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CHAPTER # 2

Semi-ring:-

Let X be a non-empty set i.e. $X \neq \phi$ and S be the collection of subsets of X, then S is said to be semi-ring if it satisfies the following conditions:

(i) $\phi \in S$.

(ii) For any $A, B \in S \Rightarrow A \cap B \in S$.

(iii) For any $A, B \in S$

$$\Rightarrow A \setminus B = \bigcup_{i=1}^{n} C_i \quad , \quad C_i \in S \; ; \; i = 1, 2, 3, \ldots, n$$

$$\text{and } C_i \cap C_j = \phi \quad ; \; j \neq i.$$

We say that (X, S) is semi-ring.

Sigma set (σ-set):-

Let (X, S) is a semi-ring and $A \subseteq X$ (not necessary that $A \in S$) then A is said to be a sigma set if $A = \bigcup_{i=1}^{\infty} C_i$, $C_i \in S$, $C_i \cap C_j = \phi$; $i \neq j$.

Theorem:-

Let (X, S) is a semi-ring and $A_1, A_2, A_3, \ldots, A_n \in S$, then show that

$$A \setminus \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} C_i \quad , \quad C_i \cap C_j = \phi \; , \; C_i \in S.$$

is σ-set.

Proof:-

We prove that by Mathematical induction:

For $n=1$:-

$A \setminus A_1$ is a σ-set . (proved)

Let us suppose that

$A \setminus \bigcup_{i=1}^{n} A_i$ is σ-set .

And we need to prove that

$A \setminus (\bigcup_{i=1}^{n+1} A_i)$ is σ-set .

Now ; $A \setminus (\bigcup_{i=1}^{n+1} A_i) = A \setminus (\bigcup_{i=1}^{n} A_i \cup A_{n+1})$

$= (A \setminus \bigcup_{i=1}^{n} A_i) \setminus A_{n+1}$

$= (\bigcup_{j=1}^{\infty} B_j) \setminus A_{n+1}$

where $B_j \in S$, $B_j \cap B_k = \phi$; $j \neq k$.

By hypothesis $A \setminus (\bigcup_{i=1}^{n} A_i)$ is a σ-set.

$= \bigcup_{j=1}^{\infty} (B_j \setminus A_{n+1})$ —— ①

$\Rightarrow B_j \in S$, $A_{n+1} \in S$

$\Rightarrow B_j \setminus A_{n+1}$ is a σ-set .

$\Rightarrow B_j \setminus A_{n+1} = \bigcup_{i=1}^{\infty} C_{ji}$; $C_{ji} \in S$

; $C_{ji} \cap C_{jk} = \phi$, $i \neq$

eq ① $\Rightarrow A \setminus (\bigcup_{i=1}^{n+1} A_i) = \bigcup_{j=1}^{\infty} (\bigcup_{i=1}^{\infty} C_{ji})$; $C_{ji} \in S$

$C_{ji} \cap C_{jk} = \phi$; $i \neq$

So, $A \setminus (\bigcup_{i=1}^{n+1} A_i)$ is a σ-set.

Thus all the result of mathematical induction are satisfied.

Hence $A \setminus (\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{\infty} C_i$; $C_i \in S$, $C_i \cap C_j = \phi$; $i \neq j$

is a δ-set.

Theorem:-

If (X, S) is semi-ring and $A_1, A_2, A_3, \ldots \in S$ then $\bigcup_{i=1}^{\infty} A_i$ is δ-set.

OR

If (X, S) is semi-ring and $\{A_i\}_{i=1}^{\infty}$ be a sequence of sets from "S". Then show that $\bigcup_{i=1}^{\infty} A_i$ is a δ-set.

Proof:-

let (X, S) be a semi-ring such that $A_1, A_2, A_3, \ldots \in S$.

we need to show that $\bigcup_{i=1}^{\infty} A_n$ is a δ-set.

let us denote $\bigcup_{i=1}^{\infty} A_i$ by A i.e. $A = \bigcup_{i=1}^{\infty} A_i$.

So we will show that A is a δ-set.

let us construct a sequence of sets as;

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus A_1 \cup A_2$$
$$B_4 = A_4 \setminus A_1 \cup A_2 \cup A_3$$
$$\vdots \qquad \vdots$$
$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

Now $A = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$; $B_i \cap B_j = \phi$; $i \neq j$

Since we know that "if (X, S) is a semi-ring and $A_1, A_2, A_3, \ldots, A_n \in S$ then

Date: (34)

$A \setminus \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} C_i \;;\; C_i \cap C_j = \phi \;;\; i \neq j$ "

So this theorem is also applicable here, because each $A_i \in S \;\forall\; i$ and each B_i is in the form of "$A_i \setminus \bigcup_j A_j$", so we can say that

$$B_i = \bigcup_{j=1}^{n} C_{ij} \;;\; C_{ij} \cap C_{ik} = \phi \;,\; C_{ij} \in S$$

So $A = \bigcup_{i=1}^{\infty}\left(\bigcup_{j=1}^{n} C_{ij}\right) \;;\; C_{ij} \cap C_{kl} = \phi$
$C_{ij} \in S$

So A is a σ-set.
Thus $\bigcup_{i=1}^{\infty} A_i$ is σ-set.

Algebra:-

If $X \neq \phi$ and F is the collection of subsets of X, then F is said to be algebra.

If the following conditions are satisfied.

(i) If $A \in F$ then $A^c \in F$
(ii) for any $A, B \in F \Rightarrow A \cap B \in F$

2^{nd} definition:-

If $X \neq \phi$ and F is the collection of subsets of X, then F is said to be algebra

If the following conditions are satisfied.

(i) If $A \in F$ then $A^c \in F$
(ii) For any $A, B \in F \Rightarrow A \cup B \in F$

Theorem:-

Show that Def:(1) and Def:(2) are equivalent.

Proof:-

Since Def:(1) ⇒ " let $X \neq \phi$ and F be any collection of subsets of X, satisfying the two conditions i-e

$$\left.\begin{array}{l}\text{(i) for any } A, B \in F \Rightarrow A \cap B \in F \\ \text{(ii) if } A \in F \text{ then } A^c \in F\end{array}\right\} \text{— (A)}$$

Def:(2) ⇒ " let $X \neq \phi$ and F be any collection of subsets of X, satisfying the two conditions i-e

$$\left.\begin{array}{l}\text{(i) for any } A, B \in F \Rightarrow A \cup B \in F \\ \text{(ii) If } A \in F \text{ then } A^c \in F\end{array}\right\} \text{— (B)}$$

let Def:(1) is true.

we need to prove ~~that~~ Def:(2).

let $A, B \in F$, we prove that $A \cup B \in F$

$\Rightarrow A, B \in F \Rightarrow A^c, B^c \in F$ (by eq (A) (ii))

$\Rightarrow A^c \cap B^c \in F$ (by eq (A) (i))

By Demorgan's law

$\Rightarrow (A \cup B)^c \in F$

$\Rightarrow ((A \cup B)^c)^c \in F$ (by eq (A) (ii))

$\Rightarrow A \cup B \in F$

Now, let Def:(2) is true.

we need to prove Def:(1)

let $A, B \in F$, we prove that $A \cap B \in F$

$\Rightarrow A, B \in F \Rightarrow A^c, B^c \in F$ (by eq (B) (ii))

$\Rightarrow A^c \cup B^c \in F$ (by eq ⓑ (i))

By Demorgan's law

$\Rightarrow (A \cap B)^c \in F$

$\Rightarrow ((A \cap B)^c)^c \in F$ (by eq ⓑ (iii))

$\Rightarrow A \cap B \in F$

which is the required result.

Note:-

From the definition of algebra it is clear that $\phi, X \in F$ and $A \setminus B = A \cap B^c \in F$ always.

↳ Consequences from the Def: of Algebra

Theorem:-

If (X, F) is an algebra then

(i) $\phi, X \in F$

(ii) Intersection and union of finite number of sets from F belong to F.

(iii) (X, F) is a semi-ring.

Proof:-

(i) Since $X \neq \phi$, So let $A \in F$.

by def: of algebra $\Rightarrow A^c \in F$

$\Rightarrow A \cup A^c \in F$ ($\because$ by def: of algebra)

$\Rightarrow \boxed{X \in F}$

Now since $X \in F$, so $X^c \in F$

$\Rightarrow \boxed{\phi \in F}$

So $\phi, X \in F$

ii) Next let $A_1, A_2, A_3, \dots, A_n \in F$, then for
$A_1, A_2 \in F \Rightarrow A_1 \cap A_2 \in F$ (by def of algebra)
$\Rightarrow A_1 \cap A_2 \cap A_3 \in F$
$\Rightarrow A_1 \cap A_2 \cap A_3 \cap A_4 \in F$

Continuing on the same way we can write that $A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n \in F$
$\Rightarrow \bigcap_{i=1}^{n} A_i \in F$

Similarly, let $A_1, A_2, A_3, \dots, A_n \in F$, then
for $A_1, A_2 \Rightarrow A_1 \cup A_2 \in F$ (by def. of algebra)
$\Rightarrow A_1 \cup A_2 \cup A_3 \in F$
$\Rightarrow A_1 \cup A_2 \cup A_3 \cup A_4 \in F$

Continuing on the same way we can write that $A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n \in F$
$\Rightarrow \bigcup_{i=1}^{n} A_i \in F$.

iii) We show that (X, F) is semi-ring.
(a) $\phi \in F$ (proved)
(b) Also by def: of algebra, for any
$A, B \in F \Rightarrow A \cap B \in F$.

(c) Next we show that for any $A, B \in F$;
$A \setminus B = \bigcup_{i=1}^{n} C_i$; $C_i \in F$; $C_i \cap C_j = \phi$; $i \neq j$

Now, since $A \setminus B = A \cap B^c$

$\Rightarrow A, B^c \in F$
$\Rightarrow A \cap B^c \in F$ (by def of algebra)
$\Rightarrow A \setminus B \in F$

Mon Tue Wed Thu Fri Sat

Date: (21)

$\Rightarrow A \setminus B = A \cap B^c$

$= A \cap B^c \cup \phi \cup \phi \cup \cdots \cup \phi$

$= C_1 \cup C_2 \cup \cdots \cup C_n$; where $C_1 = A \cap B^c$

and $C_i = \phi$

$\forall\ i = 2, 3, \ldots, n$

$A \setminus B = \dot{\bigcup\limits_{i=1}^{n}} C_i$; $C_i \in F$; $C_i \cap C_j = \phi$; $i \neq j$

Thus (X, F) is a semi-ring.

Examples:-

1) let $X \neq \phi$ and $F = \{\phi, X\}$, show that F is an algebra.

Sol:-

Since $X \in F$

$\Rightarrow X^c \in F$ (by def. of algebra)

$\Rightarrow \boxed{\phi \in F}$

Now since $\phi \in F$

$\Rightarrow \phi^c \in F$ (by def. of algebra)

$\Rightarrow \boxed{X \in F}$

$\Rightarrow \phi \cap X = \phi \in F$

$\Rightarrow \phi \cup X = X \in F$

Thus F is an algebra.

And this is the smallest algebra of all possible algebras defined on X.

2) let $X \neq \phi$ and $A \subseteq X$ and $F = \{\phi, X, A, A^c\}$, then show that

F is an algebra.

Sol:-

Since $\phi \in F$

$\Rightarrow \phi^c \in F$ (by def of algebra)

$\Rightarrow \boxed{X \in F}$

Now since $X \in F$

$\Rightarrow X^c \in F$ (by def of algebra)

$\Rightarrow \boxed{\phi \in F}$

And $\boxed{A \in F}$

$\Rightarrow \boxed{A^c \in F}$

$\Rightarrow X \cup \phi = X \in F$ $\Rightarrow X \cap \phi = \phi \in F$

$\Rightarrow X \cup A = X \in F$ $\Rightarrow X \cap A = A \in F$

$\Rightarrow X \cup A^c = X \in F$ $\Rightarrow X \cap A^c = A^c \in F$

$\Rightarrow A \cup \phi = A \in F$ $\Rightarrow A \cap \phi = \phi \in F$

$\Rightarrow \phi \cup A^c = A^c \in F$ $\Rightarrow \phi \cap A^c = \phi \in F$

$\Rightarrow A \cup A^c = X \in F$ $\Rightarrow A \cap A^c = \phi \in F$.

Thus F is an algebra.
And F is the smallest algebra containing A.

3) $P(X) = \{$All possible subsets of $X\}$ and $X \neq \phi$, then show that $P(X)$ is an algebra.

Sol:-

$P(X)$ is also an algebra and this is the largest algebra of all possible algebras defined on X.

Date:

(40)

4) Let $X \neq \phi$ and F be the collection of pairwise disjoint subset of X. Show that F is semi-ring.

Sol:-

(i) Since for any set $A \in F$, we can write $A \cap \phi = \phi \quad \forall A \subseteq X$.

So $\phi \in F$.

(ii) Next, we show that for any $A, B \in F$ $A \cap B \in F$.

For this, since $A, B \in F$, so we can write

$$A \cap B = A \in F \text{ if } A = B$$
$$A \cap B = \phi \in F \text{ if } A \neq B.$$

So in both cases "$A \cap B \in F$".

(iii) Let $A, B \in F \Rightarrow A \setminus B = \phi$ if $A = B$.

$\Rightarrow A \setminus B = A$ if $A \neq B$.

So, $\phi, A \in F$

So in both cases "$A \setminus B \in F$".

Since $A \setminus B = A \setminus B \cup \phi \cup \phi \cup \cdots \cup \phi$

$= C_1 \cup C_2 \cup \cdots \cup C_m$

$$A \setminus B = \bigcup_{i=1}^{m} C_i \quad ; C_i \in F, C_i \cap C_j = \phi.$$

Hence (X, F) is a semi-ring.

which completes the proof.

5, Give an example of semi-ring, which is not algebra.

Sol:-

let $X = \mathbb{R}$ and

let $S = \{[a,b) : a,b \in \mathbb{R} \; ; a < b\} \cup \{\phi\}$

First we show that S is a semi-ring.

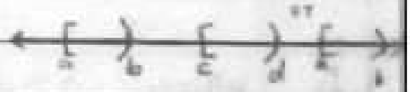
i) $\phi \in S$ (given)

ii) Next we show that the intersection of any two sets from S is again in S.

For this let $A, B \in S$, then

$A = [a,b)$, $B = [c,d)$ for some $a,b,c,d \in \mathbb{R}$

Now there are cases i-e.

(a) when $b < c$ or $a > d$ i-e. 

then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = \phi \in S$.

(b) When $c < a < d < b$ i-e. 

then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [a,d) \in S$

(c) When $a < c < b < d$ i-e. 

then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [c,b) \in S$

(d) When $[a,b) \subseteq [c,d)$ i-e 

then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [a,b) \in S$

(e) When $[c,d) \subseteq [a,b)$ i-e. 

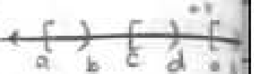
then $A \cap B = [a,b) \cap [c,d)$

$A \cap B = [c,d) \in S$

So all the cases we see that $A \cap B \in S$, so "S" is closed under finite intersection.

(iii) Next we show that the difference of any two sets from S can be written as the finite union of disjoint sets from S.

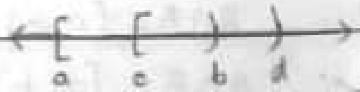
For this let $A, B \in S$
$\Rightarrow A = [a, b)$; $B = [c, d)$
where $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$.
Now there are cases i-e.

(f) When $a > d$ or $c > b$ i-e. 

then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = [a, b) \in S$

(g) When $c < a < d < b$ i-e. 

then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = [d, b) \in S$

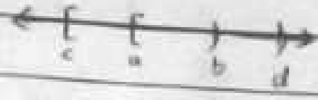
(h) When $a < c < b < d$ i-e. 

then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = [a, c) \in S$

(i) When $[a, b) \subseteq [c, d)$ i-e 

then $A \setminus B = [a, b) \setminus [c, d)$
$A \setminus B = \phi \in S$

(j) when $[c,d) \subseteq [a,b)$ i.e.



then $A \setminus B = [a,b) \setminus [c,d)$

$$A \setminus B = [a,b) \in S$$

So $\forall$ cases we see that $A \setminus B \in S$

So $A \setminus B = A \setminus B \cup \phi \cup \phi \cup \dots \cup \phi$

$= C_1 \cup C_2 \cup \dots \cup C_n$; $C_1 = A \setminus B$, $C_i = \phi$; $i = 2, 3, \dots$

$$A \setminus B = \bigcup_{i=1}^{m} C_i \; ; \; C_i \cap C_j = \phi \; ; \; C_i \in S \; ; \; i \neq j$$

This shows that S is a semi-ring.

→ Now we will show that S is not algebra.

For this let $\phi \in S$.

So $\phi^c = \mathbb{R} = (-\infty, \infty) \notin S$

this means that S is not closed under complementation. So "S" does not satisfy the property of algebra. Thus "S" is not an algebra.

OR e.g $\Rightarrow [1,2) \in S$

$\Rightarrow [7,9) \in S$

$\Rightarrow [1,2) \cup [7,9) \notin S$

So "S" is not an algebra.

σ- Algebra:-

An algebra F is said to be a σ-Algebra, if for any sequence

$A_1, A_2, A_3, \dots \in F$, we have $\bigcup_{i=1}^{\infty} A_i \in F$

OR

for any $A_1, A_2, A_3, \dots \in F \Rightarrow \bigcap_{i=1}^{\infty} A_i \in F$

Remark:-

If F is a σ-Algebra, then $\bigcap_{i=1}^{\infty} A_i \in F$.

Proof:-

If $A_1, A_2, A_3, \dots \in F$

$\Rightarrow A_1^c, A_2^c, A_3^c, \dots \in F$

$\Rightarrow A_1^c \cup A_2^c \cup A_3^c \cup \dots \in F$ (by def. of σ-al

By Demorgan's law

$\Rightarrow (A_1 \cap A_2 \cap A_3 \cap \dots)^c \in F$

$\Rightarrow \left(\bigcap_{i=1}^{\infty} A_i\right)^c \in F$

$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in F$

Example:-

Let X be an uncountable set and F be the collection as
$F = \{E \subseteq X : E \text{ or } E^c \text{ is countable}\}$
then show that "F" is σ-algebra

Sol:-

i) let $A \in F$, we show that $A^c \in F$.

$\Rightarrow A$ or A^c is countable

$\Rightarrow (A^c)^c$ or A^c is countable.

Let $A^c = B$

$\Rightarrow B^c$ or B is countable

$\Rightarrow B \in F$ ($\because$ by def. of F)

$\Rightarrow A^c \in F$ ($\because A^c = B$)

$\Rightarrow$ F is closed under complementation.

ii) Now let $A_1, A_2, A_3, \ldots$ be any collection from F.

we will show that $\bigcup_{i=1}^{\infty} A_i \in F$

Case-I:-

If all A_i's are countable, then we know that $\bigcup_{i=1}^{\infty} A_i$ is always countable.

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$$

Case-II:-

let there exist one A_i say A_k such that A_k^c is countable, then since

$$\Rightarrow \bigcap_{i=1}^{\infty} A_i^c \subseteq A_k^c$$

$$\Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq A_k^c$$

Since A_k^c is countable

$\Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right)^c$ is countable.

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$$

Thus F is σ-Algebra.

Example:-

let X be a topological space and $S = \{C \cap O \,;\, C$ is closed set and O is open set$\}$. Show that S is a semi-ring.

Sol:-

i) Since ϕ is both open and closed such that $\phi = \phi \cap \phi$

$\Rightarrow \phi \in S$

ii) let $A, B \in S$

we will show that $A \cap B \in S$.

For this, since $A, B \in S$

$\Rightarrow A = C_1 \cap O_1$ and $B = C_2 \cap O_2$

where C_1, C_2 are closed sets.

and O_1, O_2 are open sets

Now $A \cap B = (C_1 \cap O_1) \cap (C_2 \cap O_2)$

$= (C_1 \cap C_2) \cap (O_1 \cap O_2) \in S$

So $A \cap B \in S$

iii) let $A, B \in S$, we will show that

$A \setminus B = \bigcup_{i=1}^{n} C_i \;;\; C_i \in S$ and $C_i \cap C_j = \phi \;;\; i \neq j$

Here, since $A, B \in S$

$\Rightarrow A = C_1 \cap O_1$ and $B = C_2 \cap O_2$

where C_1, C_2 are closed sets

and O_1, O_2 are open sets.

Now $A \setminus B = (C_1 \cap O_1) \setminus (C_2 \cap O_2)$

$A \setminus B = (C_1 \cap O_1) \cap (C_2 \cap O_2)^c$ $\quad (\because A \setminus B = A \cap B^c)$

By demorgan's law

$= (C_1 \cap O_1) \cap (C_2^c \cup O_2^c)$

$= ((C_1 \cap O_1) \cap C_2^c) \cup ((C_1 \cap O_1) \cap O_2^c)$

$A \setminus B = (C_1 \cap (O_1 \cap C_2^c)) \cup ((C_1 \cap O_2^c) \cap O_1)$ — ⊛

Since "C_1" is closed and $(O_1 \cap C_2^c)$ is open.

$\Rightarrow C_1 \cap (O_1 \cap C_2^c) \in S$

Also $(C_1 \cap O_2^c)$ is closed and O_1 is open

$\Rightarrow (C_1 \cap O_2^c) \cap O_1 \in S$

So eq ⊛ $\Rightarrow$ $A \setminus B = \bigcup_{i=1}^{2} G_i$; where $G_i \in S$

$G_i \cap G_j = \phi \quad ; i \neq j$

Thus S is a semi-ring.

Example:-

Let S be the collection of all sub-sets of $[0,1)$ that we can written as finite union of subsets of $[0,1)$ of the form $[a,b)$.

Show that S is an algebra of set but not σ-algebra.

Sol:-

i) Let $A \in S$, we will show that $A^c \in S$.

So, here $A \in S$

$\Rightarrow A = \bigcup_{i=1}^{n} [a_i, b_i)$ (by def of S)

$A^c = [0,1) \setminus A$

$A^c = [0,1) \setminus \bigcup_{i=1}^{n} [a_i, b_i)$

Date: (48)

$$A^c = \bigcup_{i=1}^{n} \{ [0,1) \setminus [a_i, b_i) \}$$

$$A^c = \bigcup_{i=1}^{n} \{ [0, a_i) , [b_i, 1) \} \in S$$

So $A^c \in S$.

ii) let $A, B \in S$. we show that $A \cup B \in S$

$\Rightarrow$ $A = \bigcup_{i=1}^{n} [a_i, b_i)$ (by def. of S)

and

$$B = \bigcup_{i=1}^{n} [c_i, d_i)$$

Now $A \cup B = \left[\bigcup_{i=1}^{n} [a_i, b_i) \right] \cup \left[\bigcup_{i=1}^{n} [c_i, d_i) \right]$

$$= [a_1, b_1) \cup [a_2, b_2) \cup \cdots \cup [a_n, b_n)$$

$$\cup$$

$$[c_1, d_1) \cup [c_2, d_2) \cup \cdots \cup [c_n, d_n)$$

$$A \cup B = \bigcup_{i=1}^{n} \left([a_i, b_i) \cup [c_i, d_i) \right) \in S$$

So $A \cup B \in S$.

Thus S is an algebra.

Now we will show that S is not a σ-algebra.

For this, Since $[0, 1/n) \in S \quad \forall n \in \mathbb{N}$

but $\bigcap_{n=1}^{\infty} [0, 1/n) = \{0\} \notin S$

$\Rightarrow$ S is not a σ-algebra.

which completes the proof.

Date: (49) Mon Tue Wed Thu Fri Sat

Problem:-

let S be a semi-ring of subsets of X and $Y \subseteq X$, and $S_Y = \{Y \cap A : A \in S\}$. Show that S_Y form a semi-ring on Y.

Sol:-

i) Since S is a semi-ring, so $\phi \in S$

$\Rightarrow \phi = Y \cap \phi$

$\Rightarrow Y \cap \phi \in S_Y$

$\Rightarrow \phi \in S_Y$

ii) let $A, B \in S_Y$. we need to show that $A \cap B \in S_Y$.

For this, since $A, B \in S_Y$, so by def of S_Y we can write $A = Y \cap A_1$, $B = Y \cap B_1$ where $A_1, B_1 \in S$

Now $A \cap B = (Y \cap A_1) \cap (Y \cap B_1)$

$= Y \cap (A_1 \cap B_1)$, $A_1 \cap B_1 \in S$

$= Y \cap (A_1 \cap B_1) \in S_Y$

So $A \cap B \in S_Y$.

iii) we need to show that $A \setminus B \in S_Y$.

$A \setminus B = (Y \cap A_1) \setminus (Y \cap B_1)$

$A \setminus B = Y \cap (A_1 \setminus B_1)$

Since $A_1, B_1 \in S \Rightarrow A_1 \setminus B_1 = \bigcup_{i=1}^{n} C_i$, $C_i \in S$, $C_i \cap C_j = \phi$; $i \neq j$

$A \setminus B = Y \cap \left(\bigcup_{i=1}^{n} C_i\right)$

$= \bigcup_{i=1}^{n} (Y \cap C_i)$; $C_i \in S$; $Y \cap C_i \in S_Y$; $(Y \cap C_i) \cap (Y \cap C_j) = \phi$

So $A \setminus B \in S_Y$

Thus S_Y is a semi-ring.

Measure :-

Let (X, S) be a semi-ring and $\mu : S \to [0, \infty]$ be a function such that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ where $A_1, A_2, A_3, \ldots \in S$; $A_i \cap A_j = \phi$ for $i \neq j$

and $\bigcup_{i=1}^{\infty} A_i \in S$. Also $\mu(\phi) = 0$.

Examples:-

1. let $X \neq \phi$ and $S = P(X) = \{$All subset of $X\}$ and $\mu : S \to [0, \infty]$ be defined by
$\mu(A) =$ number of elements in A if A is finite
$\mu(A) = \infty$, if A is infinite set.
Show that μ is a measure.

Sol:-

Clearly $\mu(\phi) = 0$

let $A_1, A_2, A_3, \ldots \in S$ such that $A_i \cap A_j = \phi$

We show that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

we discuss the following cases.

Case - 1 :-

If A_i's say A_k is infinite, then $A = \bigcup_{i=1}^{\infty} A_i$ will be infinite and

$$\mu(A) = \infty$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty \quad \text{---} \quad (i)$$

Also $\sum_{i=1}^{\infty} \mu(A_i) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_k) + \cdots$

$= \mu(A_1) + \mu(A_2) + \cdots + \infty + \cdots$

$= \infty$

$$\sum_{i=1}^{\infty} \mu(A_i) = \infty \quad \text{---} \quad (ii)$$

From eq (i) and eq (ii), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Case-2:-

let $A_{i_1}, A_{i_2}, A_{i_3}, \ldots, A_{i_n}$ be non-empty, finite and disjoint sets.

Let $A_{i_j} = \phi$ for $i_j \neq i_1, i_2, i_3, \ldots, i_n$

then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{j=1}^{\infty} A_{i_j}\right)$$

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) =$ no. of elements in "$\bigcup_{j=1}^{\infty} A_{i_j}$" $\left\{\text{because } \bigcup_{j=1}^{\infty} A_{i_j} \text{ is finite}\right.$

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) =$ no of elements in A_{i_1} + no of elements in A_{i_2} + no of elements in A_{i_3} + ⋯ + no of elements in A_{i_n} + no of elements in ϕ + ⋯ (ϕ is Finite)

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(A_{i_1}) + \mu(A_{i_2}) + \mu(A_{i_3}) + \cdots + \mu(A_{i_n}) + \mu(\phi) + \cdots$$

$$= \sum_{i=1}^{\infty} \mu(A_i)$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

In each case "μ" satisfies the properties of measure.

So μ is a measure.

This type of measure is called Discrete or

Date: (52)

or counting measure.

2. Let $X \neq \phi$ and $S = P(X)$ and fix $a \in X$ and define a function

$\mu : S \to [0, \infty]$ by

$\mu(A) = 1$ if $a \in A$

$\mu(A) = 0$ if $a \notin A$

Show that μ is a measure.

Sol:-

Let $A_1, A_2, A_3, \ldots \in S$ such that $A_i \cap A_j = \phi$; $i \neq j$

we need to prove that μ is a measure.

For this we need to show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

we discuss the following cases.

Case-1:-

When $a \notin A_i \quad \forall i$

then $\mu(A_i) = 0$

$$\Rightarrow \sum_{i=1}^{\infty} \mu(A_i) = 0 \quad \text{--- (i)}$$

Also $a \notin \bigcup_{i=1}^{\infty} A_i$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 \quad \text{--- (ii)}$$

From eq (i) and (ii), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Date: (57)

Mon Tue Wed Thu Fri Sat

Case-2

When $a \in \bigcup_{i=1}^{\infty} A_i$, then there exist only one A_i, say A_k such that $a \in A_k$ (because A_i's are disjoint).

Now since $a \in \bigcup_{i=1}^{\infty} A_i$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \quad \text{—— (iii)}$$

Also $\sum_{i=1}^{\infty} \mu(A_i) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_k) + \cdots$

$= 0 + 0 + \cdots + 1 + 0 + \cdots$

$= 1$

$$\sum_{i=1}^{\infty} \mu(A_i) = 1 \quad \text{—— (iv)}$$

From eq (iii) and (iv), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

And clearly $\mu(\phi) = 0$. In each case "μ" satisfies the properties of measure.

Thus μ is a measure.

This type of measure is called Dirac measure.

3. let $X \neq \phi$ and "S" be the collection of pair-wise disjoint collection of sub-set of X and for each $A \in S$. let us choose $m_A \in [0, \infty]$ and define

$\mu : S \to [0, \infty]$ by $\mu(A) = m_A$; $\forall A \in S$, $A \neq \phi$

$\mu(\phi) = 0$; $\forall A \in S$, $A = \phi$

Show that μ is a measure.

Sol:-

let $A_1, A_2, A_3, \dots \in S$, such that $\bigcup_{i=1}^{\infty} A_i \in S$

we need to prove that "μ" is a measure.

For this we need to show that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Now, Here we see that $A_1, A_2, A_3, \dots$ $A = \bigcup_{i=1}^{\infty} A_i \in S$.

Now since "S" contains disjoint sets. So at a time it is not possible that $A_1, A_2, A_3, \dots \in S$ and $\bigcup_{i=1}^{\infty} A_i \in S$ because these sets (i-e A_i's and $\bigcup_{i=1}^{\infty} A_i$) are not disjoint.

So $A_1, A_2, A_3, \dots$ must be one set say A_k. i-e. $\bigcup_{i=1}^{\infty} A_i = A_k$

So
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(A_k)$$
$$= m_{A_k} \quad \text{(by def of "}\mu\text{")}$$
$$= 0 + 0 + \dots + m_{A_k} + 0 + 0 + \dots$$
$$= \mu(\phi) + \mu(\phi) + \dots + \mu(A_k) + \mu(\phi) + \dots$$
$$= \sum_{i=1}^{\infty} \mu(A_i) \text{ ; where all } A_i\text{'s are empty except } A_k$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

which shows that "μ" is a measure.

Measure Space:-

If (X, S) is semi-ring and μ is a measure on S then

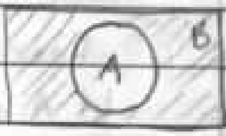
(X, S, μ) is called measure space.

Theorem:-

If (X, S, μ) is a measure space and $A, B \in S$ such that $A \subseteq B$ then $\mu(A) \leq \mu(B)$.

Proof:-

Since $A \subseteq B$



$\Rightarrow \quad B = A \cup (B \setminus A)$

$\Rightarrow \quad \mu(B) = \mu(A \cup (B \setminus A)) \longrightarrow (1)$

But $A, B \in S$

$\Rightarrow \quad B \setminus A = \bigcup_{i=1}^{n} C_i \quad ; \quad C_i \in S \; ; \; C_i \cap C_j = \phi \; ; i \neq j$

$\Rightarrow \quad \mu(B) = \mu\left[A \cup \left(\bigcup_{i=1}^{n} C_i\right)\right]$

$\Rightarrow \quad \mu(B) = \mu(A) + \mu\left(\bigcup_{i=1}^{n} C_i\right)$

by definition of measure.

$\Rightarrow \quad \mu(B) = \mu(A) + \sum_{i=1}^{n} \mu(C_i) \geq \mu(A)$

as $\sum_{i=0}^{n} \mu(C_i) \geq 0$

$\Rightarrow \quad \mu(B) \geq \mu(A)$

$\Rightarrow \quad \mu(A) \leq \mu(B)$.

6- Additive :-

let $\mu : S \longrightarrow [0, \infty]$ is a function such that for $A_1, A_2, A_3, \cdots \in S$; $A_i \cap A_j = \phi$;

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$i \neq j$ for $\bigcup_{i=1}^{\infty} A_i \in S$, if $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, then μ is said to be σ-additive function.

Finitly Additive:-

let $\mu : S \to [0, \infty]$ is a function such that for $A_1, A_2, A_3, \ldots \in S$, where $A_i \cap A_j = \phi \ \forall \ i \neq j$ and for $\bigcup_{i=1}^{n} A_i \in S$, if $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$, then μ is said to be finitly additive function.

σ - Sub Additive:-

let $\mu : S \to [0, \infty]$ is a function such that for $A_1, A_2, A_3, \ldots \in S$ where $A_i \cap A_j = \phi \ \forall \ i \neq j$ and for $\bigcup_{i=1}^{\infty} A_i \in S$, if $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$, then μ is said to be σ-sub additive function.

Finitly Sub Additive:-

Let $\mu : S \to [0, \infty]$ is a function such that for $A_1, A_2, A_3, \ldots \in S$ where $A_i \cap A_j = \phi \ \forall \ i \neq j$ and for $\bigcup_{i=1}^{n} A_i \in S$, If $\mu(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mu(A_i)$, then μ is said to be finitly sub additive function.

Theorem:-

let (X, S, μ) be a measure space then show that μ is finitly additive i.e. If μ is measure then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

Proof:-

let μ is measure.
we need to show that μ is finitly additive i.e. $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

For this let us choose disjoint collections from "S" i.e. $A_1, A_2, A_3, \ldots, A_n \in S$ such that $A_i \cap A_j = \phi$ $i \neq j$ and $\bigcup_{i=1}^{n} A_i \in S$

Now $\mu(\bigcup_{i=1}^{n} A_i) = \mu(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \cup \phi \cup \phi \cdots)$

$= \mu(\bigcup_{i=1}^{\infty} A_i)$ where $A_{n+1} = A_{n+2} = \cdots = \phi$

$= \sum_{i=1}^{\infty} \mu(A_i)$ ($\because \mu$ is measure)

$= \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n) + \mu(A_{n+1}) + \cdots$

$= \sum_{i=1}^{n} \mu(A_i) + \mu(\phi) + \mu(\phi) + \cdots$

$= \sum_{i=1}^{n} \mu(A_i) + 0 + 0 + \cdots$

$\Rightarrow \mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$

which shows that "μ" is finitly additive.

Theorem:-

If μ is measure then $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Proof:-

let μ is measure such that $A_1, A_2, A_3, \ldots \in S$ and let $A = \bigcup_{i=1}^{\infty} A_i$.
We need to prove that $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$
let us contruct a sequence of a set as:

Date: (58)

$B_1 = A_1$

$B_2 = A_2 \setminus A_1 = A_2 \cap A_1^c \subseteq A_2$

$B_3 = A_3 \setminus (A_1 \cup A_2) = A_3 \cap (A_1 \cup A_2)^c \subseteq A_3$

$\vdots \qquad \vdots \qquad \vdots$

Clearly $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \quad ; B_i \cap B_j = \phi ; i \neq j$

$B_1 \subseteq A_1 \Rightarrow \mu(B_1) \leq \mu(A_1)$

$B_2 \subseteq A_2 \Rightarrow \mu(B_2) \leq \mu(A_2)$

$B_3 \subseteq A_3 \Rightarrow \mu(B_3) \leq \mu(A_3)$

$\vdots \qquad \vdots \qquad \vdots$

Since $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \quad \text{(using property of measure)}$$

$$\leq \sum_{i=1}^{\infty} \mu(A_i)$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

which shows that "μ" is σ- sub additive

Example:-

let $S = \{A \subseteq R : A \text{ is countable}\}$ be a semi-ring and $\mu : S \rightarrow [0, \infty]$ be a function defined by

$\mu(A) = 0 \quad ;$ if A is finite set.

$\quad\;\; = \infty \quad ;$ if A is infinite set

Show that "μ" is finitly additive

but not σ-additive.

Sol:-

let $A_1, A_2, A_3, \ldots, A_n \in S$ such that $A_i \cap A_j = \phi$; $i \neq j$

we will prove that $\mu(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} \mu(A_i)$

For this there are cases:

Case-1:-

If all A_i's are finite, then obviously "$\bigcup_{i=1}^{n} A_i$" will be finite, so by definition of μ,

$$\Rightarrow \mu(\bigcup_{i=1}^{n} A_i) = 0 \text{ ——— (i)}$$

Also $\sum_{i=1}^{n} \mu(A_i) = \mu(A_1) + \mu(A_2) + \cdots$

$= 0 + 0 + \cdots$

$= 0$

$$\sum_{i=1}^{n} \mu(A_i) = 0 \text{ ——— (ii)}$$

From eq (i) and (ii), we get

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$$

Case-2:-

If one of A_i's is infinite, say A_k is infinite, then $\bigcup_{i=1}^{n} A_i$ is an infinite set.

So by definition of "μ", we can write

$$\mu(\bigcup_{i=1}^{n} A_i) = \infty \text{ ——— (iii)}$$

Also $\sum_{i=1}^{n} \mu(A_i) = \mu(A_1) + \mu(A_2) + \mu(A_3) + \cdots + \mu(A_k) + \cdots$

$= 0 + 0 + 0 + \cdots + \infty + \cdots = \infty$

Date: (60)

$$\sum_{i=1}^{n} \mu(A_i) = \infty \text{ —— (iv)}$$

From eq (iii) and eq (iv), we get

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

Hence "μ" is finitly additive.

Next, we need to show that "μ" is not σ-additive.

Let us suppose that

$A_1 = \{1\} \Rightarrow \mu(A_1) = 0$

$A_2 = \{2\} \Rightarrow \mu(A_2) = 0$

$A_3 = \{3\} \Rightarrow \mu(A_3) = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\sum_{i=1}^{\infty} \mu(A_i) = 0 \text{ —— (v)}$$

Also $A = \bigcup_{i=1}^{\infty} A_i = \{1, 2, 3, 4, \ldots\}$

$\Rightarrow \mu(A) = \infty$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty \text{ —— (vi)}$$

From eq (v) and (vi), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \neq \sum_{i=1}^{\infty} \mu(A_i)$$

Thus "μ" is not σ-additive.

Theorem:-

Let (X, S) be a semi-ring and

$\mu : S \to [0,\infty]$ is a finitly additive such that "μ" is a σ- sub additive, then prove that "μ" is a measure.

Proof:-

let $A_1, A_2, A_3, \ldots \in S$ such that $A = \bigcup_{i=1}^{\infty} A_i \in S$; $A_i \cap A_j = \phi$ for $i \neq j$.

we need to show that "μ" is measure.

Equivalently we will show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad \text{—— (A)}$$

Now since "μ" is σ- sub additive, so

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i) \longleftarrow (1)$$

Now to prove (A), it is enough to show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i)$$

For this let $A = \bigcup_{i=1}^{\infty} A_i$, then

$$A \setminus A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \bigcup_{i=1}^{m} B_i \quad (2) \quad \forall B_i \in S$$

where $\forall\ i \neq j$, $B_i \cap B_j = \phi$

Now eq (2) $\Rightarrow$ $A = \left[\bigcup_{i=1}^{n} A_i \cup \left(\bigcup_{i=1}^{m} B_i\right)\right]$

$$\mu(A) = \mu\left[\bigcup_{i=1}^{n} A_i \cup \left(\bigcup_{i=1}^{m} B_i\right)\right]$$

$$\mu(A) = \mu\left(\bigcup_{i=1}^{n} A_i\right) + \mu\left(\bigcup_{i=1}^{m} B_i\right) \quad (\because \mu \text{ is finitly additive})$$

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i) + \sum_{i=1}^{m} \mu(B_i)$$

$$\Rightarrow \mu(A) \geq \sum_{i=1}^{n} \mu(A_i) \quad \text{as } \sum_{i=1}^{m} \mu(B_i) \geq 0$$

Date: (62)

Taking $n \to \infty$, we have

$$\mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)$$

$\left(\text{as } A = \bigcup_{i=1}^{\infty} A_i\right)$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i) \longrightarrow (3)$$

from eq(1) and eq(3), we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Thus "μ" is a measure.

Example:-

let $X \neq \phi$ and $f : X \to [0,\infty]$ be a function and $\mu : P(X) \to [0,\infty]$ be defined by

$$\mu(A) = \begin{cases} \sum_{x \in A} f(x) & ; \text{ if } A \text{ is countable} \\ \infty & ; \text{ if } A \text{ is uncountable} \end{cases}$$

and $\mu(\phi) = 0$.

Show that "μ" is a measure.

Sol:-

Let $A_1, A_2, A_3, \dots \in P(X)$ such that $A_i \cap A_j = \phi$, for $i \neq j$

we need to show that "μ" is measure.

Equivalently we need to show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

For this there are cases:

Case-1:-

If all A_i's are countable

Date: (63) Mon Tue Wed Thu Fri Sat

So. $\bigcup_{i=1}^{\infty} A_i$ is countable.

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{x \in \left(\bigcup_{i=1}^{\infty} A_i\right)} f(x)$$ [by definition of function "μ"

$$= \sum_{x \in A_1} f(x) + \sum_{x \in A_2} f(x) + \sum_{x \in A_3} f(x) + \cdots$$

$$= \mu(A_1) + \mu(A_2) + \mu(A_3) + \cdots$$

$$= \sum_{i=1}^{\infty} \mu(A_i)$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Case-2:-

If one of A_i's is uncountable, say A_k is uncountable $\Rightarrow \mu(A_k) = \infty$

$$\Rightarrow \sum_{i=1}^{\infty} \mu(A_i) = \infty \quad \text{—— (i)}$$

Also since A_k is uncountable.

$\Rightarrow \bigcup_{i=1}^{\infty} A_i$ is uncountable.

So by def: of μ,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty \quad \text{—— (ii)}$$

From eq (i) and (ii), we get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

And $\mu(\phi) = 0$

So in all cases "μ" satisfies the

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Date: (56)

property of measure.

$\Rightarrow$ Thus "μ" is a measure.

Outer Measure (Carathedory Measure).

let $X \neq \phi$ and $\mu : P(X) \to [0,\infty]$ is a function such that

i) $\mu(\phi) = 0$.

ii) $\mu(A) \leq \mu(B)$ if $A \subseteq B$.

iii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ where $A_1, A_2, A_3, \ldots \in P(X)$

then "μ" is called outer measure.

Measurable Set:-

If $E \subseteq X$ and μ is an outer measure and

$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$ for any $A \subseteq X$.

then E is said to be measurable set.

Note:-

Since $A = (A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \ldots$

$\Rightarrow \mu(A) = \mu\left((A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \ldots\right)$

Now Since "μ" is outer measure, so by property (iii), since "μ" is σ-sub additive, so

$\Rightarrow \mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + \mu(\phi) + \mu(\phi) + \ldots$

$\Rightarrow \mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + 0 + 0 + \ldots$

$\Rightarrow \mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c)$ always holds.

So from here we can say that "E" will be measurable, with respect to the outer measure "μ" if

$$\mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c) \quad \text{(i)}$$

if "μ" is measure on $P(X)$ then every set (i-e. subset of X) is measurable.

Because "$\leq$" reduces to "=" in above eq (i).

Null set:-

Let "μ" be an outer measure, then any set $E \subseteq X$ is said to be null if $\mu(E) = 0$.

Now, Since $\mu(\phi) = 0$ (by 1st property of outer measure)

So, empty set is a null set.
But not every null set is empty, because there exist non-empty sets, which are null sets.

Theorem:-

Every null set is measurable, under the outer measure μ.

Proof:-

Let "μ" be an outer measure, and E is a null set under μ, i-e. $\mu(E) = 0$.

we need to prove that E is measurable.

Date: (16)

i.e. we will show that for any $A \subseteq X$, $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

For this, since
$$A = (A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \cdots$$

$$\mu(A) = \mu\left((A \cap E) \cup (A \cap E^c) \cup \phi \cup \phi \cup \cdots\right)$$

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + \mu(\phi) + \mu(\phi) + \cdots$$

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) + 0 + 0 + \cdots$$

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) \quad \text{--- (1)}$$

Now, since $A \cap E \subseteq E$
$$\mu(A \cap E) \leq \mu(E) = 0$$

$$\mu(A \cap E) \leq 0 \quad \text{--- (i)}$$

and $A \cap E^c \subseteq A$
$$\Rightarrow \mu(A \cap E^c) \leq \mu(A) \quad \text{--- (ii)}$$

Adding eq (i) and (ii), we get

$$\mu(A \cap E) + \mu(A \cap E^c) \leq 0 + \mu(A)$$

$$\mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A) \quad \text{--- (2)}$$

From eq (1) and eq (2), we have

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A)$$

$\Rightarrow \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

Hence E is measurable.

Theorem:-

If E is measurable, then prove that E^c is measurable.

Proof:-

Since E is measurable, so

$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

$\Rightarrow \mu(A) = \mu(A \cap (E^c)^c) + \mu(A \cap E^c)$

$\Rightarrow \mu(A) = \mu(A \cap E^c) + \mu(A \cap (E^c)^c)$

Thus E^c is measurable.

Note:- Every complemente of measurable set is measurable.

Theorem:-

If E_1 and E_2 are measurable sets, then prove that $E_1 \cup E_2$ is measurable set.

Proof:-

Since E_1 and E_2 are measurable

let $E = E_1 \cup E_2$

we need to prove that E is measurable

i.e. we prove that $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$.

Now, $E = E_1 \cup E_2$

$E = E_1 \cup (E_1^c \cap E_2)$

$E = (E_1 \cup E_1^c) \cap (E_1 \cup E_2)$

$E = E_1 \cup (E_1^c \cap E_2)$

Now $\mu(A) = \mu[(A \cap E) \cup (A \cap E^c)]$

$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

put $E = E_1 \cup (E_1^c \cap E_2)$

$\mu(A) = \mu[A \cap (E_1 \cup (E_1^c \cap E_2))] + \mu[A \cap (E_1 \cup (E_1^c \cap E_2))^c]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (E_1^c \cap (E_1^c \cap E_2)^c)]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (E_1^c \cap ((E_1^c)^c \cup E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (E_1^c \cap (E_1 \cup E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap ((E_1^c \cap E_1) \cup (E_1^c \cap E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu[A \cap (\phi \cup (E_1^c \cap E_2^c))]$

$\mu(A) = \mu[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu(A \cap E_1^c \cap E_2^c)$

$\mu(A) \leq \mu(A \cap E_1) + \mu(A \cap E_1^c \cap E_2) + \mu(A \cap E_1^c \cap E_2^c)$

Since E_2 is measurable.

$\mu(A) \leq \mu(A \cap E_1) + \mu(A \cap E_1^c)$

$= \mu(A)$ as E_1 is measurable

$\Rightarrow \mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c) \leq \mu(A)$

$\Rightarrow \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

which shows that "E" is measurable.

So $E_1 \cup E_2 = E$ is measurable.

Note:-

→ Every union set is measurable set

→ Ground set is measurable set.

$\phi^c = X$ is measurable set.

Lemma:-

let $E_1, E_2, E_3, \ldots, E_n$ be pairwise disjoint and measurable sets, then

$$\mu\left(\dot{\bigcup_{i=1}^{n}}(A \cap E_i)\right) = \sum_{i=1}^{n} \mu(A \cap E_i)$$

where "μ" is an outer measure, where $A \subseteq X$.

Proof:-

We will prove this by mathematical induction;

let $n=1$, then

$\mu(A \cap E_1) = \mu(A \cap E_1)$, it is clearly true for $n=1$.

Suppose it is true for $n=k$.

$$\mu\left[\dot{\bigcup_{i=1}^{k}}(A \cap E_i)\right] = \sum_{i=1}^{k} \mu(A \cap E_i) \quad \text{--- (1)}$$

where $E_1, E_2, E_3, \ldots, E_n$ are pairwise disjoint and measurable.

We need to prove for $n = k+1$,

$$\mu\left[\dot{\bigcup_{i=1}^{k+1}}(A \cap E_i)\right] = \sum_{i=1}^{k+1} \mu(A \cap E_i)$$

For this let us consider

$$\mu\left[\dot{\bigcup_{i=1}^{k+1}}(A \cap E_i)\right] = \mu\left[(A \cap E_1) \cup (A \cap E_2) \cup \cdots \cup (A \cap E_k) \cup (A \cap E_{k+1})\right]$$

$$= \mu\left(\left[\dot{\bigcup_{i=1}^{k}}(A \cap E_i)\right] \cup (A \cap E_{k+1})\right)$$

$$= \mu\left[\dot{\bigcup_{i=1}^{k}}(A \cap E_i) \cup (A \cap E_{k+1})\right]$$

Date: (70)

$$= \mu\left[\bigcup_{i=1}^{k}(A\cap E_i)\right] + \mu(A\cap E_{k+1})$$

$$= \sum_{i=1}^{k} \mu(A\cap E_i) + \mu(A\cap E_{k+1}) \quad (\text{using}\ldots$$

$$\mu\left[\bigcup_{i=1}^{k+1}(A\cap E_i)\right] = \sum_{i=1}^{k+1} \mu(A\cap E_i)$$

it is true for $n=k+1$.
So all the cases of mathematical induction are true.
Thus $\mu\left[\bigcup_{i=1}^{n}(A\cap E_i)\right] = \sum_{i=1}^{n} \mu(A\cap E_i)$.

Hence proved!

Theorem:-

Let $E_1, E_2, E_3, \ldots$ be measurable sets, then prove that $\bigcup_{i=1}^{\infty} E_i$ is measurable set.

Proof:-

Let $E_1, E_2, E_3, \ldots$ be measurable sets.
We need to prove that $\bigcup_{i=1}^{\infty} E_i$ is measurable set.

let $G_1 = E_1$
$G_2 = E_2 / E_1$
$G_3 = E_3/(E_1 \cup E_2)$
$\vdots$

So let $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} G_i$, clearly $G_i \cap G_j$ ...

And let $F_n = \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} G_i$

Since E_i's are measurable set, so F_n is measurable set.

So by definition of measurable set for any set $A \subseteq X$.

$$\mu(A) = \mu(A \cap F_n) + \mu(A \cap F_n^c) \quad ; \text{ --- } (1) \quad \forall A \subseteq X$$

Since $F_n \subseteq E$

$\Rightarrow F_n^c \supseteq E^c$

$\Rightarrow A \cap F_n^c \supseteq A \cap E^c$

$\Rightarrow \mu(A \cap F_n^c) \geq \mu(A \cap E^c)$

eq (1), becomes

$$\mu(A) \geq \mu(A \cap F_n) + \mu(A \cap E^c)$$

$$\Rightarrow \mu(A) \geq \mu\left[A \cap \left(\bigcup_{i=1}^{n} G_i\right)\right] + \mu(A \cap E^c)$$

$$\Rightarrow \mu(A) \geq \mu\left(\bigcup_{i=1}^{n} (A \cap G_i)\right) + \mu(A \cap E^c)$$

$$\Rightarrow \mu(A) \geq \sum_{i=1}^{n} \mu(A \cap G_i) + \mu(A \cap E^c) \quad (2) \text{ (using lemma)}$$

Now $E = \bigcup_{i=1}^{\infty} G_i$

$$A \cap E = \bigcup_{i=1}^{\infty} (A \cap G_i)$$

$$\Rightarrow \mu(A \cap E) = \mu\left[\bigcup_{i=1}^{\infty} (A \cap G_i)\right]$$

$$\Rightarrow \mu(A \cap E) \leq \sum_{i=1}^{\infty} \mu(A \cap G_i) \text{ --- } (*)$$

Taking $n \to \infty$ in eq (2), we get

$$\Rightarrow \mu(A) \geq \sum_{i=1}^{\infty} \mu(A \cap G_i) + \mu(A \cap E^c)$$

$$\Rightarrow \mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c) \quad \text{(using eq (*))}$$

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$\Rightarrow$ E is measurable set.

$\Rightarrow E = \bigcup_{i=1}^{n} E_i$ is measurable set.

Theorem:-

let Δ be the class of all measurable set and μ is outer measure, then μ restrickted to Δ is measure $\{(X, \Delta, \mu)$ is measure space$\}$.

Proof:-

let $E_1, E_2, E_3, \ldots$ be measurable sets such that $E_i \cap E_j = \phi$

We need to prove that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Since "μ" is an outer measure.
So by definition of outer measure "μ".

$$\mu(E) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i) \quad \text{--- (i)}$$

So to prove the required result, it is enough to show that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

For this consider

$$\sum_{i=1}^{k} \mu(E_i) = \sum_{i=1}^{k} \mu(E \cap E_i)$$

$$= \mu\left(\bigcup_{i=1}^{k} (E \cap E_i)\right)$$

$$\sum_{i=1}^{k} \mu(E_i) = \mu\left[E \cap \left(\bigcup_{i=1}^{k} E_i\right)\right] \quad \text{--- (ii)}$$

But $E \cap (\bigcup_{i=1}^{k} E_i) \subseteq E$

$$\Rightarrow \mu\left[E \cap (\bigcup_{i=1}^{k} E_i)\right] \leq \mu(E)$$

So eq (ii) $\Rightarrow \sum_{i=1}^{k} \mu(E_i) = \mu\left[E \cap (\bigcup_{i=1}^{k} E_i)\right] \leq \mu(E)$

$$\Rightarrow \sum_{i=1}^{k} \mu(E_i) \leq \mu(E)$$

Taking $k \to \infty$, we get

$$\Rightarrow \sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E) = \mu(\bigcup_{i=1}^{\infty} E_i) \quad \text{--- (iii)}$$

From eq (i) and eq (iii), we get

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$$

Thus μ is measure.

Hence proved!

Question:-

let $A \subseteq \mathbb{R}^2$ and define μ^* by $\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i\right\}$, where A_i's are rectangles and μ represents area and infimum is taken over possible of covers of rectangles.

We show that μ^* is an outer measure.

Sol:-

We have to show that μ^* satisfies three properties of outer measure.

Clearly $\mu^*(A) \geq 0$

Since $\phi \subseteq \phi \cup \phi \cup \phi \cup ---$

$$\mu^*(\phi) \le \mu(\phi) + \mu(\phi) + \mu(\phi) + \cdots$$

$$\mu^*(\phi) \le 0 + 0 + 0 + \cdots$$

$$\mu^*(\phi) \le 0$$

$$\mu^*(\phi) = 0$$

ii) If $A \subseteq B$, we show that

$$\mu^*(A) \le \mu^*(B)$$

let $L = \left\{ \sum_{i=1}^{\infty} \mu(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$

and $M = \left\{ \sum_{i=1}^{\infty} \mu(B_i) : B \subseteq \bigcup_{i=1}^{\infty} B_i \right\}$

If $B \subseteq \bigcup_{i=1}^{\infty} B_i$ and $A \subseteq B$

$$\Rightarrow A \subseteq \bigcup_{i=1}^{\infty} B_i$$

Every cover of B is also cover of A.

Thus $M \subseteq L$.

$\inf L \le \inf M$.

$$\mu^*(A) \le \mu^*(B).$$

Hence μ^* is monotonic.

iii) let $A_1, A_2, A_3, \cdots \subseteq \mathbb{R}^2$, we show that

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i)$$

By definition of μ^*, for every $\epsilon > 0$, there exists covers A_i^n of A_i such that

$$\mu^*(A_i) + \epsilon 2^{-i} > \sum_{n=1}^{\infty} \mu(A_i^n)$$

$$\sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \sum_{i=1}^{\infty} 2^{-i} > \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A^n_i).$$

$$\sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon > \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A^n_i) \quad \text{--- } (1)$$

Since $A_i \subseteq \bigcup_{n=1}^{\infty} A^n_i$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A^n_i$$

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A^n_i)$$

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) < \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \quad \text{(using eq (1))}$$

Since ϵ is an arbitrary, so takes $\epsilon \to 0$.

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + 0$$

$$\Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Thus all the conditions of outer measure are satisfied.

Hence μ^* is an outer measure.

Q:- Let $A = \{x\}$ then $\mu^*(A) = 0$.

Sol:-

$$x \in \left(x - \epsilon/2,\ x + \epsilon/2\right)$$

$$\Rightarrow \{x\} \subseteq \left(x - \epsilon/2,\ x + \epsilon/2\right)$$

$$\Rightarrow \mu^*\{x\} \leq \mu\left(x - \epsilon/2,\ x + \epsilon/2\right)$$

$$\Rightarrow \mu^*\{x\} \leq x + \epsilon/2 - \left(x - \epsilon/2\right)$$

Date: (76)

$$\mu^*\{x\} \le \cancel{x} + \epsilon/2 - \cancel{x} + \epsilon/2$$

$$\mu^*\{x\} \le \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\mu^*\{x\} \le \frac{2\epsilon}{2}$$

$$\mu^*\{x\} \le \epsilon$$

Since $A = \{x\}$

$$\mu^*(A) \le \epsilon$$

but taking $\epsilon \to 0$

$$\mu^*(A) = 0$$

Q:- Let $A = \{x_1, x_2, x_3, ----\}$ be countable set then $\mu^*(A) = 0$.

Sol:-

Given

$$A = \{x_1, x_2, x_3, ----\}$$

$$A = \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup ----$$

$$\mu^*(A) = \mu^*\{x_1\} + \mu^*\{x_2\} + \mu^*\{x_3\} + ----$$

$$\mu^*(A) = 0 + 0 + 0 + ----$$

$$\mu^*(A) = 0$$

Lemma:-

Show that each countable set is measurable.

Proof:-

let A be any countable set,

so $\mu^*(A) = 0$

$\Rightarrow$ A is null set.

Since we know that "every null set is measurable".

So A is measurable.

Hence each countable set is measurable.

Question:-

Show that (a,∞) is measurable set.

Sol:-

To show that the set $E = (a,\infty)$ is measurable.

we have to show that for any set $A \subseteq \mathbb{R}$, we have

$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c) \quad \text{——— } (*)$$

For this we discuss the following two cases:

Case-1:-

If $\mu(A) = \infty$, then

$$\mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c)$$

but

$$\mu(A) \leq \mu(A \cap E) + \mu(A \cap E^c)$$

always holds, so

$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c).$$

Case-2:-

If $\mu(A) < \infty$, then for any cover $\{I_n : n \in N\}$ of A, by def. of Lebsegue measure.

$$\mu(A) = \inf\left\{\sum_{n=1}^{\infty} l(I_n) \; : \; A \subseteq \bigcup_{n=1}^{\infty} I_n\right\}$$

$$\mu(A) \leq \sum_{n=1}^{\infty} l(I_n) \quad , \quad A \subseteq \bigcup_{n=1}^{\infty} I_n$$

Date: (78)

let $\epsilon > 0$ be any arbitrary small +ve value.

So by definition of infimum

$$\mu(A)+\epsilon > \sum_{n=1}^{\infty} l(I_n) \quad , A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\mu(A)+\epsilon \geq \sum_{n=1}^{\infty} l(I_n) \text{ --- (1) (by property of inequal}$$

Now $I_n = (I_n \cap E) \cup (I_n \cap E^c)$

$\because l$ is measurable

$$\Rightarrow l(I_n) = l(I_n \cap E) + l(I_n \cap E^c)$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \left[l(I_n \cap E) + l(I_n \cap E^c) \right]$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} l(I_n \cap E) + \sum_{n=1}^{\infty} l(I_n \cap E^c)$$

therefore eq (1) becomes,

$$\mu(A)+\epsilon \geq \sum_{n=1}^{\infty} l(I_n \cap E) + \sum_{n=1}^{\infty} l(I_n \cap E^c) \text{ --- (A)}$$

Now Since

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$A \cap E \subseteq \bigcup_{n=1}^{\infty} I_n \cap E$$

$$= \bigcup_{n=1}^{\infty} (I_n \cap E)$$

$$\Rightarrow A \cap E \subseteq \bigcup_{n=1}^{\infty} (I_n \cap E)$$

$\Rightarrow \{I_n \cap E : n \in N\}$ form cover for $A \cap E$.

So by def. of infimum and lebesgue measure.

$$\mu(A \cap E) \leq \sum_{n=1}^{\infty} l(I_n \cap E) \text{ --- (2)}$$

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On the similar way, since $A\cap E^c \subseteq \bigcup_{n=1}^{\infty}(I_n \cap E^c)$,
So,

$$\mu(A\cap E^c) \leq \sum_{n=1}^{\infty} l(I_n \cap E^c) \quad \text{---} \ (3)$$

Adding eq (2) and (3), we have

$$\mu(A\cap E) + \mu(A\cap E^c) \leq \sum_{n=1}^{\infty} l(I_n \cap E) + \sum_{n=1}^{\infty} l(I_n \cap E^c)$$

$$\mu(A\cap E) + \mu(A\cap E^c) \leq \mu(A) + \epsilon \quad (\text{using eq } (A))$$

Since ϵ is an arbitrary, so $\epsilon \to 0$

$$\mu(A\cap E) + \mu(A\cap E^c) \leq \mu(A) + 0$$
$$\mu(A\cap E) + \mu(A\cap E^c) \leq \mu(A)$$

$$\Rightarrow \mu(A) \geq \mu(A\cap E) + \mu(A\cap E^c) \quad \text{---} \ (B)$$

But $\mu(A) \leq \mu(A\cap E) + \mu(A\cap E^c)$ — (C) is obviously

From eq (B) and eq (C), we get

$$\Rightarrow \mu(A) = \mu(A\cap E) + \mu(A\cap E^c)$$

$\Rightarrow E$ is measurable.
i.e. (a, ∞) is measurable set.

Lebesgue Measure:-

Let X be any non-empty set usually we take $X = \mathbb{R}^n$, and F be the collection of subsets of X, and $\mu : F \to [0, \infty]$, be a set function such that $\mu(\phi) = 0$.

Date: (80)

Let us define $\mu^*: P(X) \to [0, \infty]$, by

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i \right\},$$
where $\{A_i\}$ is a cover for A, $A_i \in F$ $\forall$

$= \infty$, if there is no cover for A

Where infimum is taken over all possible covers of A.

Q:- Show that $(-\infty, a]$ is measurable set.

Sol:-

Since (a, ∞) is measurable set.

$\Rightarrow (a, \infty)^c$ is measurable set.

$\Rightarrow (-\infty, a]$ is measurable set.

Q:- Show that $(-\infty, a)$ is measurable set.

Sol:-

Since $(-\infty, a]$ is measurable, $\forall a \in \mathbb{R}$.

$\Rightarrow (-\infty, a - 1/n]$ is measurable set for each $n \in N$

let $E_n = (-\infty, a - 1/n]$

$\Rightarrow E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$

Since each E_i is measurable,

So $\bigcup_{n=1}^{\infty} E_n$ is measurable

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} (-\infty, a - 1/n]$$

$$= (-\infty, a)$$

$\Rightarrow (-\infty, a)$ is measurable set.

Q:- Show that $[a,b]$ is measurable set.

Sol:-

Since $(-\infty, b]$ and $[a,\infty)$ are measurable set, so

$\Rightarrow (-\infty, b] \cap [a,\infty)$ is measurable set.

$\Rightarrow [a,b]$ is measurable set.

Q:- Show that $\mathbb{R}$ is measurable set.

Sol:-

Since $(-\infty, a]$ and $[a,\infty)$ are measurable set

$\Rightarrow (-\infty, a] \cup [a,\infty)$ is measurable set.

$\Rightarrow (-\infty, \infty)$ is measurable set.

$\Rightarrow \mathbb{R}$ is measurable set.

Theorem:-

If A is not null set and $B \subseteq A$. If B is null set, prove that $A \setminus B$ is not null set.

Proof:-

Let us suppose that $A \setminus B$ is null set.

As B is null set.

$\Rightarrow (A \setminus B) \cup B$ is null set.

$\Rightarrow A$ is null set. $(\because B \subseteq A)$

But A is not null set (given)

So which is contradiction.

Hence $A \setminus B$ is not null set.

Date: (52)

Theorem:-

Show that for any set E,
$\mu(E) = \mu(E+x)$, where $E+x = \{e+x : e \in E\}$

Proof:-

By def. of lebesgue measure

$$\mu(E) = \inf\{ \sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \}$$

$$\mu(E) \leq \sum_{n=1}^{\infty} l(I_n) \quad , \quad E \subseteq \bigcup_{n=1}^{\infty} I_n$$

for every $\epsilon > 0$

$$\mu(E) + \epsilon > \sum_{n=1}^{\infty} l(I_n) \quad \text{for some covers } \{I_n\} \text{ of}$$

$$\mu(E) + \epsilon \geq \sum_{n=1}^{\infty} l(I_n) \quad \text{—— (i)}$$

Now since

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$E + x \subseteq \bigcup_{n=1}^{\infty} I_n + x$$

$$E + x \subseteq \bigcup_{n=1}^{\infty} (I_n + x)$$

$$\mu(E+x) \leq \sum_{n=1}^{\infty} l(I_n + x)$$

$$\Rightarrow \mu(E+x) \leq \sum_{n=1}^{\infty} l(I_n + x) = \sum_{n=1}^{\infty} l(I_n)$$

$$\Rightarrow \mu(E+x) \leq \sum_{n=1}^{\infty} l(I_n) \quad \text{—— (ii)}$$

Combining eq (i) and (ii), we get

$$\mu(E+x) \leq \sum_{n=1}^{\infty} l(I_n) \leq \mu(E) + \epsilon$$

$$\Rightarrow \mu(E+x) \leq \mu(E) + \epsilon$$

but ϵ is any arbitrary small +ve number

i.e. $\epsilon \to 0$

$$\mu(E+x) \leq \mu(E) \text{ --- } (A)$$

Since, $E = E + x - x$

$$E = (E+x) - x$$

$$\Rightarrow \mu(E) = \mu((E+x) - x)$$

$$\Rightarrow \mu(E) = \mu((E+x) + (-x))$$

So by eq (A), we get

$$\mu(E) = \mu((E+x)+(-x)) \leq \mu(E+x)$$

$$\Rightarrow \mu(E) \leq \mu(E+x)$$

$$\Rightarrow \mu(E+x) \geq \mu(E) \text{ --- } (B)$$

From eq (A) and (B), we get

$$\mu(E) = \mu(E+x)$$

which is the required proof.

Theorem:-

let $\{E_n\}$ be a sequence of measurable sets, then the following statements holds.

i) If $E_n \uparrow E$, then $\mu(E_n) \uparrow \mu(E)$.

ii) If $E_n \downarrow E$ and $\mu(E_k) < \infty$ for some k, then $\mu(E_n) \downarrow \mu(E)$.

Proof:-

Since $E_n \uparrow E \Rightarrow E_n$ is increasing sequence approches to E.

$\Rightarrow E_1 \subseteq E_2 \subset E_3 \subset E_4 \subseteq \text{----}$ and $E = \bigcup_{n=1}^{\infty} E_n$

let $B_1 = E_1$, $B_2 = E_2 \setminus E_1$, $B_3 = E_3 \setminus E_2$, $B_4 = E_4 \setminus E_3$ ----

Date: (84)

then $B_i \cap B_j = \Phi$ for $i \neq j$.
Further more since each $E_i \ \forall \ i$ is measurable.
$\Rightarrow B_i \ \forall i$ is measurable.

Now since $E_n = \bigcup_{i=1}^{n} B_i$

$$\Rightarrow \lim_{n \to \infty} E_n = \lim_{n \to \infty} \bigcup_{i=1}^{n} B_i$$

But $E_n \uparrow E$. So from above

$$E = \lim_{n \to \infty} \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{\infty} B_i$$

$$\Rightarrow \quad E = \bigcup_{i=1}^{\infty} B_i$$

$$\mu(E) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$$\mu(E) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$\mu(E) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i)$$

$$\mu(E) = \lim_{n \to \infty} \mu(E_n)$$

$$\Rightarrow \lim_{n \to \infty} \mu(E_n) = \mu(E) \longrightarrow (A)$$

But $\mu(E_1) \leq \mu(E_2) \leq \mu(E_3) \leq \cdots$
then

$$\mu(E_n) \uparrow \mu(E)$$

ii) Since $E_n \downarrow E$
$\Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq \cdots$.
Since $\mu(E_k) < \infty$, for some K.
So without of loss of generality

Date: (85)

Mon Tue Wed Thu Fri Sat

let

$$\mu(E_1) < \infty$$

Now since $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

$\Rightarrow E_1 \setminus E_1 \subseteq E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq \cdots$

$\Rightarrow E_1 \setminus E_n \nearrow E_1 \setminus E$

by eq (A), we have

$$\lim_{n\to\infty} \mu(E_1 \setminus E_n) = \lim_{n\to\infty} \mu(E_1 \setminus E)$$

$$\lim_{n\to\infty} \mu(E_1 - E_n) = \lim_{n\to\infty} \mu(E_1 - E)$$

$$\Rightarrow \mu(E_1) - \lim_{n\to\infty} \mu(E_n) = \mu(E_1) - \mu(E)$$

$$\Rightarrow -\lim_{n\to\infty} \mu(E_n) = -\mu(E)$$

$$\Rightarrow \lim_{n\to\infty} \mu(E_n) = \mu(E).$$

Thus $\mu(E_n) \downarrow \mu(E)$.

Zermelo's Axiom:-

Consider a family of arbitrary non-empty disjoint sets indexed by set A, $\{E_\alpha : \alpha \in A\}$. Then there exist a set consisting of exactly one element from each E_α, $\alpha \in A$.

Lemma:-

Let E be a measurable subset of $\mathbb{R}$ with $\mu(E) > 0$. Then the set of difference $\{d : d = x - y, x \in E, y \in E\}$ contain an interval centre at the origin.

Theorem:- (Vitali): there exists Non-lebesgue measurable subset of $\mathbb{R}$.

Proof:-

Define a relation T on $\mathbb{R}$ by xTy if $x-y=r$ (rational), $x,y\in\mathbb{R}$.

We show that T is an equivalence relation.

i) **Reflexive:-**

let $x\in\mathbb{R}$

$\Rightarrow x-x=0$ (rational)

$\Rightarrow xTx$

$\Rightarrow T$ is reflexive.

ii) **Symmetric:-**

let $x,y\in\mathbb{R}$

there exists xTy

$\Rightarrow x-y=r$ (rational)

$\Rightarrow -(y-x)=r$

$\Rightarrow y-x=-r$ (rational)

$\Rightarrow yTx$

$\Rightarrow T$ is symmetric.

iii) **Transitive:-**

let $x,y,z\in\mathbb{R}$

there exists xTy and yTz

$\Rightarrow x-y=r_1$ and $y-z=r_2$

$\Rightarrow x-\cancel{y}+\cancel{y}-z=r_1+r_2$ (rational)

$\Rightarrow x-z=r_1+r_2$ (rational)

$\Rightarrow xTz$

$\Rightarrow T$ is transitive.

Hence T is an equivalence relation.

let E_x be an equivalence w.r.t $x \in \mathbb{R}$.

$$E_x = \{y \in \mathbb{R} : y T x\}$$

$$E_x = \{y \in \mathbb{R} : y - x = r, r \in \mathbb{Q}\}$$

$$E_x = \{y \in \mathbb{R} : y = x + r, r \in \mathbb{Q}\}$$

$$E_x = \{x + r : r \in \mathbb{Q}\}$$

Any two classes are either identical or disjoint.

let E_x and E_y be two classes such that $E_x \cap E_y \neq \phi$

$\Rightarrow z \in E_x \cap E_y$
$\Rightarrow z \in E_x$ and $z \in E_y$
$\Rightarrow z T x$ and $z T y$
$\Rightarrow x T y$

let $x_1 \in E_x$
$\Rightarrow x_1 T x$
$\Rightarrow x_1 T y$
$\Rightarrow x_1 \in E_y$
$\Rightarrow E_x \subseteq E_y$

Similarly $x_1 \in E_y$
$\Rightarrow x_1 T y \Rightarrow x_1 T x$
$\Rightarrow x_1 \in E_x$
$\Rightarrow E_y \subseteq E_x$

So, $E_x = E_y$

$E_x = \{x+r : r \in \mathbb{Q}\}$

there exists one class which contain all rational number.

Each class is countable.

As the union of all these classes is equal to $\mathbb{R}$.

So the number of class is uncountable.

By zermelo's axiom, we can find a set E which contain exactly one element from each different class.

Now we show that E is Non-measurable set.

If E is measurable and consider $D = \{d : d = x-y, \ x, y \in E\}$, then D cannot contain rational numbers.

So D cannot contain any interval.

let us suppose that E has lebesgue measure is zero.

$x \in E \quad , \quad y \in E$

$x+r \quad , \quad y+r$

$\bigcup_r (E+r) = \mathbb{R}$

$\mu(\mathbb{R}) = 0 \quad$ as $\quad \mu(E) = \mu(E+r) = 0$

which is contradiction and hence E is non-lebesgue measurable subset of $\mathbb{R}$.

Assignment:-

Every interval contain non-lebesgue measurable subset.

Date: ______ 29

Question:-

Show that the lebesgue measure of cantour set C is zero.

Sol:-

Since cantour set C is a subset of [0,1] i.e $C \subseteq [0,1]$

$\Rightarrow \mu(C) = l[0,1] -$ total length of removed open interval.

$$= \left[1 - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots\right)\right]$$

$$= 1 - 1 \qquad \left(S_\infty = \frac{a_1}{1-r}\right)$$

$$= 0$$

$$\Rightarrow \boxed{\mu(C) = 0}$$

Example:-

$[1,3] \cup [5,7]$

$\Rightarrow \mu[1,3] + \mu[5,7]$

$\Rightarrow (3-1) + (7-5)$

$\Rightarrow 2 + 2$

$\Rightarrow 4$

Question:-

If μ is an outer measure and A is a null set then show that $\mu(B) = \mu(B \cup A) = \mu(B \setminus A)$. holds for any $B \subseteq X$.

Sol:-

Since $B \subseteq B \cup A$

$\Rightarrow \mu(B) \leq \mu(B \cup A)$ —— ①

as μ is an outer measure

and $B \cup A = (B \setminus A) \cup A$

$\Rightarrow \mu(B \cup A) = \mu[(B \setminus A) \cup A]$

Date:

$$\mu(B \cup A) = \mu(B \setminus A) + \mu(A)$$
$$\mu(B \cup A) = \mu(B \setminus A) + 0 \quad \text{as } \mu(A) = 0$$

$$\mu(B \cup A) = \mu(B \setminus A)$$

So eq ① ⇒
$$\mu(B) \leq \mu(B \cup A) = \mu(B \setminus A) \quad \text{——— ②}$$

Since $B \setminus A \subseteq B$
$$\mu(B \setminus A) \leq \mu(B)$$

So eq ② ⇒
$$\mu(B) \leq \mu(B \cup A) = \mu(B \setminus A) \leq \mu(B)$$

$$\Rightarrow \mu(B) = \mu(B \cup A) = \mu(B \setminus A).$$

Question:-

Let μ be an outer measure and if a sequence $\{A_n\}$ satisfies $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then show that the set $E = \{x \in X : x \in A_n \text{ for infinitly many}\}$ is null set.

Sol:-

let us suppose that
$$E_n = \bigcup_{n=1}^{\infty} A_n$$

$$\mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$
$$\mu(E_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

Clearly $E \subseteq E_n$ for each n.
$$\mu(E) \leq \mu(E_n)$$

Also $\mu(E) \geq 0$

$$\Rightarrow 0 \leq \mu(E) \leq \mu(E_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

$$\Rightarrow 0 \leq \mu(E) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

If $n \to \infty$, $\sum_{n=1}^{\infty} \mu(A_n) \longrightarrow 0$

$$\Rightarrow 0 \leq \mu(E) \leq 0$$

$$\Rightarrow \mu(E) = 0$$

Hence E is the null set.

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