Written by;
Asim Marwat
MSc Mathematics (UOP)

Notes;
Measure Theory
Students for;
MSc And BS

Special Thanks to;
DR.Adil Sir (UOP)

Contact and Wtssp
03151949572
asimmarwat41@gmail.com

CHAPTER : 1

SET THEORY

Equivalent Set:-

If A and B are two non-empty sets then A is said to be equivalent to B if there is bijective function from A to B.

If A is equivalent to B then we write as $A \approx B$ or $A \sim B$.

e.g: $A = \{1, 2, 3\}$

$B = \{a, b, c\}$

So $A \sim B$

<u>Note</u>: If A and B are finite then the number of elements of A and B are finite.

Infinite Set:-

A set which is equivalent to its proper subset is called the infinite set.

e.g:

$N = \{1, 2, 3, \ldots\}$ is infinite because N is equivalent to its proper subset $E = \{2, 4, 6, 8, \ldots\}$

that is if we define the function

$f : N \longrightarrow E$ by $f(n) = 2n \;\; ; \forall \; n \in N$

then f is clearly bijective.

Denumerable set:-

A set which is equivalent to natural number $\mathbb{N}$.

e.g:

The sets of $\mathbb{N}, \mathbb{Z}, E, \mathbb{Q}$,
$A = \{2^n, n \in N\}$, $B = \{1/n, n \in N\}$ etc
are denumerable sets

Non-Denumerable set:-

An infinite set is said to be non-denumerable set if it is not denumerable.

e.g:

$[1, 2]$, $R = (-\infty, \infty)$ are non-denumerable sets.

Countable set:-

A set which is finite or denumerable is called countable set.

e.g:

The sets of $\mathbb{N}, E, \mathbb{Q}$,
$A = \{1, 2, 3\}$ and $B = \{-1/2, 0, 1/2\}$ etc
are countable sets.

$A = \{1, 2, 3, ---, 1000\}$
$B = \{a, b, c, ----, z\}$
$C = \{a, e, i, o, u\}$ all are countable sets because these are finite.

Note: → Every interval is uncountable.
→ A is said to countable set if there exists injective (one-one) function.

from S to N.

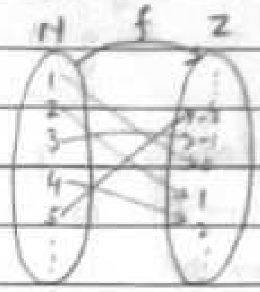
Examples:-

Q1 Show that the sets $Z = \{\text{all integers}\}$ and $X = \{2^n ; n \in N\}$ are countable.

Sol:-

Given that $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ is the set of all integers.

we have to show that Z is countable.



let us define

$f : N \to Z$ by

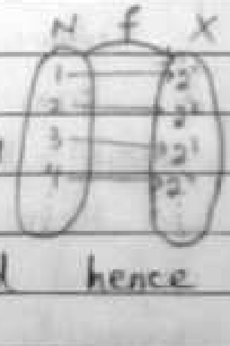
$$f(n) = \begin{cases} n/2 & \text{if } n = \text{even} \\ \dfrac{-(n-1)}{2} & \text{if } n = \text{odd} \end{cases}$$

Clearly f is bijective and hence $Z \sim N$ or $N \sim Z$.

$\Rightarrow$ Z is denumerable and hence Z is countable.

Next, $X = \{2^n ; n \in \mathbb{N}\}$

we will show that X is countable.



let us define

$g : \mathbb{N} \longrightarrow X$ by

$g(n) = 2^n \quad ; \forall \ n \in \mathbb{N}$

clearly g is bijective and hence $\mathbb{N} \sim X$ or $X \sim \mathbb{N}$.

$\Rightarrow$ X is denumerable and hence X is countable.

Q2:- Show that the set $M = \{1/n \,;\, n \in \mathbb{N}\}$ is countable.

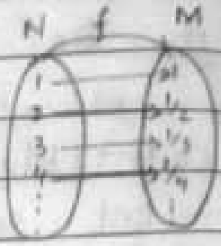
Sol:-

Given that

$$M = \{1/n \,;\, n \in \mathbb{N}\}$$

we will show that M is countable.

let us define



$$g : N \longrightarrow M \text{ by}$$
$$g(n) = 1/n \quad ; \forall\, n \in \mathbb{N}$$

Clearly g is bijective and hence $\mathbb{N} \sim M$ or $M \sim \mathbb{N}$.

$\Rightarrow M$ is denumerable and hence M is countable.

Theorem:-

Every subset of denumerable set is either finite or denumerable

Proof:-

Let A be any denumerable set, and let B be any subset of A.

we need to prove that B is finite or denumerable.

Now since A is denumerable, so A can be written as $A = \{a_1, a_2, a_3, \ldots\}$ where $A \sim N$.

Here the function $f : N \longrightarrow A$ is defined as $f(k) = a_k \;\forall\, k \in \mathbb{N}$; which is obviously bijective.

Now since $B \subseteq A$.

If $B = \phi$ then B is finite.

If $B \neq \phi$ then since $B \subseteq A$, so we can find an element say $a_{n_1} \in A$ such that $a_{n_1} \in B$. If $B = \{a_{n_1}\}$ then B is finite. If $B \neq \{a_{n_1}\}$, then there is $a_{n_2} \in A$ such that $a_{n_2} \in B$.

If $B = \{a_{n_1}, a_{n_2}\}$ then B is finite, but if $B \neq \{a_{n_1}, a_{n_2}\}$ then we can have $a_{n_3} \in A$ such that $a_{n_3} \in B$.

If $B = \{a_{n_1}, a_{n_2}, a_{n_3}\}$ then B is finite. But if $B \neq \{a_{n_1}, a_{n_2}, a_{n_3}\}$ then we continue the above process and as such we can have $B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ where $n_i \in Z^+$.

If the indexing set $\{n_1, n_2, n_3, \ldots\}$ is bounded then it is finite otherwise it is denumerable because we can find a function $g : N \to B$ defined by $g(i) = a_n$ where $i \in N$.

then g as defined above is obviously bijective.

Thus B is not finite, but in this case B is denumerable i-e. $B \sim N$

Hence every subset of denumerable set is either finite or denumerable.

Theorem:-

Every subset of countable set is countable.

Proof:-

Let A be a countable set. We need to show that each subset of A is countable.

Now since A is countable, so by def: of countable set A is either finite or countably infinite (denumerable).

So two cases arises:

i) When A is finite:-

In this case each subset of A is finite and we know that every finite set is countable.

So in this case each subset of A is countable.

ii) When A is denumerable:-

let A is denumerable and let B be a subset of A then B can either be

(a) $B = \phi$ or

(b) B = finite and non-empty. or

(c) B is infinite.

(a) and (b) cases are clear because in these cases B is finite and so B is countable.

(c) Here B is a subset of A. we will show that B is countable.

For this let "n_1" be the least +ve integer such that $a_{n_1} \in B$ then $B \neq \{a_{n_1}\}$, because B is infinite.

let n_2 be the least +ve integer

such that $n_2 > n_1$ and $a_{n_2} \in B$, then $B = \{a_{n_1}, a_{n_2}\}$ because B is infinite.
Continuing this process we get A subset
$B = \{a_{n_1}, a_{n_2}, a_{n_3}, ----\}$
clearly B is countable because if we define $f: N \to B$ by
$f(K) = a_{n_K}$, then f is bijective.

$\Rightarrow B \sim N \Rightarrow B$ is denumerable and hence B is countable.
So in each case each subset of the countable set A is countable. which is the required result.

2^ND method:-
let A is any countable set. Then A is either finite or denumerable.
If B is any subset of A then B will also be finite if A is finite.
Now if B is finite, and we know that every finite set is countable. So B being a subset of A is countable.
Now if A is countable, but not finite means that A is denumerable. Now A is denumerable and B is a subset of A.
$\Rightarrow$ B is also denumerable.
(i.e. by a result 'the subset of a

denumerable set is again denumerable.

So in both cases "B" being a subset of A is countable.

Hence every subset of countable set is countable.

Theorem:-

Show that $[0,1]$ is uncountable.

Proof:-

let us suppose that $I=[0,1]$ is countable, then we can write as $I=[0,1]=\{x_1, x_2, x_3, ----\}$.

To prove the required result we construct a sequence of nested intervals of I.

Divide $[0,1]$ in three equal parts as $[0, 1/3], [1/3, 2/3], [2/3, 1]$

Since $x_1 \in I$, so there exists one interval say $[a_1, b_1]$ such that $x_1 \notin [a_1, b_1] = I_1$, $l(I_1) = 1/3$

Divide $[a_1, b_1]$ in three equal parts $[a_1, a_1 + 1/9], [a_1 + 1/9, a_1 + 2/9], [a_1 + 2/9, b_1]$

So there exists one interval say $I_2 = [a_2, b_2]$ such that $x_2 \notin I_2$, $l(I_2) = 1/3^2$

Similarly $x_3 \notin I_3$, $l(I_3) = 1/3^3$

$\vdots \qquad \vdots$

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$x_n \notin I_n$, $l(I_n) = 1/3^n$

$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n$

$\lim_{n\to\infty} l(I_n) = 0$

By using Nested Interval Theorem there exists only one element (say) $x \in \bigcap_{n=1}^{\infty} I_n$.

So this result, $\exists$ x_m (say) in I such that $x_m \in \bigcap_{n=1}^{\infty} I_n$

$\Rightarrow x_m \in I_m$, but according to our conclusion $x_m \notin I_m$.

Which is contradiction to our supposition. So our supposition was wrong and this contradiction arises due to our wrong supposition that [0,1] is countable.

Thus [0,1] is uncountable.

Note $\rightarrow$ If $A \subseteq B$ and A is countable then B is countable.

$\rightarrow$ If $A \subseteq B$ and A is uncountable then B is uncountable.

Theorem:-

Show that $\mathbb{R}$ is uncountable.

Proof:-

Let us suppose that $\mathbb{R}$ is countable, then we can write $\mathbb{R}$ in the form

$\mathbb{R} = \{x_1, x_2, x_3, \cdots\}$

Now let $I_1 = \left(x_1 - \frac{1}{2^2}, x_1 + \frac{1}{2^2}\right)$

$x_1 \in I_1$, $l(I_1) = \frac{1}{2}$

and $I_2 = \left(x_2 - \frac{1}{2^3}, x_2 + \frac{1}{2^3}\right)$

$x_2 \in I_2$, $l(I_2) = \frac{1}{2^2}$

$\vdots \qquad \vdots$

$I_n = \left(x_n - \frac{1}{2^{n+1}}, x_n + \frac{1}{2^{n+1}}\right)$

$x_n \in I_n$, $l(I_n) = \frac{1}{2^n}$

$\Rightarrow x_n \in I_n \quad \forall\, n$

$\Rightarrow \{x_n\} \subset I_n \quad \forall\, n$

$\Rightarrow \bigcup_{n=1}^{\infty} \{x_n\} \subseteq \bigcup_{n=1}^{\infty} I_n \quad \forall\, n$

$\Rightarrow \mathbb{R} \subseteq \bigcup_{n=1}^{\infty} I_n$ —— (1)

(Geometric series $\frac{a}{1-r}$)

$l(I_1) + l(I_2) + \cdots = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

$\mathbb{R}$ is contained in the union of interval where sum of length is 1. But the length of $\mathbb{R}$ is infinite, which is contradiction.

Hence $\mathbb{R}$ is uncountable.

Q:- Show that $[-1, 5] \sim [0, 1]$.

Sol:-

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Since we know that $[a,b] \sim [c,d]$ iff $f:[a,b] \longrightarrow [c,d]$ is defined as

$$f(x) = \left(\frac{b-x}{b-a}\right)c + \left(\frac{x-a}{b-a}\right)d \quad \text{— (1)}$$

Here let us define $f:[-1,5] \longrightarrow [0,1]$

i.e. putting $a=-1$, $b=5$, $c=0$, $d=1$ in eq(1)

$$f(x) = \left(\frac{5-x}{5-(-1)}\right)(0) + \left(\frac{x-(-1)}{5-(-1)}\right)(1)$$

$$= 0 + \left(\frac{x+1}{5+1}\right)$$

$$f(x) = \frac{x+1}{6}$$

$$f(x) = x/6 + 1/6 \quad \forall \ x \in [-1,5]$$

Clearly f is linear polynomial and we can easily show that "f" is bijective.

Thus we have defined bijective function between $[-1,5]$ and $[0,1]$

$$\Rightarrow [-1,5] \sim [0,1].$$

→ **Bijective function b/w the interval:-**

Q1:- Show that $[0,1] \sim [a,b]$.

Sol:-

Define a function $f:[0,1] \longrightarrow [a,b]$ by

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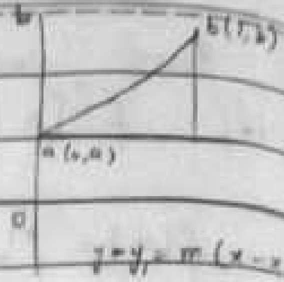
$$f(x) = a + (b-a)x$$



$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - a = \frac{b-a}{1-0}(x - 0)$$

$$y - a = (b-a)x$$

$$y = a + (b-a)x$$

$$f(x) = a + bx - ax$$

$$f(x) = bx + a - ax$$

$$f(x) = bx + a(1-x)$$

$$f(0) = a$$

$$f(1) = b$$

$$f'(x) = b - a > 0$$

Thus f is bijective function.

Q2:- Find bijective function from $[a,b]$ to $[0,1]$.

Sol:-

Since $f(x) = bx + (1-x)a$ is bijective function from $[0,1]$ to $[a,b]$.

let $y = bx + a(1-x)$

$\Rightarrow bx + a(1-x) = y$

$\Rightarrow bx + a - ax = y$

$\Rightarrow bx - ax = y - a$

$\Rightarrow x(b-a) = y - a$

$\Rightarrow x = \dfrac{y-a}{b-a}$

$\Rightarrow f(x) = \dfrac{y-a}{b-a}$

Thus f is bijective function from $[a,b]$ to $[0,1]$.

Q3:- Find a bijective function from $[a,b]$ to $[c,d]$.

Sol:-

Since $f:[a,b] \longrightarrow [0,1]$ is define by $f(x) = \dfrac{x-a}{b-a}$ is bijective.

And $g:[0,1] \longrightarrow [c,d]$ is define by $g(x) = dx + (1-x)c$ is bijective.

$$h = g \circ f : [a,b] \longrightarrow [c,d]$$

$$h(x) = g \circ f(x) : [a,b] \longrightarrow [c,d]$$

$$h(x) = g\left(\frac{x-a}{b-a}\right) = d\left(\frac{x-a}{b-a}\right) + \left(1 - \left(\frac{x-a}{b-a}\right)\right)c$$

$$= d\left(\frac{x-a}{b-a}\right) + \left(\frac{b-\cancel{a}-x+\cancel{a}}{b-a}\right)c$$

$$h(x) = d\left(\frac{x-a}{b-a}\right) + \left(\frac{b-x}{b-a}\right)c$$

Hence this is bijective.

Q4:- Show a bijective function $[1,3] \rightarrow [4,5]$.

Sol:-

Since we know that the function $f:[a,b] \longrightarrow [c,d]$ is defined as

$$f(x) = \left(\frac{b-x}{b-a}\right)c + \left(\frac{x-a}{b-a}\right)d \quad \text{---} \quad (1)$$

$\forall \; x \in [a,b]$

Here let us define $f:[1,3]\to[4,5]$
i.e.

putting $a=1, b=3, c=4, d=5$ in eq

$$f(x)=\left(\frac{3-x}{3-1}\right)4+\left(\frac{x-1}{3-1}\right)5$$

$$f(x)=\left(\frac{3-x}{\cancel{2}}\right)\overset{2}{\cancel{4}}+\left(\frac{x-1}{2}\right)5$$

$$f(x)=(3-x)2+(x-1)\frac{5}{2}$$

$$f(x)=6-2x+\frac{5x}{2}-\frac{5}{2}$$

$$f(x)=\frac{5x}{2}-2x+6-\frac{5}{2}$$

$$f(x)=\frac{5x-4x}{2}+\frac{12-5}{2}$$

$$f(x)=\frac{x}{2}+\frac{7}{2} \quad \forall\ x\in[1,3]$$

Clearly f is linear polynomial. Thus f is bijective function b/w $[1,3]$ and $[4,5]$.

Q 5:- Show a bijective function $[-1,5]\longrightarrow[\frac{1}{4},\frac{1}{2}]$.

Sol:-
Since we know that
$f:[a,b]\longrightarrow[c,d]$ is defined as

$$f(x)=\left(\frac{b-x}{b-a}\right)c+\left(\frac{x-a}{b-a}\right)d \quad \text{---} \ (1)$$

$\forall\ x\in[a,b]$

Here let us define $f: [-1, 5] \to [1/4, 1/2]$

i.e. putting $a = -1$, $b = 5$, $c = 1/4$, $d = 1/2$ in eq (1)

$$f(x) = \left(\frac{5-x}{5-(-1)}\right)\frac{1}{4} + \left(\frac{x-(-1)}{5-(-1)}\right)\frac{1}{2}$$

$$= \left(\frac{5-x}{5+1}\right)\frac{1}{4} + \left(\frac{x+1}{5+1}\right)\frac{1}{2}$$

$$= \left(\frac{5-x}{6}\right)\frac{1}{4} + \left(\frac{x+1}{6}\right)\frac{1}{2}$$

$$= \frac{5-x}{24} + \frac{x+1}{12}$$

$$= \frac{5-x+2x+2}{24}$$

$$= \frac{x+7}{24}$$

$$f(x) = \frac{x}{24} + \frac{7}{24} \quad \forall \ x \in [-1, 5]$$

clearly f is linear polynomial. Thus f is bijective function b/w $[-1, 5]$ and $[1/4, 1/2]$.

Q6: Show a bijective function $[3, 4] \sim [1, 100]$

Sol:-

Since we know that $[a, b] \sim [c, d]$ iff $f: [a, b] \to [c, d]$ is defined as

$$f(x) = \left(\frac{b-x}{b-a}\right)c + \left(\frac{x-a}{b-a}\right)d \longleftarrow (1)$$

$$\forall \ x \in [a, b]$$

Here let us define $f:[3,4]\to[1,100]$

i.e. putting $a=3, b=4, c=1, d=100$ in eq (i)

$$f(x)=\left(\frac{4-x}{4-3}\right)(1)+\left(\frac{x-3}{4-3}\right)(100)$$

$$f(x)=\frac{4-x}{1}+\frac{(x-3)}{1}100$$

$$f(x)=4-x+100x-300$$

$$f(x)=99x-296 \quad \forall \; x\in[3,4]$$

Clearly 'f' is linear polynomial and we can easily show that "f" is bijective.

Thus we have defined bijective function between $[3,4]$ and $[1,100]$

Theorem:-

Let A be an uncountable set and B be countable (or denumerable) subset of A then prove that $A\setminus B\sim A$.

Proof:-

Let B is a countable subset of A, so $B=\{b_1, b_2, b_3, ---\}$ and let $C=\{c_1, c_2, c_3, ---\}$ be a countable subset of $A\setminus B$.

Define a function

$f: A\setminus B\longrightarrow A$ by

$f(x) = x \quad \text{if} \quad x \notin C$

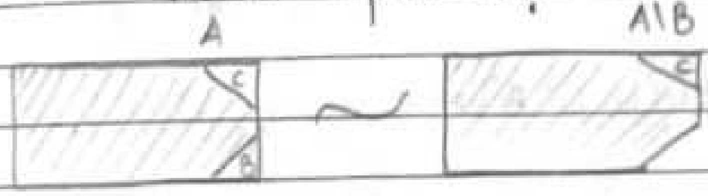
$f(C_{2n+1}) = C_{n+1} \quad \text{if} \quad n = 0, 1, 2, 3, \ldots$

$f(C_{2n}) = b_n \quad \text{if} \quad n = 1, 2, 3, \ldots$

OR (see figure below)

$$f(x) = \begin{cases} x & \text{if } x \notin C \\ c_n & \text{if } x \in C_{2n-1} \\ b_n & \text{if } x \in C_{2n} \end{cases}$$

$\Rightarrow f$ is bijective and so $A \setminus B \sim A$.
Hence proved!



$C \subseteq A \setminus B$

and $C \longrightarrow B \cup C$

$C = \{c_1, c_2, c_3, c_4, c_5, c_6, \ldots\}$

$B \cup C = \{b_1, c_1, b_2, c_2, b_3, c_3, \ldots\}$

Examples:-

Q1:- Show that $[0,1] \sim [0,1)$.

Sol:-

We have to show that $[0,1] \sim [0,1)$.

let $A = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$

then clearly $A \subseteq [0,1]$.

and $B = \{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\}$

then clearly $B \subseteq [0,1)$

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Since $[0,1]\setminus A = [0,1)\setminus B$

let us define a function $f:[0,1] \longrightarrow [0,1)$ by

$f(x) = x$ if $x \notin A$.
$f(0) = 0$
$f\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$ if $n = 0, 1, 2, \cdots$

Clearly f is bijective function or this relation of 1-1 correspondence can already shown as

$$[0,1] = A \cup ([0,1]\setminus A) = \{0, 1, \tfrac{1}{2}, \tfrac{1}{2^2}, \tfrac{1}{2^3}, \tfrac{1}{2^4}, \cdots$$

$$[0,1) = B \cup ([0,1)\setminus B) = \{0, \tfrac{1}{2}, \tfrac{1}{2^2}, \tfrac{1}{2^3}, \tfrac{1}{2^4}, \cdots\}$$

Hence $[0,1] \sim [0,1)$.

Q:- 2 Show that $[0,1] \sim (0,1)$.

Sol:-

We need to show that $[0,1] \sim (0,1)$
let $A = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\}$

then clearly $A \subseteq [0,1]$

and let $B = \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \cdots\}$

then clearly $B \subseteq (0,1)$.

Since $[0,1]\setminus A = (0,1)\setminus B$

Let us define a function
$f: [0,1] \to (0,1)$ by

$f(x) = x$ if $x \notin A$.

$f(0) = 1/2$

$f\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+2}}$ if $n = 0, 1, 2, 3, \ldots$

Clearly f is bijective function or this can already shown as

$[0,1] = A \cup ([0,1] \setminus A) = \{0, 1, 1/2, 1/2^2, 1/2^3, \ldots\}$

$(0,1) = B \cup ((0,1) \setminus B) = \{1/2, 1/2^2, 1/2^3, 1/2^4, \ldots\}$

Hence $[0,1] \sim (0,1)$.

OR Let $A = \{0, 1, 1/2, 1/3, 1/4, \ldots\} \subseteq [0,1]$

and $B = \{1/2, 1/3, 1/4, 1/5, 1/6, \ldots\} \subseteq (0,1)$

Since $[0,1] \setminus A = (0,1) \setminus B$

Define a function
$f: [0,1] \to (0,1)$ by

$f(x) = x$ if $x \notin A$
$f(0) = 1/2$

$f(1/n) = 1/n+2$ if $n = 1, 2, 3, \ldots$

Clearly f is bijective, so $[0,1] \sim (0,1)$.

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Q3:- Show that $[1,0) \sim (1,0]$.

Sol:-

We need to show that $[1,0) \sim (1,0]$.

Let us define a function $f : [1,0) \longrightarrow (1,0]$ by

$$f(x) = 1 - x \quad \forall \quad x \in [1,0).$$

Here f is a polynomial, so continuous and hence f is bijective function.

Thus $[1,0) \sim (1,0]$.

Note:-

Since $[a,b] \sim [0,1] \sim [0,1) \sim [c,d)$

$$\Rightarrow [a,b] \sim [c,d)$$

Also $[a,b] \sim [0,1] \sim (0,1) \sim (c,d)$

$$\Rightarrow [a,b] \sim (c,d).$$

Q4:- Show that $[0,1] \setminus \{1/2, 1/3, 1/4\} \sim [0,1]$

Sol:-

let $A = \{0, 1, \frac{1}{5}, \frac{1}{6}, \frac{1}{7} \dots\}$

then clearly $A \subseteq [0,1] \setminus \{1/2, 1/3, 1/4\}$

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and let $B = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$

then clearly $B \subseteq [0,1]$.

Since $([0,1] \setminus \{1/2, 1/3, 1/4\}) \setminus A = [0,1] \setminus B$.

let us define a function
$f: [0,1] \setminus \{1/2, 1/3, 1/4\} \longrightarrow [0,1]$ by

$f(x) = x \quad \forall \; x \notin A$
$f(0) = 0$
$f(1) = 1$

$f\left(1/_{n+3}\right) = \frac{1}{n} \quad$ if $n = 2, 3, 4, \ldots$

clearly f is bijective function.
Thus $[0,1] \setminus \{1/2, 1/3, 1/4\} \sim [0,1]$.

Q5:- Show that $[2,10] \sim [2,10)$.

Sol:-
let $A = \{2, 10, 2\tfrac{1}{2}, 2\tfrac{1}{2^2}, \ldots\}$

then clearly $A \subseteq [2,10]$

and let $B = \{2, 2\tfrac{1}{2}, 2\tfrac{1}{2^2}, 2\tfrac{1}{2^3}, \ldots\}$

then clearly $B \subseteq [2,10)$

Since $[2,10] \setminus A = [2,10) \setminus B$

let us define a function
$f: [2,10] \longrightarrow [2,10)$ by

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$f(x) = x$ if $x \notin A$

$f(2) = 2$

$f(10) = 2\frac{1}{2}$

$f\left(2\frac{1}{2^n}\right) = 2\frac{1}{2^{n+1}}$ if $n = 1, 2, 3, \ldots$

Clearly f is bijective function.

Thus $[2, 10] \sim [2, 10)$.

Q6:- Show that $[5,10] \sim [5,10] \setminus \{6,7,8,9\}$.

Sol:-

let $A = \{5, 10, 6, 7, 8, \ldots\}$

then clearly $A \subseteq [5, 10]$

and $B = \{5, 10, 5\frac{1}{2}, 5\frac{1}{2^2}, 5\frac{1}{2^3}, \ldots\}$

then clearly $B \subseteq [5, 10] \setminus \{6,7,8,9\}$

Since $[5,10] \setminus A = ([5,10] \setminus \{6,7,8,9\}) \setminus B$

let us define a function.

$f: [5,10] \to [5,10] \setminus \{6,7,8,9\}$ by

$f(x) = x$ if $x \notin A$

$f(5) = 5$

$f(10) = 10$

$f(n+5) = 5\frac{1}{2^n}$ if $n = 1, 2, 3, \ldots$

Clearly f is bijective function or this relation of 1-1 correspondence can already shown as

$[5,10] = A \cup ([5,10] \setminus A) = \{5, 10, 6, 7, 8, \dots\}$

$[5,10] \setminus \{6,7,8,9\} = B \cup ([5,10] \setminus \{6,7,8,9\}) = \{5, 10, 5\frac{1}{2}, 5\frac{1}{2^2}, 5\frac{1}{2^3}, \dots\}$

Hence $[5,10] \sim [5,10] \setminus \{6,7,8,9\}$.

Q7. Show that $(-\pi/2, \pi/2) \sim \mathbb{R}$.
means $(-\infty, \infty)$

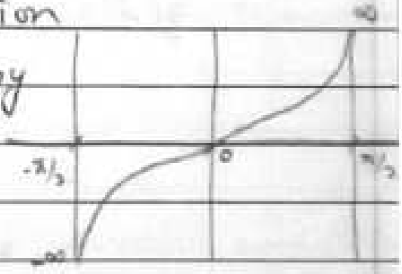
Sol:-

We need to show that $(-\pi/2, \pi/2) \sim \mathbb{R}$

Let us define a function

$f : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ by

$f(x) = \tan x$



→ $\tan x$ $\forall$ $x \in (-\pi/2, \pi/2)$ is an increasing function because $f'(x) = \sec^2 x \geq 0$; $\forall$ $x \in \text{Dom} f'$.

So this can be increased upto $\mathbb{R}$.
Hence f is bijective function.
Thus $(-\pi/2, \pi/2) \sim \mathbb{R}$.

Q8:- Show that $(a,b) \sim \mathbb{R}$.

Sol:-

Since $(a,b) \sim (-\pi/2, \pi/2) \sim \mathbb{R}$ —— (1)

we need to show that $(a,b) \sim \mathbb{R}$.

the function will be for eq ① is defined as the composition of "f" and "g", where

$f : (a,b) \longrightarrow (-\pi/2, \pi/2)$ defined by

$$f(x) = \left(\frac{b-x}{b-a}\right)\left(\frac{-\pi}{2}\right) + \left(\frac{x-a}{b-a}\right)\pi/2$$

and $g : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ defined by

$$g(x) = \tan x$$

$\Rightarrow$ we will show that $g \circ f : (a,b) \longrightarrow \mathbb{R}$

$$\Rightarrow g(f(x)) = g\left[\left(\frac{b-x}{b-a}\right)(-\pi/2) + \left(\frac{x-a}{b-a}\right)\pi/2\right]$$

$$= \tan\left[\left(\frac{b-x}{b-a}\right)(-\pi/2) + \left(\frac{x-a}{b-a}\right)(\pi/2)\right]$$

$$= \tan\left[\pi/2\left(\frac{-(b-x) + x-a}{b-a}\right)\right]$$

$$= \tan\left[\pi/2\left(\frac{-b+x+x-a}{b-a}\right)\right]$$

$$= \tan\left[\pi/2\left(\frac{2x-b-a}{b-a}\right)\right]$$

$$g \circ f = \tan\left[\pi/2\left(\frac{2x-b-a}{b-a}\right)\right]$$

$$h(x) = \tan\left[\pi/2\left(\frac{2x-b-a}{b-a}\right)\right] \quad \forall x \in (a,b)$$

So $h(x)$ is bijective function.
Thus $(a,b) \sim \mathbb{R}$.

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Q9:- Find a bijective function $(0,1) \sim \mathbb{R}$.

Sol:-

Since we know that $(a,b) \sim \mathbb{R}$ iff $f: (a,b) \to \mathbb{R}$ defined by

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-b-a}{b-a}\right)\right] \text{ — (A)}$$

Here let us define $f: (0,1) \to \mathbb{R}$.

i.e. putting $a=0$, $b=1$ in eq (A), we get

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-1-0}{1-0}\right)\right]$$

$$= \tan\left[\frac{\pi}{2}\left(\frac{2x-1}{1}\right)\right]$$

$$= \tan\left(\frac{\pi}{2}(2x-1)\right)$$

$$= \tan\left(\frac{\cancel{2}x\pi}{\cancel{2}} - \frac{\pi}{2}\right)$$

$$f(x) = \tan\left(x\pi - \frac{\pi}{2}\right)$$

Thus f is bijective function.

Hence $(0,1) \sim \mathbb{R}$.

Note:- $(-\infty, \infty) = \mathbb{R}$ is an unbounded interval which is equivalent to bounded interval.

Q10:- Show that $(-1,1) \sim \mathbb{R}$.

Sol:-

Since we know that $(a,b) \sim \mathbb{R}$ iff $f: (a,b) \to \mathbb{R}$ defined by

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$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-b-a}{b-a}\right)\right] \quad \text{—— (A)}$$

Here let us define a function $f:(-1,1) \to \mathbb{R}$ i.e. putting $a=-1$, $b=1$ in eq (A), we get

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-1-(-1)}{1-(-1)}\right)\right]$$

$$= \tan\left[\frac{\pi}{2}\left(\frac{2x-\cancel{1}+\cancel{1}}{1+1}\right)\right]$$

$$= \tan\left[\frac{\pi}{2}\left(\frac{\cancel{2}x}{\cancel{2}}\right)\right]$$

$$f(x) = \tan\left(\frac{\pi x}{2}\right) \quad \forall\ x \in (-1,1)$$

Thus f is bijective function.
Hence $(-1,1) \sim \mathbb{R}$.

Q:- Show that $(1,100) \sim \mathbb{R}$.

Sol:-
Since we know that $(a,b) \sim \mathbb{R}$ iff $f:(a,b) \to \mathbb{R}$ defined by

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-b-a}{b-a}\right)\right] \quad \text{—— (A)}$$

Here let us define a function $f:(1,100) \to \mathbb{R}$ i.e. putting $a=1$, $b=100$ in eq (A), we get

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-100-1}{100-1}\right)\right]$$

$$f(x) = \tan\left[\frac{\pi}{2}\left(\frac{2x-101}{99}\right)\right] \quad \forall\ x \in (1,100)$$

Thus f is bijective function.
Hence $(1,100) \sim \mathbb{R}$.

Q12:- Show that $[0,\infty) \sim (0,\infty)$.

Sol:-
We need to show that $[0,\infty) \sim (0,\infty)$.

let $A = \{0,1,2,3,----\}$
then clearly $A \subseteq [0,\infty)$

and $B = \{1,2,3,4,----\}$
then clearly $B \subseteq (0,\infty)$.

Here $[0,\infty) \setminus A = (0,\infty) \setminus B$.

let us define a function
$f:[0,\infty) \to (0,\infty)$ by

$f(x) = x$ if $x \notin A$

$f(n) = n+1$ if $n = 0,1,2,3,---$

Clearly f is bijective function.
Thus $[0,\infty) \sim (0,\infty)$.

Q13:- Show that $(0,\infty) \sim (-\infty,0)$.
Sol:-
let $A = \{1,2,3,4,----\}$
then $A \subseteq (0,\infty)$
and $B = \{-1,-2,-3,-4,----\}$
then $B \subseteq (-\infty,0)$

Since $(0,\infty)\setminus A = (-\infty,0)\setminus B$

Let us define a function
$f:(0,\infty)\to(-\infty,0)$ by

$$f(n) = -n \quad \text{if } n = 1,2,3,4,\ldots$$

Clearly f is bijective function.
Thus $(0,\infty)\sim(-\infty,0)$.

Q14:- Show that $(0,\infty)\sim(-\infty,\infty)$.

Sol:-
we need to show that $(0,\infty)\sim(-\infty,\infty)$

let us define a function.
$f:(0,\infty)\longrightarrow(-\infty,\infty)$ by

$$f(x) = x - \frac{1}{x}$$

if $x\to 0$ then $f\to -\infty$
if $x\to\infty$ then $f\to\infty$

So clearly f is bijective function.
Thus $(0,\infty)\sim(-\infty,\infty)$.

Q15:- Show that $[a,\infty)\sim[0,\infty)$.

Sol:-
we need to show that $[a,\infty)\sim[0,\infty)$

let us define a function
$f:[a,\infty)\longrightarrow[0,\infty)$ by

$f(x) = x - a$

Clearly f is bijective function.
Thus $[a,\infty) \sim [0,\infty)$.

Prepared by : Asim Marwat
MSc Mathematics (Final)
University of Peshawer
Mob : 03151949572
Session [redacted] 2020-2021
gmail : asimmarwat41@gmail.com