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Notes;
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CHAPTER: 5

L_p - SPACES (L^p)

Let (X, S, μ) be a Lebesgue measure space and consider the set
$$L_p = \{f : X \to \mathbb{R} : \int_X |f|^p dx < \infty\} \quad 0 < p < \infty.$$

i.e. L_p contains those functions 'f' such that $|f|^p$ is integrable. (Also known as p integrable).

Question:-

Show that L_p is a linear space.

Proof:-

Let $f, g \in L_p = \{f : X \to \mathbb{R} ; \int_X |f|^p dx < \infty\}$

i) $\Rightarrow \int_X |f|^p dx < \infty$ and $\int_X |g|^p dx < \infty$

We will show that $f+g \in L_p$.
i.e. we will show that $\int_X |f+g|^p dx < \infty$. (∴ by def of L_p)

Now $|f+g| \leq |f| + |g|$
If $|f| \leq |g|$

$$\Rightarrow |f(x) + g(x)| \leq |f(x)| + |f(x)|$$
$$\leq 2|f(x)| = 2\max\{|f(x)|, |g(x)|\}$$

Hence
$$|f(x) + g(x)| \leq 2\max\{|f(x)|, |g(x)|\}$$
$$|f(x) + g(x)|^p \leq 2^p \max\{|f(x)|^p, |g(x)|^p\}$$

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$$|f(x)+g(x)|^p \le 2^p\{|f(x)|^p+|g(x)|^p\}$$

$$\Rightarrow \int_X |f(x)+g(x)|^p\,dx \le 2^p\left\{\int_X |f(x)|^p dx+\int_X |g(x)|^p dx\right\}$$

$$\Rightarrow \int_X |f(x)+g(x)|^p\,dx < \infty$$

$\Rightarrow f+g \in L_p$ if $p>0$

$\Rightarrow L_p$ is closed under addition.

Next we show that L_p is closed under scalar multiplication.
i.e. for any scalar (say) α and $f\in L_p$.
we will show that $\alpha f\in L_p$.

For this we will show that $\int_X |\alpha f|^p dx < \infty$.

Now since $f\in L_p \Rightarrow \int_X |f|^p dx < \infty$

Consider

$$\int_X |\alpha f|^p dx = \int_X |\alpha|^p |f|^p dx$$

$$= |\alpha|^p \int_X |f|^p dx < \infty$$

$$\Rightarrow \int_X |\alpha f|^p dx < \infty$$

$\Rightarrow \alpha f \in L_p$.

$\Rightarrow L_p=\{f: X\to \mathbb{R};\ \int_X |f|^p dx<\infty\}$ is a vector space for $p>0$.

Lp as a normed linear space:-

Definition:-

Let us define a function

$\|\cdot\| : L_p \to \mathbb{R}$ by

$$\|f\|_p = \left(\int_E |f(x)|^p \, d\mu\right)^{1/p}$$

i) $\|f\|_p \geq 0$ (ii) $\|\alpha f\|_p = |\alpha| \, \|f\|_p$.

$$\Rightarrow \|\alpha f\|_p = \left(\int_E |\alpha f|^p \, d\mu\right)^{1/p}$$

$$= \left(\int_E |\alpha|^p . |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha|^{p \times 1/p} \left(\int_E |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha| \left(\int_E |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha| \, \|f\|_p .$$

Thus

$$\|\alpha f\|_p = |\alpha| \, \|f\|_p .$$

Note:- $\|x\| = 0 \iff x = 0$

(i) If $f(x) = 0 \quad \forall \ x$

$$\Rightarrow |f(x)| = 0$$

$$\Rightarrow |f(x)|^p = 0$$

$$\Rightarrow \int_E |f(x)|^p \, dx = 0$$

$$\Rightarrow \left(\int_E |f(x)|^p \, dx\right)^{1/p} = 0$$

$$\Rightarrow \|f\|_p = 0$$

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Conversely; let $\|f\|_p = 0$

$\Rightarrow \left(\int_E |f|^p dx\right)^{1/p} = 0$

$\Rightarrow \int_E |f(x)|^p dx = 0$

$\Rightarrow |f(x)|^p = 0$ a.e.

$\Rightarrow |f(x)| = 0$ a.e.

$\Rightarrow f(x) = 0$ a.e

$\Rightarrow f = 0$

we have define the relation "$\approx$" in L by $f = g$ if $f(x) = g(x)$ a.e.

Thus $\|f\|_p = 0$

$\Rightarrow f = 0$.

$\|f+g\|_p \leq \|f\|_p + \|g\|_p$

$$\left(\int_E |f(x)+g(x)|^p d\mu\right)^{1/p} \leq \left(\int_E |f|^p d\mu\right)^{1/p} + \left(\int_E |g|^p d\mu\right)^{1/p}.$$

State and prove that lebesgue integral version of Holder Inequality.

Holder Inequality:-

Statement:-

Let $p, q > 1$ such that $1/p + 1/q = 1$. If $f \in L_p$ and $g \in L_q$, then $f g \in L_1$ and $\int_E |f \cdot g| d\mu \leq \left(\int_E |f|^p d\mu\right)^{1/p} \cdot \left(\int_E |g|^q d\mu\right)^{1/q}$ —— (1)

$\Rightarrow \|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$.

Proof:-

If $f = 0$ a.e and $g = 0$ a.e, then clearly eq (1) holds.

Let us assume that $f \neq 0$, $g \neq 0$ a.e, then clearly $\|f\|_p > 0$ and $\|g\|_q > 0$.

Since we know that

$a^{\lambda} \cdot b^{1-\lambda} \leq \lambda a + (1-\lambda) b$ for $\lambda \in (0,1)$, $a,b > 0$

put $\lambda = 1/p$, $1-\lambda = 1/q$, $a = \left(\frac{|f(x)|^p}{\|f\|_p}\right)^{p}$,

$b = \left(\frac{|g(x)|^q}{\|g\|_q}\right)^{1/q}$

$$\Rightarrow \frac{|f(x)|}{\|f\|_p} \cdot \frac{|g(x)|}{\|g\|_q} \leq \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

$$\Rightarrow \frac{|f(x) \cdot g(x)|}{\|f\|_p \cdot \|g\|_q} \leq \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}$$

Taking integral

$$\frac{1}{\|f\|_p \cdot \|g\|_q}\int_E |f(x) \cdot g(x)|\, d\mu \leq \frac{1}{p\|f\|_p^p}\int_E |f(x)|^p d\mu + \frac{1}{q\|g\|_q^q}\int_E |g(x)|^q d\mu$$

$$= \frac{1}{p\|f\|_p^p}\|f\|_p^p + \frac{1}{q\|g\|_q^q}\|g\|_q^q$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{\|f\|_p \cdot \|g\|_q}\int_E |f(x) \cdot g(x)|\, d\mu \leq 1$$

$$\Rightarrow \int_E |f(x) \cdot g(x)|\, d\mu \leq \|f\|_p \cdot \|g\|_q$$

$$\Rightarrow \|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q \quad . \text{ proved!}$$

Minkowski's Inequality:-

Lebesgue integral version of Minkowski's inequality.

Let $1 \le p < \infty$ and $f, g \in L_p$.

Then $\|f+g\|_p \le \|f\|_p + \|g\|_p$

i.e.
$$\left(\int_E |f+g|^p du\right)^{1/p} \le \left(\int_E |f|^p du\right)^{1/p} + \left(\int_E |g|^p du\right)^{1/p}.$$

Proof:-

Since $f, g \in L^p$

$$\Rightarrow \int_E |f|^p du < \infty \quad , \quad \int_E |g|^p du < \infty$$

Now as $|f+g| \le |f| + |g|$

$$\Rightarrow |f+g| \le 2 \max\{|f|, |g|\}$$

$$\Rightarrow |f+g|^p \le 2^p \max\{|f|^p, |g|^p\}$$

$$\Rightarrow |f+g|^p \le 2^p \{|f|^p + |g|^p\}$$

$$\Rightarrow \int_E |f+g|^p du \le 2^p \left\{\int_E |f|^p du + \int_E |g|^p du\right\} < \infty$$

as $f, g \in L_p$.

$$\Rightarrow f+g \in L_p \quad (\because \text{by def: of } L_p)$$

Now consider

$$\|f+g\|_p^p = \int_E |f+g|^p du = \int_E |f+g|^{p-1} . |f+g| \, du$$

$$\le \int_E |f+g|^{p-1} (|f| + |g|) \, du.$$

$$= \int_E |f| \, |f+g|^{p-1} du + \int_E |g| \, |f+g|^{p-1} du$$

$$\|f+g\|_p^p \le \int_E |f| \, |f+g|^{p-1} du + \int_E |g| \, |f+g|^{p-1} du$$

Applying Holder inequality.

$$\|f+g\|_p^p \le \left(\int_E |f|^p d\mu\right)^{1/p} \cdot \left(\int_E |f+g|^{q(p-1)} d\mu\right)^{1/q} + \left(\int_E |g|^p d\mu\right)^{1/p} \cdot \left(\int_E |f+g|^{q(p-1)} d\mu\right)^{1/q}$$

$\frac{1}{p}+\frac{1}{q}=1$

$\frac{q+p}{pq}=1$

$\frac{q}{q}+\frac{p}{q}=p$

$1+\frac{p}{q}=p$

$\frac{p}{q}=p-1$

$p=(p-1)q$

where $1/p + 1/q = 1$

$$\Rightarrow \|f+g\|_p^p \le \|f\|_p \cdot \left(\int_E |f+g|^p d\mu\right)^{1/q} + \|g\|_p \left(\int_E |f+g|^p d\mu\right)^{1/q}$$

$$\Rightarrow \|f+g\|_p^p \le \left(\int_E |f+g|^p d\mu\right)^{1/q} \left[\|f\|_p + \|g\|_p\right]$$

$$\le \left(\int_E |f+g|^p d\mu\right)^{\frac{1}{p}\times\frac{p}{q}} \left[\|f\|_p + \|g\|_p\right]$$

$$\le \left(\|f+g\|_p\right)^{p/q} \left[\|f\|_p + \|g\|_p\right]$$

$$\Rightarrow \|f+g\|_p^p \le \|f+g\|_p^{p/q} \left[\|f\|_p + \|g\|_p\right]$$

$$\frac{\|f+g\|_p^p}{\|f+g\|_p^{p/q}} \le \|f\|_p + \|g\|_p$$

$p - p/q = p(1-1/q)$
$= p(1/p)$
$= 1$

$$\|f+g\|_p^{p-p/q} \le \|f\|_p + \|g\|_p$$

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

which is the required proof

Note:- $L_p = \{f : \int_E |f|^p d\mu < \infty\}$

L_p is vector space if $p > 0$

L_p is normed linear space if $p \ge 1$.

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Essential Bounded:-

Let f be a function defined on E, then a number $M \in \mathbb{R}$ is said to be essential bound if $|f(x)| \leq M$ a.e. If f has essential bound then f is said to be essential bounded function.

The essential supremum of a function f is denoted by $\|f\|_\infty$ and is defined by $\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e}\}$.

The collection of all measurable and essential bounded function is denoted by L_∞ or $L_\infty(\mu)$ or $L_\infty(E)$.

And L_∞ is normed linear space under the normed defined

$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e}\}$.

Note:-

→ $L_p = \{f : \int_E |f|^p d\mu < \infty\}$ for $p \geq 1$,

L_p is normed linear space.

$$\|f\|_p = \left(\int_E |f|^p d\mu\right)^{1/p}$$

→ $L_\infty = \{f : f \text{ is essential bounded}\}$

$$\|f\|_\infty = \inf\{M : |f(x)| < M \text{ a.e}\}$$

Theorem:-

If $\mu(E) < \infty$, then $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$.

Proof:-

Since $\|f\|_\infty = \inf\{\alpha : \mu\{x \in E : |f(x)| > \alpha\} = 0\}$

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let us suppose $\|f\|_\infty = M$

$$\mu\{x \in E : |f(x)| > M\} = 0$$

i.e $|f(x)| < M$ a.e.

we need to prove that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$

Clearly $M - 1/p < M$ for any $p > 0$

By definition of essential supremum that M is the infimum value such that

$$\mu\{x ; |f(x)| > M\} = 0$$

But since $M - 1/p < M$, so

$$\Rightarrow \mu\{x ; |f(x)| > M - 1/p\} > 0$$ (because $\mu(A) \nless 0$ for any set A)

let $A = \{x \in E : |f(x)| > M - 1/p\}$

Now $\|f\|_p = \left(\int_E |f|^p d\mu\right)^{1/p} \geq \left(\int_A |f|^p d\mu\right)^{1/p}$ ($\because A \subseteq E$)

$$> \left(\int_A \left(M - \frac{1}{p}\right)^p d\mu\right)^{1/p}$$

$$= \left(M - 1/p\right)^{p \times 1/p} \left(\int_A d\mu\right)^{1/p}$$

$$= \left(M - \frac{1}{p}\right)(\mu(A))^{1/p}$$

$$\Rightarrow \|f\|_p > \left(M - \frac{1}{p}\right)(\mu(A))^{1/p}$$

Taking limit inf on b.s

$$\liminf \|f\|_p \geq \liminf\left(M - 1/p\right) \cdot \liminf (\mu(A))^{1/p}$$

$$\geq (M - 0) \cdot 1 = M$$

$$\liminf \|f\|_p \geq M \quad \text{———} \ (1)$$

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Also $\|f\|_p = \left(\int_E |f|^p d\mu\right)^{1/p}$

$$\leq \left(\int_E M^p d\mu\right)^{1/p}$$

$$\leq M^{p \times 1/p} \left(\int_E d\mu\right)^{1/p}$$

$$\leq M(\mu(E))^{1/p}$$

$\Rightarrow \|f\|_p \leq M(\mu(E))^{1/p}$

Taking limit Sup on both sides

$$\lim\sup \|f\|_p \leq \lim\sup \left[M(\mu(E))^{1/p}\right]$$

$$\leq M \lim\sup(\mu(E))^{1/p}$$

$$= M(\mu(E))^{\lim_{p\to\infty}(1/p)}$$

$$= M(\mu(E))^{1/\infty}$$

$$= M(\mu(E))^0$$

$$= M \cdot 1$$

$$= M$$

$\Rightarrow \lim\sup \|f\|_p \leq M$ —— (2)

But $\lim\inf \|f\|_p \leq \lim\sup \|f\|_p$ —— (3)

From eq (1), (2) and (3), we get

$\lim\sup \|f\|_p \leq M \leq \lim\inf \|f\|_p \leq \lim\sup \|f\|_p$

$\Rightarrow \lim\sup \|f\|_p \leq M \leq \lim\sup \|f\|_p$.

$\Rightarrow \lim\sup \|f\|_p = M$

$$\lim_{p\to\infty} \|f\|_p = M = \|f\|_\infty$$

Thus

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$$

proved!

Example:-

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p \quad ; \text{ if } \mu(E) < \infty$$

If $\mu(E) < \infty$, then the above result fails.

Sol:-

Let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = c \quad ; \forall\ x \in (0,\infty)$ and $c \neq 0$

where $E = (0,\infty)$

$\mu(E) = \infty$

Then

$$\text{ess Sup } f = \|f\|_\infty = \inf\{M ; |f(x)| \leq M \text{ a.e}\}$$
$$= c$$

$$\Rightarrow \text{ess Sup } f = \|f\|_\infty = c \quad \text{—— (1)}$$

Also since

$$\|f\|_p = \left(\int_E |f(x)|^p \, d\mu\right)^{1/p}$$

$$\lim_{p\to\infty} \|f\|_p = \left(\int_0^\infty |c|^p \, d\mu\right)^{1/p}$$

$$= |c|^{p \cdot 1/p} \cdot \left(\int_0^\infty d\mu\right)^{1/p}$$

$$= |c| \cdot \left(\mu(E)\Big|_0^\infty\right)^{1/p}$$

$$= \infty$$

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$\lim_{p\to\infty} \|f\|_p = \infty$ —— (2)

From eq (1) and (2), we get

$$\lim_{p\to\infty} \|f\|_p \neq \|f\|_\infty .$$

Holder Inequality for L^∞:-

If $f \in L^\infty$ and $g \in L^1$, then
$\|f \cdot g\| \leq \|f\|_\infty \cdot \|g\|_1$.

Proof:-

Let us suppose $\|f\|_\infty = M$

Since $\|f(x)\| \leq \|f\|_\infty = M$ a.e

$\Rightarrow \|f\| \leq M$ a.e.

Now

$|f \cdot g| \leq |f(x)| \cdot |g(x)|$ a.e.

$|fg| \leq M \, |g(x)|$ a.e.

$\int_E |f \cdot g| \, d\mu \leq M \int_E |g(x)| \, d\mu$

$\Rightarrow \left(\int_E |f(x)\, g(x)|^1 d\mu \right)^1 \leq M \left(\int_E |g(x)|^1 d\mu \right)^1$

$\Rightarrow \|f \cdot g\|_1 \leq M \|g\|_1$

$\Rightarrow$ But $M = \|f\|_\infty$

So $\|f \cdot g\|_1 \leq \|f\|_\infty \|g\|_1$.

Proved!!

OR Holder Inequality for L_p:-

$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q$

If $p \to \infty$ then $q = 1$

$\|f \cdot g\|_1 \leq \|f\|_\infty \|g\|_1$ proved

$\because \frac{1}{p} + \frac{1}{q} = 1$

$\frac{1}{\infty} + \frac{1}{q} = 1$

$0 + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1$

$\Rightarrow \boxed{q = 1}$

Minkowski's Inequality for L^∞ :-

If $f, g \in L_p$; $p \geq 1$, then

$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Proof:-

Since $|f(x)| \leq \|f\|_\infty$ a.e

and $|g(x)| \leq \|g\|_\infty$ a.e

Now

$$|f(x)+g(x)| \leq |f(x)| + |g(x)|$$

$$\leq \|f\|_\infty + \|g\|_\infty$$

$$\Rightarrow |f(x)+g(x)| \leq \|f\|_\infty + \|g\|_\infty \quad \text{a.e} \quad \text{--- (1)}$$

But $|f(x)+g(x)| \leq \|f+g\|_\infty$ a.e --- (2)

From eq (1) and (2), we get

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

proved!

Note:-

Holder inequality for L_p.

$$\int_E |f \cdot g| \, d\mu \leq \left(\int_E |f|^p d\mu\right)^{1/p} \cdot \left(\int_E |g|^q d\mu\right)^{1/q}$$

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$$

put $p = q = 2$

$$\int_E |f \cdot g| \, d\mu \leq \left(\int_E |f|^2 d\mu\right)^{1/2} \cdot \left(\int_E |g|^2 d\mu\right)^{1/2}$$

$$\|f \cdot g\|_1 \leq \|f\|_2 \cdot \|g\|_2$$

It is known as Cauchy Schwarz inequality

Theorem:-

Let (X, S, μ) be finite Lebesgue measure space i.e. $\mu(X) < \infty$, then for $1 \leq p < q < \infty$, we have

$L_q(\mu) \subseteq L_p(\mu)$.

Proof:-

Since $p < q \Rightarrow q > p$

$\Rightarrow \frac{q}{p} > 1$

let $r = q/p$.

let us choose "s" such that

$\frac{1}{r} + \frac{1}{s} = 1$

we need to prove that $L_q(\mu) \subseteq L_p(\mu)$.

let us suppose that $f \in L_q(\mu)$

For this we need to show that $f \in L_p(\mu)$.

If $f \in L_q(\mu)$.

$\Rightarrow \int_X |f|^q d\mu < \infty$ $\quad \because r = q/p$, $rp = q$

$\Rightarrow \int_X |f|^{rp} d\mu < \infty$

$\Rightarrow \int_X (|f|^p)^r d\mu < \infty$

$\Rightarrow |f|^p \in L_r$

and $1 \in L_s(X)$ as $\int_X 1 \, d\mu = \mu(X) < \infty$

$\Rightarrow 1 \in L_s$ and $|f|^p \in L_r$ and $\frac{1}{r} + \frac{1}{s} = 1$.

$f \in L_p$, $g \in L_q$
$\lvert f \cdot g \rvert \in L_1$

$\Rightarrow |f|^p \cdot 1 \in L_1$

$\Rightarrow |f|^p \in L_1$

$$\Rightarrow \int_X (|f|^r)^r du < \infty$$

$$\Rightarrow \int_X |f|^p du < \infty$$

$$\Rightarrow f \in L_p$$

or

$$\Rightarrow f \in L_p(\mu).$$

So $L_q(\mu) \subseteq L_p(\mu)$.

proved!

Convergence in L_p (or Mean Convergence):-

Let $\{f_n\}$ be a sequence of functions in L_p then we say f_n converge to $f \in L_p$ or f_n converge in measure to $f(x)$, if

$$\lim_{n\to\infty} \int_E |f_n(x) - f(x)|^p d\mu = 0$$

or

$$f_n \to f \quad \text{in } L_p$$

If $\|f_n(x) - f(x)\|_p < \epsilon \quad \forall \quad n \geq N_0$.

We say that $\{f_n\}$ is cauchy sequence in L_p.

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_E |f_n(x) - f_m(x)|^p dx = 0 \quad ; \quad m, n \geq N.$$

Theorem:-

If $f_n \in L_p$ and $\{f_n\}$ converges in mean or L_p then $\lim_{n\to\infty} f_n(x)$ is unique.

Proof:-

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let us suppose that

$\lim_{n\to\infty} f_n(x) = f(x)$ and $\lim_{n\to\infty} f_n(x) = g(x)$

where $f \neq g$.

Now $\left(\int_E |f-g|^p d\mu\right)^{1/p} = \left(\int_E |f - f_n + f_n - g|^p d\mu\right)^{1/p}$

$\leq \left(\int_E |f-f_n|^p d\mu\right)^{1/p} + \left(\int_E |f_n - g|^p d\mu\right)^{1/p}$

$\lim_{n\to\infty}\left(\int_E |f-g|^p d\mu\right)^{1/p} \leq \lim_{n\to\infty}\left(\int_E |f-f_n|^p d\mu\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |f_n-g|^p d\mu\right)^{1/p}$

$\left(\lim_{n\to\infty} \int_E |f-g|^p d\mu\right)^{1/p} \leq \left(\lim_{n\to\infty} \int_E |f_n-f|^p d\mu\right)^{1/p} + \left(\lim_{n\to\infty} \int_E |f_n-g|^p d\mu\right)^{1/p}$

$\Rightarrow \left(\int_E |f-g|^p d\mu\right)^{1/p} \leq (0)^{1/p} + (0)^{1/p} = 0$

$\Rightarrow \left(\int_E |f-g|^p d\mu\right)^{1/p} = 0$

$\Rightarrow \int_E |f-g|^p d\mu = 0$

$\Rightarrow |f-g|^p = 0$

$\Rightarrow |f-g| = 0$ a.e.

$\Rightarrow f-g=0$ a.e.

$\Rightarrow f=g$ a.e.

$\Rightarrow f(x) = g(x)$ a.e.

Thus the $\lim_{n\to\infty} f_n(x)$ is unique.

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Theorem:-

Show that L^p is a metric space.

Proof:-

Let us define D on $L^p \times L^p$ by

$D(f,g) = \|f-g\|_p$

We need to show that D is a metric space.

i) Clearly $D \geq 0$.

ii) $D(f,g) = 0 \Leftrightarrow \|f-g\|_p = 0$

$$\Leftrightarrow \left(\int_E |f-g|^p du\right)^{1/p} = 0$$

$$\Leftrightarrow \int_E |f-g|^p du = 0$$

$$\Leftrightarrow |f-g|^p = 0$$

$$\Leftrightarrow |f-g| = 0$$

$$\Leftrightarrow f-g = 0$$

$$\Leftrightarrow f = g$$

Thus $D(f,g) = 0 \Leftrightarrow f = g$.

iii) $D(f,g) = \|f-g\|_p$

$$= \|g-f\|_p$$

$$= D(g,f)$$

Thus $D(f,g) = D(g,f)$

iv) If $f, g, h \in L_p$, then

$D(f,g) \le D(f,h) + D(h,g)$

$\Rightarrow D(f,g) = \|f-g\|_p$
$= \|f-h+h-g\|_p$
$\le \|f-h\|_p + \|h-g\|_p$

$\le D(f,h) + D(h,g)$

$\Rightarrow D(f,g) \le D(f,h) + D(h,g)$.

Thus D satisfies all the conditions of metric space.
Hence D is a metric space.

Theorem:-
If $f_n \to f$ in L^p and $g_n \to g$ in L^p then $f_n+g_n \to f+g$ in L^p.

Proof:-
Given that
$\Rightarrow f_n \to f$ in L^p
i.e $\lim_{n\to\infty} \int_E |f_n(x) - f(x)|^p dx = 0$
and
$\Rightarrow g_n \to g$ in L^p
i.e $\lim_{n\to\infty} \int_E |g_n(x) - g(x)|^p dx = 0$

we need to prove that $f_n+g_n \to f+g$ in L^p

Now
$$\left(\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} = \left(\int_E |f_n-f+g_n-g|^p dx\right)^{1/p}$$
$$\left(\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} \le \left(\int_E |f_n-f|^p dx\right)^{1/p} + \left(\int_E |g_n-g|^p dx\right)^{1/p}$$

Taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty}\left(\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n-f|^p dx\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |g_n-g|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n-f|^p dx\right)^{1/p} + \left(\lim_{n\to\infty}\int_E |g_n-g|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} \le (0)^{1/p} + (0)^{1/p} = 0$$

$$\left(\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx\right)^{1/p} = 0$$

$$\lim_{n\to\infty}\int_E |f_n+g_n-(f+g)|^p dx = 0$$

Hence $f_n+g_n \longrightarrow f+g$ in L_p.

Theorem:-

If $\lim_{n\to\infty} f_n(x) = f(x)$ in L_p (or mean converges), then prove that $\lim_{n\to\infty}\int_E |f_n(x)|^p dx = \int_E |f(x)|^p dx$.

Equivalently $\lim_{n\to\infty} \|f_n\| = \|f\|$.

Proof:-

Given that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ in } L_p$$

i.e.

$$\lim_{n\to\infty}\int_E |f_n(x) - f(x)|^p dx = 0$$

We need to prove that

$$\lim_{n\to\infty}\int_E |f_n(x)|^p dx = \int_E |f(x)|^p dx.$$

Now

$$\left(\int_E |f_n(x)|^p dx\right)^{1/p} = \left(\int_E |f_n(x) - f(x) + f(x)|^p dx\right)^{1/p}$$

$$\le \left(\int_E |f_n(x) - f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \left(\int_E |f_n(x)|^p dx\right)^{1/p} \le \left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

Taking limit $n \to \infty$ on b.s

$$\lim_{n\to\infty}\left(\int_E |f_n(x)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p} \le (0)^{1/p} + \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p} \le \left(\int_E |f(x)|^p dx\right)^{1/p}$$

$$\lim_{n\to\infty}\int_E |f_n(x)|^p dx \le \int_E |f(x)|^p dx \quad \text{——— } ①$$

Now

$$\left(\int_E |f(x)|^p dx\right)^{1/p} = \left(\int_E |f(x)-f_n(x)+f_n(x)|^p dx\right)^{1/p}$$

$$\le \left(\int_E |f(x)-f_n(x)|^p dx\right)^{1/p} + \left(\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \left(\int_E |f(x)|^p dx\right)^{1/p} \le \left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\int_E |f_n(x)|^p dx\right)^{1/p}$$

Taking limit $n \to \infty$ on b.s

$$\lim_{n\to\infty}\left(\int_E |f(x)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \lim_{n\to\infty}\left(\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\left(\int_E |f(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\left(\int_E |f(x)|^p dx\right)^{1/p} \le (0)^{1/p} + \left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\left(\int_E |f(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \int_E |f(x)|^p dx \le \lim_{n\to\infty}\int_E |f_n(x)|^p dx \quad \text{——— } ②$$

From eq ① and eq②, we get

$$\lim_{n\to\infty} \int_E |f_n(x)|^p dx = \int_E |f(x)|^p dx$$

proved!

Theorem:-

If $\{f_n(x)\}$ converges to $f(x)$ in L^p (f_n in mean convergent). Prove that $\{f_n(x)\}$ is cauchy sequence in L^p.

Proof:-

Since $\{f_n(x)\}$ converges to $f(x)$ in L^p

i.e $f_n \to f$ in L^p

$$\Rightarrow \lim_{n\to\infty} \int_E |f_n - f|^p dx = 0$$

Let $f_m \to f$ in L^p.

$$\Rightarrow \lim_{m\to\infty} \int_E |f_m - f|^p dx = 0$$

We need to prove that $\{f_n(x)\}$ is cauchy sequence in L^p.

Consider

$$\left(\int_E |f_n(x) - f_m(x)|^p dx\right)^{1/p} = \left(\int_E |f_n(x) - f(x) + f(x) - f_m(x)|^p dx\right)^{1/p}$$

$$\leq \left(\int_E |f_n(x) - f(x)|^p dx\right)^{1/p} + \left(\int_E |f(x) - f_m(x)|^p dx\right)^{1/p}$$

$$\Rightarrow \left(\int_E |f_n(x) - f_m(x)|^p dx\right)^{1/p} \leq \left(\int_E |f_n(x) - f(x)|^p dx\right)^{1/p} + \left(\int_E |f_m(x) - f(x)|^p dx\right)^{1/p}$$

Taking limit $n\to\infty$, $m\to\infty$ on b.s

$$\lim_{\substack{n\to\infty \\ m\to\infty}}\left(\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} \le \lim_{n\to\infty}\left(\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \lim_{m\to\infty}\left(\int_E |f(x)-f_m(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{\substack{n\to\infty \\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} \le \left(\lim_{n\to\infty}\int_E |f_n(x)-f(x)|^p dx\right)^{1/p} + \left(\lim_{m\to\infty}\int_E |f_m(x)-f(x)|^p dx\right)^{1/p}$$

$$\left(\lim_{\substack{n\to\infty \\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} \le (0)^{1/p} + (0)^{1/p} = 0$$

$$\left(\lim_{\substack{n\to\infty \\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx\right)^{1/p} = 0$$

$$\Rightarrow \lim_{\substack{n\to\infty \\ m\to\infty}}\int_E |f_n(x)-f_m(x)|^p dx = 0$$

$$\Rightarrow \lim_{\substack{n\to\infty \\ m\to\infty}}\int_E |f_n(x)-f(x)|^p dx = 0$$

Hence $\{f_n(x)\}$ is cauchy sequence in L^p.

proved!

Convergent in Measure:-

Let $M = \{f : f \text{ is measurable function}\}$

A sequence $\{f_n(x)\}$ converges to $f \in M$ is measure, we write $f_n \xrightarrow{\mu} f$ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

Note:-

$\Rightarrow x \in \{x : f(x) + g(x) \geq \epsilon\}$

$\Rightarrow x \in \{x : f(x) \geq \epsilon\} \cup \{x : g(x) \geq \epsilon\}$

Theorem:-

If $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ (f_n converges to f in measure, g_n converges to g in measure), then prove that

$\alpha f_n(x) + \beta g_n(x) \xrightarrow{\mu} \alpha f(x) + \beta g(x)$

Proof:-

Consider

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let $A = \{x : |\alpha f_n(x) + \beta g_n(x) - (\alpha f(x) + \beta g(x))| \geq \epsilon\}$ where $\alpha, \beta \neq 0$

$A = \{x : |\alpha f_n(x) - \alpha f(x) + \beta g_n(x) - \beta g(x)| \geq \epsilon\}$

$A = \{x : |\alpha (f_n(x) - f(x)) + \beta (g_n(x) - g(x))| \geq \epsilon\}$

$A = \{x : |\alpha| |f_n(x) - f(x)| + |\beta| |g_n(x) - g(x)| \geq \epsilon\}$

$A \subseteq \{x : |\alpha| |f_n(x) - f(x)| \geq \epsilon/2\} \cup \{x : |\beta| |g_n(x) - g(x)| \geq \epsilon/2\}$

$A \subseteq \{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\} \cup \{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2|\beta|}\}$

Taking measure (μ) on b.s.

$\mu(A) \leq \mu\left(\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\}\right) + \mu\left(\{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2|\beta|}\}\right)$

Taking limit $n \to \infty$ on b.s

$\lim_{n\to\infty} \mu(A) \leq \lim_{n\to\infty} \mu\left(\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2|\alpha|}\}\right) + \lim_{n\to\infty} \mu\left(\{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2|\beta|}\}\right)$

$\lim_{n\to\infty} \mu(A) \leq 0 + 0 = 0$

$\lim_{n\to\infty} \mu(A) = 0$

$\lim_{n\to\infty} \mu\left(\{x : |\alpha f_n(x) + \beta g_n(x) - (\alpha f(x) + \beta g(x))| \geq \epsilon\}\right) = 0$

Thus $\alpha f_n(x) + \beta g_n(x) \xrightarrow{\mu} \alpha f(x) + \beta g(x)$ in L'

Proved!

Note:-

If $f(x) \leq g(x)$

let $A = \{x : f(x) \geq \epsilon\}$, $B = \{x : g(x) \geq \epsilon\}$

If $x \in A \Rightarrow f(x) \geq \epsilon$

but $|g(x)| \geq f(x) \geq \epsilon$

So $g(x) \geq \epsilon$

$\Rightarrow x \in B$

Thus $A \subseteq B$.

Theorem:-

If $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, then $f = g$ a.e.

Proof:-

Given that

$\Rightarrow f_n \xrightarrow{\mu} f$ i.e. $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$

and

$\Rightarrow g_n \xrightarrow{\mu} g$ i.e. $\lim_{n\to\infty} \mu(\{x : |g_n(x) - g(x)| \geq \epsilon\}) = 0$

Consider

$|f(x) - g(x)| = |f(x) - f_n(x) + f_n(x) - g(x)|$

$|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$

$|f(x) - g(x)| \leq |f_n(x) - f(x)| + |f_n(x) - g(x)|$ ——— (1)

Now

$\{x : |f(x) - g(x)| \geq 2\epsilon\} \subseteq \{x : |f_n(x) - f(x)| + |f_n(x) - g(x)| \geq 2\epsilon\}$ using eq (1)

$\{x : |f(x) - g(x)| \geq 2\epsilon\} \subseteq \{x : |f_n(x) - f(x)| \geq \epsilon\} \cup \{x : |f_n(x) - g(x)| \geq \epsilon\}$

Taking measure (μ) on b/s

$\mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) + \mu(\{x : |f_n(x) - g(x)| \geq \epsilon\})$

Taking limit $n \to \infty$.

$\lim_{n\to\infty} \mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) \leq \lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\})$
$+ \lim_{n\to\infty} \mu(\{x : |f_n(x) - g(x)| \geq \epsilon\})$

$$\lim_{n\to\infty} \mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) < 0 + 0 = 0$$

$\because f_n \to f, \quad f_n \to g$

$$\lim_{n\to\infty} \mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) = 0$$

$$\mu(\{x : |f(x) - g(x)| \geq 2\epsilon\}) = 0$$

$\Rightarrow \quad f(x) - g(x) = 0 \quad a.e$

$\Rightarrow \quad f(x) = g(x) \quad a.e.$

$\Rightarrow \quad f = g \quad a.e$

proved!

Theorem:-

Let $\langle f_n(x)\rangle$ be a sequence of integrable function such that $f_n(x)$ converges to $f(x)$ in means, then show that $f_n \xrightarrow{\mu} f$.

Proof:-

Since $\{f_n\}$ converges to $f(x)$ in means

So $\lim_{n\to\infty} \int_E |f_n(x) - f(x)|^p \, dx = 0$

We show that $f_n \xrightarrow{\mu} f$.

i.e we show that $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \delta\}) = 0$

Let $E_n = \{x : |f_n(x) - f(x)| \geq \delta\}$

Clearly $E_n \subseteq E$

$\Rightarrow |f_n(x) - f(x)| \geq \delta \quad \forall \quad x \in E_n$

$\Rightarrow |f_n(x) - f(x)|^p \geq \delta^p \quad \forall \quad x \in E_n$

$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p \, dx \geq \int_{E_n} \delta^p \, dx$

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$$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p dx \geq \delta^p \int_{E_n} dx = \delta^p \mu(E_n)$$

$$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p dx \geq \delta^p \mu(E_n) \quad \text{——— (1)}$$

Since $E_n \subseteq E$

$$\Rightarrow \int_{E_n} |f_n(x) - f(x)|^p dx \leq \int_{E} |f_n(x) - f(x)|^p dx$$

$$\Rightarrow \delta^p \mu(E_n) \leq \int_{E_n} |f_n(x) - f(x)|^p dx \leq \int_{E} |f_n(x) - f(x)|^p dx$$

using eq (1).

$$\Rightarrow \delta^p \mu(E_n) \leq \int_{E} |f_n(x) - f(x)|^p dx$$

Taking limit $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \delta^p \mu(E_n) \leq \lim_{n\to\infty} \int_{E} |f_n(x) - f(x)|^p dx$$

$$\Rightarrow \lim_{n\to\infty} \delta^p \mu(E_n) \leq 0$$

$$\Rightarrow \delta^p \lim_{n\to\infty} \mu(E_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} \mu(E_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \delta\}) = 0$$

Hence $f_n \xrightarrow{\mu} f$.

proved!

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