DIFFERENTIAL GEOMETRY

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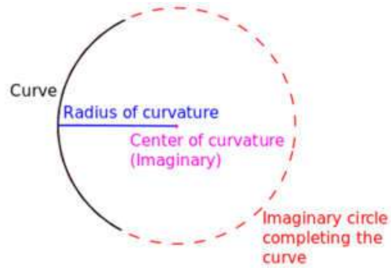
Lecture # 1

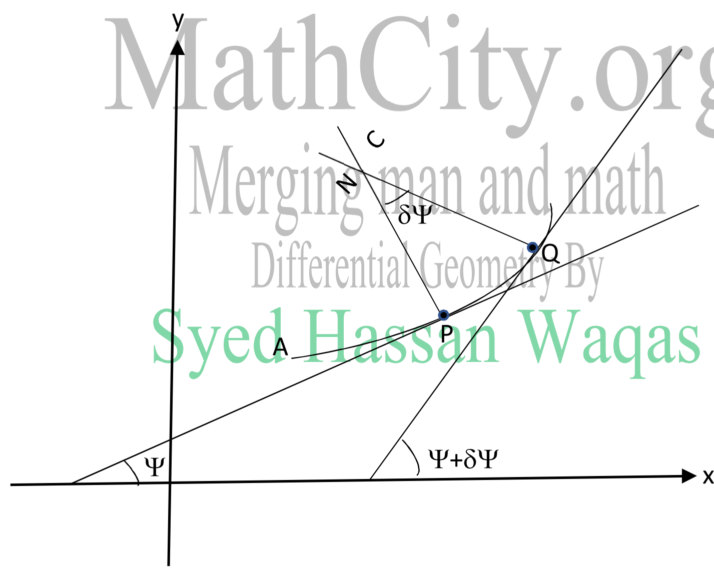
Curvature:



More bending less curvature.

Less bending more curvature.

Centre, Radius of Curvature:



Let $AP = s$

$QP = \delta s$

$\angle PNQ = \delta\Psi$

If $P \to Q$ then $N \to C$

Also $\delta\Psi \to 0$

C is the Centre of curvature.

$$\lim_{\delta s \to 0} \frac{\delta\Psi}{\delta s} = \frac{d\Psi}{ds} = K \text{ (Kappa)}$$

Radius of Curvature $= \frac{1}{K}$

The curvature, radius of curvature, Centre of curvature, circle of curvature is different at any points on the curve.

If we change the points. Then these all can be change. If the curvature is constant at every point then it is called **Circle**.

$$\text{Radius of Curvature} = \frac{1}{K} = \rho$$

Formula for Radius of Curvature:

Let tangent of curve at point makes angle Ψ with x-axis then by the definition of the derivative

$$\frac{dy}{dx} = \tan\Psi$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\Psi.\frac{d\Psi}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (1+\tan^2\Psi)\frac{d\Psi}{ds}\frac{ds}{dx}$$

$$= [1+(\frac{dy}{dx})^2].\frac{1}{\rho}.\sqrt{1+(\frac{dy}{dx})^2} \qquad \because \frac{d\Psi}{ds} = K$$

$$\frac{d^2y}{d^2x} = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\rho} \qquad \therefore \frac{ds}{dx} = \text{length of the curve}$$

$$\Rightarrow \quad \rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \text{Radius of Curvature}$$

Centre of Curvature:

Let $y = f(x)$ be the given curve. And (α, β) be the Centre of the curvature.

Then at point P (x_1, y_1) of the curve the values of α and β given as

$$\alpha = x_1 - \frac{\frac{dy}{dx}[1+(\frac{dy}{dx})^2]}{\frac{d^2y}{dx^2}}$$

$$\beta = y_1 + \frac{[1+(\frac{dy}{dx})^2]}{\frac{d^2y}{dx^2}}$$

Circle of Curvature:

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2$$

If sometime parametric equations of the curve $y = f(t)$, $x = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

Question:

Prove that curvature of the circle $x^2 + y^2 = a^2$ is constant.

Given

$$x^2 + y^2 = a^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \frac{d^2y}{d^2x} = \frac{-y-(-x)\frac{dy}{dx}}{y^2}$$

$$= \frac{-y+x(-\frac{x}{y})}{y^2} \Rightarrow \frac{-y^2-x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-a^2}{y^3}$$

$$\rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \Rightarrow = \frac{[1+(-\frac{x}{y})^2]^{\frac{3}{2}}}{\frac{-a^2}{y^3}}$$

$$\rho = \frac{[x^2+y^2]^{\frac{3}{2}}}{(-a^2)} = \frac{a^3}{-a^2} = -a$$

Question:

Prove that Radius of the curvature at point $x = \frac{\pi}{2}$ then $y = 4\sin x - \sin 2x$ is $\rho = \frac{5\sqrt{5}}{4}$

$$\frac{dy}{dx} = 4\cos x - 2\cos 2x \qquad \because \text{ at } x = \frac{\pi}{2}, \frac{dy}{dx} = 2$$

$$\frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x \qquad \because \text{ at } x = \frac{\pi}{2}, \frac{d^2y}{dx^2} = -4$$

$$\rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \Rightarrow \frac{[1+(2)^2]^{\frac{3}{2}}}{-4} = -\frac{5\sqrt{5}}{4}$$

$$= \frac{5\sqrt{5}}{4}$$

Note:

Here (-) is neglected because radius cannot be negative.

Question:

Find the Centre of curvature at point x, y of parabola $y^2 = 4ax$.

Solution

Given $y^2 = 4ax.$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = 2ay^{-1}$$

$$\frac{d^2y}{dx^2} = -4a^2y^{-3}$$

$$\alpha = x - \frac{\frac{dy}{dx}[1+(\frac{dy}{dx})^2]}{\frac{d^2y}{dx^2}}$$

$$\alpha = x - \frac{(2ay^{-1})[1+(2ay^{-1})^2]}{(-4a^2y^{-3})}$$

$$\alpha = x - \frac{y^2[1+4a^2y^{-2}]}{(2a)}$$

$$\alpha = x - \frac{[y^2+4a^2]}{(2a)}$$

$$\alpha = x + \frac{4ax}{2a} + \frac{4a^2}{2a}$$

$$\alpha = 3x + 2a$$

$$\beta = y + \frac{[1+(\frac{dy}{dx})^2]}{\frac{d^2y}{dx^2}}$$

$$\beta = y - \frac{[1+(2ay^{-1})^2]}{(-4a^2y^{-3})}$$

$$\beta = y - \frac{[1+4a^2y^{-2}]}{(-4a^2y^{-3})}$$

$$\beta = y - [\frac{1}{4a^2y^{-3}} + \frac{4a^2y^{-2}}{4a^2y^{-3}}]$$

$$\beta = y - \frac{1}{4a^2y^{-3}} - y$$

$$\beta = -\frac{1}{4a^2y^{-3}} \text{ or } -\frac{y^3}{4a^2}$$

Question: Find the radius of curvature for the curve.

$$y = 3a\sin t - a\sin 3t$$

$$x = 3a\cos t - a\cos 3t$$

$\frac{dy}{dt} = 3a\cos t - 3a\cos 3t$, $\frac{dx}{dt} = -3a\sin t + 3a\sin 3t$

$\frac{d^2y}{dt^2} = -3a\sin t + 9a\sin 3t$, $\frac{d^2x}{dt^2} = -3a\cos t + 9a\cos 3t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, $\frac{dy}{dx} = \frac{3a\cos t - 3a\cos 3t}{-3a\sin t + 3a\sin 3t}$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$\sin A - \sin B = \cos 2t.\sin t$, $\cos A - \cos B = \sin 2t.\sin t$

$$\frac{dy}{dx} = \tan 2t$$

$$\frac{d^2y}{dx^2} = 2\sec^2 2t \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{2\sec^2 2t}{-3a\sin t + 3a\sin 3t}$$

$$\frac{d^2y}{dx^2} = \frac{2\sec^2 2t}{3a(2\cos 2t\sin t)}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2 2t}{3a(\cos 2t\sin t)}$$

$$\rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{[1+(\tan 2t)^2]^{\frac{3}{2}}}{\frac{\sec^2 2t}{3a(\cos 2t\sin t)}}$$

$$\rho = \frac{\frac{\sec^3 2t}{\sec^2 2t}}{3a(\cos 2t\sin t)} = (3a\cos 2t.\sin t)(\sec 2t)$$

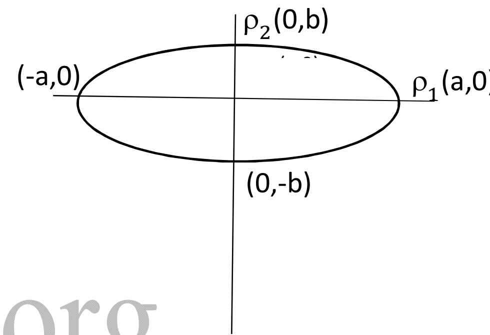
$$\rho = 3a\sin t$$

Question:

If ρ_1 and ρ_2 be the radii of curvature at the Extremities of two conjugate diameters then prove that

$$\left((\rho_1)^{\frac{2}{3}}+(\rho_2)^{\frac{2}{3}}\right)(ab)^{\frac{2}{3}}=a^2+b^2$$

Solution: Use an equation of Ellipse



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

$$\Rightarrow \qquad \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{y}{b^2}\frac{dy}{dx}=\frac{-x}{a^2} \text{ or } \frac{dy}{dx}=\frac{-xb^2}{ya^2}$$

$$\frac{d^2y}{dx^2}=\frac{b^2}{a^2}\left[\frac{-y-(-x)\frac{dy}{dx}}{y^2}\right]$$

$$\frac{d^2y}{dx^2}=\frac{b^2}{a^2}\left[\frac{-y+x(\frac{-xb^2}{ya^2})}{y^2}\right] \Rightarrow \frac{d^2y}{dx^2}=\frac{b^2}{a^2}\left[\frac{-a^2y^2-x^2b^2}{a^2y^3}\right]$$

$$=-\frac{b^2}{a^2}\left[\frac{a^2b^2}{a^2y^3}\right] \Rightarrow \frac{d^2y}{dx^2}=\frac{-b^4}{y^3a^2}$$

$$\rho=\frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \Rightarrow \frac{[1+(\frac{-xb^2}{ya^2})^2]^{\frac{3}{2}}}{\frac{-b^4}{y^3a^2}}$$

$$\rho=y^3a^2\frac{\frac{[y^2a^4+x^2b^4]^{\frac{3}{2}}}{y^2a^4}}{-b^4} \Rightarrow \rho=y^3a^2\frac{[y^2a^4+x^2b^4]^{\frac{3}{2}}}{-b^4y^3a^6}$$

$$\rho=\frac{[y^2a^4+x^2b^4]^{\frac{3}{2}}}{-b^4a^4}$$

At $\rho_1(a,0) \Rightarrow \rho_1=\frac{b^2}{a}$, At $\rho_2(0,b) \Rightarrow \rho_2=\frac{a^2}{b}$

$$\left((\rho_1)^{\frac{2}{3}}+(\rho_2)^{\frac{2}{3}}\right)(ab)^{\frac{2}{3}}=a^2+b^2 \Rightarrow \left(\left(\frac{b^2}{a}\right)^{\frac{2}{3}}+\left(\frac{a^2}{b}\right)^{\frac{2}{3}}\right)(ab)^{\frac{2}{3}}=a^2+b^2$$

$$(ab)^{\frac{2}{3}}\left(\left(\frac{b^{\frac{4}{3}}}{b^{\frac{2}{3}}}\right)+\left(\frac{a^{\frac{4}{3}}}{b^{\frac{2}{3}}}\right)\right)=a^2+b^2 \Rightarrow a^2+b^2=a^2+b^2$$

Lecture # 2

Curvature and Radius of Curvature in Polar Form:

Let $r = f(\theta)$

Let $x = r\cos\theta \quad , \quad y = r\sin\theta$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \text{(i)}$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \qquad \text{(ii)}$$

$$\frac{d^2x}{d\theta^2} = -r\cos\theta - \frac{dr}{d\theta}\sin\theta - \frac{dr}{d\theta}\sin\theta + \cos\theta\frac{d^2r}{d\theta^2}$$

$$\frac{d^2x}{d\theta^2} = \frac{d^2r}{d\theta^2}\cos\theta - 2\frac{dr}{d\theta}\sin\theta - r\cos\theta \qquad \text{(iii)}$$

$$\frac{d^2y}{d\theta^2} = -r\sin\theta + \cos\theta\frac{dr}{d\theta} + \sin\theta\frac{d^2r}{d\theta^2} + \frac{dr}{d\theta}\cos\theta$$

$$\frac{d^2y}{d\theta^2} = \frac{d^2r}{d\theta^2}\sin\theta + 2\frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \text{(iv)}$$

Squaring and Adding (i) and (ii)

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta) + r^2(\sin^2\theta + \cos^2\theta)$$

$$= r^2(1) + \left(\frac{dr}{d\theta}\right)^2(1)$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \qquad \text{(v)}$$

Formula for curvature if

$x = f(\theta) \quad , \quad y = g(\theta)$

$$K = \frac{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}}{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}$$

$$K = \frac{\left[\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)\cdot\left(\frac{d^2r}{d\theta^2}\sin\theta + 2\frac{dr}{d\theta}\cos\theta - r\sin\theta\right) - \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)\left(\frac{d^2r}{d\theta^2}\cos\theta - 2\frac{dr}{d\theta}\sin\theta - r\cos\theta\right)\right]}{\left[(r)^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}$$

Replace $\frac{dr}{d\theta} = r' \quad , \quad \frac{d^2r}{d\theta^2} = r''$

$$K = \frac{[(r'\cos\theta - r\sin\theta).(r''\sin\theta + 2r'\cos\theta - r\sin\theta) - (r'\sin\theta + r\cos\theta)(r''\cos\theta - 2r'\sin\theta - r\cos\theta)]}{[(r)^2 + (r')^2]^{\frac{3}{2}}}$$

$$K = \frac{(r'r''\sin\theta\cos\theta + 2r'^2\cos^2\theta - rr'\cos\theta\sin\theta - rr''\sin^2\theta - 2rr''\cos\theta\sin\theta + r^2\sin^2\theta)}{(r^2 + r'^2)^{\frac{3}{2}}}$$

$$- \frac{(r'r''\sin\theta\cos\theta - 2r'^2\sin^2\theta - rr'\cos\theta\sin\theta + rr''\cos^2\theta - 2rr'\cos\theta\sin\theta - r^2\cos^2\theta)}{(r^2 + r'^2)^{\frac{3}{2}}}$$

$$K = \frac{2r'^2(\cos^2\theta + \sin^2\theta) - rr''(\cos^2\theta + \sin^2\theta) + r^2(\cos^2\theta + \sin^2\theta)}{(r^2 + r'^2)^{\frac{3}{2}}}$$

$$K = \frac{2r'^2(1) - rr''(1) + r^2(1)}{(r^2 + r'^2)^{\frac{3}{2}}} \quad \Rightarrow \quad K = \frac{2r'^2 - rr'' + r^2}{(r^2 + r'^2)^{\frac{3}{2}}}$$

$$K = \frac{2(\frac{dr}{d\theta})^2 - r\frac{d^2r}{d\theta^2} + r^2}{(r^2 + r'^2)^{\frac{3}{2}}} \quad \Rightarrow \quad \rho = \frac{1}{K}$$

$$\rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{2(r')^2 - r\,r'' + r^2} \text{ or } \quad \rho = \frac{\left[r^2 + (\frac{dr}{d\theta})^2\right]^{\frac{3}{2}}}{2(\frac{dr}{d\theta})^2 - r\frac{d^2r}{d\theta^2} + r^2}$$

Question:

For curve $r^m = a^m \cos m\theta$ then prove that $\rho = \frac{a^m}{(m+1)r^{m-1}}$

Solution:

$$r^m = a^m \cos m\theta$$

Taking ln on both side

$$\ln r^m = \ln a^m \cos m\theta$$

$$m \ln r = \ln(a^m) + \ln(\cos m\theta)$$

Differentiate w.r.t θ

$$m.\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta}(-\sin m\theta.\, m)$$

$$\frac{m}{r}.\frac{dr}{d\theta} = -\, m\,\frac{\sin m\theta}{\cos m\theta} \quad \Rightarrow \quad \frac{1}{r}.\frac{dr}{d\theta} = -\tan m\theta \quad \text{or} \quad \frac{dr}{d\theta} = -r\tan m\theta$$

Again, differentiating w.r.t θ

$$\frac{d^2r}{d\theta^2} = -r\sec^2 m\theta.\, m - \tan m\theta \frac{dr}{d\theta} \quad \Rightarrow \quad \frac{d^2r}{d\theta^2} = -mr\sec^2 m\theta.\, m - \tan m\theta(-r\tan m\theta)$$

$$\frac{d^2r}{d\theta^2} = r\tan^2 m\theta - mr\sec^2 m\theta$$

We know that

$$\rho = \frac{\left(r^2+(\frac{dr}{d\theta})^2\right)^{\frac{3}{2}}}{2(\frac{dr}{d\theta})^2-r\frac{d^2r}{d\theta^2}+r^2} \Rightarrow \frac{[r^2+(-r\tan m\theta\ )^2]^{\frac{3}{2}}}{2(-r\tan m\theta\ )^2-r(r\tan^2 m\theta-mr\sec^2 m\theta)+r^2}$$

$$= \frac{[r^2+r^2\tan^2 m\theta]^{\frac{3}{2}}}{r^2-r^2\tan^2 m\theta+mr^2\sec^2 m\theta+2r^2\tan^2 m\theta}$$

$$= \frac{[r^2(1+\tan^2 m\theta)]^{\frac{3}{2}}}{r^2+mr^2\sec^2 m\theta+r^2\tan^2 m\theta} \Rightarrow \frac{r^3[(\sec^2 m\theta)]^{\frac{3}{2}}}{r^2+mr^2\sec^2 m\theta+r^2\tan^2 m\theta}$$

$$= \frac{r\sec^3 m\theta}{(1+m)\sec^2 m\theta} \Rightarrow \frac{r}{(1+m)\cos m\theta}$$

$$= \frac{r}{(1+m)\frac{r^m}{a^m}}$$ since $r^m = a^m\cos m\theta$ or $\cos m\theta = \frac{r^m}{a^m}$

$$\rho = \frac{a^m}{(1+m)r^{m-1}}$$

Question:

Prove that If $r = a\ (1+\cos\theta)$ then $\frac{\rho^2}{r}$ is constant.

Solution:

$r = a\ (1+\cos\theta) \Rightarrow r = a+a\cos\theta$

$\frac{dr}{d\theta} = -a\sin\theta \Rightarrow \frac{d^2r}{d\theta^2} = -a\cos\theta$

$$\rho = \frac{\left[r^2+(\frac{dr}{d\theta})^2\right]^{\frac{3}{2}}}{2(\frac{dr}{d\theta})^2-r\frac{d^2r}{d\theta^2}+r^2} \Rightarrow \frac{[(a+a\cos\theta)^2+(-a\sin\theta)^2]^{\frac{3}{2}}}{2(-a\sin\theta)^2-(a+a\cos\theta)(-a\cos\theta)+(a+a\cos\theta)^2}$$

$$= \frac{[a^2+a^2\cos^2\theta+2a^2\cos^2\theta+a^2\sin^2\theta]^{\frac{3}{2}}}{a^2+a^2\cos^2\theta+2a^2\cos\theta+2a^2\sin^2\theta+a^2\cos\theta+a^2\cos^2\theta}$$

$$= \frac{[a^2+a^2+2a^2\cos\theta]^{\frac{3}{2}}}{a^2+2a^2(\cos^2\theta+\sin^2\theta)+3a^2\cos\theta} \Rightarrow \rho = \frac{[2a^2+2a^2\cos\theta]^{\frac{3}{2}}}{3a^2+3a^2\cos\theta}$$

$$= \frac{(2a^2)^{\frac{3}{2}}[1+\cos\theta]^{\frac{3}{2}}}{3a^2(1+\cos\theta)} \Rightarrow \rho = \frac{(2a^2)^{\frac{3}{2}}[1+\cos\theta]^{\frac{1}{2}}}{3a^2}$$

Taking square on both sides $\rho^2 = \frac{(2a^2)^3(1+\cos\theta)}{9a^4}$

$$\rho^2 = \frac{8a^6\frac{r}{a}}{9a^4} \Rightarrow \frac{\rho^2}{r} = \frac{8a}{9}$$ which is constant

For Parametric Equations:

$$\rho = \frac{\left[(\frac{dx}{dt})^2+(\frac{dy}{dt})^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}$$

Question:

If $x = a(t+\sin t)$, $y = a(1-\cos t)$ then show that $\rho = 4a\cos\frac{t}{2}$

Solution:

$x = a(t+\sin t)$	$y = a(1-\cos t)$
$\frac{dx}{dt} = a(1+\cos t)$ or $(a+a\cos t)$	$\frac{dy}{dt} = a\sin t$
$\frac{d^2x}{dt^2} = -a\sin t$	$\frac{d^2y}{dt^2} = a\cos t$

We know that

$$\rho = \frac{\left[(\frac{dx}{dt})^2+(\frac{dy}{dt})^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}} \Rightarrow \frac{[(a+a\cos t)^2+(a\sin t)^2]^{\frac{3}{2}}}{(a+a\cos t)(a\cos t)-(a\sin t).(-a\sin t)}$$

$$= \frac{[a^2(1+\cos^2 t+2\cos t)+a^2\sin^2 t)]^{\frac{3}{2}}}{a^2(\sin^2 t+\cos^2 t)+a^2\cos t} \Rightarrow \frac{[a^2+a^2\cos^2 t+2a^2\cos t+a^2\sin^2 t)]^{\frac{3}{2}}}{a^2+a^2\cos t}$$

$$= \frac{[2a^2+2a^2\cos t)]^{\frac{3}{2}}}{a^2+a^2\cos t} \Rightarrow \frac{(2a^2)^{\frac{3}{2}}[1+\cos t)]^{\frac{3}{2}}}{a^2(1+\cos t)}$$

$$= \frac{2^{\frac{3}{2}}a^3[1+\cos t)]^{\frac{1}{2}}}{a^2} \Rightarrow 2^{\frac{3}{2}}\, a \left[2\cos^2\frac{t}{2}\right]^{\frac{1}{2}}$$

$$= 2^2 a\cos\frac{t}{2} \Rightarrow \rho = 4a\cos\frac{t}{2}$$

Question:

Find the curvature at $(\frac{3a}{2}, \frac{3a}{2})$ of $x^3+y^3 = 3axy$

Solution:

Let $f(x, y) = x^3+y^3 = 3axy$

$f_x(x, y) = 3x^2 - 3ay$

$f_x(x, y)/(\frac{3a}{2}, \frac{3a}{2}) = 3(\frac{3a}{2})^2 - 3a(\frac{3a}{2}) = \frac{9a^2}{4}$

$f_y(x,y) = 3y^2 - 3ax$

$f_y(x,y)/(\frac{3a}{2},\frac{3a}{2}) \quad = 3(\frac{3a}{2})^2 - 3a(\frac{3a}{2}) = \frac{9a^2}{4}$

$f_{xx}(x,y) = 6x$

$f_{xx}(x,y)/(\frac{3a}{2},\frac{3a}{2}) \quad = 6(\frac{3a}{2}) = 9a$

$f_{yy}(x,y) = 6y$

$f_{yy}(x,y)/(\frac{3a}{2},\frac{3a}{2}) \quad = 6(\frac{3a}{2}) = 9a$

$f_{xy}(x,y) = -3a$

Now formula for Curvature

$$K = \frac{[f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2]}{[(f_x)^2+(f_y)^2]^{\frac{3}{2}}}$$

$$= -\frac{\left[(9a)\left(\frac{9a^2}{4}\right)^2 - 2(\frac{9a^2}{4})(\frac{9a^2}{4})(-3a) + (9a)\left(\frac{9a^2}{4}\right)^2\right]}{\left[\left(\frac{9a^2}{4}\right)^2 + \left(\frac{9a^2}{4}\right)^2\right]^{\frac{3}{2}}}$$

$$= -\frac{\left(\frac{9a^2}{4}\right)^2[9a+6a+9a]}{\left[2\left(\frac{9a^2}{4}\right)^2\right]^{\frac{3}{2}}} \Rightarrow -\frac{\left(\frac{9a^2}{4}\right)^2(24a)}{(2)^{\frac{3}{2}}\left(\frac{9a^2}{4}\right)^3} = -\frac{(24a)}{2\sqrt{2}(\frac{9a^2}{4})}$$

$$= -\frac{(4)(24a)}{2\sqrt{2}\,(9a^2)} \Rightarrow K = -\frac{8\sqrt{2}}{3a}$$

Question:

Show that the radius of curvature at point $x = a\cos^3\theta$, $y = a\sin^3\theta$ is equal to three times length of perpendicular from origin to the tangent.

Solution:

$$x = a\cos^3\theta \quad , \quad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad , \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} \Rightarrow \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$\frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta} \Rightarrow \frac{dy}{dx} = -\tan\theta$$

$\frac{d^2y}{dx^2} = -\sec^2\theta.\frac{d\theta}{dx}$ $\Rightarrow$ $\frac{d^2y}{dx^2}$ = -$\sec^2\theta.\frac{1}{-3a\cos^2\theta\sin\theta}$

$\frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta\sin\theta}$

We Know

$\rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ $\Rightarrow$ $\rho = \frac{[1+(-\tan\theta)^2]^{\frac{3}{2}}}{\frac{1}{3a\cos^4\theta\sin\theta}}$

$\rho = \quad [1 + \tan^2\theta]^{\frac{3}{2}}.\, 3a\cos^4\theta\sin\theta$

$\rho = \quad [\sec^2\theta]^{\frac{3}{2}}.\, 3a\cos^4\theta\sin\theta$ $\Rightarrow$ $\sec^3\theta.\, 3a\cos^4\theta\sin\theta$

$\rho = \quad 3a\sin\theta\cos\theta$ (i)

Now we find the equation of tangent.

Equation of tangent at ($a\cos^3\theta$,$a\sin^3\theta$)

Since $y - y_1 = m(x - x_1)$

$y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta)$

$y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$

$y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\sin\theta\cos^3\theta$

$x\sin\theta + y\cos\theta - a\sin^3\theta\cos\theta + -a\sin\theta\cos^3\theta = 0$

$x\sin\theta + y\cos\theta - a\sin\theta\cos\theta(\cos^2\theta + \sin^2\theta) = 0$

$x\sin\theta + y\cos\theta - a\sin\theta\cos\theta = 0$ (ii)

Let d be the length of perpendicular from (0,0) to the tangent line.

Since $d = \frac{|Ax_1+By_{1+c}|}{\sqrt{A^2+B^2}}$

$d = \frac{|(0)(\sin\theta)+(0)(\cos\theta)-a\sin\theta\cos\theta|}{\sqrt{\sin^2\theta+\cos^2\theta}}$ $\Rightarrow$ $d = \frac{|-a\sin\theta\cos\theta|}{\sqrt{1}}$

$d = a\sin\theta\cos\theta$

from (i) $\rho = \quad 3a\sin\theta\cos\theta$

$\rho = \quad 3d$

Hence radius of curvature is equal to three times length of perpendicular from origin to tangent.

Lecture # 3

Space curve or Twisted curve or Skew curve:

When all the points denoted the curve lies in the same plane is said to be a plain curve otherwise is called space curve or twisted curve or skew curve.

Tangent line:

The line which cut the curve at one point is called tangent line.

Normal line:

The line which is perpendicular to the tangent line is called Normal line.

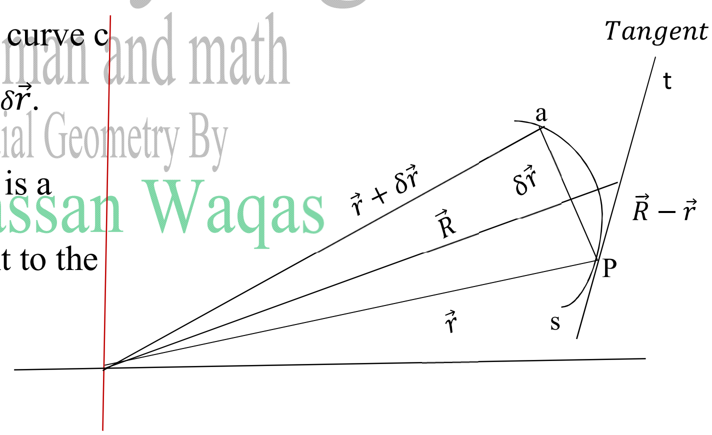
Secant line:

The line which cut the curve at two points is called secant line.

Equation of tangent of a point:

Let P and a be any two points on the curve c whose position vector are $\vec{r}$ and $\vec{r}+\delta\vec{r}$.

We note that the limiting value if $\frac{\delta\vec{r}}{\delta s}$ is a unit vector and parallel to the tangent to the curve at point P.



Let tangent = $\vec{t} = \lim_{\delta s \to 0} \frac{\delta\vec{r}}{\delta s}$

$$\vec{t} = \frac{d\vec{r}}{ds} \quad , \quad \vec{t} = \vec{r}'$$

Let $\vec{R}$ be the position vector of any point on the tangent line. If u is the variable number +ve or -ve then we know that $\vec{R}-\vec{r}$ is parallel to $\vec{t}$ then

$$\vec{R}-\vec{r} = u\vec{t}$$

$$\vec{R} = \vec{r} + u\vec{t}$$

$$\vec{R} = \vec{r} + u\vec{r}' \qquad \because \ \vec{t} = \vec{r}'$$

which is equation of tangent. Similarly, equation of normal is $\vec{R} = \vec{r} + u\vec{n}$ and equation of binomial is $\vec{R} = \vec{r} + u\vec{b}$. The line which is perpendicular on both tangent and normal line is called binomial line.

Question:

Find the equation of tangent to the curve whose coordinates are

x = acosθ , y = asinθ and z = 0

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\vec{r}' = x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k}$

And $\vec{R} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$

$x' = -a\sin\theta.\theta'$, $y' = a\cos\theta.\theta'$, $z' = 0$

We know the equation of tangent

$$\vec{R} - \vec{r} = u\vec{r}'$$

$$\Rightarrow \quad \frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'} = u$$

$$\frac{X-a\cos\theta}{-a\sin\theta.\theta'} = \frac{Y-a\sin\theta}{a\cos\theta.\theta'} = \frac{Z-0}{0} = u$$

$$\frac{X-a\cos\theta}{-a\sin\theta.\theta'} = \frac{Y-a\sin\theta}{a\cos\theta.\theta'}$$

$$\cos\theta(X - a\cos\theta) = -\sin\theta(Y - a\sin\theta)$$

$$X\cos\theta - a\cos^2\theta = -Y\sin\theta + a\sin^2\theta$$

$$X\cos\theta + Y\sin\theta = a$$

Question:

Find the equation of tangent whose coordinates are

x = -bcosθ , y = bsinθ and z = 0

Solution:

Given the coordinates are

x = -bcosθ , y = bsinθ and z = 0

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\vec{r}' = x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k}$

And $\vec{R} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$

$x' = b\sin\theta.\theta'$, $y' = b\cos\theta.\theta'$, $z' = 0$

We know the equation of tangent

$$\vec{R}-\vec{r}=u\vec{r}'$$

$$\Rightarrow \quad \frac{X-x}{x'}=\frac{Y-y}{y'}=\frac{Z-z}{z'}=\text{u}$$

$$\frac{X+b\cos\theta}{b\sin\theta.\theta'}=\frac{Y-b\sin\theta}{b\cos\theta.\theta'}=\frac{Z-0}{0}=\text{u}$$

$$\frac{X+b\cos\theta}{b\sin\theta.\theta'}=\frac{Y-b\sin\theta}{b\cos\theta.\theta'}$$

$$X\cos\theta+b\cos^2\theta = Y\sin\theta-\sin^2\theta$$

$$Y\sin\theta - X\cos\theta = b$$

Normal plane:

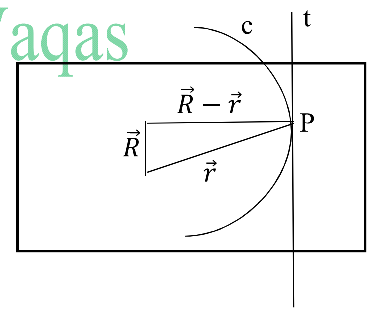
A plane passing through the point P and perpendicular to the tangent at point P is called normal plane.

Equation of Normal plane:

Let $\vec{r}$ be the position vector of point on the curve $\vec{R}-\vec{r}$ is the position vector at any line in the plane. According to the definition of normal plane $\vec{R}-\vec{r}$ and $\vec{t}$ are perpendicular to each other.

$$\vec{R}-\vec{r}\;.\;\vec{t}=0$$

Which is the equation of normal plane.



Oscillating plane or plane of Curvature:

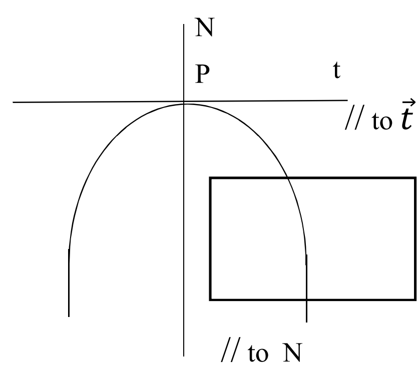
The plane parallel to the $\vec{t}$ and normal $\vec{N}$at a point P on the curve c.

If $\vec{R}$ be the position vector of any point on the plane

the $\vec{R}-\vec{r}$, $\vec{t}$ and $\vec{n}$ are coplanar vector.

$$[\,\vec{R}-\vec{r}\,,\vec{t}\,,\vec{n}]=0 \qquad (1)$$

or $\quad \vec{R}-\vec{r}\;.\;\vec{t}\;\times\vec{n}=0$



which is the equation of oscillating plane. It can also be expressed as $\vec{t}=\vec{r}'$

$$\vec{t}'=\vec{r}'' \qquad \Rightarrow \qquad K\vec{n}=\vec{r}'' \qquad \because \qquad \vec{t}'=K\vec{n}$$

$$\vec{n} = \frac{\vec{r}''}{K}$$

So, Equation (1) $\Rightarrow$ $[\,\vec{R}-\vec{r}\,,\vec{t}\,,\frac{\vec{r}''}{K}\,] = 0$

Which is the equation of oscillating plane in terms of $\vec{r}$ and its derivative.

In Cartesian form the equation of oscillating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

Question:

For the curve x =3t , y = $3t^2$ and z = $2t^3$. Find the equation of oscillating plane.

Solution:

Since $\vec{r} = [\,x, y, z]$

$\vec{r} = [\,3t, 3t^2, 2t^3]$

$\vec{r}' = [\,3, 6t, 6t^2]$

$\vec{r}'' = [\,0\,, 6\,, 12t]$

Now $\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$

$$\begin{vmatrix} X-3t & Y-3t^2 & Z-2t^3 \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix} = 0$$

$(X-3t)[72t^2 - 36t^2] - (Y-3t^2)[36t-0] + (Z-2t^3)[18-0] = 0$

$(X-3t)[36t^2] - (Y-3t^2)[36t] + (Z-2t^3)[18] = 0$

$36t^2X - 108t^3 - 36tY + 108t^3 + 18Z - 36t^3 = 0$

$36t^2X - 36tY + 18Z - 36t^3 = 0$

$18[2t^2X - 2tY + Z - 2t^3] = 0$

$2t^2X - 2tY + Z - 2t^3 = 0$

$2t^2X - 2tY + Z = 2t^3$

Question:

Find the equation of oscillating plane if

$\vec{r}(s) = acoss\hat{\imath} + asins\hat{\jmath} + 0\hat{k}$

$\vec{r}'(s) = -asins\hat{\imath} + acoss\hat{\jmath} + 0\hat{k}$

$\vec{r}''(s) = -acoss\hat{\imath} - asins\hat{\jmath} + 0\hat{k}$

We know the equation of oscillating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

$$\begin{vmatrix} X-acoss & Y-asins & Z-0 \\ -asins & acoss & 0 \\ -acoss & -asins & 0 \end{vmatrix} = 0$$

(X-acosx)(0-0)-(Y-asins)(0-0)+Z($a^2sin^2s + a^2cos^2s$) = 0

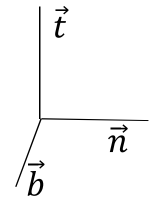
a^2z $(sin^2s + cos^2s) = 0$

a^2z = 0

z = 0

Lecture # 4

Frenet-Serret Formula:

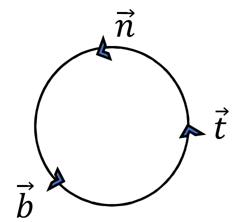


(i) $\vec{t}\,' = K\vec{n}$

(ii) $\vec{n}\,' = \tau\vec{b} - K\vec{t}$

(iii) $\vec{b}\,' = -\tau\vec{n}$

The vector $\vec{t}$, $\vec{n}$ and $\vec{b}$ are perpendicular to each other. From the right-handed system.



$\vec{b} \times \vec{t} = \vec{n}$

$\vec{t} \times \vec{n} = \vec{b}$

$\vec{n} \times \vec{b} = \vec{t}$

And $\vec{t} \cdot \vec{n} = 0$, $\vec{n} \cdot \vec{b} = 0$, $\vec{b} \cdot \vec{t} = 0$

$\vec{n} \cdot \vec{n} = 1$, $\vec{t} \cdot \vec{t} = 1$, $\vec{b} \cdot \vec{b} = 1$

or $|\vec{n}|^2 = 1$, $|\vec{t}|^2 = 1$, $|\vec{b}|^2 = 1$

$\vec{n} \times \vec{n} = 1$, $\vec{t} \times \vec{t} = 1$, $\vec{b} \times \vec{b} = 1$

Proof:

$\vec{b}\,' = -\tau\vec{n}$

Consider $\vec{b} \cdot \vec{b} = 1$

Diff. w.r.t 's'

$$\vec{b} \cdot \frac{d\vec{b}}{ds} + \frac{d\vec{b}}{ds} \cdot \vec{b} = 0$$

$$2\vec{b} \cdot \frac{d\vec{b}}{ds} = 0 \quad \Rightarrow \quad \vec{b} \cdot \frac{d\vec{b}}{ds} = 0$$

$\vec{b} \cdot \vec{b}\,' = 0 \quad \Rightarrow \quad \vec{b}$ and $\vec{b}\,'$ are $\perp$ to each other.

Now $\vec{t} \cdot \vec{b} = 0$

Diff. w.r.t 's'

$$\vec{t} \cdot \frac{d\vec{b}}{ds} + \frac{d\vec{t}}{ds} \cdot \vec{b} = 0 \quad \Rightarrow \quad \vec{t} \cdot \vec{b}\,' + \vec{t}\,' \cdot \vec{b} = 0$$

As in (i) $\vec{t}\,' = K\vec{n} \quad \Rightarrow \quad \vec{t} \cdot \vec{b}\,' + K\vec{n} \cdot \vec{b} = 0$

$\vec{t}\,.\,\vec{b}' + K(0) = 0 \qquad \because \vec{n}\,.\vec{b} = 0$

$\vec{t}\,.\,\vec{b}' = 0 \qquad \Rightarrow \vec{t}$ is $\perp$ to $\vec{b}'$.

$\vec{b}$ is $\perp$ to $\vec{b}'$and $\vec{t}$ is $\perp$ to $\vec{b}'$. But $\vec{n}$ is $\perp$ to tangent $\vec{t}$ and $\vec{n}$.

$\vec{b}'$ is parallel to $\vec{n}$

$$\vec{b}' = -\tau\vec{n}$$

Where τ measure the magnitude of arc rate of rotation of binomial and -ve sign indicate that vector along the normal but opposite direction.

$\boldsymbol{\vec{n}' = \tau\vec{b} - K\vec{t}}$

Consider $\vec{b} \times \vec{t} = \vec{n}$

Diff. w.r.t 's'

$\vec{b}\,.\,\frac{d\vec{t}}{ds} + \frac{d\vec{b}}{ds}\,.\,\vec{t} = \frac{d\vec{n}}{ds}$

$\vec{n}' = \vec{b}\,.\,\vec{t}' + \vec{b}'.\,\vec{t}$

$= \vec{b}\,.\,(K\vec{n}) + (-\tau\vec{n}).\,\vec{t}$

$= K(\vec{b} \times \vec{n}) - \tau(\vec{n} \times \vec{t})$

$= K(-\vec{t}) - \tau(-\vec{b}) \qquad \because \vec{b} \times \vec{n} = -\vec{t} \quad , \vec{n} \times \vec{t} = -\vec{b}$

$\vec{n}' = \tau\vec{b} - K\vec{t}$

Torsion:

Rate of turning of binormal is called torsion of the curve at point P.

Question:

Prove that tangent

(i) $\vec{t}'' = \vec{r}''' = K'\vec{n} - K^2\vec{t} + K\tau\vec{b}$

(ii) $\vec{r}'^{v} = \vec{t}''' = (K'' - K^3 - K\tau^2)\,\vec{n} - 3KK'\vec{t} + (2K'\tau + K\tau')\vec{b}$

Proof:

Since $\vec{r} = \vec{r}(s)$

$\Rightarrow \qquad \vec{r}' = \frac{d\vec{r}}{ds} = \vec{t} \qquad \because \vec{t} = \frac{d\vec{r}}{ds}$

$\Rightarrow \qquad \vec{r}' = \vec{t}$

$$\Rightarrow \qquad \vec{r}'' = \vec{t}' \qquad (1)$$

$$\vec{r}''' = \vec{t}'' = K\vec{n}' + K'\vec{n}$$

$$= K(\tau\vec{b} - K\vec{t}) + K'\vec{n} \qquad \because \vec{n}' = \tau\vec{b} - K\vec{t}$$

$$\vec{r}''' = \vec{t}'' = K'\vec{n} - K^2\vec{t} + K\tau\vec{b}$$

Now again diff. w.r.t 's'

$$\vec{r}'^{v} = \vec{t}''' = K'\vec{n}' + K''\vec{n} - 2KK'\vec{t} - K^2\vec{t}' + K'\tau\vec{b} + K\tau'\vec{b} + K\tau\vec{b}'$$

$$= K'(\tau\vec{b} - K\vec{t}) + K''\vec{n} - 2KK'\vec{t} - K^2(K\vec{n}) + K'\tau\vec{b} + K\tau'\vec{b} + K\tau(-\tau\vec{n})$$

$$= K'\tau\vec{b} - KK'\vec{t} + K''\vec{n} - 2KK'\vec{t} - K^3\vec{n} + K'\tau\vec{b} + K\tau'\vec{b} - K\tau^2\vec{n}$$

$$\vec{r}'^{v} = \vec{t}''' = (K'' - K^3 - K\tau^2)\,\vec{n} - 3KK'\vec{t} + (2K'\tau + K\tau')\vec{b}$$

Question:

(i) If $\vec{b}' = -\tau\vec{n}$ find $\vec{b}''$ and $\vec{b}'''$

(ii) If $\vec{n}' = \tau\vec{b} - K\vec{t}$ find $\vec{n}''$ and $\vec{n}'''$

Solution: (i)

$$\vec{b}' = -\tau\vec{n}$$

$$\vec{b}'' = -\tau'\vec{n} - \tau\vec{n}'$$

$$= -\tau'\vec{n} - \tau(\tau\vec{b} - K\vec{t})$$

$$= -\tau'\vec{n} - \tau^2\vec{b} + K\tau\vec{t}$$

$$= K\tau\vec{t} - \tau'\vec{n} - \tau^2\vec{b}$$

Again diff. w.r.t 's'

$$\vec{b}''' = K'\tau\vec{t} + K\tau'\vec{t} + K\tau\vec{t}' - \tau''\vec{n} - \tau'\vec{n}' - 2\tau\tau'\vec{b} - \tau^2\vec{b}'$$

$$\vec{b}''' = K'\tau\vec{t} + K\tau'\vec{t} + K\tau(K\vec{n}) - \tau''\vec{n} - \tau'(\tau\vec{b} - K\vec{t}) - 2\tau\tau'\vec{b} - \tau^2(-\tau\vec{n})$$

$$\vec{b}''' = K'\tau\vec{t} + K\tau'\vec{t} + K^2\tau\vec{n} - \tau''\vec{n} - \tau'\tau\vec{b} + \tau'K\vec{t} - 2\tau\tau'\vec{b} + \tau^3\vec{n}$$

$$\vec{b}''' = 2\,K\tau'\vec{t} + K'\tau\vec{t} + K^2\tau\vec{n} - \tau''\vec{n} + \tau^3\vec{n} - 3\tau\tau'\vec{b}$$

$$\vec{b}''' = (2\,K\tau' + K'\tau)\,\vec{t} + (K^2\tau - \tau'' + \tau^3)\vec{n} - 3\tau\tau'\vec{b}$$

Solution: (ii)

$$\vec{n}' = \tau\vec{b} - K\vec{t}$$

$$\Rightarrow \quad \vec{n}'' = \tau'\vec{b} + \tau\vec{b}' - K'\vec{t} - K\vec{t}'$$

$$= \tau'\vec{b} + \tau(-\tau\vec{n}) - K'\vec{t} - K(K\vec{n})$$

$$= \tau'\vec{b} - \tau^2\vec{n} - K'\vec{t} - K^2\vec{n}$$

Diff. w.r.t 's'

$$\vec{n}''' = \tau''\vec{b} + \tau'\vec{b}' - 2\tau\tau'\vec{n} - \tau^2\vec{n}' - K''\vec{t} - K'\vec{t}' - 2KK'\vec{n} - K^2\vec{n}'$$

$$= \tau''\vec{b} + \tau'(-\tau\vec{n}) - 2\tau\tau'\vec{n} - \tau^2(\tau\vec{b} - K\vec{t}) - K''\vec{t} - K'(K\vec{n}) - 2KK'\vec{n} - K^2(\tau\vec{b} - K\vec{t})$$

$$= \tau''\vec{b} - 3\tau\tau'\vec{n} - \tau^3\vec{b} + \tau^2 K\vec{t} - K''\vec{t} - 3KK'\vec{n} - \tau K^2\vec{b} + K^3\vec{t}$$

$$= (\tau'' - \tau^3 - K^2\tau)\vec{b} + (K\tau^2 + K^3 - K'')\vec{t} + (-3\tau\tau' - 3KK')\vec{n}$$

Question:

Prove that $\tau = \frac{1}{K^2}[\vec{r}', \vec{r}'', \vec{r}''']$

Solution:

Since $\vec{r} = \vec{r}(s)$

$$\vec{r}' = \frac{d\vec{r}}{ds} = \vec{t} \qquad \because \vec{t} = \frac{d\vec{r}}{ds}$$

$$\Rightarrow \quad \vec{r}'' = \vec{t}' = K\vec{n}$$

$$\Rightarrow \quad \vec{r}''' = \vec{t}'' = K\vec{n}' + K'\vec{n}$$

$$= K(\tau\vec{b} - K\vec{t}) + K'\vec{n}$$

$$= K'\vec{n} + K\tau\vec{b} - K^2\vec{t}$$

Now $[\vec{r}', \vec{r}'', \vec{r}'''] = \vec{r}' \times \vec{r}'' . \vec{r}'''$

$$= \vec{t} \times K\vec{n} . K'\vec{n} + K\tau\vec{b} - K^2\vec{t}$$

$$= K(\vec{t} \times \vec{n}) . [K'\vec{n} + K\tau\vec{b} - K^2\vec{t}]$$

$$= K(\vec{b}) . [K'\vec{n} + K\tau\vec{b} - K^2\vec{t}]$$

$$= KK'\vec{b}.\vec{n} + K.K\tau\vec{b}.\vec{b} - KK^2\vec{b}.\vec{t}$$

$$[\vec{r}', \vec{r}'', \vec{r}'''] = KK'(0) + K.K\tau(1) - KK^2(0) = K^2\tau$$

$$\tau = \frac{1}{K^2}[\vec{r}', \vec{r}'', \vec{r}''']$$

Question: Prove that position vector on the curve satisfied the differential equation.

$$\frac{d}{ds}\left[\sigma\frac{d}{ds}\left(\rho\frac{d^2\vec{r}}{ds^2}\right)\right]+\frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{d\vec{r}}{ds}\right]+\frac{\rho}{\sigma}\frac{d^2\vec{r}}{ds^2}=0$$

Solution:

Since $\vec{r}=\vec{r}(s)$

$$\vec{r}\,'=\frac{d\vec{r}}{ds}=\vec{t} \qquad \because \vec{t}=\frac{d\vec{r}}{ds}$$

$$\Rightarrow \quad \vec{r}\,''=\vec{t}\,'=K\vec{n}=\frac{d^2\vec{r}}{ds^2}$$

$$\text{L.H.S}=\frac{d}{ds}\left[\sigma\frac{d}{ds}(\rho K\vec{n})\right]+\frac{d}{ds}\left[\frac{\sigma}{\rho}\vec{t}\right]+\frac{\rho}{\sigma}K\vec{n}$$

Since $K=\frac{1}{\rho}$, $\tau=\frac{1}{\sigma}$

$$=\frac{d}{ds}\left[\sigma\frac{d}{ds}(\vec{n})\right]+\frac{d}{ds}\left[\frac{\sigma}{\rho}\vec{t}\right]+\tau\,\vec{n}$$

$$=\frac{d}{ds}\left[\sigma\vec{n}\,'\right]+\frac{d}{ds}\left[\sigma K\vec{t}\right]+\tau\,\vec{n}$$

$$=\frac{d}{ds}\left[\sigma(\tau\vec{b}-K\vec{t})\right]+\frac{d}{ds}\left[\sigma K\vec{t}\right]+\tau\,\vec{n}$$

$$=\frac{d}{ds}\left[(1)\vec{b}-\sigma K\vec{t})\right]+\frac{d}{ds}\left[\sigma K\vec{t}\right]+\tau\,\vec{n} \qquad \because \sigma\tau=1$$

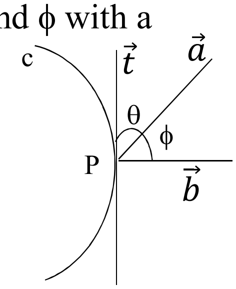
$$=\vec{b}\,'-\frac{d(\sigma K\vec{t})}{ds}+\frac{d(\sigma K\vec{t})}{ds}+\tau\,\vec{n}$$

$$=-\tau\vec{n}+\tau\,\vec{n}$$

$$=0=\text{R.H.S} \qquad \because \vec{b}\,'=-\tau\vec{n}$$

Theorem:

If tangent and binormal at a point P of a curve make angle θ and ϕ with a fix direction then prove that



$$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}=-\frac{K}{\tau}$$

Proof:

Let c be the curve with point P on it and $\vec{a}$ be a fixed direction

with magnitude making angle θ and ϕ with tangent and normal respectively.

Consider

$\vec{t}.\vec{a} = |\vec{t}||\vec{a}|\cos\theta = 1.1\cos\theta$

$\vec{t}.\vec{a} = \cos\theta$ (1)

Diff. w.r.t 's'

$\vec{t}'.\vec{a} + \vec{t}.\vec{a}' = -\sin\theta\frac{d\theta}{ds}$

$\vec{t}'.\vec{a} + \vec{t}.0 = -\sin\theta\frac{d\theta}{ds}$ $\because \vec{a}' = 0$ as $\vec{a}$ is fixed

$K\vec{n}.\vec{a} = -\sin\theta\frac{d\theta}{ds}$ (2) $\vec{t}' = K\vec{n}$

Now

$\vec{b}.\vec{a} = |\vec{b}||\vec{a}|\cos\phi = 1.1\cos\phi$

$\vec{b}.\vec{a} = \cos\phi$

Diff. w.r.t 's'

$\vec{b}'.\vec{a} + \vec{b}.0 = -\sin\phi\frac{d\phi}{ds}$ since $\vec{a}' = 0$ as $\vec{a}$ is fixed

$-\tau\vec{n}.\vec{a} = -\sin\phi\frac{d\phi}{ds}$ (3)

Divide (2) by (3)

$$\frac{K\vec{n}.\vec{a}}{-\tau\vec{n}.\vec{a}} = \frac{-\sin\theta\frac{d\theta}{ds}}{-\sin\phi\frac{d\phi}{ds}} \quad \Rightarrow \quad \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{K}{\tau}$$

Question: Prove that if K = 0 at all points then that curve is straight line.

Proof: We know that

$\vec{t}' = K\vec{n}$

$= (0)\vec{n} = 0$ On integrating

$\Rightarrow$ $\vec{t} = a$ (tangent is fix which is possible only when curve is straight line.

Question: Prove that if $\tau = 0$ at all points then that curve is plane curve.

Proof: We know that $\vec{b}' = -\tau\vec{n}$ $\Rightarrow$ $\vec{b}' = -(0)\vec{n} = 0$

On integrating $\vec{b} = a$

Binormal is fix which is possible only when curve is plane curve.

Lecture # 5

Parameters other than ‘s’

If the position vector r is a function of any other parameter u then prove that

(i) $\vec{b} = \dfrac{\vec{r}' \times \vec{r}''}{K(s')^3}$ (ii) $\vec{n} = \dfrac{s'\vec{r}'' \times s''\vec{r}}{K(s')^3}$

(iii) $\mathrm{K} = \dfrac{|\vec{r}' \times \vec{r}''|}{|s'|^3}$ (iv) $K^2 = \dfrac{(\vec{r}'')^2 - (s'')^2}{(s')^4}$

(v) $\tau = \dfrac{[\vec{r}', \vec{r}'', \vec{r}''']}{K^2(s')^6}$

Solution:

Since $\vec{r} = \vec{r}(u)$

$$\vec{r}' = \frac{d\vec{r}}{du} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{du}$$

$$\vec{r}' = \vec{t} \,.\, s' \qquad (1) \qquad \because \vec{t} = \frac{d\vec{r}}{ds} \text{ and } \frac{ds}{du} = s'$$

$$\Rightarrow \vec{r}'' = \frac{d\vec{t}}{du} \cdot s' + \vec{t} \cdot \frac{ds'}{du}$$

$$= \frac{d\vec{t}}{du} \cdot s' + \vec{t} \,.\, s''$$

$$\vec{r}'' = \vec{t}' \,.\, s' \,.\, s' + \vec{t} \,.\, s''$$

$$\vec{r}'' = \mathrm{K}\vec{n} \,.\, s'^2 + \vec{t} \,.\, s'' \qquad (2) \qquad \because \vec{t}' = \mathrm{K}\vec{n}$$

Again differentiating w.r.t. ‘u’

$$\vec{r}''' = K'\vec{n} \,.\, (s')^2 + \mathrm{K}\frac{d\vec{n}}{du} \,.\, s'^2 + \mathrm{K}\vec{n} . 2s's'' + \frac{d\vec{t}}{du} . s'' + \vec{t} \,.\, s'''$$

$$\vec{r}''' = K'\vec{n}s'^2 + \mathrm{K}\frac{d\vec{n}}{ds} . \frac{ds}{du} s'^2 + 2\mathrm{K}\vec{n} . s's'' + \frac{d\vec{t}}{ds} . \frac{ds}{du} s'' + \vec{t} \,.\, s'''$$

$$= K'\vec{n}s'^2 + \mathrm{K}\vec{n}' . s's'^2 + 2\mathrm{K}\vec{n} . s's'' + \mathrm{K}\vec{n} . s's'' + \vec{t} \,.\, s'''$$

$$= K'\vec{n}s'^2 + \mathrm{K}(\tau\vec{b} - \mathrm{K}\vec{t}) . s'^3 + 2\mathrm{K}\vec{n} . s's'' + \mathrm{K}\vec{n} . s's'' + \vec{t} \,.\, s'''$$

since $\vec{n}' = \tau\vec{b} - \mathrm{K}\vec{t}$, $\vec{t}' = \mathrm{K}\vec{n}$

$$= K'\vec{n}s'^2 + \mathrm{K}\tau\vec{b}s'^3 - K^2\vec{t}s'^3 + 3\mathrm{K}\vec{n} . s's'' + \vec{t}\ s'''$$

Now

(i) $\vec{b} = \dfrac{\vec{r}' \times \vec{r}''}{K(s')^3} \quad \Rightarrow \quad \vec{r}' \times \vec{r}'' = (\vec{t} \,.\, s') \times \mathrm{K}\vec{n}\, s'^2 + \vec{t}\, s''$

$$\vec{r}' \times \vec{r}'' = \mathrm{K}s'^3(\vec{t} \times \vec{n}) + (\vec{t} \times \vec{t})s's''$$

$\vec{r}' \times \vec{r}'' = Ks'^3(\vec{b}) + (0)s's''$ since $\vec{t} \times \vec{t} = 0$, $\vec{t} \times \vec{n} = \vec{b}$

$$\Rightarrow \quad \vec{b} = \frac{\vec{r}' \times \vec{r}''}{K(s')^3}$$

(ii) $\vec{n} = \frac{s'\vec{r}'' \times s''\vec{r}}{K(s')^3}$

$$s'\vec{r}'' \times s''\vec{r} = s'\,(K\vec{n}\,.\,s'^2 + \vec{t}\,.\,s'') - s''(\vec{t}\,.\,s')$$

$$= K\vec{n}s'^3 + \vec{t}\,s's'' - \vec{t}s's''$$

$$s'\vec{r}'' \times s''\vec{r} = K\vec{n}s'^3$$

$$\Rightarrow \quad \vec{n} = \frac{s'\vec{r}'' \times s''\vec{r}}{K(s')^3}$$

(iii) $K = \frac{|\vec{r}' \times \vec{r}''|}{|s'|^3}$

$$|\vec{r}' \times \vec{r}''| = |Ks'^3\vec{b}| = |K||s'^3||\vec{b}|$$

$$|\vec{r}' \times \vec{r}''| = K|s'^3| \quad \text{since } |\vec{b}| = 1$$

$$\Rightarrow K = \frac{|\vec{r}' \times \vec{r}''|}{|s'|^3}$$

(iv) $K^2 = \frac{(\vec{r}'')^2 - (s'')^2}{(s')^4}$

$$\Rightarrow (\vec{r}'')^2 = (K\vec{n}\,.\,s'^2 + \vec{t}\,.\,s'').\,(K\vec{n}\,.\,s'^2 + \vec{t}\,.\,s'')$$

$$= K^2(s')^4(\vec{n}.\vec{n}) + Ks'^2s''(\vec{n}.\vec{t}) + Ks'^2s''(\vec{t}.\vec{n}) + (s'')^2(\vec{t}.\vec{t})$$

$$= K^2(s')^4(1) + Ks'^2s''(0) + Ks'^2s''(0) + (s'')^2(1)$$

$$= K^2(s')^4 + (s'')^2$$

$$(\vec{r}'')^2 - (s'')^2 = K^2(s')^4$$

$$\Rightarrow \quad K^2 = \frac{(\vec{r}'')^2 - (s'')^2}{(s')^4}$$

(v) $\tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{K^2(s')^6}$

$$[\vec{r}', \vec{r}'', \vec{r}'''] = \vec{r}' \times \vec{r}''.\vec{r}'''$$

$$= Ks'^3\vec{b}.[\,K\vec{n}s'^2 + K\tau\vec{b}s'^3 - K\vec{t}s'^3 + 3K\vec{n}s's'' + \vec{t}s'''\,]$$

$$= KK's'^5(\vec{b}.\vec{n}) + K^2\tau s'^6(\vec{b}.\vec{b}) - K^3s'^6(\vec{b}.\vec{t}) + 3K^2s'^4s''(\vec{b}.\vec{n}) + Ks'^3s'''(\vec{b}.\vec{t})$$

As $\vec{b}.\vec{b} = 1$, $\vec{b}.\vec{n} = 0$, $\vec{b}.\vec{t} = 0$

$= KK's'^5(0) + K^2\tau s'^6(1) - K^3 s'^6(0) + 3K^2 s'^4 s''(0) + Ks'^3 s'''(0)$

$$[\vec{r}', \vec{r}'', \vec{r}'''] = K^2 \tau s'^6$$

$$\Rightarrow \qquad \tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{K^2 (s')^6}$$

Theorem:

For $x = a(3u - u^3) = 3au - au^3$, $y = 3au^2$, $z = a(3u + u^3) = 3au + au^3$

Then prove that K = τ.

Solution:

We know that

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

And $\tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{K^2 (s')^6}$

$$\Rightarrow \qquad \tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{K^2 (\vec{r}')^6}$$

$$\tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{K^2 |\vec{r}'|^6}$$

Note:

$\vec{r} = \vec{r}(u)$

$\vec{r}' = \frac{d\vec{r}}{du} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{du}$

$\vec{r}' = \vec{t} \cdot s'$

$= |\vec{t} \cdot s'| = |\vec{t}| . |s'|$

$|\vec{r}'| = |s'|$ since $|\vec{t}| = 1$

Now $\vec{r} = (x, y, z)$

$\vec{r} = (3au - au^3, 3au^2, 3au + au^3)$

Diff. w.r.t 'u'

$\vec{r}' = (3a - 3au^2, 6au, 3a + 3au^2)$

$\vec{r}'' = (-6au, 6a, 6au)$

$\vec{r}''' = (-6a, 0, 6a)$

Now $|\vec{r}'| = \sqrt{(3a - 3au^2)^2 + (6au)^2 + (3a + 3au^2)^2}$

$= \sqrt{18a^2 + 18a^2u^4 + 36a^2u^2} \quad \Rightarrow \sqrt{18a^2(1 + u^4 + 2u^2)}$

$= \sqrt{18a^2(1 + u^2)^2} \quad \Rightarrow \quad 3\sqrt{2}\, a(1 + u^2) \qquad (1)$

Now

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} i & j & k \\ 3a - 3au^2 & 6au & 3a + 3au^2 \\ -6au & 6a & 6au \end{vmatrix}$$

$\vec{r}'\times\vec{r}'' = 18a^2\{(u^2-1)i - 2uj + (u^2+1)k\}$

$|\vec{r}'\times\vec{r}''| = \sqrt{(18a^2)^2\{(u^2-1)^2 + (-2u)^2 + (u^2+1)^2\}}$

$= 18a^2\sqrt{\{1+u^4-2u^2+4u^2+1+u^4+2u^2\}}$

$= 18a^2\sqrt{2}(1+u^2)$

$K = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3} \Rightarrow \frac{18a^2\sqrt{2}(1+u^2)}{|3\sqrt{2}\,a(1+u^2)|^3}$

$= \frac{18\sqrt{2}}{27\times2\sqrt{2}a(1+u^2)^2} \Rightarrow \frac{18}{54a(1+u^2)^2}$

$K = \frac{1}{3a(1+u^2)^2}$ (2)

Now $[\vec{r}',\vec{r}'',\vec{r}'''] = \vec{r}'\times\vec{r}''.\vec{r}'''$

$= 18a^2\{(u^2-1)i - 2uj + (u^2+1)k\}.(-6a\, i + 0j + 6a\, k)$

$= 18a^2\{-6a(u^2-1) - 2u(0) + (u^2+1)6a\}$

$= 18a^2(12a) = 216a^3$

$|\vec{r}'|^6 = \left[3\sqrt{2}\,a(1+u^2)\right]^6 = 729\left(2^{6\times\frac{1}{2}}\right)a^6(1+u^2)^6$

$= 29\,(2^3)a^6(1+u^2)^6$

$= 5832a^6(1+u^2)^6$

$\tau = \frac{[\vec{r}',\vec{r}'',\vec{r}''']}{K^2|\vec{r}'|^6}$

$= \frac{216a^3}{\left(\frac{1}{3a(1+u^2)^2}\right)^2.5832a^6(1+u^2)^6} = \frac{216a^3\times9a^2(1+u^2)^4}{5832a^6(1+u^2)^6}$

$= \frac{1944}{5832a(1+u^2)^2}$

$\tau = \frac{1}{3a(1+u^2)^2}$ (3)

From (2) and (3)

$K = \tau$ Hence proved

Lecture # 6

Question:

For $x = 4a\cos^3 u$, $y = 4a\sin^3 u$, $z = 3c\cos 2u$

Then prove that

$\vec{n} = (\sin u, \cos u, 0)$ and $K = \frac{a}{6\sin 2u(a^2+c^2)}$

Solution:

$\vec{r} = (x, y, z)$

$= (4a\cos^3 u\ , 4a\sin^3 u\ , \ 3c\cos 2u)$

Diff. w.r.t 's'

$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{du} \cdot \frac{du}{ds}$ since $\vec{t} = \frac{d\vec{r}}{ds}$

$\Rightarrow \quad \vec{t} = \frac{d((4a\cos^3 u\ , 4a\sin^3 u\ , 3c\cos 2u)}{du} \cdot \frac{du}{ds}$

$= (-12a\cos^2 u\sin u\ , 12a\sin^2 u\cos u\ , -6c\sin 2u\ ).u'$

$= \ (-6a\cos u(2\sin u\cos u)\ , 6a(2\sin u\cos u)\sin u\ , -6c\sin 2u\ ).u'$

$= (-6a\cos u(\sin 2u)\ , 6a(\sin 2u)\sin u\ , -6c\sin 2u\ ).u'$

$\vec{t} = 6\sin 2u\ (-a\cos u, a\sin u\ , -c\ ).u'$ (1)

Taking magnitude on both sides

$|\vec{t}| = \sqrt{(6\sin 2u)^2(a^2\cos^2 u + a^2\sin^2 u + c^2)u'^2}$

$1 \quad = 6\sin 2u\sqrt{a^2(\cos^2 u + \sin^2 u) + c^2}\ .u'$ $\because\ |\vec{t}| = 1$

$1 \quad = 6\sin 2u\sqrt{a^2 + c^2}\ .u'$

$u' \quad = \frac{1}{6\sin 2u\sqrt{a^2+c^2}}$

Put in (1) $\Rightarrow$ $\vec{t} = 6\sin 2u\ (-a\cos u, a\sin u\ , -c\ ).\frac{1}{6\sin 2u\sqrt{a^2+c^2}}$

$\vec{t} = (-a\cos u, a\sin u\ , -c\ ).\frac{1}{\sqrt{a^2+c^2}}$

Diff. w.r.t 's'

$\frac{d\vec{t}}{ds} = \frac{d\vec{t}}{du} \cdot \frac{du}{ds}$ since $\vec{t}' = \frac{d\vec{t}}{ds}$

$\vec{t}' = \frac{d}{du}[(-acosu, asinu, -c).\frac{1}{\sqrt{a^2+c^2}}].\frac{du}{ds}$

$K\vec{n} = \frac{(asinu, acosu, 0)}{\sqrt{a^2+c^2}}.u'$ (2) $\because$ $\vec{t}' = K\vec{n}$

Put the value of u' in eq (2)

$K\vec{n} = \frac{(asinu, acosu, 0)}{\sqrt{a^2+c^2}}.\frac{1}{6sin2u\sqrt{a^2+c^2}}$

$K\vec{n} = \frac{(asinu, acosu, 0)}{6sin2u(a^2+c^2)}$ (3)

Taking Magnitude of both sides

$|K\vec{n}| = \sqrt{\frac{a^2sin^2u+a^2cos^2u+0}{((a^2+c^2)6sin2u)^2}}$

$|K||\vec{n}| = \frac{\sqrt{a^2(cos^2u+sin^2u)}}{(a^2+c^2)6sin2u}$

$\Rightarrow$ $K = \frac{a}{(a^2+c^2)6sin2u}$ $\because |\vec{n}| = 1, |K| = K$

From (3)

$K\vec{n} = \frac{a(sinu, cosu, 0)}{6sin2u(a^2+c^2)}$

$K\vec{n} = K(sinu, cosu, 0)$

$\vec{n} = (sinu, cosu, 0)$

Question:

For a point of curve of intersection of surfaces $x^2 - y^2 = c^2$ and y =xtanh$(\frac{z}{c})$

Then prove that $\rho = \sigma = \frac{2x^2}{c}$

Solution:

As given that

$x^2 - y^2 = c^2$ (i)

y =xtanh$(\frac{z}{c})$ (ii)

$x^2 - y^2 = c^2(1)$

$x^2 - y^2 = c^2(\cosh^2\theta - sinh^2\theta)$

$$x^2 - y^2 = c^2\cosh^2\theta - c^2 sinh^2\theta$$

On comparing

$$x^2 = c^2\cosh^2\theta \qquad , \qquad y^2 = c^2 sinh^2\theta$$

$$x = c\cosh\theta \qquad , \qquad y = c\sinh\theta$$

$$\Rightarrow \qquad \cosh\theta = \frac{x}{c} \qquad , \qquad \sinh\theta = \frac{y}{c}$$

$$\Rightarrow \qquad \tanh\theta = \frac{y}{x} \qquad \text{(iii)}$$

From (ii)

$$\frac{y}{x} = \tanh\left(\frac{z}{c}\right)$$

$$\Rightarrow \qquad \tanh\theta = \tanh\left(\frac{z}{c}\right)$$

$$\Rightarrow \qquad \theta = \frac{z}{c}$$

$$\Rightarrow \qquad z = c\theta$$

We know

$$\vec{r} = (x, y, z)$$

$$\vec{r} = (c\cosh\theta, c\sinh\theta, c\theta)$$

$$\Rightarrow \qquad \vec{r}' = (c\sinh\theta, c\cosh\theta, c)$$

$$\Rightarrow \qquad \vec{r}'' = (c\cos\theta, c\sinh\theta, 0)$$

$$\Rightarrow \qquad \vec{r}''' = (c\sinh\theta, c\cosh\theta, 0)$$

As $\qquad K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$

Now $\qquad \vec{r}' \times \vec{r}'' = \begin{vmatrix} i & j & k \\ c\sinh\theta & c\cosh\theta & c \\ c\cosh\theta & c\sinh\theta & 0 \end{vmatrix}$

$$= (0 - c^2 sinh\theta)\hat{i} - (0 - c^2 cosh\theta)\hat{j} + \left(c^2\sinh^2\theta - c^2\cosh^2\theta\right)\hat{k}$$

$$\vec{r}' \times \vec{r}'' = -c^2 sinh\theta\hat{i} + c^2 cosh\theta\hat{j} + c^2\hat{k} \qquad \text{(iv)}$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{c^4\sinh^2\theta + c^4\cosh^2\theta + c^4}$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{c^4 cosh^2\theta + c^4(\sinh^2\theta + 1)}$$

$$= \sqrt{c^4 cosh^2\theta + c^4 \cosh^2\theta} \quad \Rightarrow \sqrt{2c^4 \cosh^2\theta}$$

$$= \sqrt{2}\, c^2 \cosh\theta$$

$$|\vec{r}'| = \sqrt{c^2 \sinh^2\theta + c^2\cosh^2\theta + c^2}$$

$$= \sqrt{c^2 cosh^2\theta + c^2(\sinh^2\theta + 1)}$$

$$= \sqrt{c^2 cosh^2\theta + c^2\cosh^2\theta} \quad \Rightarrow \sqrt{2c^2\cosh^2\theta}$$

$$= \sqrt{2}\, c \cosh\theta$$

$$K = \frac{\sqrt{2}\, c^2\cosh\theta}{(\sqrt{2}\, c\cosh\theta)^3} \quad \Rightarrow \frac{\sqrt{2}\, c^2\cosh\theta}{(\sqrt{2})^3 (c)^3 \cosh^3\theta} = \frac{1}{2c\cosh^2\theta}$$

$$\rho = \frac{1}{K} \quad = \quad 2c\ cosh^2\theta$$

$$\rho = 2\,c\left(\frac{x}{c}\right)^2 \quad \Rightarrow \quad 2c\,\frac{x^2}{c^2} \quad \ldots \quad (v) \quad \text{since } \cosh\theta = \frac{x}{c}$$

And

$$\tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{K^2|\vec{r}'|^6}$$

$$\vec{r}' \times \vec{r}''.\vec{r}''' = (-c^2 sinh\theta\hat{\imath} + c^2 cosh\theta\hat{\jmath} + c^2\hat{k}).\ (c\ \sinh\theta\hat{\imath} + c\ \cosh\theta\hat{\jmath} + 0\hat{k})$$

$$= \ -c^3\sinh^2\theta + c^3\ cosh^2\theta + 0$$

$$\vec{r}' \times \vec{r}''.\vec{r}''' = \ c^3$$

$$\tau = \frac{c^3}{\left(\frac{1}{2c\cosh^2\theta}\right)^2 (\sqrt{2}\, c\cosh\theta)^6} \quad \Rightarrow \quad \frac{4c^2\cosh^4\theta . c^3}{8c^6\cosh^6\theta} \quad = \frac{1}{2c\cosh^2\theta}$$

$$\sigma = \frac{1}{\tau} = 2c\cosh^2\theta = 2c\,\frac{x^2}{c^2} \qquad \text{since } \cosh\theta = \frac{x}{c}$$

$$\sigma = \frac{2x^2}{c} \quad \Rightarrow \quad \rho = \sigma = \frac{2x^2}{c} \qquad \text{Proved}$$

Prove that

(i) $\vec{r}'.\vec{r}'' = 0$

(ii) $\vec{r}'.\vec{r}''' = -K^2$

(iii) $\vec{r}'.\vec{r}^{iv} = -3KK'$

(iv) $\vec{r}''.\vec{r}''' = KK'$

(v) $\vec{r}''.\vec{r}^{iv} = K(K' - K^3 - K\tau^2)$

(vi) $\vec{r}'''.\vec{r}^{iv} = K'K\tau^2 + K^2\tau\tau' + 2K^3K' + K'K''$

Solution:

We know that

$\vec{r}' = \vec{t}$ (Tanget)

$\vec{r}'' = K\vec{n}$

$\vec{r}''' = K\tau\vec{b} - K^2\vec{t} + K'\vec{n}$

$\vec{r}^{iv} = (K'' - K^3 - K\tau^2)\,\vec{n} - 3KK'\vec{t} + (2K'\tau + K\tau')\vec{b}$

(i) $\vec{r}'.\vec{r}'' = 0$

L.H.S $= \vec{r}'.\vec{r}'' = \vec{t}.K\vec{n}$

$= K(0) \qquad \because \vec{t}.\vec{n} = 0$

$= 0 =$ R.H.S

(ii) $\vec{r}'.\vec{r}''' = \vec{t}\,.\,(K\tau\vec{b} - K^2\vec{t} + K'\vec{n})$

$= K\tau(\vec{t}.\vec{b}) - K^2(\vec{t}.\vec{t}) + K'(\vec{t}.\vec{n})$

$= K\tau(0) - K^2(1) + K'(0)$

$= -K^2$

(iii) $\vec{r}'.\vec{r}^{iv} = \vec{t}\,.[\,(K'' - K^3 - K\tau^2)\,\vec{n} - 3KK'\vec{t} + (2K'\tau + K\tau')\vec{b}]$

$= -3KK'(\vec{t}.\vec{t}) \quad = -3KK'$

(iv) $\vec{r}''.\vec{r}''' = K\vec{n}\,.\,(K\tau\vec{b} - K^2\vec{t} + K'\vec{n})$

$= KK'(\vec{n}.\vec{n}) - K^3(\vec{n}.\vec{t}) + K^2\tau(\vec{b}.\vec{n})$

$= KK'$

(v) $\vec{r}''.\vec{r}^{iv} = K\vec{n}\,.\,[(K'' - K^3 - K\tau^2)\,\vec{n} - 3KK'\vec{t} + (2K'\tau + K\tau')\vec{b}]$

$= K(K'' - K^3 - K\tau^2)\,(\vec{n}.\vec{n}) - 3K^2K'(\vec{n}.\vec{t}) + K(2K'\tau + K\tau')(\vec{n}.\vec{b})$

$= K(K'' - K^3 - K\tau^2) \qquad$ since $\vec{n}.\vec{n} = 1$, $\quad \vec{n}.\vec{t} = 0$ $\quad, \vec{n}.\vec{b} = 0$

(vi) $\vec{r}'''.\vec{r}^{iv} = (K\tau\vec{b} - K^2\vec{t} + K'\vec{n}).[(K'' - K^3 - K\tau^2)\,\vec{n} - 3KK'\vec{t} + (2K'\tau + K\tau')\vec{b}]$

$= K\tau(K'' - K^3 - K\tau^2)\,(\vec{b}.\vec{n}) + K\tau(-3KK')\,(\vec{b}.\vec{t}) + K\tau(2K'\tau + K\tau')(\vec{b}.\vec{b})$

$- K^2(K'' - K^3 - K\tau^2)\,(\vec{t}.\vec{n}) + 3K^3K'(\vec{t}.\vec{t}) - K^2(2K'\tau + K\tau')(\vec{t}.\vec{b})$

$+ K'(K'' - K^3 - K\tau^2)\,(\vec{n}.\vec{n}) + K'(-3KK')\,(\vec{n}.\vec{t}) + K'(2K'\tau + K\tau')(\vec{n}.\vec{b})$

Since $(\vec{n}.\vec{n}) = (\vec{b}.\vec{b}) = (\vec{t}.\vec{t}) = 1$

And $(\vec{n}.\vec{b}) = (\vec{b}.\vec{t}) = (\vec{t}.\vec{n}) = 0$

$= K\tau(2K'\tau + K\tau')(1) + 3K^3K'(1) + K'(K'' - K^3 - K\tau^2)(1)$

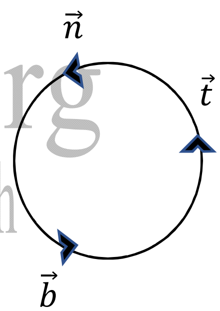
$= 2KK'\tau^2 + K^2\tau\tau' + 3K^3K' + K'K'' - K'K^3 - K'K\tau^2$

$\vec{r}'''.\vec{r}^{iv} = K'K\tau^2 + K^2\tau\tau' + 2K^3K' + K'K''$

Question:

Prove that

$[\vec{t}', \vec{t}'', \vec{t}'''] = [\vec{r}'', \vec{r}''', \vec{r}^{iv}] = K^5 \frac{d}{ds}\left(\frac{\tau}{K}\right)$



Solution:

We know that

$\vec{r}'' = \vec{t}' = K\vec{n} = 0\vec{t} + K\vec{n} + 0\vec{b}$

$\vec{r}''' = \vec{t}'' = -K^2\vec{t} + K'\vec{n} + K\tau\vec{b}$

$\vec{r}^{iv} = \vec{t}''' = -3KK'\vec{t} + (K'' - K^3 - K\tau^2)\,\vec{n} + (2K'\tau + K\tau')\vec{b}$

$[\vec{t}', \vec{t}'', \vec{t}'''] = [\vec{r}'', \vec{r}''', \vec{r}^{iv}]$

$$= \begin{vmatrix} 0 & K & 0 \\ -K^2 & K' & K\tau \\ -3KK' & K'' - K^3 - K\tau^2 & 2K'\tau + K\tau' \end{vmatrix}$$

$= 0 - K[-2K^2K'\tau - K^3\tau' + 3K^2K'\tau] + 0$

$= -K[K^2K'\tau - K^3\tau'] = K.\,K^2[K\tau' - K'\tau]$

$= \frac{K^2.K^3[K\tau' - K'\tau]}{K^2}$

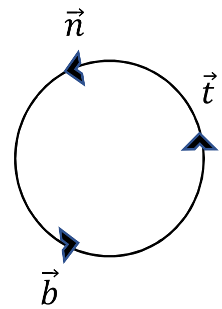
$= K^5 \frac{d}{ds}\left(\frac{\tau}{K}\right)$

Hence proved.

Question:

Prove that

$$[\vec{b}', \vec{b}'', \vec{b}'''] = \tau^5 \frac{d}{ds}\left(\frac{K}{\tau}\right)$$



Solution:

We know that

$$\vec{b}' = -\tau\vec{n} = 0\vec{t} - \tau\vec{n} + 0\vec{b}$$

$$\vec{b}'' = K\tau\vec{t} - \tau'\vec{n} - \tau^2\vec{b}$$

$$b''' = (2K\tau' + K'\tau)\vec{t} + (K^2\tau - \tau'' + \tau^3)\,\vec{n} - 3\tau\tau'\vec{b}$$

$$[\vec{b}', \vec{b}'', \vec{b}'''] = \begin{vmatrix} 0 & -\tau & 0 \\ K\tau & -\tau' & -\tau^2 \\ 2K\tau' + K'\tau & K^2\tau - \tau'' + \tau^3 & -3\tau\tau' \end{vmatrix}$$

$$= 0 - \tau\left[-3K\tau^2\tau' + 2K\tau^2\tau' + K'\tau^3\right] + 0$$

$$= -\tau\left[-K\tau^2\tau' + K'\tau^3\right] = \tau\left[K'\tau^3 - K\tau^2\tau'\right]$$

$$= \tau.\tau^2\left[K'\tau - K\tau'\right]$$

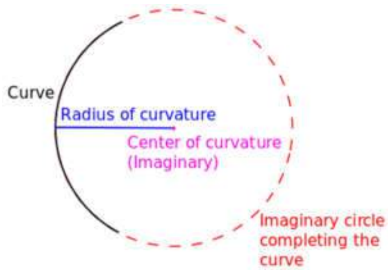
$$= \frac{\tau^2.\tau^3\left[K'\tau - K\tau'\right]}{\tau^2}$$

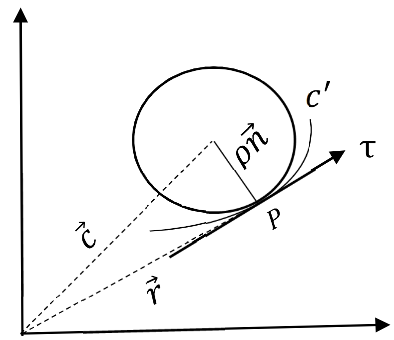
$$= \tau^5 \frac{d}{ds}\left(\frac{K}{\tau}\right)$$

Hence proved.

Lecture # 7

Equation of the Centre of Curvature:





Let $\vec{c}$ be the position vector of Centre of curvature. And $\vec{r}$ be the position vector of point p on the curve c' with respect to Centre O.

Then $\quad \vec{c} = \vec{r} + \rho\vec{n}$

Theorem:

Prove that tangent to its locus lies in the normal plane.

Proof:

We know that equation of center of curvature

$$\vec{c} = \vec{r} + \rho\vec{n} \qquad (1)$$

Diff. w.r.t 's'

$$\frac{d\vec{c}}{ds} = \frac{d\vec{r}}{ds} + \rho\frac{d\vec{n}}{ds} + \frac{d\rho}{ds}\vec{n}$$

$$= \vec{r}' + \rho\vec{n}' + \rho'\,\vec{n}$$

$$= \vec{t} + \rho(\tau\vec{b} - K\vec{t}\,) + \rho'\,\vec{n}$$

$$\frac{d\vec{c}}{ds} = \vec{t} + \rho\tau\vec{b} - \rho K\vec{t} + \rho'\,\vec{n}$$

$$= \vec{t} + \rho'\,\vec{n} + \rho\tau\vec{b} - \rho.\frac{1}{\rho}\vec{t} \qquad \because\ K = \frac{1}{\rho}$$

$$= \vec{t} + \rho'\,\vec{n} + \rho\tau\vec{b} - \vec{t}$$

$$\frac{d\vec{c}}{ds} = \rho'\,\vec{n} + \rho\tau\vec{b}$$ lies in normal plane proved

Moving Trihedral:

The triplet $(\vec{t}\ , \vec{n}, \vec{b})$ of unit tangent, unit principle normal and unit binormal are called Moving trihedral.

Equation of binormal

$\vec{R} = \vec{r} + \mu\vec{b}$

$\vec{R} = \vec{r} + \mu(\vec{t} \times \vec{n})$ $\quad \because\ \vec{b} = \vec{t} \times \vec{n}$

$\vec{R} = \vec{r} + \mu(\vec{t} \times \frac{\vec{r}''}{K})$ $\quad \because\ \vec{r}'' = K\vec{n} \Rightarrow \vec{n} = \frac{\vec{r}''}{K}$

$\vec{R} = \vec{r} + \mu(\vec{r}' \times \frac{\vec{r}''}{K})$ $\quad \because\ \vec{t} = \vec{r}'$

$= \vec{r} + \frac{\mu}{K}(\vec{r}' \times \vec{r}'')$

$\vec{R} = \vec{r} + \nu(\vec{r}' \times \vec{r}'')$ $\quad \because\ \nu = \frac{\mu}{K}$

Theorem:

If the radius of curvature is a constant for a given curve C then prove that

(i) The tangent to its locus of its center is parallel to the binormal at point P on C.

(ii) Curvature of locus c_1 is same as curvature of the given curve.

(iii) Torsion of locus of centre of curvature vary inversely as the torsion of the given curve c. $\quad \tau_1 \propto \frac{1}{\tau}$

Proof: (i) As we know that

$\vec{c} = \vec{r} + \rho\vec{n}$ (1)

Diff. w.r.t 's'

$\frac{d\vec{c}}{ds} = \frac{d\vec{r}}{ds} + \rho\frac{d\vec{n}}{ds} + \frac{d\rho}{ds}\vec{n}$

$= \vec{r}' + \rho\vec{n}' + (0).\vec{n}$ $\quad \because$ given radius is constant

$= \vec{t} + \rho(\tau\vec{b} - K\vec{t})$

$\frac{d\vec{c}}{ds} = \vec{t} + \rho\tau\vec{b} - \rho K\vec{t}$

$= \vec{t} + \rho\tau\vec{b} - \rho.\frac{1}{\rho}\vec{t}$ $\quad \because\ K = \frac{1}{\rho}$

$= \vec{t} + \rho\tau\vec{b} - \vec{t}$

$\frac{d\vec{c}}{ds} = \rho\tau\vec{b}$

$\frac{d\vec{c}}{ds} = \lambda\vec{b}$ $\qquad \because \rho\tau = \lambda$

$\frac{d\vec{c}}{ds} // \vec{b}$ $\qquad \because \vec{a} // \vec{b} \Rightarrow \vec{a} = \lambda\vec{b}$

(ii) As we know that

$\vec{c} = \vec{r} + \rho\vec{n}$

Diff. w.r.t 's_1'

$\frac{d\vec{c}}{ds_1} = \frac{d\vec{c}}{ds} \cdot \frac{ds}{ds_1}$

$= \frac{d}{ds}(\vec{r} + \rho\vec{n}) \cdot \frac{ds}{ds_1}$

$\vec{t}_1 = \rho\tau \frac{ds}{ds_1}\vec{b}$ (1)

Taking magnitude both side

$|\vec{t}_1| = \rho\tau \frac{ds}{ds_1}|\vec{b}|$

$1 = \rho\tau \frac{ds}{ds_1}$ $\qquad \because |\vec{t}_1| = 1, |\vec{b}| = 1$

$\frac{ds}{ds_1} = \frac{1}{\rho\tau}$ (2)

Put eq (2) in eq (1)

$\vec{t}_1 = \rho\tau \frac{1}{\rho\tau}\vec{b}$

$\vec{t}_1 = \vec{b}$ (3)

Now diff. (3) w.r.t 's_1'

$\frac{d\vec{t}_1}{ds_1} = \frac{d\vec{b}}{ds} \cdot \frac{ds}{ds_1}$

$\vec{t}_1' = \vec{b}' \cdot \frac{ds}{ds_1}$

$= -\tau\vec{n} \cdot \frac{1}{\rho\tau}$ $\qquad \because \vec{b}' = -\tau\vec{n}, \frac{ds}{ds_1} = \frac{1}{\rho\tau}$

$K_1\vec{n}_1 = -\vec{n}K$ $\qquad \because \vec{t}' = K\vec{n}, K = \frac{1}{\rho}$

Taking magnitude on both sides

$K_1 |\vec{n}_1| = |-\vec{n}|K \qquad \Rightarrow K_1 = K$ Proved

(iii) As we know

$\vec{t}_1 = \vec{b}$ (1)

$n_1 = -\vec{n}$ (2)

Taking cross product of (1) and (2)

$$\vec{t}_1 \times n_1 = \vec{b} \times (-\vec{n}) = -(\vec{b} \times \vec{n})$$

$$\Rightarrow \quad \vec{b}_1 = -(-\vec{t})$$

$$\Rightarrow \quad \vec{b}_1 = \vec{t}$$

Diff. w.r.t 's_1'

$$\frac{d\vec{b}_1}{ds_1} = \frac{d\vec{t}}{ds} \cdot \frac{ds}{ds_1}$$

$$\vec{b}_1' = \vec{t}' \cdot \frac{ds}{ds_1}$$

$$-\tau_1 \vec{n}_1 = K\vec{n} \cdot \frac{1}{\rho\tau} \qquad \because \vec{b}_1' = -\tau_1 \vec{n}_1 , \vec{t}' = K\vec{n} , \frac{ds}{ds_1} = \frac{1}{\rho\tau}$$

$$= K\vec{n} \cdot \frac{K}{\tau} \qquad \because K = \frac{1}{\rho}$$

$$\tau_1(-\vec{n}_1) = K^2 \cdot \frac{1}{\tau} \qquad \because -\vec{n}_1 = (\vec{n}_1)$$

$$\tau_1 \propto \frac{1}{\tau} \qquad K^2 \text{ is constant}$$

Hence proved

Theorem:

If s_1 is the arc length of locus of centre of curvature then show that

$$\frac{ds}{ds_1} = \frac{1}{K^2}\sqrt{K^2\tau^2 + (K')^2}$$

Or

$$\frac{ds}{ds_1} = \sqrt{\left(\frac{\rho}{\sigma}\right)^2 + (\rho')^2}$$

Proof:

As we know that

$$\vec{c} = \vec{r} + \rho\vec{n}$$

Diff. w.r.t 's_1'

$$\frac{d\vec{c}}{ds_1} = \frac{d\vec{c}}{ds} \cdot \frac{ds}{ds_1}$$

$$= \frac{d}{ds}(\vec{r} + \rho\vec{n}) \cdot \frac{ds}{ds_1}$$

$$= \left(\frac{d\vec{r}}{ds} + \rho\frac{d\vec{n}}{ds} + \frac{d\rho}{ds}\vec{n}\right)\frac{ds}{ds_1}$$

$$= (\vec{t} + \rho(\tau\vec{b} - K\vec{t}) + \rho'\,\vec{n})\frac{ds}{ds_1}$$

$$= (\vec{t} + \rho\tau\vec{b} - \rho K\vec{t} + \rho'\,\vec{n})\frac{ds}{ds_1}$$

$$= (\vec{t} + \rho'\,\vec{n} + \rho\tau\vec{b} - \rho.\frac{1}{\rho}\vec{t})\frac{ds}{ds_1} \qquad \because\ K = \frac{1}{\rho}$$

$$= (\vec{t} + \rho'\,\vec{n} + \rho\tau\vec{b} - \vec{t})\frac{ds}{ds_1}$$

$$\vec{t}_1 = (\rho'\,\vec{n} + \rho\tau\vec{b})\frac{ds}{ds_1}$$

Taking magnitude on both sides

$$|\vec{t}_1| = \sqrt{(\rho')^2 + (\rho\tau)^2}\,\frac{ds}{ds_1}$$

$$1 = \sqrt{(\rho')^2 + (\rho\tau)^2}\,\frac{ds}{ds_1}$$

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{(\rho')^2 + (\rho\tau)^2}}$$

$$\frac{ds_1}{ds} = \sqrt{(\rho')^2 + (\rho\tau)^2} \qquad (1)$$

$$\therefore \quad \rho = \frac{1}{K} = K^{-1}$$

Taking derivative

$$\rho' = -K^{-2}K'$$

$$\rho' = \frac{-K'}{K^2}$$

Put in (1)

$$\frac{ds_1}{ds} = \sqrt{\left(\frac{-K'}{K^2}\right)^2 + (\frac{1}{K}\tau)^2} \qquad \because \quad \rho = \frac{1}{K}$$

$$= \sqrt{\frac{K'^2}{K^4} + \frac{\tau^2}{K^2}}$$

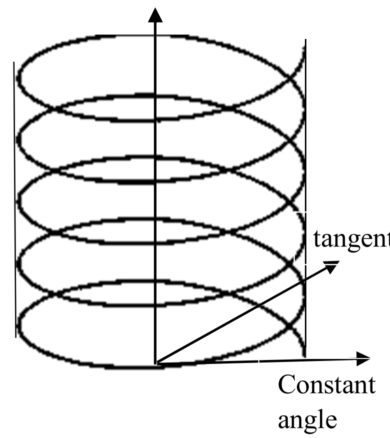
$$= \sqrt{\frac{K'^2 + K^2\tau^2}{K^4}} \qquad \Rightarrow \qquad \frac{ds}{ds_1} = \frac{1}{K^2}\sqrt{K^2\tau^2 + (K')^2} \quad \text{Proved}$$

Lecture # 8

Helix:

A curve traced on the surface of a cylinder and cutting the generator at a constant angle is called Helix.

Thus, the tangent to the Helix is inclined at a constant angle to a fix direction. If $\vec{t}$ is on its tangent to the Helix and $\vec{a}$ is a constant vector parallel to the generator of the Helix then



$$\vec{t}.\vec{a} = \text{constant}$$

$$\Rightarrow \quad \vec{t}.\vec{a} = |\vec{t}||\vec{a}|\cos\alpha$$

$$= 1.1\cos\alpha$$

$$\Rightarrow \quad \vec{t}.\vec{a} = \cos\alpha$$

Since α is fixed so the $\vec{t}.\vec{a}$ is constant

Question:

Prove that necessary and sufficient condition for the curve to be Helix is that

$$\frac{K}{\tau} = \text{constant}$$

Solution:

Let the curve is helix. Then we have to prove that $\frac{K}{\tau} = \text{constant}$

Since for a helix we know that tangent at any point to the curve makes a constant angle α with the fix direction of the cylinder.

Let $\vec{a}$ be the unit constant vector along the direction (generator). Then

$$\vec{t}.\vec{a} = |\vec{t}||\vec{a}|\cos\alpha \qquad \because \quad |\vec{t}| = |\vec{a}| = 1$$

$$\vec{t}.\vec{a} = \cos\alpha \qquad \because \ \alpha \text{ is constant}$$

Diff. w.r.t 's'

$$\frac{d\vec{t}}{ds}.\vec{a} + \vec{t}.0 = 0 \qquad \because \ \vec{a} \text{ is constant}$$

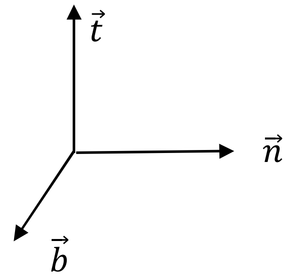
$$\Rightarrow \quad \vec{t}'.\vec{a} = 0$$

$\Rightarrow \qquad K\vec{n}.\vec{a} = 0 \qquad\qquad \because \vec{t}' = K\vec{n}$

$\Rightarrow \qquad K \neq 0, \quad \vec{n}.\vec{a} = 0$

$\Rightarrow \qquad \vec{n} \perp \vec{a}$ but $\vec{n} \perp \vec{t}$

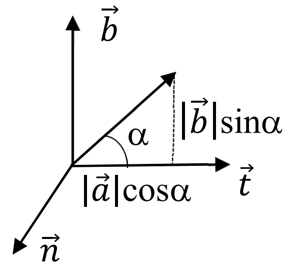
$\Rightarrow \qquad \vec{a}$ will lies in the plane formed by the tangent and binormal.



Then

$\vec{a} = \quad |\vec{t}|\cos\alpha + |\vec{b}|\sin\alpha$

$\vec{a} = \quad \vec{t}\cos\alpha + \vec{b}\sin\alpha$



Diff. w.r.t 's'

$\frac{d\vec{a}}{ds} = \vec{t}'\cos\alpha + \vec{b}'\sin\alpha \qquad \because \alpha$ is constant

$0 = K\vec{n}\cos\alpha + (-\tau\vec{n})\sin\alpha \qquad \because \vec{t}' = K\vec{n}, \ \vec{b}' = -\tau\vec{n}$

$0 = \vec{n}(K\cos\alpha - \tau\sin\alpha)$

Taking dot product with $\vec{n}$ both sides

$0.\ \vec{n} = \vec{n}.\vec{n}(K\cos\alpha - \tau\sin\alpha)$

$0 = (1)(K\cos\alpha - \tau\sin\alpha)$

$0 = K\cos\alpha - \tau\sin\alpha$

$\frac{K}{\tau} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$\frac{K}{\tau} = \text{constant} \qquad \because \alpha$ is constant

Sufficient Condition:

Let $\frac{K}{\tau} = \text{constant}$

We have to prove that curve is helix. For this it is sufficient to prove that

$\vec{t}.\vec{a} = \text{constant}$

$\frac{K}{\tau} = \frac{1}{c} \quad \text{or } K = \frac{\tau}{c} \qquad (1) \qquad \because$ c is constant

Now we consider

$\vec{t}' = K\vec{n} \qquad (2)$

Put eq (1) in eq (2)

$\Rightarrow \qquad \vec{t}' = \frac{\tau}{c}\,\vec{n} \qquad (3)$

As $\qquad \vec{b}' = -\tau\vec{n}$

Divide both sides by 'c'

$\Rightarrow \qquad \frac{\vec{b}'}{c} = -\frac{\tau}{c}\,\vec{n} \qquad (4)$

Adding (3) and (4)

$$\vec{t}' + \frac{\vec{b}'}{c} = \frac{\tau}{c}\,\vec{n} - \frac{\tau}{c}\,\vec{n}$$

$$\frac{c\vec{t}' + \vec{b}'}{c} = \vec{0}$$

$$\Rightarrow \quad c\vec{t}' + \vec{b}' = \vec{0}$$

$$\Rightarrow \quad c\,\frac{d\vec{t}}{ds} + \frac{d\vec{b}}{ds} = \vec{0}$$

$$\Rightarrow \quad \frac{d}{ds}(c\vec{t} + \vec{b}) = \vec{0}$$

Integrate both sides

$$\Rightarrow \quad \int \frac{d}{ds}(c\vec{t} + \vec{b}) = \int \vec{0}$$

$c\vec{t} + \vec{b} = \vec{a} \qquad \because$ a is constant of integration

Taking dot product with $\vec{t}$

$c\,\vec{t}.\vec{t} + \vec{b}.\vec{t} = \vec{a}\,.\,\vec{t} \qquad \because \vec{b}.\vec{t} = 0\ ,\ \vec{t}.\vec{t} = 1$

$\Rightarrow \quad c(1) + 0 = \vec{a}\,.\,\vec{t}$

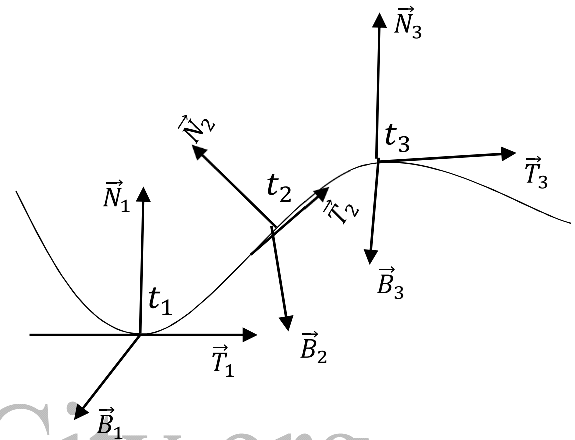
$\Rightarrow \quad \vec{t}\,.\,\vec{a} = c$

$\Rightarrow \quad \vec{t}\,.\,\vec{a} =$ constant

Spherical indicatrices:

When we move all unit tangent $\vec{T}$ of a curve c to points their extremities then describe a curve c_1 on the unit sphere. This curve c_1 is called spherical images (indicatrices). There is on-one corresponding between c and c_1. We can similarly obtain image of c when its normal and binormal move to a point to construct the spherical indicatrices of tangent line parallel to the positive direction. If the tangent at the points of the given curve from the center 'o' of unit sphere.

Let $t_1, t_2, t_3, \ldots\ldots$ where those line meets are spherical indicatrices of the tangent.



Definition:

(i) Spherical Indicatrices of tangent:
The locus of point where position vector is equal to unit tangent at any point of the given curve.

(ii) Spherical Indicatrices of Normal:
The locus of point where position vector is equal to unit normal at any point of the given curve.

(iii) Spherical Indicatrices of Binormal:
The locus of point where position vector is equal to unit binormal at any point of the given curve.

Lecture # 9

Question:

Let x = 3cosh2t , y = 3sinh2t , z =6t

Find arc length from '0' to 'π'.

Solution:

Diff. the above w.r.t 't'

$\frac{dx}{dt}$ = 6sinh2t

$\frac{dy}{dt}$ = 6cosh2t

$\frac{dz}{dt}$ = 6

Formula for arc length

$$\int_a^b \left|\frac{dr}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$\text{Now } \left|\frac{dr}{dt}\right| = \sqrt{(6\sinh 2t)^2 + (6\cosh 2t)^2 + (6)^2}$$

$$= \sqrt{(6)^2(sinh^2 2t + cosh^2 2t + 1)}$$

$$= 6\sqrt{(sinh^2 2t + 1) + cosh^2 2t}$$

$$= 6\sqrt{cosh^2 2t + cosh^2 2t}$$

$$= 6\sqrt{2cosh^2 2t}$$

$$= 6\sqrt{2}\cosh 2t$$

Taking integration from '0' to 'π'.

$$\int_0^\pi \left|\frac{dr}{dt}\right| dt = 6\sqrt{2}\int_0^\pi \cosh 2t\, dt$$

$$= 6\sqrt{2}\left.\frac{sinh2t}{2}\right|_0^\pi$$

$$= 3\sqrt{2}\,(\sinh 2\pi - \sinh 2(0))$$

$$= 3\sqrt{2}\,(\sinh 2\pi - 0)$$

$$= 3\sqrt{2}\,\sinh 2\pi$$

Question:

Given that x = (a+b)cosθ−bcos $(\frac{a+b}{b})\theta$, y = (a+b)sinθ−bsin$(\frac{a+b}{b})\theta$, z = 0

Find arc length from 0 to θ

Solution:

Diff the above equation w.r.t θ

$$\frac{dx}{d\theta} = -(a+b)\sin\theta - b\sin(\frac{a+b}{b})\theta \;.\; \frac{a+b}{b}$$

$$= -(a+b)\sin\theta + (a+b)\sin(\frac{a+b}{b})\theta$$

$$= (a+b)\left(-\sin\theta + \sin(\frac{a+b}{b})\theta\right)$$

$$\frac{dy}{d\theta} = (a+b)\cos\theta - b\cos(\frac{a+b}{b})\theta \;.\; \frac{a+b}{b}$$

$$= (a+b)\cos\theta - (a+b)\cos(\frac{a+b}{b})\theta$$

$$= (a+b)\left(\cos\theta - \cos(\frac{a+b}{b})\theta\right)$$

$$\frac{dz}{d\theta} = 0$$

Formula for arc length

$$\int_a^b \left|\frac{dr}{d\theta}\right| d\theta = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2}\; d\theta$$

Now

$$\left|\frac{dr}{d\theta}\right| = \sqrt{\left((a+b)\left(-\sin\theta + \sin(\frac{a+b}{b})\theta\right)\right)^2 + \left((a+b)\left(\cos\theta - \cos(\frac{a+b}{b})\theta\right)\right)^2 + (0)^2}$$

$$= \sqrt{\left((a+b)\left(-\sin\theta + \sin(\frac{a+b}{b})\theta\right)\right)^2 + \left((a+b)\left(\cos\theta - \cos(\frac{a+b}{b})\theta\right)\right)^2}$$

$$= \sqrt{(a+b)^2\left[\left(-\sin\theta + \sin(\frac{a+b}{b})\theta\right)^2 + \left(\cos\theta - \cos(\frac{a+b}{b})\theta\right)^2\right]}$$

=(a+b)×

$$\sqrt{\sin^2\theta + \sin^2\left(\frac{a+b}{b}\right)\theta - 2\sin\theta\sin\left(\frac{a+b}{b}\right)\theta + \cos^2\theta + \cos^2\left(\frac{a+b}{b}\right)\theta - 2\cos\theta\cos\left(\frac{a+b}{b}\right)\theta}$$

$$= (a+b)\sqrt{1 + 1 - 2\cos(\theta - \frac{a+b}{b}\theta)}$$

$$= (a+b)\sqrt{2 - 2\cos(\frac{b\theta - a\theta + b\theta}{b})}$$

$$= (a+b)\sqrt{2(1 - \cos(\frac{-a\theta}{b}))} \qquad \because \cos(-\theta) = \cos\theta$$

$$= (a+b)\sqrt{2(1 - \cos(\frac{a}{b}\theta))} \qquad \because 1-\cos\theta = 2sin^2\frac{\theta}{2}$$

$$= (a+b)\sqrt{2(2sin^2(\frac{a}{2b}\theta))}$$

$$= (a+b)\sqrt{4sin^2(\frac{a}{2b}\theta)}$$

$$= 2(a+b)\sin(\frac{a}{2b}\theta)$$

Now integrate from 0 to θ

$$\int_0^\theta \left|\frac{dr}{d\theta}\right| d\theta = \int_0^\theta 2(a+b)\sin(\frac{a}{2b}\theta)\, d\theta$$

$$= 2(a+b) \left.\frac{-\cos(\frac{a}{2b}\theta)}{(\frac{a}{2b}\theta)}\right|_0^\theta$$

$$= 2(a+b) . \frac{2b}{a}(-\cos\left(\frac{a}{2b}\theta\right) - (-\cos\left(\frac{a}{2b}0\right))$$

$$= 2(a+b) . \frac{2b}{a}(-\cos\left(\frac{a}{2b}\theta\right) + 1)$$

$$= 4(a+b) . \frac{b}{a}\{1 - \cos\left(\frac{a}{2b}\theta\right)\}$$

Question:

$x = 2a(sin^{-1}t + t\sqrt{1-t^2})$, $y = 2at^2$ and $z = 4at$ find arc length from 0 to t .

Solution:

Diff. above w.r.t 't'

$$\frac{dx}{dt} = 2a\,[\frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} + t(\frac{1}{2\sqrt{1-t^2}}(-2t)]$$

$$= 2a\,[\frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} - \frac{t^2}{\sqrt{1-t^2}}]$$

$$= 2a[\frac{1-t^2}{\sqrt{1-t^2}} + \sqrt{1-t^2}]$$

$$= 2a\,[\sqrt{1-t^2} + \sqrt{1-t^2}]$$

$$\frac{dx}{dt} = 2a(2\sqrt{1-t^2})$$

$$= 4a\sqrt{1-t^2}$$

$$\frac{dy}{dt} = 4at$$

$$\frac{dz}{dt} = 4a$$

Now

Formula for arc length

$$\int_a^b \left|\frac{dr}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$\left|\frac{dr}{dt}\right| = \sqrt{\left(4a\sqrt{1-t^2}\right)^2 + (4at)^2 + (4a)^2}$$

$$= \sqrt{(1-t^2+t^2+1)(4a)^2}$$

$$= 4a\sqrt{2}$$

Integrate from 0 to t

$$\int_0^t \left|\frac{dr}{dt}\right| dt = 4a\sqrt{2}\int_0^t dt$$

$$= 4a\sqrt{2}(t-0)$$

$$= 4a\sqrt{2}t$$

Theorem:

Show that the curvature of spherical indicatrices of the tangent is the ratio of skew curvature to circular curvature of the curve that is

$$K_1 = \frac{\sqrt{K^2+\tau^2}}{K} \qquad \text{also prove that}$$

$$\tau_1 = \frac{K\tau' + \tau K'}{K(K^2+\tau^2)}$$

Proof:

Let $\vec{r}_1$ be the position vector of a point of the spherical indicatrices of the tangent to the curve then

$$\vec{r}_1 = \vec{t} \qquad (1)$$

Diff. eq (1) w.r.t 's'

$$\frac{d\vec{r}_1}{ds} = \frac{d\vec{t}}{ds}$$

$$\frac{d\vec{r}_1}{ds_1} \cdot \frac{ds_1}{ds} = \vec{t}' \qquad \because \quad \vec{t}' = \mathrm{K}\vec{n}$$

$$\vec{r}'_1 \cdot \frac{ds_1}{ds} = \mathrm{K}\vec{n}$$

So that

$$\frac{ds_1}{ds} = \mathrm{K}$$

$$\Rightarrow \quad \vec{t}_1 = \vec{n} \qquad (2)$$

Diff. eq (2) w.r.t 's'

$$\frac{d\vec{t}_1}{ds} = \frac{d\vec{n}}{ds}$$

$$\frac{d\vec{t}_1}{ds_1} \cdot \frac{ds_1}{ds} = \vec{n}'$$

$$\vec{t}'_1 . (\mathrm{K}) = \tau\vec{b} - \mathrm{K}\vec{t} \qquad \because \quad \vec{n}' = \tau\vec{b} - \mathrm{K}\vec{t}$$

$$(K_1\vec{n}'_1).\mathrm{K} = \tau\vec{b} - \mathrm{K}\vec{t}$$

Divide by K

$$K_1\vec{n}'_1 = \frac{\tau\vec{b} - \mathrm{K}\vec{t}}{K}$$

Taking square

$$K_1{}^2(\vec{n}'_1 . \vec{n}'_1) = \frac{\tau^2\vec{b}.\vec{b} - \mathrm{K}^2\vec{t}.\vec{t}}{\mathrm{K}^2}$$

$$K_1{}^2 = \frac{\tau^2 - \mathrm{K}^2}{\mathrm{K}^2}$$

$$K_1 = \frac{\sqrt{K^2 + \tau^2}}{K}$$

Now equation of osculating sphere

$$\mathrm{R}^2 = \rho^2 + (\sigma\rho')^2 \qquad (3)$$

As the indicatrices lies on the sphere of unit radius since R =1

$$1 = \rho^2{}_1 + (\sigma_1\rho'{}_1)^2$$

$$1 = \frac{1}{K_1{}^2} + \left(\frac{1}{\tau_1}.\frac{-K_1{}'}{K_1{}^2}\right) \qquad \because \rho_1 = \frac{1}{K_1},\ \sigma_1 = \frac{1}{\tau_1},\ \rho'_1 = \frac{-K_1{}'}{K_1{}^2}$$

$$1 = \frac{1}{K_1{}^2} + \frac{1}{\tau_1{}^2}.\frac{-K_1{}'^2}{K_1{}^4}$$

$$\frac{1}{\tau_1{}^2}\frac{K_1{}'^2}{K_1{}^4} = \frac{K_1{}^2-1}{K_1{}^2}$$

$$\Rightarrow \qquad \tau_1{}^2 = \frac{K_1{}'^2}{K_1{}^2(K_1{}^2-1)}$$

$$\Rightarrow \qquad \tau_1 = \frac{K_1{}'}{K_1\sqrt{(K_1{}^2-1)}} \qquad (4)$$

Also

$$K_1 = \frac{\sqrt{K^2+\tau^2}}{K}$$

Diff. w.r.t 's_1'

$$\frac{dK_1}{ds_1} = \frac{dK_1}{ds}.\frac{ds}{ds_1}$$

$$K_1{}' = \frac{d}{ds}\frac{\sqrt{K^2+\tau^2}}{K}.\frac{ds}{ds_1}$$

$$= \left[\frac{K\left(\frac{2KK'+2\tau\tau'}{2\sqrt{K^2+\tau^2}}\right)-\sqrt{K^2+\tau^2}.K'}{K^2}\right].\frac{1}{K}$$

$$= \frac{K\left(\frac{2KK'+2\tau\tau'}{2\sqrt{K^2+\tau^2}}\right)-\sqrt{K^2+\tau^2}.K'}{K^3}$$

$$= \frac{K^2K'+K\tau\tau'-K^2K'-K'\tau^2}{K^3\sqrt{K^2+\tau^2}}$$

$$K_1{}' = \frac{K\tau\tau'-K'\tau^2}{K^3\sqrt{K^2+\tau^2}}$$

Put the value of $K_1{}'$ in equation (4)

$$\Rightarrow \tau_1 = \frac{\frac{K\tau\tau'-K'\tau^2}{K^3\sqrt{K^2+\tau^2}}}{K_1\sqrt{(K_1{}^2-1)}}$$

$$\Rightarrow \quad \tau_1 = \frac{K\tau\tau' - K'\tau^2}{\frac{\sqrt{K^2+\tau^2}}{K}\sqrt{\left(\frac{\sqrt{K^2+\tau^2}}{K}\right)^2 - 1}} \cdot \frac{1}{K^3\sqrt{K^2+\tau^2}}$$

$$\Rightarrow \quad \tau_1 = \frac{K\tau\tau' - K'\tau^2}{\sqrt{K^2+\tau^2}\left(\sqrt{K^2+\tau^2-K^2}\right)} \cdot \frac{K^2}{K^3\sqrt{K^2+\tau^2}}$$

$$\Rightarrow \quad \tau_1 = \frac{K\tau' - K'\tau}{(K^2+\tau^2)K} \quad \text{proved}$$

Lecture # 10

Theorem:

Prove that the curvature and torsion of the spherical indicatrices of the binormal is

$$K_1 = \frac{\sqrt{K^2+\tau^2}}{\tau} \quad \text{and } \tau_1 = \frac{\tau K' - \tau' K}{(K^2+\tau^2)\tau}$$

Proof:

The equation of spherical indicatrices of the binormal is

$$\vec{r}_1 = \vec{b}$$

Diff. eq (1) w.r.t 's_1'

$$\frac{d\vec{r}_1}{ds_1} = \frac{d\vec{b}}{ds} \cdot \frac{ds}{ds_1}$$

$$\frac{d\vec{r}_1}{ds} \cdot = \vec{b}' \cdot \frac{ds}{ds_1} \qquad \because \vec{b}' = -\tau\vec{n}$$

$$\vec{t}_1 = -\tau\vec{n} \cdot \frac{ds}{ds_1}$$

$$\vec{t}_1 = -\vec{n} \quad ..(1) \qquad \because \tau\frac{ds}{ds_1} = 1$$

$$|\vec{t}_1| = \left|-\tau\vec{n}\frac{ds}{ds_1}\right|$$

$$1 = \tau\frac{ds}{ds_1}|\vec{n}|$$

$$1 = \tau\frac{ds}{ds_1}$$

Diff. eq (1) w.r.t 's_1'

$$\frac{d\vec{t}_1}{ds_1} = \frac{-d\vec{n}}{ds_1}$$

$$\vec{t}'_1 = \frac{-d\vec{n}}{ds} \cdot \frac{ds}{ds_1}$$

$$\vec{t}'_1 = -\vec{n}' \cdot \frac{ds}{ds_1}$$

$$\vec{t}'_1 = K_1\vec{n}_1$$

$$\frac{ds}{ds_1} = \frac{1}{\tau}$$

$$K_1\vec{n}_1 = -\vec{n}' \cdot \frac{1}{\tau} \qquad \because \vec{n}' = \tau\vec{b} - K\vec{t}$$

$$(K_1\vec{n}_1) = -(\tau\vec{b} - K\vec{t}) \cdot \frac{1}{\tau}$$

$$K_1\vec{n}_1 = \frac{K\vec{t} - \tau\vec{b}}{K}$$

Taking square on both sides

$$K_1{}^2\vec{n}_1 \cdot \vec{n}_1 = \frac{\tau^2\vec{t}.\vec{t} + \tau^2\vec{b}.\vec{b} - 2K\tau\vec{t}.\vec{b}}{\tau^2} \qquad \because \vec{t}.\vec{b} = 0$$

$$K_1{}^2 = \frac{K^2+\tau^2}{\tau^2}$$

$$K_1 = \frac{\sqrt{K^2+\tau^2}}{\tau} \qquad (2)$$

As the indicatrices lies on the sphere of unit sphere (R =1)

$$R^2 = \rho^2{}_1 + (\sigma\rho')^2$$

$$1 = \rho^2{}_1 + (\sigma_1\rho'{}_1)^2 \qquad \because \ R = 1$$

$$1 = \frac{1}{K_1{}^2} + \left(\frac{1}{\tau_1} . \frac{-K_1{}'}{K_1{}^2}\right) \qquad \because \rho_1 = \frac{1}{K_1}, \sigma_1 = \frac{1}{\tau_1}, \rho'{}_1 = \frac{-K_1{}'}{K_1{}^2}$$

$$1 = \frac{1}{K_1{}^2} + \frac{1}{\tau_1{}^2} . \frac{-K_1{}'^2}{K_1{}^4}$$

$$\frac{1}{\tau_1{}^2} \frac{K_1{}'^2}{K_1{}^4} = \frac{K_1{}^2-1}{K_1{}^2}$$

$$\Rightarrow \qquad \tau_1{}^2 = \frac{K_1{}'^2}{K_1{}^2(K_1{}^2-1)}$$

$$\Rightarrow \qquad \tau_1 = \frac{K_1{}'}{K_1\sqrt{(K_1{}^2-1)}} \qquad (3)$$

Also

$$K_1 = \frac{\sqrt{K^2+\tau^2}}{\tau}$$

Diff. w.r.t 's_1'

$$\frac{dK_1}{ds_1} = \frac{dK_1}{ds} . \frac{ds}{ds_1}$$

$$K_1{}' = \frac{d}{ds} \frac{\sqrt{K^2+\tau^2}}{\tau} . \frac{ds}{ds_1}$$

$$= \left[\frac{\tau\left(\frac{2KK'+2\tau\tau'}{2\sqrt{K^2+\tau^2}}\right) - \sqrt{K^2+\tau^2}.\tau'}{\tau^2}\right] . \frac{1}{\tau}$$

$$= \frac{\tau\left(\frac{2KK'+2\tau\tau'}{2\sqrt{K^2+\tau^2}}\right) - \sqrt{K^2+\tau^2}.\tau'}{\tau^3}$$

$$= \frac{\tau^2\tau' + \tau K K' - \tau^2\tau' - \tau' K^2}{\tau^3\sqrt{K^2+\tau^2}}$$

$$K_1{}' = \frac{K(\tau K' - \tau' K)}{\tau^3\sqrt{K^2+\tau^2}}$$

Put the value of $K_1{}'$ in equation (3)

$$\Rightarrow \tau_1 = \frac{\frac{K(\tau K' - \tau' K)}{\tau^3\sqrt{K^2+\tau^2}}}{K_1\sqrt{(K_1{}^2 - 1)}}$$

$$\Rightarrow \quad \tau_1 = \frac{K(\tau K' - \tau' K)}{\frac{\sqrt{K^2+\tau^2}}{\tau}\sqrt{\left(\left(\frac{\sqrt{K^2+\tau^2}}{\tau}\right)^2 - 1\right)}} \cdot \frac{1}{\tau^3\sqrt{K^2+\tau^2}}$$

$$\Rightarrow \quad \tau_1 = \frac{K(\tau K' - \tau' K)}{\tau^2(K^2+\tau^2)} \cdot \frac{\tau}{K}$$

$$\Rightarrow \quad \tau_1 = \frac{\tau K' - K\tau'}{\tau(K^2+\tau^2)} \quad \text{proved}$$

Question:

Find out spherical indicatrices of circular helix.

Solution:

As $\vec{r} = (a\cos\theta\,, a\sin\theta\,, c\theta)$; $c \neq 0$

Diff. w.r.t 's'

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{d\theta} \cdot \frac{d\theta}{ds}$$

$$\vec{r}' = \frac{d}{d\theta}(a\cos\theta, a\sin\theta, c\theta) \cdot \frac{d\theta}{ds}$$

$$\vec{t} = (-a\sin\theta\,, a\cos\theta\,, c) \cdot \frac{d\theta}{ds} \quad \_\_\_\_\_(1) \qquad \because \ \vec{r}' = \vec{t}$$

Squaring both sides

$$\vec{t} \cdot \vec{t} = (a^2\sin^2\theta + a^2\cos^2\theta + c^2)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2(\sin^2\theta + \cos^2\theta) + c^2)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2 + c^2)\left(\frac{d\theta}{ds}\right)^2$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2 + c^2}$$

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{a^2+c^2}} = \frac{1}{\lambda} \qquad \because \text{ (say) } \lambda = \sqrt{a^2+c^2}$$

$$\Rightarrow \frac{ds}{d\theta} = \lambda$$

Put in (1)

$$\vec{t} = (-a\sin\theta\, ,\, a\cos\theta\, ,\, c)\,.\,\frac{1}{\lambda}$$

Diff. w.r.t 's'

$$\frac{d\vec{t}}{ds} = \frac{d\vec{t}}{d\theta}\cdot\frac{d\theta}{ds}$$

$$\vec{t}' = \frac{d}{d\theta}(-a\sin\theta\, ,a\cos\theta\, ,c)\,.\frac{1}{\lambda}\ \frac{d\theta}{ds}$$

$$K\vec{n} = (-a\cos\theta, a\sin\theta, 0)\frac{1}{\lambda}.\frac{1}{\lambda} \qquad \because \frac{d\theta}{ds} = \frac{1}{\lambda}$$

$$\Rightarrow \quad K\vec{n} = (-a\cos\theta, -a\sin\theta, 0)\left(\frac{1}{\lambda^2}\right) \qquad \ldots\ldots(4)$$

Squaring both sides

$$K^2\,(\vec{n}.\vec{n}) = (a^2\cos^2\theta + a^2\sin^2\theta).\frac{1}{\lambda^4}$$

$$K^2 = (a^2(\cos^2\theta + \sin^2\theta)).\frac{1}{\lambda^4}$$

$$K = \frac{a}{\lambda^2} \qquad \textit{put in } (4)$$

$$\frac{a}{\lambda^2}\vec{n} = a(-\cos\theta, -\sin\theta, 0).\frac{1}{\lambda^2}$$

$$\vec{n} = (-\cos\theta, -\sin\theta, 0) \qquad (5)$$

Now $\vec{b} = \vec{t}\times\vec{n}$

$$= \begin{vmatrix} i & j & k \\ \dfrac{-a\sin\theta}{\lambda} & \dfrac{a\cos\theta}{\lambda} & \dfrac{c}{\lambda} \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= i\left(0 + \frac{c}{\lambda}\sin\theta\right) - i\left(0 + \frac{c}{\lambda}\cos\theta\right) + k\left(\frac{a}{\lambda}\sin^2\theta + \frac{a}{\lambda}\cos^2\theta\right)$$

$$= \frac{c}{\lambda}\sin\theta\hat{i} - \frac{c}{\lambda}\cos\theta\hat{j} + \frac{a}{\lambda}\left(\sin^2\theta + \cos^2\theta\right)\hat{k}$$

$$\Rightarrow \quad \vec{b} = \left(\frac{c}{\lambda}\sin\theta, \frac{-c}{\lambda}\cos\theta, \frac{a}{\lambda} \right)$$

Spherical indicatrices for tangent is

$$x = -\frac{a\sin\theta}{\lambda}$$

$$y = \frac{a\cos\theta}{\lambda}$$

$$z = \frac{c}{\lambda}$$

Spherical indicatrices for normal is

$$x = -\cos\theta$$

$$y = -\sin\theta$$

$$z = 0$$

Spherical indicatrices for binormal is

$$x = \frac{c\sin\theta}{\lambda}$$

$$y = -\frac{c\cos\theta}{\lambda}$$

$$z = \frac{a}{\lambda}$$

Question:

Find the equation of tangent plane and normal to the surfaces z = x^2+y^2 at point (1, −1,2)

Solution Given that z = x^2+y^2

Let F[x,y,z] = z$-x^2 - y^2$

$\Rightarrow$ F_x[x,y,z] = −2x

At (1, −1,2) $\Rightarrow$ F_x[x,y,z] = −2(1) = −2

$\Rightarrow$ F_y[x,y,z] = −2y

At (1, −1,2) $\Rightarrow$ F_y[x,y,z] = −2(−1) = 2

$\Rightarrow$ F_z[x,y,z] = 1

At (1, −1,2) $\Rightarrow$ F_z[x,y,z] = 1

As equation of tangent is

$(X-x)\ F_x+(Y-y)\ F_y+(Z-z)\ F_z = 0$

$(X-1)\ (-2)+(Y+1)\ (2)+(Z-2)\ (1) = 0$

$-2X+2+2Y+2+Z-2 = 0$

$-2X+2Y+Z+2 = 0$

$-2X+2Y+Z = -2$

Now equation of Normal is

$$\frac{(X-x)}{F_x} = \frac{(Y-y)}{F_y} = \frac{(Z-z)}{F_z}$$

$$\frac{(X-1)}{-2} = \frac{(Y+1)}{2} = \frac{(Z-2)}{1} = \lambda$$

Question:

Find the equation of tangent plane and normal to the surfaces $a^{2/3} = x^{2/3}+y^{2/3}+z^{2/3}$ at point (1,2,2)

Solution Given that $a^{2/3} = x^{2/3}+y^{2/3}+z^{2/3}$

Let $F[x,y,z] = x^{2/3}+y^{2/3}+z^{2/3} - a^{2/3}$

$F_x[x,y,z] = \frac{2}{3}x^{-1/3}$, $F_y[x,y,z] = \frac{2}{3}y^{-1/3}$, $F_y[x,y,z] = \frac{2}{3}z^{-1/3}$

AT (1,2,2)

$F_x[x,y,z] = \frac{2}{3}(1)^{-1/3} = \frac{2}{3}$, $F_y[x,y,z] = \frac{2}{3}(2)^{-1/3} = \frac{2^{2/3}}{3}$,

$F_z[x,y,z] = \frac{2}{3}(2)^{-1/3} = \frac{2^{2/3}}{3}$

As equation of tangent is

$(X-x)\ F_x+(Y-y)\ F_y+(Z-z)\ F_z = 0$

$(X-1)\ (\frac{2}{3})+(Y-2)\ (\frac{2^{2/3}}{3})+(Z-2)\ (\frac{2^{2/3}}{3}) = 0$

$\frac{1}{3}\,[2(X-1)+ 2^{2/3}(Y-2)+ 2^{2/3}(Z-2)] = 0$

$2(X-1)+ 2^{2/3}(Y-2)+ 2^{2/3}(Z-2) = 0$

Now equation of Normal is

$$\frac{(X-x)}{F_x} = \frac{(Y-y)}{F_y} = \frac{(Z-z)}{F_z}$$

$$\frac{(X-1)}{\frac{2}{3}} = \frac{(Y+1)}{\frac{2^{2}/3}{3}} = \frac{(Z-2)}{\frac{2^{2}/3}{3}} = \lambda$$

is the equation of normal.

Surface:

A surface is said to be a locus of a point whose cartesian coordinate (x,y,z) are function of independent parameter u and v i.e.

x = f(u,v)

y = g(u,v)

z = h(u,v)

Another definition of Surface:

A surface S is locus of point whose coordinates can be expressed as the function of two independent variables i.e.

$$\left.\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}\right\} \quad \ldots(1) \quad \text{where } a \le u \le b \quad \text{and} \quad c \le v \le d$$

In vector form $\vec{r} = \vec{r}$ (u,v) denotes the equation of surfaces. Equation (1) called Gaussian form of the surface. Sometimes it is possible to eliminate u and v to get functional relation

$$f(x,y,z) = c \qquad \ldots(2)$$

which is called implicit form of surface. It is possible only if the matrix

$$M = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$$ has rank 2 those points where matrix M has rank 0 or 1 is called singular point. The implicit form of eq (2) can be written as

$$z = f(x,y) \qquad \ldots..(3)$$

Which is called as Monge's form of surface

Example:

x = acosϕsinψ , y = acosϕcosψ , z = asinϕ

find matrix

Solution:

$$\frac{\partial x}{\partial \phi} = \text{-asin}\phi\text{sin}\psi \quad , \quad \frac{\partial x}{\partial \psi} = \text{acos}\phi\text{cos}\psi$$

$$\frac{\partial y}{\partial \phi} = \text{-asin}\phi\text{cos}\psi \quad , \quad \frac{\partial y}{\partial \psi} = \text{-acos}\phi\text{sin}\psi$$

$$\frac{\partial x}{\partial \phi} = \text{acos}\phi \quad , \quad \frac{\partial x}{\partial \psi} = 0$$

$$M = \begin{bmatrix} -\text{asin}\phi\text{sin}\psi & -\text{asin}\phi\text{cos}\psi & \text{acos}\phi \\ \text{acos}\phi\text{cos}\psi & -\text{acos}\phi\text{sin}\psi & 0 \end{bmatrix}$$

Lecture # 11

Tangent plane:

Let $\underline{r} = \underline{r}$ (u,v) be the equation of surface in terms of parameters u,v. Then

$$\frac{dr}{ds} = \frac{\partial r}{\partial u} \cdot \frac{du}{ds} + \frac{\partial r}{\partial v} \cdot \frac{dv}{ds}$$

$$\frac{dr}{ds} = r_1 \cdot \frac{du}{ds} + r_2 \cdot \frac{dv}{ds} \qquad \text{where} \qquad \underline{r_1} = \frac{\partial r}{\partial u}\ , \ \underline{r_2} = \frac{\partial r}{\partial v}$$

Equation of Normal

$$\underline{N} = \frac{|\underline{r_1} \times \underline{r_2}|}{|\underline{r_1} + \underline{r_2}|}$$

N, r_1, r_2 forms right handed system.

Parametric curves:

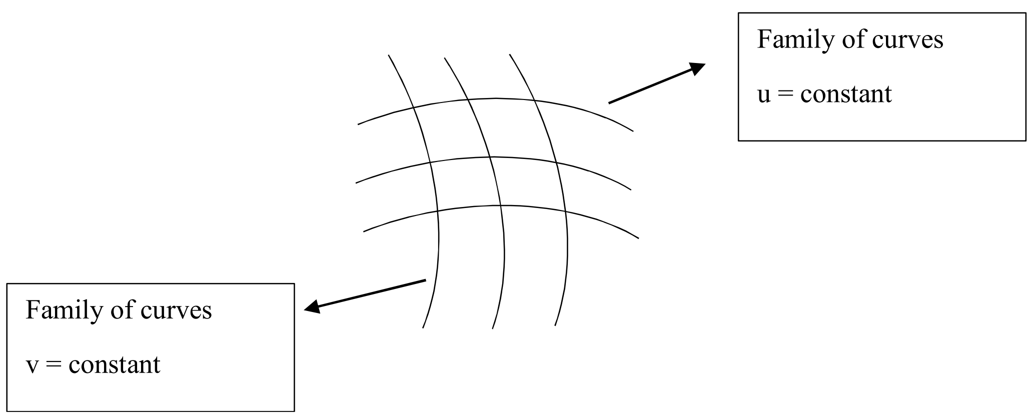
Let $\underline{r} = \underline{r}$ (u,v) be the equation of surface. Now by keeping u = c (constant) or v = c (constant) we get curves of spherical importance and are called parametric curves.

U-curves:

If v = c and u varies the points $\underline{r} = \underline{r}$ (u,c) describe a parametric curve called u-curve or parametric curve at v = c.

V-curves:

If u = c and v varies the points $\underline{r} = \underline{r}$ (c,v) describe a parametric curve called u-curve or parametric curve at u = c.



First order Fundamental magnitude:

As $\underline{r_1} = \frac{\partial r}{\partial u}$, $\underline{r_2} = \frac{\partial r}{\partial v}$

$\underline{r_{11}} = \frac{\partial^2 r}{\partial u^2}$, $\underline{r_{22}} = \frac{\partial^2 r}{\partial v^2}$

And $\underline{r_{12}} = \frac{\partial^2 r}{\partial u \partial v}$

The vector $\underline{r_1}$ is tangential to the curve v = constant at the point r_0. Its direction is that the displacement dr due to the variation du in the first parameter.

We take the positive direction along the parametric curve v = constant i.e. for which u increases. Similarly, vector r_2 is the tangent to the curve when u = constant which correspond to the increase of v. Consider the neighbouring point on the surface which position vector r and r+dr corresponding to the parametric to the parametric value u,v and u+du , v+dv respectively

Then

$$d\underline{r} = \frac{\partial r}{\partial u} . du + \frac{\partial r}{\partial v} . dv$$

$$= \underline{r_1} . du + \underline{r_2} . dv$$

Since the two points are adjacent point on the curve passing through them.

The length ds of the element of arc joining them is equal to their actual distance.

$$d\underline{r} = d\underline{s} = \underline{r_1} . du + \underline{r_2} . dv$$

$$(ds)^2 = \left(\underline{r_1} . du + \underline{r_2} . dv\right)^2$$

$$(ds)^2 = {r_1}^2 du^2 + {r_2}^2 dv^2 + 2 r_1 r_2 dudv$$

As $E = {r_1}^2$, $F = r_1 . r_2$, $G = {r_2}^2$

$$(ds)^2 = Edu^2 + 2Fdudv + Gdv^2 \qquad .....(1)$$

The quantities denoted by E,F,G are called Fundamental magnitude of first order.

The quantity EG$-F^2$ is positive on real surface in u and v are real $\sqrt{G}$ and $\sqrt{E}$ are the modulus of r_1 and r_2 and if it denotes the angle between these vectors.

Let H^2 = EG$-F^2$ and He be the positive square root to this quantity.

Question:

Calculate the 1st fundamental magnitude as x = ucosϕ , y = sinϕ , z = cϕ

Sol.

Let $\underline{r}$ (x,y,z) = $\underline{r}$ (ucosϕ, usinϕ, cϕ)

$\frac{\partial \underline{r}}{\partial u}$ (x,y,z) = r_1 = (cosϕ , sinϕ , 0)

$\frac{\partial^2 r}{\partial u^2}$ (x,y,z) = r_{11} = (0,0,0)

$\frac{\partial \underline{r}}{\partial \phi}$ (x,y,z) = r_2 = (-usinϕ , ucosϕ , c)

$\frac{\partial^2 r}{\partial \phi^2}$ (x,y,z) = r_{22} = (-ucosϕ , -usinϕ , 0)

$\frac{\partial^2 r}{\partial u \partial \phi}$ (x,y,z) = r_{12}= (-sinϕ , cosϕ , 0)

Now for first order fundamental magnitude

E = ${r_1}^2$ = (cosϕ , sinϕ , 0) . (cosϕ , sinϕ , 0)

= $cos^2\phi + sin^2\phi$ +0

E = 1

F = $r_1 r_2$

= (cosϕ , sinϕ , 0) . (-usinϕ , ucosϕ , c)

= -ucosϕsinϕ+ucosϕsinϕ+0

= 0

G = ${r_2}^2$

= (-usinϕ , ucosϕ , c) . (-usinϕ , ucosϕ , c)

= $u^2 sin^2\phi + u^2 cos^2\phi + c^2$

= $u^2 + c^2$

Question:

Take x,y as parameters. Calculate the first fundamental magnitude of

2z = ax^2+2hxy+by^2

Solution:

Given $2z = ax^2 + 2hxy + by^2$

$$z = \frac{ax^2+2hxy+by^2}{2}$$

$$\underline{r}\,(x,y,z) = \underline{r}\,\left(x,\ y,\ \frac{ax^2+2hxy+by^2}{2}\right)$$

$$\frac{\partial \underline{r}}{\partial x}\,(x,y,z) = r_1 = \left(1\ ,\ 0\ ,\ \frac{2ax+2hy}{2}\right)$$

$$= (1,0,\ ax+hy)$$

$$\frac{\partial^2 r}{\partial x^2}\,(x,y,z) = r_{11} = (0,0,a)$$

$$\frac{\partial \underline{r}}{\partial y}\,(x,y,z) = r_2 = \left(0,\ 1\ ,\ \frac{2by+2hx}{2}\right)$$

$$= (0,1,\ by+hx)$$

$$\frac{\partial^2 r}{\partial y^2}\,(x,y,z) = r_{22} = (0\ ,\ 0\ ,\ b)$$

$$\frac{\partial^2 r}{\partial x \partial y}\,(x,y,z) = r_{12} = (0\ ,\ 0\ ,\ h)$$

Now for first order fundamental magnitude

$$E = {r_1}^2 = (1,0,\ ax+hy).\ (1,0,\ ax+hy)$$

$$= 1 + (ax + hy)^2$$

$$F = r_1 r_2$$

$$= (1,0,\ ax+hy)\ .\ (0,1,\ by+hx)$$

$$= (ax+hy)\ (by+hx)$$

$$G = {r_2}^2$$

$$= (0,1,\ by+hx).\ (0,1,\ by+hx)$$

$$= 1 + (hx + by)^2$$

Question: For the surface x = ucosϕ , y = sinϕ , z = f(u). Find first fundamental magnitude.

Solution:

Let $\underline{r}\,(x,y,z) = \underline{r}\,(u\cos\phi,\ u\sin\phi,\ f(u))$

$$\frac{\partial r}{\partial u}\,(x,y,z) = r_1 = (\cos\phi\ ,\ \sin\phi\ ,\ f'(u))$$

$\frac{\partial^2 r}{\partial u^2}$ (x,y,z) = r_{11} = (0,0, $f''(u)$)

$\frac{\partial r}{\partial \phi}$ (x,y,z) = r_2 = (-usinϕ , ucosϕ , 0)

$\frac{\partial^2 r}{\partial \phi^2}$ (x,y,z) = r_{22} = (-ucosϕ , -usinϕ , 0)

$\frac{\partial^2 r}{\partial u \partial \phi}$ (x,y,z) = r_{12}= (-sinϕ , cosϕ , 0)

Now for first order fundamental magnitude

E = $r_1{}^2$ = (cosϕ , sinϕ , $f'(u)$) . (cosϕ , sinϕ , $f'(u)$)

= $cos^2\phi + sin^2\phi + f'^2(u)$

E = 1

F = $r_1 r_2$

= (cosϕ , sinϕ , $f'(u)$) . (-usinϕ , ucosϕ , 0)

= -ucosϕsinϕ+ucosϕsinϕ+0

= 0

G = $r_2{}^2$

= (-usinϕ , ucosϕ , 0) . (-usinϕ , ucosϕ , 0)

= $u^2 sin^2\phi + u^2 cos^2\phi + 0$

G = u^2

Lecture # 12

Second order Fundamental form and second order Fundamental magnitude:

Let $\vec{r} = \vec{r}$(u,v) be the equation of surface and N be the normal to this surface at the point $\vec{r}$(u,v)

$$\vec{N} = \frac{\vec{r_1} \times \vec{r_2}}{|\vec{r_1} \times \vec{r_2}|}$$

$$\Rightarrow \quad \vec{N} = \frac{\vec{r_1} \times \vec{r_2}}{H} \quad \text{....(A)} \qquad \text{where} \qquad H = |\vec{r_1} \times \vec{r_2}|$$

As we know that

$$\overrightarrow{r_{11}} = \frac{\partial^2 \vec{r}}{\partial u^2}, \quad \overrightarrow{r_{22}} = \frac{\partial^2 \vec{r}}{\partial v^2}$$

And $$\overrightarrow{r_{12}} = \frac{\partial^2 \vec{r}}{\partial u \partial v}, \quad \overrightarrow{r_{21}} = \frac{\partial^2 \vec{r}}{\partial v \partial u}$$

If $L = \overrightarrow{r_{11}} \cdot \vec{N}$, $M = \overrightarrow{r_{12}} \cdot \vec{N}$, $N = \overrightarrow{r_{22}} \cdot \vec{N}$

Then the quadratic equation form

Ldu^2+2Mdudv+Mdv^2 where du,dv is called second order fundamental form. The quantities L,M,N are called the second Fundamental magnitude

Alternative form for L,M,N since the vector $\vec{r_1}$ *and* $\vec{r_2}$ are tangential to the surface at point $\vec{r}$. So unit vector N is perpendicular to both vectors $\vec{r_1}$ *and* $\vec{r_2}$

Then we have

$\vec{N}. \ \vec{r_1} = 0$ .... (1) & $\vec{N}. \ \vec{r_2} = 0$ .... (2)

Diff. (1) w.r.t 'u'

$\vec{N}. \ \overrightarrow{r_{11}} + \vec{N_1}. \ \vec{r_1} = 0$

$\vec{N}. \ \overrightarrow{r_{11}} = -\vec{N_1}. \ \vec{r_1}$

$L = \vec{N}. \ \overrightarrow{r_{11}} = -\vec{N_1}. \ \vec{r_1}$

Diff. (2) w.r.t 'v'

$\vec{N}. \ \overrightarrow{r_{22}} + \vec{N_2}. \ \vec{r_2} = 0$

$\vec{N}. \ \overrightarrow{r_{22}} = -\vec{N_2}. \ \vec{r_2}$

$N = \vec{N}. \ \overrightarrow{r_{22}} = -\vec{N_2}. \ \vec{r_2}$

Diff. (1) w.r.t 'v'

$\vec{N}. \ \overrightarrow{r_{12}} + \vec{N_2}. \ \vec{r_1} = 0$

$\vec{N}. \ \overrightarrow{r_{12}} = -\vec{N_2}. \ \vec{r_1} \quad \Rightarrow M = \vec{N}. \ \overrightarrow{r_{12}} = -\vec{N_2}. \ \vec{r_1}$

In the case of three vectors

$[\vec{r_1}, \vec{r_2}, \vec{r_{11}}] = \vec{r_1} \times \vec{r_2} \cdot \vec{r_{11}}$

$= H\vec{N} \cdot \vec{r_{11}}$ From (A) $\vec{r_1} \times \vec{r_2} = H\vec{N}$

$= HL$ $\because \vec{N} \cdot \vec{r_{11}} = L$

$[\vec{r_1}, \vec{r_2}, \vec{r_{22}}] = \vec{r_1} \times \vec{r_2} \cdot \vec{r_{22}}$

$= H\vec{N} \cdot \vec{r_{22}}$ From (A) $\vec{r_1} \times \vec{r_2} = H\vec{N}$

$= HN$ $\because \vec{N} \cdot \vec{r_{22}} = N$

Question:

Calculate the 2nd fundamental magnitude as x = ucosϕ , y = sinϕ , z = cϕ

Sol.

Let $\vec{r}$ (x,y,z) = $\vec{r}$ (ucosϕ, usinϕ, cϕ)

$\frac{\partial \vec{r}}{\partial u}$ (x,y,z) = $\vec{r_1}$ = (cosϕ , sinϕ , 0)

$\frac{\partial^2 \vec{r}}{\partial u^2}$ (x,y,z) = $\vec{r}_{11}$ = (0,0,0)

$\frac{\partial \vec{r}}{\partial \phi}$ (x,y,z) = $\vec{r}_2$ = (-usinϕ , ucosϕ , c)

$\frac{\partial^2 \vec{r}}{\partial \phi^2}$ (x,y,z) = $\vec{r}_{22}$ = (-ucosϕ , -usinϕ , 0)

$\frac{\partial^2 \vec{r}}{\partial u \partial \phi}$ (x,y,z) = $\vec{r}_{12}$ = (-sinϕ , cosϕ , 0)

Now $\vec{N} = \frac{\vec{r_1} \times \vec{r_2}}{|\vec{r_1} \times \vec{r_2}|}$

$$\vec{r_1} \times \vec{r_2} = \begin{vmatrix} i & j & k \\ \cos\phi & \sin\phi & 0 \\ -u\sin\phi & u\cos\phi & c \end{vmatrix}$$

$= (c\sin\phi - 0)\,\hat{\imath} - (c\cos\phi - 0)\hat{\jmath} + (u\cos^2\phi + u\sin^2\phi)\hat{k}$

$= c\sin\phi\hat{\imath} - c\cos\phi\hat{\jmath} + u\hat{k}$

$\vec{r_1} \times \vec{r_2}$ = (csinϕ , -c cosϕ , u)

$|\vec{r_1} \times \vec{r_2}| = \sqrt{c^2\sin^2\phi + c^2\cos^2\phi + u^2} = \sqrt{c^2 + u^2}$

$\Rightarrow \quad \vec{N} = \frac{(\text{csin}\phi\,,-\text{c}\cos\phi\,,\text{u})}{\sqrt{c^2+u^2}}$

$L = \overrightarrow{r_{11}}\ .\vec{N}\ = (0,0,0)\ .\ \frac{(\text{csin}\phi\,,-\text{c}\cos\phi\,,\text{u})}{\sqrt{c^2+u^2}} = 0$

$M = \overrightarrow{r_{12}}\ .\vec{N}\ = (-\sin\phi\ ,\ \cos\phi\ ,\ 0)\ .\ \frac{(\text{csin}\phi\,,-\text{c}\cos\phi\,,\text{u})}{\sqrt{c^2+u^2}}$

$= \frac{-c}{\sqrt{c^2+u^2}}$

$N = \overrightarrow{r_{22}}.\vec{N}\ = (-\text{ucos}\phi\ ,\ -\text{usin}\phi\ ,\ 0)\ .\ \frac{(\text{csin}\phi\,,-\text{c}\cos\phi\,,\text{u})}{\sqrt{c^2+u^2}}$

$= \frac{-\text{ucsin}\phi\cos\phi+\text{cusin}\phi\cos\phi + 0)}{\sqrt{c^2+u^2}}$

$= 0$

Question:

Take x,y as parameters. Calculate the 2nd fundamental magnitude of

$2z = ax^2+2hxy+by^2$

Solution:

Given $\qquad 2z = ax^2+2hxy+by^2$

$z = \frac{ax^2+2hxy+by^2}{2}$

$\vec{r}\,(x,y,z) = \underline{\vec{r}}\,(x,\ y,\ \frac{ax^2+2hxy+by^2}{2})$

$\frac{\partial\vec{r}}{\partial x}\,(x,y,z) = \vec{r}_1 = (1\ ,\ 0\ ,\ \frac{2ax+2hy}{2})$

$= (1,0,\ ax+hy)$

$\frac{\partial^2\vec{r}}{\partial x^2}\,(x,y,z) = \ \vec{r}_{11} = (0,0,a)$

$\frac{\partial\vec{r}}{\partial y}\,(x,y,z) = \vec{r}_2 = (0,\ 1\ ,\ \frac{2by+2hx}{2})$

$= (0,1,\ by+hx)$

$\frac{\partial^2\vec{r}}{\partial y^2}\,(x,y,z) = \ \vec{r}_{22} = (0\ ,\ 0\ ,\ b)$

$\frac{\partial^2\vec{r}}{\partial x\partial y}\,(x,y,z) = \ \vec{r}_{12} = (0\ ,\ 0\ ,\ h)$

Now for 2nd order fundamental magnitude

Now $\vec{N} = \frac{\vec{r_1} \times \vec{r_2}}{|\vec{r_1} \times \vec{r_2}|}$

$$\vec{r_1} \times \vec{r_2} = \begin{vmatrix} i & j & k \\ 1 & 0 & ax+hy \\ 0 & 1 & by+hx \end{vmatrix}$$

$$= (0-(ax+hy))\,\hat{\imath} - (hx+by-0)\hat{\jmath} + (1-0)\hat{k}$$

$$= -(ax+hy)\hat{\imath} - (hx+by)\hat{\jmath} + \hat{k}$$

$$\vec{r_1} \times \vec{r_2} = (-(ax+hy)\,,\, -(hx+by)\,,\, 1)$$

$$|\vec{r_1} \times \vec{r_2}| = \sqrt{(ax+hy)^2 + (by+hx)^2 + (1)^2}$$

$$= \sqrt{(ax+hy)^2 + (by+hx)^2 + 1}$$

$$\Rightarrow \vec{N} = \frac{(-(ax+hy)\,,-(hx+by)\,,1)}{\sqrt{(ax+hy)^2+(by+hx)^2+1}}$$

$$L = \vec{r_{11}}\,.\vec{N} = (0,0,a)\,.\,\frac{(-(ax+hy)\,,-(hx+by)\,,1)}{\sqrt{(ax+hy)^2+(by+hx)^2+1}} = \frac{a}{\sqrt{(ax+hy)^2+(by+hx)^2+1}}$$

$$M = \vec{r_{12}}\,.\vec{N} = (0\,,0\,,h).\frac{(-(ax+hy)\,,-(hx+by)\,,1)}{\sqrt{(ax+hy)^2+(by+hx)^2+1}}$$

$$= \frac{h}{\sqrt{(ax+hy)^2+(by+hx)^2+1}}$$

$$N = \vec{r_{22}}.\vec{N} = (0\,,0\,,b).\frac{(-(ax+hy)\,,-(hx+by)\,,1)}{\sqrt{(ax+hy)^2+(by+hx)^2+1}}$$

$$= \frac{b}{\sqrt{(ax+hy)^2+(by+hx)^2+1}}$$

Question: For the surface x = ucosϕ , y = usinϕ , z = f(u). Find 2nd fundamental magnitude.

Solution:

Let $\vec{r}$ (x,y,z) = $\vec{r}$ (ucosϕ, usinϕ, f(u))

$$\frac{\partial \vec{r}}{\partial u}(x,y,z) = r_1 = (\cos\phi\,,\, \sin\phi\,,\, f'(u))$$

$$\frac{\partial^2 \vec{r}}{\partial u^2}(x,y,z) = \vec{r}_{11} = (0,0,f''(u))$$

$$\frac{\partial \vec{r}}{\partial \phi}(x,y,z) = \vec{r}_2 = (-u\sin\phi\,,\, u\cos\phi\,,\, 0)$$

$$\frac{\partial^2 \vec{r}}{\partial \phi^2}(x,y,z) = \vec{r}_{22} = (-u\cos\phi\,,\, -u\sin\phi\,,\, 0)$$

$$\frac{\partial^2\vec{r}}{\partial u\partial\phi}(x,y,z) = \vec{r}_{12} = (-\sin\phi , \cos\phi , 0)$$

Now for 2nd order fundamental magnitude

Now $\vec{N} = \frac{\vec{r_1}\times\vec{r_2}}{|\vec{r_1}\times\vec{r_2}|}$

$$\vec{r_1}\times\vec{r_2} = \begin{vmatrix} i & j & k \\ \cos\phi & \sin\phi & f'(u) \\ -u\sin\phi & u\cos\phi & 0 \end{vmatrix}$$

$= (0-u\cos\phi f'(u))\,\hat{\imath} - (0+u\sin\phi f'(u))\hat{\jmath} + (u\cos^2\phi + u\sin^2\phi)\hat{k}$

$= -(u\cos\phi f'(u))\,\hat{\imath} - (u\sin\phi f'(u))\hat{\jmath} + u\hat{k}$

$= ((-u\cos\phi f'(u)) , (-u\sin\phi f'(u)) , u)$

$\vec{r_1}\times\vec{r_2} = ((-u\cos\phi f'(u)) , (-u\sin\phi f'(u)) , u)$

$|\vec{r_1}\times\vec{r_2}| = \sqrt{((-u\cos\phi f'(u))^2 + ((-u\sin\phi f'(u))^2 + (u)^2}$

$= \sqrt{u^2 + u^2 f'^2(u)}$

$$\Rightarrow \vec{N} = \frac{((-u\cos\phi f'(u)),(-u\sin\phi f'(u)),u)}{\sqrt{u^2+u^2f'^2(u)}}$$

$$L = \vec{r_{11}}\,.\vec{N} = (0,0,f''(u)).\frac{((-u\cos\phi f'(u)),(-u\sin\phi f'(u)),u)}{\sqrt{u^2+u^2f'^2(u)}}$$

$$= \frac{u\,f''(u)}{\sqrt{u^2+u^2f'^2(u)}}$$

$$M = \vec{r_{12}}\,.\vec{N} = (-\sin\phi , \cos\phi , 0).\frac{((-u\cos\phi f'(u)),(-u\sin\phi f'(u)),u)}{\sqrt{u^2+u^2f'^2(u)}}$$

$$= 0$$

$$N = \vec{r_{22}}.\vec{N} = (-u\cos\phi , -u\sin\phi , 0).\frac{((-u\cos\phi f'(u)),(-u\sin\phi f'(u)),u)}{\sqrt{u^2+u^2f'^2(u)}}$$

$$= \frac{u\,f'(u)}{\sqrt{u^2+u^2f'^2(u)}}$$