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Notes;
Measure Theory
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CHAPTER : 3

MEASURABLE FUNCTIONS

Almost everywhere (a.e) :-

We say that a property holds almost everywhere (a.e) if that property for all point except for the set of point having lebesgue measure zero.

Example :-

let $f(x) = x+1$ if $x \in Q$
$= x^2+2$ if $x \in Q'$

and $g(x) = 4x^3+10$ if $x \in Q$
$= x^2+2$ if $x \in Q'$

Clearly $f(x) \neq g(x)$

Now let $A = \{x : f(x) \neq g(x)\}$
$\Rightarrow A = Q$
$\mu^*(A) = \mu^*(Q) = 0$
Hence $f = g$ a.e

Example :-

let $f(x) = 2x^2+1$ if $x \in Q'$
$= x-10$ if $x \in Q$

and

$g(x) = x^2$ if $x \in Q'$
$= x+20$ if $x \in Q$

Clearly $f \ngtr g$

Now let $A = \{x : f \ngtr g\}$
$\Rightarrow A = Q$

But $\mu^*(A) = \mu^*(Q) = 0$
Hence $f > g$ a.e.

Example:-
let $\{f_n(x)\}$ be a sequence of functions defined on $[0,1]$ by
$f_n(x) = x^n$ if $x \in [0,1)$
and let $f : [0,1] \longrightarrow \mathbb{R}$ be defined by $f(x) = 0$ if $x \in [0,1)$
$= 5$ if $x = 1$

then clearly $f_n(x) \not\rightarrow f(x)$

Now let $A = \{x : f_n(x) \not\longrightarrow f(x)\}$
$\Rightarrow A = \{1\}$

$\Rightarrow \mu^*(A) = 0$
Hence $f_n \longrightarrow f$ a.e.

Measurable Function:-
i) $f = g$ a.e if $\mu^*(\{x \in X : f(x) \neq g(x)\}) = 0$
ii) $f \geq g$ a.e if $\mu^*(\{x \in X : f(x) < g(x)\}) = 0$
iii) $f_n \rightarrow f$ a.e if $\mu^*(\{x \in X : f_n(x) \not\rightarrow f(x)\}) = 0$.
iv) $f_n \uparrow f$ a.e if $f_n \leq f_{n+1}$ a.e $\forall n$ and $f_n \rightarrow f$ a.e.
v) $f_n \downarrow f$ a.e if $f_{n+1} < f_n$ a.e. and $f_n \rightarrow f$ a.e.
Let $(X \in \mathbb{R}^n, S, \mu)$ be a measure space.
let f be a real valued function on E

i.e. $f : E \subset R^n \longrightarrow \mathbb{R}$

Then f is lebesgue measurable function on E or simply measurable function, if for every finite a. The set $\{x \in E : f(x) > a\}$ is measurable. $f : E \to \mathbb{R}$ if $S \subset \mathbb{R}$ then

$f^{-1}(S) = \{x \in E : f(x) \in S\}$

We know that if S is a set

$f^{-1}(S) = \{x : f(x) \in S\}$

Now

$\{x \in E : f(x) > a\}$

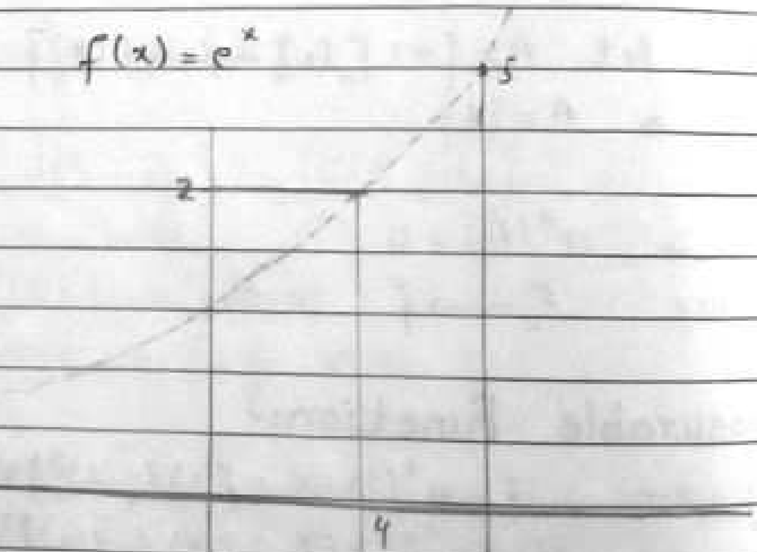
$\Rightarrow \{x \in E : f(x) \in (a, \infty)\}$

$\Rightarrow f^{-1}(a, \infty)$

if $x > 1$
$x \in (1, \infty)$
if $x > -2$
$x \in (-2, \infty)$

So a function f is measurable if $f^{-1}(a, \infty)$ is measurable set.

e.g. Let $f(x) = e^x$



$f^{-1}(1, \infty) = \{x : f(x) \in (1, \infty)\}$

$= \{x : e^x \in (1, \infty)\}$

$= (0, \infty)$ is measurable.

$f^{-1}(2, \infty) = (x, \infty)$

$f^{-1}(5,\infty) = (x_2,\infty)$ is measurable.

Lemma (i) :-

Show that the sets

$\{x; f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x; f(x) > a - \frac{1}{n}\}$

Proof:-

let $x_0 \in A = \{x : f(x) \geq a\}$

$\Rightarrow f(x_0) > a > a - \frac{1}{n} \quad \forall n$

$\Rightarrow f(x_0) > a - \frac{1}{n} \quad \forall n$

$\Rightarrow x_0 \in \{x : f(x) > a - \frac{1}{n}\} \quad \forall n$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$

$\Rightarrow A \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\} \quad \forall n$ —— (i)

Conversely; let $x_0 \in \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$

$\Rightarrow x_0 \in \{x : f(x) > a - \frac{1}{n}\} \quad \forall n$

$\Rightarrow f(x_0) > a - \frac{1}{n} \quad \forall n$ —— (*)

We claim that $f(x_0) \geq a$. because if $f(x_0) < a$, then $f(x_0) < a - \frac{1}{n}$

But $f(x_0) > a - \frac{1}{n}$ for some n.

which is contradiction for eq (*), So

$f(x_0) \geq a$

$\Rightarrow x_0 \in \{x; f(x) \geq a\}$

$\Rightarrow x_0 \in A$

So $\bigcap_{n=1}^{\infty} \{x ; f(x) > a - \frac{1}{n}\} \subseteq A$ —— (ii) $\forall$ n

From eq (i) and (ii), we get

$$A = \bigcap_{n=1}^{\infty} \{x ; f(x) > a - \frac{1}{n}\} \quad \forall n$$

Since $A = \{x ; f(x) \geq a\}$

Hence

$$\{x ; f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x ; f(x) > a - \frac{1}{n}\}$$

proved!

Lemma (ii):-

Show that

$$\{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}.$$

Proof:-

let $A = \{x ; f(x) > a\}$

and $B = \bigcup_{n=1}^{\infty} \{x ; f(x) \geq a + \frac{1}{n}\}$

We need to show that $A = B$

let $y \in A$

$\Rightarrow y \in \{x : f(x) > a\}$

$\Rightarrow f(y) > a$

$\Rightarrow f(y) > f(y) - \frac{1}{n} \geq a$ for some "n".

$\Rightarrow f(y) \geq a + \frac{1}{n}$ for some n.

$\Rightarrow y \in \{x : f(x) \geq a + \frac{1}{n}\}$

$\Rightarrow y \in \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}$

$y \in B$

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Thus $A \subseteq B$ ——— (i)

Conversely; let $y \in B$

$\Rightarrow y \in \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}$

$\Rightarrow f(y) \geq a + \frac{1}{n}$ for some n

$\Rightarrow f(y) \geq a + \frac{1}{n} > a$

$\Rightarrow f(y) > a$

$\Rightarrow y \in \{x : f(x) > a\}$

$\Rightarrow y \in A$

Thus $B \subseteq A$ ——— (ii)

From eq (i) and (ii), we get

$A = B$

Hence $\{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}$

proved!

Theorem:-

let $f : E \to \mathbb{R}$ be a function, where E is measurable ~~function~~ subset of $\mathbb{R}^n$. Then the following statements are equivalent.

f is measurable function.

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ii) $f^{-1}(-\infty, a]$ is measurable set for any $a \in \mathbb{R}$.

iii) $f^{-1}([a, \infty))$ is measurable set for any $a \in \mathbb{R}$.

iv) $f^{-1}(-\infty, a)$ is measurable set for any $a \in \mathbb{R}$.

Proof:-

Let (i) is true.

Since f is measurable function

So $\{x : f(x) > a\}$ is measurable set.

$\Rightarrow \{x : f(x) > a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) \leq a\}$ is measurable set.

$\Rightarrow \{x : f(x) \in (-\infty, a]\}$ is measurable set.

$\Rightarrow f^{-1}(-\infty, a]$ is measurable set.

Now let (ii) is true

Since $f^{-1}(-\infty, a]$ is measurable set

$\Rightarrow \{x : f(x) \leq a\}$ is measurable set

$\Rightarrow \{x : f(x) \leq a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) > a\}$ is measurable set

$\Rightarrow \{x : f(x) > a - \frac{1}{n}\}$ is measurable set for any $n \in \mathbb{N}$.

$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$ is measurable set

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$\Rightarrow \{x : f(x) \geq a\}$ is measurable set

$\Rightarrow \{x : f(x) \in [a, \infty)\}$ is measurable set

$\Rightarrow f^{-1}[a, \infty)$ is measurable set.

Now let (iii) is true.
Since $f^{-1}[a, \infty)$ is measurable set.

$\Rightarrow \{x : f(x) \geq a\}$ is measurable set

$\Rightarrow \{x : f(x) \geq a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) < a\}$ is measurable set.

$\Rightarrow \{x : f(x) \in (-\infty, a)\}$ is measurable set.

$\Rightarrow f^{-1}(-\infty, a)$ is measurable set.

Now let (iv) is true.
Since $f^{-1}(-\infty, a)$ is measurable set.

$\Rightarrow \{x : f(x) < a\}$ is measurable set.

$\Rightarrow \{x : f(x) < a\}^c$ is measurable set.

$\Rightarrow \{x : f(x) \geq a\}$ is measurable set.

$\Rightarrow \{x : f(x) \geq a + \frac{1}{n}\}$ is measurable set for any $n \in N$.

$\Rightarrow \bigcup_{n=1}^{\infty} \{x : f(x) \geq a + \frac{1}{n}\}$ is measurable set.

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$\Rightarrow \{x : f(x) > a\}$ is measurable set.

(by lemma)

Thus f is measurable function.

Imp Theorem:-

A function $f : \mathbb{R}^n \to \mathbb{R}$ is measurable iff for every open set G in $\mathbb{R}$ the inverse image $f^{-1}(G)$ is a measurable subset of $\mathbb{R}^n$.

Proof:-

Let $f^{-1}(G)$ is measurable for every open set G in R.

We will prove that f is measurable i.e. we prove that the set $\{x \in \mathbb{R}^n : f(x) > a\}$ is measurable for every finite

Since (a, ∞) is open and

$f^{-1}(a,\infty) = \{x \in \mathbb{R}^n : f(x) \in (a,\infty)\}$

i.e. $f^{-1}(a,\infty) = \{x \in \mathbb{R}^n : a < f(x) < \infty\}$ measurable

$\Rightarrow$ f is measurable function.

Conversely, let G be any open set in $\mathbb{R}$.

We prove that $f^{-1}(G)$ is measurable for every open set,

Since G can be written as $\bigcup_k (a_k, b_k)$.

But $f^{-1}(a_k, b_k) = \{x \in \mathbb{R}^n : a_k < f(x) < b_k\}$ is measurable.

$f^{-1}(G) = f^{-1}\left[\bigcup_k (a_k, b_k)\right]$

$f^{-1}(G) = \bigcup_k f^{-1}(a_k, b_k)$ it follows that $f^{-1}(G)$ is measurable.

proved!

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Theorem:-

A function $f: R^n \to R$ is measurable iff $f^{-1}(C)$ is measurable for every closed subset C of R.

Proof:-

Let $f: R^n \longrightarrow R$ is measurable.
we need to prove that for any closed set "c" $f^{-1}(c)$ is measurable.

Let us consider "c" is any arbitrary closed set.

$\Rightarrow$ c^c is open.

Since we know that "$f: \mathbb{R}^n \to \mathbb{R}$ is measurable iff $f^{-1}(G)$ is measurable for any open set G."

So $f^{-1}(c^c)$ is measurable.

$\Rightarrow$ $[f^{-1}(c^c)]^c$ is measurable

Since $f^{-1}(c) = [f^{-1}(c^c)]^c$

Thus $f^{-1}(c)$ is measurable

Conversely; Let $f^{-1}(c)$ is measurable for closed c.

let G be any open set. $\Rightarrow G^c$ is closed

Then $f^{-1}(G^c)$ is measurable.

$\Rightarrow$ $[f^{-1}(G^c)]^c$ is measurable

$\Rightarrow$ $f^{-1}(G)$ is measurable

$\Rightarrow$ f is measurable

Theorem:-

If f is measurable and $g: \mathbb{R}^n \to \mathbb{R}$ such $f = g$ a.e then g is measurable.

Proof:-

If $A = \{x \in \mathbb{R}^n : f(x) \neq g(x)\}$

then $\mu^*(A) = 0$

$\Rightarrow$ A is measurable

Let G be an open set in $\mathbb{R}$

$\Rightarrow$ $f^{-1}(G)$ is measurable as f is measurable

and hence $A^c \cap g^{-1}(G) = A^c \cap f^{-1}(G)$ is measurable set.

Also since $A \cap g^{-1}(G) \subset A$

$\Rightarrow$ $\mu^*(A \cap g^{-1}(G)) = 0$

So $g^{-1}(G) = [A \cap g^{-1}(G)] \cup [A^c \cap g^{-1}(G)]$

Since we know that "the union of two measurable sets are measurable"

So $g^{-1}(G)$ is measurable.

Thus g is measurable

Theorem:-

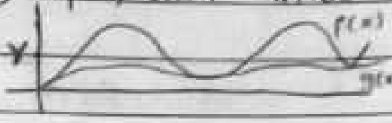
let f, g are measurable functions on set E, then prove that the sets $\{x \in E ; f(x) > g(x)\}$, $\{x \in E ; f(x) \leq g(x)\}$ and $\{x \in E : f(x) = g(x)\}$ are measurable.

Proof:-

Given that f, g are measurable functions on set E.

We are to prove that

(i) $\{x \in E;\ f(x) > g(x)\}$ is also measurable on E.
Since for each $x \in E$ we can find a rational number (say) "r_1", such that $f(x) \geq r_1 \geq g(x)$ i.e.



Let $r_1, r_2, r_3, \ldots$ be the enumeration of such rational numbers, then

$$\{x;\ f(x) > g(x)\} = \bigcup_{i=1}^{\infty} \{x \in E;\ f(x) \geq r_i \geq g(x)\}$$

$$= \bigcup_{i=1}^{\infty} \left(\{x \in E;\ f(x) \geq r_i\} \cap \{x \in E;\ g(x) \leq r_i\}\right)$$

$$= \bigcup_{i=1}^{\infty} \left(\{x;\ f(x) \in [r_i, \infty)\} \cap \{x;\ g(x) \in (-\infty, r_i]\}\right)$$

$$= \bigcup_{i=1}^{\infty} \left(f^{-1}[r_i, \infty) \cap g^{-1}(-\infty, r_i]\right)$$

$$= \text{measurable set} \quad \left(\begin{array}{l}\text{because } f^{-1}[r_i, \infty) \\ \text{and } g^{-1}(-\infty, r_i] \\ \text{are measurable.}\end{array}\right)$$

Hence $\{x \in E : f(x) > g(x)\}$ is measurable set.

(ii) We have to show that $\{x \in E;\ f(x) \leq g(x)\}$ is measurable set.
For this since $\{x \in E;\ f(x) > g(x)\}$ is measurable.

$\Rightarrow \{x;\ f(x) > g(x)\}^c$ is measurable.

$\Rightarrow \{x \in E;\ f(x) \leq g(x)\}$ is measurable set

Next we have to show that
$\{x \in E;\ f(x) = g(x)\}$ is measurable.

For this since $A = \{x;\ f(x) \geq g(x)\}$ and $B = \{x;\ f(x) \leq g(x)\}$ are measurable sets.

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$\Rightarrow A \cap B = \{x; f(x) \geq g(x)\} \cap \{x; f(x) \leq g(x)\}$

$\Rightarrow A \cap B = \{x; f(x) = g(x)\}$

$\Rightarrow A \cap B$ is measurable

$\Rightarrow \{x; f(x) = g(x)\}$ is measurable.

Theorem:-

Let f, g be two measurable functions on set E, then show that

(i) $c-f$ (ii) $f+g$ (iii) f^2 (iv) $f \cdot g$

(v) $|f|$ (vi) $f \vee g$ (vii) $f \wedge g$

(viii) f^+ (ix) f^- (x) cf

(xi) f/g ; $g \neq 0$ all are measurable.

where c is any constant.

Proof:-

(i) To show that "$c-f$" is measurable.

Let us consider

$\{x \in E ; c - f(x) \geq a\} = \{x \in E ; f(x) \leq c - a\}$; $a \in \mathbb{R}$

$\{x \in E ; c - f(x) \geq a\} =$ measurable (because "f" is measurable)

$\Rightarrow$ "$c-f$" is measurable.

(ii) To show that "$f+g$" is measurable

Let us consider

$\{x : f+g \geq a\} = \{x ; f(x) \geq a - g(x)\}$; $a \in \mathbb{R}$

Since f is measurable and g is measurable.

$\Rightarrow a - g(x)$ is measurable (by proof (i))

$\Rightarrow$ "f" and "$a-g(x)$" are measurable functions

$\Rightarrow \{x \in E : f(x) \geq a - g(x)\}$ is measurable set.

$\Rightarrow \{x \in E : f(x) + g(x) \geq a\}$ is measurable.

$\Rightarrow$ "$f+g$" is measurable.

We will show that "f^2" is measurable.
Let us consider the set
$\{x : f^2(x) \leq a\}$ where $a \in \mathbb{R}$
Now if $a < 0 \Rightarrow \{x \in E ; f^2(x) \leq a\} = \phi =$ measurable

$\Rightarrow$ for $a < 0$, "f^2" is measurable.

if $a \geq 0$, then $\{x ; f^2 \leq a\} = \{x ; -\sqrt{a} \leq f \leq \sqrt{a}\}$

$$= f^{-1}[-\sqrt{a}, \sqrt{a}]$$

$=$ measurable.

So $\forall$ $a \in \mathbb{R}$, the set $\{x \in E, f^2 \leq a\}$ is measurable.

$\Rightarrow f^2$ is measurable.

We have to show that "$f \cdot g$" is measurable.
For this since

$$(f+g)^2 = f^2 + g^2 + 2f \cdot g$$

$$2f \cdot g = (f+g)^2 - f^2 - g^2$$

$$f \cdot g = \frac{1}{2}\left[(f+g)^2 - (f^2 + g^2)\right]$$

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Now since "$f+g$" is measurable

$\Rightarrow (f+g)^2$ is measurable.

Also f and g are measurable.

$\Rightarrow f^2$ and g^2 are measurable.

$\Rightarrow f^2+g^2$ is measurable.

So $(f+g)^2$ and f^2+g^2 are measurable.

$\Rightarrow (f+g)^2-(f^2+g^2)$ is measurable.

Thus

$\Rightarrow f \cdot g$ is measurable.

v) We will show that $|f|$ is measurable.

For this let us consider the set

$\{x\in E ; |f(x)| < a\}$ where $a\in\mathbb{R}$.

if $a<0 \Rightarrow \{x : |f(x)| < a\} = \Phi$ = measurable

if $a\geq 0 \Rightarrow \{x\in E : |f(x)| < a\} = \{x : -a < f(x) < a\}$

$= f^{-1}(-a, a)$

= measurable

$\Rightarrow |f|$ is measurable.

vi) We will show that $f\vee g = \max\{f, g\}$ is measurable.

For this since

$\max\{f, g\} \geq f$ — (i) , $\max\{f, g\} \geq g$ — (ii)

Adding (i) and (ii), we get

$2\max\{f, g\} \geq f+g$

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$$\max\{f,g\} \geq \frac{1}{2}\{f+g\}$$

Since f and g are measurable

$\Rightarrow$ "$f+g$" is measurable

So $\max\{f,g\}$ is measurable

Thus

"$f \vee g$" is measurable.

We need to show that $f \wedge g = \min\{f,g\}$ is measurable.

Since

$\min\{f,g\} \leq f$ —(i) and $\min\{f,g\} \leq g$ —(ii)

Adding eq(i) and (ii), we get

$$2\min\{f,g\} \leq f+g$$

$$\min\{f,g\} \leq \frac{1}{2}\{f+g\}$$

Since f and g are measurable.

$\Rightarrow$ $f+g$ is measurable.

$\Rightarrow$ $\min\{f,g\}$ is measurable.

So $f \wedge g$ is measurable.

We have to show that f^+ is measurable.

For this since

$$f^+ = \max\{f, 0\}$$

where "f" and "$g=0$" both are measurable

$\Rightarrow$ $\max\{f,0\}$ is also measurable.

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$\Rightarrow f^+$ is measurable.

(ix) we have to show that f^- is measurable.

For this since

$f^- = \min\{f, 0\}$

where "f" and "$g=0$" are measurable.

$\Rightarrow \min\{f, 0\}$ is measurable.

$\Rightarrow f^-$ is measurable.

x) we have to show that "cf" is measurable, where "c" is any constant.

For this if $c=0 \Rightarrow cf = 0.f = 0$

$\Rightarrow cf = 0$ = constant function

= measurable

if $c>0$, then consider the set

$\{x; cf(x) \geq a\} = \{x; f(x) \geq a/c\}$

= measurable.

$\Rightarrow cf$ is measurable.

if $c<0$, then

$\{x; cf(x) \geq a\} = \{x; f(x) \leq a/c\}$

= measurable.

$\Rightarrow cf$ is measurable.

So in each case "cf" is measurable.

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We have to show that "f/g" is measurable.

If $g(x) \neq 0$, then $\frac{1}{g(x)}$ is measurable.

Let $A = \left\{ x : \frac{1}{g(x)} > a \right\}$

Case-I:- If $a = 0$ then

$$A = \left\{ x ; \frac{1}{g(x)} > 0 \right\}$$ $\left\{ \text{if } g(x) > 0 \text{ then } \frac{1}{g(x)} > 0 \right.$

$$A = \{ x : g(x) > 0 \}$$

Since $g(x)$ is measurable.

$\Rightarrow$ A is measurable.

Case-II. If $a < 0$, then

$$A = \left\{ x ; \frac{1}{g(x)} > a \right\}$$

$$A = \left\{ x : g(x) < \frac{1}{a} \right\}$$

$$A = \{ x : g(x) > 0 \} \cup \{ x : g(x) > a \}$$

Since g is measurable

$\Rightarrow$ A is measurable

Case-III:- If $a > 0$, then

$$A = \left\{ x : \frac{1}{g(x)} > a \right\}$$

$$A = \left\{ x : g(x) < \frac{1}{a} \right\}$$

$$A = \left\{ x : g(x) < \frac{1}{a} \right\} \setminus \{ x : g(x) < 0 \}$$

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Since g is measurable.
$\Rightarrow$ A is measurable.

So $\frac{1}{g(x)}$ is measurable.

and $f(x)$ is measurable.

Thus f/g is measurable.

Theorem:- (a)
let $\{f_n(x)\}$ be a sequence of functions and let $F(x) = \text{Sup}\{f_n(x)\}$, then $\{x : F(x) > k\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

Proof:-
Let $A = \{x : F(x) > k\}$
and $B = \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

we need to show that $A = B$.

For this let $x_0 \in A$
$\Rightarrow F(x_0) > k$ —— (i)
we show that $x_0 \in B$
i.e. we show that $f_n(x_0) > k$ for some n
Assume on contrary that
$f_n(x_0) \leq k \quad \forall n.$

$\Rightarrow f_n(x_0) \leq k \quad \forall n$

$\Rightarrow F(x_0) \leq k$
which is contradiction to eq (i)

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So $f_n(x_0) > k$ for some n.

$\Rightarrow x_0 \in \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

$\Rightarrow x_0 \in B$

Thus $A \subseteq B$ —— (a)

Again let $x_0 \in B$

$\Rightarrow x_0 \in \bigcup_{n=1}^{\infty} \{x : f_n(x_0) > k\}$

$\Rightarrow f_n(x_0) > k \quad \forall n$

Since $F(x) = \sup \{f_n(x)\}$

$\Rightarrow k < f_n(x_0) \leq F(x_0)$

$\Rightarrow k < F(x_0)$

$\Rightarrow F(x_0) > k$

$\Rightarrow x_0 \in \{x : F(x) > k\}$

$\Rightarrow x_0 \in A$

Thus $B \subseteq A$ —— (b)

From eq (a) and (b), we get

$A = B$

Hence

$\{x : F(x) > k\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > k\}$

Proved.

Theorem:-(b)

let $f_n(x)$ be a sequence of functions and let $G(x) = \inf\{f_n(x)\}$, then $\{x : G(x) > k\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$.

Proof:-

Let $A = \{x : G(x) > k\}$

and $B = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

We need to show that $A = B$.

i.e. $\{x : G(x) > k\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

let $x_0 \in A \Rightarrow G(x_0) > k$

$\Rightarrow k < G(x_0) \leq f_n(x_0) \quad \forall n$ (by def of inf)

$\Rightarrow k < f_n(x_0) \quad \forall n$

$\Rightarrow f_n(x_0) > k \quad \forall n$

$\Rightarrow x_0 \in \{x : f_n(x) > k\}$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

$\Rightarrow x_0 \in B$

Thus $A \subseteq B$ ——— (i)

Again let $x_0 \in B$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

$\Rightarrow f_n(x_0) > k \quad \forall n$

$\Rightarrow \inf\{f_n(x_0)\} > \inf(k) \quad \forall n$

$\Rightarrow G(x_0) > k$

$\Rightarrow x_0 \in \{x : G(x) > k\}$

$\Rightarrow x_0 \in A$

Thus $B \subseteq A$ —— (ii)

From eq (i) and (ii), we get
$A = B$

Hence $\{x : G(x) > k\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > k\}$

proved!

Theorem:-

If $\{f_n(x)\}$ is a sequence of measurable function, then prove that $F(x) = \sup\{f_n(x)\}$ and $G(x) = \inf\{f_n(x)\}$ are measurable functions.

Proof:-

Theorem (a), Theorem (b).

Theorem:-

Let $\{f_n(x)\}$ be a monotonic sequence of measurable functions, such that $\lim_{n\to\infty} f_n(x) = f(x)$, then show that $f(x)$ is measurable.

Proof:-

Since we know that

"Supremum of measurable function is measurable" —— ①

"Infimum of measurable function is measurable" —— ②

There are two case

Case-I:- when $\{f_n(x)\}$ is monotonic increasing

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sequence i.e. $f_1(x) \le f_2(x) \le f_3(x) \le \cdots$
then we know that a monotonic increasing sequence always converges to its lub (supremum).

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) = \text{supremum of } f_n(x)$$

So using eq ① $f(x)$ is measurable.

Case-II:- When $\{f_n(x)\}$ is monotonic decreasing sequence i.e.

$$f_1(x) \ge f_2(x) \ge f_3(x) \ge \cdots$$

then we know that a monotonic decreasing sequence always converges to its glb (infimum).

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) = \text{glb of } f_n(x).$$

So using eq ② $f(x)$ is measurable.

Hence in both cases $f(x)$ is measurable.

Definition:-

Limit Superior of a sequence:-

A number $\bar{l}$ is called the limit superior, greatest limit or upper limit of a sequence $\langle a_n \rangle$ iff infinitly many terms of the sequence are greater than $\bar{l}-\epsilon$, while only a finitly terms are greater than $\bar{l}+\epsilon$ for any

$\epsilon > 0$. We denote the limit superior by $\lim \sup (a_n)$ or $\overline{\lim}\, a_n$.

Limit Inferior of a sequence:-

A number l is called limit inferior, least limit or lower limit of a sequence $\langle a_n \rangle$ iff infinitly many terms of the sequence are less than $l + \epsilon$, while only a finite number of terms are less than the $l - \epsilon$, for any $\epsilon > 0$. We denote the limit inferior of $\langle a_n \rangle$ by $\lim \inf (a_n)$ or $\underline{\lim}\, a_n$.

Note:- If infinitly many terms of $\langle a_n \rangle$ exceeds any positive number M. We write $\overline{\lim}\, a_n = \infty$ and if infinitly many terms are less than $-M$, where M is any positive number. We write $\underline{\lim}\, a_n = -\infty$.

Another Definition:-

Limit Superior:-

If $\{a_n\}^{\infty}$ is a sequence of real numbers, we define a sequence $\{S_n\}_{n=1}^{\infty}$ as;

$$S_n = \sup_{k \geq n} \{a_k\}$$

$$\Rightarrow S_1 = \sup \{a_1, a_2, a_3, \cdots\}$$
$$S_2 = \sup \{a_2, a_3, a_4, \cdots\}$$
$$S_3 = \sup \{a_3, a_4, a_5, \cdots\}$$
$$\vdots$$

Here $S_1 \geq S_2 \geq S_3 \geq \cdots$

So $\langle S_n \rangle$ is decreasing sequence, which is clear from the construction of $\langle S_n \rangle$. Then the $\lim\limits_{n\to\infty} S_n$ is called the limit superior of $\langle a_n \rangle$ and it is denoted by $\overline{\lim}\, a_n$ or $\lim\sup(a_n)$.

"Now note that if we are interested to find the limit supremum of $\langle a_n \rangle$ i.e. $\overline{\lim}\, a_n = ?$ then $\overline{\lim}\, a_n = \lim\limits_{n\to\infty} S_n$ — (1)

but since $\langle S_n \rangle$ is a monotonic decreasing sequence and we know that a monotonic decreasing sequence always tends to it glb.

So $\lim\limits_{n\to\infty} S_n = glb\{S_n\}$

So eq (1) $\Rightarrow \overline{\lim}\, a_n = \lim\limits_{n\to\infty} S_n = glb\{S_n\}$

$\Rightarrow \boxed{\overline{\lim}\, a_n = glb\{S_n\}}$

Limit Inferior:-

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, we define a sequence $\{T_n\}_{n=1}^{\infty}$ as;

$$T_n = \inf_{k\geq n}\{a_k\}$$

$$\Rightarrow \quad T_1 = \inf\{a_1, a_2, a_3, \cdots\}$$
$$T_2 = \inf\{a_2, a_3, a_4, \cdots\}$$
$$T_3 = \inf\{a_3, a_4, a_5, \cdots\}$$
$$\vdots \qquad \vdots \qquad \vdots$$

Clearly from the construction of

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$\langle T_n \rangle$, we see that

$T_1 \le T_2 \le T_3 \le \cdots$

$\Rightarrow \langle T_n \rangle$ is an increasing sequence and the "$\lim_{n\to\infty} T_n$" is called the limit inferior of the original sequence $\langle a_n \rangle$ and it is denoted by $\underline{\lim}\, a_n$ or $\lim \inf (a_n)$.

"And note that $\underline{\lim}\, a_n = \lim_{n\to\infty} T_n$ —— (2)

Now since $\langle T_n \rangle$ is increasing sequence.

So $\lim_{n\to\infty} T_n = \text{lub}\{T_n\}$

eq (2) $\Rightarrow$ $\underline{\lim}\, a_n = \lim_{n\to\infty} T_n = \text{lub}\{T_n\}$

$$\Rightarrow \boxed{\underline{\lim}\, a_n = \text{lub}\{T_n\}}$$

Example:- $\langle a_n \rangle = \langle \frac{1}{n} \rangle$; find (i) $\overline{\lim}\, a_n = ?$ and (ii) $\underline{\lim}\, a_n = ?$

Solution:-

i) To find $\overline{\lim}\, a_n = ?$

Let $S_1 = \text{Sup}\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} = 1$

$S_2 = \text{Sup}\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} = \frac{1}{2}$

$\vdots$

$S_n = \text{Sup}\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \cdots\} = \frac{1}{n}$

So $\overline{\lim}\, a_n = \lim_{n\to\infty} \langle S_n \rangle = \lim_{n\to\infty} \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \rangle$

$= \lim_{n\to\infty} \langle \frac{1}{n} \rangle = 0$

$$\Rightarrow \boxed{\overline{\lim}\, a_n = 0} \quad \text{—— (1)}$$

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ii) To find $\underline{\lim}\, a_n = \lim\inf(a_n) = ?$

let $T_1 = \inf\{1, 1/2, 1/3, \ldots\} = 0$

$T_2 = \inf\{1/2, 1/3, 1/4, \ldots\} = 0$

$T_3 = \inf\{1/3, 1/4, 1/5, \ldots\} = 0$

$\vdots \qquad \vdots \qquad \vdots$

$T_n = \inf\{1/n, 1/(n+1), 1/(n+2), \ldots\} = 0$

So $\underline{\lim}\, a_n = \lim_{n\to\infty} T_n = \lim_{n\to\infty} \{0\} = 0$

$\Rightarrow \boxed{\underline{\lim}\, a_n = 0}$ —— (2)

From eq (1) and (2), we get

$\overline{\lim}\, a_n = \underline{\lim}\, a_n = 0$

Thus $\lim_{n\to\infty} a_n = 0$ (exist)

Example:-

let $\langle a_n \rangle = 1-(-1)^n$

$= \langle 2, 0, 2, 0, 2, \ldots \rangle$

Find $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

Solution:-

i) To find $\overline{\lim}\, a_n$, let

$S_1 = \sup\{2, 0, 2, 0, 2, \ldots\} = 2$

$S_2 = \sup\{0, 2, 0, 2, 0, \ldots\} = 2$

$S_3 = \sup\{2, 0, 2, 0, 2, \ldots\} = 2$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$S_n = \sup\{1-(-1)^n\} = 2$

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$\Rightarrow \overline{\lim}\, a_n = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \{2\} = 2$

$\Rightarrow \boxed{\overline{\lim}\, a_n = 2}$ —— ①

Similarly to find $\underline{\lim}\, a_n =$.

Let $T_1 = \inf\{2, 0, 2, 0, ----\} = 0$

$T_2 = \inf\{0, 2, 0, 2, 0, ----\} = 0$

$T_3 = \inf\{2, 0, 2, 0, ----\} = 0$

$\vdots$

$T_n = \inf\{1-(-1)^n\} = 0$

So $\underline{\lim}\, a_n = \lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty} (0) = 0$

$\boxed{\underline{\lim}\, a_n = 0}$ —— ②

From eq ① and ②, we get

$\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$

$\Rightarrow \lim\limits_{n\to\infty} \langle a_n \rangle$ does not exist.

Example:-

Let $\langle a_n \rangle = \{2, -2, -1, 1, -1, 1, -1, -----\}$

Find $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

Solution:-

To find $\overline{\lim}\, a_n$.

Let $S_1 = \sup\{2, -2, -1, 1, -1, 1, -1, ----\} = 2$

$S_2 = \sup\{-2, -1, 1, -1, 1, -1, ----\} = 1$

$S_3 = \sup\{-1, 1, -1, 1, -1, ------\} = 1$

$\overline{\lim}\, a_n = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \{2, -2, -1, 1, -1, 1, -1, ----\}$

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$$\boxed{\overline{\lim}\, a_n = 1} \quad \text{--- (1)}$$

Also let $T_1 = \inf\{2,-2,-1,1,-1,1,-1,\ldots\} = -2$
$T_2 = \inf\{-2,-1,1,-1,1,-1,\ldots\} = -2$
$T_3 = \inf\{-1,1,-1,1,-1,\ldots\} = -1$
$T_4 = \inf\{1,-1,1,-1,\ldots\} = -1$
$T_5 = \inf\{-1,1,-1,1,-1,\ldots\} = -1$

$\Rightarrow \underline{\lim}\, a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\{2,-2,-1,1,-1,1,-1,\ldots\}$
$= -1$

$$\Rightarrow \boxed{\underline{\lim}\, a_n = -1} \quad \text{--- (2)}$$

From eq. (1) and (2), we get
$\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$

So
$\Rightarrow \lim_{n\to\infty} a_n$ does not exist.

Theorem:-

let $\{f_n(x)\}$ be a sequence of measurable functions then $\lim\sup f_n = \overline{\lim}\, f_n$ and $\lim\inf f_n = \underline{\lim}\, f_n$ are measurable functions.

Proof:-

We need to show that $\overline{\lim}\, a_n$ is measurable function.

We proceed as:

Let $S_n(x) = \sup_{k\geq n}\{f_k(x)\}$

i.e $S_1(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$
$S_2(x) = \sup\{f_2(x), f_3(x), f_4(x), \ldots\}$
$S_3(x) = \sup\{f_3(x), f_4(x), f_5(x), \ldots\}$
$\vdots$

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$\Rightarrow \quad S_1(x) \geq S_2(x) \geq S_3(x) \geq \cdots$

Since we know that $S_n(x)$ is a decreasing sequence, so it always converges to its glb.

$\Rightarrow \quad \lim_{n\to\infty} S_n(x) = glb\{S_n(x)\}$ —— (i)

But by definition of limit superior i.e. $\overline{\lim} f_n$ we know

$$\overline{\lim} f_n = \lim_{n\to\infty}\{S_n(x)\} \text{ —— (ii)}$$

From eq (i) and (ii), we get

$$\overline{\lim} f_n = \lim_{n\to\infty} S_n(x) = glb\{S_n(x)\} \text{ —— } \circledast$$

Since we see that "$S_n(x)$" is a sequence of measurable functions.

$\Rightarrow glb\{S_n(x)\}$ is also measurable.

So eq $\circledast \Rightarrow \overline{\lim} f_n$ is measurable.

Next, we show that $\underline{\lim} f_n$ is measurable

let $T_1 = \inf\{f_1(x), f_2(x), f_3(x), \cdots\}$

$T_2 = \inf\{f_2(x), f_3(x), f_4(x), \cdots\}$

$T_3 = \inf\{f_3(x), f_4(x), f_5(x), \cdots\}$

$\vdots \quad \vdots \quad \vdots$

Since $T_1 \leq T_2 \leq T_3 \leq \cdots$

i.e. $\langle T_n \rangle$ is an increasing sequence.

So it always converges to its lub.

So $\lim_{n\to\infty} T_n(x) = lub\{T_n(x)\}$ —— (iii)

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But from definition of $\underline{\lim} f_n$, we know that

$$\underline{\lim} f_n = \lim_{n\to\infty} T_n(x) \text{ --- (iv)}$$

Combining eq (iii) and (iv), we get

$$\underline{\lim} f_n = \lim_{n\to\infty} T_n(x) = \text{lub}\{T_n(x)\} \text{ --- } (**)$$

Since we see that $\{T_n(x)\}$ is a sequence of measurable functions

$\Rightarrow$ $\text{lub}\{T_n(x)\}$ is also measurable.

So eq $(**)$ $\Rightarrow$ $\underline{\lim} f_n$ is measurable.

Hence

$\overline{\lim} f_n$ & $\underline{\lim} f_n$ are measurable functions.

Theorem:-

Let $\{f_n(x)\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$, then show that $f(x)$ is measurable.

Proof:-

Since it is given that

$\lim_{n\to\infty} f_n(x) = f(x)$ (exists)

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) = \overline{\lim} f_n = \underline{\lim} f_n \text{ --- } (*)$$

Since we know that for the sequence $\{f_n(x)\}$ $\overline{\lim} f_n(x)$ and $\underline{\lim} f_n(x)$ are always measurable.

So eq (★) $\Rightarrow$ $-f(x)$ is measurable

Characteristic Function:-

let $A \subset X$, then characteristic function on A (or w.r.t A) is defined as:

$$\chi_A : X \longrightarrow \{0,1\} \text{ by}$$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Theorem:-

A set A is measurable iff χ_A is measurable.

Proof:-

Let A is measurable.

we need to show that χ_A is measurable.

For this let us take any arbitrary open set "G" in $\mathbb{R}$, we will show that $\chi_A^{-1}(G)$ is measurable.

i) if $1 \in G$, $0 \notin G$

$\Rightarrow \chi_A^{-1}(G) = A$ = measurable.

ii) if $1 \notin G$, $0 \in G$

$\Rightarrow \chi_A^{-1}(G) = A^c$ = measurable (because the complement of measurable set is measurable)

iii) if $1, 0 \notin G$

$\Rightarrow \chi_A^{-1}(G) = \phi$ = measurable iv, if $1, 0 \in G$

$$\Rightarrow \chi_A^{-1}(G) = X = \text{measurable}.$$

So in each case the inverse image of an arbitrary open set G; i-e $\chi_A^{-1}(G)$ is measurable.

$\Rightarrow$ χ_A is measurable.

Conversely;
let χ_A is measurable.
we need to show that the set A is measurable.
For this let us take any arbitrary open set G in $\mathbb{R}$,
Since χ_A is measurable.

$\Rightarrow$ $\chi_A^{-1}(G)$ is also measurable.

Since $\chi_A^{-1}(G)$ is measurable $\forall$ open set G in $\mathbb{R}$.
So let us take G such an open set, which contains "1" but not "0", then

$$\chi_A^{-1}(G) = \text{measurable}.$$

but $\chi_A^{-1}(G) = A$

$\Rightarrow$ A is measurable.
Which is the required result.

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Question:-

Give an example of a function f, such that $|f|$ is measurable, but f is not measurable.

Proof:-

Let E be a non-measurable subset of $\mathbb{R}$, consider the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{if } x \notin E \end{cases}$$

Here we can find an open set G in $\mathbb{R}$ such that $1 \in G$ and $-1 \notin G$.

$\Rightarrow f^{-1}(G) = E$ = non-measurable.

$\Rightarrow f^{-1}(G)$ is non-measurable

So we have find an open set G in $\mathbb{R}$ such that $f^{-1}(G)$ is not measurable

$\Rightarrow f$ is not measurable

But consider $|f(x)| = g(x) = 1 \quad \forall \; x \in \mathbb{R}$ which is constant and it will be always measurable.

i.e. $|f|$ is measurable.

Theorem:-

Show that every continuous function is measurable.

Proof:-

Since f is continuous.
Let us take an arbitrary open set G in $\mathbb{R}$.
So by definition of continuity $f^{-1}(G)$ will be open in the domain of f, but we know that every open set is measurable.

$\Rightarrow f^{-1}(G)$ is measurable.

But G was an arbitrary open set.
So for any open set G, $f^{-1}(G)$ is measurable.

$\Rightarrow f$ is measurable.

Theorem:-
Let f be differentiable function then prove that f' is measurable.
Proof:-
Let us define a sequence $\{g_n(x)\}$ of function as;

$$g_n(x) = \frac{f(x+1/n) - f(x)}{1/n}$$

Clearly $\{g_n(x)\}$ is a sequence of measurable function as f is differentiable

$\Rightarrow f$ is continuous.
Since we know that "every continuous function is measurable".

So f is measurable.

Now $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \dfrac{f(x+1/n)-f(x)}{1/n}$

$= f'(x)$

$\Rightarrow f'(x)$ is the limit of sequence of measurable function.

i.e. $f'(x)$ is measurable.

proved!

Theorem:-

If $f: \mathbb{R} \to \mathbb{R}$ is continuous a.e, then show that f is lebesgue measurable function.

Proof:-

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous a.e. We need to show that f is measurable.

Let $E = \{x \in \mathbb{R} : f$ is continuous at $x\}$

then

$\mu(E^c) = 0$

$\Rightarrow E$ and E^c are measurable sets.

Let "O" be any open set in $\mathbb{R}$, then $f^{-1}(O) \cap E^c$ is measurable because it is null set

Since $f_{\setminus E} : E \to \mathbb{R}$ is continuous function,

so $f_{\setminus E}^{-1}(O)$ is an open set in E.

But

$f_{\setminus E}^{-1}(O) = f^{-1}(O) \cap E$

So $f^{-1}(O) \cap E$ is open in E.
So there exists an open set V of $\mathbb{R}$ such that
$f^{-1}(O) \cap E = V \cap E$
But V and E are measurable, so $V \cap E$ is measurable.

$\Rightarrow f^{-1}(O) \cap E$ is measurable

Hence $f^{-1}(O) \cap E$ and $f^{-1}(O) \cap E^c$ are measurable sets.

Therefore; $f^{-1}(O) = (f^{-1}(O) \cap E) \cup (f^{-1}(O) \cap E^c)$ is measurable.

$\Rightarrow f$ is measurable function.

Question:-

Let (X, S, μ) be a measure space and $f: X \longrightarrow \mathbb{R}$ is measurable function and $g: \mathbb{R} \to \mathbb{R}$ is continuous function, then show that $g \circ f$ is a measurable function.

Solution:-

We show that $g \circ f : X \longrightarrow \mathbb{R}$ is measurable function.

Let "O" be any open set in $\mathbb{R}$.
We need to show that $(g \circ f)^{-1}(O)$ is measurable set.

Since "O" is open set in $\mathbb{R}$ and $g: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

So $g^{-1}(O)$ is open set in $\mathbb{R}$.

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But $f: X \rightarrow \mathbb{R}$ is measurable.

So $f^{-1}(g^{-1}(0))$ is measurable set.

i.e. $(g \circ f)^{-1}(0)$ is measurable set.

Hence

$\Rightarrow$ $g \circ f$ is measurable function.

Question:-

Let $f: [1, 10] \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 & \text{if } x \in [1, 5] \\ 3 & \text{if } x \in (5, 8) \\ -5 & \text{if } x \in [8, 10] \end{cases}$$

clearly $f(x)$ is not continuous.

Show that $f(x)$ is measurable function.

Solution:-

Let G be any open set in $\mathbb{R}$.
We need to show that $f^{-1}(G)$ is an open set.

We discuss the following cases:

Case-I:-

If $2, 3, -5 \notin G$, then
$f^{-1}(G) = \phi$ = measurable set.

Case-2:-

If $2 \in G$ and $3, -5 \notin G$, then
$f^{-1}(G) = [1, 5]$, which is measurable.

Case-3:-

If $2, 3 \in G$ and $-5 \notin G$, then
$f^{-1}(G) = [1, 5] \cup (5, 8) = [1, 8)$, which is measurable.

Case - 4 :-

If $2, -5 \in G$ and $3 \notin G$, then

$f^{-1}(G) = [1, 5] \cup [8, 10]$ = measurable

Case - 5 :-

If $2, 3 \notin G$ and $-5 \in G$, then

$f^{-1}(G) = [8, 10]$ = measurable

Case - 6 :-

If $2, -5 \notin G$ and $3 \in G$, then

$f^{-1}(G) = (5, 8)$ = measurable

Case - 7 :-

If $2, 3, -5 \in G$, then

$f^{-1}(G) = [1, 5] \cup (5, 8) \cup [8, 10]$

$f^{-1}(G) = [1, 10]$ = measurable.

Hence in every case $f^{-1}(G)$ is measurable function.

So f is measurable function.

Question:-

Let $f: R \to \mathbb{R}$ defined by

$f(x) = x^2$ if x is irrational.

$= -5x + 2$ if x is rational.

Show that f is measurable function.

Solution:-

Let $g: R \to \mathbb{R}$ be defined by

$g(x) = x^2$

then clearly $g = f$ a.e.

But g is measurable.

g is continuous, So

So f is measurable function.

Question:-

Give an example of non-measurable function such that $|f|$ is measurable and $f^{-1}(\{a\})$ is measurable set for any $a \in \mathbb{R}$.

Solution:-

let E be non-measurable subset of $[0,1]$ and $f:[0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in E \\ -x & \text{if } x \in [0,1] \setminus E \end{cases}$$

let $f^{-1}[0,1] = E$

Since E is non-measurable.

$\Rightarrow f^{-1}[0,1]$ is non-measurable.

Thus f is non-measurable.

$\Rightarrow |f| = x$ if $x \in \mathbb{R}$

So $|f|$ is measurable.

Now we check $f^{-1}(\{a\})$ for any $a \in \mathbb{R}$.

If $a \geq 0$, then

$$f^{-1}\{a\} = a \quad \text{if } a \in E$$
$$= \phi \quad \text{if } a \notin E$$

$f\{a\} = a$ if $a \in E$
$\{a\} = f^{-1}\{a\}$

If $a \leq 0$, then

$$f^{-1}\{a\} = \{-a\} \quad \text{if } -a \in [0,1] \setminus E$$

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$f^{-1}\{-a\} = a$ if $a \in [0,1] \setminus E$

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$f^{-1}\{a\} = \phi$ if $-a \notin [0,1] \setminus E$

Hence $f^{-1}(\{a\})$ is measurable set.

Question:-

If f is measurable then the inverse image of interval (a,b) is measurable or not?

Solution:-

Given that f is measurable function and (a,b) is open interval.

We need to show that $f^{-1}(a,b)$ is measurable.

Now $(a,b) = (-\infty, b) \cap (a, \infty)$

$\Rightarrow f^{-1}(a,b) = f^{-1}[(-\infty, b) \cap (a, \infty)]$

$\Rightarrow f^{-1}(a,b) = f^{-1}(-\infty, b) \cap f^{-1}(a, \infty)$ by def:

$\Rightarrow f^{-1}(a,b) = \{x : f(x) < b\} \cap \{x : f(x) > a\}$ — ①

Since f is measurable, so $\{x : f(x) > a\}$ and $\{x : f(x) < b\}$ is also measurable.

So there intersection is also measurable.

So $f^{-1}(a,b)$ is measurable.

Hence the inverse image of open interval (a,b) is measurable.

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