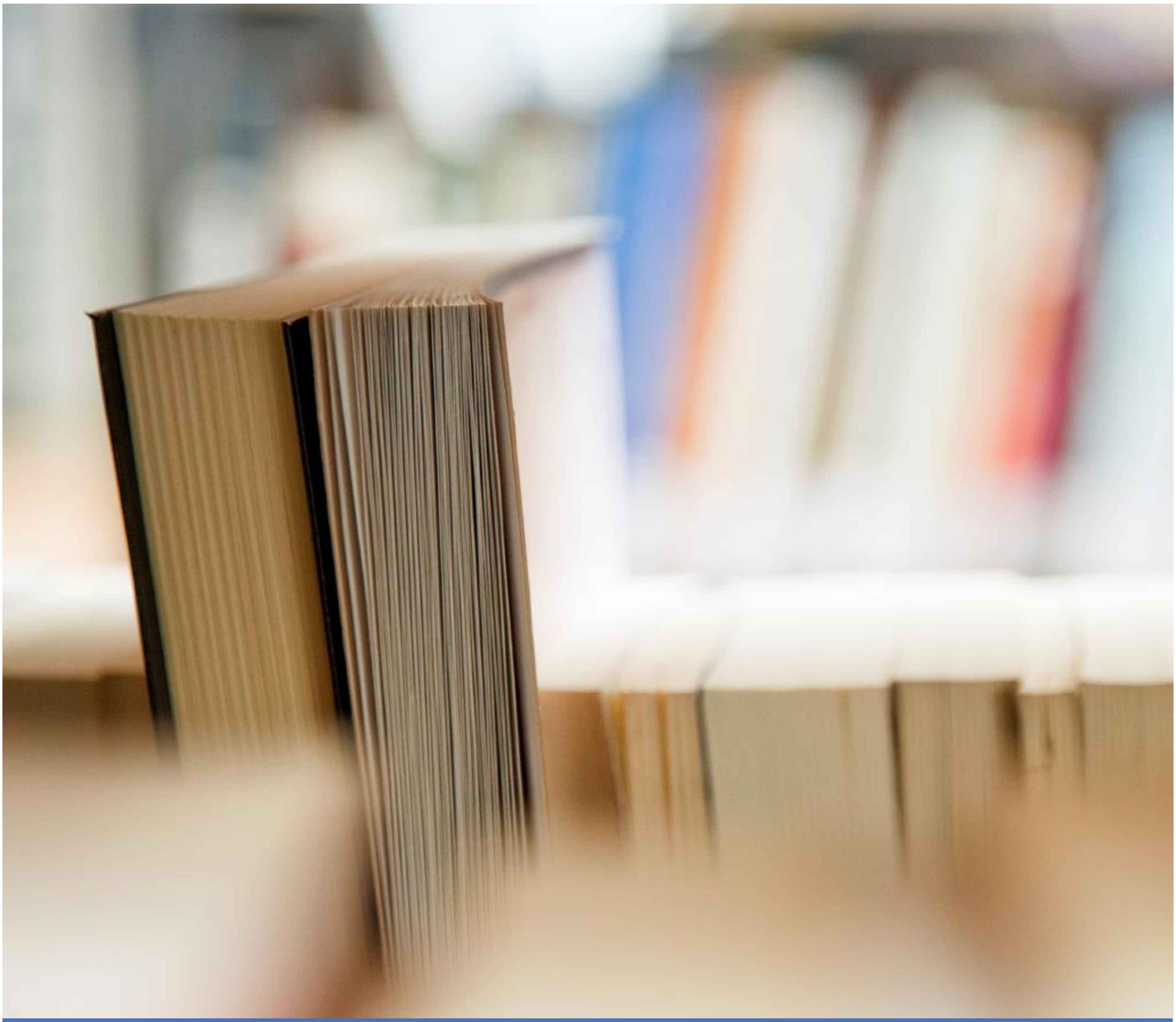

Affine & Euclidean Geometry

Short Notes

These notes are available at MathCity.org

BS-Mathematics
4th Semester

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Title of the Course: Affine and Euclidean Geometry
Credit Hours: 3

Course Outline:
Vector Spaces and Affine Geometry: Collinearity of three points, ratio AB/BC. Linear combinations and linear dependent set versus affine combinations and affine dependent sets. Classical theorems in affine geometry: Thales, Menelaus, Ceva, Desargues. Affine subspaces, affine maps. Dimension of a linear subspace and of an affine subspace.

Euclidean Geometry: Scalar product, Cauchy-Schwartz inequality: norm of a vector, distance between two points, angles between two non-zero vectors. Pythagoras theorem, parallelogram law, cosine and sine rules. Elementary geometric loci.

Orthogonal Transformations: Isometries of plane (four types), Isometries of space (six types). Orthogonal bases.

Platonic Polyhedra: Euler's theorem on finite planar graphs. Classification of regular polyhedra in space. Isometries of regular polygons and regular polyhedra.

Recommended Books:
1. E. Rees, Notes on Geometry, Springer, 2004.
2. M. A. Armstrong, Groups and Symmetry, Springer, 1998.
3. H. Eves, Fundamentals of Modern Elementary Geometry, Jones and Bartlett Publishers International, 1992
4. S. Stahl, The Poincare Half-Plane A Gateway to Modern Geometry, Jones and Bartlett Publishers International, 1993.

CONTENTS

3 Classical Theorems in Affine Geometry

4 Platonic Polyhedra

Linear & Affine Subspaces

1.1 Linear Spaces & Subspaces

We give a brief account of subspaces, linear span, linear independence and dependence, basis and dimension of linear spaces.

1.1.1 Linear Spaces: Let V be a non-empty set and F be a field. Suppose that for every $\lambda \in F$ and every $v \in V,\ \lambda v \in V$. Then V is called a linear space or vector space over F if

(i) V is an abelian group under addition.
(ii) $\lambda(u+v) = \lambda u + \lambda v$
(iii) $(\lambda + \mu)u = \lambda u + \mu u$
(iv) $\lambda(\mu u) = (\lambda\mu)u$
(v) $1.u = u.1 = u$, 1 being the unity (multiplicative identity) of F
$\forall\ \lambda, \mu \in F;\ u, v \in V$.

The elements of F are called scalars and the elements of V are called vectors.

Examples of Linear Spaces:

1. Let F be an arbitrary field and F^n be the set of all n-tuples of elements in F. Then F^n is a vector space over F under the following operations:

   (i) Vector Addition:
   $$(\lambda_1, \lambda_2, \dots, \lambda_n) + (\mu_1, \mu_2, \dots, \mu_n) = (\lambda_1, +\mu_1, \lambda_2 + \mu_2, \dots, \lambda_n + \mu_n)$$
   (ii) Scalar Multiplication:
   $$\lambda(\lambda_1, \lambda_2, \dots, \lambda_n) = (\lambda\lambda_1, \lambda\lambda_2, \dots, \lambda\lambda_n)$$

   In particular, R^n over R is a vector space called the n-dimensional Euclidean space.

2. Let $P(t)$ denote the set of all polynomials of the form
   $$p(t) = \lambda_o + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_s t^s\ ; \quad \lambda_i \in F, \qquad s \in N$$

   Then $P(t)$ is a vector space over F using the following operations:

   (i) Vector Addition:
   Here $p(t) + q(t)$ in $P(t)$ is the usual addition of polynomials.
   (ii) Scalar Multiplication:
   Here $\lambda p(t)$ in $P(t)$ is the usual operation of the product of a scalar a and a polynomial $p(t)$.

   The zero polynomial 0 is the zero vector in $P(t)$.

3. Let $P_n(t)$ denote the set of all polynomials $p(t)$ over a field F, where the degree of $p(t)$ is less than or equal to n along with the zero polynomial. Then $P_n(t)$ is a vector space with respect to the usual operations of addition of polynomials and of multiplication of a polynomial by a scalar. The degree of the zero polynomial is undefined.
4. The set M_n of all $n \times n$ matrices with entries from a field F is a linear space over F.
5. Every field is a linear space over itself.

1.1.2 Linear Subspaces:

Let V be a linear space over F and W be a non-empty subset of V. Then W is a subspace of V if W is itself a linear space over F with respect to the operations of vector addition and scalar multiplication on V.

Theorem: A non-empty subset W of V is a subspace of V iff W is closed under vector addition and scalar multiplication i.e.,

(i) $w_1, w_2 \in W$ implies $w_1 + w_2 \in W$
(ii) $\lambda \in F, w \in W$ implies $\lambda w \in W$

Proof: Let W be a subspace of V. Then W is itself a linear space and hence closed under vector addition and scalar multiplication.

Conversely, suppose that W is closed under vector addition and scalar multiplication. We have to show that W is a vector space. Let $w_1, w_2 \in W$. Since W is closed under scalar multiplication, $(-1)w_2 = -w_2 \in W$. Since W is closed under vector addition, $w_1 + (-w_2) = w_1 - w_2 \in W$. Therefore, W is a subgroup of V. Since V is abelian, W is also abelian. The remaining axioms of a vector space hold in W because they hold in the larger set V. Hence W is a linear space over F.

Both properties (i) and (ii) may be combined into the following equivalent single statement:

For every $w_1, w_2 \in W; \lambda, \mu \in F$, the linear combination $\lambda w_1 + \mu w_2 \in W$. So, a linear subspace may be defined as:

A subset W of a vector space V is called a subspace of V if $\lambda w_1 + \mu w_2 \in W$ for all $w_1, w_2 \in W$ and all $\lambda, \mu \in F$.

Any vector space V automatically contains two subspaces: the set $\{0\}$ consisting of the zero vector alone and the whole space V itself. These are sometimes called the trivial subspaces of V.

Example: Consider a system of homogeneous equations $Mx = 0$ in n-unknowns, i.e., a system of the form

$$\alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = 0$$

$$\alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n = 0$$

Let S be the solution set of this system, then S is a subspace of R^n because linear combination of any two solutions is again a solution and hence belongs to S.

1.1.3 Linear Combination of Vectors:

Let V be a linear space over a field F. A vector $v \in V$ is a linear combination of vectors $v_1, v_2, \dots, v_m \in V$ if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_m \in F$ such that $v = \lambda_1 v + \lambda_2 v_2 + \cdots + \lambda_m v_m$. The coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ are called weights.

A linear combination is said to be trivial if each $\lambda_i = 0$, it is called non-trivial if at least one of $\lambda_i \neq 0$. A subspace may also be defined as:

A subset W of a vector space V is called a subspace of V if it is closed under linear combination.

Remarks: If $v \in V$ is a linear combination of $v_1, v_2, \dots, v_m \in V$, then the representation of v in terms of $v_1, v_2, \dots, v_m$ may not be unique. That is, v may be expressible as a linear combination of these vectors in more than one way.

Examples:

1. Every vector in R^3 is a linear combination of the vectors

$$e_1 = (1, 0, 0), \qquad e_2 = (0, 1, 0), \qquad e_3 = (0, 0, 1)$$

For if $x = (x_1, x_2, x_3) \in R^3$, then

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

2. The vector $v = (2, 7, 8) \in R^3$ is not a linear combination of the vectors

$$v_1 = (1, 2, 3), \qquad v_2 = (1, 3, 5), \qquad v_3 = (1, 5, 9)$$

For if

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$$

or

$$(2, 7, 8) = \lambda_1 (1, 2, 3) + \lambda_2 (1, 3, 5) + \lambda_3 (1, 5, 9)$$

Then

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 2 \\ 2\lambda_1 + 3\lambda_2 + 5\lambda_3 &= 7 \\ 3\lambda_1 + 5\lambda_2 + 9\lambda_3 &= 8 \end{aligned}$$

In matrix form

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 9 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

This system of equations is inconsistent and does not have a solution. Hence v cannot be written as a linear combination of v_1, v_2, v_3.

3. The vector $v = (1, -1, -3) \in R^3$ is a linear combination of the vectors

$$v_1 = (2, 1, 0), \;\; v_2 = (1, 1, 1), \;\; v_3 = (0, 1, 1)$$

because

$$v = (-1)v_1 + 3v_2 + (-3)v_3$$

Note that v can also be written as

$$v = 2v_1 + (-3)v_2 + (0)v_3$$

1.1.4 Linear Span:
If $S = \{v_1, v_2, \dots, v_m\} \subset V$. Linear span of S, denoted by Span(S), is the set of all linear combinations of elements of S. That is,

$$\text{Span}(S) = \{\textstyle\sum_{i=1}^{m} \lambda_i v_i \mid v_i \in V, \lambda_i \in F\}$$

Vectors $v_1, v_2, \dots, v_m$ are said to span V or to form a spanning set of V if every $v \in V$ is a linear combination of the vectors $v_1, v_2, \dots, v_m$. That is, if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_m \in F$ such that $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m$. Spanning set for $\{0\}$ is the empty set.

Remarks:

(i) Suppose $v_1, v_2, \dots, v_m$ span V. Then for any vector w, the set $w, v_1, v_2, \dots, v_m$ also spans V.

(ii) Suppose $v_1, v_2, \dots, v_m$ span V and suppose v_k is a linear combination of some of the other v's. Then the v's without v_k also span V.

(iii) Suppose $v_1, v_2, \dots, v_m$ span V and suppose one of the v's is the zero vector. Then the v's without the zero vector also span V.

Examples:

1. The vectors $w_1 = (1, 1, 1), w_2 = (1, 1, 0), w_3 = (1, 0, 0)$ form a spanning set of R^3.

   For $x = (x_1, x_2, x_3) \in R^3$, let

   $$x = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3$$

   $$= \lambda_1(1, 1, 1) + \lambda_2(1, 1, 0) + \lambda_3(1, 0, 0)$$

   Then

   $$(x_1, x_2, x_3) = (\lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_2, \lambda_1)$$

   This implies that

   $$\lambda_1 + \lambda_2 + \lambda_3 = x_1,\ \ \lambda_1 + \lambda_2 = x_2,\ \ \lambda_1 = x_3$$

   or

   $$\lambda_1 = x_3,\ \lambda_2 = x_2 - x_3,\ \lambda_3 = x_1 - x_2$$

   Hence

   $$x = x_3 w_1 + (x_2 - x_3) w_2 + (x_1 - x_2) w_3$$

2. The vectors e_1, e_2, e_3 form a spanning set of R^3.
3. The vectors $(1, 2, 3), (1, 3, 5), (1, 5, 9)$ does not form a spanning set of R^3.

   Here, e.g., $(2, 7, 8) \in R^3$ cannot be written as a linear combination of these vectors.

1.1.5 Linear Independence:
The vectors $v_1, v_2, \dots, v_m \in V$ are said to be linearly independent over F if

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = 0$$

implies each $\lambda_i = 0,\ \lambda_i \in F,\ i = 1, 2, \dots, m$. Otherwise, they are said to be linearly dependent.

In other words, the vectors $v_1, v_2, \dots, v_m \in V$ are said to be linearly dependent over F if

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = 0$$

and not all λ_i are zero, $\lambda_i \in F,\ i = 1, 2, \dots, m$.

A subset $\{v_1, v_2, \dots, v_m\}$ is said to be linearly independent or linearly dependent according as the vectors $v_1, v_2, \dots, v_m$ are linearly independent or linearly dependent.

Remarks:

1. A nonzero vector v, by itself is linearly independent. This is because $av = 0$ and $v \neq 0$ imply $a = 0$.

2. The zero vector is linearly dependent.

3. The empty set is defined to be linearly independent.

Examples:

1. The vectors $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ belonging to R^3 are linearly independent over R.

2. The vectors $(3,0,-3), (-1,1,2), (4,2,-2), (2,1,1)$ belonging to R^3 are linearly dependent over R.

3. Let $V = P_3(t)$, then the vectors

$$t^3 - 3t^2 + 5t + 1, \qquad t^3 - t^2 + 8t + 2, \qquad 2t^3 - 4t^2 + 9t + 5$$

belonging to $P_3(t)$ are linearly independent.

4. Let V be the vector space of all functions defined on R to R. Then the vectors

$$2, \;\; 4\sin^2 t, \;\; \cos^2 t$$

are linearly dependent in V.

Exercises:

1. What conditions must $\alpha, \beta, \gamma, \delta$ satisfy so that the matrices

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

in M_{22} are linearly dependent?

2. Determine whether the following vectors in R^4 are linearly independent or linearly dependent:

(i) $(1,3,-1,4), (3,8,-5,7), (2,9,4,23)$

(ii) $(1,-2,4,1), (2,1,0,-3), (1,-6,1,4)$

3. Determine k so that the vectors

$$(1,-1,k-1), (2,k,-4), (0,2+k,-8) \in R^3$$

are linearly dependent.

Theorem: Let $S = \{v_1, v_2, \ldots, v_m\} \subset V$ and $v \in \text{Span}(S)$, then v is uniquely expressible as a linear combination of elements of S iff S is linearly independent.

Proof: Suppose S is linearly independent. Since $v \in \text{Span}(S)$, there exist $\lambda_1, \lambda_2, \dots, \lambda_m \in F$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$$

Suppose v is also expressible as

$$v = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_m v_m$$

for $\mu_1, \mu_2, \dots \mu_m \in F$.

Then the above two equations give

$$(\lambda_1 - \mu_1)v_1 + (\lambda_2 - \mu_2)v_2 + \cdots + (\lambda_m - \mu_m)v_m = 0$$

Since $v_1, v_2, \dots, v_m$ are linearly independent

$$\lambda_1 - \mu_1 = \lambda_2 - \mu_2 = \cdots = \lambda_m - \mu_m = 0$$

or

$$\lambda_1 = \mu_1, \lambda_2 = \mu_2, \dots, \lambda_m = \mu_m$$

i.e., v is uniquely expressible as a linear combination of elements of S.

Conversely, suppose that every $v \in \text{Span}(S)$ has a unique representation as a linear combination of elements of S. Consider

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = 0$$

Since $0 \in \text{Span}(S)$ and

$$(0)v_1 + (0)v_2 + \cdots + (0)v_m = 0$$

the uniqueness of representation of 0 in terms of $v_1, v_2, \dots, v_m$ implies that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$$

That is, S is linearly independent.

1.1.6 Basis and Dimension:

A set B of linearly independent vectors spanning a vector space V is called a basis for V. The basis vectors of a linear space are non-zero. A vector space V is said to be of finite dimension n or n-dimensional, written $\dim V = n$ if V has a basis with n elements. For example, the vectors e_1, e_2, e_3 are linearly independent and span R^3. So, the set $\{e_1, e_2, e_3\}$ forms a basis of R^3 and R^3 is a 3-dimensional vector space.

Basis for $\{0\}$ is the empty set and it is a 0-dimensional vector space.

Since a basis B is linearly independent and spans V, every $v \in V$ is uniquely expressible as a linear combination of elements of B.

Remarks: The definitions and theorems discussed above for V hold for subspaces of V as well.

Exercise: Find a basis and dimension of the subspace W of R^4 spanned by:

(i) $(1, 4, -1, 3), (2, 1, -3, -1), (0, 2, 1, -5)$

(ii) $(1, -4, -2, 1), (1, -3, -1, 2), (3, -8, -2, 7)$

1.2 Affine Subspaces

Let V be a linear space and W be a subspace of V, then W contains the origin (the zero vector). But its coset $W + v = A$(say) where $v \notin W$ does not pass through the origin and hence is not a subspace of V. In fact, A is not closed under linear combination. Still, there are certain kinds of linear combinations of elements of A that belong to A. More specifically, if $\lambda, \mu \in F$ and $a, b \in A$ then $\lambda a + \mu b \in A$ with the condition that $\lambda + \mu = 1$. In order to see this, note that the elements of A are of the form $w + v$ where $w \in W, v \in V$.

Now, if $\lambda a + \mu b \in A$, there is a $w \in W$ such that

$$\lambda a + \mu b = w + v$$

Also, there are $w_1, w_2 \in W$ such that

$$a = w_1 + v, b = w_2 + v$$

So that

$$\lambda(w_1 + v) + \mu(w_2 + v) = w + v$$

$$(\lambda w_1 + \mu w_2) + (\lambda + \mu)v = w + v$$

Since $v \notin W$, we must have

$$\lambda + \mu = 1$$

Which is the required condition.

If $v \in W$, we have $W + v = W$. So that, $\lambda a + \mu b \in W$ for all $a, b \in W$ and $\lambda, \mu \in R$ and in particular, for $\lambda + \mu = 1$.

The above discussion motivates the following intrinsic definition of an affine subspace of a linear space V over F.

1.2.1 Definition:
A subset A of a linear space V is called an affine subspace of V if $\lambda a + \mu b \in A$ for all $a, b \in A$ and all $\lambda, \mu \in F$ such that $\lambda + \mu = 1$.

Since, $\mu = 1 - \lambda$, the above definition may also be written as:

A subset A of a linear space V is called an affine subspace of V if $\lambda a + (1 - \lambda)b \in A$ for all $a, b \in A$ and all $\lambda \in F$.

Elements of an affine space are called points instead of vectors.

Example: Consider a system of non-homogeneous equations $Mx = c$ in n-unknowns, i.e., a system of the form

$$\alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = c_1$$

$$\alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = c_2$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n = c_m$$

Then the solution set S of this system is an affine subspace. To see this, let x, x' be any two solutions, i.e., $x, x' \in S$ and $\lambda, \mu \in F$, then

$$M(\lambda x + \mu x') = \lambda Mx + \mu Mx' = \lambda c + \mu c = (\lambda + \mu)c$$

So that $\lambda x + \mu x'$ is also a solution, i.e., $\lambda x + \mu x' \in S$ iff $\lambda + \mu = 1$.

Theorem: A subset $A \subset V$ is an affine subspace of V if and only if A is of the form $W + v$ for some $v \in V$ and a subspace $W \subset V$.

Proof: It has already been shown in the discussion above, that any subset A of the form $W + v$ is an affine subspace. Now suppose that A is an affine subspace. For any $a \in A$, consider the set

$$W = A - a = \{x - a : x \in A\}$$

We show that W is a linear subspace. Let $x - a, y - a \in W$ then,

$$(x - a) + (y - a) + a = x + y - a$$

is a linear combination of elements of A, the sum of coefficients being $1 + 1 - 1 = 1$. So

$$(x - a) + (y - a) + a = x + y - a \in A$$

and hence

$$(x - a) + (y - a) \in W$$

i.e., W is closed under vector addition.

Now, let $\lambda \in R$ and $x - a \in W$ then

$$\lambda(x - a) + a = \lambda x + (1 - \lambda)a$$

Since $\lambda + (1 - \lambda) = 1$, so $\lambda(x - a) + a \in A$ and hence $\lambda(x - a) \in W$ i.e., W is closed under scalar multiplication. Hence $W = A - a$ is a linear subspace and $A = W + a$.

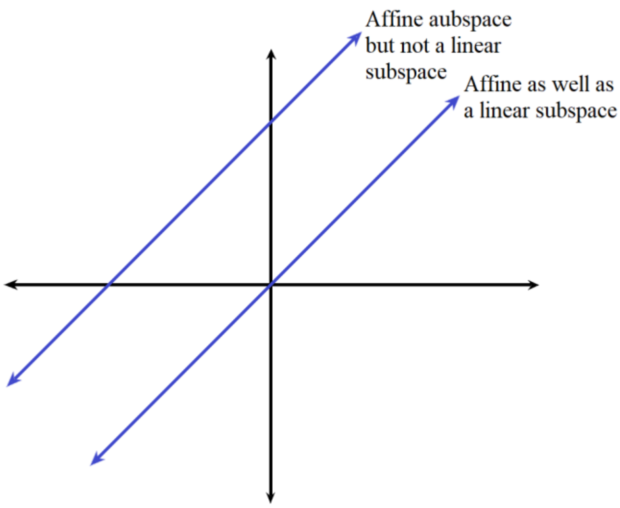
This theorem allows us to alternatively define an affine subspace as follows:

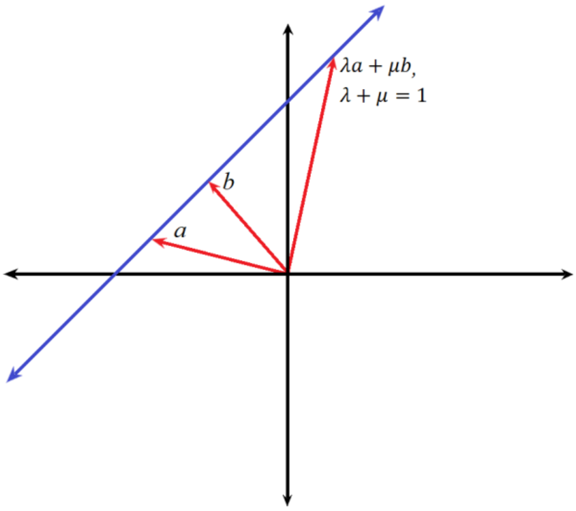
A subset A of V is called an affine subspace of V if $A = W + v$, where W is a subspace of V and $v \in V$.

In fact, affine subspaces are translates of linear subspaces.

Examples:

1. Lines in R^2 passing through the origin are affine as well as linear subspaces. However, the lines which do not pass through the origin are affine subspace but not linear subspaces.





2. Consider the 3-dimentional Euclidean space R^3. The plane $W = \{(x, y, 0): x, y \in R\}$ contains two of the three coordinate axes and passes through the origin and is a subspace of R^3. If we translate W by any vector $x \in R^3$, we get a plane A parallel to W which may not pass through the origin. Such a plane is an affine subspace of R^3. If $x \in W$, the affine subspace A coincides with W and is a linear subspace. Points, lines, planes and R^3 itself are affine subspaces of R^3.
An affine subspace of R^n is also called a flat in R^n.

3. Let S' be the solution set of a system of non-homogeneous equations $Mx = c$ in n-unknowns and S be the solution set of the associated homogeneous equations $Mx = 0$. If x' is a particular solution of $Mx = c$, then

$$S' = x' + S$$

Solutions of a non-homogeneous system of equations are obtained by translating solutions of the corresponding homogeneous system using a particular solution of the non-homogeneous system. So, S' is an affine subspace of R^n obtained by translating the linear subspace S of R^n.

Theorem: If $a, b \in A$ and A is an affine subspace, then $A - a = A - b$.

Proof: Since A is an affine subspace and $a, b \in A$, $W_1 = A - a$ and $W_2 = A - b$ are linear subspaces and $W_1 + a = A = W_2 + b$. We show that $W_1 = W_2$.
For any $x \in A$, $x - a \in W_1$ and in particular,

$$b - a \in W_1 \qquad (\because b \in A)$$

Since W_1 is a linear subspace

$$W_1 + b - a = W_1$$

This implies that

$$W_1 + b = W_1 + a$$

or

$$W_1 + b = W_2 + b \qquad (\because W_1 + a = A = W_2 + b)$$

So,

$$W_1 = W_2$$

i.e.,

$$A - a = A - b$$

Theorem: An affine subspace A is a linear subspace iff $0 \in A$. In other words, a subset A is a linear subspace iff

(i) $0 \in A$

(ii) for any $a, b \in A$ and $\lambda \in F$, $a\lambda + (1-\lambda)b \in A$

Proof: Suppose A is an affine subspace and $0 \in A$. For any $a \in A$,

$$\lambda a = \lambda a + (1-\lambda)(0) \in A$$

So that A is closed under scalar multiplication.

Also, for any $a, b \in A$, we have

$$\frac{1}{\lambda}a, \frac{1}{1-\lambda}b \in A$$

because A is closed under scalar multiplication. So,

$$a + b = \lambda\left(\frac{1}{\lambda}a\right) + (1-\lambda)\left(\frac{1}{1-\lambda}b\right) \in A$$

Hence A is a linear subspace.

Conversely, suppose that A is a linear subspace. Then $0 \in A$ and A is closed under all linear combinations and in particular, for any $a, b \in A$ and $\lambda \in F$, $a\lambda + (1-\lambda)b \in A$.

With the above theorem, a linear subspace may also be defined as follows:

A subset W of a linear space V is a subspace of V if

(i) $0 \in W$

(ii) for any $u, v \in W$ and $\lambda \in F$, $u\lambda + (1-\lambda)v \in W$

We see that the difference between linear subspaces and affine subspaces is that linear subspaces necessarily contain the zero vector, whereas affine subspaces may not i.e., linear subspaces are special types of affine subspaces which contain the zero vector.

Remarks:

1. The empty set $\{\} \subset V$ is not a linear subspace because $0 \notin \{\}$ but it is trivially an affine subspace of V.
2. Any singleton $\{v\} \subset V$ is an affine subspace of V.
3. The only singleton which is a linear subspace of V is $\{0\}$.

1.2.2 Affine Combination: Let V be a linear space over a field F. A vector (or a point) $v \in V$ is an affine combination of $v_1, v_2, \dots, v_m \in V$ if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_m \in F$ such that $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$.

If A is an affine subspace of R^n, then the set of all affine combinations of two points $a, b \in A$ is the line passing through a and b.

We may also define an affine subspace as follows:

A subset A of V is called an affine subspace of V if A is closed under affine combination.

Theorem: A point $v \in V$ is an affine combination of $v_0, v_1, \dots, v_m \in V$ iff $v - v_0$ is a linear combination of the translated points $v_1 - v_o, v_2 - v_0, \dots, v_m - v_0$.

Proof: The following two equations

$$v - v_0 = \lambda_1(v_1 - v_0) + \lambda_2(v_2 - v_0) + \cdots + \lambda_m(v_m - v_0)$$

$$v = (1 - \lambda_1 - \lambda_2 - \cdots - \lambda_m)v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$$

are equivalent, where $\lambda_1, \lambda_2, \dots, \lambda_m \in R$.

So, any $v \in R^n$ is an affine combination of $v_0, v_1, \dots, v_m \in R^n$ iff $v - v_0$ is a linear combination of the translated points $v_1 - v_0, v_2 - v_0, \dots, v_m - v_0$.

Examples:

1. For $v = (4,1), v_0 = (1,2), v_1 = (2,5), v_2 = (1,3), v_3 = (-2,2) \in R^2$, we see if v is expressible as an affine combination of v_0, v_1, v_2, v_3. For this, consider the linear combination

$$\lambda_1(v_1 - v_0) + \lambda_2(v_2 - v_0) + \lambda_3(v_3 - v_0) = v - v_0$$

This implies that

$$\lambda_1(1,3) + \lambda_2(0,1) + \lambda_3(-3,0) = (3,-1)$$

Which gives the following system of two equations in three unknowns

$$\lambda_1 - 3\lambda_3 = 3$$

$$3\lambda_1 + \lambda_2 = -1$$

and in matrix form

$$\begin{bmatrix} 1 & 0 & -3 & 3 \\ 3 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & 9 & -10 \end{bmatrix}$$

So, the above system of equations is consistent, and the general solution of this system of equations is $\lambda_1 = 3\lambda_3 + 3$, $\lambda_2 = -9\lambda_3 - 10$, with λ_3 free. Setting $\lambda_3 = 0$, we get $\lambda_1 = 3$ and $\lambda_2 = -10$. So,

$$3(v_1 - v_0) - 10(v_2 - v_0) + 0(v_3 - v_0) = v - v_0$$

Which gives v as an affine combination of v_0, v_1, v_2 and v_3 i.e.,

$$v = 8v_0 + 3v_1 - 10v_2 + 0v_3$$

Note that v is not uniquely expressible as an affine combination of v_0, v_1, v_2, v_3 because, for example, if we take $\lambda_3 = 1$ we get,

$$v = 13v_0 + 6v_1 - 19v_2 + v_3$$

However, if we write v as an affine combination of basis vectors, then the representation is unique because basis vectors are linearly independent. In fact, there is no need to follow the above procedure. Instead, we write v as a linear combination of the given basis vectors. Now this linear combination is an affine combination of the given basis vectors iff the weights sum to 1. The following example elaborates this.

2. Consider the vectors $v = (1,2,2), v_1 = (4,0,3), v_2 = (0,4,2), v_3 = (5,2,4) \in R^3$. We see if v is an affine combination of the other vectors.

It may be easily verified that v_1, v_2, v_3 form a basis for R^3. So, we write v as a linear combination of these vectors. Let,

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$$

or

$$(1,2,2) = \lambda_1(4,0,3) + \lambda_2(0,4,2) + \lambda_3(5,2,4)$$

Which gives

$$\begin{bmatrix} 4 & 0 & 5 & 1 \\ 0 & 4 & 2 & 2 \\ 3 & 2 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$$

So that

$$v = \frac{2}{3}v_1 + \frac{2}{3}v_2 - \frac{1}{3}v_3$$

We see that the sum of the weights is not 1. So, v is not expressible as an affine combination of v_1, v_2, v_3.

1.2.3 Convex Combination: A convex combination of $v_1, v_2, \dots v_m \in V$ is a linear combination $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$ and each $\lambda_i \geq 0$. In other words, a convex combination of $v_1, v_2, \dots v_m$ is an affine combination with non-negative weights.

Example: For $v_1, v_2 \in R^n$, the set of convex combinations of v_1 and v_2 is the line segment joining v_1 and v_2.

1.2.4 Dimension of an Affine Subspace: If A is an affine subspace of V and $a \in A$ then the dimension of A is the dimension of the linear subspace $A - a$ of V.

1.2.5 Affine Span: If $S = \{v_1, v_2, \dots, v_m\} \subset V$. Affine span of S, denoted by $\text{Aff}(S)$, is the set of all affine combinations of elements of S. That is,

$$\text{Aff}(S) = \{\textstyle\sum_{i=1}^{m} \lambda_i v_i | \; v_i \in V, \; \sum_{i=1}^{m} \lambda_i = 1\}$$

A subset S is said to span an affine subspace A or to form a spanning set of A if every $a \in A$ is an affine combination of the elements of S. That is, if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_m \in F$ such that $a = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$.

1.2.6 Affine Independence: The vectors (or points) $v_0, v_1, \dots, v_m \in V$ are said to be affine independent if $\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_m v_m = 0$ with $\lambda_0 + \lambda_1 + \cdots + \lambda_m = 0$ implies

$$\lambda_o = \lambda_1 = \cdots = \lambda_m = 0$$

Otherwise, they are said to be affine dependent.

A subset $\{v_o, v_1, \dots, v_m\}$ is said to be affine independent or affine dependent according as the vectors $v_o, v_1, \dots, v_m$ are affine independent or affine dependent.

Exercise: Show that a subset $\{v_o, v_1, \dots, v_m\}$ is affinely independent if and only if $\{ v_1 - v_o, v_2 - v_o, \dots, v_m - v_o\}$ is linearly independent subset.

Solution: Note that the following two equations

$$\lambda_1(v_1 - v_0) + \lambda_2(v_2 - v_0) + \cdots + \lambda_m(v_m - v_0) = 0$$

$$-(\lambda_1 + \lambda_2 + \cdots + \lambda_m)v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = 0$$

are equivalent. Now it is easy to see that $\{v_o, v_1, \dots, v_m\}$ is affinely independent if and only if $\{ v_1 - v_o, v_2 - v_o, \dots, v_m - v_o\}$ is linearly independent.

Remarks: As in case of linear spaces, an element $a \in \mathrm{Aff}(S)$ is uniquely expressible as an affine combination of elements of S iff S is affine independent.

1.2.7 Affine Basis:
An affine independent subset B which spans an affine subspace A is called an affine basis of A. If B has $r+1$ elements, then the dimension of A is r.

If $\{e_1, e_2, \dots, e_r\}$ is a basis of a linear subspace W, then an affine basis for W is $\{0, e_1, e_2, \dots, e_r\}$.

Since a basis B is affine independent and spans A, every $a \in A$ is uniquely expressible as an affine combination of elements of B.

1.2.8 Barycentric or Affine Coordinates:
If B is an affine basis of the affine subspace A, then each point $a \in A$ has a unique representation as an affine combination of the elements of B. The coefficients of this affine combination are called barycentric or affine coordinates of a.

Let $a = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$, $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, then these two equations are equivalent to the single equation

$$\begin{bmatrix} a \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} v_2 \\ 1 \end{bmatrix} + \cdots + \lambda_m \begin{bmatrix} v_m \\ 1 \end{bmatrix}$$

or

$$[a'] = \lambda_1 [v_1'] + \lambda_2 [v_2'] + \cdots + \lambda_m [v_m']$$

and row reduction of the augmented matrix $[v_1' \;\; v_2' \;\; \dots \;\; v_m' \;\; a']$ produces the barycentric coordinates of a.

Example: If $a = (5, 3) \in \mathrm{Aff}\,(\{(1,7), (3,0), (9,3)\})$ and $(1,7), (3,0), (9,3)$ are affine independent, we can find the barycentric coordinates of a. Consider the augmented matrix

$$\begin{bmatrix} 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{12} \end{bmatrix}$$

Thus, the barycentric coordinates of a are $1/4$, $1/3$ and $5/12$.

So,

$$a = \frac{1}{4}(1,7) + \frac{1}{3}(3,0) + \frac{5}{12}(9,3)$$

1.2.9 Hyperplane in R^n:
An affine subspace H of R^n whose dimension is $n-1$ is called a hyperplane in R^n.

Hyperplanes in Euclidean spaces arise as the perpendicular bisectors of line segments.

Remarks: If H is a linear hyperplane of R^n then there is a non-zero $x \in R^n$ such that $H = \{x\}^{\perp}$ This is because the dimension of H is $n-1$ and any orthonormal basis of H can be extended by a vector $x \in R^n$ to an orthonormal basis of R^n.

Theorem: If $a, b \in R^n$ with $a \neq b$, then $B = \{x|\ d(x,a) = d(x,b)\}$ is a hyperplane in R^n.

Proof: Since

$$d\left(\frac{a+b}{2}, a\right) = d\left(\frac{a+b}{2}, b\right)$$

So,

$$\frac{a+b}{2} \in B$$

We show that $H = B - (a+b)/2$ is a linear subspace. Since d is translation invariant and $d(x,a) = d(x,b)$, this implies that in H,

$$d\left(x - \frac{a+b}{2}, a - \frac{a+b}{2}\right) = d\left(x - \frac{a+b}{2}, b - \frac{a+b}{2}\right)$$

This implies that

$$d\left(x - \frac{a+b}{2}, \frac{a-b}{2}\right) = d\left(x - \frac{a+b}{2}, -\frac{a-b}{2}\right)$$

or

$$d\left(x - \frac{a+b}{2}, c\right) = d\left(x - \frac{a+b}{2}, -c\right)$$

where

$$c = \frac{a-b}{2}$$

Hence

$$H = \left\{x - \frac{a+b}{2} : d\left(x - \frac{a+b}{2}, c\right) = d\left(x - \frac{a+b}{2}, -c\right)\right\} = \{x | d(x,c) = d(x,-c)\}$$

Now, if $c, e_2, e_3, \dots, e_n$ is an orthogonal basis for R^n, then $e_2, e_3, \dots, e_n$ is a basis for H.

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Inner Product and Euclidean Geometry

2.1. Inner Product Spaces

Inner product spaces are vector spaces with an inner product defined on them. We discuss inner product and norm induced by an inner product.

2.1.1 Inner Product: Let V be a vector space over the field R of real numbers. A mapping denoted by $<.,.>: V \times V \to R$ is said to be a real inner product or simply an inner product on V if it satisfies the following axioms:

(i) $< v, v > \geq 0$ and $< v, v > = 0 \Leftrightarrow v = 0$ (Positive definite property)

(ii) $< u, v > = < v, u >$ (Symmetric property)

(iii) $< \alpha u + \beta v,\ w > = \alpha < u, w > + \beta < v, w >$ (Linear property)

The vector space V with an inner product is called an inner product space.

2.1.2 Euclidean Inner Product: Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in R^n$. Then the product

$$< x, y > = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

satisfies all the properties of inner product and is called the Euclidean inner product on R^n. This product is also called the scalar product.

R^n with this inner product is called the n-dimensional Euclidean space.

Exercises:

1. Show that the Euclidean inner product defined above satisfies the axioms of an inner product.

2. Let $x = (x_1, x_2), y = (y_1, y_2) \in R^2$. Show that the following is an inner product on R^2

$$< x, y > = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3 x_2 y_2$$

2.1.3 Norm of a Vector: Let V be an inner product space and $v \in V$. Then the norm or length of v is a real number denoted by $\|v\|$ and is defined as

$$\|v\| = \sqrt{< v, v >}$$

If $\|v\| = 1$, then v is called a unit vector or is said to be a normalized vector. Any nonzero vector $v \in V$ can be normalized by multiplying it by $1/\|v\|$. Thus $v/\|v\|$ is a normalized vector.

Norm of any $x = (x_1, x_2, \dots, x_n) \in R^n$ with respect to the Euclidean inner product or scalar product is

$$\|x\| = \sqrt{< x, x >} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

This is the length or magnitude of x in R^n.

Example: If we consider the inner product $< x, y > = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ in R^2 where $x = (x_1, x_2), y = (y_1, y_2)$, the norm of $x = (3, 4)$ with respect to this inner product is

$$\|x\| = \sqrt{< x, x >} = 9 - 24 + 48 = 33$$

However, with respect to the Euclidean inner product the norm of the same vector x is

$$\|x\| = \sqrt{9 + 16} = 5$$

which represents the length or magnitude of x in R^2.

Note: R^n will always represent n-dimensional Euclidean space if no inner product is specified.

Exercise: Normalize $y = (1, 2, 1) \in R^3$.

2.1.4 Distance Between Two Points: In R^n each vector $x = (x_1, x_2, \dots, x_n)$ corresponds to a point P and is called the position vector of P. The real numbers $x_1, x_2, \dots, x_n$ are called the coordinates of P.

If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are the position vectors of P and Q respectively then the distance between P and Q denoted by PQ or QP is the length of the line segment joining P and Q and is given by

$$PQ = d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

2.1.4 Cauchy-Schwarz Inequality: Let u, v be elements of an inner product space V over R. Then

$$|< u, v >| \le \|u\| \, \|v\|$$

Proof: If $v = 0$, then both sides are zero and the equality holds. Now let $v \neq 0$.

Then, $\forall\, t \in R$,

$$\begin{aligned} 0 &\le \|u - tv\|^2 \\ &= < u - tv, u - tv > \\ &= < u, u > - t < u, v > - t < v, u > + t^2 < v, v > \\ &= < u, u > - 2t < u, v > + t^2 < v, v > \ (\because < u, v > = < v, u >) \end{aligned}$$

Let $t = < u, v > / \|v\|^2$. Then,

$$0 \leq \|u\|^2 - \frac{2 < u, v >< u, v >}{\|v\|^2} + \frac{| < u, v > |^2}{\|v\|^4} \quad (\because t^2 = |t|^2)$$

$$= \|u\|^2 - \frac{| < u, v > |^2}{\|v\|^2}$$

Multiplying both sides by $\|v\|^2 > 0$, we have

$$0 \leq \|u\|^2 \, \|v\|^2 - | < u, v > |^2$$

That is,

$$| < u, v > |^2 \leq \|u\|^2 \, \|v\|^2$$

Taking square root of both the sides, we get

$$| < u, v > | \leq \|u\| \, \|v\|$$

2.1.5 Angle Between Two Vectors:

Let V be an inner product space over R and $u, v \in V$. Then the angle θ between u and v is defined as

$$\cos\theta = \frac{< u, v >}{\|u\| \, \|v\|}, \quad 0 \leq \theta \leq \pi$$

If $< u, v > = 0$, we say that u is orthogonal to v and write as $u \perp v$. The relation of being orthogonal is symmetric, for $< u, v > = 0 \Leftrightarrow < v, u > = 0$. Thus if $u \perp v$, then $v \perp u$.

Two vectors u and v are said to be orthogonal if and only if $< u, v > = 0$ or $\theta = \pi/2$.

Examples:

1. Let $x = (1, -1, 2)$, $y = (-1, 1, 1) \in R^3$. Then $< x, y > = -1 - 1 + 2 = 0$.

So, $x \perp y$. Similarly, $(1, -1, 1, -1), (-1, 2, 2, 1) \in R^4$ are orthogonal.

2. A vector x orthogonal to $(1, 1, 2), (0, 1, 3) \in R^3$ is

$$x = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= e_1 - 3e_2 + e_3$$

$$= (1, -3, 1)$$

and a unit vector in the direction of x is

$$\frac{x}{\|x\|} = \left(\frac{1}{11}, -\frac{3}{11}, \frac{1}{11}\right)$$

Theorem: Let V be an inner product space and $u, v \in V$, $k \in R$, then the norm in V satisfies the following axioms:

(i) $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$

(ii) $\|kv\| = |k|\|v\|$

(iii) $\|u + v\| \leq \|u\| + \|v\|$ (The Triangle Inequality)

Proof:

(i) Since

$$\|v\| = \sqrt{< v, v >} \text{ and } < v, v > \geq 0,$$

We have

$$\|v\| \geq 0.$$

Also,

$$\|v\| = 0 \iff v = 0$$

(ii) Here,

$$\begin{aligned} \|kv\|^2 &= < kv, kv > \\ &= k^2 < v, v > \\ &= |k|^2 \|v\| \end{aligned}$$

Taking square root of both sides, we get

$$\|kv\| = |k|\|v\|$$

(iii) In this case

$$\begin{aligned} \|u + v\|^2 &= < u + v, u + v > \\ &= < u, u > + < u, v > + < v, u > + < v, v > \\ &= \|u\|^2 + 2 < u, v > + \|v\|^2 \\ &\leq \|u\|^2 + 2|< u, v >| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\| \, \|v\| + \|v\|^2 \end{aligned}$$

That is,

$$\|u + v\|^2 \leq (\|u\| + \|v\|)^2$$

Taking square root of both sides, we have

$$\|u + v\| \leq \|u\| + \|v\|$$

Exercises:

1. Let $(2, -1, 2)$, $(3, -1, 4)$, $(2, 1, 1) \in R^3$. Find a unit vector orthogonal to $(2, -1, 2)$ and $(3, -1, 4)$ and a unit vector orthogonal to $(2, -1, 2)$ and $(2, 1, 1)$.

2. If u is orthogonal to v, show that every scalar multiple of u is also orthogonal to v.

3. Show that 0 is the only vector which is orthogonal to every vector.

2.2. The Laws of Cosine and Sines

We can prove the laws of cosine and sines using Cauchy-Schwarz inequality.

2.2.1 The Law of Cosine: In any triangle ABC,

$$(BC)^2 = (CA)^2 + (AB)^2 - 2(CA)(AB)\cos\alpha$$

$$(CA)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos\beta$$

$$(AB)^2 = (BC)^2 + (CA)^2 - 2(BC)(CA)\cos\gamma$$

where, $m\angle CAB = \alpha, m\angle ABC = \beta, m\angle BCA = \gamma$.

Proof: Let x, y, z be the vectors along the sides of the triangle ABC such that $x + y = z$ and $\|x\| = BC, \|y\| = CA$, $\|z\| = AB$.

Now,

$$\begin{aligned} \|x\|^2 &=< x, x > \\ &= < z - y, z - y > \\ &= \|z\|^2 + \|y\|^2 - 2 < z, y > \\ &= \|z\| + \|y\|^2 - 2\,\|z\|\|y\|\cos\alpha \quad (\because < z, y > = \|z\|\;\|y\|\cos\alpha) \end{aligned}$$

i.e.,

$$(BC)^2 = (CA)^2 + (AB)^2 - 2(CA)(AB)\cos\alpha$$

Similarly,

$$(CA)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos\beta$$

$$(AB)^2 = (BC)^2 + (CA)^2 - 2(BC)(CA)\cos\gamma$$

2.2.2 Pythagoras Theorem: If ABC is a right-angled triangle with $m\angle BCA = \gamma = \pi/2$, then

$$(AB)^2 = (BC)^2 + (CA)^2$$

Proof: With $\gamma = \pi/2$ in the law of cosine $(AB)^2 = (BC)^2 + (CA)^2 - 2(BC)(CA)\cos\gamma$, we get the result.

2.2.3 Parallelogram Law: For any two elements $x, y \in R^n$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Proof: Here,

$$\begin{aligned}\|x+y\|^2 &= < x+y, x+y > \\ &= < x,x > + < x,y > + < y,x > + < y,y > \\ &= \|x\|^2 + 2 < x,y > + \|y\|^2\end{aligned}$$

Similarly,

$$\|x-y\|^2 = \|x\|^2 - 2 < x,y > + \|y\|^2$$

Adding these two equations we get the result.

2.2.4 The Law of Sines: In any tringle ABC,

$$\frac{AB}{\sin\gamma} = \frac{BC}{\sin\alpha} = \frac{CA}{\sin\beta}$$

where, $m\angle CAB = \alpha, m\angle ABC = \beta, m\angle BCA = \gamma.$

Proof: Since none of the angles α, β, γ exceeds π, we can write

$$\begin{aligned}\frac{\sin\alpha}{\sin\beta} &= \frac{\sqrt{1-\cos^2\alpha}}{\sqrt{1-\cos^2\beta}} \\ &= \sqrt{\frac{1-\left(\frac{(BC)^2-(CA)^2-(AB)^2}{2(CA)(AB)}\right)^2}{1-\left(\frac{(CA)^2-(AB)^2-(BC)^2}{2(AB)(BC)}\right)^2}} \\ &= \frac{BC}{CA}\end{aligned}$$

This implies that

$$\frac{BC}{\sin\alpha} = \frac{CA}{\sin\beta}$$

Similarly,

$$\frac{AB}{\sin\gamma} = \frac{BC}{\sin\alpha}$$

Hence,

$$\frac{AB}{\sin\gamma} = \frac{BC}{\sin\alpha} = \frac{CA}{\sin\beta}$$

2.3 Euclidean Constructions

A construction is, in some sense, a physical substantiation of the abstract. In Greek times, geometric constructions of figures and lengths were restricted to the use of only a straightedge and compass (or in Plato's case, a compass only). No markings could be placed on the straightedge to be used to make measurements.

Because of the prominent place Greek geometric constructions held in Euclid's Elements, these constructions are also known as Euclidean constructions.

2.3.1 The Euclidean Tools:

Following are the first three postulates in Euclid's Elements (an ancient mathematical treatise consisting of thirteen books) based on which all Euclidean constructions are made:

(i) A straight line can be drawn from any point to any point.

(ii) A finite straight line can be produced continuously in a straight line.

(iii) A circle may be described with any center and distance(radius).

These postulates restrict constructions to only those that can be made in a permissible way with straightedge and compass and these two instruments are called the Euclidean tools.

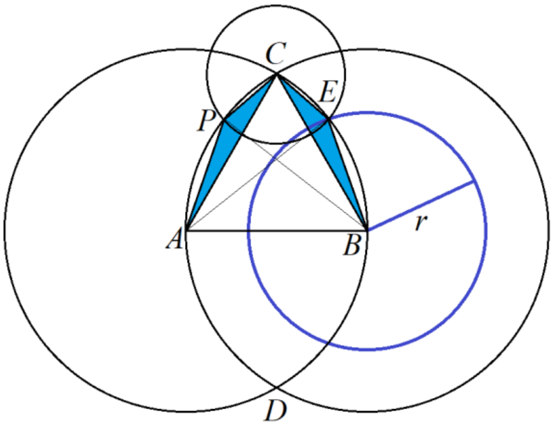
The first two postulates tell us what we can do with the Euclidean straightedge; we are permitted to draw as much as may be desired of the straight line determined by any two given points. The third postulate tells us what we can do with the Euclidean compass; we are permitted to draw the circle of given center and having any straight line segment radiating from that center as a radius-that is, we are permitted to draw the circle of given center and passing through a given point.

Note that neither instrument is to be used for directly transferring distances. This means that the straightedge cannot be marked, and the compass must be regarded as having the characteristic that if either leg is lifted from the paper, the instrument immediately collapses. For this reason, a Euclidean compass is often referred to as the collapsing compass; it differs from the modern or fixed compass which has a fixable aperture and retains its opening and hence can be used as a divider for transferring distances and copying circles directly without any further steps.

It may seem that the modern compass is more powerful than the collapsing compass. Curiously enough, such turns out not to be the case; any construction performable with the modern compass can also be carried out (in perhaps a longer way) by means of the collapsing compass. We prove this in the following theorem.

The Compass Equivalence Theorem: The collapsing and fixed compasses are equivalent.

Proof: To prove the theorem, it suffices to show that any circle $C(B;\ r)$ can be congruently copied using only the straightedge and collapsing compass so that a given point A serves as the centre of the copy. Consider the given circle and a point A as shown.



We need to construct a circle of radius r centered at A, using only the straightedge and collapsing compass. First construct the circle $C(A; AB)$ centered at A with radius AB and the circle $C(B; BA)$ centered at B with radius BA. These circles intersect in two points C and D. The circles $C(B; r)$ and $C(A; AB)$ intersect in a point E and the circles $C(C; CE)$ and $C(B; BA)$ intersect in a point P. We claim that $AP = r$.

Note that, by SSS

$$\Delta PCB \equiv \Delta ECA$$

Thus,

$$\angle PCB \equiv \angle ECA$$

and

$$m\angle PCB - m\angle ACB = m\angle ECA - m\angle ACB.$$

Therefore,

$$\angle PCA \equiv \angle ECB.$$

Now,

$$CP = CE \text{ and } AC = BC$$

Thus, by SAS

$$\Delta APC \equiv \Delta BEC$$

This implies that,

$$AP = BE = r.$$

We have transferred the distance r and now the circle with center A and radius $AP = r$ is constructible.

Remark: There are constructions which cannot be made with the Euclidean tools alone. Three famous constructions of this sort are:

1. The duplication of the cube, or the problem of constructing the edge of a cube having twice the volume of a given cube.
2. The trisection of an angle, or the problem of dividing a given arbitrary angle into three equal parts.
3. The quadrature of the circle, or the problem of constructing a square having an area equal to that of a given circle.

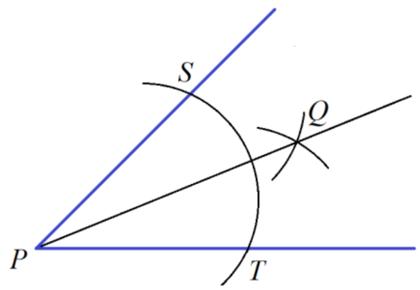
Note: We shall always use the modern or fixed compass for Euclidean constructions.

2.3.2 The Method of Loci:
Before we discuss the method of loci to the solution of geometric constructions, it is of great value to know a considerable number of loci that are constructible straight lines and circles e.g., the locus of points at a given distance from a given point is the circle having the given point as center and the given distance as radius.

Here are a few such loci.

1. Angle Bisector: The locus of points equidistant from two given intersecting lines consists of the bisector of the angle formed by the two given lines.

Construction: Let P be the vertex of the given angle.



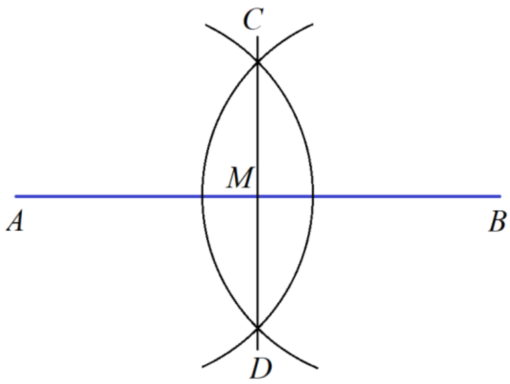
With centre P, draw an arc cutting the arms of the angle at S and T.

With centres S and T, draw arcs of the same radius meeting at Q.

Since triangles SPQ and TPQ are congruent by SSS, $\angle SPQ \equiv \angle TPQ$. Then PQ is the bisector of the given angle.

2. Perpendicular Bisector: The locus of points equidistant from two given points is the perpendicular bisector of the segment joining the given points.

Construction: Given points A and B.



With centres A and B, draw two arcs of the same radius meeting at C and D.
Let M be the point where CD meets AB. Note that, by SSS

$$\Delta ACD \equiv \Delta BCD$$

So,

$$\angle ACD \equiv \angle BCD \text{ and } \angle ACM \equiv \angle BCM$$

Now, by SAS

$$\Delta ACM \equiv \Delta BCM$$

This implies that,

$$AM = BM$$

and

$$m\angle AMC = m\angle BMC = \pi/2$$

which means that CM is the right bisector of AB.

Exercise: Construct a perpendicular to a line from a point not on the line.

3. Parallel Lines: The locus of points at a given distance from a given line consists of the two lines parallel to the given line and at the given distance from it.

Construction: Let l be the given line.
Draw a perpendicular k on l meeting at a point A on l.
With center A, draw an arc of radius $2r$ meeting k at B and C.
Draw perpendicular bisectors m and n of AB and AC respectively.
Then m and n are the required parallel lines.

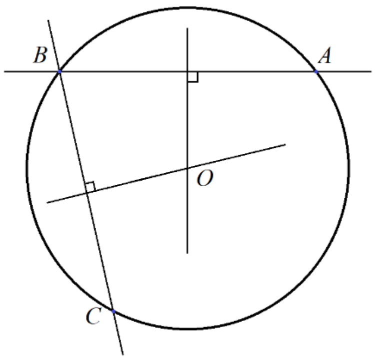
Now we discuss the method of loci. The solution of a construction problem very often depends upon first finding some key point e.g., the problem of drawing a circle through three given points is essentially solved once the center of the circle is located. Similarly, the problem of drawing a circle of a given radius and tangent to two intersecting lines is essentially solved once the center of the circle has been found. The key point satisfies certain conditions, and each condition considered alone generally restricts the position of the key point to a certain locus. The key point is found at the intersections of certain loci.

Thus, the most basic method used to solve geometric construction problems is to locate the key points by using the intersection of loci, which is usually referred to as the method of loci. We illustrate this method in the following.

Examples:

1. We construct the circle passing through three non-collinear points A, B, C.

The sought center O of the circle through A, B, C must be equidistant from A, B and from B, C. The first condition places O on the perpendicular bisector of AB, and the second condition places O on the perpendicular bisector of BC. The point O is thus found at the intersection, if it exists, of these two perpendicular bisectors. Since the three given points are not collinear, there is exactly one solution.



So, we may write the solution as:

Draw perpendicular bisectors l and m of AB and BC respectively.

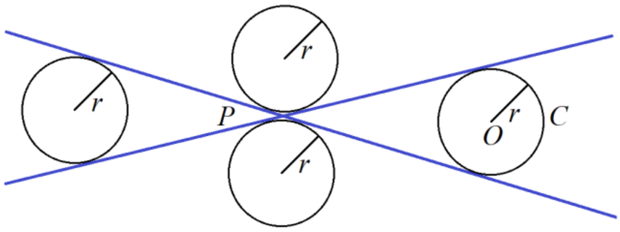
Let $O = l \cap m$.

With center O and radius OA, draw the circle $C(O; OA)$.

Note: If the three given points are collinear, there is no solution.

2. Given two intersecting lines l and m and a fixed radius r, we construct a circle of radius r that is tangent to the two given lines.

It is often useful to sketch the expected solution. We refer to this sketch as an analysis figure which include all possible solutions. The given lines l and m are intersecting at P.



The analysis figure indicates that there are four solutions. The constructions of all four solutions are basically the same, so in this case it suffices to show how to construct one of the four circles.

Since we are given the radius of the circle, it is enough to construct O, the center of the circle C. Since we only have the straightedge and compass, there are three ways to construct a point, namely, as the intersection of two lines, two circles, or a line and a circle.

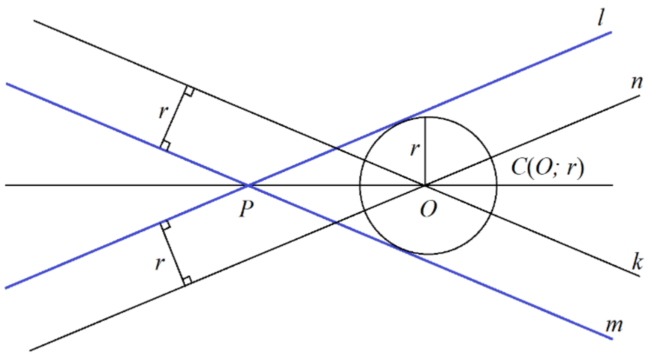
The centre O of circle C is equidistant from both l and m and therefore lies on the following constructible loci:

(i) an angle bisector,

(ii) a line parallel to l at distance r from l,

(iii) a line parallel to m at distance r from m.

and any two of these loci determine the point O.



Having done the analysis, now we write up the solution:

Construct line n parallel to l at distance r from l.

Construct line k parallel to m at distance r from m.

Let $O = n \cap k$.

With centre O and radius r, draw the circle $C(O;\ r)$.

Classical Theorems in Affine Geometry

3.1 Sensed Magnitudes

One of the innovations of modern elementary geometry is the employment of signed magnitudes. It was the extension of the number system to include both positive and negative numbers that led to this forward step in geometry.

We start a study of sensed magnitudes with some definitions and a notation.

3.1.1 Positive and Negative Segments:

Consider a line passing through the points A and B. If we choose one direction along the line as the positive direction and the other direction as the negative direction, then a segment AB on the line is called positive or negative according as the direction from A to B is the positive or negative direction of the line. The symbol $\overline{AB}$ instead of AB is used to denote the signed distance from A to B. The segment $\overline{AB}$ is called a sensed or directed segment with A as its initial point and B as its terminal point.

$\overline{AB}$ and $\overline{BA}$ are equal in magnitude but opposite in direction i.e., $\overline{AB} = -\overline{BA}$ or $\overline{AB} + \overline{BA} = 0$.

Note that for any point $A,$ we have $\overline{AA} = 0$.

3.1.2 Range of Points and Complete Range:

A set of collinear points is said to constitute a range of points, and the straight line on which they lie is called the base of the range.

A range which consists of all the points of its base is called a complete range.

3.1.3 Basic Theorems:

We are now in a position to establish a few basic theorems about sensed line segments.

Theorem: If $A,\ B,\ C$ are any three collinear points, then

$$\overline{AB} + \overline{BC} + \overline{CA} = 0.$$

Proof: If the three points are distinct, three cases may arise.

(i) The point C lies between the points A and B. Then

$$\overline{AB} = \overline{AC} + \overline{CB}$$

or

$$\overline{AB} - \overline{AC} - \overline{CB} = 0$$

or

$$\overline{AB} + \overline{BC} + \overline{CA} = 0$$

(ii) The point C lies on the prolongation of $\overline{AB}$. Then

$$\overline{AB} + \overline{BC} = \overline{AC}$$

or

$$\overline{AB} + \overline{BC} - \overline{AC} = 0$$

or

$$\overline{AB} + \overline{BC} + \overline{CA} = 0$$

(iii) The point C lies on the prolongation of $\overline{BA}$. Then

$$\overline{CA} + \overline{AB} = \overline{CB}$$

or

$$\overline{AB} - \overline{CB} + \overline{CA} = 0$$

or

$$\overline{AB} + \overline{BC} + \overline{CA} = 0$$

It is simple to get the same result when one or more points coincide.

Corollary: Let O be any point on the line of segment AB. Then $\overline{AB} = \overline{OB} - \overline{OA}$.

Proof: Since $A,\ B,\ O$ are collinear, we have $\overline{AB} + \overline{BO} + \overline{OA} = 0$. This implies that

$\overline{AB} = -\overline{BO} - \overline{OA} = \overline{OB} - \overline{OA}$.

Euler's Theorem: If $A,\ B,\ C,\ D$ are any four collinear points, then

$$\overline{AD}.\overline{BC} + \overline{BD}.\overline{CA} + \overline{CD}.\overline{AB} = 0$$

Proof:

$$\begin{aligned}
&\overline{AD}.\overline{BC} + \overline{BD}.\overline{CA} + \overline{CD}.\overline{AB}\\
&= \overline{AD}(\overline{DC} - \overline{DB}) + \overline{BD}(\overline{DA} - \overline{DC}) + \overline{CD}(\overline{DB} - \overline{DA})\\
&= \overline{AD}.\overline{DC} - \overline{AD}.\overline{DB} + \overline{BD}.\overline{DA} - \overline{BD}.\overline{DC} + \overline{CD}.\overline{DB} - \overline{CD}.\overline{DA}\\
&= \overline{AD}.\overline{DC} - \overline{AD}.\overline{DB} + \overline{AD}.\overline{DB} - \overline{BD}.\overline{DC} + \overline{BD}.\overline{DC} - \overline{AD}.\overline{DC}\\
&= 0
\end{aligned}$$

3.2 MENELAUS, CEVA AND DESARGUES THEOREMS

The theorems of Menelaus and Ceva when stated in terms of sensed magnitudes, in contrast to their original versions, deal elegantly with many problems involving collinearity of points and concurrency of lines. We now study these remarkable theorems.

3.2.1 The ratio $\overline{\boldsymbol{AP}}/\overline{\boldsymbol{PB}}$: If A, B, P are distinct collinear points, we define the ratio in which P divides the segment $\overline{AB}$ to be the ratio $\overline{AP}/\overline{PB}$.

The value of this ratio is independent of any direction assigned to the line AB. If P lies between A and B, the division is said to be internal, otherwise the division is said to be external.

Denoting $\overline{AP}/\overline{PB}$ by r, note that:

(i) If P lies on the prolongation of $\overline{BA}$, then $-1 < r < 0$.

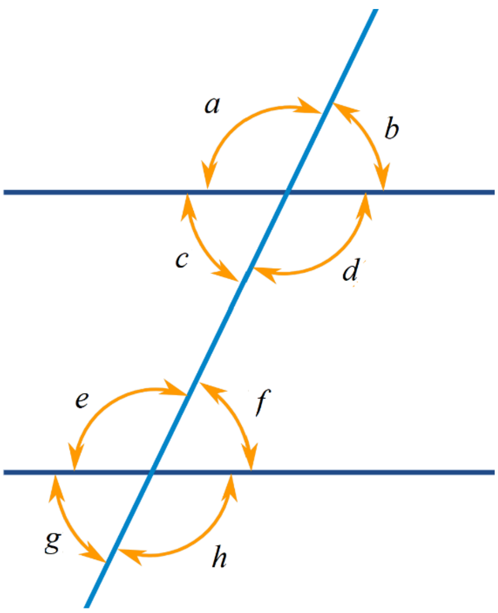
(ii) If P lies between A and B, then $0 < r < \infty$.

(iii) If P lies on the prolongation of $\overline{AB}$, then $-\infty < r < -1$.

(iv) If P coincides with A but not with B, then $r = 0$.

(v) If P coincides with B but not with A, then r is undefined. We indicate this by writing $r = \infty$.

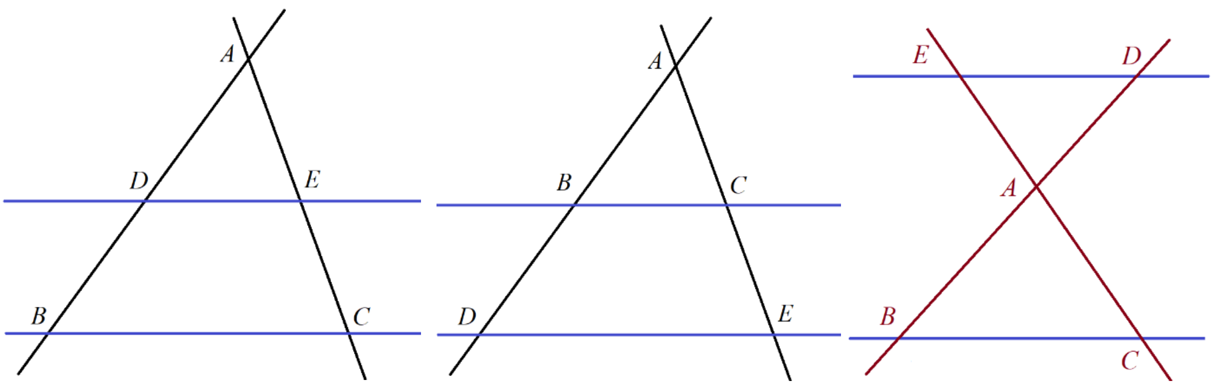
3.2.2 Angles Associated with Parallel Lines: If a transversal cuts two parallel lines, we refer as follows to the angles formed:

Vertically Opposite Angles (Congruent)	$\angle a \equiv \angle d$ $\angle b \equiv \angle c$ $\angle f \equiv \angle g$ $\angle e \equiv \angle h$	
Corresponding Angles (Congruent)	$\angle a \equiv \angle e$ $\angle b \equiv \angle f$ $\angle c \equiv \angle g$ $\angle d \equiv \angle h$	
Alternate Interior Angles (Congruent)	$\angle c \equiv \angle f$ $\angle d \equiv \angle e$	
Alternate Exterior Angles (Congruent)	$\angle a \equiv \angle h$ $\angle b \equiv \angle g$	
Consecutive Angles (Supplementary)	$m\angle c + m\angle e = 180^0$ $m\angle d + m\angle f = 180^0$	

3.2.3 Thales' Theorem (Basic Proportionality Theorem): In a triangle ABC if D and E lie on AB and AC respectively in such a way that DE is parallel to BC then $\Delta ABC \sim \Delta ADE$ and

$$\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}$$

Proof: It is easy to see that the congruency between the angles formed by the intersection of transversals and parallel lines is as follows



$$\angle CAB \equiv \angle EAD, \qquad \angle ABC \equiv \angle ADE, \qquad \angle BCA \equiv \angle DEA$$

So, the triangles ABC and ADE are similar.

Now let

$$m\angle CAB = \alpha, \qquad m\angle ABC = \beta, \qquad m\angle BCA = \gamma$$

then

$$m\angle EAD = \alpha, \qquad m\angle ADE = \beta, \qquad m\angle DEA = \gamma$$

By the law of sines,

$$\frac{AB}{\sin\gamma} = \frac{BC}{\sin\alpha} = \frac{CA}{\sin\beta}$$

$$\frac{AD}{\sin\gamma} = \frac{DE}{\sin\alpha} = \frac{EA}{\sin\beta}$$

Hence,

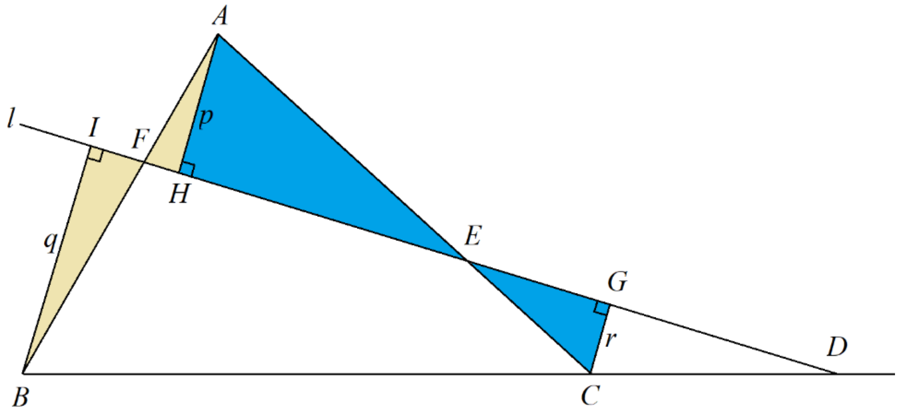
$$\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}$$

3.2.4 Menelaus Point: A point lying on a side line of a triangle, but not coinciding with a vertex of the triangle, is called a menelaus point of the triangle for this side.

3.2.5 Menelaus' Theorem: A necessary and sufficient condition for three Menelaus points D, E, F for the sides BC, CA, AB of an ordinary triangle ABC to be collinear is that

$$(\overline{BD}/\overline{DC})\ (\overline{CE}/\overline{EA})\ (\overline{AF}/\overline{FB}) = -1$$

Proof:



Necessity: Suppose D, E, F are collinear on a line l. Drop perpendiculars p, q, r on l from A, B, C respectively. Then, disregarding signs,

$$BD/DC = q/r \qquad (\because\ \ \Delta BID \sim \Delta CGD)$$

$$CE/EA = r/p \qquad (\because\ \ \Delta CGE \sim \Delta AHE)$$

$$AF/FB = p/q \qquad (\because\ \ \Delta AHF \sim \Delta BIF)$$

It follows that

$$(\overline{BD}/\overline{DC})\ (\overline{CE}/\overline{EA})\ (\overline{AF}/\overline{FB}) = \pm[(q/r)(r/p)(p/q)] = \pm 1$$

Since however, l must cut one or all three sides externally, we can have only the – sign.

Sufficiency: Suppose

$$(\overline{BD}/\overline{DC})\ (\overline{CE}/\overline{EA})\ (\overline{AF}/\overline{FB}) = -1$$

and let EF cut BC in D' then D' is a menelaus point. From the first half of the theorem, we have

$$(\overline{BD'}/\overline{D'C})\ (\overline{CE}/\overline{EA})\ (\overline{AF}/\overline{FB}) = -1$$

It follows that $\overline{BD}/\overline{DC} = \overline{BD'}/\overline{D'C}$, or that $D \equiv D'$. That is, D, E, F are collinear.

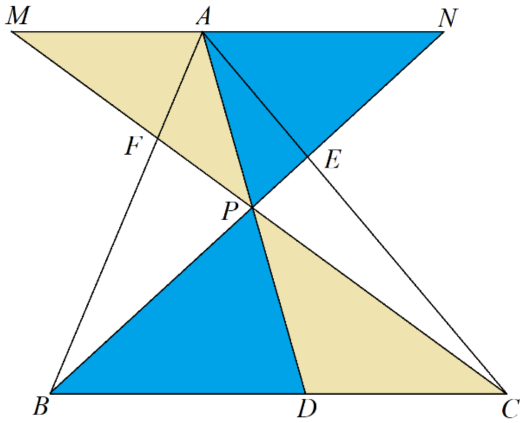
3.2.6 Cevian Line: A line passing through a vertex of a triangle, but not coinciding with a side of the triangle, will be called a cevian line of the triangle for this vertex.

A cevian line will be identified by the vertex to which it belongs and the point in which it cuts the opposite side.

3.2.7 Ceva's Theorem: A necessary and sufficient condition for three cevian lines AD, BE, CF of an ordinary triangle ABC to be concurrent is that

$$(\overline{BD}/\overline{DC})(\overline{CE}/\overline{EA})(\overline{AF}/\overline{FB}) = +1$$

Proof:



Necessity: Suppose AD, BE, CF are concurrent in P. Without loss of generality we may assume that P does not lie on the parallel through A to BC. Let BE, CF intersect this parallel line in N and M. Then, disregarding signs,

$$BD/AN = DP/PA \qquad (\because \ \Delta BPD \sim \Delta NPA)$$

and

$$DP/PA = DC/MA \qquad (\because \ \Delta DPC \sim \Delta APM)$$

This implies that

$$BD/AN = DC/MA$$

or

$$BD/DC = AN/MA$$

Moreover,

$$CE/EA = BC/AN \qquad (\because \ \Delta CBE \sim \Delta ANE)$$

and

$$AF/FB = MA/BC \qquad (\because \ \Delta CBF \sim \Delta MAF)$$

Whence

$$(\overline{BD}/\overline{DC})(\overline{CE}/\overline{EA})(\overline{AF}/\overline{FB})$$

$$= (\overline{AN}/\overline{MA})(\overline{BC}/\overline{AN})(\overline{MA}/\overline{BC}) = \pm 1$$

That sign must be $+$ follows from the fact that either none or two of the points D, E, F divide their corresponding sides externally.

Sufficiency: Suppose

$$(\overline{BD}/\overline{DC})(\overline{CE}/\overline{EA})(\overline{AF}/\overline{FB}) = +1$$

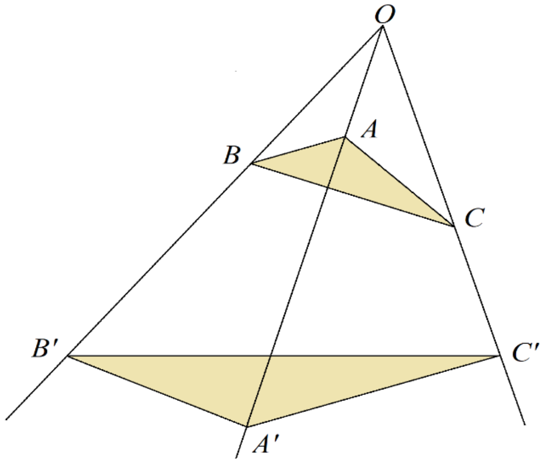
and let BE, CF intersect in P and draw AP to cut BC in D'. Then AD' is a cevian line. Hence, from the first half of the theorem, we have

$$(\overline{BD'}/\overline{D'C})(\overline{CE}/\overline{EA})(\overline{AF}/\overline{FB}) = +1$$

It follows that $\overline{BD'}/\overline{D'C} = \overline{BD}/\overline{DC}$, or that $D \equiv D'$. That is, AD, BE, CF are concurrent.

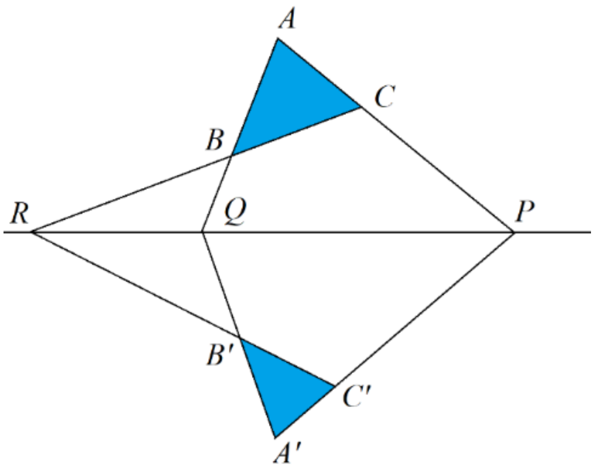
3.2.8 Copolar Triangles:

Two triangles ABC and $A'B'C'$ are said to be copolar (or perspective from a point) if AA', BB', CC' are concurrent. The point of concurrency is called the pole (or perspector).



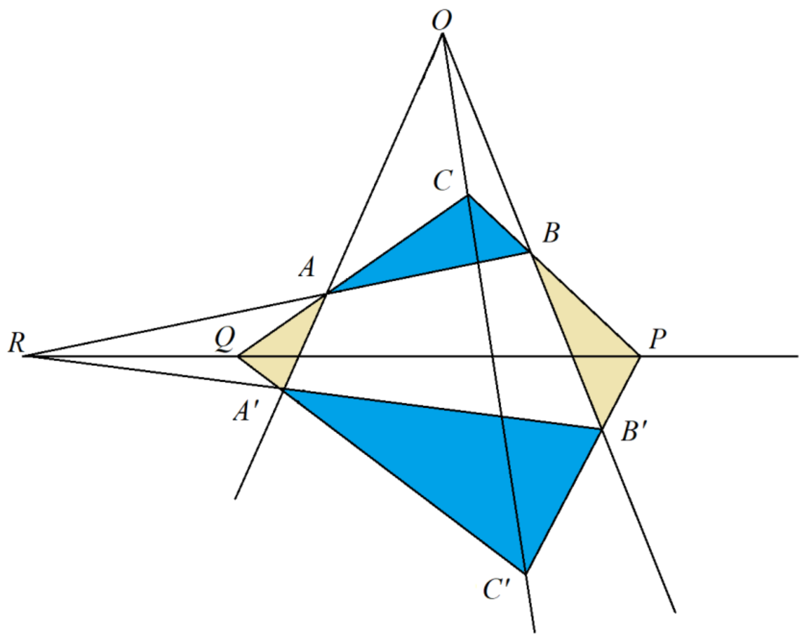
3.2.9 Coaxial Triangles:

Two triangles ABC and $A'B'C'$ are said to be coaxial (or perspective from a line) if the points of intersection of BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ are collinear. The line of collinearity is called the Desargues' line (or perspectrix).



3.2.10 Desargues' Theorem: Copolar triangles are coaxial, and conversely.

Proof: Let the two triangles be ABC and $A'B'C'$.



Suppose AA', BB', CC' are concurrent in a point O. Let P, Q, R be the points of intersection of BC and $B'C'$, CA and $C'A'$, AB and $A'B'$. Considering the triangles BCO, CAO, ABO in turn, with the respective transversals $B'C'P, C'A'Q, A'B'R$, we find by Menelaus' Theorem,

$$(\overline{BP}/\overline{PC})(\overline{CC'}/\overline{C'O})(\overline{OB'}/\overline{B'B}) = -1$$
$$(\overline{CQ}/\overline{QA})(\overline{AA'}/\overline{A'O})(\overline{OC'}/\overline{C'C}) = -1$$
$$(\overline{AR}/\overline{RB})(\overline{BB'}/\overline{B'O})(\overline{OA'}/\overline{A'A}) = -1$$

Setting the product of the three left members of the above equations equal to the product of the three right members, we obtain

$$(\overline{BP}/\overline{PC})(\overline{CQ}/\overline{QA})(\overline{AR}/\overline{RB}) = -1$$

whence P, Q, R are collinear. Thus, copolar triangles are coaxial.

Conversely, suppose P, Q, R are collinear and let O be the point of intersection of AA' and BB'. We show that CC' also passes through O. Now triangles AQA' and BPB' are copolar, and therefore coaxial (from the first half of the theorem). That is, O, C, C' are collinear and hence CC' passes through O. Thus, coaxial triangles are copolar.

Remarks: Desargues' theorem can also be stated as:
If two triangles are perspective from a point, they are perspective from a line.

Platonic Polyhedra

4.1 Platonic Solids

Geometers have studied the platonic solids for thousands of years. They are named for the ancient Greek philosopher Plato. These solids were studied by the Platonic school and played a role in their philosophy.

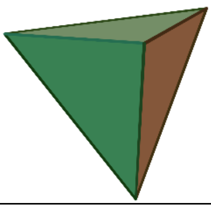
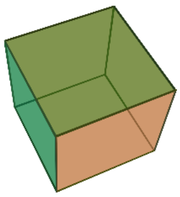
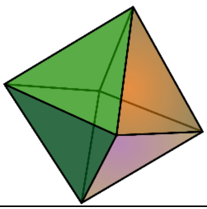
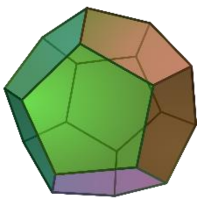
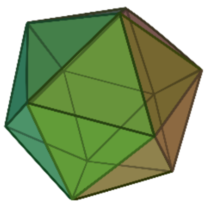
4.1.1 Convex Set:
A subset X of R^n is convex if for every pair of points $x, y \in X$ the line segment joining them lies them entirely within X, i.e., for each $t \in [0, 1]$, $tx + (1 - t)y \in X$. That is, X is closed under convex combination (non-negative affine combination).

4.1.2 Convex Polyhedron:
A subset of R^n defined by a finite set of linear inequalities is convex and is called a convex polyhedron. So, a convex polyhedron is a figure composed of finitely many planar polygons.

We shall study convex polyhedra in R^3 with non-empty interiors i.e., convex solid polyhedra. A convex polyhedron X in R^3 is bounded by finitely many planes. The intersection of X with any of these planes is a two-dimensional subset of X and is called a face of X. The intersection of two faces is called an edge and the intersection of two edges is a vertex. A face is homeomorphic is a closed disk, an edge to a closed interval and a vertex is a point.

4.1.3 Regular Polyhedron:
A polyhedron is called regular if all its faces, edges and vertices are identical to each other. That is, there are the same number of edges at each vertex and that the angles between then are the same.

4.1.4 Classification of Platonic Solids:
A convex regular polyhedron is called a platonic solid. There are only five platonic solids. A platonic solid is described by the number of its faces.

Solid	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Faces	4 Triangles	6 Squares	8 Triangles	12 Pentagons	20 Triangles
Edges	6	12	12	30	30
Vertices	4	8	6	20	12
Shape					
Dual	Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron

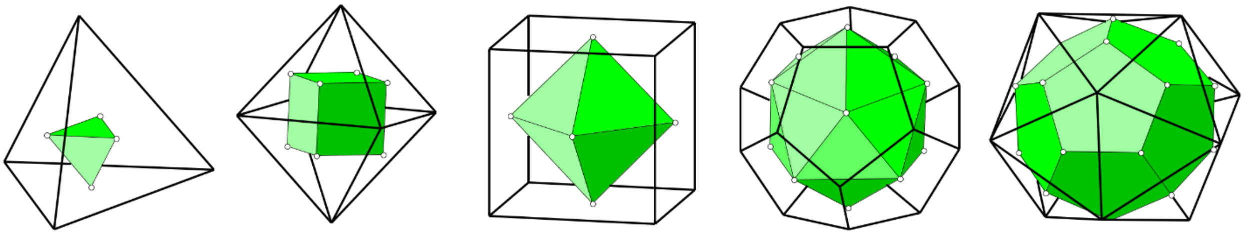
4.1.5 Euler's Formula:
If V, E and F respectively are the number of vertices, edges and faces of a convex polyhedron then $V - E + F = 2$.

It is easy to verify this formula for the platonic solids.

Exercise: Construct models of these solids.

4.1.6 Duality: If one takes a platonic solid, joins the mid points of adjacent faces by a new edge and fills in the resulting solid, one obtains the dual solid which is again platonic. If this process is done twice, the original platonic solid is recovered (although smaller in size).

The tetrahedron is self-dual, the cube and octahedron are dual, as are dodecahedron and icosahedron.



Exercise: Show that a platonic solid and its dual have the same number of edges.

Solution: By definition, corresponding to each face of a platonic solid there is a vertex of its dual. So, a platonic solid and its dual interchange the number of faces and vertices. Euler's formula then implies that the number of edges must remain the same.

As was known to the ancient Greeks these five are the only Platonic solids.

Theorem: There are precisely five platonic solids.

Proof: There are five known platonic solids. We prove that there are no more. Suppose that r faces meet at each vertex and that each face is a regular n-gon. It is clear that each of r and n is at least 3. The sum of the angles at a vertex is less than 2π and each angle is $(n-2)\pi/n$, being the angle of a regular n-gon. Hence, we see that

$$\frac{r(n-2)\pi}{n} < 2\pi$$

or

$$rn - 2r - 2n < 0$$

or

$$rn - 2r - 2n + 4 < 4$$

This implies that

$$(r-2)(n-2) < 4$$

The only integral solutions of this inequality with $r, n \geq 3$ are

$$(r, n) = (3,3), (3,4), (3,5), (4,3), (5,3)$$

If one knows the shape of a face (that is n) and how many faces meet at a vertex (that is r) then there is only one possible solid with that particular (r, n). So, corresponding to each solution (r, n) there is exactly one platonic solid. Hence there are only five platonic solids.