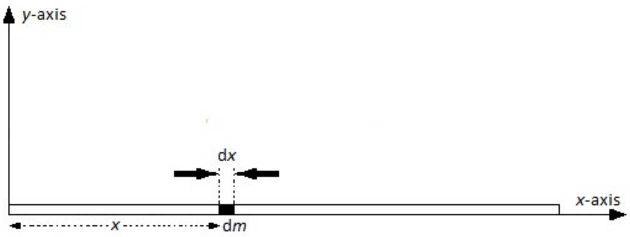
Example 1: ***Find the moment of inertia*** of a (uniform) rigid rod of length l ***about*** an axis perpendicular to the rod and passing through one of its end points.

Solution: Let M, l and λ, respectively, be the mass, length and linear mass density of the rod. We choose x-axis and y-axis as shown in the figure, so that we have to find moment of inertia of the rod about y-axis. We divide rod into large number of elements of infinitesimal width. One typical element of mass dm and length dx, at distance x from the origin, is shown in the figure.



Moment of inertia of typical mass element about y-axis is given by

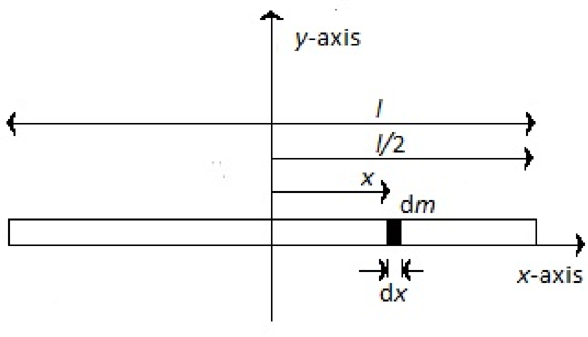
$$\mathrm{d}I_{yy} = x^2 \mathrm{d}m$$

Thus, moment of inertia of rod about y-axis is

$$\begin{aligned} I_{yy} &= \int_{\text{Rod}} x^2 \mathrm{d}m \\ &= \lambda \int_{\text{Rod}} x^2 \mathrm{d}x && \boxed{\because \ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}x} = \text{constant}} \\ &= \frac{M}{l}\int_{x=0}^{l} x^2 \mathrm{d}x = \frac{M}{l}\left(\frac{l^3}{3}\right) = \frac{1}{3}Ml^2 && \boxed{\because \ \lambda = \tfrac{M}{l} \quad \text{(for rod)}} \end{aligned}$$

Example 2: ***Find the moment of inertia*** of a (uniform) rigid rod of length l ***about*** an axis perpendicular to the rod and passing through its centre.

Solution: Let M, l and λ, respectively, be the mass, length and linear mass density of the rod. Choose y-axis as axis of rotation, as shown in the figure. We divide rod into large number of elements of infinitesimal width. One typical element of mass dm and length dx, at distance x from the origin, is shown in the figure.



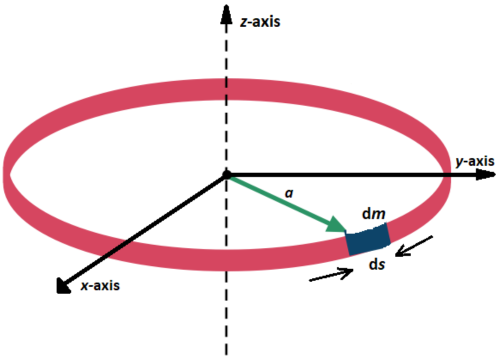
Moment of inertia of typical element about y-axis is given by

$$\mathrm{d}I_{yy} = x^2 \mathrm{d}m$$

Thus, moment of inertia of rod about y-axis is

$$\begin{aligned} I_{yy} &= \int_{\text{Rod}} x^2 \mathrm{d}m \\ &= \lambda \int_{\text{Rod}} x^2 \mathrm{d}x && \boxed{\because \ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}x} = \text{constant}} \\ &= \frac{M}{l}\int_{x=-l/2}^{l/2} x^2 \mathrm{d}x = \frac{M}{l}\left(\frac{l^3}{12}\right) = \frac{1}{12}Ml^2 && \boxed{\because \ \lambda = \tfrac{M}{l} \quad \text{(for rod)}} \end{aligned}$$

Example 3: ***Find the moment of inertia*** of a (uniform) circular ring of radius a ***about***
(i) an axis passing through its centre and perpendicular to its plane,
(ii) its diameter.



Solution: (i) **Moment of inertia about central axis:** Let M, a and λ, respectively, be the mass, radius and linear mass density of the ring. Choose coordinate axes as shown in the figure. We divide ring into large number of elements of infinitesimal width. One typical element of mass dm and length ds is shown in the figure.

Moment of inertia of typical element about z-axis is given by

$$\mathrm{d}I_{zz} = a^2\mathrm{d}m$$

Thus, moment of inertia of ring about z-axis is

$$\begin{aligned} I_{zz} &= a^2\int_{\text{Ring}}\mathrm{d}m \\ &= \lambda a^2\int_{\text{Ring}}\mathrm{d}s && \boxed{\because\ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}s} = \text{constant}} \\ &= \frac{Ma}{2\pi}\int_{s=0}^{2\pi a}\mathrm{d}s = \frac{Ma}{2\pi}(2\pi a) = Ma^2 && \boxed{\because\ \lambda = \tfrac{M}{2\pi a}\quad \text{(for ring)}} \end{aligned}$$

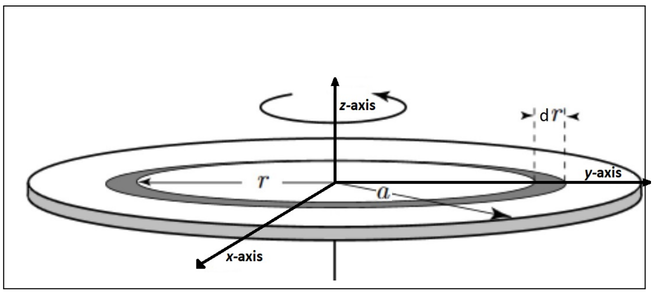
(ii) **Moment of inertia about diameter:**
By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx}, \qquad \boxed{\because\ I_{xx} = I_{yy}\ \text{(by symmetry)}}$$

$$\Rightarrow I_{xx} = \frac{1}{2}Ma^2$$

Example 4: ***Find the moment of inertia*** of a (uniform) circular disc of Mass M and radius a ***about***
(i) an axis passing through its centre and perpendicular to its plane,
(ii) its diameter.
Solution: (i) **Moment of inertia about central axis:** Let M, a and σ, respectively, be the mass, radius and surface (areal) mass density of the disc. Choose axis of rotation as z-axis, as shown in figure.

We divide disc into large number of concentric circular rings of infinitesimal width. One typical elementary ring of mass dm, radius r, width dr and area dA is shown in the figure.



Moment of inertia of typical elementary ring about z-axis is given by

$$\mathrm{d}I_{zz} = r^2\mathrm{d}m$$

Thus, moment of inertia of disc about z-axis is

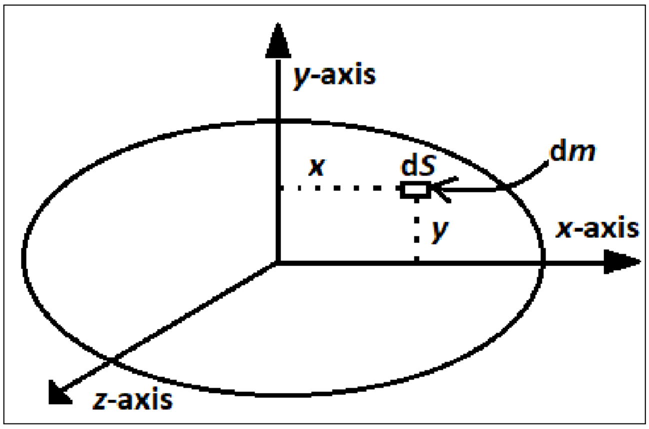
$$\begin{aligned} I_{zz} &= \int_{\text{Disc}} r^2 \mathrm{d}m \\ &= 2\pi\sigma \int_{\text{Disc}} r^3 \mathrm{d}r && \boxed{\because\ \ \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}A} = \tfrac{\mathrm{d}m}{(2\pi r)\mathrm{d}r} = \text{constant}} \\ &= \frac{2M}{a^2}\int_{r=0}^{a} r^3 \mathrm{d}r = \frac{2M}{a^2}\left(\frac{a^4}{4}\right) = \frac{1}{2}Ma^2 && \boxed{\because\ \ \sigma = \tfrac{M}{\pi a^2}\quad \text{(for disc)}} \end{aligned} \tag{1}$$

(*ii*) <u>Moment of inertia about diameter</u>:
By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx}, \qquad \boxed{\because\ \ I_{xx} = I_{yy}\ \text{(by symmetry)}}$$

$$\Rightarrow I_{xx} = \frac{1}{4}Ma^2 \tag{2}$$

Example 5: ***Find the moment of inertia*** of a (uniform) elliptical plate with semi-major axis and semi minor axis a and b, respectively ***about***
(i) major axis,
(ii) minor axis,
(iii) an axis passing through centre of plate and perpendicular to its plane.



Solution: Consider an elliptical plate in xy-plane whose boundary curve is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$

Let M and σ, respectively, be the mass and surface (areal) mass density of the elliptical plate. To find moment of inertia about major axis (x-axis), we proceed as follows. We divide plate into large number of elementary rectangular pieces of infinitesimal area with sides parallel to x and y axis. One typical area element having mass dm, area dS, length dx and width dy is shown in the figure at point (x, y).

Moment of inertia of typical area element about x-axis is given by

$$\mathrm{d}I_{xx} = y^2 \mathrm{d}m$$

Thus, moment of inertia of elliptical plate about x-axis is

$$\begin{aligned} I_{xx} &= \int_{\text{Elliptical plate}} y^2 \mathrm{d}m \\ &= \sigma \int_{\text{Elliptical plate}} y^2\, \mathrm{d}x\, \mathrm{d}y && \boxed{\because\ \ \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}S} = \tfrac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y} = \text{constant}} \\ &= \frac{M}{\pi ab}\int_{\text{Elliptical plate}} y^2 \mathrm{d}x\, \mathrm{d}y && \boxed{\because\ \ \sigma = \tfrac{M}{\pi a\, b}\quad \text{(for elliptical plate)}} \\ &= \frac{M}{\pi ab}\int_{x=-a}^{a}\left(\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 \mathrm{d}y\right)\mathrm{d}x \\ &= \frac{M}{\pi ab}\left(\frac{2b^3}{3a^3}\right)\int_{x=-a}^{a} (a^2 - x^2)^{3/2}\mathrm{d}x \end{aligned}$$

$$I_{xx} = \frac{4Mb^2}{3\pi a^4}\int_{x=0}^{a}(a^2-x^2)^{3/2}\mathrm{d}x \qquad \boxed{\because \text{ integrand is even}}$$

Put $x = a\sin\theta \Rightarrow \mathrm{d}x = a\cos\theta\mathrm{d}\theta,\ x=0 \Rightarrow \theta = 0,\ x = a \Rightarrow \theta = \pi/2$

$$I_{xx} = \frac{4Mb^2}{3\pi a^4}\int_{x=0}^{\pi/2} a^4\cos^4\theta\mathrm{d}\theta$$

Using Wallis cosine formula, we get

$$I_{xx} = \frac{4Mb^2}{3\pi}\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{1}{4}Mb^2 \tag{3}$$

Similarly, moment of inertia about minor axis is

$$I_{yy} = \frac{1}{4}Ma^2$$

By perpendicular axis theorem, the moment of inertia about the axis passing through centre of the elliptical plate and perpendicular to its plane, is

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{4}M(a^2+b^2) \tag{4}$$

Corollary: ***The moment of inertia*** of a (uniform) circular disc of radius a ***about*** (i) its diameter and (ii) an axis passing through its centre and perpendicular to its plane can be obtained by putting $b = a$ in (3) and (4), to give (respectively)

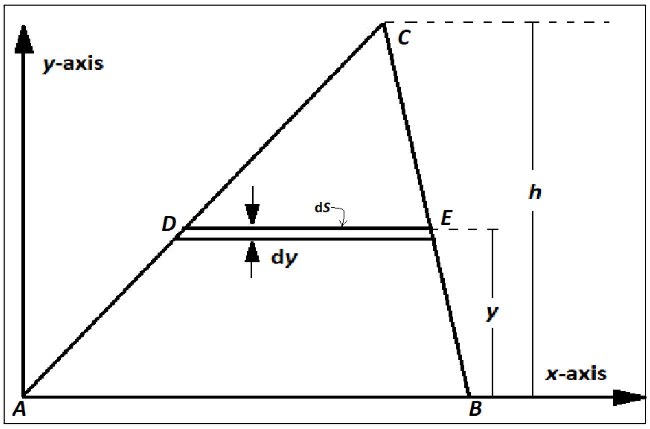
$$I_{xx} = \frac{1}{4}Ma^2 \tag{5}$$

and

$$I_{zz} = \frac{1}{2}Ma^2 \tag{6}$$

Note that, the results obtained in (5) and (6) are in accordance (as they should be) with the results, obtained in (2) and (1), respectively.

Example 6: ***Find the moment of inertia*** of a (uniform) triangular lamina (i.e., two dimensional triangular plate) of mass M ***about*** one of its sides.



Solution: Let M and σ, respectively, be the mass and surface (areal) mass density of the triangular lamina in xy-plane. Choose x-axis and y-axis as shown in figure. We divide lamina into large number of strips of infinitesimal width parallel to the base AB of lamina. One typical elementary strip DE of mass dm, width dy and area dS is shown in the figure.

Moment of inertia of typical elementary strip about side AB (x-axis) is given by

$$\mathrm{d}I_{xx} = y^2\mathrm{d}m$$

Thus, moment of inertia of triangular lamina about x-axis is

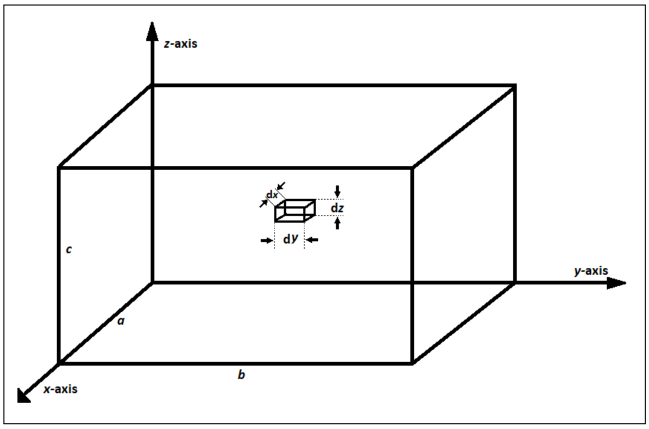
$$I_{xx} = \int_{\text{Triangular lamina}} y^2 \mathrm{d}m$$
$$= \sigma \int_{\text{Triangular lamina}} y^2 |\mathrm{DE}| \mathrm{d}y \qquad \boxed{\because\ \sigma = \frac{\mathrm{d}m}{\mathrm{d}S} = \frac{\mathrm{d}m}{|\mathrm{DE}|\mathrm{d}y} = \text{constant}}$$
$$= \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \frac{|\mathrm{DE}|}{|\mathrm{AB}|} \mathrm{d}y \qquad \boxed{\because\ \sigma = \frac{M}{\frac{1}{2}|\mathrm{AB}|h} \quad \text{(for tiangular lamina)}}$$

From equivalent triangles ABC and DEC, we have are equivalent triangles, therefore

$$\frac{|\mathrm{DE}|}{|\mathrm{AB}|} = \frac{\text{height of DEC}}{\text{height of ABC}} = \frac{h-y}{h}$$

$$\Rightarrow I_{xx} = \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \left(\frac{h-y}{h}\right) dy$$
$$= \frac{2M}{h^2} \int_{y=0}^{h} y^2 (h-y) dy = \frac{2M}{h^2} \left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{1}{6} M h^2$$

Example 7: ***Calculate the inertia matrix*** of a (uniform solid) rectangular box (rectangular parallelopiped or cuboid) of mass M ***at*** one of its corners, by taking coordinate axes along its edges.



Solution: Let M and ρ, respectively, be the mass and volume mass density of the rectangular box. Let the lengths of adjacent edges be a, b and c. Choose coordinate axis along the edges of box, as shown in figure. We divide lamina into large number of elementary rectangular boxes of infinitesimal volume. One typical elementary volume element of mass $\mathrm{d}m$, volume $\mathrm{d}V$ and dimensions $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, is shown in the figure.

Moment of inertia of typical elementary volume element about x-axis is given by

$$\mathrm{d}I_{xx} = (y^2 + z^2)\mathrm{d}m$$

Thus, moment of inertia of triangular lamina about x-axis is

$$I_{xx} = \int_{\text{Rectangular box}} (y^2 + z^2)\mathrm{d}m$$
$$= \rho \int_{\text{Rectangular box}} (y^2 + z^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \boxed{\because\ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}}$$
$$= \frac{M}{a\,b\,c} \int_{\text{Rectangular box}} (y^2 + z^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \boxed{\because\ \rho = \frac{M}{a\,b\,c} \quad \text{(for rectangular box)}}$$
$$= \frac{M}{a\,b\,c} \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (y^2 + z^2) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$
$$= \frac{M}{a\,b\,c} \left[\int_{x=0}^{a} \mathrm{d}x\right] \left[\int_{z=0}^{c} \int_{y=0}^{b} (y^2 + z^2)\, \mathrm{d}y\, \mathrm{d}z\right]$$

$$
\begin{aligned}
I_{xx} &= \frac{M}{b\,c}\int_{z=0}^{c}\int_{y=0}^{b}\left(y^2+z^2\right)\mathrm{d}y\,\mathrm{d}z \\
&= \frac{M}{b\,c}\int_{z=0}^{c}\left(\frac{b^3}{3}+bz^2\right)\mathrm{d}z \\
&= \frac{M}{b\,c}\left(\frac{b^3c}{3}+\frac{b\,c^3}{3}\right) = \frac{M}{3}\left(b^2+c^2\right)
\end{aligned}
$$

Similarly,

$$
I_{yy} = \frac{M}{3}\left(a^2+c^2\right) \qquad \text{and} \qquad I_{zz} = \frac{M}{3}\left(a^2+b^2\right)
$$

For product of inrtia

$$
\begin{aligned}
I_{xy} &= \int_{\text{Rectangular box}} x\,y\,\mathrm{d}m \\
&= -\frac{M}{abc}\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} x\,y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\frac{M}{abc}\left(\frac{a^2}{2}\right)\left(\frac{b^2}{2}\right)c = -\frac{1}{4}Mab
\end{aligned}
$$

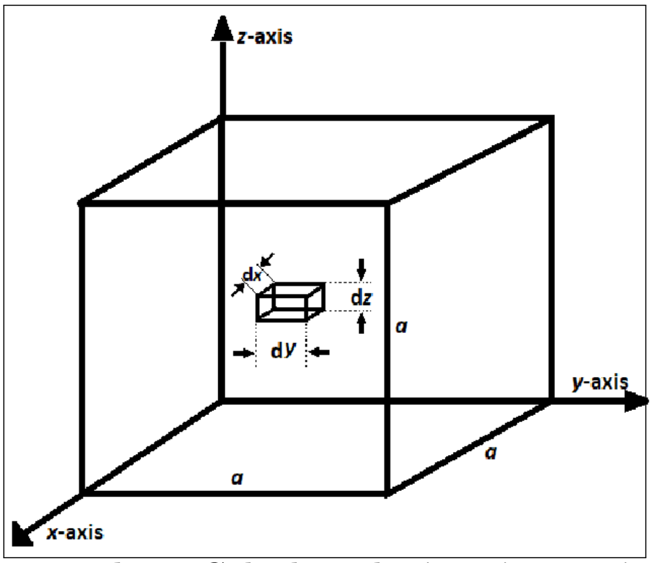
Similarly,

$$
I_{yz} = -\frac{1}{4}Mbc \qquad \text{and} \qquad I_{xz} = -\frac{1}{4}Mac
$$

The required inertia matrix is given by

$$
[I_O] = \begin{bmatrix} (1/3)M(b^2+c^2) & -(1/4)Mab & -(1/4)Mac \\ -(1/4)Mab & (1/3)M(a^2+c^2) & -(1/4)Mbc \\ -(1/4)Mac & -(1/4)Mbc & (1/3)M(a^2+b^2) \end{bmatrix}
$$

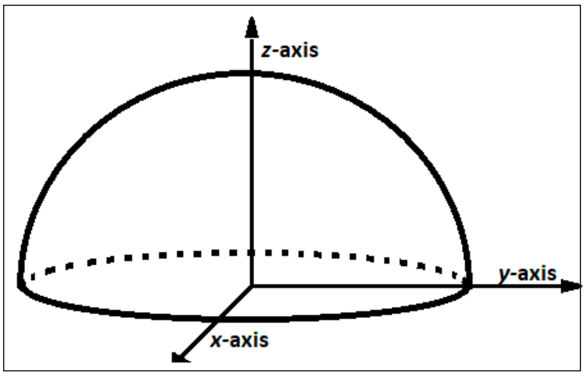
$$
\Rightarrow \boxed{[\mathrm{I}_O] = \tfrac{1}{12}M\begin{bmatrix} 4(b^2+c^2) & -3ab & -3ac \\ -3ab & 4(a^2+c^2) & -3bc \\ -3ac & -3bc & 4(a^2+b^2) \end{bmatrix}}
$$



Example 8: ***Calculate the inertia matrix*** of a (uniform solid) cube of mass M ***at*** one of its corners, by taking coordinate axes along its edges.
Solution: Repeat example 7 for $a = b = c$ and get

$$
[I_O] = \frac{1}{12}Ma^2\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}
$$

Example 9: ***Find the moment of inertia*** of a (uniform solid) hemisphere of mass M ***about***
(i) its axis of symmetry
(ii) an axis perpendicular to the axis of symmetry and passing through the centre of the base.



Solution:

(i) **Moment of inertia about axis of symmetry:**
Let M, a and ρ, respectively, be the mass, radius and volume mass density of the hemisphere. Choose coordinate axes as shown in figure.
Moment of inertia of typical volume element of hemisphere, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of hemisphere about z-axis is

$$\begin{aligned} I_{zz} &= \int_{\text{Hemisphere}} (x^2+y^2)\mathrm{d}m \\ &= \rho \int_{\text{Hemisphere}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because \quad \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \tfrac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}} \\ &= \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} (y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because \quad \rho = \tfrac{M}{\frac{2}{3}\pi a^3} \quad \text{(for hemisphere)}} \end{aligned}$$

To make the computation simpler, we transform the problem from Cartesian coordinates to spherical coordinates (r, θ, ϕ) by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2+y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta$$

$$\text{For hemisphere}: \quad 0 \le r \le a, \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r \int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi \quad (7)$$

Where,

$$\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Bigg|_{\theta=0}^{\pi/2} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3} \qquad (8)$$

Using (8) in (7), we get

$$I_{zz} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

(ii) **Moment of inertia about a diameter through the base:**

$$I_{xx} = \int_{\text{Hemisphere}}(y^2+z^2)\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}}(y^2+z^2)\,\mathrm{d}V$$

Transforming problem in spherical coordinates $(r,\, \theta,\, \phi)$, we get

$$\begin{aligned} I_{xx} &= \frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 (\sin^3\theta \sin^2\phi + \cos^2\theta \sin\theta) \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \\ &= \frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4\, \mathrm{d}r \left(\int_{\theta=0}^{\pi/2} \sin^3\theta\, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin^2\phi\, \mathrm{d}\phi + \int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta\, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi \right) \end{aligned} \tag{9}$$

Where,

$$\int_{\phi=0}^{2\pi} \sin^2\phi\, \mathrm{d}\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 - \cos 2\phi)\, \mathrm{d}\phi = \frac{1}{2} \left(\phi - \frac{1}{2} \sin 2\phi \right) \Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi \tag{10}$$

and

$$\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta\, \mathrm{d}\theta = -\frac{1}{3} \cos^3\theta \Big|_{\theta=0}^{\pi/2} = \frac{1}{3} \tag{11}$$

Using (8), (10) and (11), (9) gives

$$I_{xx} = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5} \right) \left(\frac{2\pi}{3} + \frac{2\pi}{3} \right) = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5} \right) \left(\frac{4\pi}{3} \right) = \frac{2}{5} M a^2$$

Example 10: ***Find three products of inertia*** of a (uniform) solid hemisphere of mass M ***with respect to*** coordinate axes as in figure of example 9.

Solution:

$$\begin{aligned} I_{xy} &= -\int_{\text{Hemisphere}} x\, y\, \mathrm{d}m = \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} x\, y\, \mathrm{d}V \\ &= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^2\theta \sin\phi \cos\phi\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \\ &= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4\, \mathrm{d}r \int_{\theta=0}^{\pi/2} \sin^2\theta\, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin\phi \cos\phi\, \mathrm{d}\phi \end{aligned}$$

But

$$\int_{\phi=0}^{2\pi} \sin\phi \cos\phi\, \mathrm{d}\phi = \frac{1}{2} \sin^2\phi \Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Now,

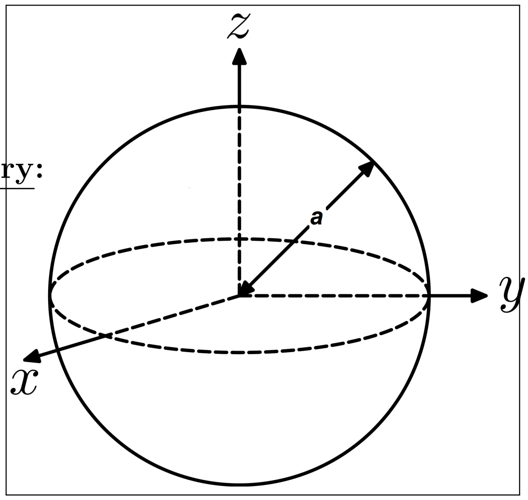
$$\begin{aligned} I_{xz} &= -\int_{\text{Hemisphere}} x\, z\, \mathrm{d}m = \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} x\, y\, \mathrm{d}V \\ &= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin\theta \cos\theta \cos\phi\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \\ &= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4\, \mathrm{d}r \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta\, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \cos\phi\, \mathrm{d}\phi \end{aligned}$$

But

$$\int_{\phi=0}^{2\pi} \cos\phi\, \mathrm{d}\phi = \sin\phi \Big|_{\phi=0}^{2\pi} = 0 \quad \Longrightarrow \quad I_{xz} = 0 = I_{yz}, \qquad \boxed{\because\ I_{xz} = I_{yz}\ \text{(by symmetry)}}$$

Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0$$



Example 11: ***Find the moments and products of inertia*** of a (uniform solid) sphere of mass M and radius a ***with respect to*** its axes of symmetry.

Solution: (i) **Moment of inertia about axis of symmetry:** Let M, a and ρ, respectively, be the mass, radius and volume mass density of the sphere. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of sphere, with mass dm and volume dV, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of sphere about z-axis is

$$\begin{aligned} I_{zz} &= \int_{\text{Sphere}} (x^2+y^2)\mathrm{d}m \\ &= \rho \int_{\text{Sphere}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because\ \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \tfrac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}} \\ &= \frac{3M}{4\pi a^3}\int_{\text{Sphere}} (y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because\ \rho = \tfrac{M}{\frac{4}{3}\pi a^3}\quad \text{(for sphere)}} \end{aligned}$$

To make the computation simpler, we transform the problem from Cartesian coordinates to spherical coordinates (r, θ, ϕ) by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\, d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2+y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta$$

$$\text{For sphere}: \qquad 0 \le r \le a, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi$$

Where,

$$\begin{aligned} \int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta &= \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) && \boxed{\because\ \sin 3\theta = 3\sin\theta - 4\sin^3\theta} \\ &= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Bigg|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3} \end{aligned}$$

Thus,

$$I_{zz} = \frac{3M}{4\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

Similarly,

$$I_{xx} = I_{yy} = \frac{2}{5}Ma^2 \qquad \boxed{\because\ I_{xx} = I_{yy} = I_{zz}\ \text{(by symmetry)}}$$

(ii) **Products of inertia with respect to axes of symmetry:**

$$\begin{aligned} I_{xy} &= -\int_{\text{Sphere}} x\,y\,\mathrm{d}m = \frac{3M}{4\pi a^3}\int_{\text{Sphere}} x\,y\,\mathrm{d}V \\ &= -\frac{3M}{4\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^2\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi \end{aligned}$$

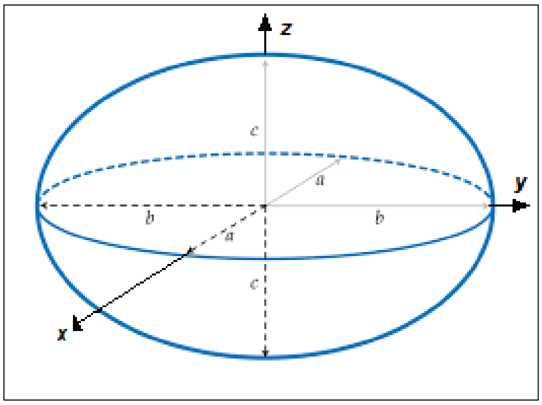
But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad\Rightarrow\qquad I_{xy}=0$$

Similarly,

$$I_{yz} = I_{xz} = 0 \qquad \boxed{\because\ I_{xy} = I_{yz} = I_{xz}\ \text{(by symmetry)}}$$

Example 12: ***Find the moments and products of inertia*** of a (uniform) solid ellipsoid



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

of mass M ***with respect to*** its axes of symmetry.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M and ρ, respectively, be the mass and volume mass density of the ellipsoid. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of ellipsoid, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of ellipsoid about z-axis is

$$\begin{aligned} I_{zz} &= \int_{\text{Ellipsoid}} (x^2+y^2)\mathrm{d}m \\ &= \rho\int_{\text{Ellipsoid}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because\ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}} \\ &= \frac{3M}{4\pi a\,b\,c}\int_{\text{Sphere}} (y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because\ \rho = \frac{M}{\frac{4}{3}\pi a\,b\,c}\quad\text{(for ellipsoid)}} \end{aligned}$$

$$I_{zz} = \int_{\text{Ellipsoid}} (x^2 + y^2)\mathrm{d}m \tag{12}$$

Let us substitute

$$x/a = x', \quad y/a = y', \quad z/a = z'$$

$$\Rightarrow \quad \mathrm{d}x/a = \mathrm{d}x', \quad \mathrm{d}y/a = \mathrm{d}y', \quad \mathrm{d}z/a = \mathrm{d}z', \qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

Under the above transformation, the given ellipsoid is transformed into the unit sphere

$$S : x'^2 + y'^2 + z'^2 = 1.$$

$$\begin{aligned}
\Rightarrow \quad I_{zz} &= \frac{3M}{4\pi a\,b\,c}\int_S (a^2x'^2 + b^2y'^2)(\,a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z') \\
&= \frac{3M}{4\pi}\int_S (a^2x'^2 + b^2y'^2)\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'
\end{aligned}$$

$$\because \quad \int_S x'^2\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \int_S y'^2\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' \qquad \text{(by symmetry)}$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\int_S x'^2\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x', y', z') to spherical coordinates (r, θ, ϕ) by using

$$x' = r\sin\theta\cos\phi, \qquad y' = r\sin\theta\sin\phi, \qquad z' = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

For unit sphere,

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

$$\begin{aligned}
\Rightarrow \quad I_{zz} &= \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\cos^2\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos^2\phi\,\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
\text{Where,} \quad \int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta &= \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) \qquad \because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta \\
&= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{4}{3}
\end{aligned}$$

and

$$\int_{\phi=0}^{2\pi}\cos^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1+\cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\left(\frac{1}{5}\right)\left(\frac{4}{3}\right)(\pi) = \frac{1}{5}M(a^2+b^2)$$

Similarly,

$$I_{xx} = \frac{1}{5}M(b^2+c^2) \qquad \text{and} \qquad I_{yy} = \frac{1}{5}M(a^2+c^2)$$

(ii) **Products of inertia with respect to axes of symmetry:**

$$
\begin{aligned}
\Rightarrow \quad I_{xy} &= -\int_{\text{Ellipsoid}} x\, y\, \mathrm{d}m = -\frac{3M}{4\pi a\, b\, c}\int_{\text{Ellipsoid}} x\, y\, \mathrm{d}V \\
&= -\frac{3M}{4\pi a\, b\, c}\int_{\mathrm{S}} (\, a\, b\, x'\, y')(\, a\, b\, c\, \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}z') \\
&= -\frac{3a\, b\, M}{4\pi}\int_{\mathrm{S}} x'\, y' \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}z' \\
&= -\frac{3\, a\, b\, M}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4 \sin^2\theta \sin\phi\cos\phi\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi
\end{aligned}
$$

$$
I_{xy} = -\frac{3\, a\, b\, M}{4\pi}\int_{r=0}^{1} r^4\, \mathrm{d}r \int_{\theta=0}^{\pi} \sin^2\theta\, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin\phi\cos\phi\, \mathrm{d}\phi
$$

But

$$
\int_{\phi=0}^{2\pi} \sin\phi\cos\phi\, \mathrm{d}\phi = \frac{1}{2}\sin^2\phi \Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0
$$

Similarly, it is not difficult to show that

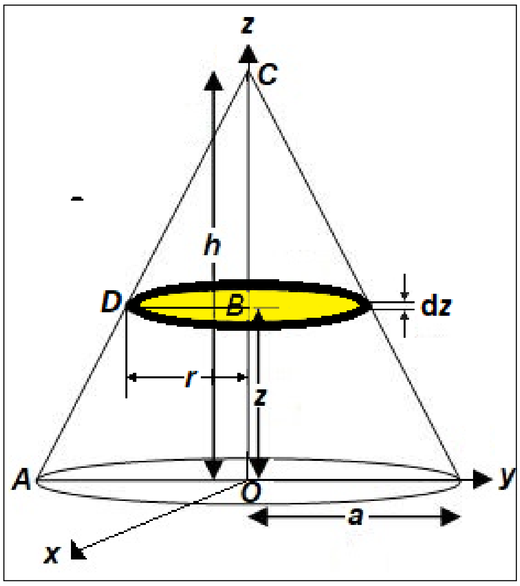
$$
I_{yz} = I_{xz} = 0
$$

Corollary: ***The moment and product of inertia*** of a (uniform) solid sphere of mass M and radius a ***with respect to*** its axis of symmetry can be obtained by putting $a = b = c$ in results of above example 12. The obtained results are in accordance (as they should be) with the results of example 11.

Example 13: ***Find the moment of inertia*** of a (uniform) right circular solid cone ***about*** (i) its axis of symmetry and
(ii) any diameter of the base.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M, h, a and ρ, respectively, be the mass, height, radius of base and volume mass density of a (uniform) right circular solid cone. Choose coordinate axes as shown in figure. Let us divide cone into large number of elementary solid discs parallel to the base of the cone. One such elementary disc of radius r, mass $\mathrm{d}m$, thickness $\mathrm{d}z$ and volume $\mathrm{d}V$ is shown in the figure, at a distance z from the base of the cone.



Moment of inertia of elementary disc about z-axis is given by

$$
\mathrm{d}I_{zz} = \frac{1}{2}\, r^2 \mathrm{d}m
$$

Thus, moment of inertia of cone about z-axis is

$$\begin{aligned} I_{zz} &= \frac{1}{2}\int_{\text{Cone}} r^2 \mathrm{d}m \\ &= \frac{\rho}{2}\int_{\text{Cone}} r^2\left(\pi r^2\right)\mathrm{d}z && \boxed{\because\ \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \tfrac{\mathrm{d}m}{(\pi r^2)\,\mathrm{d}z} = \text{constant}} \\ &= \frac{3M}{2a^2h}\int_{\text{Cone}} r^4\,\mathrm{d}z && \boxed{\because\ \rho = \tfrac{M}{\frac{1}{3}\pi a^2 h}\quad \text{(for cone)}} \end{aligned}$$

From similar triangles AOC and DBC

$$\frac{r}{a} = \frac{h-z}{h} \qquad \text{or} \qquad r = \frac{a(h-z)}{h}$$

$$\begin{aligned} \Rightarrow I_{zz} &= \frac{3M}{2a^2h}\int_{\text{Cone}}\left[\frac{a(h-z)}{h}\right]^4 \mathrm{d}z \\ &= \frac{3M\,a^2}{2\,h^5}\int_{z=0}^{h}(h-z)^4\,\mathrm{d}z \\ &= -\frac{3M\,a^2}{10\,h^5}\,(h-z)^5\Big|_{z=0}^{h} = \frac{3}{10}\,M\,a^2 \end{aligned}$$

(*ii*) <u>Moment of inertia about diameter of the base</u>:
In this case, the moment of inertia of the elementary disc of mass $\mathrm{d}m$ about a diameter, along DB, is given by

$$\mathrm{d}I_o = \frac{1}{4}\,r^2\mathrm{d}m$$

We note that the diameter passes through the center (which is also the centroid) of the elementary disc. Hence, by parallel axis theorem, the moment of inertia of the elementary disc about a parallel axis along AO (through the centre of the base of cone) is given by

$$\begin{aligned} \mathrm{d}I_{yy} &= \mathrm{d}I_o + (\mathrm{d}m)\,z^2 \\ &= \frac{1}{4}\,r^2\mathrm{d}m + (\mathrm{d}m)\,z^2 = \left(\frac{1}{4}\,r^2 + z^2\right)\mathrm{d}m \\ &= \frac{3M}{a^2h}\left(\frac{1}{4}\,r^4 + r^2z^2\right)\mathrm{d}z, && \boxed{\because \mathrm{d}m = \rho\,\mathrm{d}V = \tfrac{3M}{a^2h}(r^2\,\mathrm{d}z)} \end{aligned}$$

Therefore, the moment of inertia of whole cone about diameter of the base is given by

$$\begin{aligned} I_{yy} &= \frac{3M}{a^2h}\int_{z=0}^{h}\left[\frac{1}{4}\left(\frac{a(h-z)}{h}\right)^4 + \left(\frac{a(h-z)}{h}\right)^2 z^2\right]\mathrm{d}z && \boxed{\because r = \tfrac{a(h-z)}{h}} \\ &= \frac{3M}{a^2h}\int_{z=0}^{h}\left[\frac{a^4}{4\,h^4}(h-z)^4 + \frac{a^2}{h^2}(h^2z^2 - 2h\,z^3 + z^4)\right]\mathrm{d}z \\ &= \frac{3M}{a^2h}\left[-\frac{a^4}{20\,h^4}(h-z)^5 + \frac{a^2}{h^2}\left(h^2\frac{z^3}{3} - h\,\frac{z^4}{2} + \frac{z^5}{5}\right)\right]\Big|_{z=0}^{h} \\ &= \frac{3M}{a^2h}\left[\frac{a^4\,h}{20} + \frac{a^2}{h^2}\left(\frac{h^5}{3} - \frac{h^5}{2} + \frac{h^5}{5}\right)\right] = \frac{3M}{a^2h}\left[\frac{a^4\,h}{20} + a^2\,h^3\left(\frac{10-15+6}{30}\right)\right] \\ &= \frac{3M}{a^2h}\left[\frac{a^4\,h}{20} + \frac{a^2\,h^3}{30}\right] = \frac{3M}{a^2h}\left[\frac{3a^4\,h + 2a^2\,h^3}{60}\right] = \frac{1}{20}M(3a^2 + 2h^2) \end{aligned}$$