Chapter 6

CURVILINEAR COORDINATES

SOME IMPORTANT FORMULAS OF THIS CHAPTER

➢ Curvilinear Coordinates

$P(u_1 \ , u_2 \ , u_3)$

Unit vectors= $(\hat{e}_1 \ , \hat{e}_2 \ , \hat{e}_3)$

Gradient

in curvilinear Coordinate System

$$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3 \text{ ------------------(1)}$$

Divergence

in curvilinear system

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right] \text{ ------------------(2)}$$

Curl

in curvilinear coordinates system

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \text{ ------------------(3)}$$

Laplacian

in Curvilinear coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right] \text{ ------------------(4)}$$

Jacobian

in Curvilinear coordinate system.

$J\left(\frac{x\ ,\ \ y\ ,\ \ z}{u_1\ ,u_2\ ,\ u_3}\right) = h_1h_2h_3$-----------(5)

Arc length Element

in Curvilinear coordinate system.

$(ds)^2 = (h_1)^2(du_1)^2 + (h_1)^2(du_2)^2 + (h_3)^2(du_3)^2$--------(6)

➢ Cylindrical Polar Coordinate System:

P(x, y,z) = P(r , θ , z)

x=rcosθ , y=rsinθ , z=z

P(x, y,z) = P(r , θ , z)

Relationship between units vectors in Rectangular and Cylindrical Ploar Coordinates

In matrix form

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$ **(from Cylindrical to Rectangular)----------(7)**

In Equations form

$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$

$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta$

$\hat{e}_z = \hat{k}$

In matrix form

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix}$$ **(from Rectangular to Cylindrical) ---------(8)**

In Equations form

$\hat{i} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta$

$\hat{j} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$

$\hat{k} = \hat{e}_z$

Gradient

in curvilinear Coordinate System

$$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Cylindrical Coordinate System

$$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$$

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z \text{-------------(9)}$$

Divergence

We know that in curvilinear system

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

Hence In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Spherical Coordinate System

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial}{\partial\theta}(A_\theta) + \frac{\partial}{\partial\phi}(rA_z)\right] \text{----------------(10)}$$

Curl

We know that in curvilinear system

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} \text{------------------}(11)$$

Laplacian

in Curvilinear coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in cylindrical Coordinate system

$$\nabla^2\psi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(r\frac{\partial\psi}{\partial \phi}\right)\right] \text{--------------}(12)$$

Jacobian

in Curvilinear coordinate system.

$$J\left(\frac{x\ ,\ y\ ,\ z}{u_1\ ,u_2\ ,\ u_3}\right) = h_1h_2h_3$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in cylindrical Coordinate system

$$J\left(\frac{x\ ,\ y\ ,\ z}{r\ ,\theta\ ,\ z}\right) = h_1h_2h_3 = r \text{------------------}(13)$$

Arc length Element

in Curvilinear coordinate system.

$$(ds)^2 = (h_1)^2(du_1)^2 + (h_1)^2(du_2)^2 + (h_3)^2(du_3)^2$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in cylindrical Coordinate system

$(ds)^2 = (h_r)^2(dr)^2 + (h_\theta)^2(d\theta)^2 + (h_z)^2(dz)^2$

$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (1)^2(dz)^2$

$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$--------------------(14)

➢ Spherical Polar Coordinate System

P(x, y,z) = P(r , θ , ϕ)

x=rsinθcos ϕ , y=rsinθsinϕ , z=rcosθ ,

P(x, y,z) = P(r , θ , ϕ)

Relationship between units vectors in Rectangular and Spherical Coordinates

In matrix form

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$ (from Spherical to Rectangular)-----------(15)

In Equations form

$\hat{e}_r = \sin\theta\cos\phi\hat{i} + \sin s\sin\phi\hat{j} + \cos\theta\hat{k}$

$\hat{e}_\theta = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$

$\hat{e}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j}$

In matrix form

$$\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\phi\end{bmatrix}$$ **(from Rectangular to Spherical)-**

-----------(16)

In Equations form

$\hat{i}= \sin\theta\cos\phi\hat{e}_r + \cos\theta\cos\phi\hat{e}_\theta - \sin\phi\hat{e}_\phi$

$\hat{j}= \sin\theta\sin\phi\hat{e}_r + \cos\theta\sin\phi\hat{e}_\theta + \cos\phi\hat{e}_\phi$

$\hat{k}= \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta$

Gradient

in curvilinear Coordinate System

$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Spherical Coordinate System

$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_\phi$

$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial\phi}\right)\hat{e}_\phi$ **-----------(17)**

Divergence

We know that in curvilinear system

$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Spherical Coordinate System

$$\nabla.\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial \phi}(rA_\phi)\right] \text{---------(18)}$$

Curl

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \text{---------(19)}$$

Laplacian

in Curvilinear coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Spherical Coordinate system

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right] \text{-------(20)}$$

Jacobian

in Curvilinear coordinate system.

$J\left(\frac{x\ ,\ y\ ,\ z}{u_1\ ,u_2\ ,\ u_3}\right) = h_1 h_2 h_3$

In Spherical Coordinate System

$h_1 = h_r = 1\ ,\ h_2 = h_\theta = r\ ,\ h_3 = h_\phi = r\sin\theta\ ,\ (u_1\ ,u_2\ ,u_3) = (r\ ,\theta\ ,\phi)$

Hence in Spherical Coordinate system

$J\left(\frac{x\ ,\ y\ ,\ z}{r\ ,\ \theta\ ,\ \phi}\right) = h_1 h_2 h_3 = r^2\sin\theta$ -------------(21)

Arc length Element

in Curvilinear coordinate system.

$(ds)^2 = (h_1)^2(du_1)^2 + (h_1)^2(du_2)^2 + (h_3)^2(du_3)^2$

In Spherical Coordinate System

$h_1 = h_r = 1\ ,\ h_2 = h_\theta = r\ ,\ h_3 = h_\phi = r\sin\theta\ ,\ (u_1\ ,u_2\ ,u_3) = (r\ ,\theta\ ,\phi)$

$(ds)^2 = (h_r)^2(dr)^2 + (h_\theta)^2(d\theta)^2 + (h_\phi)^2(d\phi)^2$

$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2$

$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$-------------(22)

➢ Rectangular Coordinates

$P(u_1\ ,u_2\ ,u_3) = P(x, y, z)$

$(\hat{e}_1\ ,\hat{e}_2\ ,\hat{e}_3) = (\hat{i}\ ,\hat{j}\ ,\hat{k})$ from curvilinear to Rectangular

Gradient

in curvilinear Coordinate System

$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$

In Rectangular Coordinate System

$h_1 = 1\ ,\ h_2 = 1\ ,\ h_3 = 1\ ,\ (u_1\ ,u_2\ ,u_3) = (x\ ,y\ ,z)\ ,\ (\hat{e}_1\ ,\hat{e}_2\ ,\hat{e}_3) = (\hat{i}\ ,\hat{j}\ ,\hat{k})$

Hence in Rectangular Coordinate System

$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial x}\right)\hat{i} + \frac{1}{1}\left(\frac{\partial\psi}{\partial y}\right)\hat{i} + \frac{1}{1}\left(\frac{\partial\psi}{\partial z}\right)\hat{k}$

$$\nabla\psi = \left(\frac{\partial\psi}{\partial x}\right)\hat{\imath} + \left(\frac{\partial\psi}{\partial y}\right)\hat{\jmath} + \left(\frac{\partial\psi}{\partial z}\right)\hat{k} \text{-----------(23)}$$

Divergence

in curvilinear system

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

In Rectangular Coordinate System

$h_1 = 1$, $h_2 = 1$, $h_3 = 1$, $(u_1, u_2, u_3) = (x, y, z)$, $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{\imath}, \hat{\jmath}, \hat{k})$

Hence in Rectangular Coordinate System

$$\nabla.\vec{A} = \frac{1}{1}\left[\frac{\partial}{\partial x}(A_1) + \frac{\partial}{\partial y}(A_2) + \frac{\partial}{\partial z}(A_3)\right]$$

$$\nabla.\vec{A} = \left[\frac{\partial}{\partial x}(A_1) + \frac{\partial}{\partial y}(A_2) + \frac{\partial}{\partial z}(A_3)\right] \text{-----------------(24)}$$

Curl

We know in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Rectangular Coordinate System

$h_1 = 1$, $h_2 = 1$, $h_3 = 1$, $(u_1, u_2, u_3) = (x, y, z)$, $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{\imath}, \hat{\jmath}, \hat{k})$

Hence in Rectangular Coordinate System

$$\nabla \times \vec{A} = \frac{1}{1}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{1}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \text{------------(25)}$$

Laplacian

in Curvilinear coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

In Rectangular Coordinate System

$h_1 = 1$, $h_2 = 1$, $h_3 = 1$, $(u_1, u_2, u_3) = (x, y, z)$, $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{i}, \hat{j}, \hat{k})$

Hence in Rectangular Coordinate System

$$\nabla^2\psi = \frac{1}{1}\left[\frac{\partial}{\partial x}\left(\frac{1}{1}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{1}\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{1}\frac{\partial\psi}{\partial z}\right)\right]$$

$$\nabla^2\psi = \left[\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right)\right] \text{-----------(26)}$$

Jacobian

in Curvilinear coordinate system.

$$J\left(\frac{x\ ,\ y\ ,\ z}{u_1\ ,u_2\ ,\ u_3}\right) = h_1h_2h_3$$

In Rectangular Coordinate System

$h_1 = 1$, $h_2 = 1$, $h_3 = 1$, $(u_1, u_2, u_3) = (x, y, z)$, $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{i}, \hat{j}, \hat{k})$

Hence in Rectangular Coordinate System

$$J\left(\frac{x\ ,\ y\ ,\ z}{x,\ y\ ,\ z}\right) = h_1h_2h_3 = 1 \text{-----------------(27)}$$

Arc length Element

in Curvilinear coordinate system.

$$(ds)^2 = (h_1)^2(du_1)^2 + (h_1)^2(du_2)^2 + (h_3)^2(du_3)^2$$

In Rectangular Coordinate System

$h_1 = 1$, $h_2 = 1$, $h_3 = 1$, $(u_1, u_2, u_3) = (x, y, z)$, $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{i}, \hat{j}, \hat{k})$

Hence in Rectangular Coordinate System

$$(ds)^2 = (1)^2(dx)^2 + (1)^2(dy)^2 + (1)^2(dz)^2$$

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \text{----------------(28)}$$

Some other formulas in Cylindrical Polar and Spherical Polar Coordinates System

$$\frac{\partial}{\partial r}(\hat{e}_r) = 0 \quad , \frac{\partial}{\partial \theta}(\hat{e}_r) = \hat{e}_\theta \quad , \quad \frac{\partial}{\partial z}(\hat{e}_z) = 0$$

$$\frac{\partial}{\partial r}(\hat{e}_\theta) = 0 \, , \quad \frac{\partial}{\partial \theta}(\hat{e}_\theta) = -\hat{e}_r \quad \frac{\partial}{\partial z}(\hat{e}_\theta) = 0$$

$$\frac{\partial}{\partial r}(\hat{e}_z) = 0 \quad , \quad \frac{\partial}{\partial \theta}(\hat{e}_z) = 0 \quad , \quad \frac{\partial}{\partial z}(\hat{e}_z) = 0$$

And

$$\frac{\partial}{\partial r}(\hat{e}_r) = 0 \, , \quad \frac{\partial}{\partial \theta}(\hat{e}_r) = \hat{e}_\theta \, , \quad \frac{\partial}{\partial \phi}(\hat{e}_r) = \sin\theta \, \hat{e}_\phi$$

$$\frac{\partial}{\partial r}(\hat{e}_\theta) = 0 \, , \quad \frac{\partial}{\partial \theta}(\hat{e}_\theta) = -\hat{e}_r \, , \quad \frac{\partial}{\partial \theta}(\hat{e}_\theta) = \cos\theta \, \hat{e}_\phi$$

$$\frac{\partial}{\partial r}(\hat{e}_\phi) = 0 \quad , \frac{\partial}{\partial \theta}(\hat{e}_\phi) = 0 \, , \quad \frac{\partial}{\partial \phi}(\hat{e}_\phi) = -\sin\theta \, \hat{e}_r - \cos\theta \, \hat{e}_\theta$$

Exercise

Q1:Consider the curvilinear coordinate system define by

$u_1 = x+y$, $u_2 = x-y$, $u_3 = 2z$,

(i)Solve for x , y , z in term of u_1 , u_2 , u_3

(ii)Show that system is orthogonal and left handed .

Compute the scale factors.

(iii)Find expression for $(ds)^2$ in this system

(iv)Find $\nabla\psi$ for $\psi(u_1, u_2, u_3) = u_1 + u_2 + 2u_3$

(v)Find $\nabla^2\psi$ relative to this coordinate system.

Solution:

(i)

Solve for x , y , z in term of u_1 , u_2 , u_3

Given that

u_1= x+y -------------(1)

u_2= x- y -------------(2)

u_3= 2z -------------(2)

Adding (1) , (2)

$x = \frac{u_1+u_2}{2}$

and subtracting (1) , (2)

$y = \frac{u_1-u_2}{2}$

from (3)

$z = \frac{u_3}{2}$

(ii)

Show that system is orthogonal and left handed . Compute the scale factors.

We know that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

From part (i)

$x = \frac{u_1+u_2}{2}$, $y = \frac{u_1-u_2}{2}$, $z = \frac{u_3}{2}$

Now

$\vec{r} = \left(\frac{u_1+u_2}{2}\right)\hat{i} + \left(\frac{u_1-u_2}{2}\right)\hat{j} + \left(\frac{u_3}{2}\right)\hat{k}$

$\frac{\partial \vec{r}}{\partial u_1} = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}$

and

$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$

And

$\hat{e}_1 = \frac{\frac{\partial \vec{r}}{\partial u_1}}{\left|\frac{\partial \vec{r}}{\partial u_1}\right|} = \frac{\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}}{\frac{1}{\sqrt{2}}}$

$\hat{e}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Now

$$\frac{\partial \vec{r}}{\partial u_1} = \frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}$$

and

$$h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

And

$$\hat{e}_2 = \frac{\frac{\partial \vec{r}}{\partial u_2}}{\left|\frac{\partial \vec{r}}{\partial u_2}\right|} = \frac{\frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}}{\frac{1}{\sqrt{2}}} =$$

$$\hat{e}_2 = \frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$$

Now

$$\frac{\partial \vec{r}}{\partial u_3} = \frac{1}{2}\hat{k}$$

and

$$h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right| = \sqrt{\left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

And

$$\hat{e}_3 = \frac{\frac{\partial \vec{r}}{\partial u_3}}{\left|\frac{\partial \vec{r}}{\partial u_3}\right|} = \frac{\frac{1}{2}\hat{k}}{\frac{1}{2}} = \hat{k}$$

Now

$$\hat{e}_1.\hat{e}_2 = \left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right).\left(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}\right) = 0$$

$$\hat{e}_2.\hat{e}_3 = \left(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}\right).\left(\frac{1}{2}\hat{k}\right) = 0$$

$$\hat{e}_3.\hat{e}_1 = \left(\frac{1}{2}\hat{k}\right).\left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right) = 0$$

Proved that system is orthogonal

Now

$$\hat{e}_1 \times \hat{e}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - \hat{k} = -\hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - \hat{k} = -\hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{-1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + 0\hat{k} = -\hat{e}_1$$

$$\hat{e}_3 \times \hat{e}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} = \frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + 0\hat{k} = -\hat{e}_2$$

Proved that system is left handed

And scale factors

$h_1 = \frac{1}{\sqrt{2}}$, $h_2 = \frac{1}{\sqrt{2}}$, $h_3 = \frac{1}{2}$

(iii)

Find expression for $(ds)^2$ in this system

From part (i)

$x = \frac{u_1+u_2}{2}$, $y = \frac{u_1-u_2}{2}$, $z = \frac{u_3}{2}$

$dx = \frac{1}{2}(du_1+du_2)$, $dy = \frac{1}{2}(du_1-du_2)$, $dz = \frac{1}{2}du_3$

We know that

$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

$(ds)^2 = \left(\frac{1}{2}(du_1 + du_2)\right)^2 + \left(\frac{1}{2}(du_1 + du_2)\right)^2 + \left(\frac{1}{2}du_3\right)^2$

$= \frac{1}{4}[(du_1)^2 + (du_2)^2 + 2du_1du_2 \quad + (du_1)^2 + (du_2)^2 -$

$2du_1du_2) \; + (du_3)^2]$

$=\frac{1}{2}(du_1)^2+\frac{1}{2}(du_2)^2+\frac{1}{4}(du_3)^2$

Altarnate Method

We know that

$(ds)^2= (dx)^2+(dy)^2+(dz)^2$ **in rectangular Cartesian coordinates system**

And

$(ds)^2=(h_1)^2(du_1)^2+(h_1)^2(du_2)^2+(h_3)^2(du_3)^2$ **in curvilinear coordinates system**

From part (ii)

$h_1=\frac{1}{\sqrt{2}}$, $h_2=\frac{1}{\sqrt{2}}$, $h_3=\frac{1}{2}$

hence

$$(ds)^2=\frac{1}{2}(du_1)^2+\frac{1}{2}(du_2)^2+\frac{1}{4}(du_3)^2$$

(iv)

Find $\nabla\psi$ **for** $\psi(u_1, u_2, u_3)=u_1+u_2+2u_3$

We know that

$$\nabla\psi=\frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1+\frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2+\frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

From part (ii)

$h_1=\frac{1}{\sqrt{2}}$, $h_2=\frac{1}{\sqrt{2}}$, $h_3=\frac{1}{2}$

Given that

$\psi(u_1, u_2, u_3)=u_1+u_2+2u_3$

Now

$$\nabla\psi=\frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1+\frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2+\frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

$$\nabla\psi=\frac{1}{\frac{1}{\sqrt{2}}}(1)\hat{e}_1+\frac{1}{\frac{1}{\sqrt{2}}}(1)\hat{e}_2+\frac{1}{\frac{1}{2}}(2)\hat{e}_3$$

$$\nabla\psi=\sqrt{2}\,\hat{e}_1+\sqrt{2}\,\hat{e}_2+4\hat{e}_3$$

(v)

Find $\nabla^2\psi$ relative to this coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

From part (ii)

$h_1 = \frac{1}{\sqrt{2}}$, $h_2 = \frac{1}{\sqrt{2}}$, $h_3 = \frac{1}{2}$

$$\nabla^2\psi = \frac{1}{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{2}}\left[\frac{\partial}{\partial u_1}\left(\frac{\frac{1}{\sqrt{2}}\frac{1}{2}}{\frac{1}{\sqrt{2}}}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{\frac{1}{\sqrt{2}}\frac{1}{2}}{\frac{1}{\sqrt{2}}}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{\frac{1}{2}}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$\nabla^2\psi = 4\left[\frac{\partial}{\partial u_1}\left(\frac{1}{2}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$\nabla^2\psi = 2\frac{\partial}{\partial u_1}\left(\frac{\partial\psi}{\partial u_1}\right) + 2\frac{\partial}{\partial u_2}\left(\frac{\partial\psi}{\partial u_2}\right) + 4\frac{\partial}{\partial u_3}\left(\frac{\partial\psi}{\partial u_3}\right)$$

$$\nabla^2\psi = 2\frac{\partial^2\psi}{(\partial u_1)^2} + 2\frac{\partial^2\psi}{(\partial u_2)^2} + 4\frac{\partial^2\psi}{(\partial u_3)^2}$$

Q2:Consider the curvilinear coordinate system for which $x = {u_1}^2 - {u_2}^2$, $y = 2u_1u_2$, $z = u_3$

(i)Show that system is orthogonal and Right handed. Compute the scale factors.

(ii)Find $\nabla.\vec{A}$ and $\nabla \times \vec{A}$ for $\vec{A} = u_3\,\hat{e}_1 + u_1\hat{e}_2 + u_2\hat{e}_3$

(iii)Find the expression for $\nabla^2\psi\,(u_1, u_2, u_3)$

Solution:

(i)

We know that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

From part (i)

$x = u_1{}^2 - u_2{}^2$, $y = 2u_1u_2$, $z = u_3$

Now

$$\vec{r} = (u_1{}^2 - u_2{}^2)\,\hat{i} + 2u_1u_2\,\hat{j} + u_3\,\hat{k}$$

$$\frac{\partial\vec{r}}{\partial u_1} = 2u_1\hat{i} + 2u_2\,\hat{j}$$

and

$$h_1 = \left|\frac{\partial\vec{r}}{\partial u_1}\right| = \sqrt{(2u_1)^2 + (2u_2)^2} = 2\sqrt{(u_1)^2 + (u_2)^2}$$

And

$$\hat{e}_1 = \frac{\frac{\partial\vec{r}}{\partial u_1}}{\left|\frac{\partial\vec{r}}{\partial u_1}\right|} = \frac{2u_1\hat{i} + 2u_2\,\hat{j}}{2\sqrt{(u_1)^2 + (u_2)^2}}$$

$$\hat{e}_1 = \frac{u_1\hat{i}}{\sqrt{(u_1)^2 + (u_2)^2}} + \frac{u_2\,\hat{j}}{\sqrt{(u_1)^2 + (u_2)^2}}$$

Now

$$\frac{\partial\vec{r}}{\partial u_1} = -2u_2\hat{i} + 2u_1\,\hat{j}$$

and

$$h_2 = \left|\frac{\partial\vec{r}}{\partial u_2}\right| = \sqrt{(-2u_2)^2 + (2u_1)^2} = 2\sqrt{(u_1)^2 + (u_2)^2}$$

And

$$\hat{e}_2 = \frac{\frac{\partial\vec{r}}{\partial u_2}}{\left|\frac{\partial\vec{r}}{\partial u_2}\right|} = \frac{-2u_2\hat{i} + 2u_1\hat{j}}{2\sqrt{(u_1)^2 + (u_2)^2}}$$

$$\hat{e}_2 = \frac{-u_2\hat{i}}{\sqrt{(u_1)^2 + (u_2)^2}} + \frac{u_1\,\hat{j}}{\sqrt{(u_1)^2 + (u_2)^2}}$$

Now

$$\frac{\partial\vec{r}}{\partial u_3} = \hat{k}$$

and

$$h_3 = \left|\frac{\partial\vec{r}}{\partial u_3}\right| = \sqrt{(1)^2} = 1$$

And

$$\hat{e}_3 = \frac{\frac{\partial \bar{r}}{\partial u_3}}{\left|\frac{\partial \bar{r}}{\partial u_3}\right|} = \frac{\hat{k}}{1} = \hat{k}$$

Now

$$\hat{e}_1.\hat{e}_2 = \left(\frac{u_1\hat{i}}{\sqrt{(u_1)^2+(u_2)^2}} + \frac{u_2\hat{j}}{\sqrt{(u_1)^2+(u_2)^2}}\right).\left(\frac{-u_2\hat{i}}{\sqrt{(u_1)^2+(u_2)^2}} + \frac{u_1\hat{j}}{\sqrt{(u_1)^2+(u_2)^2}}\right) = 0$$

$$\hat{e}_2.\hat{e}_3 = \left(\frac{-u_2\hat{i}}{\sqrt{(u_1)^2+(u_2)^2}} + \frac{u_1\hat{j}}{\sqrt{(u_1)^2+(u_2)^2}}\right).\left(\hat{k}\right) = 0$$

$$\hat{e}_3.\hat{e}_1 = (\hat{k}).\left(\frac{u_1\hat{i}}{\sqrt{(u_1)^2+(u_2)^2}} + \frac{u_2\hat{j}}{\sqrt{(u_1)^2+(u_2)^2}}\right) = 0$$

Proved that system is orthogonal

Now

$$\hat{e}_1 \times \hat{e}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{u_1}{\sqrt{(u_1)^2+(u_2)^2}} & \frac{u_2}{\sqrt{(u_1)^2+(u_2)^2}} & 0 \\ \frac{-u_2}{\sqrt{(u_1)^2+(u_2)^2}} & \frac{u_1}{\sqrt{(u_1)^2+(u_2)^2}} & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \hat{k} = \hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - \hat{k} = -\hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-u_2}{\sqrt{(u_1)^2+(u_2)^2}} & \frac{u_1}{\sqrt{(u_1)^2+(u_2)^2}} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\hat{e}_2 \times \hat{e}_3 = \frac{u_1\hat{i}}{\sqrt{(u_1)^2+(u_2)^2}} + \frac{u_2\hat{j}}{\sqrt{(u_1)^2+(u_2)^2}} = \hat{e}_1$$

$$\hat{e}_3 \times \hat{e}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \frac{u_1}{\sqrt{(u_1)^2+(u_2)^2}} & \frac{u_2}{\sqrt{(u_1)^2+(u_2)^2}} & 0 \end{vmatrix}$$

$$\hat{e}_3 \times \hat{e}_1 = \frac{-u_2\hat{i}}{\sqrt{(u_1)^2+(u_2)^2}} + \frac{u_1\hat{j}}{\sqrt{(u_1)^2+(u_2)^2}} = \hat{e}_2$$

Proved that system is Right handed

And scale factors

$$\mathbf{h_1 = 2\sqrt{(u_1)^2 + (u_2)^2}} \quad , \mathbf{h_2 = 2\sqrt{(u_1)^2 + (u_2)^2}} \quad , \mathbf{h_3 = 1}$$

(ii)

Find $\nabla.\vec{A}$ and $\nabla \times \vec{A}$ for $\vec{A} = u_3\,\hat{e}_1 + u_1\hat{e}_2 + u_2\hat{e}_3$

We know that

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

Given that

$$\vec{A} = u_3\,\hat{e}_1 + u_1\hat{e}_2 + u_2\hat{e}_3$$

From part (i)

$$h_1 = 2\sqrt{(u_1)^2 + (u_2)^2} \quad , h_2 = 2\sqrt{(u_1)^2 + (u_2)^2} \quad , h_3 = 1$$

Hence

$$\nabla.\vec{A} = \frac{1}{(2\sqrt{(u_1)^2+(u_2)^2})(2\sqrt{(u_1)^2+(u_2)^2}}\left[\frac{\partial}{\partial u_1}\left(2\sqrt{(u_1)^2 + (u_2)^2}\,(1)(u_3)\right) + \right.$$

$$\frac{\partial}{\partial u_2}\left(2\sqrt{(u_1)^2 + (u_2)^2}(1)(u_1)\right) +$$

$$\left.\frac{\partial}{\partial u_3}\left((2\sqrt{(u_1)^2 + (u_2)^2}\;)(2\sqrt{(u_1)^2 + (u_2)^2}\;)(u_2)\right)\right]$$

$$\nabla.\vec{A} = \frac{1}{4[(u_1)^2+(u_2)^2]}\left[\frac{2u_1u_3}{\sqrt{(u_1)^2+(u_2)^2}} + \frac{2u_1u_2}{\sqrt{(u_1)^2+(u_2)^2}}\right]$$

$$\nabla.\vec{A} = \frac{2u_1(u_3+u_3)}{\left(\sqrt{(u_1)^2+(u_2)^2}\right)^3}$$

We know that

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{(2\sqrt{(u_1)^2+(u_2)^2}\;)(2\sqrt{(u_1)^2+(u_2)^2}\;)}$$

$$\begin{vmatrix} 2\sqrt{(u_1)^2 + (u_2)^2}\,\hat{e}_1 & 2\sqrt{(u_1)^2 + (u_2)^2}\,\hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ u_3(2\sqrt{(u_1)^2 + (u_2)^2}) & u_1(2\sqrt{(u_1)^2 + (u_2)^2}) & u_2 \end{vmatrix}$$

$\nabla \times \vec{A}$

$$=\frac{1}{4[(u_1)^2+(u_2)^2]}\Bigg[\left(2\sqrt{(u_1)^2+(u_2)^2}\right)\hat{e}_1(1)+\left(2\sqrt{(u_1)^2+(u_2)^2}\right)\hat{e}_2\left(2\sqrt{(u_1)^2+(u_2)^2}\right)+\hat{e}_3\left(2\sqrt{(u_1)^2+(u_2)^2}+\frac{2u_1u_1}{\sqrt{(u_1)^2+(u_2)^2}}-\frac{2u_1u_3}{\sqrt{(u_1)^2+(u_2)^2}}\right)\Bigg]$$

$$\nabla \times \vec{A} = \left(\frac{1}{2\sqrt{(u_1)^2+(u_2)^2}}\right)\hat{e}_1+\hat{e}_2+\left(\frac{4(u_1)^2+2(u_2)^2-2u_1u_3}{2\left(\sqrt{(u_1)^2+(u_2)^2}\right)^3}\right)\hat{e}_3$$

$$\nabla \times \vec{A} = \left(\frac{1}{2\sqrt{(u_1)^2+(u_2)^2}}\right)\hat{e}_1+\hat{e}_2+\left(\frac{2(u_1)^2+(u_2)^2-u_1u_3}{\left(\sqrt{(u_1)^2+(u_2)^2}\right)^3}\right)\hat{e}_3$$

(iii)

Find the expression for $\nabla^2\psi(u_1, u_2, u_3)$

We know that

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

From part (i)

$$h_1 = 2\sqrt{(u_1)^2+(u_2)^2},\ h_2 = 2\sqrt{(u_1)^2+(u_2)^2},\ h_3 = 1$$

Hence

$$\nabla^2\psi = \frac{1}{\left(2\sqrt{(u_1)^2+(u_2)^2}\right)\left(2\sqrt{(u_1)^2+(u_2)^2}\right)}\left[\frac{\partial}{\partial u_1}\left(\frac{\left(2\sqrt{(u_1)^2+(u_2)^2}\right)(1)}{2\sqrt{(u_1)^2+(u_2)^2}}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{2\sqrt{(u_1)^2+(u_2)^2}(1)}{2\sqrt{(u_1)^2+(u_2)^2}}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{2\sqrt{(u_1)^2+(u_2)^2}\left(2\sqrt{(u_1)^2+(u_2)^2}\right)}{1}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$\nabla^2\psi = \frac{1}{4[(u_1)^2+(u_2)^2]}\left[\frac{\partial}{\partial u_1}\left(\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(4[(u_1)^2+(u_2)^2\frac{\partial\psi}{\partial u_3}\right)\right]$$

Q3:Suppose u_1 , u_2 , u_3 are orthogonal curvilinear coordinate system define by

$$(ds)^2 = {u_2}^2{du_1}^2+{u_1}^2{du_2}^2+{du_3}^2$$

(i)Find $\nabla.\hat{e}_1$ where $\hat{e}_1$ is unit vector tangent to u_1-Curve

(ii)Find $\nabla^2\psi$ if $\psi = u_1 u_2 u_3$

Solution:

(i)

Find $\nabla.\hat{e}_1$ where $\hat{e}_1$ is unit vector tangent to u_1-Curve

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

Given that

$$(ds)^2 = u_2{}^2du_1{}^2 + u_1{}^2du_2{}^2 + du_3{}^2$$

Comparing it with

$$(ds)^2 = (h_1)^2(du_1)^2 + (h_1)^2(du_2)^2 + (h_3)^2(du_3)^2$$

We get

$$h_1 = u_2, \quad h_2 = u_1, \quad h_3 = 1$$

Given that

$$\vec{A} = \hat{e}_1$$

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$\nabla.\vec{A} = \frac{1}{(u_2)(u_1)(1)}\left[\frac{\partial}{\partial u_1}((u_1)(1)(1))\right] = \frac{1}{u_1u_2}$$

(ii)

Find $\nabla^2\psi$ if $\psi = u_1 u_2 u_3$

We know that

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$\nabla^2\psi = \frac{1}{u_1 u_2}\left[\frac{\partial}{\partial u_1}\left(\frac{(u_1)(1)}{u_2}\frac{\partial u_1u_2u_3}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{(u_2)(1)}{u_1}\frac{\partial u_1u_2u_3}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{u_2u_1}{1}\frac{\partial u_1u_2u_3}{\partial u_3}\right)\right]$$

$$\nabla^2\psi = \frac{1}{u_1 u_2}\left[\frac{\partial}{\partial u_1}\left(\frac{u_1 u_2 u_3}{u_2}\right) + \frac{\partial}{\partial u_2}\left(\frac{u_2 u_1 u_3}{u_1}\right) + \frac{\partial}{\partial u_3}(u_2{}^2 u_1{}^2)\right]$$

$$\nabla^2\psi = \frac{1}{u_1 u_2}\left[\frac{\partial}{\partial u_1}(u_1 u_3) + \frac{\partial}{\partial u_2}(u_2 u_3) + 0\right]$$

$$\nabla^2\psi = \frac{1}{u_1 u_2}[u_3 + u_3] =$$

$$\nabla^2\psi = \frac{2u_3}{u_1 u_2}$$

➢ Cylindrical Coordinate System

P(x, y,z) = P(r , θ , z)

x=rcosθ , y=rsinθ , z=z

P(x, y,z) = P(r , θ , z)

Relationship between units vectors in Rectangular and Cylindrical Polar Coordinates

In matrix form

$$\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_z\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}$$ **(from Cylindrical to Rectangular)-----(7)**

In Equations form

$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$

$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta$

$\hat{e}_z = \hat{k}$

In matrix form

$$\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_z\end{bmatrix}$$ **(from Rectangular to Cylindrical) -----(8)**

In Equations form

$\hat{i} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta$

$\hat{j} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$

$\hat{k}=\hat{e}_z$

Gradient

in curvilinear Coordinate System

$$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1+\frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2+\frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1 , u_2 , u_3)=(r , \theta , z)$

Hence in Cylindrical Coordinate System

$$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r+\frac{1}{r}\left(\frac{\partial\psi}{\partial \theta}\right)\hat{e}_\theta+\frac{1}{1}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$$

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r+\frac{1}{r}\left(\frac{\partial\psi}{\partial \theta}\right)\hat{e}_\theta+\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z \text{--------------(9)}$$

Divergence

We know that in curvilinear system

$$\nabla.\vec{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1)+\frac{\partial}{\partial u_2}(h_1h_3A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

Hence In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1 , u_2 , u_3)=(r , \theta , z)$

Hence in Spherical Coordinate System

$$\nabla.\vec{A}=\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r)+\frac{\partial}{\partial \theta}(A_\theta)+\frac{\partial}{\partial \phi}(rA_z)\right]\text{-------------(10)}$$

Curl

We know that in curvilinear system

$$\nabla\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1 , u_2 , u_3)=(r , \theta , z)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} \text{---------------(11)}$$

Laplacian

in Curvilinear coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in cylindrical Coordinate system

$$\nabla^2\psi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(r\frac{\partial\psi}{\partial\phi}\right)\right] \text{---------------(12)}$$

Jacobian

in Curvilinear coordinate system.

$$J\left(\frac{x\ ,\ y\ ,\ z}{u_1\ ,u_2\ ,\ u_3}\right) = h_1h_2h_3$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in cylindrical Coordinate system

$$J\left(\frac{x\ ,\ y\ ,\ z}{r\ ,\theta\ ,z}\right) = h_1h_2h_3 = r \text{-----------(13)}$$

Arc length Element

in Curvilinear coordinate system.

$$(ds)^2 = (h_1)^2(du_1)^2 + (h_1)^2(du_2)^2 + (h_3)^2(du_3)^2$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in cylindrical Coordinate system

$$(ds)^2 = (h_r)^2(dr)^2 + (h_\theta)^2(d\theta)^2 + (h_z)^2(dz)^2$$

$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (1)^2(dz)^2$

$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$--------------------(14)

Q4:Transform $\vec{A} = \frac{x}{y}\hat{i}$ into cylindrical poplar coordinates:

Solution:

Given that

$\vec{A} = \frac{x}{y}\hat{i}$

In Cylindrical Polar coordinates x= rcosθ , y = rsinθ , $\hat{i} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta$

Hence

$\vec{A} = \frac{x}{y}\hat{i}$

$\vec{A} = \frac{r\cos\theta}{r\sin\theta}[\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta]$

$\vec{A} = \frac{\cos\theta}{\sin\theta}[\cos\theta\hat{e}_r - \text{Sin}\theta\hat{e}_\theta]$

$\vec{A} = (\cot\theta\cos\theta)\hat{e}_r - (\cos\theta)\hat{e}_\theta]$

Q5: In cylindrical coordinates $\vec{A} = r\hat{e}_r + r\hat{e}_\theta$ .Transform $\vec{A}$ in rectangular Cartesian coordinates.

Solution:

Given that

$\vec{A} = r\hat{e}_r + r\hat{e}_\theta$

In Relationship between unit vectors of Cylindrical polar coordinates and Rectangular Cartesian coordinates

$\hat{e}_\theta = \cos\theta\hat{i} + \text{Sin}\theta\hat{j}$, $\hat{e}_r = -\text{Sin}\theta\hat{i} + \text{Cos}\theta\hat{j}$

Hence

$\vec{A} = r\hat{e}_r + r\hat{e}_\theta$

$\vec{A} = r[\cos\theta\hat{i} + \sin\theta\hat{j} - \sin\theta\hat{i} + \text{Cos}\theta\hat{j}]$

$\vec{A} = r[(\cos\theta - \sin\theta)\hat{\imath} + (\sin\theta + \cos\theta)\hat{\jmath}]$

$\vec{A} = (r\cos\theta - r\sin\theta)\hat{\imath} + (r\sin\theta + r\cos\theta)\hat{\jmath}$

Now In Relationship between Cylindrical polar coordinates and Rectangular Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$,

hence

$\vec{A} = (x - y)\hat{\imath} + (y + x)\hat{\jmath}$

Q6: In cylindrical Polar coordinates if $\psi = r^2 z \sin\theta\cos\theta$. Find $\nabla\psi$ at $r=1$, $\theta = \frac{\pi}{4}$, $z=2$

Solution:

Given that

$\psi = r^2 z \sin\theta\cos\theta$

We know that in curvilinear system

$$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Cylindrical Coordinate System

$$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$$

$$\nabla\psi = \frac{1}{1}\left(\frac{\partial r^2 z\sin\theta\cos\theta}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial r^2 z\sin\theta\cos\theta}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial r^2 z\sin\theta\cos\theta}{\partial z}\right)\hat{e}_z$$

$$\nabla\psi = (2rz\sin\theta\cos\theta)\hat{e}_r + \frac{r}{2}(z\cos2\theta)\hat{e}_\theta + (r^2\sin\theta\cos\theta)\hat{e}_z$$

at $r=1$, $\theta = \frac{\pi}{4}$, $z=2$

$$\nabla\psi = \left(4\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right)\hat{e}_r + \frac{1}{2}\left(2\cos\frac{\pi}{2}\right)\hat{e}_\theta + \left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right)\hat{e}_z$$

$$\nabla\psi = 2\hat{e}_r + \frac{1}{2}\hat{e}_z$$

Q7: Verify in cylindrical Polar coordinates

$\nabla (\ln r) = \nabla \times (\theta \hat{e}_z)$

Solution:

We know that in curvilinear system

$$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Cylindrical Coordinate System

$$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial \theta}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$$ -------(Students may use direct this formula)

LHS=

$$\nabla (\ln r) = \frac{1}{1}\left(\frac{\partial \ln r}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial \ln r}{\partial \theta}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial \ln r}{\partial z}\right)\hat{e}_z$$

$$\nabla (\ln r) = \frac{1}{r}\hat{e}_r$$

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$ $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_1 & r\hat{e}_z & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$ ------(Students may use direct this formula)

$\nabla \times (\theta\hat{e}_z) = ?$

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_1 & r\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \theta \end{vmatrix}$$

$\nabla \times \vec{A} = \frac{1}{r}\hat{e}_r$

Q8–In cylindrical Polar coordinates $\vec{A}$ =rcos$\theta\hat{e}_r$+rsin$\theta\hat{e}_\theta$. Find $\nabla.\vec{A}$

Solution:

We know that in curvilinear system

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3=1$ $(u_1 , u_2 , u_3) = (r , \theta , z)$

Hence

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_1) + \frac{\partial}{\partial \theta}(A_2) + \frac{\partial}{\partial u_3}(rA_3)\right]$$

Given that

$\vec{A}$ =rcos$\theta\hat{e}_r$+rsin$\theta\hat{e}_\theta$

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos\theta) + \frac{\partial}{\partial \theta}(r\sin\theta) + \frac{\partial}{\partial z}(0)\right]$$

$$\nabla.\vec{A} = \frac{1}{r}[\,2r\cos\theta + r\cos\theta]$$

$=3\cos\theta$

Q9:In cylindrical Polar coordinates $\vec{A}$ =r$\hat{e}_r$+zsin$\theta\hat{e}_\theta$+ rz$\hat{e}_z$. Find (i) $\nabla.\vec{A}$ (ii) $\nabla \times \vec{A}$

Solution:

(i)

We know that in curvilinear system

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$ $(u_1 , u_2 , u_3) = (r , \theta , z)$

Hence

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_1) + \frac{\partial}{\partial \theta}(A_2) + \frac{\partial}{\partial u_3}(rA_3)\right]$$

Given that

$\vec{A} = r\hat{e}_r + z\sin\theta\,\hat{e}_\theta + r\,z\hat{e}_z$

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2) + \frac{\partial}{\partial \theta}(z\sin\theta) + \frac{\partial}{\partial z}(r^2 z)\right]$$

$$\nabla.\vec{A} = \frac{1}{r}\left[2r + z\cos\theta + r^2\right]$$

$$= 2 + \frac{z\cos\theta}{r} + r$$

(ii)

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$ $(u_1 , u_2 , u_3) = (r , \theta , z)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$ --------(Students may use direct this formula)

Given that

$\vec{A} = r\hat{e}_r + z\sin\theta\,\hat{e}_\theta + r\,z\hat{e}_z$

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ r & rz\sin\theta & rz \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r}\left[(\hat{e}_r)(0 - r\sin\theta) + r\hat{e}_\theta(0 - z) + \hat{e}_\theta(z\sin\theta - 0)\right]$$

$=\frac{1}{r}[-r\sin\theta\hat{e}_r - rz\hat{e}_\theta + (z\sin\theta)\hat{e}_\theta]$

Q10:Express the following vector fields in cylindrical polar coordinates and Find $\nabla.\vec{A}$ and $\nabla\times\vec{A}$ if

(i) $\vec{A} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$ (ii) $\vec{A} = \frac{-y\hat{i}+x\hat{j}}{x^2+y^2}$

Solution:

(i)

$\vec{A} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$

In Cylindrical Polar coordinates x= rcosθ , y = rsinθ , $x^2 + y^2 = r^2$

, $\hat{i} = \cos\theta\hat{e}_r - (\sin\theta)\hat{e}_\theta$, $\hat{j} = (\sin\theta)\hat{e}_r + (\cos\theta)\hat{e}_\theta$

Hence

$\vec{A} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$

$\vec{A} = \frac{r\cos\theta[(\cos\theta)\hat{e}_r-(\sin\theta)\hat{e}_\theta] + r\sin\theta[(\sin\theta)\hat{e}_r+(\cos\theta)\hat{e}_\theta]}{r^2}$

$\vec{A} = \frac{\cos\theta[(\cos\theta)\hat{e}_r-(\sin\theta)\hat{e}_\theta]+\sin\theta[(\sin\theta)\hat{e}_r+(\cos\theta)\hat{e}_\theta]}{r}$

$\vec{A} = \frac{1}{r}\hat{e}_r$

$\nabla.\vec{A}$= ? and $\nabla\times\vec{A}$=?

We know that in curvilinear system

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3=1$ (u_1 , u_2 , u_3)=(r , θ , z)

Hence

$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial u_3}(rA_z)\right]$---(Students may use direct this formula)

Given that $\vec{A} = \frac{1}{r}\hat{e}_r$

Now

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{r}{r}\right) + \frac{\partial}{\partial \theta}(0) + \frac{\partial}{\partial u_3}(0)\right] = 0$$

Now

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$ $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$ ------(Students may use direct this formula)

Given that

Given that $\vec{A} = \frac{1}{r}\hat{e}_r$

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 & 0 \end{vmatrix}$$

$\nabla \times \vec{A} = \frac{1}{r}[(\hat{e}_r)(0-0) + r\hat{e}_\theta(0-0) + \hat{e}_\theta(0-0)]$

$\nabla \times \vec{A} = 0\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_\theta$

$\nabla \times \vec{A} = \vec{0}$

(ii)

$\vec{A} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$

In Cylindrical Polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

, $\hat{i} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta$, $\hat{j} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$

Hence

$$\vec{A} = \frac{-y\hat{i}+x\hat{j}}{x^2+y^2}$$

$$\vec{A} = \frac{-r\sin\theta[(\cos\theta)\hat{e}_r-(\sin\theta)\hat{e}_\theta] + r\cos\theta[(\sin\theta)\hat{e}_r+(\cos\theta)\hat{e}_\theta]}{r^2}$$

$$\vec{A} = \frac{-\sin\theta[(\cos\theta)\hat{e}_r-(\sin\theta)\hat{e}_\theta] + \cos\theta[(\sin\theta)\hat{e}_r+(\cos\theta)\hat{e}_\theta]}{r}$$

$$\vec{A} = \frac{1}{r}\hat{e}_\theta$$

We know that in curvilinear system

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$ $(u_1 , u_2 , u_3) = (r , \theta , z)$

Hence

$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}(rA_z)\right]$ -------(Students may use direct this formula)

Given that $\vec{A} = \frac{1}{r}\hat{e}_\theta$

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(0) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\right) + \frac{\partial}{\partial z}(0)\right] = 0$$

Now

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Cylindrical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = 1$ $(u_1 , u_2 , u_3) = (r , \theta , z)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$ ------(Students may use direct this formula)

Given that

$$\vec{A} = \frac{1}{r}\hat{e}_\theta$$

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 1 & 0 \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r}[(\hat{e}_r)(0\text{-}0) + r\hat{e}_\theta(0\text{-}0) + \hat{e}_\theta(0\text{-}0)]$$

$$\nabla \times \vec{A} = 0\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_\theta$$

$$\nabla \times \vec{A} = \vec{0}$$

Q11: In cylindrical polar coordinates $\vec{A} = r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta$, Evaluate $(\vec{A}.\nabla)\vec{A}$

Solution:

We know that in curvilinear system

$$\nabla = \frac{1}{h_1}\left(\frac{\partial}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial}{\partial u_3}\right)\hat{e}_3$$

In Cylindrical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in Cylindrical Coordinate System

$$\nabla = \frac{1}{1}\left(\frac{\partial}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial}{\partial z}\right)\hat{e}_z$$

$$\nabla = \left(\frac{\partial}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)\hat{e}_\theta + \left(\frac{\partial}{\partial z}\right)\hat{e}_z$$ ---------(Students may use direct this formula)

Now

Given that

$$\vec{A} = r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta$$

$$(\vec{A}.\nabla)\vec{A} = ?$$

$(\vec{A}.\nabla) = (r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta).\left[\left(\frac{\partial}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial\theta}\right)\hat{e}_\theta + \left(\frac{\partial}{\partial z}\right)\hat{e}_z\right]$

$(\vec{A}.\nabla) = \left[r\cos\theta\left(\frac{\partial}{\partial r}\right) + \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}\right)\right]$

And now

$(\vec{A}.\nabla)\vec{A} == \left[r\cos\theta\left(\frac{\partial}{\partial r}\right) + \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}\right)\right](r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta)$

$= r\cos\theta\left(\frac{\partial}{\partial r}[r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta]\right) + \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}[r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta]\right)$

$= r\cos\theta\left(\frac{\partial}{\partial r}[r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta]\right) + \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}[r\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta]\right)$

$(\vec{A}.\nabla)\vec{A} = r\cos\theta\,[\cos\theta\hat{e}_r + \cos\theta\frac{\partial}{\partial r}(\hat{e}_r) + \sin\theta\frac{\partial}{\partial r}(\hat{e}_\theta)]$

$+ \frac{\sin\theta}{r}[-r\sin\theta\hat{e}_r + r\cos\theta\frac{\partial}{\partial\theta}(\hat{e}_r) + \cos\theta\hat{e}_\theta + \sin\theta\frac{\partial}{\partial\theta}(\hat{e}_\theta)]$

We know that

$\frac{\partial}{\partial r}(\hat{e}_r) = 0 \quad , \quad \frac{\partial}{\partial r}(\hat{e}_\theta) = 0$

$\frac{\partial}{\partial\theta}(\hat{e}_r) = \hat{e}_\theta \quad , \quad \frac{\partial}{\partial\theta}(\hat{e}_\theta) = -\hat{e}_r$

Hence

$(\vec{A}.\nabla)\vec{A} = r\cos\theta\,[\cos\theta\hat{e}_r + 0 + 0] + \frac{\sin\theta}{r}[-r\sin\theta\hat{e}_r + r\cos\theta\,\hat{e}_\theta + \cos\theta\hat{e}_\theta + \sin\theta\,(-\hat{e}_r)]$

$(\vec{A}.\nabla)\vec{A} = [r^2\cos^2\theta - \sin^2\theta - \frac{\sin^2\theta}{r}]\hat{e}_r + [\sin\theta\text{Cos}\theta + \frac{\sin\theta\cos\theta}{r}]\,\hat{e}_\theta$

$(\vec{A}.\nabla)\vec{A} = [r^2\cos^2\theta - \sin^2\theta(1 + \frac{1}{r})]\hat{e}_r + [\sin\theta\cos\theta(1 + \frac{1}{r})]\,\hat{e}_\theta$

Q12:In cylindrical polar coordinates if

$\vec{A} = (r\sin\theta\cos\theta + z\cos\theta)\hat{e}_r + (r\cos^2\theta - z\sin\theta)\,\hat{e}_\theta + r\sin\theta\,\hat{e}_z$.

Verify $\nabla\times\nabla\times\vec{A} = \vec{0}$

Solution:

We know Cylindrical Coordinate System

$$\nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & rA_\theta & A_z\end{vmatrix}$$

Given that

$\vec{A}=(r\sin\theta\cos\theta+z\cos\theta)\hat{e}_r+(r\cos^2\theta-z\sin\theta)\,\hat{e}_\theta+r\sin\theta\,\hat{e}_z$

hence

$$\nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ r\sin\theta\cos\theta+z\cos\theta & r(r\cos^2\theta-z\sin\theta) & r\sin\theta\end{vmatrix}$$

$$\nabla\times\vec{A}=\frac{1}{r}[(r\cos\theta+r\sin\theta)\hat{e}_r+(\cos\theta-\sin\theta)r\hat{e}_\theta$$

$$+\{2r\cos^2\theta-z\sin\theta-r\cos2\theta+z\sin\theta\}\hat{e}_z]$$

$$\nabla\times\vec{A}=(\cos\theta+\sin\theta)\hat{e}_r+(\cos\theta-\sin\theta)\hat{e}_\theta+[(2\cos^2\theta-\cos2\theta)]\hat{e}_z\text{-------------}(1)$$

Now

$\nabla\times\nabla\times\vec{A}=?$

We know

in Cylindrical Coordinate System

$$\nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & rA_\theta & A_z\end{vmatrix}$$

From (1) Taking

$\vec{A}=\nabla\times\vec{A}$

Then

$$\nabla\times\nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ (\cos\theta+\sin\theta) & r(\cos\theta-\sin\theta) & (2\cos^2\theta-\cos2\theta)\end{vmatrix}$$

$$\nabla\times\nabla\times\vec{A}=\frac{1}{r}[\{(-4\cos\theta\sin\theta+2\sin2\theta-0)\hat{e}_r+\{0-0\}r\hat{e}_\theta$$

$$+\{(\cos\theta-\sin\theta)+\sin\theta-\cos\theta\}\hat{e}_z]$$

$\nabla\times\nabla\times\vec{A}=\frac{1}{r}[\{(-2\sin2\theta+2\sin2\theta-0\}\hat{e}_r+\{0-0\}r\hat{e}_\theta$

$+\{(\cos\theta-\sin\theta)+\sin\theta-\cos\theta)\}\hat{e}_z]$

$\nabla\times\nabla\times\vec{A}=\frac{1}{r}[0\hat{e}_r+0\hat{e}_\theta+0\hat{e}_z]$

$\nabla\times\nabla\times\vec{A}=\vec{0}$

Q13: In cylindrical polar coordinates if

$\vec{A}=\hat{e}_\theta$ Verify the identity $\nabla\times\nabla\times\vec{A}=\nabla(\nabla.\vec{A})-\nabla^2\vec{A}$

Solution:

We know

in Cylindrical Coordinate System

$$\nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & rA_\theta & A_z\end{vmatrix}$$

Given that

$\vec{A}=\hat{e}_\theta$

Hence

$$\nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ 0 & r(1) & 0\end{vmatrix}$$

$$\nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ 0 & r & 0\end{vmatrix}$$

$=\frac{1}{r}[(0-0)\hat{e}_r+(0-0)r\hat{e}_\theta+(1-0)\hat{e}_z]$

$\nabla\times\vec{A}=\frac{1}{r}\hat{e}_z$------(1)

Now

LHS=$\nabla\times\nabla\times\vec{A}$=?

We know

in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

From (1)

$$\nabla \times \vec{A} = \frac{1}{r}\hat{e}_z$$

Taking

$$\nabla \times \vec{A} = \vec{A}$$

Then

$$\nabla \times \nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{r} \end{vmatrix}$$

$$\nabla \times \nabla \times \vec{A} = \frac{1}{r}\left[(0-0)\hat{e}_r + (0+\frac{1}{r^2})r\hat{e}_\theta + (1-0)\hat{e}_z\right]$$

$$LHS == \nabla \times \nabla \times \vec{A} = \frac{1}{r^2}\hat{e}_\theta$$

Now

$$RHS = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A} = ?$$

In Cylindrical Coordinate System

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial u_3}(rA_z)\right]$$

Given that

$$\vec{A} = \hat{e}_\theta$$

Hence

$$\nabla.\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(0) + \frac{\partial}{\partial \theta}(1) + \frac{\partial}{\partial u_3}(0)\right]$$

$$\nabla.\vec{A} = 0 \text{---------}(2)$$

Now

We know in Cylindrical Coordinate System

$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$

From (2) $\nabla.\vec{A} = 0$

Taking

$\nabla.\vec{A} = \psi$

Then

$\nabla(\nabla.\vec{A}) = \frac{1}{1}\left(\frac{\partial}{\partial r}\nabla.\vec{A}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial\theta}\nabla.\vec{A}\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial}{\partial z}\nabla.\vec{A}\right)\hat{e}_z$

$\nabla(\nabla.\vec{A}) = \frac{1}{1}\left(\frac{\partial}{\partial r}0\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial\theta}0\right)\hat{e}_\theta + \frac{1}{1}\left(\frac{\partial}{\partial z}0\right)\hat{e}_z$

$\nabla(\nabla.\vec{A}) = \vec{0}$ -----------(3)

We know in Curvilinear coordinate system.

$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, $(u_1, u_2, u_3) = (r, \theta, z)$

Hence in cylindrical Coordinate system

$\nabla^2\psi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial u_z}\left(r\frac{\partial\psi}{\partial u_z}\right)\right]$ -------**(students can use direct this formula also)**

$\nabla^2\vec{A} = ?$

hence

$\nabla^2\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\vec{A}}{\partial r}\right) + \frac{\partial}{\partial u_\theta}\left(\frac{1}{r}\frac{\partial\vec{A}}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\vec{A}}{\partial z}\right)\right]$

Given that

$\vec{A} = \hat{e}_\theta$

Hence

$\nabla^2\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\hat{e}_\theta\right) + \frac{\partial}{\partial u_\theta}\left(\frac{1}{r}\frac{\partial}{\partial u_\theta}\hat{e}_\theta\right) + \frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\hat{e}_\theta\right)\right]$

$\nabla^2\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\hat{e}_\theta\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\hat{e}_\theta\right) + \frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\hat{e}_\theta\right)\right]$

$\nabla^2\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(0) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}(-\hat{e}_r)\right) + \frac{\partial}{\partial z}(0)\right]$

$\nabla^2\vec{A}=\frac{1}{r}\left[\ \frac{\partial}{\partial\theta}\left(\frac{1}{r}(-\hat{e}_r)\right)\right]$

$\nabla^2\vec{A}=\frac{-1}{r^2}\hat{e}_\theta$ -------(4)

From (3) and (4)

RHS= $\nabla(\nabla.\vec{A})-\nabla^2\vec{A}=0+\frac{1}{r^2}\hat{e}_\theta=\frac{1}{r^2}\hat{e}_\theta$

Hence proved

$\nabla\times\nabla\times\vec{A}=\nabla(\nabla.\vec{A})-\nabla^2\vec{A}=\frac{1}{r^2}\hat{e}_\theta$

➢ Spherical Coordinate System

P(x, y,z) = P(r , θ, ϕ)

x=rsinθcos ϕ , y=rsinθsinϕ , z=rcosθ ,

P(x, y,z) = P(r , θ, ϕ)

Relationship between units vectors in Rectangular and Spherical Polar Coordinates

In matrix form

$$\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\phi\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}$$ (from Spherical to Rectangular

)-----------(15)

In Equations form

$\hat{e}_r=\sin\theta\cos\phi\hat{i}+\text{sinssin}\phi\hat{j}+\cos\theta\hat{k}$

$\hat{e}_\theta=\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k}$

$\hat{e}_\phi=-\sin\phi\hat{i}+\cos\phi\hat{j}$

In matrix form

$$\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\text{S}\text{in}\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\phi\end{bmatrix}$$ **(from Rectangular to Spherical)-**

--------(16)

In Equations form

$\hat{i}= \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi$

$\hat{j}= \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi$

$\hat{k}= \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$

Gradient

in curvilinear Coordinate System

$$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1+\frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2+\frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1 , u_2 , u_3)=(r , \theta , \phi)$

Hence in Spherical Coordinate System

$$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r+\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta+\frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_\phi$$

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r+\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta+\frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial\phi}\right)\hat{e}_\phi \quad \text{------------(17)}$$

Divergence

We know that in curvilinear system

$$\nabla.\vec{A}= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Spherical Coordinate System

$$\nabla.\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial\theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial\phi}(rA_\phi)\right] \text{---------(18)}$$

Curl

We know that in curvilinear system

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Spherical Coordinate System

$$\nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \text{---------(19)}$$

Laplacian

in Curvilinear coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Spherical Coordinate system

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right] \text{-------(20)}$$

Jacobian

in Curvilinear coordinate system.

$J\left(\frac{x\ ,\ y\ ,\ z}{u_1\ ,u_2\ ,\ u_3}\right) = h_1h_2h_3$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1 , u_2 , u_3) = (r , \theta , \phi)$

Hence in Spherical Coordinate system

$J\left(\frac{x\ ,\ y\ ,\ z}{r\ ,\ \theta\ ,\ \phi}\right) = h_1h_2h_3 = r^2\sin\theta$ -------------(21)

Arc length Element

in Curvilinear coordinate system.

$(ds)^2 = (h_1)^2(du_1)^2 + (h_1)^2(du_2)^2 + (h_3)^2(du_3)^2$

In Spherical Coordinate System

$h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$, $(u_1 , u_2 , u_3) = (r , \theta , \phi)$

$(ds)^2 = (h_r)^2(dr)^2 + (h_\theta)^2(d\theta)^2 + (h_\phi)^2(d\phi)^2$

$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2$

$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$-------------(22)

Q14: Express the vector field $\vec{A} = -y\hat{i} + x\hat{j}$ in Spherical polar coordinates

<u>Solution:</u>

We know that

Relationship between Rectangular and Spherical coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Hence

$\vec{A} = -y\hat{i} + x\hat{j}$

$\vec{A} = -r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j}$

And Relationship between units vectors of Rectangular and Spherical corinate systems

$\hat{i} = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi$

$\hat{j} = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi$

$\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$

Hence

$\vec{A} = -r\sin\theta\sin\phi(\sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi)$

$+r\sin\theta\cos\phi(\sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi)$

$\vec{A} = (-r\sin\theta\sin\phi\sin\theta\cos\phi + r\sin\theta\sin\phi\sin\theta\cos\phi)\hat{e}_r + (-\sin\theta\sin\phi\cos\theta\cos\phi + r\sin\theta\sin\phi\cos\theta\cos\phi)\hat{e}_\theta + (r\sin\theta\sin^2\phi + r\sin\theta\cos^2\phi)\hat{e}_\phi$

$\vec{A} = 0\hat{e}_r + 0\hat{e}_\theta + r\sin\theta\,\hat{e}_\phi$

Q15: In Spherical Polar coordinates $\vec{A} = \frac{1}{r}\hat{e}_r$ .Transform $\vec{A}$ in rectangular Cartesian coordinates.

Solution:

We know that relation between unit vector of Spherical polar and Rectangular coordinates

$\hat{e}_r = \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath}+\cos\theta\hat{k}$

$\hat{e}_\theta = \cos\theta\cos\phi\hat{\imath} + \cos\theta\sin\phi\hat{\jmath}-\sin\theta\hat{k}$

$\hat{e}_\phi = -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}$

Hence

$\vec{A} = \frac{1}{r}\hat{e}_r$

$\vec{A} = \frac{1}{r}(\sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath}+\cos\theta\hat{k})$

$\vec{A} = \frac{1}{r^2}(r\sin\theta\cos\phi\hat{\imath} + r\sin\theta\sin\phi\hat{\jmath}+ r\cos\theta\hat{k})$

Also We know that relation between Spherical and polar Rectangular coordinates

$r\sin\theta\cos\phi = x$, $r\sin\theta\sin\phi = y$, $r\cos\theta = z$, $r^2=x^2+y^2+z^2$,

Therefoe

$\vec{A} = \frac{1}{x^2+y^2+z^2}(x\hat{\imath} + y\hat{\jmath}+z\hat{k})$

Q16: In Spherical Polar coordinates $\vec{A} = r\hat{e}_\theta + r\hat{e}_\phi$. Find in Spherical polar coordinates. $\frac{\partial\vec{A}}{\partial r}$, $\frac{\partial\vec{A}}{\partial\theta}$, $\frac{\partial\vec{A}}{\partial\phi}$

<u>Solution:</u>

$\vec{A} = r\hat{e}_\theta + r\hat{e}_\phi$

$\vec{A} = r(\hat{e}_\theta + \hat{e}_\phi)$

Now

$\frac{\partial\vec{A}}{\partial r} = \frac{\partial}{\partial r}[r(\hat{e}_\theta + \hat{e}_\phi)]$

$\frac{\partial\vec{A}}{\partial r} = (\hat{e}_\theta + \hat{e}_\phi)\frac{\partial r}{\partial r} + r[\frac{\partial}{\partial r}(\hat{e}_\theta) + \frac{\partial}{\partial r}(\hat{e}_\phi)]$

We know that

$\frac{\partial}{\partial r}(\hat{e}_\theta) = 0$, $\frac{\partial}{\partial r}(\hat{e}_\phi) = 0$

Hence

$\frac{\partial \vec{A}}{\partial r} = (\hat{e}_\theta + \hat{e}_\phi) + r[0+0]$

$\frac{\partial \vec{A}}{\partial r} = \hat{e}_\theta + \hat{e}_\phi$

Now

$\vec{A} = r(\hat{e}_\theta + \hat{e}_\phi)$

$\frac{\partial \vec{A}}{\partial \theta} = \frac{\partial}{\partial \theta}[r(\hat{e}_\theta + \hat{e}_\phi)]$

$\frac{\partial \vec{A}}{\partial \theta} = (\hat{e}_\theta + \hat{e}_\phi)\frac{\partial r}{\partial \theta} + r[\frac{\partial}{\partial \theta}(\hat{e}_\theta) + \frac{\partial}{\partial \theta}(\hat{e}_\phi)]$

We know that

$\frac{\partial}{\partial \theta}(\hat{e}_\theta) = -\hat{e}_r$, $\frac{\partial}{\partial \theta}(\hat{e}_\phi) = 0$

Hence

$\frac{\partial \vec{A}}{\partial \theta} = (\hat{e}_\theta + \hat{e}_\phi)(0) + r[-\hat{e}_r + 0]$

$\frac{\partial \vec{A}}{\partial \theta} = = -\hat{e}_r$

Now

$\vec{A} = r(\hat{e}_\theta + \hat{e}_\phi)$

$\frac{\partial \vec{A}}{\partial \phi} = \frac{\partial}{\partial \phi}[r(\hat{e}_\theta + \hat{e}_\phi)]$

$\frac{\partial \vec{A}}{\partial \phi} = (\hat{e}_\theta + \hat{e}_\phi)\frac{\partial r}{\partial \phi} + r[\frac{\partial}{\partial \phi}(\hat{e}_\theta) + \frac{\partial}{\partial \phi}(\hat{e}_\phi)]$

We know that

$\frac{\partial}{\partial \phi}(\hat{e}_\theta) = \cos\theta\,\hat{e}_\phi$, $\frac{\partial}{\partial \phi}(\hat{e}_\phi) = -\sin\theta\hat{e}_r - \cos\theta\hat{e}_\theta$

Hence

$\frac{\partial \vec{A}}{\partial \phi} = (\hat{e}_\theta + \hat{e}_\phi)(0) + r[\cos\theta\,\hat{e}_\phi + (-\sin\theta\hat{e}_r - \cos\theta\hat{e}_\theta)]$

$\frac{\partial \vec{A}}{\partial \phi}$ = - rsinθ$\hat{e}_r$- rcosθ$\hat{e}_\theta$+rcosθ$\hat{e}_\phi$

Q17: Find ∇ ψ in Spherical Polar coordinates if

$\psi(r, \theta, \phi) = \frac{\cos\theta}{r^2}$

Solution:

We know that

In spherical Polar Coordinates

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial\phi}\right)\hat{e}_\phi$$

Given that

$$\psi(r, \theta, \phi) = \frac{\cos\theta}{r^2}$$

Hence

$$\nabla\psi = \left(\frac{\partial}{\partial r}\frac{\cos\theta}{r^2}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial\theta}\frac{\cos\theta}{r^2}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\frac{\cos\theta}{r^2}\right)\hat{e}_\phi$$

$$\nabla\psi = \frac{-2\cos\theta}{r^3}\hat{e}_r - \frac{\sin\theta}{r^3}\hat{e}_\theta + \frac{1}{r\sin\theta}(0)\hat{e}_\phi$$

$$\nabla\psi = \frac{-2\cos\theta}{r^3}\hat{e}_r - \frac{\sin\theta}{r^3}\hat{e}_\theta + 0\hat{e}_\phi$$

Q18:Verify in Spherical Polar coordinates: $\nabla\phi = \nabla \times \left(\frac{r\nabla\theta}{\sin\theta}\right)$

Solution:

Prove $\nabla\phi = \nabla \times \left(\frac{r\nabla\theta}{\sin\theta}\right)$

We know that

In spherical Polar Coordinates

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial\phi}\right)\hat{e}_\phi \text{ -------(1)}$$

Put $\psi = \phi$

then

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\phi}{\partial\phi}\right)\hat{e}_\phi$$

$\nabla \phi = 0\hat{e}_r + 0\hat{e}_\theta + \frac{1}{r\sin\theta}(1)\hat{e}_\phi$

LHS$=\nabla \phi = 0\hat{e}_r + 0\hat{e}_\theta + \frac{1}{r\sin\theta}\hat{e}_\phi$

Now

$\nabla \times \left(\frac{r\nabla\theta}{\sin\theta}\right) = ?$

Put Put $\psi = \theta$

then

$\nabla\theta = \left(\frac{\partial\theta}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\theta}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\theta}{\partial\phi}\right)\hat{e}_\phi$

$\nabla\theta = (0)\hat{e}_r + \frac{1}{r}(1)\hat{e}_\theta + \frac{1}{r\sin\theta}(0)\hat{e}_\phi$

$\nabla\theta = 0\hat{e}_r + \frac{1}{r}\hat{e}_\theta + 0\hat{e}_\phi$

Now

$\frac{r\nabla\theta}{\sin\theta} = ?$

$\frac{r\nabla\theta}{\sin\theta} = \frac{r}{\sin\theta}\left(0\hat{e}_r + \frac{1}{r}\hat{e}_\theta + 0\hat{e}_\phi\right)$

$\frac{r\nabla\theta}{\sin\theta} = \frac{r}{\sin\theta}\left(0\hat{e}_r + \frac{1}{r}\hat{e}_\theta + 0\hat{e}_\phi\right)$

$\frac{r\nabla\theta}{\sin\theta} = 0\hat{e}_r + \frac{1}{\sin\theta}\hat{e}_\theta + 0\hat{e}_\phi$ --------(2)

Now

We know in Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

$\vec{A} = \frac{r\nabla\theta}{\sin\theta}$

From (2)

$\vec{A} = \frac{r\nabla\theta}{\sin\theta} = 0\hat{e}_r + \frac{1}{\sin\theta}\hat{e}_\theta + 0\hat{e}_\phi$ --------(3)

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

Hence

RHS=

$$\nabla \times \frac{r\nabla\theta}{\sin\theta} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \frac{r}{\sin\theta} & 0 \end{vmatrix}$$

$$\nabla \times \frac{r\nabla\theta}{\sin\theta} = \frac{1}{r^2\sin\theta}\left[(0-0)\,\hat{e}_r + (0-0)r\hat{e}_\theta + \left(\frac{1}{\sin\theta}\right)r\sin\theta\,\hat{e}_\phi\right]$$

$$\text{RHS} = \nabla \times \frac{r\nabla\theta}{\sin\theta} = 0\hat{e}_r + 0\hat{e}_\theta + \frac{1}{r\sin\theta}\hat{e}_\phi$$

Hence proved

$$\nabla\phi = \nabla \times \frac{r\nabla\theta}{\sin\theta} = 0\hat{e}_r + 0\hat{e}_\theta + \frac{1}{r\sin\theta}\hat{e}_\phi$$

Q19: In Spherical Polar coordinates

$\vec{A}(r, \theta, \phi) = \hat{e}_r + r\,\hat{e}_\theta + r\cos\phi\,\hat{e}_\phi$ **. Find (i)** $\nabla.\vec{A}$ **(ii)** $\nabla \times \vec{A}$

Solution:

(i)

In Spherical Polar Coordinate System

$$\nabla.\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial\theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial\phi}(rA_\phi)\right]$$

Given that

$\vec{A}(r, \theta, \phi) = \hat{e}_r + r\,\hat{e}_\theta + r\cos\phi\,\hat{e}_\phi$

Hence

$$\nabla.\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta) + \frac{\partial}{\partial\theta}(r^2\sin\theta) + \frac{\partial}{\partial\phi}(r^2\cos\phi)\right]$$

$$\nabla.\vec{A} = \frac{1}{r^2\sin\theta}\left[2r\sin\theta + r^2\cos\theta - r^2\sin\phi\right]$$

$\nabla.\vec{A} = \frac{2}{r} + \cot\theta - \frac{\sin\phi}{\sin\theta}$

(ii)

$\nabla \times \vec{A} = ?$

We know that

in Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

Given that

$\vec{A}(r, \theta, \phi) = \hat{e}_r + r\,\hat{e}_\theta + r\cos\phi\,\hat{e}_\phi$

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 1 & r(r) & r\sin\theta\,(r\cos\phi) \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 1 & r^2 & r^2\sin\theta\cos\phi \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\left[(r^2\cos\theta\cos\phi - 0)\,\hat{e}_r + (0 - 2r\sin\theta\cos\phi)r\hat{e}_\theta + (2r - 0)r\sin\theta\hat{e}_\phi\right]$$

$\nabla \times \vec{A} = \cot\theta\cos\phi\,\hat{e}_r - 2\cos\phi\,\hat{e}_\theta + 2\hat{e}_\phi$

Q20: In Spherical Polar coordinates if $\vec{A} = \hat{e}_\theta$, Find the components of $\nabla(\nabla.\vec{A})$.

<u>**Solution:**</u>

In Spherical Polar Coordinate System

$$\nabla.\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial \phi}(rA_\phi)\right]$$

Given that

$\vec{A} = \hat{e}_\theta$

Hence

$$\nabla.\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(0) + \frac{\partial}{\partial \theta}(r\sin\theta) + \frac{\partial}{\partial\phi}(0)\right]$$

$$\nabla.\vec{A} = \frac{\cot\theta}{r} \text{ ---------(1)}$$

Now

We know

in Spherical Coordinate System

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_\phi \text{ --------(2)}$$

From (1) $\nabla.\vec{A} = \frac{\cot\theta}{r}$

Taking

$\nabla.\vec{A} = \psi$

Then

$$\nabla(\nabla.\vec{A}) = \left(\frac{\partial}{\partial r}\nabla.\vec{A}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial\theta}\nabla.\vec{A}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial z}\nabla.\vec{A}\right)\hat{e}_\phi$$

$$\nabla(\nabla.\vec{A}) = \left(\frac{\partial}{\partial r}\frac{\cot\theta}{r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial\theta}\frac{\cot\theta}{r}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial z}\frac{\cot\theta}{r}\right)\hat{e}_\phi$$

$$\nabla(\nabla.\vec{A}) = -\frac{\cot\theta}{r^2}\hat{e}_r + \frac{1}{r^2}(-\operatorname{cosec}^2\theta)\hat{e}_\theta + \frac{1}{r\sin\theta}(0)\hat{e}_\phi$$

$$\nabla(\nabla.\vec{A}) = -\frac{\cot\theta}{r^2}\hat{e}_r - \frac{\operatorname{cosec}^2\theta}{r^2}\hat{e}_\theta + 0\hat{e}_\phi$$

Q21: In Spherical Polar coordinates $\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta$ where A_r, A_θ are independent of ϕ, Prove that $\nabla\times\nabla\times\vec{A}$ is similar to $\vec{A}$

Solution:

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Spherical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$ $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$ -----(Students can use direct this formula

also)

Given that

$\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta$ where A_r , A_θ are independent of ϕ ,

hence

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & 0 \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\left[(0-0)\hat{e}_r + (0-0)(r\hat{e}_\theta) + \left(A_\theta + \frac{r\,\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)r\sin\theta\hat{e}_\phi\right]$$

$(0-0)\hat{e}_r + (0-0)(r\hat{e}_\theta) +$

$$\nabla \times \vec{A} = \left(\frac{1}{r}A_\theta + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\hat{e}_\phi \text{ -------(1)}$$

Now

$\nabla \times \nabla \times \vec{A} = ?$

We know that in Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

From (1)

$$\nabla \times \vec{A} = \left(\frac{1}{r}A_\theta + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\hat{e}_\phi \text{ -------(1)}$$

Taking $\vec{A} = \nabla \times \vec{A}$ in (2) we get

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta\left(\frac{1}{r}A_\theta + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right) \end{vmatrix}$$

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \sin\theta\left(A_\theta + r\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right) \end{vmatrix}$$

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{r^2 \sin\theta}\left[\left\{\sin\theta\left(\frac{\partial}{\partial \theta}A_\theta + r\frac{\partial^2 A_\theta}{\partial\theta\partial r} + \frac{\partial^2 A_r}{(\partial\theta)^2}\right) + \cos\theta\left(A_\theta + r\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\right\}\hat{e}_r + \left\{\sin\theta\left(-\frac{\partial}{\partial r}A_\theta - r\frac{\partial^2 A_\theta}{(\partial r)^2} - \frac{\partial A_\theta}{\partial r} + \frac{\partial^2 A_r}{\partial r\partial\theta}\right)\right\}(r\hat{e}_\theta)\right]$$

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{r^2}\left[\left\{\left(\frac{\partial}{\partial \theta}A_\theta + r\frac{\partial^2 A_\theta}{\partial\theta\partial r} + \frac{\partial^2 A_r}{(\partial\theta)^2}\right) + \cot\theta\left(A_\theta + r\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\right\}\hat{e}_r + \left\{\left(-2\frac{\partial}{\partial r}A_\theta - r\frac{\partial^2 A_\theta}{(\partial r)^2} + \frac{\partial^2 A_r}{\partial r\partial\theta}\right)\right\}(r\hat{e}_\theta)\right]$$

which is Similar to $\vec{A}$

Hence Proved $\nabla \times (\nabla \times \vec{A})$ is Similar to $\vec{A}$

Q22: In Spherical Polar coordinates $\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_\phi\hat{e}_\phi$ if $A_r = A_\theta = \frac{\partial\phi}{\partial\phi} = 0$ and $\nabla \times \nabla \times \vec{A} = \vec{0}$ then prove that

$$r^2\frac{\partial^2 A_\phi}{\partial r^2} + 2r\frac{\partial A_\phi}{\partial r} + \frac{\partial}{\partial\theta}\left(\frac{\partial A_\phi}{\partial\theta} + A_\phi\cot\theta\right) = 0$$

Solution:

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Spherical Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$ $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence in Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\mathrm{Sin}\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\mathrm{Sin}\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\mathrm{Sin}\theta A_\phi \end{vmatrix}$$

------**(Students can use direct this formula also)**

Given that $A_r = A_\theta = \frac{\partial\phi}{\partial\phi} = 0$ **and** $\nabla \times \nabla \times \vec{A} = \vec{0}$

hence

$$\nabla \times \vec{A} = \frac{1}{r^2\mathrm{Sin}\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\mathrm{Sin}\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\mathrm{Sin}\theta A_\phi \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\left([r\cos\theta A_\phi + r\sin\theta\frac{\partial A_\phi}{\partial\theta}]\hat{e}_r + [-\sin\theta A_\phi - r\sin\theta\frac{\partial A_\phi}{\partial r}](r\hat{e}_\theta)\right)$$

$$\nabla \times \vec{A} = \left(\frac{\cot\theta A_\phi}{r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta}\right)\hat{e}_r + \left(-\frac{A_\phi}{r} - \frac{\partial A_\phi}{\partial r}\right)\hat{e}_\theta \text{ -------(1)}$$

Now

$\nabla \times \nabla \times \vec{A} = ?$

Hence in Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla\times(\nabla\times\vec{A})=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

From (1)

$$\nabla\times\vec{A}=\left(\frac{\cot\theta A_\phi}{r}+\frac{1}{r}\frac{\partial A_\phi}{\partial\theta}\right)\hat{e}_r+\left(-\frac{A_\phi}{r}-\frac{\partial A_\phi}{\partial r}\right)\hat{e}_\theta$$

Taking $\vec{A}=\nabla\times\vec{A}$ **in (2) we get**

$$\nabla\times(\nabla\times\vec{A})=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ \frac{\cot\theta A_\phi}{r}+\frac{1}{r}\frac{\partial A_\phi}{\partial\theta} & r\left(-\frac{A_\phi}{r}-\frac{\partial A_\phi}{\partial r}\right) & 0\end{vmatrix}$$

$$=\frac{1}{r^2\sin\theta}\left[\left(0-\frac{\partial A_\phi}{\partial\phi}-r\frac{\partial^2A_\phi}{\partial\phi\partial r}\right)\hat{e}_r+\left(\frac{\cot\theta}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{1}{r}\frac{\partial^2A_\phi}{\partial\phi\partial\theta}\right)(r\hat{e}_\theta)+\left(-\frac{\partial A_\phi}{\partial r}-\frac{\partial A_\phi}{\partial r}-r\frac{\partial^2A_\phi}{(\partial r)^2}-\frac{\partial}{\partial\theta}\left(\frac{\cot\theta A_\phi}{r}+\frac{1}{r}\frac{\partial A_\phi}{\partial\theta}\right)\right)r\sin\theta\hat{e}_\phi\right]$$

Given that $A_r=A_\theta=\frac{\partial\phi}{\partial\phi}=0$ **and** $\nabla\times\nabla\times\vec{A}=\vec{0}$

$$\vec{0}=\frac{1}{r^2\sin\theta}\left[(-0+0)\hat{e}_r+(0+0)(r\hat{e}_\theta)-\left(r\frac{\partial^2A_\phi}{(\partial r)^2}+2\frac{\partial A_\phi}{\partial r}+\frac{\partial}{\partial\theta}\left(\frac{\cot\theta A_\phi}{r}+\frac{1}{r}\frac{\partial A_\phi}{\partial\theta}\right)\right)r\sin\theta\hat{e}_\phi\right]$$

$$0\hat{e}_r+0\hat{e}_\theta+0\hat{e}_\phi=\frac{1}{r^2\sin\theta}\left[0\hat{e}_r+0\hat{e}_\theta)+\left(-r\frac{\partial^2A_\phi}{(\partial r)^2}-2\frac{\partial A_\phi}{\partial r}-\frac{\partial}{\partial\theta}\left(\frac{\cot\theta A_\phi}{r}+\frac{1}{r}\frac{\partial A_\phi}{\partial\theta}\right)\right)r\sin\theta\hat{e}_\phi\right]$$

$$0\hat{e}_r+0\hat{e}_\theta+0\hat{e}_\phi=0\hat{e}_r+0\hat{e}_\theta-\left(\frac{\partial^2A_\phi}{(\partial r)^2}+\frac{2}{r}\frac{\partial A_\phi}{\partial r}+\frac{\partial}{\partial\theta}\left(\frac{\cot\theta A_\phi}{r^2}+\frac{1}{r^2}\frac{\partial A_\phi}{\partial\theta}\right)\right)\hat{e}_\phi$$

Comparing $\hat{e}_r,\hat{e}_\theta,\hat{e}_\phi$ **we get**

$$\left[\frac{\partial^2 A_\phi}{(\partial r)^2}+\frac{2}{r}\frac{\partial A_\phi}{\partial r}+\frac{\partial}{\partial\theta}\left(\frac{\cot\theta A_\phi}{r^2}+\frac{1}{r^2}\frac{\partial A_\phi}{\partial\theta}\right)\right]=0$$

$$\left[r^2\frac{\partial^2 A_\phi}{(\partial r)^2}+2r\frac{\partial A_\phi}{\partial r}+\frac{\partial}{\partial\theta}\left(\cot\theta A_\phi+\frac{\partial A_\phi}{\partial\theta}\right)\right]=0$$

$$r^2\frac{\partial^2 A_\phi}{(\partial r)^2}+2r\frac{\partial A_\phi}{\partial r}+\frac{\partial}{\partial\theta}\left(\cot\theta A_\phi+\frac{\partial A_\phi}{\partial\theta}\right)=0$$

Q23: Prove (i) $\nabla\times\nabla\psi = 0$ (ii) $\nabla\cdot\nabla\times\vec{A}=0$ in Cylindrical and Spherical coordinate Systems:

Solution:

(i)

We know that in curvilinear system

$$\nabla\psi=\frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1+\frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2+\frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

In Cylindrical Polar Coordinate System

$h_1=1$, $h_2=r$, $h_3=1$, $(u_1, u_2, u_3)=(r, \theta, z)$

Hence in cylindrical polar Coordinate System

$$\nabla\psi=\frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r+\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta+\frac{1}{1}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$$

$\nabla\psi=\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r+\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta+\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$------**(Students can write direct this formula also)**

Now

We know that in curvilinear system

$$\nabla\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Cylindrical polar Coordinate System

$h_1=1$, $h_2=r$, $h_3=1$ $(u_1, u_2, u_3)=(r, \theta, z)$

Hence in Spherical polar Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$
------(Students can use direct this formula also)

Find $\nabla \times \nabla \psi = ?$

From(1)

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \left(\frac{\partial\psi}{\partial z}\right)\hat{e}_z$$

Taking $\nabla\psi = \vec{A}$

$$\nabla \times \nabla \psi = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{\partial\psi}{\partial r} & \frac{\partial\psi}{\partial \theta} & \frac{\partial\psi}{\partial z} \end{vmatrix}$$

$$\nabla \times \nabla \psi = \frac{1}{r}\left[\left(\frac{\partial^2\psi}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial z\partial\theta}\right)\hat{e}_r + \left(\frac{\partial^2\psi}{\partial z\partial r} - \frac{\partial^2\psi}{\partial r\partial z}\right)(r\hat{e}_\theta) + \left(\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{\partial^2\psi}{\partial\theta\partial r}\right)\hat{e}_z\right]$$

$$= 0\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_z$$

$$= \vec{0}$$

(ii)

We know that in curvilinear system

$$\nabla\psi = \frac{1}{h_1}\left(\frac{\partial\psi}{\partial u_1}\right)\hat{e}_1 + \frac{1}{h_2}\left(\frac{\partial\psi}{\partial u_2}\right)\hat{e}_2 + \frac{1}{h_3}\left(\frac{\partial\psi}{\partial u_3}\right)\hat{e}_3$$

In Spherical Polar Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$, $(u_1 , u_2 , u_3) = (r , \theta , \phi)$

Hence in Spherical Ploar Coordinate System

$$\nabla\psi = \frac{1}{1}\left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_\phi$$

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial z}\right)\hat{e}_\phi \text{------(2)}$$

Now

We know that in curvilinear system

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_r\hat{e}_r & h_2\hat{e}_2 & h_3\hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

In Spherical Polar Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$, $(u_1 , u_2 , u_3) = (r , \theta , \phi)$

Hence in Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

Find $\nabla \times \nabla\psi = ?$

From(2)

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial\phi}\right)\hat{e}_\phi \text{------}(2)$$

Taking $\nabla\psi = \vec{A}$

$$\nabla \times \nabla\psi = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{\partial\psi}{\partial r} & \frac{\partial\psi}{\partial\theta} & \frac{\partial\psi}{\partial\phi} \end{vmatrix}$$

$$\nabla \times \nabla\psi = \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial^2\psi}{\partial\theta\partial\phi} - \frac{\partial^2\psi}{\partial\phi\partial\theta}\right)\hat{e}_r + \left(\frac{\partial^2\psi}{\partial\phi\partial r} - \frac{\partial^2\psi}{\partial r\partial\phi}\right)(r\hat{e}_\theta) + \left(\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{\partial^2\psi}{\partial\theta\partial r}\right)\hat{e}_\phi\right]$$

$$= 0\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_z$$

$$= \vec{0}$$

Q24: Express heat equation $\frac{\partial u}{\partial r} = C^2\,\nabla^2 u$ in Spherical Polar coordinates if u is independent of

(i) , (ii) ϕ and θ (iii) r and t (iv) θ , ϕ and t

Solution:

(i)

We know that in curvilinear system

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

In Spherical Polar Coordinate System

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$, $(u_1, u_2, u_3) = (r, \theta, \phi)$

Hence

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(\frac{r(r\sin\theta)}{1}\frac{\partial\psi}{\partial u_r}\right)+\frac{\partial}{\partial u_\theta}\left(\frac{(1)(r\sin\theta)}{r}\frac{\partial\psi}{\partial u_\theta}\right)+\frac{\partial}{\partial u_\phi}\left(\frac{(1)(r)}{r\sin\theta}\frac{\partial\psi}{\partial u_\phi}\right)\right]$$

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial\psi}{\partial u_r}\right)+\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial\psi}{\partial u_\theta}\right)+\frac{\partial}{\partial u_\phi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial u_\phi}\right)\right]$$

$$\frac{\partial u}{\partial t} = C^2\nabla^2 u = ?$$

Hence

Put $\psi = u$

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right)+\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{\partial}{\partial u_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial u_\phi}\right)\right]$$

Given that u is independent of ϕ

hence

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right)+\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{\partial}{\partial u_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial u_\phi}\right)\right] \text{ becomes}$$

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right)+\frac{\partial}{\partial u_\theta}\left(r^2\sin\theta\frac{\partial u}{\partial u_\theta}\right)\right]$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)$$

Therefore

$$\frac{\partial u}{\partial t} = C^2\nabla^2 u = ?$$

$$\frac{\partial u}{\partial t} = C^2\left[\frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)\right]$$

(ii)

Given that u is independent of ϕ and θ

hence

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right)+\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{\partial}{\partial u_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial u_\phi}\right)\right]$$ **becomes**

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right)\right]$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)$$

Therefore

$$\frac{\partial u}{\partial t} = C^2\,\nabla^2 u \;=?$$

$$\frac{\partial u}{\partial t} = C^2\left[\frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)\right]$$

(iii)

Given that u is independent of r and t

hence

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right)+\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{\partial}{\partial u_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial u_\phi}\right)\right]$$ **becomes**

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{\partial}{\partial u_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial u_\phi}\right)\right]$$

$$\nabla^2 u == \frac{1}{r^2\sin\theta}\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial u_\phi}\left(\frac{\partial u}{\partial u_\phi}\right)$$

Therefore

$$\frac{\partial u}{\partial t} = C^2\,\nabla^2 u =?$$

$$\frac{\partial u}{\partial t} = C^2\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial u_\phi}\left(\frac{\partial u}{\partial u_\phi}\right)\right]$$

$$\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial u_\phi}\left(\frac{\partial u}{\partial u_\phi}\right)\right]=0$$

(iv)

Given that u is independent of θ , ϕ and t

hence

$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right) + \frac{\partial}{\partial u_\theta}\left(\sin\theta\frac{\partial u}{\partial u_\theta}\right) + \frac{\partial}{\partial u_\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial u_\phi}\right)\right]$ becomes

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial u_r}\left(r^2\sin\theta\frac{\partial u}{\partial u_r}\right)\right]$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)$$

Therefore

$$\frac{\partial u}{\partial t} = C^2\,\nabla^2 u \ =?$$

$$\frac{\partial u}{\partial t} = C^2\left[\frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)\right]$$

$$0 = C^2\left[\frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)\right]$$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)\right]=0$$

$$\frac{1}{r^2}\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)=0$$

$$\frac{\partial}{\partial u_r}\left(r^2\frac{\partial u}{\partial u_r}\right)=0$$

Q25: Using cylindrical polar coordinates , Evaluate

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\right)dzdydx$$

<u>**Solution:**</u>

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\right)dzdydx=?$$

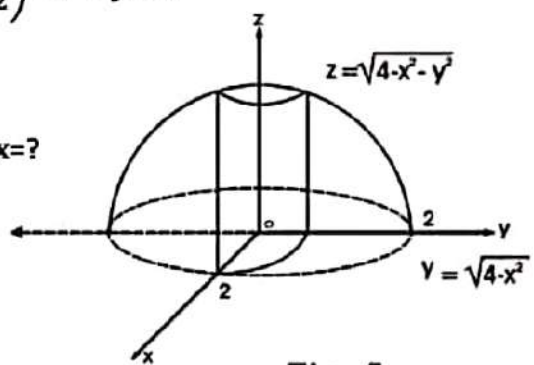

Fig 1

Using cylindrical Polar Coordinates

$x=r\cos\theta$, $y=r\sin\theta$, $z= z$,

$0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le z \le 2$,

$x^2 + y^2 = r^2$, $dzdydx = rdzdrd\theta$

then

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\right)dzdydx = \int_0^{\frac{\pi}{2}}\int_0^2\int_0^{\sqrt{4-r^2}}\left(\frac{r\cos\theta}{r}\right)rdzdrd\theta$$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \left(\frac{x}{\sqrt{x^2+y^2}}\right) dzdydx = \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{\sqrt{4-r^2}} (\cos\theta) rdzdrd\theta$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \left(\frac{x}{\sqrt{x^2+y^2}}\right) dzdydx = \int_0^{\frac{\pi}{2}} \int_0^2 \left(\sqrt{4-r^2} - 0\right) rdrd\theta$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \left(\frac{x}{\sqrt{x^2+y^2}}\right) dzdydx = \int_0^{\frac{\pi}{2}} \int_0^2 \left(\sqrt{4-r^2}\right) rdrd\theta$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \left(\frac{x}{\sqrt{x^2+y^2}}\right) dzdydx = \left|\frac{(4-r^2)^{3/2}}{(-2)(3/2)}\right|_0^2 (\sin\frac{\pi}{2} - \text{Sin}0)$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \left(\frac{x}{\sqrt{x^2+y^2}}\right) dzdydx = (0 + \frac{(4-0^2)^{3/2}}{3})(1)$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \left(\frac{x}{\sqrt{x^2+y^2}}\right) dzdydx = \frac{8}{3}$

Q26: Using cylindrical polar coordinates, Evaluate $\iiint_R x\, dV$ where R is the region in 1st octant bounded by cylinder $x^2 + y^2 = 8$ and planes z=0 , z = 6

Solution:

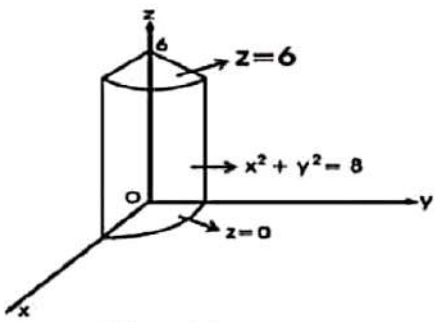

Fig 2

$\iiint_R x\, dV = ?$ =?

where R is the region in 1st octant bounded by cylinder $x^2 + y^2 = 8$ and planes z=0 , z = 6

Hence

$$\iiint_R x\, dV = \int_0^{\sqrt{8}} \int_0^{\sqrt{8-x^2}} \int_0^6 (x) dzdydx$$

Using cylindrical Polar Coordinates x=rcosθ , y=rsinθ , z= z ,

$0 \le r \le \sqrt{8}, 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le z \le 2 \quad$, dzdydx =rdzdrdθ

$$\iiint_R x\,dV = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{8}} \int_0^{6} (r\cos\theta) r\,dz\,dr\,d\theta$$

$$\iiint_R x\,dV = (6\text{-}0)\left(\frac{(\sqrt{8})^3}{3} - 0\right)(\sin\frac{\pi}{2} - \sin 0)$$

$$\iiint_R x\,dV = = 32\sqrt{2}$$

Q27: Using cylindrical polar coordinates, Evaluate

$\iiint_R (x^2+y^2)\,dV$ **where R is the region bounded by two cylinders $r=1$, $r=2$ for $0 \le z \le 2$**

Solution:

$\iiint_R (x^2+y^2)\,dV = ?$

where R is the region bounded

by two cylinders

$r=1$, $r=2$ for $0 \le z \le 2$

Now

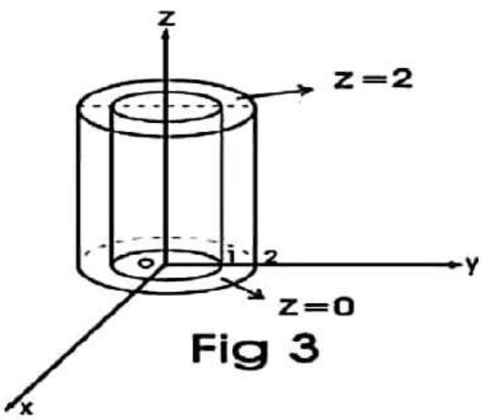

Fig 3

$$\iiint_R (x^2+y^2)\,dV = \iiint_R (x^2+y^2)\,dz\,dy\,dx$$

Using cylindrical Polar Coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$,

$1 \le r \le 2$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, dzdydx =rdzdrdθ

$$\iiint_R (\mathbf{x}^2+\mathbf{y}^2)\,dV = \iiint_R (\mathbf{x}^2+\mathbf{y}^2)\,dz\,dy\,dx$$

$$\iiint_R (\mathbf{x}^2+\mathbf{y}^2)\,dV = \int_0^{2\pi}\int_1^2\int_0^2 (r^2)\, r\,dz\,dr\,d\theta$$

$$=(2-0)\left(\frac{2^4}{4}-\frac{1^4}{4}\right)(2\pi-0)$$

$$\iiint_R (\mathbf{x}^2+\mathbf{y}^2)\,dV = 15\pi$$

Q28–Using Spherical polar coordinates, Evaluate

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dz\,dy\,dx$$

Solution:s

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dz\,dy\,dx = ?$$

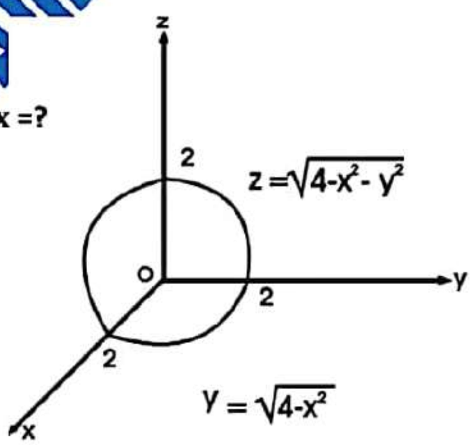

Fig 4

Using Spherical Coordinates

$x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$,

$0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$,

$x^2+y^2+z^2=r^2$, $dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$

Hence

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dz\,dy\,dx$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2\left(\frac{1}{1+r^2}\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dzdydx = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2\left(\frac{1+r^2-1}{1+r^2}\right)\sin\theta\, drd\theta d\phi$$

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dzdydx = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2\left(1-\frac{1}{1+r^2}\right)\sin\theta\, drd\theta d\phi$$

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dzdydx = (2-\tan^{-1}2)\left(-\cos\frac{\pi}{2}+\cos 0\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dzdydx = (2-\tan^{-1}2)(-0+1)\left(\frac{\pi}{2}\right)$$

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\left(\frac{1}{1+x^2+y^2+z^2}\right)dzdydx = (2-\tan^{-1}2)(-0+1)\left(\frac{\pi}{2}\right)$$

Q29: Using Spherical polar coordinates, Evaluate

$\iiint\limits_R \sqrt{x^2+y^2+z^2}\,dV$ Where R is the regin bounded by xy plane and hemi sphere

$x^2+y^2+z^2 = 9$, $z \geq 1$

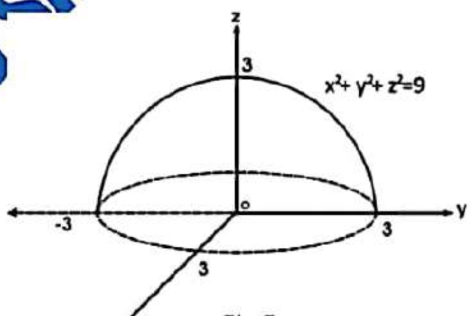

Fig 5

Solution:

$\iiint\limits_R \sqrt{x^2+y^2+z^2}\,dV = ?$

Using Spherical Coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq 2\pi$, $x^2+y^2+z^2 = r^2$,

$dzdydx = r^2\sin\theta\, drd\theta d\phi$

hence

$$\iiint\limits_R \sqrt{x^2+y^2+z^2}\,dV = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}}\left(\sqrt{x^2+y^2+z^2}\right)dzdydx$$

$$\iiint_R \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 (r)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\iiint_R (x^2+y^2+z^2)\, dV = \left(\frac{3^4}{4} - 0\right)\left(-\cos\frac{\pi}{2} + \cos 0\right)(2\pi - 0)$$

$$\iiint_R \sqrt{x^2+y^2+z^2}\, dV = \left(\frac{81}{4}\right)(0+1)(2\pi)$$

$$\iiint_R (x^2+y^2+z^2)\, dV = \frac{81\pi}{2S}$$

Q30:Using Spherical Polar coordinates, Evaluate $\iiint_R x^2\, dV$

where R is the region bounded by cone $z = \sqrt{x^2+y^2}$

sphere $x^2 + y^2 + z^2 = 1$

Solution:

$\iiint_R x^2\, dV = ?$

where **R** is the region bounded by cone $z = \sqrt{x^2+y^2}$

and sphere $x^2 + y^2 + z^2 = 1$

as shown in fig 6

hence

for point of intersection of

cone and sphere

Cone : $z = \sqrt{x^2+y^2}$ ------(1)

Sphere: $x^2 + y^2 + z^2 = 1$------(2)

From (1)

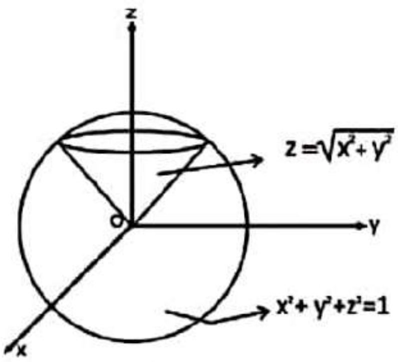

Fig 6

sphere $x^2 + y^2 = z^2$ put in (2)

$x^2 + y^2 + z^2 = 1$

$z^2 + z^2 = 1$

$2z^2 = 1$

$z = \frac{1}{\sqrt{2}}$

Using Spherical Coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$0 \le r \le 1$, $0 \le \phi \le 2\pi$, $dzdydx = r^2\sin\theta drd\theta d\phi$

For limit of θ

$z = r\cos\theta \Longrightarrow \frac{1}{\sqrt{2}} = 1\cos\theta \implies \theta = \frac{\pi}{4}$ Hence $\frac{\pi}{4} \le \theta \le \pi$

$$\iiint_R x^2\, dV = \iiint_R x^2\, dzdydx$$

$$\iiint_R x^2\, dV = ? = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\pi} \int_0^1 (r^2\sin^2\theta\cos^2\phi)\, r^2\sin\theta\, drd\theta d\phi$$

$$\iiint_R x^2\, dV = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\pi} \int_0^1 (r^4\sin^3\theta\cos^2\phi)\, drd\theta d\phi$$

$$\iiint_R x^2\, dV = \int_0^1 r^4 dr \int_0^{2\pi} \frac{1+\cos 2\phi}{2} d\phi \int_{\frac{\pi}{4}}^{\pi} \sin\theta(1-\cos^2\theta)d\theta$$

$$\iiint_R x^2\, dV = \left(\frac{1}{5}\right)\left(\frac{1}{2}\right)\left(2\pi + \frac{\sin 2(2\pi)}{2} - 0 - \frac{\sin 2(0)}{2}\right)\left(-\cos\pi + \cos\frac{\pi}{4} + \frac{(\cos^3\pi)}{3} - \frac{[\cos^3(\frac{\pi}{4})]}{3}\right)$$

$$\iiint_R x^2\, dV = \left(\frac{1}{10}\right)(2\pi)\left(1 + \frac{1}{\sqrt{2}} - \frac{1}{3} - \frac{(\frac{1}{\sqrt{2}})^3]}{3}\right)$$

$$\iiint_R x^2\,dV = \left(\frac{\pi}{10}\right)\left(\frac{2}{3} + \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)$$

$$\iiint_R x^2\,dV = \left(\frac{\pi}{10}\right)\left(\frac{4\sqrt{2}+6-1}{6\sqrt{2}}\right)$$

$$\iiint_R x^2\,dV = \left(\frac{4\sqrt{2}+5}{60\sqrt{2}}\right)\pi$$

$$\iiint_R x^2\,dV = \left(\frac{8+5\sqrt{2}}{60}\right)\pi$$

------------------BEST OF LUCK----------------------

May all the Students

come off through their

Examinations with

flying colours by

the Grace of

ALLAH ALMIGHTY

&

MUHAMMAD (خاتم النبیین صلی اللہ علیہ وآلہ وسلم)

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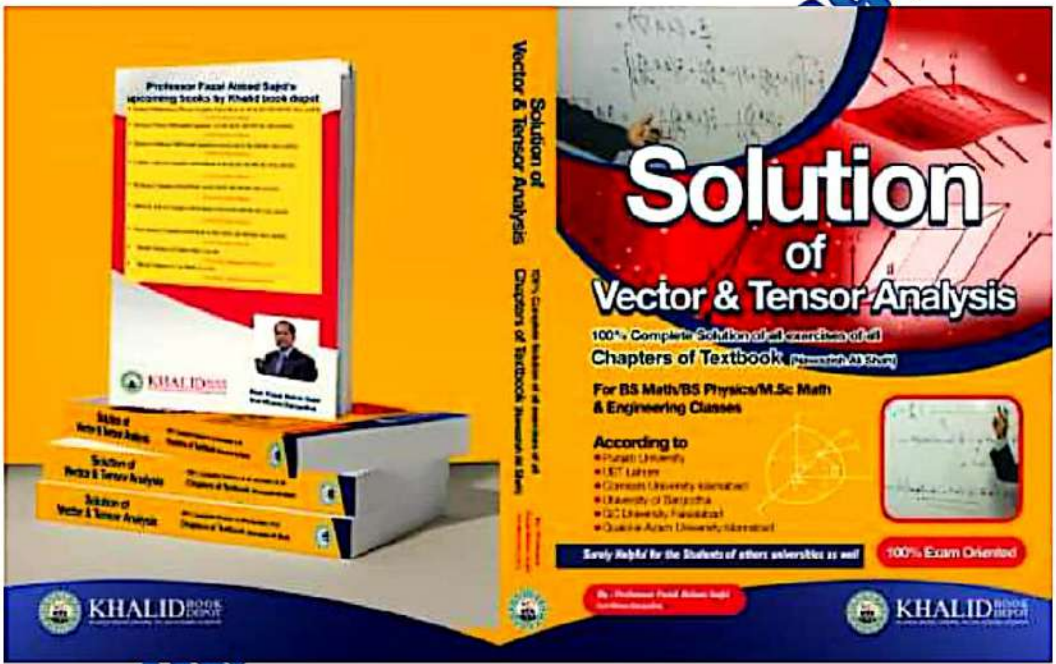

I am infinity time grateful to

ALLAH ALMIGHTY

&

MUHAHAMMAD (خاتم النبیین صلی اللہ تعالیٰ علیہ وآلہ واصحابہ وسلم)

for these big achievements

and want to say thanks to Directors of

www.mathcity.org

for share my data on their website

Regards: Prof.FAZAL ABBAS SAJID
