Written by;
Asim Marwat
MSc Mathematics (UOP)

Notes;
Measure Theory

Students for;
MSc And BS

Special Thanks to;
DR.Adil Sir (UOP)

Contact and Wtssp
03151949572
asimmarwat41@gmail.com

CHAPTER: 4

LEBESGUE INTEGRAL

(for bounded functions)

Explanation:-

For a bounded measurable function $f(x)$ over a measurable set "E" having finite measure, the lebesgue integral is defined as follows;

let "U" is the upper bound of $f(x)$ over E, and "L" is the lower bound of $f(x)$ over E.

The interval $[L, U]$ is divided into "n" subintervals by numbers

$$L = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = U$$

The set E is divided into sets $E_1, E_2, \ldots\ldots$, where E_1 is the set of points "x" for which

$$y_0 \leq f(x) < y_1$$

Also

$$E_2 = \{x;\ y_1 \leq f(x) < y_2\}$$

$$E_3 = \{x;\ y_2 \leq f(x) < y_3\}$$

- - - - - - - - - - -

- - - - - - - - - - -

Generally

$$E_k = \{x;\ y_{k-1} \leq f(x) < y_k\}$$

The Lebesgue measure of the set E_k are measure as $\mu^*(E)$.

Two sum can be formed

$$S = \sum_{k=1}^{n} y_k\, \mu^*(E_k) \text{ and } s = \sum_{k=1}^{n} y_{k-1}\, \mu^*(E_k)$$

Date: (93) Mon Tue Wed Thu Fri Sat

S is called the upper sum over the partitioning set $\{y_0, y_1, y_2, \ldots, y_m\}$ and Similarly s is called the lower sum of the same partition.

Now corresponding to different partition s, we obtain different upper and lower sums.

Thus we get

Upper lebesgue Integral $= I = \inf\{S\}$ or glb

lower Lebesgue Integral $= J = \sup\{s\}$ or lub

If $I = J$, we say that $f(x)$ is lebesgue integrable on E. and denote the common value by

$I = J = \int_E f(x)\,dx$ called the lebesgue definite integral of $f(x)$ over E.

Theorem:-

Let $f: E \to \mathbb{R}$ be bounded and measurable function (mean lebesgue integrable), then for any partition the upper sum has lower bound and the lower sum has upper bound.

Proof:-

let $P = \{y_0, y_1, y_2, \ldots, y_n\}$ be the partition and since f is bounded, so

$\alpha \le y_k \le \beta \quad \forall \; k = 0, 1, 2, \ldots, n$

Since the upper sum $= S = \sum_{i=1}^{n} y_i \, \mu^*(E_i)$

and lower sum $= s = \sum_{i=1}^{n} y_{i-1} \, \mu^*(E_i)$

Where $E_i = \{x ; y_{i-1} \leq f(x) < y_i\} \; \forall \; i = 1, 2, \ldots$

Since f is measurable so each "E_i" is measurable, Also since

$$y_i \geq \alpha \quad \forall \; i = 0, 1, 2, \ldots, n$$

and

$$\mu^*(E_i) \geq 0 \Rightarrow y_i \mu^*(E_i) \geq \alpha \mu^*(E_i)$$

$$\Rightarrow \sum_{i=1}^{n} y_i \mu^*(E_i) \geq \alpha \sum_{i=1}^{n} \mu^*(E_i)$$

$$\Rightarrow S \geq \alpha \mu^*(E)$$

$\boxed{\because E = \bigcup_{i=1}^{n} E_i \Rightarrow \mu^*(E) = \sum_{i=1}^{n} \mu^*(E_i)}$

Since $\alpha \mu^*(E) \in \mathbb{R} \Rightarrow \alpha \mu^*(E)$ is a lower bound of S.

Next since

$$y_{i-1} \leq \beta \quad \forall \; i = 0, 1, 2, \ldots, n$$

$$\Rightarrow y_{i-1} \mu^*(E_i) \leq \beta \mu^*(E_i)$$

$$\Rightarrow \sum_{i=1}^{n} y_{i-1} \mu^*(E_i) \leq \beta \sum_{i=1}^{n} \mu^*(E_i)$$

$$\Rightarrow s \leq \beta \mu^*(E)$$

Since $\beta \mu^*(E) \in \mathbb{R}$

$\Rightarrow s$ has an upper bound.

Theorem:-

Prove that for any partition the upper sum has glb, and the lower sum has lub.

Proof:-

Since in above theorem we have

Date: (95) Mon Tue Wed Thu Fri Sat

proved that for any partition the upper sum has lower bound.

⇒ i.e. corresponding to different partitions we get different upper sums and if we make a set of all these upper sum i.e. $\{S\}$

⇒ the set $\{S\}$ has lower bound.

But according to completeness property of $\mathbb{R}$, "every non-empty set of real numbers that has a lower bound also has an infimum in $\mathbb{R}$.

⇒ For any partition the upper sum has glb, and similarly on the same lines replacing upper by lower and infimum by supremum, we can say that the lower sum has lub.

Theorem:-

Prove that the upper sum cannot increase and the lower sum cannot decrease.

OR

If "S" is an upper sum to the given partition P, and S_1 is an upper sum corresponding to refienment of the same partition P, say P_1, then $S_1 \leq S$.

Also if "s" is a lower sum to the given partition P and s_1 is a lower sum corresponding to the refienment of the same partition P, say P_1 then $s_1 \geq s$.

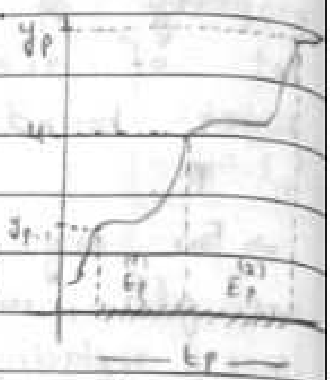
Proof:-

Let $P = \{y_0, y_1, y_2, \ldots, y_n\}$ be the partition.

And let $P_1 = \{y_0, y_1, \ldots, y_{p-1}, u, y_p, \ldots, y_n\}$ be the refined partition or refinement of P.

Let s, S and s_1, S_1 be the lower and upper sum corresponding to P and P_1 respectively.

Then



$$S = \sum_{i=1}^{n} y_i \mu^*(E_i)$$

$$= y_1 \mu^*(E_1) + y_2 \mu^*(E_2) + \cdots + y_n \mu^*(E_n) \quad \text{(1)}$$

$$\Rightarrow S_1 = y_1 \mu^*(E_1) + \cdots + y_{p-1} \mu^*(E_{p-1}) + u \mu^*(E_p^{(1)}) + y_p \mu^*(E_p^{(2)}) + \cdots + y_n \mu^*(E_n) \quad \text{(2)}$$

where

$$E_p = \{x;\ y_{p-1} \le f(x) < y_p\}$$
$$E_p^{(1)} = \{x;\ y_{p-1} \le f(x) < u\}$$
$$E_p^{(2)} = \{x;\ u \le f(x) < y_p\}$$

Also $E_p = E_p^{(1)} \cup E_p^{(2)}$ and $E_p^{(1)} \cap E_p^{(2)} = \phi$

$$\Rightarrow \mu^*(E_p) = \mu^*(E_p^{(1)}) + \mu^*(E_p^{(2)})$$

Now from eq (1)

$$S = y_1 \mu^*(E_1) + \cdots + y_p \mu^*(E_p^{(1)}) + y_p \mu^*(E_p^{(2)}) + \cdots + y_n \mu^*(E_n) \quad \text{(3)}$$

eq (2) $\Rightarrow$ $S_1 = y_1 \mu^*(E_1) + \cdots + y_{p-1} \mu^*(E_{p-1}) + u \mu^*(E_p^{(1)}) + y_p \mu^*(E_p^{(2)}) + \cdots + y_n \mu^*(E_n)$

Subtract eq (2) and (3), we get

$$S_1 - S = u \mu^*(E_p^{(1)}) - y_p \mu^*(E_p^{(1)})$$

$$S_1 - S = \mu^*(E_p^{(1)})\,(u - y_p) \quad \text{— (A)}$$

From fig $(u - y_p) < 0$ and we know $\mu^*(E_p^{(1)}) \geq 0$.

$$\Rightarrow S_1 - S < 0$$

$$\Rightarrow \boxed{S_1 < S}$$

on the similar way since.

$$s = y_0\mu^*(E_1) + y_1\mu^*(E_2) + \cdots + y_{p-1}\mu^*(E_p) + \cdots + y_{n-1}\mu^*(E_n)$$

and

$$s_1 = y_0\mu^*(E_1) + \cdots + y_{p-1}\mu^*(E_p^{(1)}) + u\mu^*(E_p^{(2)}) + y_p\mu^*(E_{p+1}) + \cdots + y_{n-1}\mu^*(E_n)$$

Now from above

$$s - s_1 = y_{p-1}\mu^*(E_p) - \left[y_{p-1}\mu^*(E_p^{(1)}) + u\mu^*(E_p^{(2)})\right]$$

$$= y_{p-1}\mu^*(E_p^{(1)} \cup E_p^{(2)}) - y_{p-1}\mu^*(E_p^{(1)}) - u\mu^*(E_p^{(2)})$$

$$= \cancel{y_{p-1}\mu^*(E_p^{(1)})} + y_{p-1}\mu^*(E_p^{(2)}) - \cancel{y_{p-1}\mu^*(E_p^{(1)})} - u\mu^*(E_p^{(2)})$$

$$s - s_1 = \mu^*(E_p^{(2)})\,(y_{p-1} - u)$$

From fig $(y_{p-1} - u) < 0$ and we know $\mu^*(E_p^{(2)}) \geq 0$.

$$\Rightarrow s - s_1 < 0$$

$$\Rightarrow \boxed{s < s_1}$$

Theorem:-

Prove that an upper sum for any

Date: (98)

partition is not less than a lower sum corresponding to the same or any other partition.

Proof:-

Case-1:-

let S and s be the upper and lower sum corresponding to the same partition.

Then $S \leq s$ is obvious.

Case: 2:-

Let S_1 and s_1 be the upper and lower sum corresponding to a partition (say) P_1, and let S_2 and s_2 be the upper and lower sum corresponding to an another partition (say) P_2.

Then we have to show that

$$S_1 \geq s_2 \quad \text{and} \quad S_2 \geq s_1$$

If S_3 and s_3 are the upper and lower sum corresponding to the union of above partitions.

Then

$$S_3 \leq S_1 \quad \text{and} \quad S_3 \leq S_2$$

and $s_3 \geq s_1$ and $s_3 \geq s_2$

and $S_3 \geq s_3$ is obvious.

Also we know that

$$S_1 \geq s_1 \quad \text{and} \quad S_2 \geq s_2$$

Now as

$$S_1 \geq S_3 \geq s_3 \geq s_1$$

$$\Rightarrow \boxed{S_1 \geq s_1}$$

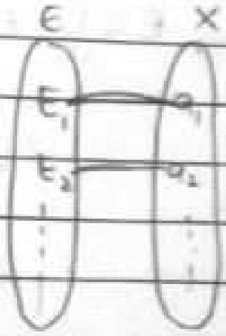
Also $S_2 \geq S_3 \geq s_3 \geq s_2$

$$\Rightarrow \boxed{S_2 \geq s_2}$$

Simple Function:-

let $f : E \to \mathbb{R}$ be a function then f is said to be simple function if it assumes only finite number of values which are finite.

If $f : E \to \mathbb{R}$ is simple function and taking distinct values $a_1, a_2, \cdots, a_n$.



Then there exists disjoint sub-sets $E_1, E_2, \cdots, E_n$ of E such that

$$f(E_i) = \{a_i\}$$

and $E = \bigcup_{i=1}^{n} E_i$

then we can write f as:

$$\boxed{f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)}$$

e.g: If $x \in E_3 \Rightarrow f(x) = a_3$ as

$$f(x) = a_1 \chi_{E_1}(x) + a_2 \chi_{E_2}(x) + a_3 \chi_{E_3}(x) + \cdots + a_n \chi_{E_n}(x).$$

$$\Rightarrow f(x) = 0 + 0 + a_3 \cdot 1 + \cdots + 0$$

$$\Rightarrow f(x) = a_3$$

Also it is well known that f is measurable iff $E_1, E_2, \ldots, E_n$ are measurable sets.

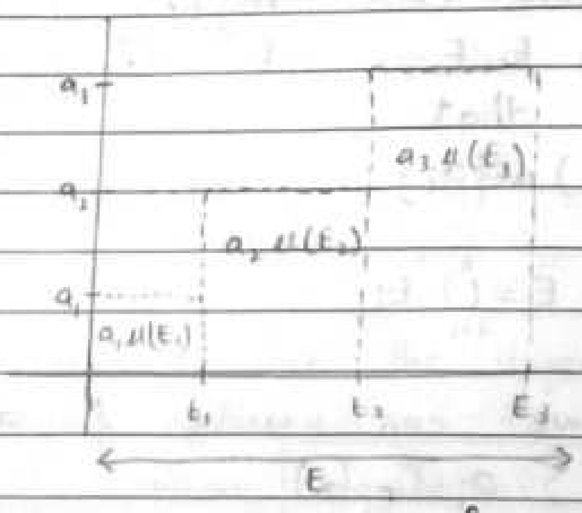
Lebesgue Integral of Simple Function:-

Let $f: E \to \mathbb{R}$ be simple measurable function then we can write

$$f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

where $f(E_i) = a_i$, $E_i \cap E_j = \phi$

$$E = \bigcup_{i=1}^{n} E_i \quad , \quad E_i\text{'s are measurable.}$$



the lebesgue integral of simple function f is defined by

$$\int_E f\, d\mu = \int_E f(x)\, dx = \int_E f\, dx = \sum_{i=1}^{n} a_i \mu(E_i)$$

Q:- Let $f: E \to \mathbb{R}$ by defined by $f(x) = 2$ $\forall\ x \in E$. Find the lebesgue integral of f.

Sol:-

Clearly f is simple function with $a_1 = 2$ and $E_1 = E$.

then $\int_E f(x)\,dx = \sum_{i=1}^{n} a_i \mu(E_i)$

$= a_1 \mu(E_1)$

$= 2\mu(E)$

$\Rightarrow \int_E f(x)\,dx = 2\mu(E)$

If $E = [2, 10]$

i.e. $f : [2, 10] \to \mathbb{R}$

$\Rightarrow f(x) = 2 \quad \forall \; x \in [2, 10]$

then

$\int_E f(x)\,dx = 2\mu[2, 10]$

$= 2(10 - 2)$

$= 2(8)$

$\int_E f(x)\,dx = 16$: Ans.

Note:- $\int_E dx = \mu(E)$.

Q:- let $f : [1, 10] \to \mathbb{R}$ be defined by

$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in } [1, 10]. \\ 0 & \text{if } x \text{ is irrational in } [1, 10]. \end{cases}$

i) Find lebesgue integral of "f" over $[1, 10]$

ii) Is f Riemann integral.

Solution:-

Clearly f is simple function.

Date: (102)

Let A be the sub-set of all rational number in $[1,10]$ and B be the sub-set of all irrational number in $[1,10]$ then $f(A)=\{1\}$ and $f(B)=\{0\}$

So $a_1 = 1$, $E_1 = A$
$a_2 = 0$, $E_2 = B$

then

$$\int_{[1,10]} f\,d = a_1 \mu(E_1) + a_2 \mu(E_2)$$

$$= 1\,\mu(A) + 0.\,\mu(B)$$

$$= 1.(0) + 0\,(9)$$

$$= 0$$

So $\int_E f(x)\,dx = 0$

Now $A \cup B = [1,10]$

$$\mu(A \cup B) = \mu[1,10]$$

$$\mu(A) + \mu(B) = (10-1)$$

$$0 + \mu(B) = 9$$

$$\mu(B) = 9$$

Next we find Riemann integrable
Then

$$S = \text{upper sum} = 1 = \sum_{i=1}^{n} M_i \Delta x_i$$

$$s = \text{lower sum} = 0 = \sum_{i=1}^{n} m_i \Delta x_i .$$

$$\Rightarrow \quad S \neq s$$

So.
f is not Riemann integrable.

Q:- let $f:[2,20] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3 & \text{if } x \in [2,10] \\ 5 & \text{if } x \in (10,15) \\ 4 & \text{if } x \in [15,20) \end{cases}$$

Find lebesgue integral of "f" over $[2,20]$. Is f Riemann integral.

Solution:-
Clearly f is simple function
with $a_1 = 3$, $E_1 = [2,10]$
$a_2 = 5$, $E_2 = (10,15)$
$a_3 = 4$, $E_3 = [15,20)$

$\mu(E_1) = \mu[2,10] = 10-2 = 8$
$\mu(E_2) = \mu(10,15) = 15-10 = 5$
$\mu(E_3) = \mu[15,20) = 20-15 = 5$

then $\int_{[2,20]} f\,d = a_1\mu(E_1) + a_2\mu(E_2) + a_3\mu(E_3)$

$= 3(8) + 5(5) + 4(5)$
$= 24 + 25 + 20$
$= 69$

$\Rightarrow \int_{[2,20]} f\,d = 69$ Ans.

Next we find Riemann Integrable.
Then
S = upper sum $= 5 = \sum_{i=1}^{n} M_i \Delta x_i$
s = lower sum $= 3 = \sum_{i=1}^{n} m_i \Delta x_i$

→ rational countable hota hai
→ any countable ka measurable (μ) hamesha zero hota hai

Mon Tue Wed Thu Fri

Date:

$$\bar{S} \neq \underline{S}$$

So, f is not Riemann integrable

Q:- let $f : [1, 10] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in } [1,10] \\ 3 & \text{if } x \text{ is irrational in } [1,10] \end{cases}$$

i) Find lebesgue integral of "f" over [1, 10]

ii) Is f Riemann integrable.

Solution:-

Clearly f is simple function. let A be the sub-set of all rational number in $[1,10]$ and B be the sub-set of all irrational number in $[1,10]$ then $f(A) = \{1\}$ and $f(B) = \{3\}$

So

$a_1 = 1$, $E_1 = A$

$a_2 = 3$, $E_2 = B$

then

$$\int_{[1,10]} f d = a_1 \mu(E_1) + a_2 \mu(E_2)$$

$$= 1 \cdot \mu(A) + 3 \cdot \mu(B)$$

$$= 1 \cdot 0 + 3(9)$$

$$= 0 + 27$$

$$\int_{[1,10]} f d = 27$$

So $\int_E f(x)\,dx = 27$ Ans.

Now $A \cup B = [1, 10]$

$$\mu(A \cup B) = \mu([1,10])$$

$$\mu(A) + \mu(B) = 10 - 1$$

Scanned By Camera Scanner

$0 + \mu(B) = 9$

$\mu(B) = 9$

Now we find Riemann integrable then

S = upper sum $= 3 = \sum_{i=1}^{n} M_i \Delta x_i$

s = lower sum $= 1 = \sum_{i=1}^{n} m_i \Delta x_i$

$S \neq s$

So,
f is not Riemann integrable.

Theorem:-
Prove that $f(x)$ is lebesgue integrable iff for any $\epsilon > 0$, there exist a partition with upper and lower sum, "S" and "s" such that $S - s < \epsilon$.

Proof:-
let us choose for any $\epsilon > 0$, $\exists$ S and s such that $S - s < \epsilon$.
we have to show that $f(x)$ is lebesgue integrable i.e. we prove that $I = J$.

As we know that $S \geq I \geq J \geq s$

$\Rightarrow J \geq s$

$\Rightarrow -J \leq -s$ ——— (i)

and $S \geq I \Rightarrow I \leq S$ ——— (ii)

Now adding eq (i) and (ii), we get

$S - s \geq I - J$

$\Rightarrow I - J \leq S - s < \epsilon$

$\Rightarrow I - J < \epsilon$

Date:

$\Rightarrow I - J = 0$

$\Rightarrow I = J$

Thus f is lebesgue integrable.

Conversely,

$I = \underset{P}{\text{glb}}\ S$ and $J = \underset{P}{\text{lub}}\ s$

Then for any $\epsilon > 0$, there exists partition with upper sum and lower sum, such that

$I + \epsilon/2 > S$ and $J - \epsilon/2 < s$ (by def of lub and glb)

$\Rightarrow -J + \epsilon/2 > -s$

$\Rightarrow I - J + \frac{\epsilon}{2} + \frac{\epsilon}{2} > S - s$

$\Rightarrow I - J + \epsilon > S - s$

But $I = J \Rightarrow I - J = 0$

$\Rightarrow 0 + \epsilon > S - s$

$\Rightarrow S - s < \epsilon + 0$

$\Rightarrow S - s < \epsilon$

which is the required proof.

Theorem:-

If $f: E \to \mathbb{R}$ is bounded and measurable function on E, with finite measure of E, then prove that "f" is lebesgue integrable.

Proof:-

To show that f is lebesgue integrable it is enough to show that for any $\epsilon > 0$, there exist a partition with upper and lower sum's S and s respectively, such that

$$S - s < \epsilon.$$

Now since $f : E \to \mathbb{R}$ is bounded, Therefore take a partition $\{y_0, y_1, \dots, y_n\}$, such that

$$y_k - y_{k-1} < \epsilon \quad \forall \quad k = 1, 2, 3, \dots, n$$

Consider the upper and lower sums, w.r.t the above partition

$$S = \sum_{k=1}^{n} y_k \, \mu^*(E_k) \quad , \quad s = \sum_{k=1}^{n} y_{k-1} \, \mu^*(E_k)$$

where $E_i \cap E_j = \phi \quad \forall \quad i \neq j$, and

$$E_k = \{x ; y_{k-1} \leq f(x) < y_k\}$$

and $\bigcup_{k=1}^{n} E_k = E$, E_k's are measurable

Now $S - s = \sum_{k=1}^{n} y_k \, \mu^*(E_k) - \sum_{k=1}^{n} y_{k-1} \, \mu^*(E_k)$

$$= \sum_{k=1}^{n} (y_k - y_{k-1}) \, \mu^*(E_k)$$

$$< \sum_{k=1}^{n} \epsilon \, \mu^*(E_k)$$

$$< \epsilon \sum_{k=1}^{n} \mu^*(E_k)$$

$$< \epsilon \, \mu^*(E) \qquad (\because \mu^*(E) \text{ is finite})$$

$\Rightarrow \quad S - s < \epsilon_0$, where $\epsilon_0 = \epsilon \mu^*(E)$

So by necessary and sufficient condition $f(x)$ is Lebesgue integrable on E.

Simple function hota hai

Problem:-

If $\phi : E \to \mathbb{R}$, be constant function defined by $\phi(x) = c \quad \forall \; x \in E$
Then $\int_E \phi(x)\,dx = \int_E c\,dx = c\,\mu^*(E)$.

Proof:-

Since we know that
If $\phi : E \to \mathbb{R}$ be a simple function then

$$\int_E \phi\, d = \sum_{i=1}^{n} a_i\, \mu(E_i)$$

where $\phi(E_i) = a_i$

Also since $\phi(E) = \{c\} =$ finite $\quad \forall x \in E$
$\Rightarrow \phi$ is simple function, so

$$\int_E \phi\, d = c\,\mu^*(E) + 0 + 0 + \cdots$$

$$= c\,\mu^*(E)$$

$$\Rightarrow \int_E \phi\, d = c\,\mu^*(E)$$

Theorem:-

Let $f : E \to \mathbb{R}$ be bounded and measurable functions with finite measure of E, Then $\int_E (f+c)\,dx = \int_E f\,dx + c\mu^*(E)$.

Proof:-
Given that $f(x)$ is bounded and measurable on E, So $f(x)$ is lebesgue integrable on E.
Thus $\int_E f(x)\,dx$ exists.

Now due to boundness of $f(x)$ we can write $\alpha < f(x) < \beta$.
let us choose mode of partition of $[\alpha,\beta]$ as

$$\alpha = y_0 < y_1 < y_2 < \dots < y_n = \beta$$

Then the upper sum of $f(x)$ for the above mode of partition is

$$S = \sum_{k=1}^{n} y_k \mu^*(E_k) \quad \text{, where } E_k = \{x;\ y_{k-1} < f(x) < y_k\}$$

Therefore the upper sum "S_1" for the function $f(x)+c$ for the above partition is

$$S_1 = \sum_{k=1}^{n} (y_k + c)\,\mu^*(E_k)$$

$$S_1 = \sum_{k=1}^{n} y_k \mu^*(E_k) + \sum_{k=1}^{n} c\,\mu^*(E_k)$$

$$S_1 = \sum_{k=1}^{n} y_k \mu^*(E_k) + c\sum_{k=1}^{n} \mu^*(E_k)$$

$$S_1 = \sum_{k=1}^{n} y_k \mu^*(E_k) + c\,\mu^*(E)$$

$$\Rightarrow S_1 = S + c\,\mu^*(E)$$

Date: (112)

Taking infimum, we get

$\Rightarrow \inf\{S_1\} = \inf\{S\} + c\mu^*(E)$

lebesgue integral of f

$\Rightarrow \int_E (f(x)+c)\,dx = \int_E f(x)\,dx + c\mu^*(E)$

$\Rightarrow \int_E (f+c)\,dx = \int_E f\,dx + c\mu^*(E)$

proved!

Theorem:-

If $f, g : E \to \mathbb{R}$ are bounded and measurable (lebesgue integrable) functions with finite measure of E, then $\int_E (f+g)\,d = \int_E f\,d + \int_E g\,d$

Proof:-

Let $\{y_0, y_1, y_2, \ldots, y_n\}$ be partition choosen for $f(x)$.

and $E_k = \{x;\ y_{k-1} < f(x) < y_k\}$

where $E_i \cap E_j = \Phi \quad \forall\ i \neq j$

and $\bigcup_{i=1}^{n} E_i = E$

and let "S" and "s" be the upper and lower sum defined for "f(x)".

Since we know that if $A \cap B = \Phi$

then $\int_{A\cup B} f = \int_A f + \int_B f$

Now let $\int_E (f+g)\,dx = \int_{\bigcup_{k=1}^{n} E_k} (f+g)\,dx$

$$\int_E (f+g)\,dx = \int_{E_1} (f+g)\,dx + \int_{E_2} (f+g)\,dx + \dots + \int_{E_n} (f+g)\,dx$$

$$\int_E (f+g)\,dx = \sum_{k=1}^{n} \int_{E_k} (f+g)\,dx \quad \text{—— (A)}$$

$\because y_{k-1} \le f(x) < y_k$

$$\geq \sum_{k=1}^{n} \int_{E_k} \left[y_{k-1} + g(x)\right] dx$$

$$\int_E \left[f(x)+g(x)\right] dx \geq \sum_{k=1}^{n} y_{k-1} \int_{E_k} dx + \sum_{k=1}^{n} \int_{E_k} g(x)\,dx$$

$$= \sum_{k=1}^{n} y_{k-1}\, \mu^*(E_k) + \int_E g(x)\,dx$$

$$\int_E \left[f(x)+g(x)\right] dx \geq s + \int_E g(x)\,dx$$

Taking supremum (glb), we get

$$\int_E (f+g)\,d \geq \sup\{s\} + \int_E g(x)\,dx$$

$$\int_E (f+g)\,d \geq \int_E f\,d + \int_E g\,d \quad \text{—— (i)}$$

Similarly,

(A) $\Rightarrow$ $$\int_E (f+g)\,d = \sum_{k=1}^{n} \int_{E_k} (f+g)\,dx$$

$\because y_{k-1} \le f(x) < y_k$

$$\leq \sum_{k=1}^{n} \int_{E_k} (y_k + g)\,dx$$

$$\leq \sum_{k=1}^{n} y_k \int_{E_k} dx + \sum_{k=1}^{n} \int_{E_k} g(x)\,dx$$

$$\leq \sum_{k=1}^{n} y_k\, \mu^*(E_k) + \int_E g\,d$$

Date: (112)

$$\int_E (f+g)\,d \leq S + \int_E g(x)\,d$$

Taking infimum, we get

$$\int_E (f+g)\,d \leq \inf S + \int_E g\,d$$

$$\int_E (f+g)\,d \leq \int_E f\,d + \int_E g\,d \quad \text{(ii)}$$

From eq (i) and (ii), we get

$$\int_E (f+g)\,d = \int_E f\,d + \int_E g\,d$$

which is the required result.

Theorem:-

If $f: E \to \mathbb{R}$ is lebesgue integrable function, and A, B be two measurable sets such that $A \cap B = \phi$ and $A \cup B = E$, Show that

$$\int_{A\cup B} f\,d = \int_A f\,d + \int_B f\,d.$$

Proof:-

Since we know that

characteristic function of $A\cup B$

$$\chi_{A\cup B} = \chi_A + \chi_B$$

Now

$$\int_{A\cup B} f\chi_{A\cup B}\,d = \int_{A\cup B} f(\chi_A + \chi_B)\,d$$

$\therefore \chi_{A\cup B} = 1$ because all $x \in A\cup B = E$

$$\Rightarrow \int_{A\cup B} f\,d = \int_{A\cup B} (f\chi_A + f\chi_B)\,d$$

$$= \int_{A\cup B} f\chi_A\,d + \int_{A\cup B} f\chi_B\,d$$

$$= \int_A f\chi_A\,d + \int_B f\chi_A\,d + \int_A f\chi_B\,d + \int_B f\chi_B\,d$$

Since $\int_B f\chi_A\,d = 0$ and $\int_A f\chi_B\,d = 0$

So;

$$\int_{A\cup B} f\,d = \int_A f\chi_A\,d + 0 + 0 + \int_B f\chi_B\,d$$

$$\int_{A\cup B} f\,d = \int_A f\chi_A\,d + \int_B f\chi_B\,d$$

Since $\chi_A = 1$, $\chi_B = 1$; $\forall$ $x \in A$ and B.

$$\Rightarrow \int_{A\cup B} f.d = \int_A f.(1)\,d + \int_B f.(1)\,d$$

$$\Rightarrow \int_{A\cup B} f\,d = \int_A f\,d + \int_B f\,d.$$

which is the required proof.

Theorem:-

If $f: E \to \mathbb{R}$ is lebesgue integrable (bounded and measurable) function

with $\mu^*(E)=0$, then
$$\int_E f(x)\,dx = 0.$$

Proof:-
Given that the set "E" has measure zero.
Let us choose the points of partitions
$$\alpha = y_0 < y_1 < y_2 < \cdots < y_n = \beta$$

and $E_k = \{x \in E \,;\, y_{k-1} < f(x) < y_k\}$

$$E_i \cap E_j = \phi \quad \text{and} \quad \bigcup_{i=1}^{n} E_i = E$$

Now all E_i's are disjoint

$$\Rightarrow \mu^*(E) = \mu^*\left(\bigcup_{i=1}^{n} E_i\right)$$

$$\Rightarrow 0 = \sum_{i=1}^{n} \mu^*(E_i)$$

$$\Rightarrow \sum_{i=1}^{n} \mu^*(E_i) = 0$$

$$\Rightarrow \mu^*(E_i) = 0 \quad \forall\ i = 1, 2, \cdots, n$$

Now,
$$S = \sum_{k=1}^{n} y_k \,\mu^*(E_k)$$
$$= \sum_{k=1}^{n} y_k\,(0)$$
$$S = 0$$
Taking infimum, we get
$$\inf\{S\} = \inf\{0\}$$

$I = 0$

Also

$$s = \sum_{k=1}^{n} y_{k-1}\, \mu^*(E_k)$$

$$s = \sum_{k=1}^{n} y_{k-1} (0)$$

$s = 0$

Taking supremum, we get

$\Rightarrow \operatorname{Sup}\{s\} = \operatorname{Sup}\{0\}$

$\Rightarrow J = 0$

$\Rightarrow I = J = 0$

$$\Rightarrow \int_E f(x)\, dx = 0$$

Theorem:-

If $f, g : E \to \mathbb{R}$ are lebesgue integrable functions, such that $f = g$; a.e then $\int_E f d = \int_E g d$ is converse true.

Proof:-

let $E_1 = \{x \in E ; f(x) \neq g(x)\}$

and $E_2 = \{x \in E ; f(x) = g(x)\}$

Since $f = g$; a.e, So

$\mu^*(E_1) = \mu^*\{x \in E ; f(x) \neq g(x)\} = 0$

Also since E_1 is measurable (because

Mon Tue Wed Thu Fri Sat

Date:

it is null).

Also E is measurable (because f is measurable so domain is measurable)

$\Rightarrow$ $E \setminus E_1 = E_2$ is measurable.

Now E_1 and E_2 are measurable sets and $E_1 \cap E_2 = \phi$.

So by previous theorem

Here since "f" is lebesgue integrable, so

$$\int_E (f-g)\, d = \int_{E_1 \cup E_2} (f-g)\, d$$

$$= \int_{E_1} (f-g)\, d + \int_{E_2} (f-g)\, d$$

Since we know that $\lambda^*(E_1) = 0$ (use theorem)

and $f(x) = g(x) \Rightarrow f-g = 0 \;\forall\; x \in E$

$$\int_E (f-g)\, d = 0 + 0$$

$$\int_E (f-g)\, d = 0$$

$$\int_E f\, d - \int_E g\, d = 0$$

$$\Rightarrow \int_E f\, d = \int_E g\, d \quad \text{proved!}$$

Theorem:-

If $f, g : E \to \mathbb{R}$ are lebesgue

Scanned By Camera Scanner

integrable and $f(x) \leq g(x)$ on E, then

$$\int_E f(x)\,dx \leq \int_E g(x)\,dx.$$

The result is also true if $f(x) \leq g(x)$ a.e on E.

Proof:-

Since $g(x) \geq f(x)$ on E

$$\Rightarrow g(x) - f(x) \geq 0 \quad \text{on } E$$

$$\Rightarrow \int_E [g(x) - f(x)]\,dx \geq 0$$

$$\Rightarrow \int_E g(x)\,dx - \int_E f(x)\,dx \geq 0$$

$$\Rightarrow \int_E g(x)\,dx \geq \int_E f(x)\,dx$$

$$\Rightarrow \int_E f(x)\,dx \leq \int_E g(x)\,dx.$$

proved!

Theorem:-

If $f : E \to \mathbb{R}$ is lebesgue integrable and $A \leq f(x) \leq B$, then

$$A\,\mu^*(E) \leq \int_E f(x)\,dx \leq B\,\mu^*(E)$$

Proof:-

Given $A \leq f(x) \leq B$

$$\Rightarrow \int_E A\,dx \leq \int_E f(x)\,dx \leq \int_E B\,dx$$

$$\Rightarrow A\int_E dx < \int_E f(x)\,dx < B\int_E dx$$

$$\Rightarrow A\,\mu^*(E) \le \int_E f(x)\,dx \le B\,\mu^*(E)$$

proved!

↳ Results for non-negative measurable functions:

Definition:-

If $f: E \to [0,\infty]$ is measurable function, then the set

$$R(f,E) = \{(x,y) : 0 \le y \le f(x) \text{, iff } f(x) < \infty \text{ and } 0 \le y < f(x) \text{, iff } f(x) = \infty\}$$

is called the Region of the function defined on set E. Then the Lebesgue measure of $R(f,E)$ is called Lebesgue Integral of $f(x)$. i.e.

$$\mu^*[R(f,E)] = \int_E f\,d = \int_E f(x)\,d\mu(x) = \int_E f(x)\,dx$$

* Tchebeshev's Inequality:-

Statment:-

Let f be a non-negative measurable function, on the set E, if $\alpha > 0$, then

$$\mu^*(\{x \in E ; f(x) > \alpha\}) \le \frac{1}{\alpha}\int_E f\,d$$

Proof:-

Let $E_1 = \{x \in E ; f(x) > \alpha\}$

then $E_1 \subseteq E$

$$\Rightarrow \int_{E_1} f(x)\,dx \le \int_E f(x)\,dx$$

$$\Rightarrow \int_E f(x)\,dx \geq \int_{E_1} f(x)\,dx > \int_{E_1} \alpha\,dx = \alpha\,\mu^*(E_1)$$

$$\Rightarrow \int_E f(x)\,dx \geq \alpha\,\mu^*(E_1)$$

$$\Rightarrow \alpha\,\mu^*(E_1) \le \int_E f(x)\,dx$$

$$\Rightarrow \mu^*(E_1) \le \frac{1}{\alpha}\int_E f(x)\,dx$$

$$\Rightarrow \mu^*(\{x \in E ; f(x) > \alpha\}) \le \frac{1}{\alpha}\int_E f(x)\,dx$$

$$\Rightarrow \mu^*(\{x \in E ; f(x) > \alpha\}) \le \frac{1}{\alpha}\int_E f\,d.$$

proved!

Theorem:-
Let f be non-negative measurable function on the set E, such that $\int_E f\,d < \infty$, then prove that $f(x) < \infty$ a.e. i.e. f is finite a.e.

Proof:-
Let $E_1 = \{x \in E ; f(x) = \infty\}$, then $E_1 \subseteq E$
Now there are two cases:

Date: (120)

Case-1:-

If E is null set, then since each subset of nullset is null, so E_1 is also null and so $\mu^*(E_1)=0$, which shows that $f(x) < \infty$ a.e.

Case-2:-

Let E is not a null set

$\Rightarrow \mu^*(E) \neq 0$.

So obviously $\mu^*(E) > 0$.

We will show that $\mu^*(E_1)=0$.

Let on contrary suppose that $\mu^*(E_1) \neq 0$, so obviously $\mu^*(E_1) > 0$

Now

$$E_1 \subseteq E$$

$$\Rightarrow \int_{E_1} f d \leq \int_{E} f d$$

$$\Rightarrow \int_{E} f d \geq \int_{E_1} f d = \infty > \int_{E_1} a d \text{; where } a \in E_1$$

$$= a\mu^*(E_1)$$

$$\Rightarrow \int_E f d > a\mu^*(E_1) \text{ ; if } a \text{ is sufficient large i.e } a \to \infty$$

$$\Rightarrow \int_E f d = \infty$$

which is contradiction of statement given. So this contradiction arises due to our wrong supposition and hence $\mu^*(E_1)=0$

which shows that $f(x) < \infty$ a.e.

Theorem:-
let f be non-negative and measurable on E. Then $\int_E f\,d = 0$ iff $f = 0$ a.e in E.

Proof:-
Let $f = 0$ a.e, then

$$\int_E f\,d = \int_E (0)\,d = 0$$

$$\Rightarrow \int_E f\,d = 0$$

Conversely,
let $\int_E f\,d = 0$

we need to prove that $f = 0$ a.e.
For this we show that for any $\alpha > 0$ the $\{x \in E : f(x) > \alpha\}$ has lebesgue measure zero.

let $E_1 = \{x \in E : f(x) > \alpha\}$
So $E_1 \subseteq E$

therefore $\int_{E_1} f\,d \leq \int_E f\,d$

But $f > \alpha \quad \forall \ x \in E_1$
So,

$$\int_{E_1} f\,d \geq \int_{E_1} \alpha\,d$$

$$\Rightarrow \int_{E_1} \alpha\,d \leq \int_{E_1} f\,d \leq \int_E f\,d = 0$$

Scanned By Camera Scanner

$\Rightarrow \alpha \int_{E_1} d \le 0$

$\Rightarrow \alpha \mu(E_1) \le 0$

$\Rightarrow \alpha \mu(\{x \in E : f(x) > \alpha\}) \le 0$

$\Rightarrow \alpha \mu(\{x \in E : f(x) > \alpha\}) = 0$

$\Rightarrow \mu(\{x \in E : f(x) > \alpha\}) = 0$, for any $\alpha > 0$

$\Rightarrow \mu(\{x \in E : f(x) > \frac{1}{k}\}) = 0$, for $k = 1, 2, 3, \ldots$

But

$\{x \in E : f(x) \geq 1\} \subseteq \{x \in E : f(x) \geq \frac{1}{2}\} \subseteq \ldots$
$\ldots \{x \in E : f(x) > \frac{1}{k}\}$

let $f_1 = \{x \in E : f(x) > 1\}$, $f_2 = \{x \in E : f(x) \geq \frac{1}{2}\}$

$\ldots \quad f_k = \{x \in E : f(x) > \frac{1}{k}\}$

$\Rightarrow f_1 \subseteq f_2 \subseteq \ldots \subseteq f_k$

$\Rightarrow \bigcup_{k=1}^{\infty} f_k = \bigcup_{k=1}^{\infty} \{x \in E : f(x) > \frac{1}{k}\}$

$\bigcup_{k=1}^{\infty} f_k = \{x \in E : f(x) \geq 0\}$

But $\mu(f_k) = 0 \quad \forall \quad k = 1, 2, 3, \ldots$

$\Rightarrow \mu\left(\bigcup_{k=1}^{\infty} f_k\right) = \mu\{x \in E : f(x) > 0\} = 0$

$\Rightarrow f = 0 \quad$ a.e.

Date: (123) Mon Tue Wed Thu Fri Sat

Convergence Theorem:- (for non-negative functions)

Let $\{f_k\}$ be a sequence of function then is it true that

$$\lim_{k\to\infty} \int_E f_k(x)\, dx = \int_E \lim_{k\to\infty} f_k(x)\, dx.$$

Ans: Not it is not always true.
leqin agar sequence pe hum different conditions put kar le, tho different conditions put karne se pir ye equal asakta hai.

Example:-

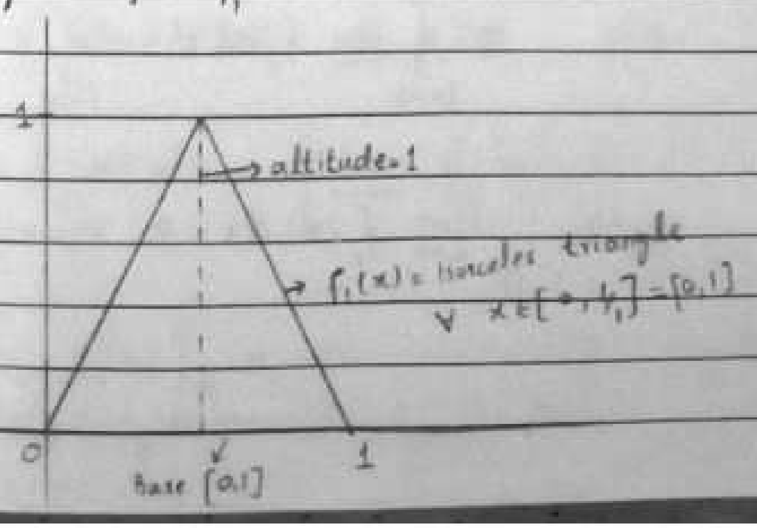
Let f_n be a sequence of functions defined on $[0,1]$, as $f_k(x)$ is an isosceles triangle with base $[0, 1/k]$ and altitude k, $\forall\ x \in [0, 1/k]$, and $f_k(x) = 0\ \forall\ x \in (1/k, 1]$

i.e $f_n : [0,1] \to \mathbb{R}$ by

$f_k(x)$ = Isosceles triangle with base $[0, 1/k]$ and altitude k. iff $x \in [0, 1/k]$

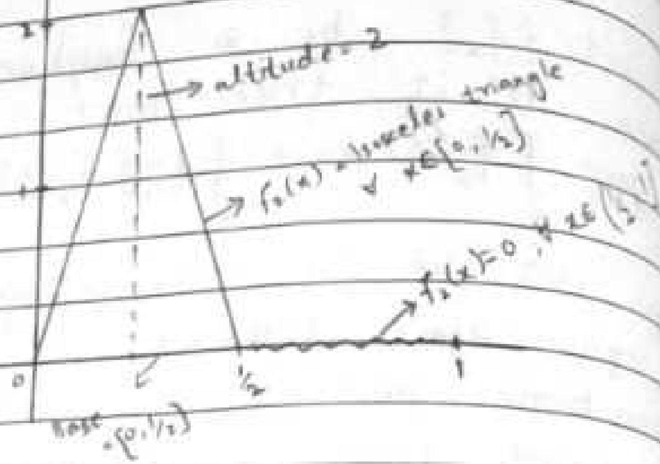
$= 0$ iff $x \in (1/k, 1]$

Then graph of $f_1(x)$ is

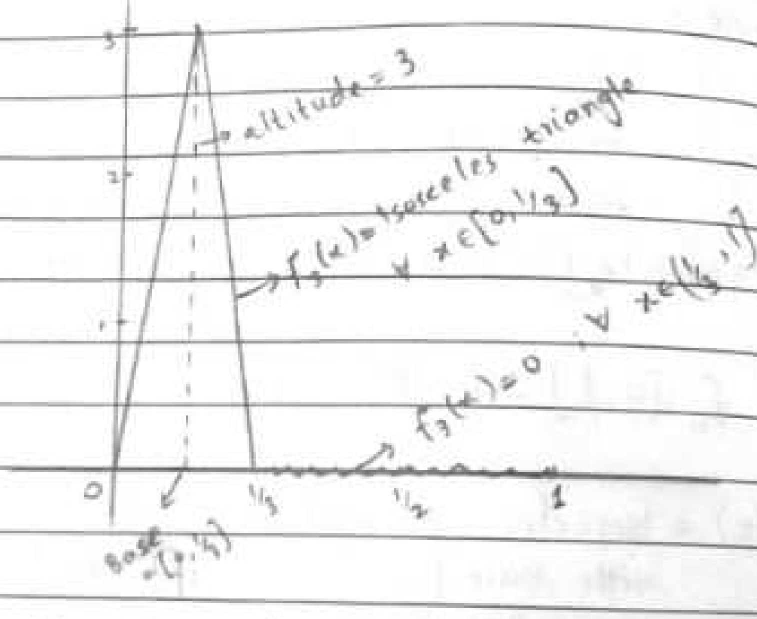


Date:

Graph of $f_2(x)$:-



Graph of $f_3(x)$:-



Here it is clear that as $k \to \infty$

$f_k(x) = 0$ i.e. $\lim_{k \to \infty} f_k(x) = 0$

$$\Rightarrow \int_{[0,1]} \lim_{k \to \infty} f_k(x)\, dx = \int_{[0,1]} 0\, dx$$

$$\Rightarrow \int_{[0,1]} \lim_{k \to \infty} f_k(x)\, dx = 0 \quad \text{--- (i)}$$

Now $\int_{[0,1]} f_k(x)\,dx = \int_{[0,1/k]} f_k(x)\,dx + \int_{(1/k,1]} f_k(x)\,dx$

$$= \frac{1}{2} \cdot K \cdot \frac{1}{K} + 0$$

$$\int_{[0,1]} f_k(x)\,dx = \frac{1}{2}$$

$$\lim_{k\to\infty} \int_{[0,1]} f_k(x)\,dx = \lim_{k\to\infty} \frac{1}{2}$$

$$\lim_{k\to\infty} \int_{[0,1]} f_k(x)\,dx = \frac{1}{2} \quad \text{--- (ii)}$$

From eq (i) and eq (ii), we get

$$\int_{[0,1]} \lim_{k\to\infty} f_k(x)\,dx \neq \lim_{k\to\infty} \int_{[0,1]} f_k(x)\,dx.$$

Monotone Convergence Theorem:-
(for non-negative functions)

Theorem:-

Let $\{f_n(x)\}$ be an increasing sequence of non-negative measurable functions, and if $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall$ $x \in E$, then

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx.$$

Proof:-

Since $\{f_n(x)\}$ is an increasing sequence

i.e. $f_1 \leq f_2 \leq f_3 \leq$ ------

Mon Tue Wed Thu Fri

(126)

Date:

Therefore $R(f_1, E) \subseteq R(f_2, E) \subseteq \cdots$ and

$$\bigcup_{n=1}^{\infty} R(f_n, E) = R(f, E)$$

and $$\int_E f \, d = \mu^*(R(f, E))$$

$$= \lim_{n \to \infty} \mu^*(R(f_n, E))$$

$$= \lim_{n \to \infty} \int_E f_n \, d$$

$$\Rightarrow \int_E f \, d = \lim_{n \to \infty} \int_E f_n \, d$$

Since $\lim_{n \to \infty} f_n = f$

$$\Rightarrow \int_E \lim_{n \to \infty} f_n \, d = \lim_{n \to \infty} \int_E f_n \, d$$

$$\Rightarrow \lim_{n \to \infty} \int_E f_n(x) \, dx = \int_E \lim_{n \to \infty} f_n(x) \, dx$$

proved!

Theorem:-

let $\{f_k\}$ be a sequence of non-negative measurable function, defined on E, then prove that

$$\int_E \sum_{k=1}^{\infty} f_k \, d = \sum_{k=1}^{\infty} \int_E f_k \, d$$

i.e. $$\int_E (f_1 + f_2 + f_3 + \cdots) \, d = \int_E f_1 \, d + \int_E f_2 \, d + \cdots$$

i.e. linearity of lebesgue integration.

Proof:-

Let us define a function $F_n(x)$ a[s]

$$F_n(x) = \sum_{k=1}^{n} f_k(x).$$

Since $\{f_k\}$ is a sequence of non-negative functions.

Therefore $\{F_n(x)\}$ is an increasing sequence. Also $F_n(x)$ is measurable $\forall$ n.

$\Rightarrow \{F_n(x)\}$ is an $\uparrow$ sequence of measurable functions. Therefore by using monotone-convergence theorem for non-negative functions we have

$$\int_E \lim_{n\to\infty} F_n(x)\,dx = \lim_{n\to\infty}\int_E F_n(x)\,dx$$

$$\Rightarrow \int_E \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)\,dx = \lim_{n\to\infty}\int_E \sum_{k=1}^{n} f_k(x)\,dx$$

$$\text{''} \qquad = \lim_{n\to\infty}\sum_{k=1}^{n}\int_E f_k(x)\,dx$$

$$\Rightarrow \int_E \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty}\int_E f_k(x)\,dx$$

proved!

Fatou's Lemma:- $(f_k \geq 0)$

If $\{f_k\}$ is a sequence of non-negative measurable functions, defined on E, then

$$\int_E (\lim\inf f_k)\, d \leq \lim\inf \int_E f_k\, d.$$

In particular if $\lim_{k\to\infty} f_k(x) = f(x)$, then

$$\int_E f(x)\,dx \leq \lim\inf \int_E f_k(x)\,dx.$$

Proof:-

As we know that a non-negative measurable function is lebesgue integrable, therefore $\int_E \liminf f_k \, dx < \infty$

i.e "$\liminf f_k$" is lebesgue integrable.

Now let $g_1 = \inf\{f_1, f_2, f_3, \ldots\}$

$$g_2 = \inf\{f_2, f_3, f_4, \ldots\}$$

$$g_3 = \inf\{f_3, f_4, f_5, \ldots\}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Then we know from definition of limit inf that

$$\lim_{k\to\infty} g_k = \lim_{k\to\infty} \inf f_k \quad \text{and}$$

also $0 \leq g_k \leq f_k$

Also $g_1 \leq g_2 \leq g_3 \leq \ldots$

Clearly $\{g_k\}$ is a monotonic increasing sequence of non-negative measurable functions. Therefore by using "monotone-convergence theorem for non-negative functions" we have

$$\lim_{k\to\infty} \int_E g_k(x)\, dx = \int_E \lim_{k\to\infty} g_k(x)\, dx$$

$$\Rightarrow \lim_{k\to\infty} \int_E g_k(x)\, dx = \int_E \liminf f_k(x)\, dx \quad \text{--- (1)}$$

But $g_k \leq f_k$

$$\Rightarrow \int_E g_k \, d \le \int_E f_k \, d$$

$$\Rightarrow \lim \inf \int_E g_k \, d \le \lim \inf \int_E f_k \, d$$

here since $\lim_{k\to\infty} g_k(x)$ exist, so

$$\Rightarrow \lim_{k\to\infty} = \lim \inf = \lim \sup.$$

So,

$$\Rightarrow \lim_{k\to\infty} \int_E g_k(x)\, dx \le \lim \inf \int_E f_k \, d \quad \text{—— (2)}$$

put eq (1) in eq (2), we get

$$\Rightarrow \int_E \lim \inf f_k(x)\, dx \le \lim \inf \int_E f_k(x)\, dx.$$

$$\Rightarrow \int_E \lim \inf f_k \, d \le \lim \inf \int_E f_k \, d.$$

proved!

Theorem:-

Let $\{f_k\}$ be a sequence of non-negative measurable functions, defined on E.

If $\lim_{k\to\infty} f_k(x) = f(x)$, and if $\int_E f_k(x)\, dx \le M \ \forall k$

Then prove that $\int_E f(x)\, dx \le M$.

Proof:-

Since we know from "Fatous lemma" that

$$\int_E \lim \inf f_k \, d \le \lim \inf \int_E f_k \, d \quad \text{—— (1)}$$

But here $\lim_{k\to\infty} f_k(x) = f(x)$ —— (2)

Therefore

Date:

Since $\lim\limits_{k\to\infty} f_k(x) = f(x)$ exist

$\Rightarrow \liminf f_k = \limsup f_k = \lim f_k(x)$

$$eq\ (1) \Rightarrow \int_E f(x)\,dx = \int_E \lim_{k\to\infty} f_k(x)\,dx = \int \liminf f_k$$

$$\leq \liminf \int f_k(x)\,dx \quad (\text{by}$$

$$\leq \liminf M \quad ; \forall k$$

$$= M$$

$$\Rightarrow \int_E f(x)\,dx \leq M$$

proved!

Lebesgue Dominated Convergence theorem for non-negative measurable functions:

Theorem:-

let $\{f_k(x)\}$ be a sequence of non-negative measurable functions defined on E, if $\lim\limits_{k\to\infty} f_k(x) = f(x)$, and if there exist a measurable function "$\phi(x)$" such that $f_k(x) \leq \phi(x)$ $\forall k$ and $\int_E \phi\, d$ is finite, then

$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$$

Proof:-

By "Fatous lemma" for the sequence $\{f_k\}$, we have $\int_E \liminf f_k\, d \leq \liminf \int_E f_k$

But since $\lim_{k\to\infty} f_k(x) = f(x)$

$\Rightarrow$ $$f(x) = \lim_{k\to\infty} f_k(x)$$

$\Rightarrow$ $$\int_E f(x)\,dx = \int_E \lim_{k\to\infty} f_k(x)\,dx$$

$$= \int_E \liminf f_k(x)\,dx \le \liminf \int_E f_k(x)\,dx$$

$\Rightarrow$ $$\int_E f(x)\,dx \le \liminf \int_E f_k(x)\,dx \quad \text{--- (A)}$$

To prove the required result it is enough to show that

$$\int_E f(x)\,dx \geq \liminf \int_E f_k(x)\,dx$$

For this, since it is given that

$$f_k(x) \le \Phi(x) \quad ; \forall\, x, k$$

$\Rightarrow$ $$\Phi(x) - f_k(x) \geq 0 \quad ; \forall\, x, k$$

$\Rightarrow$ $\{\Phi(x) - f_k(x)\}$ is a sequence of non-negative measurable functions.

Therefore by using the "Fatous lemma" for the sequence $\{\Phi - f_k\}$ we have

$$\int_E \liminf (\Phi - f_k)\,d \le \liminf \int_E (\Phi - f_k)\,d \quad \text{--- (1)}$$

But $\lim_{k\to\infty} f_k(x) = f(x) \Rightarrow -\lim_{k\to\infty} f_k = -f$

$\Rightarrow$ $$\lim_{k\to\infty} (\Phi - f_k) = \Phi - f$$

$\Rightarrow$ $\{\Phi - f_k\}$ converges to $\Phi - f$.

Mon Tue Wed Thu Fri Sat

Date: (B2)

Therefore eq (1) $\Rightarrow$

$$\int_E \liminf (\phi - f_k)\, d = \int_E \lim (\phi - f_k)\, d$$

$$= \int_E (\phi - f)\, d \leq \liminf \int_E (\phi - f_k)\, d$$

$$\Rightarrow \int_E (\phi - f)\, d \leq \liminf \int_E (\phi - f_k)\, d$$

$$= \liminf \left[\int_E \phi\, d - \int_E f_k\, d \right]$$

$$= \liminf \int_E \phi\, d + \liminf \left(-\int_E f_k\, d \right)$$

Note: $\lim(-x_n) = \liminf(-x_n) = -\limsup(x_n)$

$$= \int_E \phi\, d - \limsup \int_E f_k\, d \qquad \left[\because \int_E \phi\, d \text{ is constant} \right]$$

$$\Rightarrow \int_E (\phi - f)\, d \leq \int_E \phi\, d - \limsup \int_E f_k\, d$$

$$\Rightarrow \int_E \phi\, d - \int_E f\, d \leq \int_E \phi\, d - \limsup \int_E f_k\, d$$

$$\Rightarrow -\int_E f\, d \leq -\limsup \int_E f_k\, d$$

$$\Rightarrow \int_E f\, d \geq \limsup \int_E f_k\, d \quad \text{— (B)}$$

From eq (A) and (B), we get

$$\Rightarrow \limsup \int_E f_k\, d \leq \int_E f\, d \leq \liminf \int_E f_k\, d$$

$$\leq \limsup \int_E f_k\, d$$

$$\Rightarrow \limsup \int_E f_k\, d = \int_E f\, d \quad \text{——— (a)}$$

Also from eq (A) and (B), we get

$$\liminf \int_E f_k\, d \le \limsup \int_E f_k\, d \le \int_E f\, d \le \liminf \int_E f_k\, d$$

$$\Rightarrow \liminf \int_E f_k\, d = \int_E f\, d \quad \text{——— (b)}$$

Using eq (a) and (b), we get

$$\int_E f\, d = \limsup \int_E f_k\, d = \limsup \int_E f_k\, d = \lim_{k\to\infty} \int_E f_k$$

$$\Rightarrow \int_E f\, d = \lim_{k\to\infty} \int_E f_k\, d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k(x)\, dx = \int_E f(x)\, dx.$$

proved!

* $f^+ = \text{Max}\{f(x), 0\}$
  $f^- = -\text{Min}\{f(x), 0\}$

$$f = f^+ - f^-$$

$$\Rightarrow \int_E f\, d = \int_E f^+\, d - \int_E f^-\, d$$

$$\Rightarrow L(E) = L(\mu) = L'(\mu) = L_1(\mu) = L'(E) = \{f : \int_E f(x)\, dx < \infty\}$$

Q: $|f| = f^+ + f^-$

Sol:-

Case-1:- If $f(x) \ge 0$

then $f^+(x) = f(x)$

Date: (134)

$$f^-(x) = -\text{Min}\{f(x), 0\} = -0 = 0$$

$$\Rightarrow \quad f^+(x) + f^-(x) = f(x)$$

Case - II :-

If $f(x) \leq 0$

then $f^+(x) = 0$

$$f^-(x) = -\text{Min}\{f(x), 0\} = -f(x)$$

$$f^+(x) + f^-(x) = -f(x) \quad \text{--- (1)}$$

$$\Rightarrow \quad |f(x)| = -f(x) \quad \text{--- (2)}$$

From eq (1) and (2), we get

$$\Rightarrow \quad |f(x)| = f^+(x) + f^-(x).$$

Theorem:-

Let $f \in L(E)$ then prove that

$$\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx.$$

Proof:-

Since $\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$

$$\left|\int_E f(x)\,dx\right| = \left|\int_E f^+(x)\,dx - \int_E f^-(x)\,dx\right|$$

$$\because |a-b| \leq |a| + |b|$$

$$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \left|\int_E f^+(x)\,dx\right| + \left|\int_E f^-(x)\,dx\right| \quad \text{(A)}$$

as $f^+(x) \geq 0$ and $f^-(x) \geq 0$

Scanned By Camera Scanner

So $\int_E f^+(x)\,dx \geq 0$ and $\int_E f^-(x)\,dx \geq 0$

Hence

$\Rightarrow \left|\int_E f^+(x)\,dx\right| = \int_E f^+(x)\,dx$

$\Rightarrow \left|\int_E f^-(x)\,dx\right| = \int_E f^-(x)\,dx$

eq Ⓐ $\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$

$$\leq \int_E \left[f^+(x) + f^-(x)\right] dx$$

$$\leq \int_E |f(x)|\,dx$$

$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$

Theorem:-

let f be measurable function then f is lebesgue integrable iff $|f|$ is lebesgue integrable.

Proof:- OR $(f \in L(E) \Leftrightarrow |f| \in L(E))$

Suppose that

$|f| \in L(E)$ i.e. $|f|$ is integrable.

We prove that $f \in L(E)$.

Since

$\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$ —— ①

Also $|f| \in L(E) \Rightarrow \int_E |f(x)|\,dx < \infty$

eq ① $\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx < \infty$

$\Rightarrow \left|\int_E f(x)\,dx\right| < \infty$

$\Rightarrow \int_E f(x)\,dx < \infty$

$\Rightarrow f \in L(E)$

$\Rightarrow f$ is lebesgue integrable function.

Conversely :-

let $f \in L(E)$

we prove that $|f| \in L(E)$

Since $\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$

Also

$\int_E f(x)\,dx < \infty$

$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx < \infty$

$\Rightarrow \int_E f^+(x)\,dx < \infty$ and $\int_E f^-(x)\,dx < \infty$

$\Rightarrow \int_E f^+(x)\,dx + \int_E f^-(x)\,dx < \infty$

$\Rightarrow \int_E [f^+(x) + f^-(x)]\,dx < \infty$

$\Rightarrow \int_E |f(x)|\,dx < \infty$

$\Rightarrow |f| \in L(E)$

proved!

Theorem:-

If $f \in L(E)$ then f is finite a.e

Proof:-

Since we prove that "$f(x) \geq 0$

and $\int_E f(x)\,dx < \infty$ then f is finite a.e"

Given that $f \in L(E)$

$\Rightarrow$ $|f| \in L(E)$ (proved)

i.e $|f(x)|$ is non-negative and integrable.

so by above result

$\Rightarrow$ $|f(x)|$ is finite a.e.

$\Rightarrow$ $f(x)$ is finite a.e.

which is the required result

Theorem:-

If $f \in L(E)$ and g are measurable such that $|g| \leq M$, then prove that $f.g \in L(E)$.

Proof:-

Since f is integrable (on set E) means that $\int_E |f|\,du < \infty$

we need to show that $\int_E |fg|\,du < \infty$,

But if $|g(x)| \leq M$ for all x, then the function $|fg|$ is bounded from above by $M|f|$, so we have

$$\int_E |fg|\,du = \int_E |f|\,|g|\,du \leq \int_E |f| M\,du$$

$$\int_E |fg|\,du \leq M\int_E |f|\,du < \infty$$

$$\Rightarrow \int_E |fg|\,du < \infty$$

Date: (138

$$\Rightarrow \int_E fg \, d\mu < \infty$$

$$\Rightarrow f \cdot g \in L(E)$$

proved!

* Convergence Theorem:-

→ Monotone Convergence Theorem:-

let $\{f_k\}$ be a sequence of measurable function defined on E

i) If $f_k \uparrow f$ a.e on E and there exists $\phi \in L(E)$ such that $f_k \geq \phi$ a.e on E $\forall$ k, then

$$\lim_{k\to\infty} \int_E f_k \, dx = \int_E f d$$

ii) If $f_k \downarrow f$ a.e on E and there exists $\phi \in L(E)$ such that $f_k \leq \phi$ a.e on E $\forall$ k, then

$$\lim_{k\to\infty} \int_E f_k(x) \, dx = \int_E f(x) \, dx$$

Proof:-

i) Since $f_k \geq \phi$

$$\Rightarrow f_k - \phi \geq 0$$

$\Rightarrow \{f_k - \phi\}$ is a sequence of non-negati measurable function as $f_1 \leq f_2 \leq f_3 \leq \cdots$

$$\Rightarrow f_1 - \phi \leq f_2 - \phi \leq f_3 - \phi \leq \cdots$$

$\Rightarrow \{f_k - \phi\}$ is increases sequence.

Apply monotone convergence for non-negative function, then

$$\lim_{k\to\infty} \int_E (f_k - \phi) \, dx = \int_E \lim_{k\to\infty} (f_k - \phi) \, d$$

$$\lim_{k\to\infty} \int_E (f_k - \phi)\, dx = \int_E (f - \phi)\, d$$

$$\lim_{k\to\infty} \int_E f_k\, dx - \int_E \phi\, d = \int_E f\, d - \int_E \phi\, d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k\, dx = \int_E f\, d$$

Since $f_k \le \phi$

$\Rightarrow -f_k \ge -\phi$

$\Rightarrow -f_k + \phi \ge 0$

$\Rightarrow \{-f_k + \phi\}$ is a sequence of non-negative measurable function as $f_1 \le f_2 \le f_3 \le \cdots$

$\Rightarrow -f_1 \ge -f_2 \ge -f_3 \ge \cdots$

$\Rightarrow -f_1 + \phi \ge -f_2 + \phi \ge -f_3 + \phi \ge \cdots$

$\Rightarrow \{-f_k + \phi\}$ is decreases sequence.

Apply monotone convergence for non-negative function, then

$$\lim_{k\to\infty} \int_E (-f_k + \phi)\, dx = \int_E \lim_{k\to\infty} (-f_k + \phi)\, dx$$

$$\lim_{k\to\infty} \int_E (-f_k)\, dx + \int_E \phi\, dx = \int_E (-f + \phi)\, dx$$

$$-\lim_{k\to\infty} \int_E f_k\, dx + \int \phi\, dx = -\int_E f\, dx + \int_E \phi\, dx$$

$$-\lim_{k\to\infty} \int_E f_k\, dx = -\int_E f\, dx$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k\, dx = \int_E f\, dx$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k(x)\, dx = \int_E f(x)\, dx.$$

Fatou's Lemma :- (for arbitrary function).
Let $\{f_k\}$ be a sequence of measurable function and if there exists $\phi \in L(E)$ such that $\phi \in L(E)$, $f_k \geq \phi \quad \forall\, k$, then prove that

$$\int_E \lim_{k\to\infty} \inf f_k \, d \leq \lim_{k\to\infty} \inf \int_E f_k \, d.$$

Proof:-
Since $f_k \geq \phi$
$\Rightarrow f_k - \phi \geq 0$
$\Rightarrow \{f_k - \phi\}$ is a sequence of non-negative measurable function.
Apply Fatou's lemma for non-negative function, then

$$\int_E \lim \inf (f_k - \phi)\, d \leq \lim \inf \int_E (f_k - \phi)\, d$$

$$\Rightarrow \int_E (\lim \inf f_k - \phi)\, d \leq \lim \inf \left[\int_E f_k \, d - \int_E \phi \, d\right]$$

$$\Rightarrow \int_E \lim \inf f_k \, d - \int_E \phi \, d \leq \lim \inf \int_E f_k \, d - \int_E \phi \, d$$

$$\Rightarrow \int_E \lim \inf f_k \, d \leq \lim \inf \int_E f_k \, d$$

proved//

Corollary:-
Let $\{f_k\}$ be a sequence of measurable functions defined on E. If there exist a function $\phi(x) \in L(E)$ such that $f_k(x) \leq \phi(x)$ on E, $\forall\, k$, then $\int_E \lim \sup f_k \, d \geq \lim \sup \int_E f_k \, d$.

Date: (141)

Mon Tue Wed Thu Fri Sat

Proof:-

Since $f_k \le \phi$

$\Rightarrow -f_k \ge -\phi$

$\Rightarrow -f_k + \phi \ge 0$

$\Rightarrow \{-f_k + \phi\}$ is a sequence of non-negative measurable function.

Apply Fatou's lemma for non-negative function, then

$$\int_E \liminf (-f_k + \phi) d \le \liminf \int_E (-f_k + \phi) d$$

$$\int_E [\liminf (-f_k) + \phi] d \le \lim \inf \left[\int_E (-f_k) d + \int_E \phi d \right]$$

$$\int_E \liminf (-f_k) d + \cancel{\int \phi d} \le \liminf \int_E (-f_k) d + \cancel{\int_E \phi d}$$

$$\int_E \liminf (-f_k) d \le \liminf \int_E (-f_k) d \quad \text{--- (A)}$$

But Since $\liminf (-a_n) = -\limsup (a_n)$.

So eq (1) becomes:

$$\Rightarrow \int_E [-\limsup (f_k)] d \le -\limsup \int_E f_k d$$

$$\Rightarrow \cancel{-}\int_E \limsup f_k d \le \cancel{-} \limsup \int_E f_k d$$

$$\Rightarrow \int_E \limsup f_k d \ge \limsup \int_E f_k d. \text{ proved!}$$

→ **Lebesgue Dominated Convergence Theorem:-**
(for arbitrary sign)

Statement:-

let $\{f_k\}$ be a sequence of measurable functions such that $f_k \to f$ a.e on E. If there exists $\phi \in L(E)$ such that

Date: (142)

$|f_k| \le \phi \quad \forall\ k$, then

$$\lim_{k\to\infty} \int_E f_k \, d = \int_E f \, d.$$

Proof:-

Since $|f_k| \le \phi \quad \forall\ k$

$\Rightarrow \quad -\phi \le f_k \le \phi$

$\Rightarrow \quad -\phi+\phi \le f_k+\phi \le \phi+\phi$

$\Rightarrow \quad 0 \le f_k+\phi \le 2\phi$

$\Rightarrow \{f_k+\phi\}$ is a sequence of non-negative functions such that $f_k+\phi \le 2\phi$

So applying lebesgue dominated convergence theorem for non-negative function, we have

$$\lim_{k\to\infty} \int_E (f_k+\phi) d = \int_E \lim_{k\to\infty} (f_k+\phi) \, d$$

Since $\lim_{k\to\infty} (f_k+\phi) = f+\phi$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k \, d + \int_E \phi \, d = \int_E (f+\phi) d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k \, d + \int_E \phi \, d = \int_E f \, d + \int_E \phi \, d$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k \, d = \int_E f \, d$$

proved!

↳ **Lebesgue bounded convergence theorem**

Statement:-

let $\{f_k(x)\}$ be a sequence of measurable function defined on E, with the finite measure of E, such that $\lim_{k\to\infty} f_k(x) = f(x)$ and if $|f_k| \le M$, then

$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$$

Proof:-

Since we know from lebesgue dominated convergence theorem that

"If $\{f_k\}$ be a sequence of measurable functions defined on E, such that $\lim_{k\to\infty} f_k(x) = f(x)$ and if $|f_k| \le \phi$ on E, then $\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$" — (A)

Replacing "ϕ" by "M" in this statement.

Also $\int_E \phi(x)\,dx = \int_E M\,dx = M\int_E dx = M\mu(E) = \text{finite}$

$\Rightarrow$ $\phi = M$ is lebesgue integrable

i.e. $M \in L(E)$

So applying eq (A), we get

$|f_k| \le M$

Thus $\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$

proved!

Comparison between lebesgue and Riemann Integrals:-

Theorem:-

If the Riemann integrals $J = R\int_a^b f(x)\,dx$

exists, then $f(x)$ is lebesgue integrable on $[a,b]$ and $\int_{[a,b]} f(x)\,dx = J$.

where $\int_{[a,b]} f(x)\,dx$ shows lebesgue integrable

Proof:-

Consider the partition of $[a,b]$ into 2^n sub-interval by the point

$x_k = a + \frac{k}{2^n}(b-a)$; $k = 1,2,3,\dots,2^n$

Consider the Darboux sums.

$$\bar{S}_n = \sum_{k=1}^{2^n} \Delta x_k M_{nk} = \sum_{k=1}^{2^n} \frac{b-a}{2^n} M_{nk}$$

$$\bar{S}_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} M_{nk} \quad \text{———} \quad (a)$$

and

$$\underline{S}_n = \sum_{k=1}^{2^n} \Delta x_k m_{nk} = \sum_{k=1}^{2^n} \frac{b-a}{2^n} m_{nk}$$

$$\underline{S}_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} m_{nk} \quad \text{———} \quad (b)$$

where M_{nk} and m_{nk} are lub and glb of $f(x)$ in $[x_{k-1}, x_k]$.

So by definition:

$$\lim_{n\to\infty} \bar{S}_n = \lim_{n\to\infty} \underline{S}_n = J$$

Define sequences $\{\bar{f}_n\}$ and $\{\underline{f}_n\}$ as $\bar{f}_n(x) = M_{nk}$, $\underline{f}_n(x) = m_{nk}$.

clearly $\bar{f}_1 \geq \bar{f}_2 \geq \bar{f}_3 \geq \dots$

and

Similarly $\underline{f}_1 \leq \underline{f}_2 \leq \underline{f}_3 \leq \dots$

$\int_a^b \bar{f}_n(x)\,dx = \bar{S}_n$ and $\int_a^b \underline{f}_n(x)\,dx = \underline{S}_n$

Now since $\{\bar{f}_n\}$ is a non-increasing sequence and $\{\underline{f}_n\}$ is a non-decreasing sequence.

$$\left.\begin{array}{l}\lim_{n\to\infty} \bar{f}_n(x) = \bar{f}(x) \geq f(x) \\ \text{and} \\ \lim_{n\to\infty} \underline{f}_n(x) = \underline{f}(x) \leq f(x)\end{array}\right\} \text{——} (\star)$$

$$\therefore \lim_{n\to\infty} \bar{S}_n = \lim_{n\to\infty} \int_a^b \bar{f}_n(x)\,dx = \int_a^b \lim_{n\to\infty} \bar{f}_n(x)\,dx = \int_a^b \bar{f}(x)\,dx$$

$$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \int_{[a,b]} \lim_{n\to\infty} \bar{f}_n(x)\,dx = \lim_{n\to\infty} \int_{[a,b]} \bar{f}_n(x)\,dx$$

$$= \lim_{n\to\infty} \bar{S}_n = J = \lim_{n\to\infty} \underline{S}_n$$

$$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \lim_{n\to\infty} \underline{S}_n$$

$$= \lim_{n\to\infty} \int_{[a,b]} \underline{f}_n(x)\,dx$$

$$= \int_{[a,b]} \lim_{n\to\infty} \underline{f}_n(x)\,dx$$

$$= \int_{[a,b]} \underline{f}(x)\,dx$$

$$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \int_{[a,b]} \underline{f}(x)\,dx.$$

Therefore

$$\int_{[a,b]} |\bar{f}(x) - \underline{f}(x)|\,dx = \int_{[a,b]} (\bar{f}(x) - \underline{f}(x))\,dx \quad [\because \bar{f}(x) \geq \underline{f}(x)]$$

$$= 0$$

$$\Rightarrow \int_{[a,b]} |\bar{f}(x) - \underline{f}(x)|\,dx = 0$$

$\Rightarrow |\bar{f}(x) - \underline{f}(x)| = 0$ a.e.

$\Rightarrow \bar{f}(x) - \underline{f}(x) = 0$ a.e

$\Rightarrow \bar{f}(x) = \underline{f}(x)$ a.e.

So by eq ⊛

$\Rightarrow \bar{f}(x) = \underline{f}(x) = f(x)$ a.e.

$$\Rightarrow \int_{[a,b]} \bar{f}(x)\,dx = \int_{[a,b]} \underline{f}(x)\,dx = \int_{[a,b]} f(x)\,dx = J$$

$$\Rightarrow J = R\int_a^b f(x)\,dx$$

$$\Rightarrow J = \int_{[a,b]} f(x)\,dx$$

which is the required result. proved!

LEBESGUE INTEGRAL:

Uniform Convergence Theorem :-

let $f_n \in L(E)$ and let $\{f_n\}$ converges uniformly to "f" on E and let $\mu^*(E) < \infty$. Then prove that $f \in L(E)$ and

$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E \lim_{k\to\infty} f_k(x)\,dx$$

$$= \int_E f(x)\,dx$$

Proof:-

Since $|f(x)| = |f(x) - f_k(x) + f_k(x)|$

$$|f(x)| \leq |f(x) - f_k(x)| + |f_k(x)|$$

Now since $\langle f_k \rangle$ converges to f, so by definition of convergence, for each $\epsilon > 0$ and in particular for $\epsilon = 1$, $\exists$ +ve integer n, such that

$$|f_k(x) - f(x)| < 1 \quad ; \text{ for } k \geq n.$$

so eq (1) $\Rightarrow$ $|f(x)| < 1 + |f_k(x)|$

$$\Rightarrow \int_E |f(x)|\,dx < \int_E 1\,dx + \int_E |f_k(x)|\,dx$$

$$= \mu^*(E) + \int_E |f_k(x)|\,dx$$

$$= \text{finite} \qquad \text{as } f_k(x) \in L(E)$$

$\Rightarrow$ $|f(x)|$ is lebesgue integrable.

$\Rightarrow$ $f(x)$ is lebesgue integrable

$\Rightarrow$ $f(x) \in L(E)$.

Next we need to show that

$$\lim_{k \to \infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$$

For this consider

$$\left|\int_E f(x)\,dx - \int_E f_k(x)\,dx\right| = \left|\int_E (f(x) - f_k(x))\,dx\right|$$

$$< \int_E |f(x) - f_k(x)|\,dx$$

$$\leq \int_E \sup_{x \in E} |f(x) - f_k(x)|\,dx$$

$$\leq \sup_{x \in E} |f(x) - f_k(x)| \int_E dx$$

$$\leq \sup_{x \in E} |f(x) - f_k(x)|\, \mu^*(E)$$

Date: (M.S.)

Taking limit $k \to \infty$

$$\lim_{k\to\infty}\left|\int_E f(x)\,dx - \int_E f_k(x)\,dx\right| \le \lim_{k\to\infty}\sup_{x\in E}\left|f(x)-f_k(x)\right| \mu^*(E)$$

$$= 0\ \mu^*(E) \qquad \text{as } \lim_{k\to\infty} f_k(x) = f(x)$$

$$\lim_{k\to\infty}|f_k(x) - f(x)| = 0$$

$$= 0$$

$$\Rightarrow \lim_{k\to\infty}\left|\int_E f(x)\,dx - \int_E f_k(x)\,dx\right| \le 0 \qquad \text{because } |x| \geq 0 \ ;\ \forall x$$

$$\Rightarrow \lim_{k\to\infty}\left|\int_E f(x)\,dx - \int_E f_k(x)\,dx\right| = 0$$

$$\Rightarrow \lim_{k\to\infty}\left(\int_E f(x)\,dx - \int_E f_k(x)\,dx\right) = 0$$

$$\Rightarrow -\lim_{k\to\infty}\int_E f_k(x)\,dx + \int_E f(x)\,dx = 0$$

$$\Rightarrow \int_E f(x)\,dx = \lim_{k\to\infty}\int_E f_k(x)\,dx$$

$$\Rightarrow \lim_{k\to\infty}\int_E f_k(x)\,dx = \int_E f(x)\,dx$$

which is the required result.

* Lebesgue Integral for unbounded function:-

* let us consider a non-negative, an unbounded measurable function (say) "f" i-e.

$$f : E \to \mathbb{R} \quad , \quad E \subseteq \mathbb{R}^n .$$

$$f(x) \geq 0 \quad \forall\ x \in E$$

let "p" is a natural number and

define an another function (say) "$[f(x)]_p$", such that

$$[f(x)]_p = \begin{cases} f(x) & ; \forall\ x \in E \text{ such that } f(x) \leq p \\ p & ; \forall\ x \in E \text{ such that } f(x) \geq p. \end{cases}$$

Then for each p, the function $[f(x)]_p$ is bounded and measurable (prove it), and thus lebesgue integrable.

We define the lebesgue integral of $f(x)$ on E as

$$\int_E f(x)\,dx = \lim_{p \to \infty} \int_E [f(x)]_p\,dx \quad \text{——— } (*)$$

(prove it)

Q1:- Show that $[f(x)]_p = \min\{f(x), p\}$.

Sol:-

Case-1:- when $f(x) \leq p$,

then $[f(x)]_p = f(x)$ ——— (a)

Also since $f(x) \leq p$

$\Rightarrow \min\{f(x), p\} = f(x)$ ——— (b)

eq (a) and (b), we get

$[f(x)]_p = \min\{f(x), p\}$

Case-2:-

when $f(x) > p$,

then $[f(x)]_p = p$ ——— (c)

Also since $f(x) > p$

$\Rightarrow \min\{f(x), p\} = p$ ——— (d)

Date: (150)

From eq (c) and (d), we get

$$[f(x)]_p = \min\{f(x), p\}$$

proved!

Q2:- Show that $\lim_{p\to\infty} [f(x)]_p = f(x)$.

Sol:-

Since $[f(x)]_p = \min\{f(x), p\}$

$$\lim_{p\to\infty} [f(x)]_p = \lim_{p\to\infty} \min\{f(x), p\}$$

$$= \min\{f(x), \infty\}$$

$$= f(x)$$

$$\Rightarrow \lim_{p\to\infty} [f(x)]_p = f(x)$$

proved!

Q3:- Show that $\int_E f(x)\,dx = \lim_{p\to\infty} \int_E [f(x)]_p\,dx$

Sol:-

Using monotone convergence theorem
Since "$[f(x)]_p$" is an increasing (↑) sequence of non-negative measurable function, such that

$$\lim_{p\to\infty} [f(x)]_p = f(x)$$

So,

$$\lim_{p\to\infty} \int_E [f(x)]_p\,dx = \int_E \lim_{p\to\infty} [f(x)]_p\,dx$$

$$\lim_{p\to\infty} \int_E [f(x)]_p\,dx = \int_E f(x)\,dx$$

$$\Rightarrow \int_E f(x)\,dx = \lim_{p\to\infty} \int_E [f(x)]_p\,dx$$

proved!

Lebesgue Intergral for arbitrary unbounded functions:-

let $f(x) \le 0$

$$\Rightarrow |f(x)| = -f(x)$$

$$\Rightarrow -|f(x)| = f(x)$$

$$\Rightarrow f(x) = -|f(x)|$$

$$\Rightarrow \int_E f(x)\,dx = \int_E -|f(x)|\,dx$$

$$= -\int_E |f(x)|\,dx$$

So when $f(x) \le 0$, then

$$\boxed{\int_E f(x)\,dx = -\int_E |f(x)|\,dx}$$

where the integral on the right is obtained as above since $|f(x)| \ge 0$.

→ In General Case when $f(x)$ may have arbitrary sign (+ve, or -ve or zero), let us define

$$f^+(x) = \begin{cases} f(x) ; & \forall\ x \in E \text{ such that } f(x) \ge 0 \\ 0 ; & \forall\ x \in E \text{ such that } f(x) < 0 \end{cases}$$

and

$$f^-(x) = \begin{cases} 0 ; & \forall\ x \in E \text{ such that } f(x) \ge 0 \\ -f(x) ; & \forall\ x \in E \text{ such that } f(x) < 0 \end{cases}$$

where it is to be noted that both

$f^+(x)$ and $f^-(x)$ are non-negative, then it follows that

$$f(x) = f^+(x) - f^-(x) \quad \text{(prove it)} \text{ — } A$$

$$\Rightarrow \int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$$

Also we can easily prove that

$$|f(x)| = f^+(x) + f^-(x) \text{ — } B$$

$$\Rightarrow \int_E |f(x)|\,dx = \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$$

(proved it)

Proof:- eq A and eq B

eq A $\Rightarrow$ we have to prove that

$$f(x) = f^+(x) - f^-(x)$$

Case-1:- when $f(x) \geq 0$, then $f^+(x) = f(x)$ and $f^-(x) = 0$

$$\Rightarrow f^+(x) - f^-(x) = f(x) - 0$$

$$\Rightarrow f^+(x) - f^-(x) = f(x)$$

$$\Rightarrow \boxed{f(x) = f^+(x) - f^-(x)}$$

Case-2:- when $f(x) < 0$, then $f^+(x) = 0$ and $f^-(x) = -f(x)$

$$\Rightarrow f^+(x) - f^-(x) = 0 - (-f(x))$$

$$\Rightarrow f^+(x) - f^-(x) = f(x)$$

$$\Rightarrow \boxed{f(x) = f^+(x) - f^-(x)}$$

So. eq A proved!

eq (B) $\Rightarrow$ We have to prove that
$$|f(x)| = f^{+}(x) + f^{-}(x)$$

Case-1:- when $f(x) \geq 0 \Rightarrow |f(x)| = f(x)$ —— (a)
$f^{+}(x) = f(x)$ and $f^{-}(x) = 0$

$\Rightarrow f^{+}(x) + f^{-}(x) = f(x) + 0$
$\Rightarrow f^{+}(x) + f^{-}(x) = f(x)$ —— (b)

From (a) and (b), we get
$$\boxed{|f(x)| = f^{+}(x) - f^{-}(x)}$$

Case-2:- when $f(x) < 0$, then $|f(x)| = -f(x)$ —— (c)
$f^{+}(x) = 0$ and $f^{-}(x) = -f(x)$

$\Rightarrow f^{+}(x) - f^{-}(x) = 0 - [-f(x)]$

$\Rightarrow f^{+}(x) - f^{-}(x) = f(x)$ —— (d)

From eq (c) and (d), we get

$$\boxed{|f(x)| = f^{+}(x) - f^{-}(x)}$$
So eq (B) proved!

Theorem:-
Let $f(x)$ be an unbounded, non-negative, measurable function i.e $f(x) \geq 0$, then $\int_E f(x)dx$ exist iff $\int_E [f(x)]_p dx$ is uniformly bounded.

Proof:-
let $f(x)$ is an unbounded,

non-negative and measurable function
i.e $f(x) \geq 0 \quad \forall \ x \in Dom f$.

let $\int_E f(x)\,dx$ exists

$\Rightarrow \int_E f(x)\,dx = M$

Since we know that

$[f(x)]_p = \min\{f(x), p\}$

$[f(x)]_p \leq f(x)$

$\int_E [f(x)]_p\,dx \leq \int_E f(x)\,dx = M$

$\int_E [f(x)]_p\,dx \leq M$

$\Rightarrow \langle \int_E [f(x)]_p\,dx \rangle$ is uniformly bounded sequence.

Conversely, let $\int_E [f(x)]_p\,dx$ is uniformly bounded.

we need to prove that $\int_E f(x)\,dx$ exists

Since $\left| \int_E [f(x)]_p\,dx \right| \leq M$

$\Rightarrow \int_E [f(x)]_p\,dx \leq M$

$\because \int_E [f(x)]_p\,dx \geq 0$

and $|x| = x$ iff $x \geq 0$

Date:

155

Mon Tue Wed Thu Fri Sat

But $\int_E [f(x)]_p \, dx$ is monotonic increasing sequence.

So as it is also bounded.

$\Rightarrow \langle \int_E [f(x)]_p \, dx \rangle$ is convergent.

$\Rightarrow \lim_{p \to \infty} \int_E [f(x)]_p \, dx$ exist.

$\Rightarrow \int_E [f(x)]_p \, dx$ exist.

proved!

Theorem:-

If $|f(x)| \leq g(x)$, where $g(x)$ is integrable on E, then $f(x)$ is also integrable on E and $\int_E |f(x)| \, dx \leq \int_E g(x) \, dx$.

Prove this result for

i) $f(x) \geq 0$ (ii) $f(x)$ having arbitrary sign.

Proof:-

Given that $|f(x)| \leq g(x)$

we need to prove that $f(x)$ is integrable.

Now

i) when $f(x) \geq 0$

$\Rightarrow |f(x)| = f(x)$ and $|f(x)| \leq g(x)$

$\Rightarrow f(x) \leq g(x)$

$\Rightarrow [f(x)]_p \leq [g(x)]_p$

$$\Rightarrow \int_E [f(x)]_p \, dx \leq \int_E [g(x)]_p \, dx$$

$$\Rightarrow \lim_{p \to \infty} \int_E [f(x)]_p \, dx \leq \lim_{p \to \infty} \int_E [g(x)]_p \, dx$$

Date:

$$\Rightarrow \int_E f(x)\,dx \le \int_E g(x)\,dx$$

Since $g(x)$ is integrable

So $\int_E g(x)\,dx$ exist

$\Rightarrow \int_E f(x)\,dx$ exist.

$\Rightarrow f(x)$ is integrable on E.

$$\Rightarrow \int_E f(x)\,dx \le \int_E g(x)\,dx$$

$$\Rightarrow \int_E |f(x)|\,dx \le \int_E g(x)\,dx.$$

ii) Now when $f(x)$ has arbitrary sign, then since we know that

$$\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx \quad \text{— (A)}$$

Also $|f(x)| \le g(x)$

$\Rightarrow f^+(x) - f^-(x) \le g(x)$

$\Rightarrow f^+(x) \le g(x)$ and $f^-(x) \le g(x)$

So from above case (i) $f^+(x)$ and $f(x)$ are both integrable.

$\Rightarrow \int_E f^+(x)\,dx$ and $\int_E f^-(x)\,dx$ exists

$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$ exists.

eq (A)

$\Rightarrow \int_E f(x)\,dx$ exists.

$\Rightarrow f(x)$ is integrable.

Date: 17

Mon Tue Wed Thu Fri Sat

Also $\int_E f^+(x)\,dx + \int_E f^-(x)\,dx$ exist

$\Rightarrow \int_E \left(f^+(x) + f^-(x)\right) dx$ exist.

$\Rightarrow \int_E |f(x)|\,dx$ exist

Now since $|f(x)| \le g(x)$

$$\int_E |f(x)|\,dx \le \int_E g(x)\,dx$$

which is the required result.

Theorem:-

A function $f(x)$ is integrable on E iff $|f(x)|$ is integrable on E and in such a case $\left|\int_E f(x)\,dx\right| \le \int_E |f(x)|\,dx$, where $f(x)$ is an unbounded function.

Proof:-

Let $f(x)$ is integrable.

Since $f(x) = f^+(x) - f^-(x)$, this shows that $f^+(x) - f^-(x)$ is integrable.

So,

$\Rightarrow \int_E \left(f^+(x) - f^-(x)\right) dx$ exist

$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$ exist.

$\Rightarrow \int_E f^+(x)\,dx$ and $\int_E f^-(x)\,dx$ exist.

$\Rightarrow \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$ exist.

$\Rightarrow \int_E \left(f^+(x) + f^-(x)\right) dx$ exist.

Date: (58)

$\Rightarrow \int_E |f(x)|\,dx$ exist.

$\Rightarrow |f(x)|$ is integrable.

Conversely;

Let $|f(x)|$ is integrable on E.

But $|f(x)| = f^+(x) + f^-(x)$

$\Rightarrow f^+(x) + f^-(x)$ is integrable.

$\Rightarrow \int_E (f^+(x) + f^-(x))\,dx$ exist.

$\Rightarrow \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$ exist

$\Rightarrow \int_E f^+(x)\,dx$ and $\int_E f^-(x)\,dx$ exist

$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$ exist

$\Rightarrow \int_E (f^+(x) - f^-(x))\,dx$ exist.

$\Rightarrow \int_E f(x)\,dx$ exist

$\Rightarrow f(x)$ is lebesgue integrable.

Also $\left|\int_E f(x)\,dx\right| = \left|\int_E (f^+(x) - f^-(x))\,dx\right|$

$= \left|\int_E f^+(x)\,dx - \int_E f^-(x)\,dx\right|$

$\because |a-b| \leq |a| + |b|$

$\leq \left|\int_E f^+(x)\,dx\right| + \left|\int_E f^-(x)\,dx\right|$

$$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$$

$$\leq \int_E \left(f^+(x) + f^-(x)\right) dx$$

$$\leq \int_E |f(x)|\,dx$$

$$\Rightarrow \left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$$

proved!

Question:-

Prove that $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$ exists and find its value.

Solution:-

Since we know that for an unbounded, non-negative function $f(x)$.

$$\int_E f(x)\,dx = \lim_{p\to\infty} \int_E [f(x)]_p\,dx \quad \text{—— (1)}$$

We have to find $[f(x)]_p$

let us define

$$[f(x)]_p = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{for } \frac{1}{\sqrt[3]{x}} \leq p \text{ or } x \geq \frac{1}{p^3} \\ p & \text{for } \frac{1}{\sqrt[3]{x}} > p \text{ or } x < \frac{1}{p^3} \end{cases}$$

Then eq (1) $\Rightarrow \int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{p\to\infty} \int_0^8 [f(x)]_p\,dx$

$$\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{p\to\infty}\left[\int_0^{1/p^3} p\,dx + \int_{1/p^3}^{8} \frac{1}{\sqrt[3]{x}}\,dx\right]$$

$$\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{p\to\infty}\left[px\Big|_0^{1/p^3} + \int_{1/p}^8 x^{-1/3}\,dx \right]$$

$$= \lim_{p\to\infty}\left[p\left(\frac{1}{p^3} - 0\right) + \frac{x^{-1/3+1}}{-1/3+1}\Big|_{1/p^3}^8 \right]$$

$$= \lim_{p\to\infty}\left[p\left(1/p^3\right) + \frac{x^{2/3}}{2/3}\Big|_{1/p^3}^8 \right]$$

$$= \lim_{p\to\infty}\left[\frac{1}{p^2} + \frac{3}{2}\, x^{2/3}\Big|_{1/p^3}^8 \right]$$

$$= \lim_{p\to\infty}\left\{ \frac{1}{p^2} + \frac{3}{2}\left[8^{2/3} - \left(\frac{1}{p^3}\right)^{2/3}\right] \right\}$$

$$= \lim_{p\to\infty}\left[\frac{1}{p^2} + \frac{3}{2}\left(4 - \frac{1}{p^{2/3\times 3}}\right) \right]$$

$$= \lim_{p\to\infty}\left[\frac{1}{p^2} + \frac{3}{2}\left(4 - \frac{1}{p^2}\right) \right]$$

Applying limit

$$= \frac{1}{\infty^2} + \frac{3}{2}\left(4 - \frac{1}{\infty^2}\right)$$

$$= 0 + \frac{3}{2}(4 - 0)$$

$$= \frac{3}{\cancel{2}}(4)^2$$

$$\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = 6 \qquad \text{: Ans}$$

Thus the lebesgue integral exists and has the value 6.

Note:- The above Lebesgue integral can be evaluated as for Riemann integrals.

Also that the Riemann integral $R\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$, exists as an improper integral defined as

$$\lim_{\epsilon\to 0} R\int_\epsilon^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{\epsilon\to 0} R\int_\epsilon^8 x^{-1/3}\,dx$$

$$= \lim_{\epsilon\to 0} \left.\frac{x^{-1/3+1}}{-1/3+1}\right|_\epsilon^8$$

$$= \lim_{\epsilon\to 0} \left.\frac{x^{2/3}}{2/3}\right|_\epsilon^8$$

$$= \lim_{\epsilon\to 0} \left.\frac{3}{2}x^{2/3}\right|_\epsilon^8$$

$$= \lim_{\epsilon\to 0} \frac{3}{2}\left(8^{2/3} - \epsilon^{2/3}\right)$$

$$= \frac{3}{2}\lim_{\epsilon\to 0}\left(4 - \epsilon^{2/3}\right)$$

$$= \frac{3}{2}(4-0)$$

$$= \frac{3}{\cancel{2}_1}(\cancel{4})^2$$

$$\lim_{\epsilon\to 0} R\int_\epsilon^8 \frac{1}{\sqrt[3]{x}}\,dx = 6 \quad \text{Ans.}$$

Question:-

Prove that $\int_0^4 \frac{1}{\sqrt{x}}\,dx$ exists as a lebesgue integral and find its value.

Solution:-

Since we know that for an unbounded, non-negative function $f(x)$.

Mon Tue Wed Thu Fri Sat

Date: 1162

$$\int_{\epsilon} f(x)\,dx = \lim_{p\to\infty} \int_{\epsilon} [f(x)]_p\,dx \quad \text{--- (1)}$$

We have to find $[f(x)]_p$.

let us define

$$[f(x)]_p = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{for } 1/\sqrt{x} < p \text{ or } x \geq \dfrac{1}{p^2} \\ p & \text{for } 1/\sqrt{x} \geq p \text{ or } x < \dfrac{1}{p^2} \end{cases}$$

Then eq (1) $\Rightarrow \displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{p\to\infty} \int_0^4 [f(x)]_p\,dx$

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{p\to\infty}\left[\int_0^{1/p^2} p\,dx + \int_{1/p^2}^4 \frac{1}{\sqrt{x}}\,dx\right]$$

$$= \lim_{p\to\infty}\left[px\Big|_0^{1/p^2} + \int_{1/p^2}^4 x^{-1/2}\,dx\right]$$

$$= \lim_{p\to\infty}\left[p\left(\frac{1}{p^2} - 0\right) + \frac{x^{-1/2+1}}{-1/2+1}\Big|_{1/p^2}^4\right]$$

$$= \lim_{p\to\infty}\left[p\left(1/p^2\right) + \frac{x^{1/2}}{1/2}\Big|_{1/p^2}^4\right]$$

$$= \lim_{p\to\infty}\left[\frac{1}{p} + 2\,x^{1/2}\Big|_{1/p^2}^4\right]$$

$$= \lim_{p\to\infty}\left[\frac{1}{p} + 2\left\{4^{1/2} - \left(1/p^2\right)^{1/2}\right\}\right]$$

$$= \lim_{p\to\infty}\left[\frac{1}{p} + 2\left(2 - \frac{1}{p}\right)\right]$$

Applying limit

$$= \frac{1}{\infty} + 2\left(2 - \frac{1}{\infty}\right)$$

$$= 0 + 4 - 0$$

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = 4 \quad \text{Ans}$$

Date:

Thus the Lebesgue integral exists and has the value $\frac{1}{4}$.

Question:—

Prove that $\int_0^2 \frac{1}{x^2} dx$ does not exist.

Solution:—

Since we know that for an unbounded, non-negative function $f(x)$.

$$\int_0 f(x)\, dx = \lim_{p\to\infty} \int_0 [f(x)]_p \, dx$$

We have to find $[f(x)]_p$.

Let us define.

$$[f(x)]_p = \begin{cases} \frac{1}{x^2} & \text{for } \frac{1}{x^2} \le p \text{ or } x \ge \frac{1}{\sqrt{p}} \\ p & \text{for } \frac{1}{x^2} > p \text{ or } x < \frac{1}{\sqrt{p}} \end{cases}$$

Then eq ① $\Rightarrow$ $\int_0^2 \frac{1}{\sqrt{x}} dx = \lim_{p\to\infty} \int_0^2 [f(x)]_p \, dx$

$$\int_0^2 \frac{1}{\sqrt{x}} dx = \lim_{p\to\infty} \left[\int_0^{1/\sqrt{p}} p\, dx + \int_{1/\sqrt{p}}^2 \frac{1}{x^2} dx \right]$$

$$= \lim_{p\to\infty} \left[p x \Big|_0^{1/\sqrt{p}} + \int_{1/\sqrt{p}}^2 x^{-2} dx \right]$$

$$= \lim_{p\to\infty} \left[p\left(\frac{1}{\sqrt{p}}\right) + \frac{x^{-3}}{-3} \Big|_{1/\sqrt{p}}^2 \right]$$

$$= \lim_{p\to\infty} \left[p(p^{-1/2}) - \frac{1}{3} \frac{1}{x^3} \Big|_{1/\sqrt{p}}^2 \right]$$

$$= \lim_{p\to\infty} \left[p^{1-1/2} - \frac{1}{3}\left(\frac{1}{2^3} - \frac{1}{(1/\sqrt{p})^3} \right) \right]$$

$$= \lim_{p\to\infty} \left[p^{1/2} - \frac{1}{3}\left(\frac{1}{8} - (\sqrt{p})^3 \right) \right]$$

Applying limit

Date: (164)

Thus $\int_0^2 \frac{1}{x^2}\, dx = \infty$

$\Rightarrow \int_0^2 \frac{1}{x^2}\, dx$ does not exists.

Theorem:-

Show that $[f(x)]_p$ is bounded and measurable, and thus lebesgue integrable, for each p if $f(x)$ is measurable.

Proof:-

Since $[f(x)]_p = \min\{f(x), p\}$ —— (1)

$\Rightarrow [f(x)]_p \leq p \quad \forall\ p \in N$

$\Rightarrow [f(x)]_p$ is bounded for each p

Also since we know a result which states "If $f_1(x)$ and $f_2(x)$ are measurable on E, then (a) $\max\{f_1(x), f_2(x)\}$ (b) $\min\{f_1(x), f_2(x)\}$ are measurable on E". —— (*)

Using this result (*), Here since $f(x)$ is given to be measurable and p being a constant function is measurable, so by eq (*) $\min\{f(x), p\}$ is also measurable

$\Rightarrow [f(x)]_p$ is measurable

Note:- The result can be shown directly from the definition of the function $[f(x)]_p$.

Note:- What is the difference b/w Riemann integral and Lebesgue integral.

Ans:-

Both integral are used to calulate the area under curve and the difference is

Riemann Integral	Lebesgue Integral
(i) Function should be continuous.	(i) Continuous / Discontinuous.
(ii) Bounded	(ii) Bounded / unbounded
(iii) Countable (in case of sets).	(iii) Countable / uncountable (In case of sets).

Theorem:-

If $\int_E f(x)\,dx$ exists then prove that $f(x)$ is finite almost everywhere on E, where $f(x) \geq 0$ and unbounded.

Proof:-

let $f(x)$ be an unbounded, measurable function such that $f(x) \geq 0$.

let $\int_E f(x)\,dx$ exists.

We have to show that $f(x) < \infty$ a.e.

For this let $A = \{x \in E \; ; \; f(x) = \infty\}$

we will show that $\mu^*(A) = 0$

Since $A \subseteq E$

So,

$$\int_E [f(x)]_p \, dx \geq \int_A [f(x)]_p \, dx$$

Date: (166)

$$\Rightarrow \int_E [f(x)]_p\, dx \geq \int_A [f(x)]_p\, dx \geq \int_A p\, dx = p\mu(A)$$

$$\Rightarrow \int_E [f(x)]_p\, dx \geq p\mu(A)$$

$$\Rightarrow \lim_{p\to\infty} \int_E [f(x)]_p\, dx \geq \lim_{p\to\infty} p\mu(A)$$

$$\Rightarrow \int_E f(x)\, dx \geq \lim_{p\to\infty} p\mu(A) \quad \text{--- (1)}$$

Now if $\mu(A) > 0$

$$\Rightarrow \lim_{p\to\infty} p\mu(A) = \infty$$

So eq (1) $\Rightarrow \int_E f(x)\, dx = \infty$

which is contradiction to the given that $\int_E f(x)\, dx$ exists.

So $\mu(A) \not> 0$ and $\mu(A) \not< 0$

$$\Rightarrow \mu(A) = 0$$

$$\Rightarrow \mu(\{x : f(x) = \infty\}) = 0$$

$\Rightarrow f(x)$ is finite almost everywhere.

Theorem:-

Let $f(x)$ be an unbounded, measurable function, such that $\int_E f(x)dx$ exists, if A is a measurable subset of E, then $\int_A f(x)dx$ also exists and in such a case

$$\int_A |f(x)|\, dx \leq \int_E |f(x)|\, dx.$$

Proof:-
Given that $\int_E f(x)\,dx$ exist and $A \subseteq E$, where A is measurable. We are to prove that $\int_A f(x)\,dx$ exist and that $\int_A |f(x)|\,dx \le \int_E |f(x)|\,dx$.

Consider $f(x) \ge 0$, since $A \subseteq E$ and $[f(x)]_p$ is bounded.

So, $\int_A [f(x)]_p\,dx \le \int_E [f(x)]_p\,dx$

$$\lim_{p\to\infty} \int_A [f(x)]_p\,dx \le \lim_{p\to\infty} \int_E [f(x)]_p\,dx$$

$$\Rightarrow \int_A f(x)\,dx \le \int_E f(x)\,dx \quad \text{--- (1)}$$

But $\int_E f(x)\,dx$ exists, so $\int_A f(x)\,dx$ also exist.

Since $f(x) \ge 0 \Rightarrow |f(x)| = f(x)$

So eq (1) $\Rightarrow \int_A |f(x)|\,dx \le \int_E |f(x)|\,dx$.

So in this case the required answer proved!.

Now when $f(x)$ has arbitrary sign in such a case $f(x) = f^+(x) - f^-(x)$

$$\int_A f(x)\,dx = \int_A f^+(x)\,dx - \int_A f^-(x)\,dx$$

Also $\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$

Date: (168)

Since $f^{+}(x) \geq 0$ and $f^{-}(x) \geq 0$

So from above case (i)

$$\text{eq } (1) \Rightarrow \int_A f^{+}(x)dx + \int_A f^{-}(x)dx \leq \int_E f^{+}(x)dx + \int_E f^{-}(x)dx$$

$$\Rightarrow \int_A (f^{+}(x) + f^{-}(x))dx \leq \int_E (f^{+}(x) + f^{-}(x))dx$$

$$\Rightarrow \int_A |f(x)|\, dx \leq \int_E |f(x)|\, dx.$$

proved!

Theorem:-

If $E = E_1 \cup E_2$ where E_1 and E_2 are disjoint, then if $f(x)$ is an unbounded function on E, then

$$\int_E f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)\, dx.$$

In general prove that if $E = \bigcup_{k=1}^{n} E_k$, where all E_k are disjoint then

$$\int_E f(x)dx = \sum_{k=1}^{n} \int_{E_k} f(x)\, dx$$

Proof:-

We prove the above result in general and that deduce from the particular for two sets.

Also we first consider $f(x) \geq 0$.

Since $[f(x)]_p$ is always a bounded and measurable function and we know from lebesgue integration theory of bounded function that " if $f(x)$

Date: (169) Mon Tue Wed Thu Fri Sat

is bounded and measurable on E where $E = E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n$ where all E_i's are disjoint then

$$\int_E g(x)\,dx = \int_{E_1} g(x)\,dx + \int_{E_2} g(x)\,dx + \dots + \int_{E_n} g(x)\,dx \text{ — } (*)$$

Using this result eq (*).

$$\int_E [f(x)]_p\,dx = \int_{E_1}[f(x)]_p\,dx + \int_{E_2}[f(x)]\,dx + \dots + \int_{E_n}[f(x)]_n\,dx$$

$$\lim_{p\to\infty}\int_E [f(x)]_p\,dx = \lim_{p\to\infty}\int_{E_1}[f(x)]_p\,dx + \lim_{p\to\infty}\int_{E_2}[f(x)]_p\,dx + \dots + \lim_{p\to\infty}\int_{E_n}[f(x)]_p\,dx$$

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx + \dots + \int_{E_n} f(x)\,dx$$

or

$$\boxed{\int_E f(x)\,dx = \sum_{k=1}^{n}\int_{E_k} f(x)\,dx.} \text{ — } (A)$$

proved for $f(x) \geq 0$.

Now we consider the case when $f(x)$ has arbitrary sign and in such a case

$$f(x) = f^+(x) - f^-(x) \quad \text{and} \quad |f(x)| = f^+(x) + f^-(x)$$

Since $f^+(x) \geq 0$ and $f^-(x) \geq 0$

So using the above result eq (A) proved for $f(x) \geq 0$ we have

$$\int_E f^+(x)\,dx = \sum_{k=1}^{n}\int_{E_k} f^+(x)\,dx$$

and also $\int_E f^-(x)\,dx = \sum_{k=1}^{n}\int_{E_k} f^-(x)\,dx$

Date: (170)

$$\Rightarrow \int_E f^+(x)\,dx - \int_E f^-(x)\,dx = \sum_{k=1}^{n} \int_{E_k} f^+(x)\,dx - \sum_{k=1}^{n} \int_{E_k} f^-(x)\,dx$$

$$\Rightarrow \int_E (f^+(x) - f^-(x))\,dx = \sum_{k=1}^{n} \int_{E_k} (f^+(x) - f^-(x))\,dx$$

$$\Rightarrow \boxed{\int_E f(x)\,dx = \sum_{k=1}^{n} \int_{E_k} f(x)\,dx}$$

In particular when $E = E_1 \cup E_2$, then

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx$$

which is the required result.

Theorem:-

Let $f(x)$ be an unbounded and measurable function on E, such that $\mu(E) = 0$, then $\int_E f(x)\,dx = 0$

Proof:-

Let $f(x) \geq 0$, then since

$$\lim_{p \to \infty} \int_E [f(x)]_p\,dx = \int_E f(x)\,dx.$$

Now we have a result which states "If E has measure zero and $f(x)$ is bounded then $\int_E f(x)\,dx = 0$".

By this result, since here E has measure zero, and $[f(x)]_p$ is bounded, so

$$\int_E [f(x)]_p \, dx = 0$$

$$\lim_{p \to \infty} \int_E [f(x)]_p \, dx = 0$$

$$\Rightarrow \int_E f(x) \, dx = 0 \quad ; \text{ For } f(x) \geq 0$$

Now if $f(x)$ has arbitrary sign, then

$$f(x) = f^+(x) - f^-(x)$$

$$\Rightarrow \int_E f(x) \, dx = \int_E f^+(x) \, dx - \int_E f^-(x) \, dx \quad \text{——} \; (1)$$

Since $f^+(x)$ and $f^-(x)$ are non-negative, and E has measure zero, so from above

$$\int_E f^+(x) \, dx = 0 \quad \text{and} \quad \int_E f^-(x) \, dx = 0$$

$$\Rightarrow \int_E f^+(x) \, dx - \int_E f^-(x) \, dx = 0$$

eq (1)

$$\Rightarrow \boxed{\int_E f(x) \, dx = 0}$$

which is the required result.

x——x——x——x——x

Prepared by : Asim Marwat

MSc Mathematics (Final)

University of Peshawer

Mob : 03151949572

Session [redacted] 2020 - 2021

gmail : asimmarwat41@gmail.com