $\sum_{i=1}^{\infty} x_i$ be any absolutely convergent series in X ie $\sum_{i=1}^{\infty} \|x_i\|$ is Convergent ie $\sum_{i=1}^{\infty} \|x_i\| < \infty$. we shall show that $\sum_{i=1}^{\infty} x_i$ is Convergent.

Since $\sum_{i=1}^{\infty} \|x_i\|$ is convergent; So by definition of Convergent series, the sequence of its partial sums $\{t_n\}$ is Convergent, where $t_n = \sum_{i=1}^{n} \|x_i\|$. Thus $\{t_n\}$ must be a Cauchy sequence ($\because$ every cgt sequence is cauchy seq.) let $\{S_n\}$ be the sequence of partial sums of the series $\sum_{i=1}^{\infty} x_i$, where $S_n = \sum_{i=1}^{n} x_i$.

Now for $m > n$, we have:

$$\|S_m - S_n\| = \|x_1 + x_2 + \cdots + x_n + x_{n+1} + \cdots + x_m - x_1 - x_2 - \cdots - x_n\|$$

$$= \|x_{n+1} + x_{n+2} + \cdots + x_m\|$$

$$= \left\|\sum_{i=n+1}^{m} x_i\right\| \leq \sum_{i=n+1}^{m} \|x_i\| = \|t_m - t_n\| \quad \left(\because t_n = \sum_{i=1}^{n} \|x_i\|\right)$$

$$\longrightarrow 0 \text{ as } m, n \to \infty \quad (\because t_n \text{ is a Cauchy sequence})$$

ie $\|s_m - s_n\| \to 0$ as $m, n \to \infty$

ie $\{s_n\}$ is a Cauchy sequence in X. But X is Complete ($\because X$ is a Banach space), So $\{s_n\}$ is Convergent in X. Thus by definition, The series $\sum_{i=1}^{\infty} x_i$ is Convergent in X (because The seq: of its partial sum cgs).

Conversely, let X be a normedlinear space and let every absolutely convergent series in X is Convergent. we have to show that X is a Banach space ie X is Complete. For this let $\{x_n\}$ be a Cauchy sequence in the n.l.space X; then by induction, it is possible to select a subsequence say $\{u_K\}$ of $\{x_n\}$ such that:

$$\|u_{K+1} - u_K\| < 2^{-K} \; ; \quad K = 1, 2, \ldots\ldots$$

$$\Rightarrow \sum_{K=1}^{\infty} \|u_{K+1} - u_K\| < \sum_{K=1}^{\infty} 2^{-K} < \infty$$

$$\text{ie } \sum_{K=1}^{\infty} \|u_{K+1} - u_K\| < \infty$$

$\sum_{K=1}^{\infty} 2^{-K}$, $K=1,2,\ldots$
$= 2^{-1} + 2^{-2} + 2^{-3} + \cdots$
$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$
$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$
$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1$

$\Rightarrow$ The series $\sum_{K=1}^{\infty} (u_{K+1} - u_K)$ is absolutely Convergent. Since every absolutely Convergent series in X is Convergent, So it follows That $\sum_{K=1}^{\infty} (u_{K+1} - u_K)$ is convergent. So by definition, the sequence of its partial sums $\{y_n\}$ is Convergent in X, where

$$y_n = \sum_{K=1}^{n} (u_{K+1} - u_K) = u_{n+1} - u_1 \quad [\text{After expanding}]$$

This gives that the subsequence $\{u_K\}$ converges to some $x \in X$.

We see that the Cauchy sequence $\{x_n\}$ has a convergent subsequence $\{u_K\}$ and therefore the whole sequence $\{x_n\}$ converges ($\because$ if a Cauchy sequence has a cgt subsequence, then the whole seq. is convergent). Consequently X is a complete norm linear space ie X is a Banach space. which completes the proof.

Assignment (2.7): Show that the space l^p, where $p \geq 1$ is complete ie a Banach space.

Proof: The space l^p, where $p \geq 1$ is defined to consist of all sequences $x = \{\xi_n\}$ such that $\sum_{n=1}^{\infty} |\xi_n| < \infty$.

The norm in l^p is defined by:

$$\|x\| = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p}$$

Then with this norm, l^p, $p \geq 1$ is a normed linear space. Thus to show that l^p space is a Banach space, it only remains to show that it is complete.

Let $\{x_n\}$ be a Cauchy sequence in l^p with $x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \ldots)$. For each K, $\{\xi_K^{(n)}\}$ is a Cauchy sequence, because

$$\left| \xi_K^{(n)} - \xi_K^{(m)} \right| \leq \left(\sum_{i=1}^{\infty} \left| \xi_i^{(n)} - \xi_i^{(m)} \right|^p \right)^{1/p} = \|x_n - x_m\| \to 0 \text{ as } m, n \to \infty$$

Let us suppose that $\lim_{n \to \infty} \xi_K^{(n)} = \xi_K^{(m)} \to (1)$

we shall show that the sequence $\{\xi_k\}$ is an element of l^p. We know that $\{x_n\}$ is bounded, so $\|x_n\| \leq M$ ($\because$ Every Cauchy sequence is bounded).

Now for any k

$$\left(\sum_{i=1}^{k} |\xi_i^{(n)}|^p\right)^{1/p} \leq \|x_n\| \leq M.$$

Letting $n \to \infty$, we obtain

$$\left(\sum_{i=1}^{k} |\xi_i|^p\right)^{1/p} \leq M \quad (\text{by } ①)$$

Since k is arbitrary, it follows that $\{\xi_k\}$ is an element of l^p and that its norm does not exceed M. Let $x = \{\xi_k\}$. It remains to prove that $\|x_n - x\| \to 0$ as $n \to \infty$.

Suppose $\varepsilon > 0$, then there exists an integer N such that

$\|x_n - x_m\| < \varepsilon$ if $m, n \geq N$ ($\because$ $\{x_n\}$ is a Cauchy seq.)

therefore for any k,

$$\left(\sum_{i=1}^{k} |\xi_i^{(n)} - \xi_i^{(m)}|^p\right)^{1/p} \leq \|x_n - x_m\|$$

$$< \varepsilon \text{ if } m, n \geq N.$$

Keeping k and n fixed, let $m \to \infty$, we then get

$$\left(\sum_{i=1}^{k} |\xi_i^{(n)} - \xi_i|^p\right)^{1/p} < \varepsilon \text{ if } n \geq N$$

This is true for all k, we can let $k \to \infty$ and obtain the result that $\|x_n - x\| < \varepsilon$ if $n \geq N$

This shows that the Cauchy sequence is convergent in l^p. So that l^p is complete. Hence l^p, $p \geq 1$ is a Banach space.

Dual spaces:

Definition (2.8): Let X be a n.l.s and let $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ be a scaler field associated with X. This field is also a normed linear space with norm defined as: $\|x\| = |x|$; $x \in \mathbb{K}$, Then

① A linear operator $x': X \longrightarrow \mathbb{K}$ (scaler field assoc: with X) is called functional.

② A functional $x': X \longrightarrow \mathbb{K}$ is said to be Continuous at a point x_0 of X if for each $\varepsilon > 0$, there exists $\delta > 0$ such that:

$\|x - x_0\| < \delta \Rightarrow |x'(x) - x'(x_0)| < \varepsilon$.

we say that x' is Continuous on X iff it is continuous at each point of X.

③ A functional $x': X \longrightarrow \mathbb{K}$ is said to be linear if

(i) $x'(x_1 + x_2) = x'x_1 + x'x_2$.

(ii) $x'(\alpha x) = \alpha x' x$, where $x, x_1, x_2 \in X$ and α is any scaler.

④ A linear functional x' is said to be bounded if there exists a constant $M > 0$ such that

$|x'(x)| \leq M\|x\|$; $\forall x \in X$.

$x'(x)$ is a number (real or complex). So we take absolute in place of norm

⑤ the set of all linear functionals defined on X is itself a linear space, if addition and scaler multiplication are defined by:

$(x_1' + x_2')(x) = x_1'(x) + x_2'(x)$.

$(\alpha x')(x) = \alpha x'(x)$

and is denoted by X^f, called the Algebraic dual (or Conjugate) space of X.

(6) A norm of a linear functional $x' \in X^f$ is defined as:

$$\|x'\| = \sup_{\|x\|=1} |x'x|$$

(As in case of T, $\|T\| = \sup_{\|x\|=1} \|Tx\|$)

$$= \sup_{\|x\| \le 1} |x'x|$$

$$= \sup_{x \neq 0} \frac{|x'x|}{\|x\|}$$

Note that $|x'x| \le \|x'\|\|x\|$; $\forall x \in X$.

(7) the set of all bounded (continuous) linear functionals defined on X is a linear subspace of X^f and is denoted by X'.

A norm on X' is given by (6). the linear space X' normed in this way is called normed conjugate of X. Sometimes it is denoted by X^*.

Remark: Since X^f is a linear space, we may also consider its algebraic dual (or conjugate) space, which we denote by $(X^f)^f$ or X^{ff}. That is the class of all linear functionals on X^f. we shall denote elements of X^{ff} by x'' (i.e. $x'' : X^f \to \mathbb{K}$, the scaler field associated with X^f) and we shall use the notation $x''(x')$ for the value of x'' at x'.

<u>Theorem (2.9)</u>: Let X be a norm linear space. Then the norm conjugate space X' of X is complete.

<u>Proof</u>: Let $\{x'_n\}$ be a Cauchy sequence in X', then by definition of Cauchy sequence; for every $\varepsilon>0$, there exists +ve integer N such that

$\|x'_m - x'_n\| < \varepsilon$ whenever $m,n \geq N \longrightarrow (1)$

Consequently for each $x \in X$

$$|x'_m(x) - x'_n(x)| = |(x'_m - x'_n)x| \leq \|x'_m - x'_n\|\|x\|$$
$$< \varepsilon\|x\|, \ \forall\ m,n \geq N \longrightarrow (2)$$

which shows that $\{x'_n(x)\}$ is a Cauchy sequence in the space $\mathbb{R}$ or $\mathbb{C}$ ($\because x' : X \to K(=\mathbb{R} \text{ or } \mathbb{C})$) for each $x \in X$. Since the scaler field $\mathbb{R}$ or $\mathbb{C}$ are complete, so $\{x'_n(x)\}$ converges to a limit depending on X which we denote by $x'(x)$.

i.e. $\lim\limits_{n\to\infty} x'_n(x) = x'(x) \longrightarrow (3)$

Thus defining a functional x' on X. We show that $x' \in X'$ and for this it is enough to show that x' is linear and bounded.

<u>x' is linear</u>: Since for scalers λ_1, λ_2 and vectors x_1, x_2 in X, we have:

$$x'(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n\to\infty} x'_n(\lambda_1 x_1 + \lambda_2 x_2) \quad [\text{using } (3)].$$
$$= \lim_{n\to\infty} x'_n(\lambda_1 x_1) + \lim_{n\to\infty} x'_n(\lambda_2 x_2).$$
$$= \lambda_1 \lim_{n\to\infty} x'_n(x_1) + \lambda_2 \lim_{n\to\infty} x'_n(x_2).$$
$$= \lambda_1 x'(x_1) + \lambda_2 x'(x_2) \quad [\text{using } (3)].$$

which shows that x' is linear. (12)

<u>x' is bounded and hence Continuous:</u>

Since $\{x_n'\}$ is a Cauchy sequence, so it is bounded ($\because$ every Cauchy sequence is a bounded sequence)
Therefore by definition, There exists a Constant $K>0$ such that $\|x_n'\| \leq K$; $\forall n$.

For $x \in X$, we have:

$|x_n'(x)| \leq \|x_n'\| \, \|x\|$ ($\because \|Tx\| \leq \|T\|\|x\|$)

$\leq K\|x\|$; $\forall n$ (by above)

Taking limit as $n \to \infty$, we get:

$|x'(x)| \leq K\|x\|$; $\forall n$ (using ③).

which shows that x' is bounded and hence Continuous. Hence $x' \in X'$.

To complete the proof, it remains to show that $x_n' \longrightarrow x'$.

By ②, we have:

$|x_m'(x) - x_n'(x)| < \epsilon\|x\|$; $\forall m, n \geq N$.

Since the result holds for every $m \geq N$ and $x_m'(x) \longrightarrow x'(x)$ [by ③]

we may let $m \to \infty$. Thus letting $\lim\limits_{n\to\infty}$, we get

$|x'(x) - x_n'(x)| \leq \epsilon\|x\|$; $\forall n \geq N$ [by ③].

$\Rightarrow |(x' - x_n')(x)| \leq \epsilon\|x\|$; $\forall n \geq N$. $\boxed{< \text{ Changes to } \leq \text{ because of } m \to \infty}$

By taking Sup over all x of norm, we have

$\|x' - x_n'\| \leq \epsilon$; $\forall n \geq N$.

which shows that $\{x_n'\}$ converges to x'

ie $x_n' \longrightarrow x'$. Consequently X' is Complete.

Quotient Spaces:

Definition (2.10): let M be a subspace of a linear space X. We say that any two elements x, y in X are equivalent modulu M if $x - y \in M$ and we write $x \equiv y \mod M$.

Remark (2.11): It is easy to verify that "equivalence modulu M" is actually an equivalence relation.

ie (i) $x \equiv x \mod M$ for every x (Reflexive property)
($\because x - x = 0 \in M$ as M is a subspace)

(ii) if $x \equiv y \mod M$, Then $y \equiv x \mod M$ (Symmetric property).
($\because x - y \in M \Rightarrow -(x-y) \in M \Rightarrow y - x \in M$)

(iii) if $x \equiv y \mod M$ and $y \equiv z \mod M$, Then

$x \equiv z \mod M$ (Transitive property).
($\because x - y \in M$, $y - z \in M \Rightarrow x - y + y - z \in M \Rightarrow x - z \in M$)

Furthermore equivalence modulu M can be added and multiplied as if they were ordinary equations.

if $x_1 \equiv x_2 \mod M$ and $y_1 \equiv y_2 \mod M$.

Then $x_1 + y_1 = x_2 + y_2 \mod M$.

and $x_1 y_1 = x_2 y_2 \mod M$.

$x_1 \equiv x_2 \Rightarrow x_1 - x_2 \in M$
$y_1 \equiv y_2 \Rightarrow y_1 - y_2 \in M$
$(x_1 + y_1) - (x_2 + y_2)$
$= x_1 - x_2 + y_1 - y_2$
$\in M$.

Definition (2.12): let $x \in X$. An equivalence class of x is denoted by $[x]$ and is defined as:

$$[x] = \{y \in X : x \equiv y \mod M\}.$$

ie $[x] = \{y \in X : x \equiv y \bmod M\}$.

$= \{y \in X : y \equiv x \bmod M\}$ (Symmetry).

$= \{y \in X : y - x \in M\}$.

$= \{y \in X : y - x = m$ for some $m \in M)$.

$= \{y \in X : y = x + m$ for some $m \in M)$.

$= \{x + m : m \in M\}$

$= x + M$.

The collection of all equivalence classes of elements of X is defined as the set :

$$X/_M = \{[x] : x \in X\}.$$

Then $X/_M$ is a linear space with addition and scaler multiplication is defined by:

$$[x] + [y] = [x+y]$$

and $[\alpha x] = \alpha [x]$.

The linear space $X/_M$ is called a Quotient space.

<u>Note</u>: (i) $[x] = [y]$ iff $x - y \in M$.

(ii) If $y \in [x]$, Then $[y] = [x]$ ✓

(iii) The zero element of $X/_M$ is $[0]$, which is the same as M.

(iv) $x \in M$ iff $[x]$ is the zero element of $X/_M$.

(v) The quotient norm on $X/_M$ is defined by:

$$\|[x]\| = \inf_{y\in[x]} \|y\| .$$
$$= \inf_{m\in M} \|x-m\| .$$

(vi) $\|[x]\| \le \|x\|$.

<u>Theorem</u> (2.13):- Let M be a closed linear subspace in the linear space X. For each $[x] \in X/M$, define

$$\|[x]\| = \inf_{y\in[x]} \|y\| = \inf_{m\in M}\|x-m\| \longrightarrow (1)$$

$y\in[x] \Rightarrow x\equiv y$
$\Rightarrow x-y\in M$
$\Rightarrow x-y = m$
$\Rightarrow x-m = y$

Then $\|[x]\|$ is a norm on X/M ie X/M is a norm linear space.

<u>Proof</u>: To prove that (1) defines norm on X/M, we shall show that for $[x],[y]$ in X/M and α in $\mathbb{K}$, the following conditions are true.

(I) $\|[x]\| \geqslant 0$. (II) $\|[x]\| = 0$ iff $[x]$ is the zero element of X/M ie $[0] = M$.

(III) $\|[\alpha x]\| = |\alpha|\,\|[x]\|$ (IV) $\|[x]+[y]\| \le \|[x]\| + \|[y]\|$.

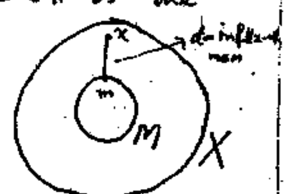
Now clearly by definition:

$$\|[x]\| = \inf_{y\in[x]}\|y\| = \inf_{m\in M}\|x-m\| \geqslant 0 \quad (\text{using } (1)).$$

Therefore $\|[x]\| \geqslant 0$, which proves (I).

Now we have by definition (1),

$\|[x]\| = \inf_{m\in M}\|x-m\|$, which shows that $\|[x]\|$ is the distance from x to M.



Since M is closed, so (A set is closed iff [illegible] all of its [illegible])

$\|[x]\| = 0$ iff $x \in M$

iff $[x]$ is the zero element of X/M

ie $[0] = M$. (by note (iv))

which proves (II).

Next for $\alpha \in \mathbb{K}$ and $x \in X$, we have:

$$\|[\alpha x]\| = \inf_{y \in [x]} \|\alpha y\| \quad (\text{by } ①).$$

$$= |\alpha| \inf_{y \in [x]} \|y\|$$

$$= |\alpha| \, \|[x]\|, \text{ which proves (III).}$$

To prove (IV); For any x, y in X, we have,

$$\|[x]+[y]\| = \|[x+y]\| = \inf_{\substack{u \in [x] \\ v \in [y]}} \|u+v\|$$

$$\leq \inf_{u \in [x]} \|u\| + \inf_{v \in [y]} \|v\|$$

$$= \|[x]\| + \|[y]\|$$

ie $\|[x]+[y]\| \leq \|[x]\| + \|[y]\|$

Since all the conditions of a norm are satisfied. Thus X/M is a norm linear space with norm defined by ①.

Theorem (2-14): Let M be a closed subspace of a Banach space X. Then X/M with the norm defined by ① in Thm (2.13) is also a Banach space.

Proof: Suppose that X is a Banach space ie X is complete as a metric space and we show that X/M is also complete.

If we start with a Cauchy sequence in X/M, then by a Theorem which states, "A Cauchy sequence is convergent iff it has a convergent subsequence", it is enough to show that this sequence has a convergent subsequence. Then by induction it is possible to find a subsequence $\{[x_n]\}$ of the original Cauchy sequence in X/M such that

$$\|[x_1]-[x_2]\| < \frac{1}{2},$$

$$\|[x_2]-[x_3]\| < \frac{1}{2^2}, \text{ ------ }, \text{ in general}$$

$$\|[x_n]-[x_{n+1}]\| < \frac{1}{2^n} \quad \forall n.$$

we prove that this sequence is convergent in X/M.

Choose $y_1 \in [x_1]$ and select $y_2 \in [x_2]$ such that

$$\|y_1 - y_2\| < \frac{1}{2}.$$

we choose $y_3 \in [x_3]$ such that $\|y_2 - y_3\| < \frac{1}{2^2}$.

Continuing this process, we obtain a sequence $\{y_n\}$ in X, where $y_n \in [x_n]$, such that

$$\|y_n - y_{n+1}\| < \frac{1}{2^n} \text{ for all } n. \longrightarrow ①$$

First we show that $\{y_n\}$ is a Cauchy sequence in X.

If $m < n$, then

$$\|y_m - y_n\| = \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \cdots + (y_{n-1} - y_n)\|$$

$$\leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \cdots + \|y_{n-1} - y_n\|.$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} \quad (\text{by } ①).$$

$$< \frac{1}{2^{m-1}} \quad \left(\begin{array}{c}\text{Geometric series with common ratio} \\ r = \tfrac{1}{2},\ a = \tfrac{1}{2^m},\ S_m = \tfrac{a(1-r^n)}{1-r}\end{array}\right).$$

$$\to 0 \text{ as } m \to \infty.$$

which shows that $\{y_n\}$ is a Cauchy sequence in X. But X is Complete, so $\{y_n\}$ Converges in X.

let $y_n \to y \in X$.

Note that $y_n \in [x_n]$, so $[y_n] = [x_n]$ (by Note).

Therefore $\|[x_n] - [y]\| = \|[y_n] - [y]\| = \|[y_n - y]\| \leq \|y_n - y\|$

$(\because \|[x]\| \leq \|x\|)$

ie $\|[x_n] - [y]\| \leq \|y_n - y\|$

$\to 0$ as $n \to \infty$.

ie $[x_n] \to [y]$ as $n \to \infty$.

This shows that the sequence $\{[x_n]\}$ is Convergent ie the subsequence $\{[x_n]\}$ of the original sequence in X/M is convergent and thus the original Cauchy sequence in X/M is Convergent.

Consequently X/M is complete and hence a Banach space.

Written by: Tahir Hussain Bangash of Parachinar.

University of Peshawar.

T.H. Hussain
15/12/2003

CHAPTER # 3 {Fundamental Theorems of Functional Analysis}

Definition (3.1): (1) If A is a subset of a topological space X, then the interior of A is defined as the union of all open sets of X that are contained in A i.e.

$$A^\circ = \bigcup_i O_i \text{ , where each } O_i \text{ is open and } O_i \subseteq A.$$

(2) we say that A has an empty interior i.e. $A^\circ = \phi$ if and only if A does not contain any non-empty open set.

Definition: A subset M of a topological space X is said to be:

(a) nowhere dense (or rare) in X if $\overline{M}$ has empty interior i.e. $\overline{M}$ does not contain any non-empty open set.

(b) of the first catagory (or meager) in X if M is the countable union of nowhere dense sets.

(c) of the second catagor (Non-meager) in X if M is not of the first catagor. i.e. M cannot be expressed as a countable union of nowhere dense sets.

Theorem (3.2) (Baire's Catagory Theorem)

Every complete metric ~~metric~~ space is of the second catagory.

Hence if $X \neq \phi$ is complete and $X = \bigcup_{k=1}^{\infty} A_k$ (A_k is closed) then at least one A_k contains a non-empty open set.

Theorem 3.3 (The principal of uniform Boundedness).

Let $\{T_n\}$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed linear space Y such that $\{T_n x\}$ is bounded in Y for all $x \in X$. Then $\{\|T_n\|\}$ is a bounded subset of real nos ie There is a constant $c > 0$ such that

$\|T_n\| \le c$; $n = 1, 2, \ldots$

Proof: For each $k \in \mathbb{N}$, we define

$A_k = \{x \in X : \|T_n x\| \le k ; \forall n\}$. we show that A_k is closed in X.

let $\{x_j\}$ be a sequence in A_k such that $x_j \to x$.

By continuity of T_n (because T_n is bounded), we have $T_n x_j \to T_n x$. Also by continuity of a norm we can write: $\|T_n x_j\| \to \|T_n x\|$

ie $\|T_n x\| = \lim\limits_{j \to \infty} \|T_n x_j\| \le k$. ($\because x_j \in A_k$).

ie $\|T_n x\| \le k$, which shows that $x \in A_k$ (by def. of A_k)

and hence A_k is closed ($\because x_j \to x \Rightarrow$ we proved $x \in X$)

Since by hypothesis, each $x \in X$ is in some A_k, because $\{\|T_n x\|\}$ is bounded for each $x \in X$ ie $\|T_n x\| \le K$ for some constant K. Hence $X = \bigcup\limits_{k=1}^{\infty} A_k$

Since X is complete, so by Baire's catagory theorem X must be of the second catagory. so there exists at least one A_k, namely A_{k_0}, which contains an open ball $B_r(x_0) = \{x \in X : \|x - x_0\| < r\}$ ie $B_r(x_0) \subset A_{k_0}$.

let x be an arbitrary non-zero vector in X. we set

$z = \gamma' x + x_0$, where $\gamma' = \frac{\gamma}{2\|x\|} \longrightarrow$ ①

Since $z - x_0 = \gamma' x \Rightarrow \|z - x_0\| = \|\gamma' x\| = \left\|\frac{\gamma}{2\|x\|} \cdot x\right\|$

$$= \frac{\gamma}{2\|x\|} \cdot \|x\|$$
$$= \frac{\gamma}{2}$$
$$< \gamma$$

So $\|z - x_0\| < \gamma$, which shows that $z \in B_\gamma(x_0)$.

But $B_\gamma(x_0) \subset A_{k_0}$, so that $z \in A_{k_0}$. and from the definition of A_{k_0}, we have: $\|T_n z\| \leq k_0 ; \forall n \longrightarrow$ ②

Also since $x_0 \in B_\gamma(x_0) \subset A_{k_0}$, so that $x_0 \in A_{k_0}$

so $\|T_n x_0\| \leq k_0 \; ; \forall n \longrightarrow$ ③ [by def: of A_{k_0}].

By ①, we have $z = \gamma' x + x_0$, where $\gamma' = \frac{\gamma}{2\|x\|}$, so we can write: $x = \frac{z - x_0}{\gamma'}$. This yields for all n.

So $T_n x = T_n \frac{(z - x_0)}{\gamma'}$

$$\Rightarrow \|T_n x\| = \left\| T_n \cdot \frac{(z - x_0)}{\gamma'} \right\|$$
$$= \frac{1}{\gamma'} \|T_n (z - x_0)\| \leq \frac{1}{\gamma'} \left[T_n z + T_n(-x_0)\right]$$
$$\leq \frac{1}{\gamma'} \cdot \left[\|T_n z\| + \|T_n x_0\|\right]. \quad (\because |-1| = 1)$$
$$\leq \frac{1}{\gamma'} \left[k_0 + k_0\right] \quad (\text{using ② \& ③}).$$
$$= \frac{2k_0}{\gamma'}$$
$$= \frac{2k_0}{\frac{\gamma}{2\|x\|}}$$
$$= \frac{4k_0 \|x\|}{\gamma} = \frac{4\|x\| k_0}{\gamma}$$

Hence for all x, $\|T_n x\| \leq \frac{4\|x\| k_0}{\gamma}$

$\Rightarrow \|T_n\| \le \frac{4K_0}{\gamma}$ (by taking sup: over both sides with $\|x\|=1$)

Assume that $\frac{4K_0}{\gamma} = c$, then

$\|T_n\| \le c$; $\forall n$.

which completes the required proof.

Note: The principle of uniform boundedness is often called Banach-Steinhaus Theorem.

Theorem (3.4): Let $\{T_n\}$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a n.l.s Y such that $\lim_{n\to\infty} T_n x = Tx$ exists for each $x \in X$. Then T is a bounded linear operator.

Proof: Since $\lim_{n\to\infty} T_n x = Tx$ exists for each $x \in X$, so $\{T_n x\}$ is a convergent sequence and hence bounded, because every convergent sequence is bounded. Therefore by ~~////~~ the principle of uniform boundedness $\{\|T_n\|\}$ is a bounded sequence of real number, that is there exists a constant $k > 0$ such that:

$$\|T_n\| \le k \; ; \forall n. \longrightarrow ①$$

we prove that T is a bounded linear operator.

T is linear, because for $x, y \in X$ and a scaler α we have,

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) \quad [\because Tx = \lim_{n\to\infty} T_n x \text{ (from above)}]$$

$$= \lim_{n\to\infty} (T_n x + T_n y) \quad [\because T_n \text{ is linear}]$$

$$= \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty \quad [\text{'' ''}]$$

Also $T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x)$ $\quad [\because Tx = \lim_{n\to\infty} T_n x]$

$= \alpha \lim_{n\to\infty} T_n x$ $\quad [\because T_n$ is linear$]$

$= \alpha Tx.$ $\quad [\because Tx = \lim_{n\to\infty} T_n x]$.

Also T is bounded, because since $\{\|T_n\|\}$ is a bounded sequence; so from (1), we have:

$\|T_n\| \leq K$; $\forall n$.

Now $\|T_n x\| \leq \|T_n\|\|x\|$ $\quad (\because T_n$ is bounded$)$.

$\leq K\|x\|$ $\quad (\because \|T_n\| \leq K)$.

if T is bounded Then $\|T_x\| \leq \|T\|\|x\|$

ie $\|T_n x\| \leq K\|x\|$.

Taking limit over both sides as $n\to\infty$, we get.

$$\lim_{n\to\infty} \|T_n x\| \leq \lim_{n\to\infty} K\|x\|$$

$$\Rightarrow \left\|\lim_{n\to\infty} T_n x\right\| \leq \lim_{n\to\infty} K\|x\|$$

$$\Rightarrow \|Tx\| \leq K\|x\| ; \forall n.$$

which shows that T is bounded.

Hence T is a bounded linear operator.

<u>Definition (3.5)</u>. Let X be a linear space. A real valued function p defined on X (ie $p: X\to\mathbb{R}$) is called subadditive if

$$p(x+y) \leq p(x) + p(y) ; \forall x, y \in X.$$

and positive homogeneous if $p(\alpha x) = \alpha(px)$, where $\alpha \geq 0$ in $\mathbb{R}$ and $x \in X$.

<u>Definition (3.6)</u>. A real valued function p on a linear space X is called sublinear functional, if it is both subadditive and positive homogeneous.

Definition (3.9): Let X be a linear space (real or complex). A semi-norm on X is a real valued function p defined on X such that

(i) $p(x) \geq 0$; $\forall x \in X$ (ii) $p(x+y) \leq p(x) + p(y)$; $\forall x, y \in X$.

(iii) $p(\alpha x) = |\alpha| (p x)$; for all scalers α and $x \in X$.

If p has the further property that $p(x) \neq 0$ if $x \neq 0$. (or equivalently $p(x) = 0$ iff $x = 0$), then p is a norm on X.

Remark: As with a norm, the properties of p imply the further properties.

(i) $p(0) = 0$ (ii) $|p(x) - p(y)| \leq p(x-y)$.

(iii) $p(-x) \geq -p(x)$.

Proof: (i) we show that $p(0) = 0$.

$$p(0) = p(0 \cdot x) = 0 \cdot p(x) = 0 \quad [\because p(\alpha x) = |\alpha| p(x)].$$

(ii) since $x = x - y + y$

$$\Rightarrow p(x) = p(x - y + y)$$
$$\leq p(x-y) + p(y) \quad [\because p(x+y) \leq p(x) + p(y)].$$

$$\Rightarrow p(x) - p(y) \leq p(x-y) \longrightarrow ①$$

Now again $y = x + y - x \Rightarrow p(y) = p(x + y - x)$
$\leq p(x) + p(y-x)$

$\Rightarrow p(y) \leq p(x) + p(y-x) \Rightarrow p(y) - p(x) \leq p(y-x)$

$\Rightarrow p(y) - p(x) \leq p(-(x-y)) \Rightarrow p(y) - p(x) \leq |-1| \, p(x-y)$

$\Rightarrow p(y) - p(x) \leq p(x-y) \Rightarrow -(p(x) - p(y)) \leq p(x-y)$

$\Rightarrow p(x) - p(y) \geq -p(x-y) \Rightarrow -p(x-y) \leq p(x) - p(y) \to ②$

① & ② $\Rightarrow |p(x) - p(y)| \leq p(x-y)$

(III) we have: $p(-x) = p(-2x + x) \leq p(-2x) + p(x)$
$= 2\, p(-x) + p(x)$

$\Rightarrow p(-x) - 2p(-x) \leq p(x) \Rightarrow -p(-x) \leq p(x)$
$\Rightarrow p(-x) \geq -p(x)$ #.

Definition (3.8):

If X is a set, M a proper subset of X and f is a function defined on M (ie $f: M \to M$), then a function F defined on X (ie $F: X \to X$) is called an extension of f if $F(x) = f(x)$ for all $x \in M$. and f is called the restriction of F to M.

(3.9) Theorem (Hahn - Banach Theorem - Real version)

Let M be a proper subspace of real linear space X. Let p be a sublinear functional on X and let f be a linear functional defined on M such that

$$f(x) \le p(x) \; ; \; \forall x \in M.$$

Then there exists a linear functional $\hat{f}$ on X, which extends f and that $-p(-x) \le \hat{f}(x) \le p(x)$; $\forall x \in X$.

(3.10) Theorem [Hahn Banach Theorem - Complex version]:

Let X be a complex linear space and M a linear subspace of X. Let p be a semi-norm defined on X. Let f be a linear functional on M such that

$$|f(x)| \le p(x) \; ; \; \forall x \in M.$$

Then there exists a linear functional F on X such that $F(x) = f(x)$; $\forall x \in X$ [Extension] and that

$$|F(x)| \le p(x) \; ; \; \forall x \in X.$$

(3.11) Theorem [Hahn - Banach Thm for n.l.s].

Statement:- Let X be a n.l.s over a field $\mathbb{K}$ and let M be a subspace of X. If $m' \in M'$ (ie m' is a linear functional defined on M); Then there exists a $x' \in X'$ such that $\|x'\| = \|m'\|$ and $m'(x) = x'(x)$; $\forall x \in M$ (extension).

Proof: Let p be a real valued function defined on X by: $p(x) = \|m'\|\|x\|$; $\forall x \in X \longrightarrow (1)$

First we show that p is a semi-norm on X.

(i) For $x \in X$, $p(x) \geq 0$ because $p(x) = \|m'\|\|x\| \geq 0$.

(ii) For $x, y \in X$, we have:

$$p(x+y) = \|m'\|\|x+y\| \quad (\text{by } (1)).$$
$$\leq \|m'\|\{\|x\| + \|y\|\} \qquad [\because \|x+y\| \leq \|x\| + \|y\|].$$
$$= \|m'\|\|x\| + \|m'\|\|y\|.$$
$$= p(x) + p(y). \quad [\text{by } (1)].$$

ie $p(x+y) \leq p(x) + p(y)$.

(iii) For any scaler α and $x \in X$, we have,

$$p(\alpha x) = \|m'\|\|\alpha x\| \quad [\text{by } (1)].$$
$$= |\alpha|\|m'\|\|x\| \quad [\text{by definition of norm}].$$
$$= |\alpha| p(x) \quad [\text{by } (1)].$$

Hence p is a semi-norm on X.

Now $|m'(x)| \leq \|m'\|\|x\| \qquad (\because |x'(x)| \leq \|x'\|\|x\|)$

$$= p(x) \quad (\text{by } (1)].$$

ie $|m'(x)| \leq p(x)$ for all $x \in M$.

Hence by "Hahn-Banach theorem for complex space", There exists a linear functional x' on X such that:

(i) $x'(x) = m'(x)$ $\forall x \in M$ (ie x' is extension of m')

and (ii) $|x'(x)| \leq p(x)$; $\forall x \in X$.

To prove the required result, we will prove that $\|m'\| = \|x'\|$ and $x'(x) = m'(x)$; $\forall x \in M$.

But $x'(x) = m'(x)$ $\forall x \in M$ is satisfied from (i) above:

Also from (ii), we have:

$|x'(x)| \leq p(x) = \|m'\| \|x\|$; $\forall x \in X$.

Taking supremum over both sides with $\|x\|=1$, we obtain

$\|x'\| \leq \|m'\| \longrightarrow ②$

Also $\|x'\| = \sup_{\substack{x\neq 0 \\ x\in X}} \dfrac{|x'(x)|}{\|x\|} \geq \sup_{\substack{x\neq 0 \\ x\in X}} \dfrac{|m'(x)|}{\|x\|} = \|m'\|$

$\Rightarrow \|m'\| \leq \|x'\|$.

Since x' is the extension of m', so that $\|m'\|$ cannot be ~~less~~ greater than $\|x'\|$ and so $\|x'\| \geq \|m'\| \longrightarrow ③$

From ② and ③, we get:

$\|x'\| = \|m'\|$.

Thus completing the proof.

Theorem (3.12): Let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Then there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'(x_0) = \|x_0\|$.

Proof:- We consider the subspace M of X containing of all elements $x = \alpha x_0$, where α is a scaler

ie $M = \{x \in X : x = \alpha x_0$, where α is a scaler$\}$.

Define a linear functional m' on M (ie $m' : M \to R$) such that:

$m'(x) = m'(\alpha x_0) = \alpha \|x_0\| \longrightarrow ①$.

Then m' is bounded ie $m' \in M'$, because

$|m'(x)| = |m'(\alpha x_0)| \leq \|m'\| \|\alpha x_0\| = \|m'\| \|x\|$ (by def: of M)

ie $|m'(x)| \leq \|m'\| \|x\|$

Also $|m'(x)| = |m'(\alpha x_0)| = |\alpha \|x_0\||$ (by ①).

$= |\alpha| \|x_0\| = \|\alpha x_0\| = \|x\|$ (by def: of M).

That is $|m'(x)| = \|x\|$; $\forall\, x \in M$.

Taking supremium over both sides, we have

$$\sup_{x \neq 0} \frac{|m'(x)|}{\|x\|} = 1 \quad \text{ie} \quad \|m'\| = 1 \longrightarrow (2)$$

since m' is a bounded linear functional defined on M. so by "Hahn-Banach Theorem for n.l.s", m' has a linear extension x' from M to X

ie $x'(x) = m'(x)$; $\forall\, x \in M \longrightarrow (3)$

and $\|x'\| = \|m'\| \longrightarrow (4)$

But by (2), $\|m'\| = 1$. so that (4) $\Rightarrow \|x'\| = 1$.

It remains to show that $x'(x_0) = \|x_0\|$.

From (1), we have $m'(x) = \alpha\|x_0\|$, where $x = \alpha x_0$.

and so by (3), we have $x'(x) = m'(x) = \alpha\|x_0\|$, where $x = \alpha x_0$

$\Rightarrow x'(x) = \alpha\|x_0\|$, where $x = \alpha x_0$.

$\Rightarrow x'(\alpha x_0) = \alpha\|x_0\|$

$\Rightarrow \alpha x'(x_0) = \alpha\|x_0\|$ ($\because x'$ is a bounded linear functional)

$\Rightarrow x'(x_0) = \|x_0\|$

Thus completing the proof of the theorem.

Recall (3.13) ① A mapping $f: X \to Y$, where X and Y are topological spaces, is said to be open mapping if f maps open subsets of X into open subsets of Y.

② In connection with subset A of X, we define for scaler α and $x_0 \in X$, as follows:

$$\alpha A = \{x \in X : x = \alpha a, \text{ where } a \in A\}$$

and $x_0 + A = \{x \in X : x = x_0 + a, \text{ where } a \in A\}$.

Lemma (3.14): Let X be a n.l.s and $B(x_0; r)$ be an open ball in X. Then $B(x_0;r) = x_0 + r B(0,1)$.

Proof: By definition, $B(x_0; r) = \{x \in X : \|x - x_0\| < r\}$

$$= \{x \in X : \|z\| < r, \text{ where } z = x - x_0\}$$

$$= \{x \in X : \|z\| < r, \text{ where } x = z + x_0\}$$

$$= \{x_0 + z \in X : \|z\| < r\}$$

$$= x_0 + \{z \in X : \|z\| < r\} \quad (\text{by Recall ②})$$

$$= x_0 + \{z \in X : \|\tfrac{z}{r}\| < 1\}$$

$$= x_0 + \{z \in X : \|z'\| < 1, \text{ where } z' = \tfrac{z}{r}\}$$

$$= x_0 + \{z \in X : \|z'\| < 1, \text{ where } z = z'r\}$$

$$= x_0 + \{rz' \in X : \|z'\| < 1\}$$

$$= x_0 + r\{z' \in X : \|z'\| < 1\} \quad (\text{by Recall ②})$$

$$= x_0 + r\{z' \in X : \|z' - 0\| < 1\}$$

$$= x_0 + r\, B(0,1).$$

ie $\boxed{B(x_0; r) = x_0 + r B(0,1)}$

Remark: In particular $B(0;r) = r B(0,1)$.

Ch-03

Note: The proof of the "open mapping theorem" depends upon the following lemma, however we omit the proof, because it is too lengthy.

Lemma (3.15): let T be a bounded linear operator from a Banach space X into a Banach space Y. Then for each open ball $B_0 = B(0,1) \subset X$, the image $T(B_0)$ contains an open ball in Y with centre at the origin.

Theorem (3.16) [The open mapping Theorem]

Statement: A bounded linear operator T from a Banach space X into a Banach space Y is an open mapping.

Proof: let $T: X \longrightarrow Y$ be a bounded linear operator from a Banach space X into a Banach space Y.
In order to show that T is an open mapping, we need to show that for any open set $A \subseteq X$, the image of A under T is open in Y ie $T(A)$ is open in Y.
For this let $y \in T(A)$: since T is an operator, so there exists $x \in A$ such that $y = Tx \in T(A)$.
It is enough to show that $T(A)$ contains an open ball around $y = Tx$.
Since A is open in X; and $x \in A$ so by definition, it contains an open ball with centre x and radius r
ie $B(x; r) \subseteq A$.
we know by Lemma (3.14) that:

$$B(x; r) = x + r B(0; 1) \longrightarrow ①$$

By Lemma (3.15), for the open ball $B(0; 1)$ in X, there is an open ball $B'(0; r)$, with centre at origin, in Y such that:

$$B'(0;r') \subseteq \overline{T(B(0;1))}$$
$$\subseteq r\,\overline{T(B(0;1))}$$
$$= \overline{T(B(0;r))} \quad [\text{using ① Taking } x=0].$$

Remark

Hence $B'(y;r') = y + r'B'(0;1) \xrightarrow{②}$
$$= y + B'(0;r') \quad [\text{by ①}]$$
$$\subseteq y + \overline{T(B(0;r))} \quad [\text{by ②}].$$
$$= Tx + \overline{T(B(0;r))} \quad [\because y = Tx].$$
$$= \overline{T(x + B(0;r))} \quad [\because T \text{ is linear}].$$
$$= \overline{T(x + rB'(0,1))}$$
$$\subseteq T(A) \quad [\text{by ①}].$$

i.e. $B'(y;r') \subseteq T(A)$.

this shows that $T(A)$ contains an open ball around $y = Tx$. Consequently $T(A)$ is open in Y and hence T is an open mapping.

1/ <u>Corrolary (3.17)</u>: Let $T: X \longrightarrow Y$ be a bijective bounded linear operator from a Banach space X into a Banach space Y, Then T is a homeomorphism.

<u>Proof:</u> Recall that T is a homeomorphism if
(i) T is continuous (II) T is bijective.
(III) T^{-1} is continuous.

Since T is continuous ($\because T$ is bounded). and bijective, so by the latter condition, $T^{-1}: Y \to X$ exists.

To show that T^{-1} is continuous, let U be an open set in X; Then $(T^{-1})^{-1}U = TU$, which is open in Y because T is open by above Thm. So that the image of any open set in Y is open in X under T^{-1}, showing that T^{-1} is continuous.

Hence T is a homeomorphism. # proved

Definition (3.18): Let X and Y be norm linear spaces, Then $X \times Y$ is also a norm linear space, where the two algebraic operations of addition and scaler multiplication are defined by,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

and $\alpha(x, y) = (\alpha x, \alpha y)$

and the norm on $X \times Y$ is defined by: $\|(x,y)\| = \|x\| + \|y\|$.

(Check it).

Theorem (3.19): Suppose that X and Y are Banach spaces, then $X \times Y$ is also a Banach space.

Proof:- We show that the product space $X \times Y$ with norm defined by $\|(x,y)\| = \|x\| + \|y\| \longrightarrow ①$ is Complete.

Let $\{z_n\}$ be a Cauchy sequence in $X \times Y$, where $z_n = (x_n, y_n)$ then by definition of Cauchy sequence, for every $\varepsilon > 0$ there exists an N such that:

$$\|z_n - z_m\| < \varepsilon \text{ for } m, n \geq N \longrightarrow ②$$

Now $\|x_n - x_m\| + \|y_n - y_m\| = \|(x_n - x_m, y_n - y_m)\|$ [using ①].

$= \|(x_n, y_n) - (x_m, y_m)\|$ [by the operation of add: in $X \times Y$]

$= \|z_n - z_m\|$ $(\because z_n = (x_n, y_n))$.

$< \varepsilon$ for $m, n \geq N$ [by ②].

$\Rightarrow \|x_n - x_m\| < \varepsilon$ and $\|y_n - y_m\| < \varepsilon$ for $n, m \geq N$ [$\because \|x_n - x_m\|$ & $\|y_n - y_m\|$ are +ve]

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X and Y respectively and since X and Y are Banach spaces. So that $\{x_n\}$ and $\{y_n\}$ Converges, say $x_n \to x$ and $y_n \to y$.

Since norm is a Continuous function, Therefore

$$\|x_n - x\| \to 0 \text{ and } \|y_n - y\| \to 0 \text{ as } n \to \infty \quad \longrightarrow (3)$$

$\| \ \|$ is Contin: if $x_n \to x$ $\Rightarrow \|x_n\| \to \|x\|$.

we show that $\{z_n\}$ is Convergent in $X \times Y$

ie $z_n = (x_n, y_n) \longrightarrow (x, y) = z$ (say).

Now $\|z_n - z\| = \|(x_n, y_n) - (x, y)\|$

$= \|(x_n - x, y_n - y)\|$ [by def. of operation of add: in $X \times Y$]

$= \|x_n - x\| + \|y_n - y\|$

$\longrightarrow 0$ as $n \to \infty$ [by (3)].

This shows that the Cauchy sequence $\{z_n\}$ in $X \times Y$ is Convergent. Since $\{z_n\}$ was choosen arbitrary, therefore $X \times Y$ is complete. Consequently $X \times Y$ is a Banach space.

Definitions (3.20):

(1) let X and Y be normed linear spaces and $T : D(T) \subset X \longrightarrow Y$ be a linear operator from $D(T) \subset X$ into Y, then the set $G = \{(x, Tx) \in X \times Y : x \in D(T)\}$ is called the graph of T.

(2) let X and Y be normed linear spaces and $T : D(T) \subset X \longrightarrow Y$ be a linear operator from $D(T) \subset X$ into Y, then T is called a closed linear operator if its graph $G = \{(x, Tx) \in X \times Y : x \in D(T)\}$ is a closed set in the normed linear space $X \times Y$.

Ch-03

Theorem (3.21) **The Closed Graph Theorem:**

Statement: Let X and Y be Banach spaces. Let T be a closed linear operator whose domain is all of X and whose range is in Y, Then T is continuous.

OR

A closed linear operator from a Banach space X into a Banach space Y is continuous.

Proof:- Let T be a closed linear operator whose domain is all of X and whose range is in Y ie

$D(T) = X$ and $R(T) \subset Y$.

Then by above definition, the graph G of T is closed subspace of the Banach space $X \times Y$, with the norm defined by: $\|(x,y)\| = \|x\| + \|y\|$.

Since X and Y are Banach spaces and G is a closed subspace of the Banach space $X \times Y$, So G itself is complete ($\because$ A closed subspace of a complete metric space is complete) and hence G is a Banach space. Define a mapping $S: G \longrightarrow X$ by:

$$S(x, Tx) = x \quad \text{for all } x \text{ in } X.$$

First we show that S is a linear mapping:

$$\begin{aligned}
\text{Now } S[(x,Tx)+(y,Ty)] &= S(x+y, Tx+Ty) \\
&= S(x+y, T(x+y)) \quad [\because T \text{ is linear}]. \\
&= x+y \quad [\text{by def. of } S]. \\
&= S(x,Tx) + S(y,Ty) \quad [\text{'' ''}].
\end{aligned}$$

$$\begin{aligned}
\text{and } S[\alpha(x,Tx)] &= S(\alpha x, \alpha Tx) = \alpha x \quad [\text{by def. of } S] \\
&= \alpha S(x,Tx).
\end{aligned}$$

which shows that T is linear.

clearly S is one-one and onto.

Next we show that S is bounded.

Now $\|S(x,Tx)\| = \|x\| \quad (\because S(x,Tx) = x)$

$$\leq \|x\| + \|Tx\|$$

$$= \|(x,Tx)\|$$

i.e. $\|S(x,Tx)\| \leq \|(x,Tx)\|$.

T is bounded if $\|Tx\| \leq M\|x\|$. Here $M=1$ so S is bdd.

$\Rightarrow$ S is bounded and hence Continuous.

Since S is bijective, so the inverse mapping $S^{-1}: X \to G$ exists and is defined by: $S^{-1}x = (x, Tx)$; $\forall x \in X$.

Now since X and G are complete and S is bijective bounded linear operator from G to X, so by Corrolary (3.17), S^{-1} is Continuous ($\because S$ is homeomorphism).

and hence bounded.

Now $\|Tx\| \leq \|x\| + \|Tx\| \quad$ ([illegible])

$$= \|(x, Tx)\|$$

$$= \|S^{-1}x\|$$

$$\leq \|S^{-1}\| \|x\| ; \forall x \in X \quad (\because S^{-1} \text{ is bounded}).$$

i.e. $\|Tx\| \leq \|S^{-1}\| \|x\| ; \forall x \in X$.

This inequality shows that T is bounded and hence continuous as required.

The end of Chapter #3

Written by: TAHIR HUSSAIN JAFFERY

Peshawar University.

Tahir Hussain
6/1/2004.

CHAPTER # 4 [Inner Product Spaces.] ①

Definition (4.1):- By an Inner product on a complex linear space X, we mean a mapping $(x,y) \longrightarrow \langle x,y\rangle : X \times X \longrightarrow \mathbb{C}$ satisfying the conditions.

(i) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$.

(ii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$, where the bar denotes complex conjugate.

(iii) $\langle \lambda x, y\rangle = \lambda \langle x,y\rangle$

(iv) $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ iff $x=0$

for all x,y,z in X and $\lambda \in \mathbb{C}$.

when $\langle\, ,\,\rangle$ is an Inner product on a complex linear space X, then the pair $(X, \langle\, ,\,\rangle)$ is called Inner product space, and we reffer to the complex number $\langle x,y\rangle$ as the Inner product of the vectors x and y.

Consequences of the Def: of Inner product:

For all x,y,z in X and $\lambda \in \mathbb{C}$, we have,

(v) $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

(VI) $\langle x, \lambda y\rangle = \bar{\lambda}\langle x,y\rangle$.

(VII) $\langle x, 0\rangle = \langle 0, x\rangle = 0$.

Proof: (v) $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle}$ (by (ii))

$= \overline{\langle y,x\rangle + \langle z,x\rangle}$ (by (i))

$= \overline{\langle y,x\rangle} + \overline{\langle z,x\rangle}$ [Property of Comp: nos:]

$= \overline{\overline{\langle x,y\rangle}} + \overline{\overline{\langle x,z\rangle}}$ [by (ii)]

$= \langle x,y\rangle + \langle x,z\rangle$ [Property of Complex nos:].

(VI) $\langle x, \lambda y\rangle = \overline{\langle \lambda y, x\rangle}$ [by (ii)] $= \overline{\lambda\langle y,x\rangle}$ [by (iii)]

$= \bar{\lambda}\,\overline{\langle y,x\rangle}$ (Property of Complex nos:)

$= \bar{\lambda}\langle x,y\rangle$.

(VIII) $\langle x, 0 \rangle = \langle x, 0 \cdot y \rangle$

$= \bar{0} \langle x, y \rangle$ [by (VII)].

$= 0 \cdot \langle x, y \rangle$

$= 0$

Similarly we can show that $\langle 0, x \rangle = 0$.

<u>Remark</u>: For real linear spaces, the definition of Inner product is the same as the one given above, except that scalars and values $\langle x, y \rangle$ are required to be real so that the "bars" denoting the complex conjugation no longer appear in conditions (II) and (VI).

In order to avoid the complexity of terms, we shall denote an Inner product simply by (x, y) instead of $\langle x, y \rangle$.

<u>Examples (4.2)</u>:- ① The space l_2^n with the Inner product of two vectors $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ defined by: $(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}$ is an Inner product space.

② The space l_2 with the Inner product of two vectors $x = (x_1, x_2, \cdots\cdots)$ and $y = (y_1, y_2, \cdots\cdots)$ defined by

$(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}$ is an Inner product space.

③ $C[a,b]$, the linear space of continuous functions f on an interval $[a,b]$, with Inner product defined by:

$$(f, g) = \int_a^b f(t)\, \overline{g(t)}\, dt \,, \quad t \in [a,b]$$

is an Inner product space.

<u>Remark</u>: An Inner product space is also called a Pre-Hilbert space.

Proposition: If X is an Inner product space, then $x \in X$ is the zero element iff $(x,y)=0$ for all y in X.

Proof: Let $x=0$, then for all y in X, we have

$(x,y) = (0,y) = 0 \quad [\because (x,0)=(0,x)=0]$.

ie $(x,y)=0 \; ; \; \forall y$ in X.

Conversely if $(x,y)=0 \; \forall y$ in X, then let $y=x$ (particular), so

$(x,x)=0 \Rightarrow x=0 \; [\because (x,x)=0 \text{ if } x=0]$.

This completes the proof.

Theorem (4.3) [Schwarz's Inequality OR Cauchy-Bunyakoski's Inequality]

Statement: If X is an Inner product space, then

$$|(x,y)| \leq \sqrt{(x,x)} \cdot \sqrt{(y,y)} \; ; \text{ for all } x,y \text{ in } X \longrightarrow (1)$$

The equality holds in (1) iff x and y are linearly dependent.

Proof: If $y=0$, then (1) holds because $(x,0)=0$

Also if $x=0$, then (1) holds because $(0,y)=0$

Now let $y \neq 0$. For every scaler α, we have:

$$(x-\alpha y, x-\alpha y) \geq 0 \qquad (\because (x,x) \geq 0)$$

$$\Rightarrow (x,x) + (x,-\alpha y) + (-\alpha y, x) + (-\alpha y, -\alpha y) \geq 0$$

$$\Rightarrow (x,x) - \bar{\alpha}(x,y) - \alpha(y,x) + \alpha\bar{\alpha}(y,y) \geq 0 \longrightarrow (2)$$

Setting $\alpha = \dfrac{(x,y)}{(y,y)}$, we have: $\bar{\alpha} = \dfrac{\overline{(x,y)}}{(y,y)}$ $\quad \left[\because \overline{(y,y)} = (y,y)\right]$

and so $\alpha\bar{\alpha} = \dfrac{(x,y)}{(y,y)} \cdot \dfrac{\overline{(x,y)}}{(y,y)} = \dfrac{|(x,y)|^2}{|(y,y)|^2}$ $\quad (\because z\bar{z} = |z|^2 \text{ for all } z \in \mathbb{C})$

Substituting these values in (2), we obtain:

$$(x,x) - \frac{\overline{(x,y)}}{(y,y)} \cdot (x,y) - \frac{(x,y)}{(y,y)} \cdot (y,x) + \frac{|(x,y)|^2}{|(y,y)|^2} \cdot (y,y) \geq 0.$$

$$\Rightarrow (x,x) - \frac{|(x,y)|^2}{(y,y)} - \cancel{\frac{|(x,y)|^2}{(y,y)}} + \cancel{\frac{|(x,y)|^2}{(y,y)}} \geq 0 \quad \left[\because (x,y)(y,x) = (x,y)\overline{(x,y)} = |(x,y)|^2\right]$$

$$\Rightarrow (x,x) - \frac{|(x,y)|^2}{(y,y)} \geq 0 \Rightarrow \frac{|(x,y)|^2}{(y,y)} \leq (x,x). \Rightarrow |(x,y)|^2 \leq (x,x)(y,y)$$

$$\Rightarrow |(x,y)| \leq \sqrt{(x,x)} \cdot \sqrt{(y,y)}$$, which is required Inequality #.

Next we see that the equality in ① holds iff $y=0$ or equality holds in ②. But from ②, we have:

$$|(x,y)| = \sqrt{(x,x)}\cdot\sqrt{(y,y)} \text{ iff } (x-\alpha y, x-\alpha y)=0$$
$$\text{iff } x-\alpha y = 0$$
$$\text{iff } x = \alpha y.$$
iff x & y are linearly dependent.

This completes the required proof.

<u>Corollary (4.4)</u>: Let X be an Inner product space, then for any x and y in X, $\sqrt{(x+y,x+y)} \le \sqrt{(x,x)} + \sqrt{(y,y)}$.

<u>Proof:</u> we can write:

$$\begin{aligned}(x+y, x+y) &= (x,x)+(x,y)+(y,x)+(y,y)\\ &= (x,x)+(x,y)+\overline{(x,y)}+(y,y).\\ &= (x,x)+2\,\mathrm{Re}\,(x,y)+(y,y) \quad [\because z+\bar{z} = 2\,\mathrm{Re}\,z].\\ &\le (x,x)+2|(x,y)|+(y,y) \quad [\because \mathrm{Re}\,z \le |z|].\\ &\le (x,x)+2\left[\sqrt{(x,x)}\cdot\sqrt{(y,y)}\right]+(y,y) \quad [\text{by above inequality}]\\ &= \left[\sqrt{(x,x)}+\sqrt{(y,y)}\right]^2.\end{aligned}$$

$$\Rightarrow (x+y,x+y) \le \left[\sqrt{(x,x)}+\sqrt{(y,y)}\right]^2.$$

Taking square root on both sides, we get:

$$\sqrt{(x+y,x+y)} \le \sqrt{(x,x)}+\sqrt{(y,y)} \quad \text{\# proved.}$$

<u>Theorem (4.5)</u> If X is an Inner product space, Then $\sqrt{(x,x)}$ has the properties of a norm OR

An Inner product space is a norm linear space.

<u>Proof:</u> Let X be an Inner product space. Define a map $\|\cdot\|: X \to \mathbb{R}$ by $\|x\| = \sqrt{(x,x)}$; $\forall x \in X$.

In order to show that $\|x\| = \sqrt{(x,x)}$ defines a norm on the

Inner Product space X, we need to show that it satisfies all the Conditions of a norm. (5)

Now (i) For any $x \in X$, $\|x\| = \sqrt{(x,x)}$ which gives

$$\|x\|^2 = (x,x) \geq 0 \quad (\text{by definition})$$

$$\Rightarrow \|x\|^2 \geq 0 \Rightarrow \|x\| \geq 0.$$

Also $\|x\| = \sqrt{(x,x)} \Rightarrow \|x\|^2 = (x,x) = 0$ iff $x=0$.

$$\Rightarrow \|x\|^2 = 0 \text{ iff } x=0 \text{ ie } \|x\| = 0 \text{ iff } x=0.$$

(ii) By def; $\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} \Rightarrow \|\alpha x\|^2 = (\alpha x, \alpha x)$.

$$= \alpha\bar{\alpha}\,(x,x).$$

$$= |\alpha|^2 \|x\|^2 \quad (\because \|x\| = \sqrt{(x,x)})$$

$$\Rightarrow \|\alpha x\| = |\alpha|\,\|x\|$$

(III) For x, y in X, we have: $\|x\| = \sqrt{(x,x)}$ & $\|y\| = \sqrt{(y,y)}$

Now by Corollary (4.3), we have,

$$\sqrt{(x+y, x+y)} \leq \sqrt{(x,x)} + \sqrt{(y,y)}$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

We see that all the Conditions of a norm are satisfied. Thus $\|x\| = \sqrt{(x,x)}$ is a norm on X and hence $(X, \|\cdot\|)$ is a norm linear space.

Remark (4.6): The Schwarz's inequality can now be written in the form $|(x,y)| \leq \|x\|\,\|y\|$

when X is an Inner product space, we shall henceforth X as a norm linear space with norm $\|x\| = \sqrt{(x,x)}$.

Proposition (4.7) [Polarization identity]

Statement: If X is a inner product space, then for x, y in X, $4(x,y) = \sum_{r=0}^{3} i^r \|x + i^r y\|^2$

Proof: we have

$$\sum_{r=0}^{3} i^r \|x+i^r y\|^2 = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$
$$= (x+y, x+y) - (x-y, x-y) + i(x+iy, x+iy)$$
$$- i(x-iy, x-iy)$$
$$= (x,x) + (x,y) + (y,x) + (y,y) - [(x,x) - (x,y)$$
$$- (y,x) + (y,y)] + i[(x,x) + (x,iy) + (iy,x)$$
$$+ (iy,iy)] - i[(x,x) - (x,iy) - (iy,x)$$
$$+ (iy,iy)].$$
$$= \cancel{(x,x)} + (x,y) + (y,x) + \cancel{(y,y)} - \cancel{(x,x)}$$
$$+ (x,y) + (y,x) - \cancel{(y,y)} + i\cancel{(x,x)} + i(x,iy)$$
$$+ i(iy,x) + i\cancel{(iy,iy)} - i\cancel{(x,x)} + i(x,iy)$$
$$+ i(iy,x) - i\cancel{(iy,iy)}.$$
$$= 2(x,y) + 2(y,x) + 2i(x,iy) + 2i(iy,x).$$
$$= 2(x,y) + 2(y,x) + 2i\bar{i}(x,y) + 2i\bar{i}(y,x)$$
$$= 2(x,y) + 2\cancel{(y,x)} + 2(x,y) - 2\cancel{(y,x)}$$
$$= 4(x,y).$$

Hence $\sum_{r=0}^{3} i^r \|x+i^r y\|^2 = 4(x,y)$.

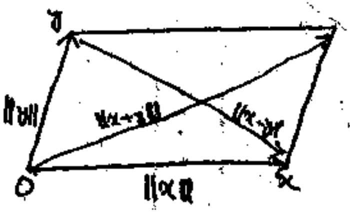
Remark (4.8):- If X is a real inner product space, then the above polarization identity becomes:

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 \text{ for } x, y \text{ in } X.$$

Proposition (4.9) [Parallelogram law for Inner product spaces]

If X is an Inner product space, then
$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for x and y in X
OR In Inner product spaces, ||gm law holds.

Proof.



$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= (x+y, x+y) + (x-y, x-y) \\
&= (x,x) + \cancel{(x,y)} + \cancel{(y,x)} + (y,y) \\
&\quad + (x,x) - \cancel{(x,y)} - \cancel{(y,x)} + (y,y) \\
&= 2(x,x) + 2(y,y) \\
&= 2\|x\|^2 + 2\|y\|^2 \quad (\because \|x\| = \sqrt{(x,x)},\ \|y\| = \sqrt{(y,y)}) \\
&= 2[\|x\|^2 + \|y\|^2]
\end{aligned}$$

Hence $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$, as required.

Definition (4.10): let X be an Inner product space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$, then we say that an Inner product (x,y) on $X \times X$ is a Continuous function or jointly Continuous if $(x_n, y_n) \to (x,y)$

Theorem: (4.11) The Inner product (x,y) is a Continuous function on $X \times X$.

Proof:- let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $x_n \to x$ and $y_n \to y$. We need to show that $(x_n, y_n) \to (x,y)$.
To prove this, it is enough to observe that

$$\begin{aligned}
|(x_n, y_n) - (x,y)| &= |(x_n, y_n) - (x_n, y) + (x_n, y) - (x,y)| \quad \left(\text{+ing and subtracting}\right) \\
&\le |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x,y)| \\
&= |(x_n, y_n - y)| + |(x_n - x, y)| \quad [\text{by def: of Inner Product}] \\
&\le \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \cdot \|y\| \quad [\text{Schwartz's inequality}].
\end{aligned}$$

$\Rightarrow |(x_n, y_n) - (x,y)| \leq \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \|y\|$.

$\longrightarrow 0$ as $n \to \infty$, because

$y_n - y \to 0$ and $x_n - x \to 0$ as $n \to \infty$.

Thus $\lim_{n\to\infty} |(x_n, y_n) - (x,y)| = 0$

which shows that Inner product is a continuous function.

Exercise (4.12): ① Show that an Inner product space is a metric space with metric defined by $d(x,y) = \sqrt{(x-y, x-y)}$; $\forall x,y \in X$

② Give some examples of n.l.s which are not Inner Product spaces.

Sol: ① Let X be an Inner product space, Then for any $x, y \in X$ we define the ~~mapping~~ metric as follows:

$$d(x,y) = \sqrt{(x-y, x-y)}$$

In order to show that X is a metric space, we have to show that $d(x,y) = \sqrt{(x-y, x-y)}$ satisfies all the properties of a metric that is:

(i) $d(x,y) \geq 0$ (ii) $d(x,y) = 0$ iff $x = y$ (iii) $d(x,y) = d(y,x)$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$.

Recall that every Inner product space is a n.l.s with norm defined by: $\|x\| = \sqrt{(x,x)}$.

Therefore (i) $d(x,y) = \sqrt{(x-y, x-y)} = \|x-y\| \geq 0$ [$\because \|x\| = \sqrt{(x,x)}$]

ie $d(x,y) \geq 0 \quad \forall x, y \in X$.

(ii) $d(x,y) = \sqrt{(x-y, x-y)} = \underbrace{\|x-y\|}_{=0 \text{ iff } x-y=0}$ [$\because \|x\| = \sqrt{(x,x)}$]

ie $d(x,y) = 0$ iff $x = y$.

(iii) $d(x,y) = \sqrt{(x-y, x-y)} = \|x-y\| \quad [\because \|x\| = \sqrt{(x,x)}]$

$= \|y-x\|$

$= \sqrt{(y-x, y-x)}$

$= d(y,x).$

(iv) $d(x,y) = \sqrt{(x-y, x-y)} = \|x-y\| \quad [\because \|x\| = \sqrt{(x,x)}].$

$= \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = \sqrt{(x-z,x-z)} + \sqrt{(z-y,z-y)}$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \;; \forall\, x,y,z \in X.$

Thus $d(x,y) = \sqrt{(x-y,x-y)}$ satisfies all the properties of a metric and hence X with the metric $d(x,y) = \sqrt{(x-y,x-y)}$ is a metric space. This completes the proof.

(2) <u>First example:</u> let $X = l_p$; $p > 1$, $p \neq 2$, Then l_p is a n.l.s with norm defined by: $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$; $\forall\, x = (x_1, x_2, \dots) \in l_p$.

We know that in an Inner product space X, Parallelogram law holds ie for any x, y in X: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

But in this case it does not hold, for let

$x = (1,1,0,0,\dots) \in l_p$ and $y = (1,-1,0,0,\dots) \in l_p$.

Then $\|x+y\| = \left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p}$

$= \left(|x_1+y_1|^p + |x_2+y_2|^p + \dots\right)^{1/p}$

$= \left(|1+1|^p + |1-1|^p + |0+0|^p + \dots\right)^{1/p}$

$= \left(|2|^p\right)^{1/p} = 2$

and similarly we can show that $\|x-y\| = 2$.

Also $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = \left(|x_1|^p + |x_2|^p + |x_3|^p + \dots\right)^{1/p}$

$= (1+1+0+0+\dots)^{1/2} = 2^{1/p}$

Similarly $\|y\| = 2^{1/p}$; where $p > 1$, $p \neq 2$.

Therefore $\|x+y\|^2+\|x-y\|^2 = (2)^2+(2)^2 = 8$

and $2\left(\|x\|^2+\|y\|^2\right) = 2\left(2^{2/p}+2^{2/p}\right) = 2.2.\,2^{2/p} = 4.2^{2/p}$; $p>1, p\neq 2$.

we see That $\|x+y\|^2+\|x-y\|^2 \neq 2\left(\|x\|^2+\|y\|^2\right)$; $\forall x,y \in l_p$.

and hence $X = l_p$; $p>1, p\neq 2$ is not an Inner product space.

Second example:- The space $C[a,b]$, The space of Continuous real valued functions on $[a,b]$, with The norm defined by: $\|x\| = \max\limits_{t\in[a,b]} |x(t)|$ is a norm linear space.

we know That in an Inner product space, The Parallelogram law holds ie for any x,y in $C[a,b]$

$$\|x+y\|^2+\|x-y\|^2 = 2\left(\|x\|^2+\|y\|^2\right)$$

But it does not hold in This Case, for if we take $x(t) = 1$ and $y(t) = \frac{t-a}{b-a}$; Then

$$\|x\| = \max_{t\in[a,b]} |x(t)| = \max_{t\in[a,b]} |1| = 1$$

$$\|y\| = \max_{t\in[a,b]} |y(t)| = \max_{t\in[a,b]} \left|\frac{t-a}{b-a}\right| = \left|\frac{b-a}{b-a}\right| = 1 \quad (\because \text{for } t=b, \text{ we get max: value})$$

Also $\|x+y\| = \max\limits_{t\in[a,b]} |(x+y)(t)| = \max\limits_{t\in[a,b]} |x(t)+y(t)|$

$$= \max_{t\in[a,b]} \left|1+\frac{t-a}{b-a}\right| = \left|1+\frac{b-a}{b-a}\right| \quad (\because \text{for } t=b, \text{ we get max: value})$$

$$= |1+1| = 2$$

and $\|x-y\| = \max\limits_{t\in[a,b]} |(x-y)(t)| = \max\limits_{t\in[a,b]} |x(t)-y(t)|$

$$= \max_{t\in[a,b]} \left|1-\frac{t-a}{b-a}\right| = \left|1-\frac{a-a}{b-a}\right| \quad (\because \text{for } t=a, \text{ we get max: value})$$

$$= 1$$

Now $\|x+y\|^2 + \|x-y\|^2 = (2)^2 + (1)^2 = 5$

and $2\left(\|x\|^2 + \|y\|^2\right) = 2(1+1) = 4$

Thus $\|x+y\|^2 + \|x-y\|^2 \neq 2\left(\|x\|^2 + \|y\|^2\right)$

and hence $C[a,b]$ is not an Inner product space.

Orthogonal and orthonormal sets:

Definition: let X be an Inner product space and let A, B be any two non-empty subsets of X, Then

(a) we say that $x, y \in X$ are orthogonal if $(x,y) = 0$. we express symbolically the orthogonal vectors x and y by $x \perp y$. (ie x perp y).

(b) we say that $x \in X$ is orthogonal to A and write $x \perp A$ if $x \perp y$ for every y in A. ie $(x,y) = 0 \ \forall y \in A$.

(c) we say that A is orthogonal to B and write $A \perp B$ if $a \perp b$ for every $a \in A$ and every $b \in B$ ie $(a,b) = 0 \ \forall a \in A, b \in B$.

(d) we say that A is orthogonal if for each pair of vectors x, y in A, $x \perp y$ $(x \neq y)$ ie $(x,y) = 0 \ \forall x, y \in A$, where $x \neq y$.

Remark: (1) Every vector is orthogonal to zero ie $x \perp 0 \ \forall x \in X$.

(2) if $x \perp y$, Then $y \perp x$ ie $(x,y) = 0 \Rightarrow (y,x) = 0$.

(3) if $x \perp y$ and $x \perp z$, Then $x \perp y+z$ and $x \perp \alpha y$ for every scalar α.

(4) If $x \perp y_n$, where $y_n \longrightarrow y$; Then $x \perp y$.

(5) 0 is the only vector orthogonal to itself.

Proof: (1) Since $(x,0) = 0$ [by def. of Inner product]; so that $x \perp 0$.

(2) Since $x \perp y$, so $(x,y) = 0$.

Now $(y,x) = \overline{(x,y)} = \overline{0} = 0$, so $y \perp x$.

(3) Since $x \perp y$ and $x \perp z \Rightarrow (x,y)=0$ and $(x,z)=0$.

Now $(x, y+z) = (x,y) + (x,z) = 0+0=0$

$\Rightarrow x \perp y+z$.

Also $(x, \alpha y) = \bar{\alpha}(x,y) = \bar{\alpha}\cdot 0 = 0 \Rightarrow x \perp \alpha y; \forall \alpha$.

(4) Since $y_n \to y \Rightarrow \lim\limits_{n\to\infty} y_n = y$.

$(x,y) = \lim\limits_{n\to\infty}(x, y_n)$ [$\because$ Inner product is a Continuous function]

$= 0$ [$\because x \perp y_n$].

So $x \perp y$.

(5). Since $(x,x) = \|x\|^2$ (by defi)

So $(0,0) = \|0\|^2 = 0 \Rightarrow 0$ is the only vector orthogonal to itself.

Example: ① $\mathbb{R}^n$ is an Inner product space with inner product defined by: $(x,y) = \sum\limits_{i=1}^{n} x_i y_i$.

Then the vectors $(1,0,0,\ldots\ldots,0)$, $(0,1,0,\ldots,0)$, $\ldots\ldots$, $(0,0,\ldots,1)$ are orthogonal, because the Inner product of any two of above vectors is 0.

(2) $\mathbb{R}^3$ is an Inner product space with Inner product defined by: $(x,y) = \sum\limits_{i=1}^{3} x_i y_i$, then the set of vectors $\{(1,1,0), (0,-1,0)\}$ is not orthogonal.

(3) l^2 is an Inner product space with Inner product defined by: $(x,y) = \sum\limits_{i=1}^{\infty} x_i \bar{y_i}$; then the vectors

$e_1 = (1,0,0,\ldots\ldots)$, $e_2 = (0,1,0,\ldots\ldots)$, $e_3 = (0,0,1,0,\ldots\ldots)$, $\ldots\ldots$,

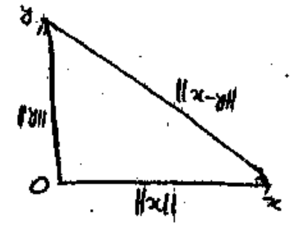
$e_i = (0,0,0,\ldots\underset{i\text{th place}}{1},0,0,\ldots\ldots)$, $\ldots\ldots$ in l^2 are orthogonal

because $e_i \perp e_j$ for all i,j with $i \neq j$.

Remark: one of the simple geometric fact about orthogonal vectors is the pythagorean theorem, which is given as follows.

Theorem: If x and y are orthogonal vector in an Inner Product space X, Then $\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x-y\|^2$.

Proof: we have:



$$\|x+y\|^2 = (x+y, x+y) \quad [\because (x,x)=\|x\|^2]$$
$$= (x,x)+(x,y)+(y,x)+(y,y).$$
$$= (x,x)+0+0+(y,y) \quad [\because x \text{ and } y \text{ are orthogonal}]$$
$$= (x,x)+(y,y)$$
$$= \|x\|^2+\|y\|^2$$

Similarly we can show that $\|x-y\|^2 = \|x\|^2+\|y\|^2$

Hence $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2+\|y\|^2$, which completes the proof.

Theorem [Generalization of Pythagorean Thm].

Let $x_1, x_2, \ldots, x_n$ be mutually orthogonal vectors in the inner Product space X, Then

$$\|x_1+x_2+\cdots+x_n\|^2 = \|x_1\|^2+\|x_2\|^2+\cdots\cdots+\|x_n\|^2$$

OR $\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n}\|x_i\|^2$

Proof:- $\|x_1+x_2+\cdots\cdots+x_n\|^2 = (x_1+x_2+\cdots+x_n, x_1+x_2+x_3+\cdots+x_n) \quad [\because (x,x)=\|x\|^2]$

$$= \underline{(x_1,x_1)}+(x_1,x_2)+\cdots\cdots+(x_1,x_n)+$$
$$(x_2,x_1)+\underline{(x_2,x_2)}+\cdots\cdots+(x_2,x_n)+\cdots\cdots$$
$$\cdots\cdots+(x_n,x_1)+(x_n,x_2)+\cdots\cdots\underline{(x_n,x_n)}$$
$$= (x_1,x_1)+(x_2,x_2)+\cdots+(x_n,x_n) \quad [\because x_1,x_2,\ldots,x_n \text{ are mutually orthogonal}].$$
$$= \|x_1\|^2+\|x_2\|^2+\cdots+\|x_n\|^2$$

OR $\left\|\sum_{i=1}^{n} x_i\right\| = \sum_{i=1}^{n}\|x_i\|^2$, as required #.

Orthonormal sets:

Definition: A set $S=\{x_i : i\in I\}$ in an Inner product space X is said to be orthonormal if

$$(x_i, x_j) = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$$

For $i=j$, $(x_i, x_j) = (x_i, x_i) = \|x_i\|^2$.

ie $(x_i, x_j) = \delta_{ij}$, the standard Kronecker delta.

In other words the set S is said to be orthonormal if it is orthogonal and $\|x\|=1$ for every $x\in S$.

Example: let $\{x_i : i\in I\}$ be an orthogonal set in an inner product space X, Then the set,

$$A=\left\{\frac{x_i}{\|x_i\|} : i\in I\right\}$$ is orthonormal.

Sol: Let $\frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|}\in A$, Then

$$\left(\frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|}\right) = \frac{1}{\|x_i\|\,\|x_j\|}(x_i, x_j) \quad [\because \|x_i\| \;\&\; \|x_j\|\in\mathbb{R}]$$

$$= \frac{1}{\|x_i\|\,\|x_j\|}\times(0) \quad [\because (x_i, x_j)=0 \text{ being orthogonal}]$$

$$=0.$$

Thus the inner product of two different element of A is zero. So that A is orthogonal.

Next we show that norm of every element of A is 1.

For this let $\frac{x_i}{\|x_i\|}\in A$, Then

$$\left\|\frac{x_i}{\|x_i\|}\right\| = \frac{\|x_i\|}{\|x_i\|} \quad (\because \|x_i\|\in\mathbb{R})$$
$$=1$$

this shows that A is an orthonormal set.

Def: (Complete orthonormal set)

An orthonormal set S in an Inner product space X is said to be complete if there exists no orthonormal set in X of which S is a proper subset.
In other words, S is complete if it is maximal w.r.t the property of being orthonormal.

Note: If S is complete orthonormal set, then there does not exists any non-zero vector such that $x \perp S$ and $\|x\| = 1$.

Example: In the space l^2, the orthonormal set composed of
$e_1 = (1,0,0,\ldots)$, $e_2 = (0,1,0,\ldots)$, $e_3 = (0,0,1,0,\ldots)$, ...
is a complete orthonormal set.

The end of CH#4.

Written by: TAHIR HUSSAIN BANGASH
Parachinar.

CHAPTER # 5 {Hilbert spaces}

Def: (5.1): ⓐ An inner product space is also called pre-Hilbert space.

ⓑ An inner product space (ie a pre-Hilbert space) X is called a Hilbert space if it is complete in the sense of a metric space.

Notation (5.2): we shall use H to represent a Hilber space.

Examples (5.3):- ① $\mathbb{R}^n$ is a Hilbert space with inner product defined by: $(x,y) = \sum_{i=1}^{n} x_i y_i$; $x, y \in \mathbb{R}^n$.

~~Because~~ Because we have proved that $\mathbb{R}^n$ is an inner product space and we know from analysis that $\mathbb{R}^n$ is also complete.

② $\mathbb{C}^n$ is ~~an inner~~ a Hilbert space with inner product ~~space~~ defined by : $(x,y) = \sum_{i=1}^{n} x_i \overline{y_i}$; $x, y \in \mathbb{C}^n$.

③ the space l_2 of all complex sequences $x = \{x_i\}$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$ is an inner product space under the inner product : $(x,y) = \sum_{i=1}^{\infty} x_i \overline{y_i}$; $y = \{y_i\} \in l_2$.

we also know that l_2 is complete, hence l_2 is a Hilbert space.

④ Every finite dimensional inner product space is a Hilbert space.

Because every finite dimensional inner product space is a finite dimensional n.l.s and we have proved in ch#4 that every finite dimensional n.l.s is complete.

Theorem (5.4) [Schwarz's Inequality]

If x and y are two vectors in a Hilbert space, then

$$|(x,y)| \le \sqrt{(x,x)}\cdot\sqrt{(y,y)}$$
$$= \|x\|\,\|y\| \longrightarrow ①$$

and equality holds in ① iff x and y are linear dependent.

Pf: See corresponding Proof in Inner product spaces, only replacing Inner product spaces by Hilbert spaces.

Theorem (5.5): The Inner product in a Hilber space H is jointly continuous ie is a continuous function.

Proof: See corresponding pf in Inner product spaces.

Theorem (5.6) (Parallelogram law)

If x and y are any vectors in a Hilbert space H, Then

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2+\|y\|^2\right).$$

Pf:- Same as for Inner product spaces.

Exercise (5.7):- ① let $X = l_p$, $p>1$, $p \neq 2$ is a norm linear space but not a Hilbert space, because we have proved in CH#4 That it is not an Inner product space.

② The space $X = C[a,b]$ is not a Hilbert space, because we have Proved That it is not an Inner product space.

Recall: The line segment joining two given elements x and y of a space X is defined to be the set of all $z \in X$, of the form: $z = tx + (1-t)y$ for every real no: t such that $0 \le t \le 1$.

A subset M of X is said to be Convex if for every $x, y \in M$ the line segment joining x and y is contained in M

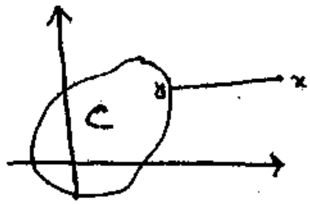
ie $z = tx + (1-t)y \in M$ for every t, where $0 \le t \le 1$.

(Convex set)

(Not convex).

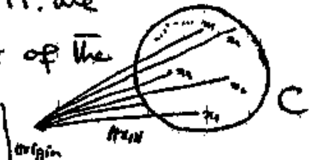
Definition (5.8): If C is any non-empty subset of a Hilbert space H, we define $d(x, C)$ (The distance from x to C) by

$$d(x,C) = \inf_{y \in C} \|x - y\|$$



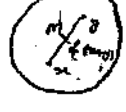
Theorem (5.9): A closed Convex subset C of a Hilbert space H Contains a unique vector of smallest norm.

Proof: let C be a closed Convex subset in H. we show that it Contains a unique vector of the smallest norm.



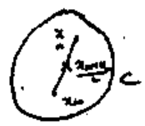
$d = \inf\{\|x_1\|, \|x_2\|, \cdots\}$
$= \|x\|$ (say)

Since C is convex, so by above definition, it is non-empty and Contains $\frac{1}{2}(x+y)$, whenever it Contains x and y.



let $d = \inf\{\|x\| : x \in C\}$, Then by def: of an infimum, There exists a sequence $\{x_n\}$ in C, Such that $x_n \to d$. (because of a result). ④

and by the Convexity of C; $\frac{1}{2}(x_n + x_m)$ is in C.



and $\left\| \frac{x_n + x_m}{2} \right\| \geq d$ (by def: of d)

$\Rightarrow \|x_n + x_m\| \geq 2d$.

Now using Parallelogram law, we have

$$\|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right).$$

$$\Rightarrow \|x_n - x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right) - \|x_n + x_m\|^2.$$

$$= 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2$$

$$\leq 2\|x_n\|^2 + 2\|x_m\|^2 - (2d)^2 \qquad \left[\because \|x_n + x_m\| \geq 2d \Rightarrow -\|x_n + x_m\| \leq -2d\right]$$

$$= 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2.$$

$$\to 2d^2 + 2d^2 - 4d^2 = 0 \text{ as } n, m \to \infty \quad (\text{by above, } \|x_n\| \to d,\ \|x_m\| \to d)$$

$\Rightarrow \|x_n - x_m\|^2 \to 0$ as $n, m \to \infty$

This shows that $\{x_n\}$ is a Cauchy sequence in C.

Now since H is Complete and C is a closed subspace of H, so C is complete, so the Cauchy sequence $\{x_n\}$ Converges in C

ie $x_n \to x \in C$ (say); Then $x = \lim\limits_{n\to\infty} x_n$.

$$\Rightarrow \|x\| = \left\| \lim_{n\to\infty} x_n \right\| = \lim_{n\to\infty} \|x_n\| \quad (\because \text{norm is a contin: function})$$

$$= d \qquad (\because \|x_n\| \to d)$$

ie x is a vector in C with smallest norm.

Now we show that x is unique. For this let us suppose that x' is another vector in C with $x' \neq x$, which also has norm d

ie $\|x'\| = d$.

Now x, x' are in C and C is Convex, so that $\frac{1}{2}(x + x')$ is also in C and by applying parallelogram law, we have,

$$\left\| \frac{x + x'}{2} \right\|^2 + \left\| \frac{x - x'}{2} \right\|^2 = 2\left(\left\| \frac{x}{2} \right\|^2 + \left\| \frac{x'}{2} \right\|^2 \right).$$

$$\Rightarrow \left\|\frac{x+x'}{2}\right\|^2 = 2\left(\left\|\frac{x}{2}\right\|^2 + \left\|\frac{x'}{2}\right\|^2\right) - \left\|\frac{x-x'}{2}\right\|^2.$$
$$= 2\left(\frac{\|x\|^2}{4} + \frac{\|x'\|^2}{4}\right) - \left\|\frac{x-x'}{2}\right\|^2$$
$$= \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} - \left\|\frac{x-x'}{2}\right\|^2.$$
$$\leq \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2}$$
$$= \frac{d^2}{2} + \frac{d^2}{2}$$
$$= d^2$$

ie $\left\|\frac{x+x'}{2}\right\| \leq d$, which is a contradiction to the definition of d $(\because d = \inf\{\|x\| : x \in C\})$. This contradiction arises due to our wrong supposition that $x \neq x'$. Hence $x = x'$ ie x is unique.

This completes the proof.

[Polarization identity]

Theorem (5.10): In any Hilbert space, the inner product is related to the norm by the following identity, called Polarization identity, which is,

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$
$$= \sum_{r=0}^{3} i^r \|x + i^r y\|^2.$$

Proof: Same as the proof in Inner product spaces.

Theorem (Assignment): If B is a complex Banach space whose norm obeys the parallelogram law and if an Inner product is defined on B by the polarization identity, then B is a Hilbert space.

Proof:-

Definitions (5.11):

① Two vectors x and y in a Hilbert space H are said to be orthogonal if $(x,y)=0$. we express symbolically the orthogonal vectors x and y by $x \perp y$.

② A vector x is said to be orthogonal to a non-empty set A if $(x,y)=0$ for every y in A. we write it as $x \perp A$.

③ Two non-empty sets A and B in a Hilbert space H are said to be orthogonal if $(x,y)=0$ for every x in A and every y in B. we write it as $A \perp B$.

④ A set A is said to be orthogonal if for every pair of elements x, y in A with $x \neq y$, we have $(x,y)=0$.

Remark: ① $x \perp 0$ for every x in a Hilbert space H.

(II) if $x \perp y$, then $y \perp x$.

(III) 0 is only vector orthogonal to itself.

(IV) if $x \perp y$, $x \perp z$, then $x \perp y+z$ and $x \perp \alpha y$ for any scaler α.

(V) if $x \perp y_n$, where $y_n \to y$; then $x \perp y$.

Proof: see proof in Inner product spaces.

Theorem (5.12) [Pythagorean Theorem]

① If x and y are orthogonal vectors in a Hilbert space H, then $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$.

② Generalized Pathagorean Thm:

If $\{x_1, x_2, \ldots, x_n\}$ is an orthogonal set in a Hilbert space H, then $\|x_1+x_2+\cdots+x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2$.

OR $\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$

Proof: see proof in Inner product spaces.

Definition (5.13):

If M is any subset of a Hilbert space H, Then the orthogonal Complement of M, denoted by $M^{\perp}$, is defined as : $M^{\perp} = \{x \in H : (x,y) = 0$ for every y in $M\}$

$$= \{x \in H : x \perp M\}$$

and also $M^{\perp\perp} = (M^{\perp})^{\perp} = \{x \in H : (x,y) = 0$ for every y in $M^{\perp}\}$

$$= \{x \in H : x \perp M^{\perp}\}.$$

Remark: From the above definition, it is clear That
① $\{0\}^{\perp} = H$ ② $H^{\perp} = \{0\}$.

Theorem (5.14):- let M_1, M_2 be subsets of a Hilbert space H, Then prove the following:

(I) $M_1 \subseteq M_1^{\perp\perp}$ ie any subset of H is contained in its double orthogonal Complement.

(II) if $M_1 \subseteq M_2$, Then $M_2^{\perp} \subseteq M_1^{\perp}$.

(III) $(M_1 \cup M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp}$ and $(M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.

(IV) $M_1^{\perp} = M_1^{\perp\perp\perp}$

(V) $M_1 \cap M_1^{\perp} \subseteq \{0\}$

(VI) $M_1^{\perp}$ is a closed linear space.

Proof:- (I)

(II) let $x \in M_2^{\perp} \Rightarrow (x,y) = 0$ for every y in M_2 (by def.)

$\Rightarrow (x,y) = 0$ for every y in M_1 ($\because M_1 \subseteq M_2$).
$\Rightarrow x \perp M_1$
$\Rightarrow x \in M_1^{\perp}$ (by def.)

So that $M_2^{\perp} \subseteq M_1^{\perp}$.

(III) Since $M_1 \subseteq M_1 \cup M_2$ and $M_2 \subseteq M_1 \cup M_2$ (always true)

$\Rightarrow (M_1 \cup M_2)^{\perp} \subseteq M_1^{\perp}$ and $(M_1 \cup M_2)^{\perp} \subseteq M_2^{\perp}$ (by (II)).

$\Rightarrow (M_1 \cup M_2)^{\perp} \subseteq M_1^{\perp} \cap M_2^{\perp} \longrightarrow$ (a)

Now let $x \in M_1^{\perp} \cap M_2^{\perp} \Rightarrow x \in M_1^{\perp}$ and $x \in M_2^{\perp}$

$\Rightarrow x \perp M_1$ and $x \perp M_2$

So by def:, $(x,u) = 0$ for every u in M_1 and

$(x,v) = 0$ for every v in M_2.

and so $(x,u) = 0$ for every u in $M_1 \cup M_2$.

$\Rightarrow x \in (M_1 \cup M_2)^{\perp}$

so that $M_1^{\perp} \cap M_2^{\perp} \subseteq (M_1 \cup M_2)^{\perp} \longrightarrow$ (b)

From (a) & (b); we get: $(M_1 \cup M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp}$ #.

Next we show that $(M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.

For this since $M_1 \cap M_2 \subseteq M_1$ and $M_1 \cap M_2 \subseteq M_2$

$\Rightarrow M_1^{\perp} \subseteq (M_1 \cap M_2)^{\perp}$ and $M_2^{\perp} \subseteq (M_1 \cap M_2)^{\perp}$ (by (II)).

$\Rightarrow M_1^{\perp} \cup M_2^{\perp} \subseteq (M_1 \cap M_2)^{\perp}$ as required.

(IV) By part (I), we have: $M_1 \subseteq M_1^{\perp\perp}$

and so part (II), we have: $(M_1^{\perp\perp})^{\perp} \subseteq M_1^{\perp}$

ie $M_1^{\perp\perp\perp} \subseteq M_1^{\perp} \longrightarrow$ (a) ([illegible] o. Complement)

Also by part (I), $M_1^{\perp} \subseteq (M_1^{\perp})^{\perp\perp}$ ie $M_1^{\perp} \subseteq M_1^{\perp\perp\perp} \longrightarrow$ (b)

From (a) and (b); we have $M_1^{\perp} = M_1^{\perp\perp\perp}$.

(V) If $M_1 \cap M_1^{\perp} = \phi$, then clearly $M_1 \cap M_1^{\perp} = \phi \subseteq \{0\}$

ie $M_1 \cap M_1^{\perp} \subseteq \{0\}$

If $M_1 \cap M_1^{\perp} \neq \phi$, then let $x \in M_1 \cap M_1^{\perp} \Rightarrow x \in M_1$ and $x \in M_1^{\perp}$.

Now since $x \in M_1^{\perp} \overset{\& x \in M_1}{\Rightarrow} (x,x) = 0$ ie $\|x\|^2 = 0$ ie $\|x\| = 0$ ie $x = 0 \in \{0\}$

ie $x \in \{0\}$. Therefore $M_1 \cap M_1^{\perp} \subseteq \{0\}$.

(VI) we show that $M_1^\perp$ is a closed linear subspace.

For this we first recall that " A subset M of a linear space X is a subspace of X if for any $x, y \in M$ and any scalars α, β, we have $\alpha x + \beta y \in M$".

Now let x, y be any two elements in $M_1^\perp$ and α, β be any scalars, then for any u in M_1, we have:

$(x,u)=0$ and $(y,u)=0$ and therefore:

$$\begin{aligned}(\alpha x+\beta y, u) &= (\alpha x, u) + (\beta y, u)\\ &= \alpha(x,u) + \beta(y,u)\\ &= \alpha \cdot 0 + \beta \cdot 0\\ &= 0\end{aligned}$$

ie $(\alpha x + \beta y, u) = 0$ for any u in M_1.

$\Rightarrow \alpha x + \beta y \in M_1^\perp$, which shows that $M_1^\perp$ is a subspace of H.

To complete the proof, it remains to show that $M_1^\perp$ is closed and in order to prove this, it is enough to show that "if $\{x_n\}$ is any convergent sequence in $M_1^\perp$ converging to a point x (say) ie $x_n \to x$, then $x \in M_1^\perp$".

Now for any $u \in M_1$, we can write:

$$\begin{aligned}(x,u) &= \left(\lim_{n\to\infty} x_n, u\right) \quad \left[\because x_n \to x \text{ ie } \lim_{n\to\infty} x_n = x\right]\\ &= \lim_{n\to\infty}(x_n, u) \quad [\because \text{Inner product is contin: function}]\\ &= 0 \text{ , because } x_n \in M_1^\perp \ \forall n \text{ as } \{x_n\} \text{ is a seq: in } M_1^\perp.\end{aligned}$$

ie $(x,u)=0$ for any $u \in M_1$. $\Rightarrow x \perp M_1$

$\Rightarrow x \in M_1^\perp$. Thus $M_1^\perp$ is closed linear subspace of H. Thus completing the proof.

<u>Theorem (5.4)</u> If M is a (closed) linear subspace of a Hilbert space H, then $M \cap M^{\perp} = \{0\}$.

<u>Proof:-</u> let $x \in M \cap M^{\perp}$ Then $x \in M$ and $x \in M^{\perp} \Rightarrow x \perp M$

$\Rightarrow (x, y) = 0$ for every y in M.

$\Rightarrow (x, x) = 0$, because $x \in M$.

$\Rightarrow \|x\|^2 = 0 \Rightarrow \|x\| = 0 \Rightarrow x = 0$.

This shows that $0 \in M \cap M^{\perp} \Rightarrow \{0\} \subseteq M \cap M^{\perp}$

But we know that part (II) of previous Thm, $M \cap M^{\perp} \subseteq \{0\}$

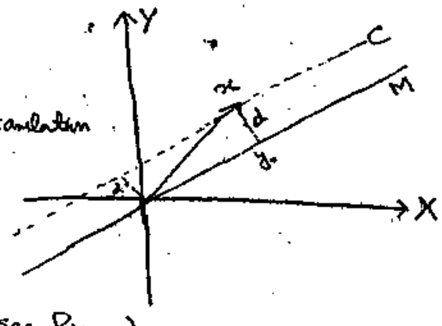
Hence $M \cap M^{\perp} = \{0\}$

<u>Remark:-</u> For sets M and $M^{\perp}$, $M \cap M^{\perp} \subseteq \{0\}$ and for subspaces M and $M^{\perp}$, $M \cap M^{\perp} = \{0\}$.
The reason is that it is not necessary for 0 to present in any subset but every subspace contains 0.

<u>Recall:</u> ① Any subspace of a linear space X is convex.

② For any subspace M of a linear space X and $x \in X$, the set $x + M = \{x + m : m \in M\}$ is convex.

<u>Theorem (5.16)</u> let M be a closed linear subspace of a Hilbert space H. let x be a vector not in M and let $d = d(x, M)$, Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.



<u>Proof:-</u> let us set $C = x + M$ (the translation of M by x), Then the set $C = x + M$ is a closed convex set and d is the distance from the origin to C (see figure).

So by Thm (5.8), There exists a unique vector say z_0 in C such That $\|z_0\| = d = \inf\{\|z\| : z \in C\}$.

Since $z_0 \in C$, so by def. of C, $z_0 = x + y$ for some $y \in M$.

let us put $x - z_0 = y_0$, Then The vector $y_0 = x - z_0$ is easily seen to be in M $(\because z_0 = x+y,\ y_0 = x - z_0 = x - x - y = -y \in M)$

and $\|z_0\| = \|x - y_0\|$ $(\because y_0 = x - z_0)$

i.e. $\|x - y_0\| = \|z_0\| = d$ (from above)

Thus there exists a vector y_0 in M such that $\|x - y_0\| = d$

It remains to prove the uniqueness of The vector y_0.

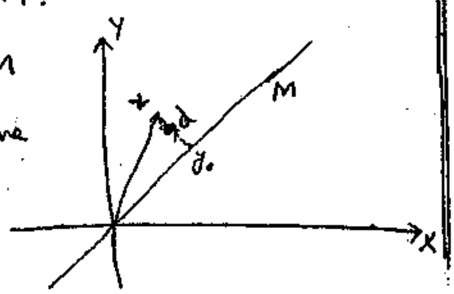
For This let y_1 be another vector in M such that $y_0 \neq y_1$, and $\|x - y_1\| = d$.

Then $z_1 = x - y_1$ is a vector in C $(\because C = x + M)$

such That $z_1 \neq z_0$ and $\|z_1\| = \|x - y_1\| = d$ i.e. $\|z_1\| = d$, which is a contradiction to the fact That "there is a unique vector z_0 in C such That $\|z_0\| = d$". This is because of our wrong supposition. so $y_0 = y_1$ and Hence there exists a unique vector y_0 in M such That $\|x - y_0\| = d$.

Theorem (5.17): If M is a proper closed linear subspace of a Hilber space H, Then there exists a non-zero vector z_0 in H such That $z_0 \perp M$.

Proof: let x be a vector not in M and let $d = d(x, M)$, Then by above Thm, there exists a unique vector y_0 in M such That $\|x - y_0\| = d$.



let us take $z_0 = x - y_0$ and observe that since $d > 0$, so z_0 is a non-zero vector in H $(\because \|z_0\| = \underset{= d > 0}{\|x - y_0\|})$

In order to show that $z_0 \perp M$, it is enough to show that $z_0 \perp y$ for every y in M.

For this let $\lambda \in \mathbb{C}$, then we have:

$$\begin{aligned}\|z_0 - \lambda y\| &= \|x - y_0 - \lambda y\| \quad (\because z_0 = x - y_0)\\ &= \|x - (y_0 + \lambda y)\|\\ &\geq d \quad [\because \text{by def. of } d]\\ &= \|z_0\| \quad (\because \|z_0\| = \|x - y_0\| = d)\end{aligned}$$

ie $\|z_0 - \lambda y\| \geq \|z_0\|$

$\Rightarrow \|z_0 - \lambda y\|^2 \geq \|z_0\|^2$

and from this, we have:

$$(z_0 - \lambda y, z_0 - \lambda y) \geq (z_0, z_0)$$

$\Rightarrow (z_0, z_0) - (z_0, \lambda y) - (\lambda y, z_0) + (\lambda y, \lambda y) \geq \cancel{(z_0, z_0)}$

$\Rightarrow -\bar{\lambda}(z_0, y) - \lambda(y, z_0) + \lambda\bar{\lambda}(y, y) \geq 0$

$\Rightarrow -\bar{\lambda}(z_0, y) - \lambda\overline{(z_0, y)} + \lambda\bar{\lambda}(y, y) \geq 0.$

$\Rightarrow -\bar{\lambda}(z_0, y) - \lambda\overline{(z_0, y)} + |\lambda|^2(y, y) \geq 0 \longrightarrow (1)$

Put $\lambda = \mu(z_0, y)$ for an arbitrary real number μ,

Then (1) becomes:

$$-\overline{\mu(z_0, y)}(z_0, y) - \mu(z_0, y)\overline{(z_0, y)} + |\mu(z_0, y)|^2 (y, y) \geq 0.$$

$\Rightarrow -\mu|(z_0, y)|^2 - \mu|(z_0, y)|^2 + \mu^2|(z_0, y)|^2(y, y) \geq 0.$ $[\because \mu \in \mathbb{R}]$

$\Rightarrow -2\mu|(z_0, y)|^2 + \mu^2|(z_0, y)|^2\|y\|^2 \geq 0 \longrightarrow (2)$

Now put $a = |(z_0, y)|^2$ and $b = \|y\|^2$, then from (2), we obtain:

$-2\mu a + \mu^2 ab \geq 0, \ \forall$ real nos: μ

$\Rightarrow \mu a(\mu b - 2) \geq 0$; $\forall$ real nos: $\mu \longrightarrow (3)$

However if $a > 0$, then (3) is impossible for all sufficiently small positive μ eg: for $a=1, b=1, \mu=1$ we get $-1 \geq 0$, which is not possible.

we see from this that $a=0$ is only possibility. which means $|(z_0,y)|^2 = 0$ $(\because a = |(z_0,y)|^2)$

$\Rightarrow |(z_0,y)| = 0 \quad \Rightarrow (z_0,y) = 0 \quad \Rightarrow z_0 \perp y$ for all y in M

$\Rightarrow z_0 \perp M$, which completes the proof.

Definition (5.18):- let M and N be two subspaces of a linear space L. we define $M+N = \{x+y : x \in M, y \in N\}$

Since M and N are subspaces, it is easy to see that $M+N$ is also a subspace spaned (generated) by all vectors in M and N together ie $M+N = [M \cup N]$.

Definition (5.19): If $M+N = L$, then we say that L is the sum of the subspaces M and N.

This means that any evector in L is expressible as the sum of a vector in M and a vector in N ie if $z \in L$, then $z = x+y$ where $x \in M$ & $y \in N$.

If each vector z in L is expressible uniquely in the form of $z = x+y$ with $x \in M$ & $y \in M$; then we say that L is the direct sum of the subspace M and N. Symbolically we write it as $L = M \oplus N$.

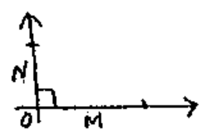
Theorem (Recall):- Let a linear space L be the sum of two subspaces M and N i.e. $L = M+N$; Then $L = M \oplus N$ iff $M \cap N = \{0\}$.

Remark:- The condition in this above Thm that the subspaces M and N have only the origin in common is often expressed by saying that M and N are disjoint. (Which is the main diff: b/w the disjoint sets and disjoint spaces).

Remark: Two non-empty sets S_1, S_2 of a Hilbert space H are said to be orthogonal (written as $S_1 \perp S_2$) if $x \perp y$ for all $x \in S_1$ and for all $y \in S_2$.

Theorem (5.20):- If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then the linear subspace $M+N$ is also closed.

Proof:- To show that $M+N$ is closed, we need to show that all the limit points of $M+N$ are in $M+N$.
Let $\{z_n\}$ be a sequence in $M+N$ converging to a limit point say z. It is enough to show that z is in $M+N$.
Since $M \perp N$, we see that M and N are disjoint (i.e. $M \cap N = \{0\}$)
So by above Thm, the sum $M+N$ can be strengthened to the direct sum $M \oplus N$ and



Thus each z_n can be expressed uniquely in the form $z_n = x_n + y_n$; where x_n is in M and y_n is in N.
Since x_n and y_n are orthogonal ($\because M \perp N$), so by Pethagorean Theorem, we have:

$$\|z_n - z_m\|^2 = \|(x_n+y_n)-(x_m+y_m)\|^2 \quad [z_n = x_n + y_n]$$
$$= \|(x_n-x_m)+(y_n-y_m)\|^2$$
$$= \|x_n-x_m\|^2 + \|y_n-y_m\|^2 \quad (\text{by Peth: Thm:})$$

ie $\|z_n - z_m\|^2 = \|x_n - x_m\|^2 + \|y_n - y_m\|^2$

Since $\{z_n\}$ is a Cauchy sequence (being cgt.), so by def:

$\|z_n - z_m\| < \varepsilon$ for $m, n \geq N$.

So from above, we have:

$\|x_n - x_m\|^2 + \|y_n - y_m\|^2 = \|z_n - z_m\|^2 < \varepsilon^2$ for $m, n \geq N$.

$\Rightarrow \|x_n - x_m\|^2 + \|y_n - y_m\|^2 < \varepsilon^2$ for $m, n \geq N$.

$\Rightarrow \|x_n - x_m\|^2 < \varepsilon^2$, $\|y_n - y_m\|^2 < \varepsilon^2$ for $m, n \geq N$.

$\Rightarrow \|x_n - x_m\| < \varepsilon$, $\|y_n - y_m\| < \varepsilon$ for $m, n \geq N$

which shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in M and N respectively.

Also M and N are closed subspace of the complete space (Hilbert space) H, ~~so~~ M and N are complete.

So by the completeness, there exists ~~two~~ vectors x and y in M and N respectively such that $x_n \to x$ and $y_n \to y$.

Since $x+y$ is a vector in $M+N$, so we have:

$z = \lim z_n = \lim (x_n + y_n) = \lim x_n + \lim y_n = x + y \in M+N$

$\Rightarrow z \in M+N$. Thus $M+N$ is closed. Thus completing the proof.

Theorem (5.21): [Projection Theorem]

Statement: If M is a closed linear subspace of a Hilbert space H, Then $H = M \oplus M^{\perp}$.

Proof: let M be a closed linear subspace of H, Then $M^{\perp}$ is also a closed linear subspace of H (Proved already).
Also M and $M^{\perp}$ are orthogonal, because if $x \in M^{\perp}$, then by def: $(x,y)=0$ for all y in M. Since x was choosen arbitrary in $M^{\perp}$, so $(x,y)=0$ for every y in M and every x in $M^{\perp}$
So that $M^{\perp} \perp M$.
Thus M and $M^{\perp}$ are orthogonal closed linear subspaces of H, Therefore $M + M^{\perp}$ is also a closed linear subspace of H (by previous result).
We need to show that $H = M \oplus M^{\perp}$.
First we show That $H = M + M^{\perp}$.
On the contrary, assume that $H \neq M + M^{\perp}$, then $M + M^{\perp}$ is a proper closed linear subspace of H. Then by Theorem (5.17), There exists a non-zero vector z_0 such that $z_0 \perp (M + M^{\perp})$. So $z_0 \in (M+M^{\perp})^{\perp}$ [by def. of orthogonal complement of a set].
Now $M \subseteq M + M^{\perp} \Rightarrow (M+M^{\perp})^{\perp} \subseteq M^{\perp}$
and $M^{\perp} \subseteq M + M^{\perp} \Rightarrow (M+M^{\perp})^{\perp} \subseteq M^{\perp\perp}$
So that $(M+M^{\perp})^{\perp} \subseteq M^{\perp} \cap M^{\perp\perp} = \{0\}$ ($\because M^{\perp}$ & $M^{\perp\perp}$ are orthog: because $M \perp M^{\perp}$)

Hence $z_0 \in (M+M^{\perp})^{\perp} \subseteq \{0\} \Rightarrow z_0 \in \{0\} \Rightarrow z_0$ is a zero vector, which is a contradiction to the fact that $z_0 \neq 0$.
So our Supposition was wrong and hence $H = M + M^{\perp}$.

To Complete the proof, it is enough to observe that since M and $M^{\perp}$ are orthogonal, so $M \cap M^{\perp} = \{0\}$. Thus by Theorem (pg 15), the statement $H = M + M^{\perp}$ can be strengthened to the $H = M \oplus M^{\perp}$.

This completes the required result.

<u>Orthonormal sets in Hilbert spaces:</u>

<u>Def (5.22)</u>: An orthonormal set in a Hilbert space H is a non-empty subset of H which consists of mutually orthogonal unit vector; that is, it is a non-empty subset $\{e_i\}$ of H with the following properties.

(i) $(e_i, e_j) = 0$ if $i \neq j$

(II) $(e_i, e_j) = 1$ if $i = j$.

<u>Examples (5.23)</u>: see examples following the definition of orthonormal sets in Inner product spaces.

<u>Remark (5.24)</u>: If $H = \{0\}$ ie H contains only the zero element, then it has no orthonormal set.

If H contains a non-zero vector x, then we can construct e by normalizing x, that is $e = \frac{x}{\|x\|}$. Then the single element set $\{e\}$ is clearly an ~~orthogon~~ orthonormal set because $(e,e) = \|e\|^2 = \left\|\frac{x}{\|x\|}\right\|^2 = \frac{\|x\|^2}{\|x\|^2} = 1$.

Generally speaking if $\{x_i\}$ is a non-empty set of mutually orthogonal non-zero vectors in H, and if the

x_i's are normalized by replacing each of them by $e_i = \dfrac{x_i}{\|x_i\|}$, Then the resulting set $\{e_i\}$ is orthonormal set.

<u>Theorem (5.25)</u>: let $S = \{e_1, e_2, \ldots, e_n\}$ be an orthonormal set in a Hilbert space H. If x is any vector in H, Then

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2. \quad \text{(Bessel's inequality)} \longrightarrow (1)$$

and $\quad x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$ for each j

i.e. $\quad x - \sum_{i=1}^{n} (x, e_i) e_i \perp S.$

<u>Proof:</u> We have: $\quad 0 \le \left\| x - \sum_{i=1}^{n} (x, e_i) e_i \right\|^2$ (obvious)

$$= \left(x - \sum_{i=1}^{n} (x, e_i) e_i \,,\, x - \sum_{j=1}^{n} (x, e_j) e_j \right)$$

$$= (x, x) - \left(x, \sum_{j=1}^{n} (x, e_j) e_j \right) - \left(\sum_{i=1}^{n} (x, e_i) e_i, x \right) + \left(\sum_{i=1}^{n} (x, e_i) e_i ; \sum_{j=1}^{n} (x, e_j) e_j \right).$$

$$= (x, x) - \sum_{j=1}^{n} \overline{(x, e_j)} (x, e_j) - \sum_{i=1}^{n} (x, e_i)(e_i, x) + \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i) \overline{(x, e_j)} (e_i, e_j).$$

$$= (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i) \overline{(x, e_j)} (e_i, e_j)$$

$$= (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 + \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)} (e_i, e_i)$$

[∵ S is an o.n.s so by putting values of i, we get terms equal to zero]

$$= (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 - \cancel{\sum_{i=1}^{n} |(x, e_i)|^2} + \cancel{\sum_{i=1}^{n} |(x, e_i)|^2} \cdot 1$$

$$\Rightarrow 0 \le (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 \Rightarrow \sum_{j=1}^{n} |(x, e_j)|^2 \le (x, x)$$

$$\Rightarrow \sum_{j=1}^{n} |(x, e_j)|^2 \le \|x\|^2,$$ which is equivalent to (1).

In order to show that $x-\sum_{i=1}^{n}(x,e_i)e_i \perp S$, Consider any e_j in S where $j=1,2,\ldots,n$.

Then
$$\left(x-\sum_{i=1}^{n}(x,e_i)e_i,\ e_j\right) = (x,e_j) - \left(\sum_{i=1}^{n}(x,e_i)e_i,\ e_j\right)$$
$$= (x,e_j) - \sum_{i=1}^{n}(x,e_i)(e_i,e_j)$$
$$= (x,e_j) - (x,e_j)(e_j,e_j)$$
(∵ for all i's the second term on R.H.S is zero because S is an O.N. set)
$$= (x,e_j) - (x,e_j)\cdot 1$$
$$= 0$$

This shows that $x-\sum_{i=1}^{n}(x,e_i)e_i \perp e_j$ for each j

$$\Rightarrow x-\sum_{i=1}^{n}(x,e_i)e_i \perp S.$$

Thus completing the proof.

<u>Theorem (5.26)</u>: If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is any vector in H, the the set $S=\{e_i : (x,e_i)\neq 0\}$ is either empty or Countable.

<u>Proof:-</u>

Theorem (5.27) [Generalization of Bessel's Inequality].

If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then

$$\sum |(x,e_i)|^2 \le \|x\|^2 \text{ for every vector } x \text{ in } H. \quad \text{①}$$

Proof: Let us define a set S as:

$$S=\{e_i : (x,e_i)\neq 0\}$$

Then by Thm (5.26), S is either empty or countable.

If S is empty, then $(x,e_i)=0$, so $\sum|(x,e_i)|^2$ is zero and so in this case ① reduces to $0\le\|x\|^2$ which is obviously true.

If S is countable, then S is finite or countably infinite.

When S is finite: let it can be written in the form

$S=\{e_1,e_2,\ldots\ldots,e_n\}$ for some positive integer n.

In this case, we denote $\sum|(x,e_i)|^2$ to be $\sum_{i=1}^{n}|(x,e_i)|^2$ which is clearly independent of the order in which the vectors of S are arranged. So Inequality ① reduces to

$\sum_{i=1}^{n}|(x,e_i)|^2\le\|x\|^2$, which is the Bessel Inequality when $\{e_i\}$ is finite ~~orthogonal~~ orthonormal set and it has been proved already in Theorem (5.25).

When S is countably infinite: let the vectors in S be arranged in some definite order ie $S=\{e_1,e_2,\ldots,e_n,\ldots\ldots\}$

Now by the Theory of "absolutely convergent series" we know that "if $\sum_{i=1}^{\infty}|(x,e_i)|^2$ converges, then every series obtained from this series by rearranging its terms also converges and all such series have the same sum".

So we therefore can define: $\sum |(x,e_i)|^2$ to be $\sum_{i=1}^{m} |(x,e_i)|^2$. and it follows from the above remark that $\sum |(x,e_i)|^2$ is a non-negative extended real number, which depends only on S and not on the arrangement of vectors in S. So in this case ① reduces to:

$$\sum_{i=1}^{\infty} |(x,e_i)|^2 \leq \|x\|^2 \longrightarrow ②$$

Now from Bessel's inequality for finite case, we have:

$$\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$$

It follows that no partial sum of the series on the left of ② can exceedes $\|x\|^2$ and so it is clear that ② is true ($\because \sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$
$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2 \Rightarrow \sum_{i=1}^{\infty} |(x,e_i)|^2 \leq \|x\|^2$)

This completes the proof.

<u>Recall</u>: ① let P be a set of elements. Suppose there is a binary relation defined between certain pairs a,b of P, expressed symbolically by $a < b$, with the properties

(I) If $a < b$ and $b < c$, then $a < c$ (Transitivity)

(II) If $a \in P$, then $a < a$ (Reflexivity)

(III) If $a < b$ and $b < a$, then $a = b$ (Antisymmetry)

Then P is said to be partially ordered set.

For example if P is the set of all subsets of a given set X, the set inclusion $(A \subseteq B)$ gives a partial ordering of P.

② If P is a partially ordered set. moreover if for any pair a,b in P either $a < b$ or $b < a$, then P is said to be completely (totally, linearly, simply) ordered set.

A completely ordered set is called a chain.

eg: The real numbers are completely ordered by the relation "a is less than or equal to b ie $a \le b$".

Zorn's Lemma: (5.28) (only Recall)

Let P be a non-empty partially ordered set with the property that every completely ordered subset of P has an upper bound. Then P contains at least one maximal element.

Theorem (5.29) Every non-zero Hilbert space H contains a complete orthonormal set.

Proof:- Let $H \neq \{0\}$ and M be the set of all subsets of H which are orthonormal. We define partially ordering in M by the usual set inclusion, so that M is a partially ordered set.

Since $H \neq \{0\}$, therefore if $x \neq 0$ is a vector in H, then $\{y\} \in M$ where $y = \frac{x}{\|x\|}$. ($\because$ by previous remark $\{y\}$ is an orthonormal set).

So the set M of all orthonormal sets is non-empty.

Now let $C = \{E_\lambda : \lambda \in \Lambda\}$ be an increasing chain of orthonormal subsets in M (ie $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$)

Then $\bigcup_i E_i$ is the upper bound of C.

Now M is a partially ordered set and every chain in M has its upper bound, so by "Zorn's Lemma", there exists a maximal element in M. Let A be that element that is the set which is maximal in M so that H contains a complete orthonormal set.

Theorem (5.30) (Assignment): Let $\{e_i\}$ be an orthonormal set in a Hilbert space H and let x be a vector in H, Then

$$x - \sum (x, e_i) e_i \perp \{e_i\}.$$

<u>Theorem (5.31)</u>: Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H, then the following are equivalent.

(a) $\{e_i\}$ is complete.

(b) $x \perp \{e_i\} \Rightarrow x = 0$

(c) if x is an arbitrary vector in H, then $x = \Sigma (x, e_i) e_i$

(d) if x is an arbitrary vector in H, then $\|x\|^2 = \Sigma |(x, e_i)|^2$.

<u>Proof:</u> (a) $\Rightarrow$ (b)

Suppose (a) is true ie $\{e_i\}$ is complete $\Rightarrow$ $\{e_i\}$ is maximal o.n.s.
On contrary suppose that (b) is not true, then there exists a vector $x \neq 0$ such that $x \perp \{e_i\}$.
Define $e = \frac{x}{\|x\|}$ (Normalization of x), then the set $\{e_i, e\}$ is an orthonormal set, which properly contains $\{e_i\}$, but this contradicts the completeness of $\{e_i\}$. Hence (b) is true.

(b) $\Rightarrow$ (c)

Suppose that (b) is true ie $x \perp \{e_i\} \Rightarrow x = 0$.
Now by (5.30), we have $x - \Sigma (x, e_i) e_i$ is orthogonal to $\{e_i\}$

ie $x - \Sigma (x, e_i) e_i \perp \{e_i\}$

So by (b), we get: $x - \Sigma (x, e_i) e_i = 0$

or $x = \Sigma (x, e_i) e_i$ for any vector x in H. Hence (c) is true.

<u>(c) $\Rightarrow$ (d)</u> Suppose that (c) is true ie $x = \Sigma (x, e_i) e_i$ for any vector x in H.

Now $x = \Sigma (x, e_i) e_i = \sum_{i=1}^{\infty} (x, e_i) e_i$

Then $\|x\|^2 = (x, x) = \left(x, \sum_{i=1}^{\infty} (x, e_i) e_i\right)$ $\quad \left[\because x = \Sigma (x, e_i) e_i = \sum_{i=1}^{\infty} (x, e_i) e_i\right]$

$= \left(x, \lim_{n\to\infty} \sum_{i=1}^{n} (x, e_i) e_i\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \overline{(x, e_i)} (x, e_i)$ $\quad$ [$\because$ Inner product is continuous].

$$\Rightarrow \|x\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |(x,e_i)|^2$$
$$= \sum_{i=1}^{\infty} |(x,e_i)|^2$$

using $\Sigma (x,e_i)e_i$ in place of $\sum_{i=1}^{\infty}(x,e_i)e_i$, we get

$\|x\|^2 = \Sigma|(x,e_i)|^2$. Hence (d) is true.

Finally (d) $\Longrightarrow$ (a)

suppose that (d) is true ie $\|x\|^2 = \Sigma|(x,e_i)|^2$.

we show that (a) is true. on the contrary assume that (a) is not true ie $\{e_i\}$ is not complete, then it is properly contained in an orthonormal set $\{e_i, e\}$.

~~Since~~ So by definition of orthonormal set, we can say that e is orthogonal to e_i's.

$$\text{Now } \|e\|^2 = \Sigma|(e,e_i)|^2 \quad (\because \text{by (d)})$$
$$= \Sigma\|0\|^2 \quad (0 \text{ is a vector, therefore we take norm})$$
$$= \|0\|$$
$$= 0$$

ie $\|e\| = 0$

and this contradicts the fact that $\|e\| = 1$

so our supposition was wrong and hence $\{e_i\}$ is complete

Hence (a) is true.

This completes the required proof.

Remark (5.32): let $\{e_i\}$ be a complete orthonormal set and let x be an arbitrary vector in a Hilbert space H Then the numbers (x,e_i) are called the Fourier coefficients of x, the expression $(x,e_i)e_i$ is called the Fourier expansion of x and the equation $\|x\|^2 = \Sigma|(x,e_i)|^2$ is called Parseval's equation or formula — all w.r.t the particular complete orthonormal set $\{e_i\}$ under consideration.

—*—*—*—*—*—*—*—

The Gram-Schmidt orthogonalization process:

It is a constructive procedure for converting a linearly independent set $\{x_1, x_2, \dots, x_n, \dots\}$ into corresponding orthonormal set $\{e_1, e_2, \dots, e_n, \dots\}$ with the property that for each n, the linear subspace spanned by $\{e_1, e_2, \dots, e_n\}$ is the same as that spaned by $\{x_1, x_2, \dots, x_n\}$.

We state this process in the form of the following theorem.

Theorem (5.33): Suppose that $\{x_1, x_2, \dots, x_n, \dots\}$ is a linearly independent set in a Hilbert space H, Then there exists an orthonormal set $\{e_1, e_2, \dots, e_n, \dots\}$ with the property that for each n, the linear subspace spanned by $\{e_1, e_2, \dots, e_n\}$ is the same as that spanned by $\{x_1, x_2, \dots, x_n\}$

Proof: Certainly $x_1 \neq 0$, because the set $\{x_1, x_2, \dots, x_n, \dots\}$ is linearly independent.

we define $y_1, y_2, \dots$ and $e_1, e_2, \dots$ recursively as follows

$$y_1 = x_1 \qquad e_1 = \frac{y_1}{\|y_1\|}$$

Clearly the subspace spanned by x_1 and e_1 are the same.

$$y_2 = x_2 - (x_2, e_1)e_1 \qquad e_2 = \frac{y_2}{\|y_2\|}$$

$$y_3 = x_3 - (x_3, e_1)e_1 - (x_3, e_2)e_2 \qquad e_3 = \frac{y_3}{\|y_3\|}$$

$$\vdots$$

$$y_n = x_n - (x_n, e_1)e_1 - (x_n, e_2)e_2 - \dots - (x_n, e_{n-1})e_{n-1} \qquad e_n = \frac{y_n}{\|y_n\|}$$

$$= x_n - \sum_{i=1}^{n-1} (x_n, e_i)e_i$$

$$y_{n+1} = x_{n+1} - \sum_{i=1}^{n} (x_{n+1}, e_i)e_i \quad , \qquad e_{n+1} = \frac{y_{n+1}}{\|y_{n+1}\|}$$

$$\vdots \qquad\qquad \vdots$$

The process terminates if $\{x_n\}$ is a finite set, otherwise it continues indefinitely.
Also note that $y_n \neq 0$ because $x_1, x_2, \ldots, x_n$ are l.I.
Thus e_n is well-definition ie the definition of e_n is valid.
From the construction, it is clear that e_2 is a linear combination of x_1 and x_2 & x_2 is a linear combination of e_1, e_2.
Similarly x_3 is a linear combination of e_1, e_2, e_3 and e_3 is a linear combination of x_1, x_2, x_3.
So by induction each x_n is a linear combination of $e_1, e_2, \ldots, e_n$ and each e_n is a linear combination of $x_1, x_2, \ldots, x_n$.
Thus the linear subspace spanned by the x's is the same as that spanned by e's.
Now it remains to show that the set of e's is an orthonormal set ie $\{e_1, e_2, \ldots, e_n, \ldots\}$ is orthonormal.

Now since $e_i = \dfrac{y_i}{\|y_i\|}$

$$\Rightarrow \|e_i\| = \frac{\|y_i\|}{\|y_i\|} = 1$$

Now we show by induction that $(e_i, e_j) = 0 \quad i,j = 1, 2, \ldots, \; i \neq j$.

Consider $(e_1, e_2) = \left(e_1, \dfrac{y_2}{\|y_2\|}\right) = \dfrac{1}{\|y_2\|}(e_1, y_2)$

$$= \frac{1}{\|y_2\|}\left(e_1, x_2 - (x_2, e_1)e_1\right)$$

$$= \frac{1}{\|y_2\|}\left[(e_1, x_2) - \left(e_1, (x_2, e_1)e_1\right)\right].$$

$$= \frac{1}{\|y_2\|}\left[(e_1, x_2) - \overline{(x_2, e_1)}(e_1, e_1)\right]$$

$$\Rightarrow (e_1, e_2) = \frac{1}{\|y_2\|}\left[(e_1, x_2) - (e_1, x_2)(1)\right] \quad (\because \|e_1\|=1)$$
$$= 0$$
$$\Rightarrow (e_1, e_2) = 0.$$

Suppose that $(e_i, e_j) = 0$ for $i, j = 1, 2, \ldots, n-1$; $i \neq j$

Now $(e_n, e_j) = \left(\frac{y_n}{\|y_n\|}, e_j\right)$

$$= \frac{1}{\|y_n\|}(y_n, e_j)$$
$$= \frac{1}{\|y_n\|}\left(x_n - \sum_{i=1}^{n-1}(x_n, e_i)e_i\,,\ e_j\right)$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - \left(\sum_{i=1}^{n-1}(x_n, e_i)e_i\,,\ e_j\right)\right].$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - \sum_{i=1}^{n-1}(x_n, e_i)(e_i, e_j)\right]$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - (x_n, e_j)(e_j, e_j)\right] \quad \left(\begin{array}{c}\text{After expansion}\\ \text{all other terms}\\ \text{vanishes by assump.}\end{array}\right)$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - (x_n, e_j)\cdot 1\right] \quad (\because \|e_j\|=1).$$
$$= 0$$

Hence by induction $\{e_1, e_2, \ldots, e_n, \ldots\}$ form an orthonormal set. Hence the result follows.

The conjugate space of a Hilbert space H :

Let H be a Hilbert space. By H^*, we denote the conjugate space of H (ie the set of all continuous linear transformations of H into $\mathbb{C}$). The elements of H^* are called continuous linear functionals or briefly functional.

One of the fundamental properties of a Hilbert space H is the fact that there is a natural correspondence between the vectors in H and the functionals in H^* as we shall see below.

If y is a vector in Hilbert space H, then the complex function f_y defined by $f_y(x) = (x, y)$ for x in H is linear, because for any x_1, x_2 in H & scalar α; we have,

$$f_y(x_1+x_2) = (x_1+x_2, y) \quad [\text{by def: of } f_y(x)].$$
$$= (x_1, y) + (x_2, y)$$
$$= f_y(x_1) + f_y(x_2)$$

and
$$f_y(\alpha x) = (\alpha x, y)$$
$$= \alpha (x, y)$$
$$= \alpha f_y(x).$$

Moreover, $|f_y(x)| = |(x, y)|$

$$\leq \|x\|\|y\| \quad [\text{by schwarz's inequality}].$$

for all x in H. This inequality shows that f_y is bounded (considering $M = \|y\|$) and hence Continuous and is therefore a functional on H ie $f_y \in H^*$.

Since $|f_y(x)| \leq \|x\|\,\|y\|$ (by above).

Thus we have: $\|f_y\| \leq \|y\|$ (Taking sup over x with $\|x\| = 1$).

Evenmore equality is attained here ie $\|f_y\| = \|y\|$, because

This is clear when $y = 0$ (if $y=0$ then $\|f_y\| \leq 0 \Rightarrow \|f_y\| = 0$ because norm is non-negative)

and if $y \neq 0$, then

$$\|y\|^2 = (y, y) = f_y(y) \quad (\because f_y(x) = (x, y)).$$
$$\leq |f_y(y)|$$
$$\leq \|f_y\|\,\|y\|$$

$$\Rightarrow \|y\|^2 \leq \|f_y\|\,\|y\| \Rightarrow \|y\| \leq \|f_y\|$$

so that $\|f_y\| = \|y\|$

We see that for every y in H, there exists a functional f_y in H^* such that $\|f_y\| = \|y\|$

In such case, we say that $y \longrightarrow f_y : H \longrightarrow H^*$ is a norm preserving mapping of H into H^*.

[If $T: X \to Y$ is a linear mapping from a n.l.s X into a n.l.s Y, then T is called norm preserving mapping if $\|Tx\| = \|x\| \ \forall x \in X$]

Theorem (5.34) [Riesz Representation Theorem]

Let H be a Hilbert space and let f be an arbitrary functional in H^*, then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H and $\|f\| = \|y\|$

Proof: Let M be the null space (Kernel) of f, that is
$$M = \{x \in H : f(x) = 0\}.$$

Since f is continuous ($\because f$ is functional), so by the continuity of f, the null space M of f is a closed subspace of H, by a result saying that "the null space of a non-zero continuous linear operator is a closed subspace".

If $M = H$, then $f(x) = 0$ (by def. of M)
$= (x, 0)$ for all x in H and the thm is proved.

If $M \neq H$, then M is a proper closed subspace of H and so there exists a non-zero vector y_0 in H which is orthogonal to M ie $y_0 \perp M$ (by 5.17).

Since y_0 is not in M, thus $f(y_0) \neq 0$. [by def. of M].

For any vector x in H, the vector $z = x - \frac{f(x)}{f(y_0)} \cdot y_0$ is in M, because $f(z) = f\left(x - \frac{f(x)}{f(y_0)} y_0\right) = f(x) - \frac{f(x)}{\cancel{f(y_0)}} \cancel{f(y_0)} = 0$.

Also since $y_0 \perp M$, so that $y_0 \perp z$ ($\because z \in M$)

$$\Rightarrow (z, y_0) = 0 \Rightarrow \left(x - \frac{f(x)}{f(y_0)} \cdot y_0, y_0\right) = 0$$

$$\Rightarrow (x, y_0) - \left(\frac{f(x)}{f(y_0)} \cdot y_0, y_0\right) = 0 \Rightarrow (x, y_0) - \frac{f(x)}{f(y_0)} (y_0, y_0) = 0$$

$$\Rightarrow \frac{f(x)}{f(y_0)} (y_0, y_0) = (x, y_0) \Rightarrow f(x) = \frac{f(y_0)}{(y_0, y_0)} \cdot (x, y_0)$$

$$\Rightarrow f(x) = \left(x, \frac{\overline{f(y_0)}}{(y_0, y_0)} y_0\right) = \left(x, \frac{\overline{f(y_0)}}{(y_0, y_0)} y_0\right)$$

Let $y = \frac{\overline{f(y_0)}}{(y_0, y_0)} y_0$; Then from we have:

$$f(x) = (x, y) \quad \text{For all } x \text{ in } H.$$

To Complete the proof, it remains to show that y is unique.

For this if we also have $f(x) = (x, y')$ for all x, Then

$$(x, y) = (x, y')$$

$$\Rightarrow (x, y) - (x, y') = 0$$

$$\Rightarrow (x, y - y') = 0 \quad \text{for all } x \text{ in } H.$$

For particular $x = y - y'$, we get:

$$(y - y', y - y') = 0 \Rightarrow \|y - y'\|^2 = 0 \Rightarrow y - y' = 0$$

$\Rightarrow y = y'$. Hence y is unique.

Next we show that $\|f\| = \|y\|$

We have: $f(x) = (x, y)$

$$\text{So } |f(x)| = |(x, y)|$$
$$\leq \|x\| \, \|y\| \quad \text{(Schwarz's inequality)}$$

and thus it follows that

$$\|f\| \leq \|y\| \quad \left(\text{Taking } \sup_{\|x\| \leq 1} \text{ over both sides}\right)$$

$$\text{Also } \|y\|^2 = (y, y) = f(y)$$
$$\leq |f(y)|$$
$$\leq \|f\| \, \|y\|$$

$$\Rightarrow \|y\| \leq \|f\|$$

So that $\|f\| = \|y\|$. This Completes the proof.