

Advanced Analysis: Handwritten Notes

provided by

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Equivalent Set:-

Two sets A and B are equivalent if there exist a bijective mapping between them and it is denoted by $A \sim B$.

On such type of sets only number of elements are same i.e. they have equal number of elements.

Example

$$\mathbb{P} \sim \mathbb{E}$$

$$f(n) = 2n$$

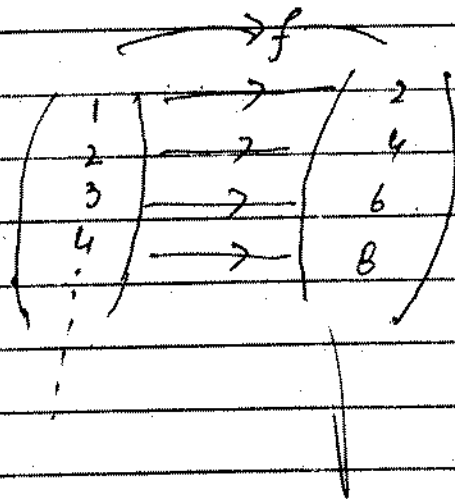
$$f(1) = 2(1) = 2.$$

$$f(2) = 2(2) = 4.$$

$$f(3) = 2(3) = 6.$$

$$f(4) = 2(4) = 8$$

⋮



Hence relation is 1-1 and onto Hence the mapping is bijective. Hence

$$\mathbb{P} \sim \mathbb{E}.$$

By the similar we can show that set of natural number is equivalent to the set of odd number by defining the mapping i.e.

$$f: \mathbb{N} \longrightarrow \mathbb{O} \text{ defined by}$$

$$f(n) = 2n + 1$$

which is one-one & onto. Hence f is bijective then

$$\mathbb{N} \sim \mathbb{O}.$$

(2)

By the similar we define a mapping b/w set of natural number and set of integer such that

$$f: \mathbb{N} \longrightarrow \mathbb{Z}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

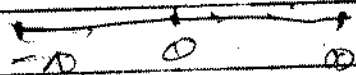
which is bijective and:

$$\mathbb{P} \sim \mathbb{Z}$$

Qn Show that unit interval $[0, 1]$ is equivalent to $[a, b]$

Proof :- First we defined a mapping b/w $[0, 1]$ and $[a, b]$ i.e

$$f: [0, 1] \longrightarrow [a, b]$$



$$f(x) = a + (b-a)x$$

$$f(0) = a$$

$$f(1) = b$$

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So unit close interval is equivalent to every closed interval of real line:

NOTE:

$$\mathbb{I} \sim [a, b]$$

$$A \subseteq \{1, 2, 3\}$$
$$A \times A = \{c\}$$

1, 2

$A \sim A$

$$f(a) = a$$

(3)

Theorem

(3)
A equivalent relation is an equivalence relation.

Proof 1. Reflexive:

For any non-empty set A \exists a mapping f such that

$f: A \rightarrow A$ defined by

$$f(a) = a$$

which is bijective. Hence

$$A \sim A.$$

Symmetric Property:-

Let A and B be two non-empty sets and $A \sim B$. Then we have to show that

$$B \sim A.$$

$f: A \sim B$ which is bijective.

$\Rightarrow f^{-1}: B \rightarrow A$ is also bijective.

NOTE:- Inverse of bijective function is also bijective.

Hence $B \sim A$.

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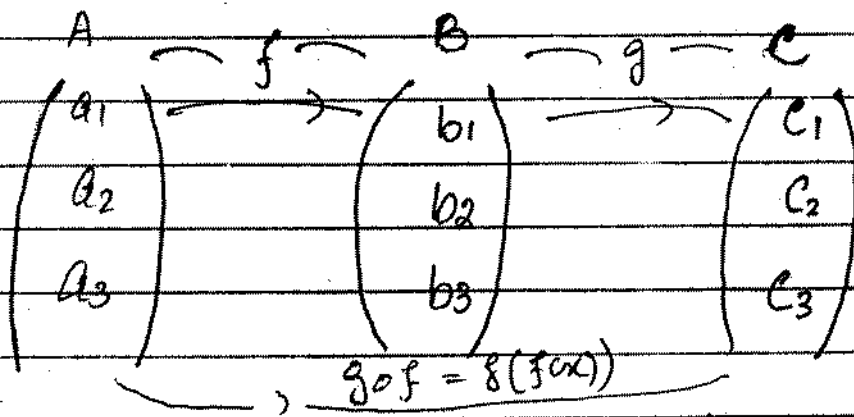
Transitive Property:-

For any three non-empty sets A, B, C if $A \sim C$ and $A \sim B$ and $B \sim C$ be two bijective mappings f and g such that

$f: A \rightarrow B$ bijective

$g: B \rightarrow C$ bijective.

(4)



$g \circ f = g(f(x))$
Composition function of f and g .

$$g(f(a_1)) = g(b_1) = c_1$$

$g \circ f : A \rightarrow C$ is bijective.
Hence $A \sim C$ and Equivalence relation.

Hence Equivalence relation is equivalence relation.

Finite SET:-

A set is said to be finite if it consists of a finite number of elements or if a set is equivalent to a set of finite natural numbers.

$$N = \{1, 2, 3, \dots, n\}$$

$$A = \{2, 4, 6, \dots, 2n\}$$

$$N \sim A$$

Hence A is finite.

Infinite Set:-

A set is said to be infinite if it is equivalent to its proper subset.

i.e

$$P \sim O$$

$$\mathbb{P} \sim \mathbb{Z}$$

$$I \sim [a, b]$$

Then

O, P, I are infinite set.

Denumerable set:-

A set A is said to be denumerable if it is equivalent to the set of natural number. i.e

$$O \sim N$$

$$E \sim N$$

$$Z \sim N$$

Then

O, E, Z are denumerable.

Denumerable set is infinite.

$$A = \{1, 2, 3, 4\}$$

$$f: A \rightarrow \mathbb{N}$$

(6)

Countable set: A set which is finite or equivalent to set of natural number is called countable set.

NOTE: Every denumerable set is countable but every countable set is not denumerable.

Thm A Countable union of finite sets is countable.

Proof: Let $\mathcal{A} = \{S_i : i \in \mathbb{N}\}$ be countable collection of finite set and let

$C = \bigcup S_i$ if C is empty then C is countable. Suppose $C \neq \emptyset$. Define
 $A_1 = S_1$, $A_2 = S_2 \setminus S_1$, $A_3 = S_3 \setminus S_2$
 $A_4 = S_4 \setminus S_3$... Then the set A_i are finite and pairwise disjoint.
Say

$$A_1 = \{a_{11}, a_{12}, a_{13}, \dots, a_{1n}\}$$

$$A_2 = \{a_{21}, a_{22}, \dots, a_{2n}\}$$

$$A_3 = \{a_{31}, a_{32}, \dots, a_{3n}\}$$

Then

union $B = \bigcup A_i$ can be written as sequence as follows

$$B = \{a_{11}, a_{12}, a_{13}, \dots, a_{1n}, a_{21}, a_{22}, \dots, a_{2n}, \dots, a_{n1}, a_{n2}, \dots, a_{nn}\}$$

That first we write down the elements of A_1 , then the elements of A_2 , and so on.

$$f: B \rightarrow \mathbb{N} \text{ as below } f(a_{ij}) = n_1 + n_2 + \dots + n_j$$

Example

(7)

Show that $\mathbb{N} \times \mathbb{N}$ is denumerable

Sol:-

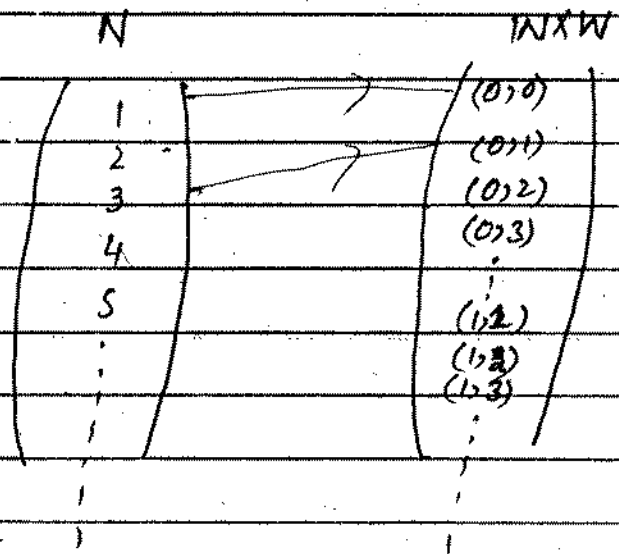
We define a function

$$f: \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$$

$$f(a) = (r, s) \quad \text{where} \quad a = 2^r(2s+1)$$

$\Rightarrow f$ is bijective

$\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ and
 $\mathbb{N} \times \mathbb{N}$ is denumerable.



Theorem:- Every subset of countable set is countable.

Proof:- Let A be any countable set. Then there are two possibilities

- if
- i) A is finite.
 - ii) A is denumerable mean which equivalent infinite to the set of natural number and infinite.

Now if A is finite and B is any subset of A . Then B be long the subset of

(B)

finite set is finite. Now if A is denumerable. if B is a subset of A and let.

$$A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$$

$$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots, a_{n_r}, \dots\}$$

if indexing set $\{n_1, n_2, n_3, \dots, n_r, \dots\}$ is finite then B is also finite \Rightarrow
 B is countable.

if the indexing set $\{n_1, n_2, n_3, \dots, n_r, \dots\}$ is infinite then B is infinite. Then B being an infinite sequence of distinct terms is denumerable.
 $\Rightarrow B$ is countable.

* Theorem 2

Every infinite set contains a denumerable subset.

Proof

Let A be an infinite set. Also let \mathcal{A} represent the collection of subsets of A .

we define a mapping $f: \mathcal{A} \rightarrow A$ such that

$$f(A) = a_1 \text{ where } a_1 \in A.$$

$$f(A - \{a_1\}) = a_2 \text{ where } a_2 \in A - \{a_1\}$$

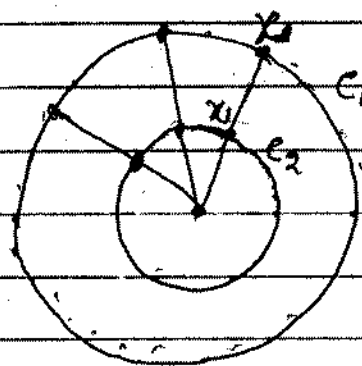
$$f(A - \{a_1, a_2\}) = a_3 \text{ where } a_3 \in A - \{a_1, a_2\}$$

$$f(A - \{a_1, a_2, a_3\}) = A \text{ where } a_4 \in A - \{a_1, a_2, a_3\}$$

Since $\{a_1, a_2, a_3, \dots\}$ is a subset of A .
Hence B being the infinite sequence of distinct element is denumerable.

Geometrical interperation of Equirelent. Set:-

Consider two circles C_1 and C_2 and defined a mapping f from



$$f: C_1 \longrightarrow C_2 \text{ s.t.}$$

$f(x) = y$ (Point of intersection of the line joining x and the Centre of circle C_1 .)

obviously 'f' is bijective and hence

$$C_1 \sim C_2.$$

Theorem:-

Set of real number \mathbb{R} is infinite.
Proof to show that \mathbb{R} is infinite we show a proper subset of \mathbb{R} is equirelent to \mathbb{R} .

let $A =]-\pi/2, \pi/2[\subset \mathbb{R}$

(10)

we define a mapping
 $f: A \rightarrow \mathbb{R}$ s.t.

$$f(x) = \tan x.$$

let $x_1, x_2 \in A$.

and $f(x_1) = f(x_2)$
 $\tan x_1 = \tan x_2$

$$\implies x_1 = x_2.$$

f is one-one.

for each $\tan x \in \mathbb{R}$ \exists an element
 $x \in A$ such that

$$f(x) = \tan x.$$

$\implies f$ is onto.

$\implies f$ is bijective

$$\implies A \sim \mathbb{R}$$

$\implies \mathbb{R}$ is infinite.

Show \mathbb{N} is equivalent to $\mathbb{N} \times \mathbb{N}$.

Proof

To show that $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$
we define a mapping f i.e.

$f: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ defined by

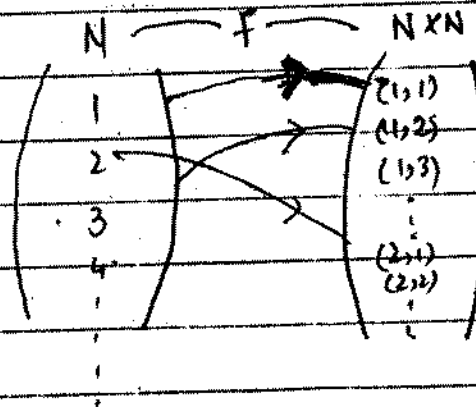
$$f(a) = (a, a)$$

where

$$a = \frac{a-1}{2} (2a-1).$$

f is bijective Hence

$$\mathbb{N} \sim \mathbb{N} \times \mathbb{N}.$$



Theorem:- Every Subset of denumerable is either denumerable or finite.

Proof let A be any denumerable set
Then $\mathbb{N} \sim A$.

let $A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$

let B be any subset of A if

$B = \{\}$ Then B is finite

let B be any non-empty sub set of A

i.e

$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots, a_{n_r}, \dots\}$

If the indexing set i.e $\{n_1, n_2, n_3, \dots, n_r, \dots\}$ is finite. Then B is finite.

If the indexing set i.e $\{n_1, n_2, n_3, \dots, n_r, \dots\}$ is infinite. Then B is infinite and clearly B is the sequence of distinct term. So

$$B \sim \mathbb{N}$$

$\Rightarrow B$ is denumerable.

————— \cdot ————— \cdot ————— \cdot ————— \cdot ————— \cdot —————

Theorem:-

Union of family of pairwise disjoint denumerable set is denumerable.

Proof let $\{A_1, A_2, A_3, \dots\}$ be the family of pairwise disjoint denumerable sets

let $A_1 = \{a_1, a_2, a_3, \dots\}$

$A_2 = \{b_1, b_2, b_3, \dots\}$

$A_3 = \{c_1, c_2, c_3, \dots\}$

|

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(12)

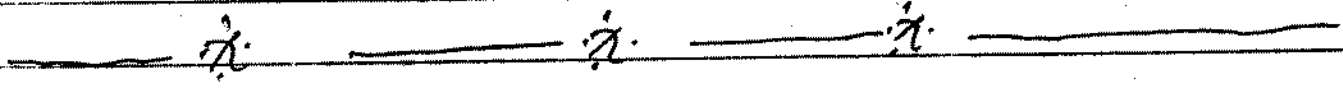
we have to show that $\cup A_i$ is denumerable where

$$\cup A_i = \{ a_1, a_2, a_3, \dots, b_1, b_2, b_3, \dots, c_1, c_2, c_3, \dots \}$$

As all A_i 's are pairwise disjoint so

$$A_i \cap A_j = \emptyset \text{ for some } i, j \in I \text{ \& } i \neq j$$

\Rightarrow Union of A_i 's is infinite sequence of disjoint distinct term so $\Rightarrow \cup A_i$'s is denumerable.



Duality :-

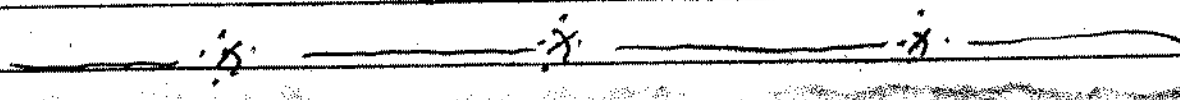
let E be an equation of set algebra then E^* of E is the equation obtain by replacing \cup, \cap, \cup and \emptyset in E by $\cap, \cup, \emptyset, \cup$ respectively.
for e.g

$$(\cup A) \cup (B \cap A) = A \text{ --- (1)}$$

By replacing \cup, \cap, \cup by \emptyset, \cup, \cap then

$$(\emptyset \cup A) \cap (B \cup A) = A \text{ --- (2)}$$

\Rightarrow equation (2) is dual of (1).



Choice function:-

Let $\{A_i\}$ where $i \in I$ be family of non-empty subsets of a set A . Then a function f

$$f: \{A_i\}_{i \in I} \longrightarrow A$$

is called a choice function if for every $i \in I$

$$f(A_i) \in A.$$

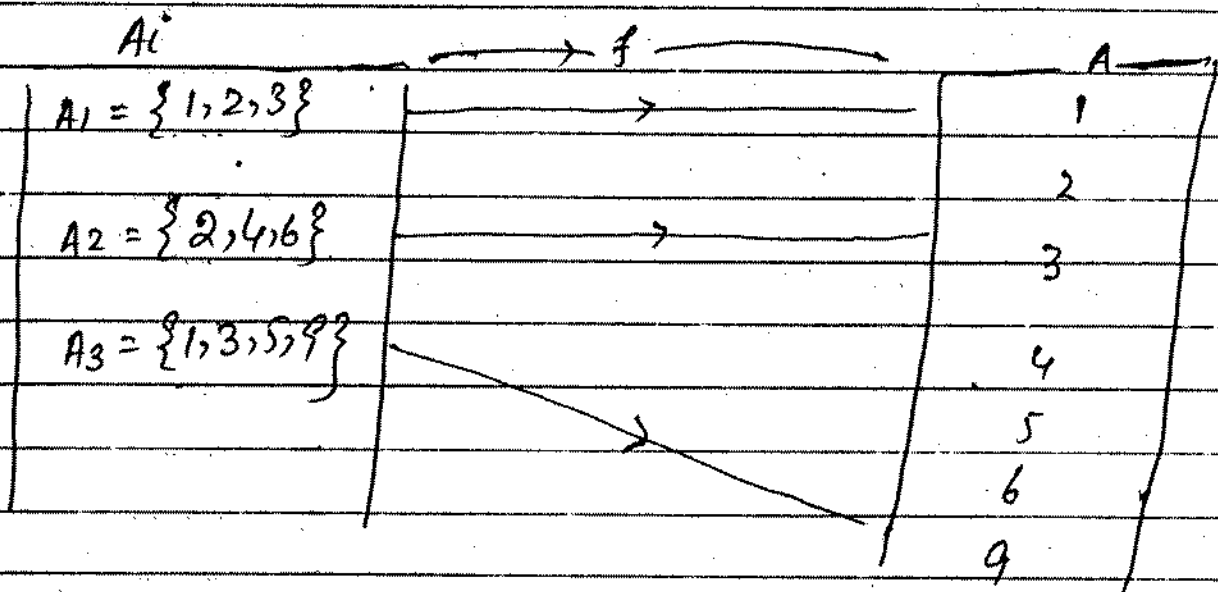
i.e. image of each set is an element of the set A .

$$\text{Let } A = \{1, 2, 3, 4, 5, 6, 9\}$$

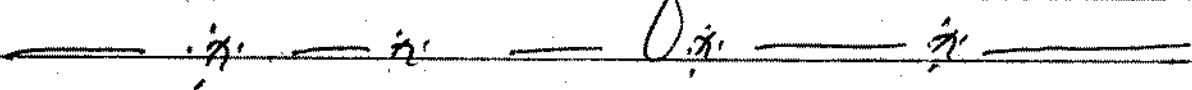
$$A_1 = \{1, 2, 3\}$$

$$A_2 = \{2, 4, 6\}$$

$$A_3 = \{1, 3, 5, 9\}$$



Then f is choice function.



Characteristic Function:-

let U be a universal set and $A \subset U$ Then characteristic function

$$\chi_A : U \longrightarrow \{0,1\}$$

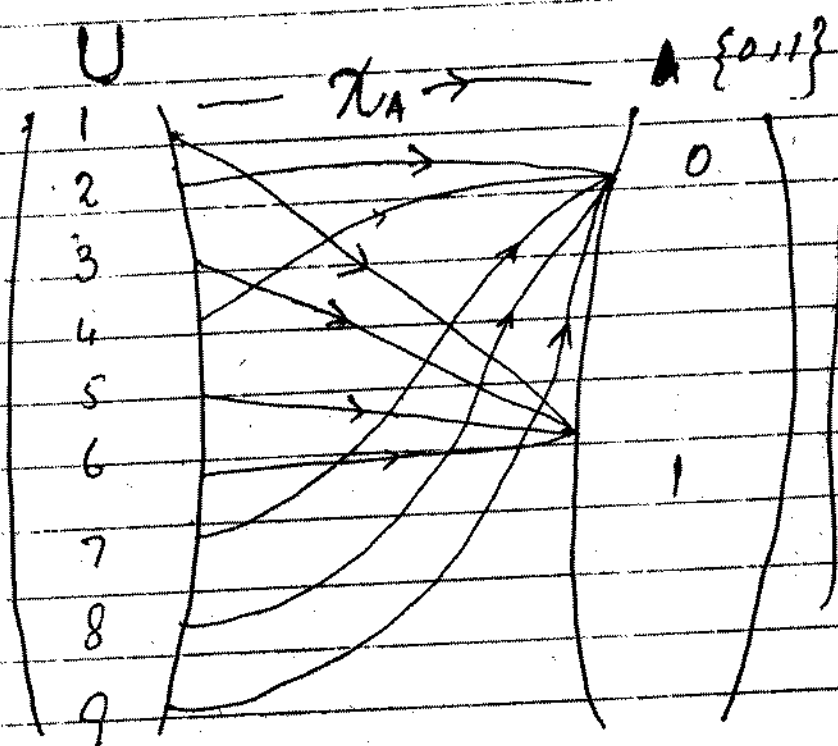
$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

e.g.

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$A = \{1, 3, 5, 6\}$$

$$\chi_A : U \longrightarrow \{0,1\}$$

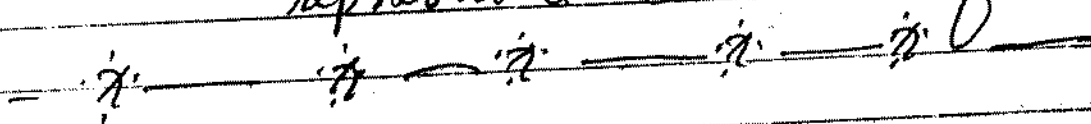


NOTE:-

$$1) \chi_{(A \cup B)} = \chi_A \vee \chi_B$$

$$\chi_{A \cap B} = \chi_A \wedge \chi_B$$

represent a characteristic function



Theorem:

Let X be non-empty set and $C(X)$ denote all characteristic functions. Then $C(X)$ is equivalent to 2^X i.e.

$$C(X) \sim 2^X$$

Proof let $A \subseteq X$ and

$f: 2^X \rightarrow C(X)$ defined by

$$f(A) = \chi_A.$$

To show that f is one-one. let $A, B \in 2^X$ and suppose that

$$f(A) = f(B)$$

$$\chi_A = \chi_B.$$

$$\Rightarrow A = B.$$

$$\Rightarrow f \text{ is 1-1.}$$

To show that f is onto

For each $\chi_A \in C(X)$

\exists an element $A \in 2^X$ such that

$$f(A) = \chi_A$$

$$\Rightarrow f \text{ is onto}$$

$$\Rightarrow f \text{ is bijective.}$$

Hence

$$2^X \sim C(X)$$

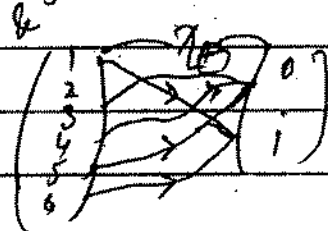
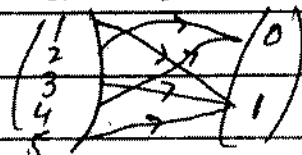
for e.g.

$$U = \{1, 2, 3, 4, 5\}$$

$$A = \{1, 3, 5\}$$

$$B = \{1, 5, 6\}$$

$$U \xrightarrow{\chi_A} \mathbb{Z}$$



$\nexists A \neq B$ then $\chi_A \neq \chi_B$.

$\times \quad \times \quad \times$

(16)

imp

Theorem

Set of rational number \mathbb{Q} is denumerable.

Proof \hookrightarrow

As the set of rational number \mathbb{Q} can be written as

$$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$$

we defined a function

$$f: \mathbb{Q}^+ \rightarrow \mathbb{N} \times \mathbb{N} \text{ such that}$$

$$f\left(\frac{p}{q}\right) = (p, q)$$

for $\frac{p}{q}, \frac{p_1}{q_1} \in \mathbb{Q}^+$

$$\text{if } f\left(\frac{p}{q}\right) = f\left(\frac{p_1}{q_1}\right)$$

$$\Rightarrow (p, q) = (p_1, q_1)$$

$$\Rightarrow p = p_1 \text{ \& } q = q_1$$

$$\Rightarrow \frac{p}{q} = \frac{p_1}{q_1}$$

Hence

f is 1-1.

For every $(p, q) \in \mathbb{N} \times \mathbb{N}$ There exist pre image $f(p/q)$ such that

$$f\left(\frac{p}{q}\right) = (p, q)$$

Hence f is onto.

Hence f is bijective.

So

$$\mathbb{Q}^+ \sim \mathbb{N} \times \mathbb{N}.$$

also

$$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$$

Therefore by transitive property

$$\mathbb{Q}^+ \sim \mathbb{N} \checkmark$$

Hence \mathbb{Q}^+ denumerable.

Again we define a function

$$g: \mathbb{Q}^+ \longrightarrow \bar{\mathbb{Q}}$$

$$g\left(\frac{p}{q}\right) = -\frac{p}{q}$$

obviously g is bijective. Hence

$$\mathbb{Q}^+ \sim \bar{\mathbb{Q}} \Rightarrow \bar{\mathbb{Q}} \sim \mathbb{N}$$

Hence

$\bar{\mathbb{Q}}$ is denumerable.

$\bar{\mathbb{Q}} \cup \{0\}$ is also denumerable being the infinite sequence of distinct term.

Also we know that union of two denumerable set is denumerable so

$\mathbb{Q}^+ \cup \bar{\mathbb{Q}} \cup \{0\} = \mathbb{Q}$ is denumerable.

π $\sqrt{2}$ $\sqrt{3}$ $\sqrt{4}$ $\sqrt{5}$ $\sqrt{6}$ $\sqrt{7}$ $\sqrt{8}$ $\sqrt{9}$ $\sqrt{10}$ $\sqrt{11}$ $\sqrt{12}$ $\sqrt{13}$ $\sqrt{14}$ $\sqrt{15}$ $\sqrt{16}$ $\sqrt{17}$ $\sqrt{18}$ $\sqrt{19}$ $\sqrt{20}$ $\sqrt{21}$ $\sqrt{22}$ $\sqrt{23}$ $\sqrt{24}$ $\sqrt{25}$ $\sqrt{26}$ $\sqrt{27}$ $\sqrt{28}$ $\sqrt{29}$ $\sqrt{30}$ $\sqrt{31}$ $\sqrt{32}$ $\sqrt{33}$ $\sqrt{34}$ $\sqrt{35}$ $\sqrt{36}$ $\sqrt{37}$ $\sqrt{38}$ $\sqrt{39}$ $\sqrt{40}$ $\sqrt{41}$ $\sqrt{42}$ $\sqrt{43}$ $\sqrt{44}$ $\sqrt{45}$ $\sqrt{46}$ $\sqrt{47}$ $\sqrt{48}$ $\sqrt{49}$ $\sqrt{50}$ $\sqrt{51}$ $\sqrt{52}$ $\sqrt{53}$ $\sqrt{54}$ $\sqrt{55}$ $\sqrt{56}$ $\sqrt{57}$ $\sqrt{58}$ $\sqrt{59}$ $\sqrt{60}$ $\sqrt{61}$ $\sqrt{62}$ $\sqrt{63}$ $\sqrt{64}$ $\sqrt{65}$ $\sqrt{66}$ $\sqrt{67}$ $\sqrt{68}$ $\sqrt{69}$ $\sqrt{70}$ $\sqrt{71}$ $\sqrt{72}$ $\sqrt{73}$ $\sqrt{74}$ $\sqrt{75}$ $\sqrt{76}$ $\sqrt{77}$ $\sqrt{78}$ $\sqrt{79}$ $\sqrt{80}$ $\sqrt{81}$ $\sqrt{82}$ $\sqrt{83}$ $\sqrt{84}$ $\sqrt{85}$ $\sqrt{86}$ $\sqrt{87}$ $\sqrt{88}$ $\sqrt{89}$ $\sqrt{90}$ $\sqrt{91}$ $\sqrt{92}$ $\sqrt{93}$ $\sqrt{94}$ $\sqrt{95}$ $\sqrt{96}$ $\sqrt{97}$ $\sqrt{98}$ $\sqrt{99}$ $\sqrt{100}$

(18)

Imp

Theorem 2 If A and B are denumerable
Then $A \times B$ (their Cartesian Product)
is denumerable.

✓ Proof: Since A and B are denumerable sets
so both can be express a infinite
sequence of distinct term.

$$\text{Let } A = \{a_1, a_2, a_3, \dots\}$$

$$B = \{b_1, b_2, b_3, \dots\}$$

Then

$$A \times B = \left\{ \begin{array}{l} (a_1, b_1), (a_1, b_2), (a_1, b_3), \dots, (a_m, b_1), \dots \\ (a_2, b_1), (a_2, b_2), (a_2, b_3), \dots \\ (a_3, b_1), (a_3, b_2), (a_3, b_3), \dots \\ \dots \\ \dots \end{array} \right\}$$

we define a function f

$$f: \mathbb{N} \times \mathbb{N} \longrightarrow A \times B. \quad \text{defined as}$$

$$f(i, j) = (a_i, b_j)$$

let $(i_1, j_1), (i_2, j_2) \in \mathbb{N} \times \mathbb{N}$ also if

$$f(i_1, j_1) = f(i_2, j_2)$$

$$\Rightarrow (a_{i_1}, a_{j_1}) = (a_{i_2}, a_{j_2})$$

$$\Rightarrow a_{i_1} = a_{i_2}, \text{ and } a_{j_1} = a_{j_2}$$

$$\Rightarrow (i_1, j_1) = (i_2, j_2)$$

Hence f is 1-1.

Also for each $(a_i, b_j) \in A \times B$ there exist
 $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that
 $f(i, j) = (a_i, b_j)$
 $\Rightarrow f$ is onto. Hence f is bijective
 and

$$A \times B \sim \mathbb{N} \times \mathbb{N}$$

and

$$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$$

Then by transitive property.

$$A \times B \sim \mathbb{N}$$

$\Rightarrow A \times B$ is denumerable.

Proposition :-

Show that the set of point with
 rational coordinates in the plane is
 denumerable.

Proof :-

Let $x \in \mathbb{Q}$ then $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$
 then p and q are also
 rationals. $q \neq 0$

$$A = \{(p, q), p, q \in \mathbb{Q}\}$$

$$= \mathbb{Q} \times \mathbb{Q}$$

We defined a mapping

$$f: \mathbb{Q} \rightarrow \mathbb{Q} \times \mathbb{Q} \text{ defined as}$$

$$f\left(\frac{p}{q}\right) = (p, q)$$

Obviously f is bijective. Hence

$$\mathbb{Q} \times \mathbb{Q} \sim \mathbb{Q} \Rightarrow \mathbb{Q} \times \mathbb{Q} \text{ is denumerable.}$$

\Rightarrow The set of points with rational coordinates in the plane is denumerable.

(20)

imp

Theorem

The unit interval $[0, 1]$ is non-denumerable.

Proof :-

Suppose that the unit interval $[0, 1]$ is denumerable. Then it can be written in finite sequence of distinct terms.

$$[0, 1] = \{x_1, x_2, x_3, x_4, x_5, \dots\}$$

where

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}a_{15}\dots$$

$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}a_{25}\dots$$

⋮

$$x_n = 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}\dots$$

⋮

Now consider $y \in [0, 1]$

$$y = 0.b_1b_2b_3b_4b_5\dots$$

where $b_1 \neq a_{11}$ and $b_1 \neq 0$

$b_2 \neq a_{22}$ and $b_2 \neq 0$

$b_3 \neq a_{33}$ and $b_3 \neq 0$

⋮

$b_n \neq a_{nn}$ and $b_n \neq 0$

$\Rightarrow y \neq x_1, y \neq x_2, y \neq x_3, \dots, y \neq x_n, \dots$

$$a_{ij} \in \{0, 1, 2, 3, \dots, 9\}$$

$$b_j \in \{1, 2, 3, \dots, 9\}$$

$\Rightarrow \gamma \notin [0, 1]$ which is contradiction.
(Impossible). Hence our supposition
is wrong. Hence unit interval is
non-denumerable.

Cardinal Number:

Let A be any set
equivalent to B i.e. $A \sim B$.
 \sim is an equivalence relation. If
 \mathcal{C} is collection of sets in which equivalent
relation is defined then the equivalence class
generated by any set $A \in \mathcal{C}$ is called
Cardinal number of A and is denoted by
Cardinal of A i.e. $\#A$. It represents size
of the set.

OR

Let A be any set and " α " denote the
collection of sets which are equivalent to A .
Then " α " is called Cardinal number of A .

e.g:- Cardinality of the following sets
 $\emptyset, \{1\}, \{1, 2\}, \{1, 2\}, \{a, b, c\}$ is
 $0, 1, 2, 3$.

As cardinality of every finite set is the
number of element of that set so these
cardinal numbers are finite cardinal numbers.
Generalization of real numbers.

Remark:-

The set A ~~set~~ which is equivalent to $[0, 1]$
has cardinal \mathbb{C} and is said to have power
of power of continuum.

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Cardinality Theorem :-

if A and B are finite sets then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof

As A and B finite sets. let
 $A = \{a_1, a_2, a_3, \dots, a_m, c_1, c_2, c_3, \dots, c_r\}$

$B = \{b_1, b_2, b_3, \dots, b_n, c_1, c_2, c_3, \dots, c_r\}$

$A \cap B = \{c_1, c_2, c_3, \dots, c_r\}$

$$n(A) = m + r, \quad n(B) = n + r.$$

$$n(A \cap B) = r.$$

$$n(A \cup B) = m + n + r.$$

$$= m + n + r + r - r$$

$$= (m + r) + (n + r) - r \quad (\text{By adding \& subtracting})$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Case : if A & B are disjoint

$$\text{ie } A \cap B = \emptyset$$

$$\therefore n(A \cap B) = 0.$$

Therefore

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= n(A) + n(B) - 0$$

$$n(A \cup B) = n(A) + n(B)$$

Case 1) If $A \subset B$.
Then $A \cap B = A$

$$n(A \cap B) = n(A)$$

As

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= n(A) + n(B) - n(A)$$

$$n(A \cup B) = n(B)$$

If $B \subset A$ Then

$$n(A \cup B) = n(A)$$

Arithmetic of Cardinals:-

Let A and B be two disjoint sets and

$$\#A = \alpha; \#B = \beta$$

Then

$$\begin{aligned} \#(A \cup B) &= \#A + \#B \\ &= \alpha + \beta \end{aligned}$$

$$\#(A \cup B) = \alpha + \beta$$

Similarly

$$\#(A \times B) = \#A \cdot \#B$$

$$\#(A \times B) = \alpha \cdot \beta$$

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* Transfinite Cardinal Numbers:-
Cardinal numbers of infinite sets are called transfinite cardinal numbers

Cardinality of \mathbb{N} is $e\mathbb{N}_0$ (aleph-naught)

That is $\text{Card } \mathbb{N}$ or $\# \mathbb{N} = e\mathbb{N}_0$

$\text{Card } (\mathbb{Z})$ or $\# \mathbb{Z} = e\mathbb{N}_0$ as $\mathbb{Z} \sim \mathbb{N}$.

$\text{Card } (\mathbb{Q}) = e\mathbb{N}_0$ as $\mathbb{Q} \sim \mathbb{N}$.

$\text{Card } [0, 1] = \mathbb{C}$

$\text{Card } \mathbb{R} = \mathbb{C} \quad \because \mathbb{R} \sim [0, 1]$

Further more cardinality of any interval on real line \mathbb{C} . Every interval on Real line is equivalent to unit interval.

Example:-
If n is a finite number then

$$n + e\mathbb{N}_0 = e\mathbb{N}_0$$

Q:- Let $A = \{1, 2, 3, \dots, n\}$

$B = \{n+1, n+2, n+3, \dots\}$

$$A \cup B = \{1, 2, 3, \dots\} = \mathbb{N}$$

$$\text{Card}(A \cup B) = \text{Card}(\mathbb{N})$$

$$\text{Card}(A) + \text{Card}(B) = \aleph_0$$

$$n + \aleph_0 = \aleph_0$$

NOTE:- If we add finite number into transfinite. Then again it is transfinite.

Remarks

It may be noted that addition and multiplication of finite cardinal number correspond to ordinary addition and multiplication of natural number.

But in the case infinite cardinal number the addition and multiplication does not correspond to ordinary addition and multiplication of natural number.

Theorem:- Show that ordinary addition & multiplication does not hold infinite cardinal number.

Also cancellation law do not hold in infinite cardinal number.

Proof in let

$$A = \{1, 3, 5, 7, \dots\}$$

$$B = \{2, 4, 6, \dots\}$$

$$\# A = \aleph_0$$

$$\# B = \aleph_0$$

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$$\text{Also } \# N = e^{\aleph_0}$$

$$\#(A \cup B) = \#A + \#B$$

$$\#N = \aleph_0 + e^{\aleph_0}$$

$$e^{\aleph_0} = \aleph_0 + e^{\aleph_0}$$

which is a contradiction to ordinary addition.

$$\#(A \times B) = \#A \cdot \#B$$

$$\#(N \times N) = \aleph_0 \cdot \aleph_0 \quad \text{Since } A \times N \rightarrow B \times N$$

$$\#N = \aleph_0 \cdot \aleph_0$$

$$\aleph_0 = \aleph_0 \cdot \aleph_0$$

which is a contradiction to ordinary multiplication.

It can be observed in the above example that e^{\aleph_0} is neither zero (0) nor 1 (ONE).

$$e^{\aleph_0 + 1} = e^{\aleph_0}$$

$$e^{\aleph_0 + 1} = e^{\aleph_0} + e^{\aleph_0} \quad \therefore e^{\aleph_0} = e^{\aleph_0} + e^{\aleph_0}$$

$$\Rightarrow e^{\aleph_0} = 1$$

\Rightarrow cancellation laws does not hold (under addition), in infinite cardinal number.

$$\aleph_0 \cdot 2 = \aleph_0$$

$$\aleph_0 \cdot 1 = \aleph_0 \cdot \aleph_0$$

$$\therefore \aleph_0 = \aleph_0 \cdot \aleph_0$$

\rightarrow $\aleph_0 = 1$.
 So cancellation laws under multiplication do not hold in infinite cardinal numbers.

Theorems

For any cardinals α, β, γ we have

$$i) \quad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

$$ii) \quad \alpha(\beta\gamma) = (\alpha\beta)\gamma$$

$$iii) \quad \alpha + \beta = \beta + \alpha$$

$$iv) \quad \alpha\beta = \beta\alpha$$

$$v) \quad \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

To show that
 $\#A = \#B$ then
 show that
 $A \times B$ is a bijective mapping.

Proof

Let A, B, C be three disjoint sets such that:

$$\#A = \alpha, \#B = \beta, \#C = \gamma$$

$$\#(A \cup (B \cup C)) = \alpha + (\beta + \gamma)$$

$$\#((A \cup B) \cup C) = (\alpha + \beta) + \gamma$$

$$\text{L.H.S.} \quad \alpha + (\beta + \gamma)$$

$$= \#(A \cup (B \cup C))$$

$$= \#((A \cup B) \cup C)$$

$$= (\alpha + \beta) + \gamma = \text{R.H.S.}$$

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Hence

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

we have to show that

$$iv) \#(A \times B) = \#(B \times A).$$

we define a function

$$s.t. f: A \times B \longrightarrow B \times A.$$

$$f(a, b) = (b, a).$$

let (a_1, b_1) and $(a_2, b_2) \in A \times B$

and let

$$f(a_1, b_1) = f(a_2, b_2).$$

$$(b_1, a_1) = (b_2, a_2).$$

$$\Rightarrow b_1 = b_2 \text{ \& } a_1 = a_2.$$

Hence

$$(a_1, b_1) = (a_2, b_2).$$

so

function is 1-1.

for every (b, a) there exist an element $(a, b) \in A \times B$ such that

$$f(a, b) = (b, a)$$

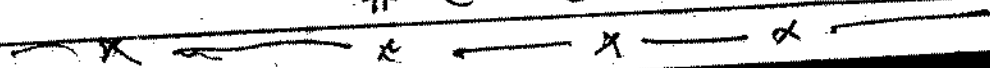
Since f is onto

Hence f is bijective. In

$$A \times B \cong B \times A$$

Hence

$$\#(A \times B) = \#(B \times A)$$



$$\text{iii)} \quad \alpha + \beta = \beta + \alpha.$$

Let A, B be disjoint sets. Such that

$$\# A = \alpha, \quad \# B = \beta.$$

$$\# (A \cup B) = \# A + \# B = \alpha + \beta.$$

$$\# (B \cup A) = \beta + \alpha.$$

L.H.S

$$\begin{aligned} \alpha + \beta &= \# A + \# B \\ &= \cancel{\# B} + \cancel{\# A} \quad \# (A \cup B) \\ &= \# (B \cup A) \\ &= \# B + \# A \\ &= \beta + \alpha \end{aligned}$$

Hence

$$\alpha + \beta = \beta + \alpha.$$

$$\text{ii)} \quad \alpha(\beta\gamma) = (\alpha\beta)\gamma.$$

we have to prove that

$$\# A \times (B \times C) = \# (A \times B) \times C.$$

we define a mapping

$$f: A \times (B \times C) \longrightarrow (A \times B) \times C \text{ defined as}$$

$$f(a, (b, c)) = ((a, b), c).$$

we are to show that f is bijective.

for this first we show that f is 1-1.

$$\text{let } (a_1, (b_1, c_1)) \text{ and } (a_2, (b_2, c_2)) \in A \times (B \times C)$$

Also let

$$f(a_1, (b_1, c_1)) = f(a_2, (b_2, c_2))$$

$$((a_1, b_1), c_1) = ((a_2, b_2), c_2)$$

$$(a_1, b_1) = (a_2, b_2) \ \& \ c_1 = c_2 \Rightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2$$

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$$\Rightarrow (a_1, (b_1, c_1)) = (a_2, (b_2, c_2))$$

For every $(a, b), c$ There exist an element $(a_1, (b_1, c_1)) \in A \times (B \times C)$ such that

$$f(a_1, (b_1, c_1)) = (a, b), c.$$

Hence f is bijective.

Then

$$\# A \times (B \times C) = \# (A \times B) \times C.$$

$$\text{Hence } \alpha(B \times C) = (\alpha B) \times C.$$

$$\textcircled{V} \quad \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

Proof ~

$$\# A \times (B \cup C) = \alpha(\beta + \gamma)$$

$$\# A \times B = \alpha\beta.$$

$$\# A \times C = \alpha\gamma.$$

$$\alpha(\beta + \gamma) = \# [A \times (B \cup C)]$$

$$= \# [A \times B \cup A \times C]$$

$$= \# A \times B + \# A \times C.$$

$$= \alpha\beta + \alpha\gamma$$

Prove that $\mathbb{C} + \mathbb{C} = \mathbb{C}$?

As we have

$$\# [0, 1/2] = \mathbb{C}$$

$$\# (1/2, 1] = \mathbb{C}$$

$$[0, 1/2] \cup (1/2, 1] = [0, 1]$$

$$\# ([0, 1/2] \cup (1/2, 1]) = \# [0, 1]$$

$$\# [0, 1/2] + \# (1/2, 1] = \# [0, 1]$$

$$\mathbb{C} + \mathbb{C} = \mathbb{C} \quad \text{Hence proved.}$$

Definition: If a set A is equivalent to a subset of B then A is said to precede B or simply A precedes B and write as $A \preceq B$.

If A is equivalent to proper subset of B . Then we say A strictly precedes B .
i.e. $A \prec B$. and if A is equivalent to exactly B . Then we say $A \sim B$.

✓ Ordering of Cardinal Numbers:
Let $A \neq B$ two sets then cardinal of A . i.e. $\text{Card } A \leq \text{Card } B$ if A is equivalent to a subset of B .

OR
If there exist 1-1 fn $f: A \rightarrow B$ then
 $\text{Card } A \leq \text{Card } B$.

(32)

* *
Therem:u Show That $\mathbb{N} < \mathbb{C}$

Define a function $f: \mathbb{N} \longrightarrow [0,1]$

$$f(n) = \frac{1}{n}$$

For 1-1 let $n_1, n_2 \in \mathbb{N}$

$$f(n_1) = f(n_2)$$

$$\frac{1}{n_1} = \frac{1}{n_2}$$

$$\Rightarrow n_1 = n_2$$

$\Rightarrow f$ is 1-1.

\nexists $0 \in [0,1]$ is not onto because $0 \in [0,1]$
and there is no such $n \in \mathbb{N}$ such
that

$$f(n) = 0. \text{ Hence}$$

\mathbb{N} strictly precedes closed unit interval
 $[0,1]$ i.e.

$$\mathbb{N} < [0,1]$$

$$\text{Card } \mathbb{N} < \text{Card } [0,1]$$

$$\mathbb{N} < \mathbb{C}$$

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imp.

Cantor's Theorem:

For any set A , $A \subset 2^A$
and hence for cardinal number
 $\alpha = \#A$, we have

$$\alpha < 2^\alpha$$

Proof Define a function

$$f: A \longrightarrow 2^A \text{ defined as}$$

$$f(a) = \{a\} \quad \forall a \in A.$$

Since every singleton set $\{a\}$ for
all $a \in A$ is a subset of A . Therefore
it can be observed A is equivalent to a
collection of all singletons in 2^A (Power set)
which form a subset of 2^A . Therefore

$$A \subset 2^A. \quad \text{--- (1)}$$

we will have to show that

$$A \not\subset 2^A.$$

on Contrary Suppose there exist a function

$$g: A \longrightarrow 2^A \text{ which is bijective.}$$

we say $a \in A$ is a bad element if $a \notin g(a)$
i.e. 'a' does not belongs to its image.

Let B be the set of all bad
elements. Then $B = \{a \in A \mid a \notin g(a)\}$

Since 'g' is onto and $B \in 2^A$ then there exist
an element say $b \in A$ such that

$$g(b) = B.$$

(34)

Case I of b is bad element

$$b \notin g(b) = B$$

$$\Rightarrow b \notin B$$

which is contradiction as B is the set of all bad elements

Case-II

of b is not bad element -

$$\Rightarrow b \in g(b) = B.$$

$\Rightarrow b \in B$ which is again a contradiction as b is not a bad element.

This concludes there exist no onto function. That is our assumption is wrong. Hence

i From ①

$$A \sim 2^A$$

$$\text{Card } A < \text{Card } 2^A$$

$$\alpha < 2^\alpha$$

Hence proved

*

Theorem: Prove that $\mathbb{Z} = \mathbb{C}$

Proof: Since $\mathbb{Q} \sim \mathbb{N}$
 $\text{Card } \mathbb{Q} = \aleph_0$

and $\text{Card } \mathbb{Z} = 2$

we define

For $a, b \in \mathbb{R}$ & of $a \neq b$ $f: \mathbb{R} \rightarrow \mathbb{Z}$ defined as such that
 $f(a) = \{x: x \in \mathbb{Q} \text{ \& } x < a\}$

$\Rightarrow a < b$ (say),

Then there exist $x \in \mathbb{Q}$ such that

$$a < x < b.$$

$$\Rightarrow x < b$$

$$\Rightarrow x \in f(b). \text{ also } x > a$$

$$\Rightarrow x \notin f(a).$$

$$\Rightarrow f(a) \neq f(b).$$

$$\Rightarrow f \text{ is 1-1.}$$

Therefore

$$\mathbb{R} \leq \mathbb{Z}^{\mathbb{Q}}$$

$$\Rightarrow \text{Card } \mathbb{R} \leq \text{Card } \mathbb{Z}^{\mathbb{Q}}$$

$$\mathbb{C} \leq 2^{\aleph_0} \text{ --- (1)}$$

$$P(X) = 2^X$$

$$P(\mathbb{N}) = 2^{\aleph_0}$$

$$C(\mathbb{N}) =$$

Now suppose that \mathbb{N} is a set of natural number and $C(\mathbb{N})$ be the set of all characteristic functions of subsets of \mathbb{N} . we also know that $C(\mathbb{N}) \sim \mathbb{Z}^{\mathbb{N}}$

(36)

$$C(N) \sim 2^N$$

i.e.

$$\text{Card } C(N) = \text{Card } 2^N$$
$$= 2^{\aleph_0}$$

we define

$$F: C(N) \longrightarrow [0,1]$$

$$F(f) = 0.f(1)f(2)f(3)\dots$$

$$f \neq g \text{ for } f, g \in C(N)$$

Then clearly

$$F(f) \neq F(g)$$

Hence

F is 1-1.

Therefore

$$C(N) \subseteq [0,1]$$

$$\text{Card } C(N) \leq \text{Card } [0,1] \quad \text{--- (2)}$$

$$2^{\aleph_0} \leq C$$

From (1) & (2)

$$C = 2^{\aleph_0}$$
$$C = 2 \quad \parallel$$



EXPONENT OF CARDINAL NUMBER:-

Let A and B be two set Then B^A represents the set of all functions from A to B . we defined

$$\# B^A = \beta^\alpha.$$

e.g:-

$$\text{let } A = \{a, b, c\}$$

$$B = \{0, 1\}$$

$$\Rightarrow \# A = 3, \# B = 2$$

$$\alpha = 3, \beta = 2.$$

now we check all functions from A to B .

$$f_1 = \{(a, 0), (b, 0), (c, 0)\}$$

$$f_2 = \{(a, 1), (b, 1), (c, 1)\}$$

$$f_3 = \{(a, 0), (b, 1), (c, 1)\}$$

$$f_4 = \{(a, 1), (b, 0), (c, 1)\}$$

$$f_5 = \{(a, 1), (b, 1), (c, 0)\}$$

$$f_6 = \{(a, 1), (b, 0), (c, 0)\}$$

$$f_7 = \{(a, 0), (b, 1), (c, 0)\}$$

$$f_8 = \{(a, 0), (b, 0), (c, 1)\}$$

$$B^A = \{f_1, f_2, f_3, \dots, f_8\}$$

$$\# = 8 = 2^3, \text{ also from } B^A = 2^3 = 8. \quad \text{--- } \textcircled{2}$$

(38)

Propositions:

Let A, B and C are sets having
Cardinal number

$$\#A = \alpha, \#B = \beta$$
$$\text{and } \#C = \gamma$$

Then show that:

$$i) (\alpha)^\beta \cdot (\alpha)^\gamma = (\alpha)^{\beta+\gamma}$$

$$ii) (\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$$

$$iii) ((\alpha\beta)^\gamma)^\delta = \alpha^{\beta\delta}$$

Proof:

As $\alpha^\beta, \alpha^\gamma$ & $(\alpha)^{\beta+\gamma}$ are the
cardinalities of A^B, A^C and $A^{B \cup C}$ respectively
So $\alpha^\beta \cdot \alpha^\gamma$ be the cardinality of $A^B \times A^C$
Now we define a function

$$f: A^{B \cup C} \longrightarrow A^B \times A^C \text{ defined as}$$

$$F(f) = (f|_B, f|_C)$$

Hence obviously " F " is 1-1 & onto
So F is bijective. So

$$\Rightarrow \# A^{B \cup C} = \# (A^B \times A^C)$$

$$\Rightarrow (\alpha)^{\beta+r} = \alpha^{\beta} \cdot \alpha^r$$

$$ii) (\alpha\beta)^r = \alpha^r \cdot \beta^r$$

Proof $\#A = \alpha$, $\#B = \beta$, $\#C = r$

$$\#(A \times B)^C = (\alpha\beta)^r, \#A^C = \alpha^r$$

and

$$\#B^C = \beta^r$$

$(A \times B)^C$ be the set of all functions from $C \rightarrow A \times B$

i.e

$$f: C \rightarrow A \times B$$

Now we define

$$g: C \rightarrow A$$

$$\Rightarrow g \in A^C$$

and

$$h: C \rightarrow B$$

$$\Rightarrow h \in B^C$$

Now we defined

$$F: (A \times B)^C \rightarrow A^C \times B^C$$

Such that

$$F(f) = (g, h)$$

For f_1

$$\text{let } g_1, g_2 \in A^C, h_1, h_2 \in B^C$$

$$\text{and } f_1, f_2 \in (A \times B)^C$$

if

$$F(f_1) = F(f_2)$$

$$\Rightarrow (g_1, h_1) = (g_2, h_2)$$

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$$\Rightarrow g_1 = g_2 \text{ and } h_1 = h_2$$

$$\Rightarrow f_1 = f_2$$

$\Rightarrow F$ is one-one.

For each $(g, h) \in A^C \times B^C$ \exists a function $f \in (A \times B)^C$ such that

$$F(f) = (g, h)$$

$\Rightarrow F$ is onto.

$\Rightarrow F$ is bijective

Hence

$$(A \times B)^C \sim A^C \times B^C$$

$$\# (A \times B)^C = \# (A^C \times B^C)$$

$$(aB)^r = a^r \cdot b^r \text{ which is required}$$

Result.

$$\text{iii) } \left((aB)^r \right) = a^{Br}$$

$$\# (A^B)^C = \left((aB)^r \right)^r, \# (A)^{B \times C}$$

$(A^B)^C$ is the set of ftns such that

$$f: C \rightarrow A^B$$

$$\Rightarrow f \in (A^B)^C$$

and

$(A)^{B \times C}$ is the set of functions

Such that

$$h: B \times C \rightarrow A$$

$$\Rightarrow h \in (A)^{B \times C}$$

now we defined a function

$$F: (A^B)^C \longrightarrow (A)^{B \times C}$$

Such that

$$F(f) = h$$

let

$$f_1, f_2 \in (A^B)^C \text{ and } h_1, h_2 \in (A)^{B \times C}$$

Then

$$\text{if } F(f_1) = F(f_2)$$

$$h_1 = h_2$$

$$\Rightarrow f_1 = f_2$$

\Rightarrow F is one-one.

Also for each $h \in (A)^{B \times C}$ There exist a function $f \in (A^B)^C$ such that

$$F(f) = h$$

Hence F is onto.

So F is bijective. Then $(A^B)^C \sim (A)^{B \times C}$

$$\Rightarrow \#(A^B)^C = \#(A)^{B \times C}$$

$$\Rightarrow (\alpha^B)^r = (\alpha)^{Br}$$

(42)

Proposition

Let A and B be two sets such that $\#A = \alpha$, $\#B = \beta$.

if $\alpha \leq \beta$ Then show that

- i) $\alpha^r \leq \beta^r$
- ii) $\alpha \leq \beta$

Proof

TO show that $\alpha^r \leq \beta^r$.

$$\alpha^r = \#A^c, \quad \beta^r = \#B^c$$

where

A^c is the set of all functions such that $f: C \rightarrow A$.

$$\Rightarrow f \in A^c$$

Since

$$\alpha \leq \beta$$

$$\Rightarrow A \subseteq B$$

Let

~~CANCELLED~~

Proposition : let A and B be two sets such that $\#A = \alpha$, $\#B = \beta$.

if $\alpha \leq \beta$. Then show that

i) $r^\alpha \leq r^\beta$ where $r = \#C$

ii) $\alpha^r \leq \beta^r$

Proof

i) To show that $r^\alpha \leq r^\beta$
 since $r^\alpha = \#C^A$, $r^\beta = \#C^B$.

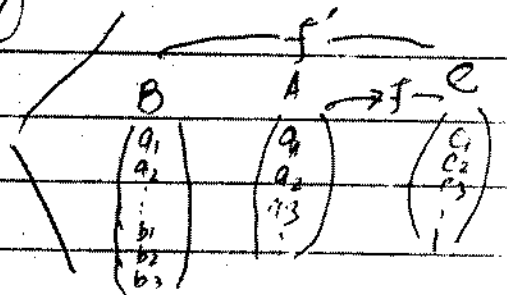
where C^A is the set of all functions such that

$$f: A \rightarrow C$$

$$\Rightarrow f \in C^A$$

Since $\alpha \leq \beta$

$$\Rightarrow A \subseteq B$$



let

$$f': B \rightarrow C \Rightarrow f' \in C^B$$

$$\text{if } f \neq g \Rightarrow f' \neq g' \quad \therefore A \subset B$$

So we defined

$$F: C^A \rightarrow C^B \text{ as}$$

$F(f) = f'$ which is one-one.

$$C^A \subset C^B$$

$$\#C^A \leq \#C^B$$

$$r^\alpha \leq r^\beta$$

(44)

ii) $\alpha^r \leq \beta^r$

Proof $\alpha^r = \# A^c, \beta^r = \# B^c$

where A^c is the set of all functions such that

$$f: C \rightarrow A$$

$$\Rightarrow f \in A^c$$

Since

$$\alpha \leq \beta$$

Therefore

$$A \subset B$$

let

$$f': C \rightarrow B$$

$$\Rightarrow f' \in B^c$$

if

$$f \neq g \Rightarrow f' \neq g'$$

So we defined

$$F: A^c \rightarrow B^c \text{ as}$$

$$F(f) = f' \text{ which is one-one}$$

$$A^c \subset B^c$$

$$\# A^c \leq \# B^c$$

$$\alpha^r \leq \beta^r$$

x ——— x

x ——— x

xx
JMP

This Theorem is applicable only for infinite set (45)

Theorem: (Schröder Bernstein Theorem:-
of $X \supset Y \supset X_1$ and $X \sim X_1$ Then
 $X \sim Y$.

Proof

Since $X \sim X_1$, Then \exists a function f such that

$f: X \rightarrow X_1$ is bijective.

Since Y is contained in X . Then restriction of f from Y to X_1 is one-one (not onto). There exist a subset Y_1 of X_1 (i.e. $X_1 \supset Y_1$) and $f: Y \rightarrow Y_1$ is bijective.

$$\Rightarrow Y \sim Y_1$$

By similar reasoning we will get a

$X_2 \subset Y_1$ Then by given hypothesis $X_1 \supset Y_1 \supset X_2$ such that

$X_1 \sim X_2$ we have

$$X \supset Y \supset X_1 \supset Y_1 \supset X_2 \supset Y_2 \dots$$

proceeding in the same way we will get sequence of equivalent sets

$$X \sim X_1 \sim X_2 \sim X_3 \sim X_4 \dots$$

also

$$Y \sim Y_1 \sim Y_2 \sim Y_3 \dots$$

Now as

$$X \sim X_1 \text{ and } Y \sim Y_1$$

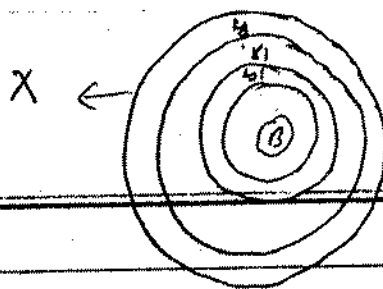
$$\Rightarrow X - Y \sim X_1 - Y_1 \quad \text{By Property of Cardinality}$$

Similarly

$$X_1 - Y_1 \sim X_2 - Y_2, X_2 - Y_2 \sim X_3 - Y_3$$

and so on.

(46)



Now we can write

$$X = (X - Y) \cup (Y - X_1) \cup (X_1 - Y_1) \cup \dots \cup B$$

where

$$B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap \dots$$

$$Y = (Y - X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \cup \dots \cup B$$

we know that

$$X_n - Y_n \sim X_{n+1} - Y_{n+1}$$

Then there exist a function g such

$g: X_n - Y_n \rightarrow X_{n+1} - Y_{n+1}$ is bijective.

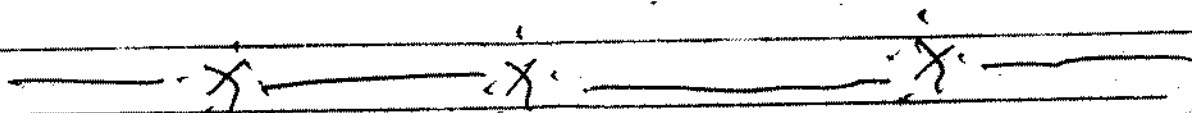
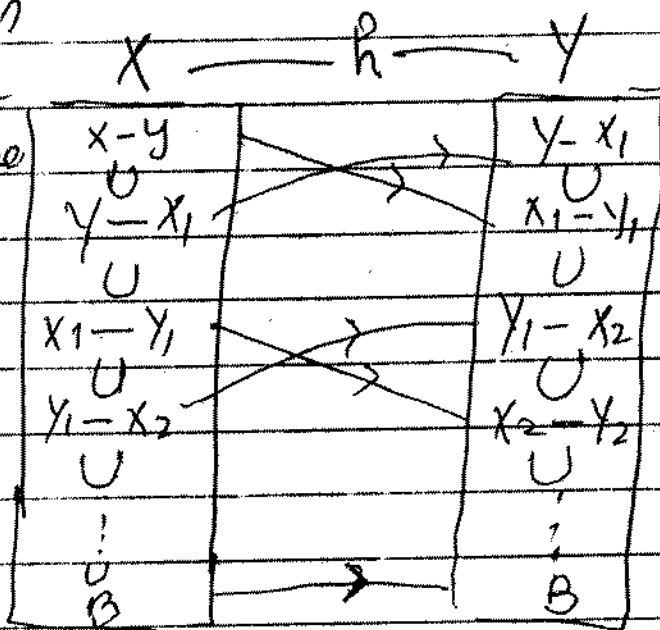
we defined a function h such that

$$h: X \rightarrow Y$$

$$h(x) = \begin{cases} x & \text{if } x \in X_i - X_{i+1} \text{ or } B \\ g(x) & \text{if } x \in X_i - Y_i \text{ and } x \in X - Y. \end{cases}$$

which shows h is both one-one and onto hence bijective
Hence

$$X \sim Y$$



Ordered Set:-

Any set in which order is defined is called ordered set.

* Partially ordered Set:-

Let A be a non-empty set and R be the relation defined in A . Such that

(1) R is reflexive. i.e. $\forall a \in A, aRa$.

(2) R is antisymmetric. i.e.

$\forall a, b$ and bRa .
Then $a = b$

(3) R is transitive. i.e.

$\forall aRb$ and bRc . Then aRc $\forall a, b, c \in A$.

Available at
www.mathcity.org

Let 'A' together with partial ordered relation is called partially ordered set. Further, if relation 'R' in 'A' defines a partial order in A. Then order pair $(a, b) \in R$ i.e. aRb . It is denoted by $a \leq b$ (a precedes b or b dominates a). we also write $a < b$ if $a \neq b$ which reads as a strictly precedes b or b strictly dominates b. Some time the term poset (partially order set) is used for (partially order set).

Ex:

Consider the set P (set of natural number) of +ve integers we say $a|b$ (a divides b) if there exist an integer $c \in \mathbb{N}$ such that

$$ac = b. \text{ For ex. } 2|4, 3|12$$

(48)

This relation of divisibility is partially ordering of \mathbb{P} .

Ex. Consider The \mathbb{Z} set of integer defined $a R b$ if there is +ve integer γ such that $b = a^\gamma$

Reflexive:

Since for all $a \in \mathbb{Z}$. we can write

R is reflexive. $a = a^1 \Rightarrow a R a$ i.e. where $\gamma = 1 \in \mathbb{Z}^+$

Asymmetric:

if $a R b$ and $b R a$

$\Rightarrow b = a^\gamma$ and $a = b^s$
where $\gamma, s \in \mathbb{Z}^+$

$$\begin{aligned} a &= b^s \\ &= (a^\gamma)^s \\ &= a^{\gamma s} \end{aligned}$$

There arises three cases

i) if $\gamma s = 1$ then $\gamma = 1$ & $s = 1$

$$\Rightarrow a = b$$

ii) $a = 1$ then $b = 1$

$$\Rightarrow a = b$$

iii) If $a = -1$ Then $b = -1$ where $s \in \mathbb{O}$.

i.e. s is odd.

In all case R is anti symmetric.

Transitive

If aRb and bRc . For
Then $\exists r, s \in \mathbb{Z}^+$ s.t. $a, b, c \in \mathbb{Z}$.

$$b = a^r \text{ and } c = b^s$$

$$\Rightarrow c = (a^r)^s$$

$$= a^{rs} \\ c = a^q \text{ where } rs = q \in \mathbb{Z}$$

$\Rightarrow aRc$ Hence R is Transitive.

Hence R is partial ordering relation and set of integer is partially order set under the relation R .

Q10 same

Let \mathcal{A} be collection of subset sets Then the relation of set inclusion is a partial order relation.

Pr:- Reflexive,

Since every set is subset of their own. Therefore for any set $A \in \mathcal{A}$

$$A \subseteq A$$

$$\Rightarrow ARA$$

$\Rightarrow R$ is reflexive.

(50)

ii) Anti Symmetry:

Let $A, B \in \mathcal{A}$ and

$A R B$ and $B R A$.

$\Rightarrow A \subseteq B$ and $B \subseteq A$.

$\Rightarrow A = B$.

Hence R is antisymmetry.

iii) Transitive property:

Let $A, B, C \in \mathcal{A}$ and

$A R B$ and $B R C$.

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq C$.

$\Rightarrow A R C$.

$\Rightarrow R$ transitive.

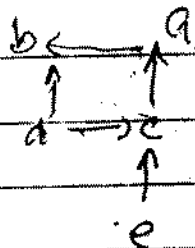
$\Rightarrow R$ is partial ordering relation &
 \mathcal{A} is partially ordered set.

Examples Let $A = \{a, b, c, d, e\}$

Then the relation defined by

$x R y$ if $x = y$ or if

one can go from x to y in the diagram moving in identical direction.



Sol:- The relation defined on A is a

$$R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (e, c), (c, a), (e, a), (b, a), (d, b), (b, d)\}$$

Reflexive

Since $\forall x \in A$

$$x = x.$$

$$\Rightarrow x \leq x.$$

$$\Rightarrow x R x$$

Antisymmetric

Now for $x, y \in A.$

and $x R y$ & $y R x$, i.e

$$x \leq y \text{ & } y \leq x.$$

$$\Rightarrow x = y.$$

$\Rightarrow R$ is antisymmetric

Transitive

For $x, y, z \in A.$

If $x R y$ & $y R z$. Then we are to show that $x R z$. i.e

$$x \leq y \text{ and } y \leq z.$$

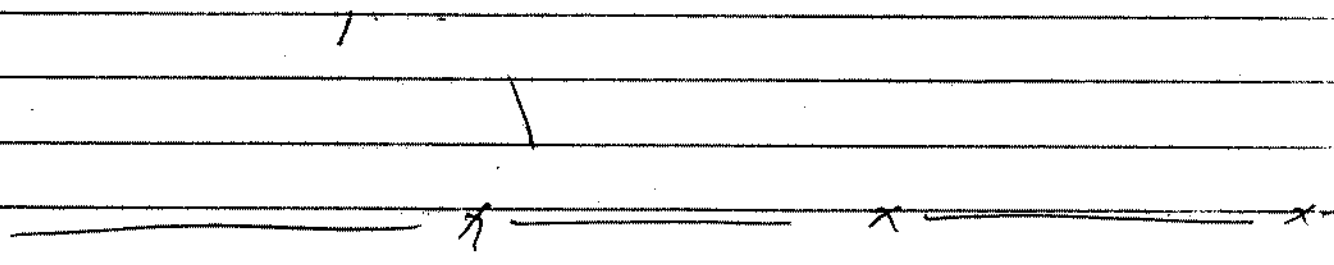
$$\Rightarrow x \leq z$$

$$\Rightarrow x R z.$$

$$\Rightarrow (x, z) \in A.$$

$\Rightarrow R$ is transitive.

Hence the relation R is partial ordering relation. so A is partial order set.



(52)

Ex:-

Let $A = \{2, 3, 4, 5, 6\}$

and R be defined by x/y .

Then R is partial order.

Q2:-

R is defined by x/y .

so

$R = \{(2,2), (3,3), (4,4), (5,5), (6,6), (2,4), (2,6), (3,6)\}$

1) Reflexive:-

since

$\forall x \in A$

x/x

$\Rightarrow x \leq x$

$\Rightarrow x R x$

$\Rightarrow R$ is reflexive

2) Antisymmetry:-

for $x, y \in A$.

if $x \leq y$ and $y \leq x$

$\Rightarrow x = y$

3) Transitive:-

for $x, y, z \in A$.

if $x R y$ and $y R z$ then

we are to show that $x R z$.

since

$x R y$ and $y R z$.

$\Rightarrow x \leq y$ and $y \leq z$.

$\Rightarrow x/y$ and y/z .

$\Rightarrow x/z$

$\Rightarrow x \leq z \Rightarrow x R z \Rightarrow R$ is transitive.

Let $x \neq y$, show that
 \subseteq is p.o.r. $A \subseteq B$

(53)

Hence R is partial ordering
relation and A is partial ordered
set.

~~Proof~~ Let A be any sub set of
 R . Then the relation R
in A is defined by $x \leq y$ is a partial
order in A . It is called natural
order or usual order relation.

Q2:- Let A be any non-empty sub
set of R .

Reflexive

Since $\forall x \in A$ such that

$$x \leq x$$

$$\Rightarrow x R x$$

$$\Rightarrow x R x$$

$\Rightarrow R$ is reflexive.

Antisymmetry:-

Let $x, y \in A$.

If $x \leq y$ and $y \leq x$

$\Rightarrow x \leq y$ and $y \leq x$

$\Rightarrow x = y$

$\Rightarrow R$ is anti symmetric.

Transitive:

Let $x, y, z \in A$.

If $x \leq y$ and $y \leq z$

$\Rightarrow x \leq z$ $\Rightarrow x R z$

(54)

$$x \leq z$$
$$x \leq z$$
$$x R z$$

Hence R is Transitive.

Hence (\mathbb{N}, R) is partially ordered set under natural order relation.

$$x \text{ --- } x$$

NOTE:- Similarly set of natural number can also be showed that it is partially ordered set.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \dots$$

Remark: let A be a partially order set and R be a partial order relation of $(a, b) \in R$ we say 'a' and 'b' are not comparable. i.e. $a \not\leq b$ and $b \not\leq a$ also $a \neq b$.

Inverse Order:-

let R be a partial order defined in a set A then R^{-1} in A is also partial order

$$R^{-1} = \{ (y, x) \mid (x, y) \in R \}$$

Ex:-

$$\text{let } A = \{a, b, c, d, e\}$$

$$\& R = \{ (a, a), (b, b), (c, c), (d, d), (e, e) \\ (e, a), (d, b), (b, a), (d, a), (d, c) \}$$

is a partial order then show that

R^{-1} is also a partial order.

Sol:-

$$\text{NOW } R^{-1} = \{ (a, a), (b, b), (c, c), (d, d), (e, e) \\ (a, e), (b, d), (a, b), (a, d), (c, d) \}$$

i) Reflexive

(since $\forall x \in A$

$$x \leq x \Rightarrow (x, x) \in R^{-1}$$

$$\Rightarrow x \leq x \\ = x R x$$

(56)

Anti symmetric :- Since, for $x, y \in A$.

$$\text{if } (x, y), (y, x) \in R^{-1} \\ \Rightarrow x = y$$

iii) Transitive property :-

For any $x, y, z \in A$.
If $(x, y), (y, z) \in R^{-1}$ Then

$$(x, z) \in R^{-1} \\ \Rightarrow x < z \\ \Rightarrow x R z$$

Hence R^{-1} is partial order.

Totally Order set :-

Let R be a partial order relation defined in A . Then R is said to be totally order if for every pair $x, y \in A$, $x < y$ or $y < x$ or $x = y$

i.e

of every element of set A is comparable with other element of the set for example the natural order is total order.

Ex:- i) set of natural number under natural order relation is totally ordered.

ii) set of Real number under natural order relation.

Totally ordered set is partially ordered. But converse may not be true.

for e.g.

$$A = \{1, 2, 3, 4, 5, 6\}$$

is P.O under the relation " $|$ " (Divisibility) but not totally order
 be $2 \nmid 3$ i.e. 2 & 3 are not comparable under that relation

*// Every open and closed interval is totally order under the natural order relation.

First Element:-

let " A " be an order set. Then $a \in A$ is called First Element of A if $a \leq x \quad \forall x \in A$
 i.e. every element of A dominate a .

For e.g., $\{1, 2, 3, 4\}$ Then 1st element under ~~set~~ natural order relation is 1.

Last Element

let " A " be an order set. Then $b \in A$ is called last element of A if

$$x \leq b \quad \forall x \in A.$$

i.e. every element of A precedes b or b dominates every element of A .

Remarks:- Every totally ordered set need not be a first element or last or both

for e.g. $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
 $\{1, 2, 3, \dots\}$ $\{\frac{1}{n}\}$

ii) If a set A has first and last element
Then 'A' needs not be totally ordered.
for e.g.:-

$\{2, 4, 6, 8, 24\}$
under the divisibility \mid

First element = 2

Last element = 24.

$4 \nmid 6$ or $6 \nmid 4$.

Hence A is not totally ordered.
for e.g.

$\{-2, 2, 4\}$ has
1st & last element but not
partially ordered.

Assignment:

Form a set have 1st &
last element which partially ordered.

$A = \{3, 6, 9, 36\}$ under relation
Divisibility

→ which is partially ordered

First element = 3.

Last element = 36. ✓

————— x ————— x —————

Theorem:-

In^{an} ordered set first and last element are unique.

Proof:- For first element:

Let A be an ordered set and $a, b \in A$ are first elements of A .

Now if $'a'$ is 1st element Then

$$a \leq x \quad \forall x \in A.$$

In particular $b \in A$.

$$a \leq b \quad \text{--- (1)}$$

If b is the 1st element Then

$$b \leq x \quad \forall x \in A.$$

In particular $a \in A$.

$$b \leq a \quad \text{--- (2)}$$

From (1) & (2)

$$a = b.$$

LAST ~~5th~~ element:-

Let A be an ordered set and $a, b \in A$ are last element of A .

If $'a'$ is last element of A Then

$$x \leq a \quad \forall x \in A.$$

In particular $b \in A$.

$$b \leq a \quad \text{--- (1)}$$

Now if b is the last element of A Then

$$x \leq b \quad \forall x \in A.$$

In particular $a \in A$.

$$a \leq b \quad \text{--- (2)}$$

From (1) & (2)

$$a = b.$$

Hence the 1st and last element of an ordered set are unique.

(6)

~~Maximal~~ element:-

Let A be an ordered

Set. Then $a \in A$ is called a maximal element of A if $a \leq x$

~~$a < x$~~ $\Rightarrow a = x$ i.e

if no element of A strictly dominates a .

Minimal element:-

An element $b \in A$

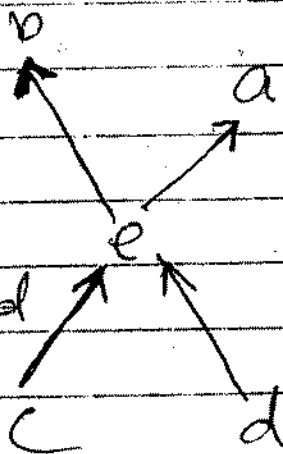
is called a minimal element of A if

$x \leq b \Rightarrow x = b$ i.e

if no element of A strictly precedes to b . Then b is called minimal element.

if

$W = \{a, b, c, d, e\}$ be ordered by following diagramme.



Maximal element = a, b

minimal element = c, d

1st element = no

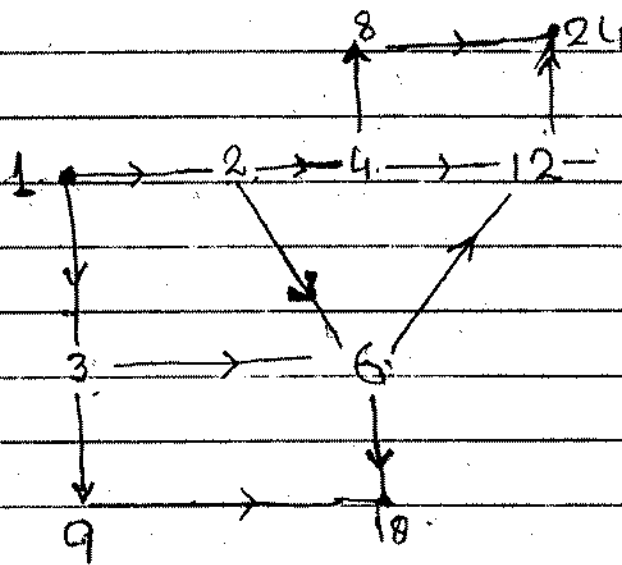
Last element = no .

$$\underline{\underline{24 \leq x \Rightarrow x = 24}} \quad x \leq 1 \Rightarrow x = 1$$

(61)

let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$
 be order by the relation $x|y$.

NOTE: Finite POSET has more than maximal and minimal element



Maximal element : 18, 24.

Minimal element :- 1

1st element :- 1

Last element :- no, ~~no~~

(62)

(c) The linearly ordered set

$$\{x, y, z, u, v\}$$

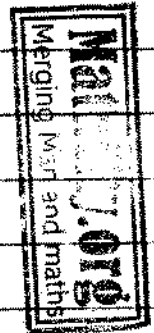
$$x \rightarrow y \rightarrow z \rightarrow u \rightarrow v. \quad (\text{chain})$$

$$\text{1st element} = x$$

$$\text{Last element} = v$$

$$\text{Max Element} = v \quad \checkmark$$

$$\text{min - element} = x.$$



^{2nd}
= a) let S be finite linearly ordered

Set i.e

$$S = \{a_1, a_2, \dots, a_m\}$$

$$\text{1st element} = \text{minimal element}$$

$$\text{1st element} = \min \{a_1, a_2, a_3, \dots, a_m\}$$

$$\text{Last element} = \text{maximal element}$$

$$= \max \{a_1, a_2, a_3, \dots, a_m\}$$

Since ' S ' is totally or linearly order. \therefore in totally order the elements becomes a chain.



* LOWER BOUND:-

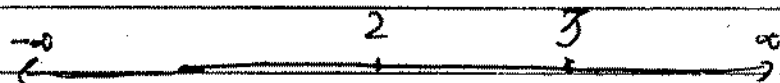
If S is an ordered set and A is the subset of S . Then an element $a \in S$ is said to be lower bound of A if

$$a \leq x \quad \forall x \in A.$$

e.g.

$$A = [2, 3]$$

$$S = \mathbb{R}$$



Set of lower bound of A is

$$= (-\infty, 2]$$

* UPPER BOUND:-

If S is an ordered set and A is the subset of S . Then an element $b \in S$ is said to be upper bound of A if

$$x \leq b \quad \forall x \in A.$$

e.g.

$$A = [2, 3]$$

$$S = \mathbb{R}$$



Then

Set of ~~the~~ upper bound of A

$$= [3, \infty).$$

NOTE :- Lower bound & upper bound of closed & open interval are same. * * *

(64)

mp

Infimum:

Let S be an ordered set and B be its sub set and element $a \in S$ is called infimum of B if 'a' is 'maximum' or last element of the set of lower bounds of B .

$$\text{if } B = (7, 11)$$

$$S = \mathbb{R}$$

max set of lower bounds of $B = (-\infty, 7]$

$$\Rightarrow \text{Infimum of } B = 7.$$

$$\text{or } \inf B = 7.$$

Suprimum:

Let S be an ordered set and B be its sub set and element $b \in S$ is called suprimum of B if b is the 'minimum' or 1st element of the set of upper bounds of B .

$$\text{if } S = \mathbb{R}$$

$$B = (7, 11)$$

Set of upper bounds of $B = [11, \infty)$

$$\text{Sup } B = 11$$

— ∞ — ∞ —

Remarks: i) Every bounded set needs not contain Supremum & infimum for e.g. if $S = \mathbb{R}$ & $A = (0, 1)$ i.e. 0 is infimum & 1 is supremum of A respectively but not contain in A .

ii) Every upper bound is not the Supremum & Every lower bound is not the infimum of A .

iii) If 'a' is the 1st element of an ordered set Then 'a' is only ~~Then a is only~~ the minimal element. i.e. let $S = \{1, 2, 3, \dots, 50\}$ Then $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \dots \rightarrow 50$. Then 1 is 1st element ^{is} also the minimal element.

iv) In case of partially ordered set the maximal and minimal elements may not be unique but in case of totally ordered set maximal and minimal elements are unique & they are exactly ^{equal} to the last and 1st element.

Theorem

In a totally ordered set maximal and minimal elements are unique.

Proof: For minimal elements. Let A be totally ordered set & $a, b \in A$ are minimal elements. Since A is totally ordered set & $a, b \in A$. Therefore

$$a \leq b \text{ or } b \leq a$$

i) $a \leq b$
 $\Rightarrow a = b$ as b is minimal element of A .

ii) $b \leq a$
 $b = a \Rightarrow$ as a is the ^{min} minimal element of A .

(66)

For maximal elements.

Let A be totally ordered set &
 $a, b \in A$ are maximal elements of A .
Since A is totally ordered then.

$$a \preceq b \text{ or } b \preceq a.$$

\Rightarrow

$$a \preceq b$$

$\Rightarrow a = b$ as a is the maximal element
of A .

\Rightarrow

$$b \preceq a$$

$b = a$ as b is the maximal element
of A .

Similar Set

Two ordered sets A &
 B are similar if there exist a mapping

$f: A \longrightarrow B$ such that

- i) f is bijective
- ii) f preserve the order.

It may be noted that the function defined
above is called similarities mapping.

Remarks

If two set are similar then they
are equivalent but converse may not
be true.

Ex:- Let $N = \{1, 2, 3, \dots\} \neq \emptyset$

$$E = \{2, 4, 6, \dots\}$$

Then we define a mapping

$$f(n) = 2n \text{ where } n \in N.$$

Since f is bijective and also for any two elements $x, y \in N$ s.t. $x < y$

$$\Rightarrow f(x) < f(y).$$

$\Rightarrow f$ preserve the order. So N and E are similar set and f is similarities mapping.

Imp Ex. Let N be the set of natural numbers and Z be the set of integers. Then $f: N \rightarrow Z$ defined by:

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

i) Since f is bijective.

ii) if $4, 5 \in N$ such that $4 < 5$

$$\text{Then } f(4) = 2, \quad f(5) = -2.$$

$$\text{so } 2 \not< -2$$

i.e

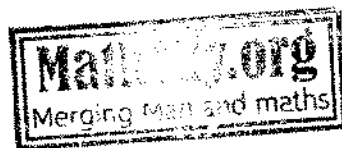
$$f(4) \not< f(5)$$

Hence f not preserve the order. So N & Z are not similar.

$x \text{ --- } x_1 \text{ --- } x \text{ --- } x$

(68)

$A = \{a_1, a_2, \dots, a_n\}$ P.O set



Theorem Finit partially ordered set has atleast one maximal and one minimal element.

Proof: Let A be a finit partially ordered set and let $a_1 \in A$ is not a maximal element then $\exists a_2 \in A$ such that

$$a_1 < a_2$$

Similarly if a_2 is not maximal element then $\exists a_3 \in A$ such that

$$a_2 < a_3$$

Continuing the same process after finit steps we obtained $a_n \in A$ s.t

$$a_n < x \Rightarrow a_n = x.$$

i.e a_n is the maximal element.

For minimal element

Let B be a finit partially ordered set and $b_1 \in B$ is not a minimal element

then \exists an element $b_2 \in B$ such that

$$b_2 < b_1.$$

Similarly if b_2 is not the minimal element of B then there exist $b_3 \in B$ such that

$$b_3 < b_2.$$

Continuing in the same process after finit steps we obtained $b_m \in B$ s.t

$$x < b_m$$

$$\Rightarrow x = b_m$$

i.e b_m is the minimal

element.

Imp. Theorem:

Let $f: A \rightarrow B$ be similarity mapping from an ordered set A to an ordered set B . Then $a \in A$ is maximal (minimal, 1st, last) element of A .

$\Rightarrow f(a)$ is the maximal (minimal, 1st, last) element of B .

Proof: a) For Maximal.

Since f is similarity mapping & $a \in A$ is maximal element of A .

$$\Rightarrow a \not\leq x \Rightarrow a = x$$

$$\text{i.e. } a \not\leq x \quad \forall x \in A.$$

$$\Rightarrow f(a) \not\leq f(x) \quad \because f \text{ is similarity mapping.}$$

$$\Rightarrow f(a) \text{ is maximal element of } B.$$

Conversely

Let $f(a)$ is maximal element of B .

i.e.

$$f(a) \not\leq f(x) \Rightarrow f(a) = f(x).$$

$$\Rightarrow f(a) \not\leq f(x).$$

Since f is similarity mapping. Therefore

$$a \not\leq x \quad \forall x \in A.$$

$$\Rightarrow a \text{ is maximal element of } A.$$

ii.

b) For 1st Element

Let f be a similarity mapping & $a \in A$ is 1st element of A .

$$\Rightarrow a \leq x \quad \forall x \in A.$$

$\Rightarrow f(a) \leq f(x) \quad \forall x \in A$. ~~Since f is similarity mapping & $f(a) \in B$~~

$$\Rightarrow f(a) \leq f(x)$$

$$\Rightarrow f(a) \text{ is 1st element of } B.$$

(70)

Conversely Suppose that $f(a)$ is 1st element of B . Then

$$f(a) \leq f(x) \quad \forall f(x) \in B.$$

Since f is similarity mapping
Therefore

$$a \leq x \quad \forall x \in A.$$

$\Rightarrow a$ is the 1st element of A .

Hence proved.

(c)

FOR LAST ELEMENT-

Let f be similarity mapping & $b \in A$ be the last element of A . Then

$$x \leq b \quad \forall x \in A.$$

Since f is similarity mapping
Therefore

$$f(x) \leq f(b) \quad \text{Then}$$

$f(b)$ is the last element of B

Conversely Suppose that $f(b) \in B$ is the last element of B . Then

$$f(x) \leq f(b) \quad \forall f(x) \in B.$$

Since

f is similarity mapping

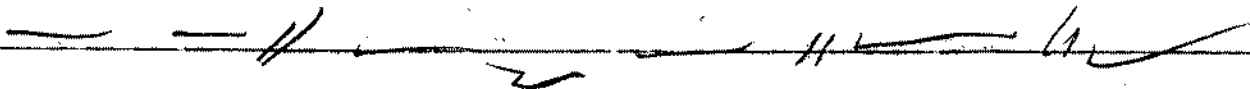
Therefore

$$f(x) \leq f(b)$$

$$\Rightarrow x \leq b \quad \forall x \in A.$$

$\Rightarrow b$ is the last element of A .

A .



Ex: - Imp.

Given example of an ordered set (A, R) which is similar to an ordered set (A, R') i.e. $(A$ with inverse order).

So: - Consider \mathbb{Q} the set of rational numbers with natural order i.e. (\mathbb{Q}, \leq)
Then if

$$f: (\mathbb{Q}, \leq) \longrightarrow (\mathbb{Q}, \geq)$$

Such that

$$f(x) = -x.$$

$\Rightarrow f$ is bijective

ii) let $x, y \in (\mathbb{Q}, \leq)$ and

$$x \leq y \quad \text{--- (1)}$$

$$\text{Then } f(x) = -x, \quad f(y) = -y.$$

$$\text{i.e. (1)} \Rightarrow -x \geq -y.$$

$$\Rightarrow f(x) \geq f(y).$$

i.e. $f(x) \leq f(y) \Rightarrow f$ preserve the ~~mapping~~ order.

Hence f is similarity mapping.

Hence! (\mathbb{Q}, \leq) and (\mathbb{Q}, \geq) are similar.

(72)

Ex // ~~NOTE~~

Set of natural number i.e. (N, \leq) and (N, \geq) are not similar.

- i) Negative numbers are not present
- ii) 1st element exist in (N, \leq) but there is not ~~1st~~ element in (N, \geq) so there exist no similarity mapping.

Theorem: The relation (\cong) is an equivalence relation in any collection of sets.

Proof

Reflexive:

Since $I: A \rightarrow A$ is a mapping defined by $I(a) = a$ which is bijective and preserve the order $\Rightarrow I$ is similarity mapping & $A \cong A$.

Symmetric Property

Suppose $f: A \rightarrow B$ be similarity mapping.

$\Rightarrow f^{-1}: B \rightarrow A$ is bijective mapping as inverse of bijective mapping is bijective. Now we have to prove f^{-1} preserve the order i.e.

if $a_1, a_2 \in A$.

$$a_1 \leq a_2 \quad (\Rightarrow) \quad f^{-1}(a_1) \leq f^{-1}(a_2)$$

$\Rightarrow f(a_1) \leq f(a_2) \quad \therefore f$ is similarity mapping.

$$f(a_1) = b_1, \quad f(a_2) = b_2$$

$$b_1 \leq b_2$$

Now f^{-1} is a mapping from B to A .

if

$$f^{-1}(b_1) \leq f^{-1}(b_2)$$

$$\Rightarrow f^{-1}(f(a_1)) \leq f^{-1}(f(a_2))$$

$$\Rightarrow a_1 \leq a_2. \quad \text{i.e.}$$

f^{-1} preserve the order.

So $B \approx A$.

Hence ' \approx ' is symmetric.

Transitive:

Suppose $f: A \rightarrow B$ and $g: B \rightarrow C$ are similarity mapping. Then

$g \circ f: A \rightarrow C$ is bijective.

Now $a_1, a_2 \in A$ and $a_1 \leq a_2$

Then $f(a_1) \leq f(a_2)$. $\therefore f$ is similarity mapping.

$\Rightarrow g(f(a_1)) \leq g(f(a_2))$ $\therefore g$ is similarity mapping.

$$\Rightarrow g \circ f(a_1) \leq g \circ f(a_2)$$

$\Rightarrow g \circ f$ preserve the order

So $g \circ f$ is similarity mapping & $A \approx C$.
Hence similar relation ' \approx ' is an equivalence relation.

(74)

Order Type: (Ordinal Number).

Let A be a totally ordered set and $\{A_i\}$ the family of those sets which are similar to 'A'. Then $\{A_i\}$ is called the order type of A .
If A and B be two set (totally ordered) set s.t. $A \cong B$, we say order type of A is equal to order type of B .
For e.g.

Order type of $\mathbb{N} = \omega$

order type of $\mathbb{Z} = \pi$

order type of $\mathbb{Q} = \eta$

Well ordered set:

An ordered set A is said to be well ordered if every subset of A has first element.
e.g.

Set of natural = \mathbb{N}

Set of Even number = E

Set of odd number = O

Counter example

Set of integers = \mathbb{Z} .
as $\{-1, -2, -3, \dots\}$ has no first element

Every open & close interval on real line are not well order.

1 — a — x

Theorem

A well ordered ^{set} is totally ordered.

Proof

let A be well ordered set.
also let $a, b \in A$. Then

$$\{a, b\} \subseteq A.$$

$\{a, b\}$ must have 1st element as A is well ordered. i.e. a & b are comparable.

Since a & b are arbitrary elements of A . Hence A is totally ordered set. (All the elements of A are comparable)

∴ ————— ∴ ————— ∴ —————

Theorem
A set which has the 1st element and the last element must be totally ordered but converse does not hold.

sol: The statement is false we show it by counterexample

$$\text{let } A = \{2, 4, 6, 8, 24\}$$

\Rightarrow 2 is the 1st and 24 is the last element of A . ~~but~~ under the relation divisibility but

$4/6 \Rightarrow 4 \nless 6$ Hence A is not totally ordered.

set of natural which totally but it has no last element.

--- CAC - CA₃CA₂CA₁CA
- If A is well order then

(76)



Theorem:

Every subset of a well order set is well ordered.

Proof:

Let A be a well ordered set and B be a subset of A .
To prove that B is well ordered
Consider $B_1 \subseteq B$.

$$\Rightarrow B_1 \subseteq A \quad \therefore B_1 \subseteq A$$

As A is well ordered so B_1 has 1st element. Therefore B is well ordered also as B is arbitrary subset of A . So every subset of A is well ordered.

Theorem:

If A is well-ordered and B is similar to A then B is also well ordered.

Let $f: A \rightarrow B$ be a similarity mapping since $A \sim B$ and $B_1 \subseteq B$.
Since f is onto such that

$$f(A_1) = B_1$$

Now A_1 is well ordered and $A_1 \subseteq A$.
Therefore A_1 has 1st element say a_0 then $f(a_0) \in B_1$ is 1st element of B_1 . Since B_1 is arbitrary therefore B is well ordered.

Theorem

All finite totally ordered sets with same number of elements are well order and similar to each other.

Proof: Let $V = \{a_1, a_2, a_3, \dots, a_n\}$ be a totally ordered set. Now we arrange its elements according to their order. Then

$V = \{a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n}\}$ as V is finite totally order. Therefore if we take any subset of V it contains 1st element and hence V is well order.

Now if $W = \{b_1, b_2, \dots, b_n\}$ then

$f: V \rightarrow W$ defined

$$f(a_{i_n}) = b_n$$

obviously f is bijective as well as preserve the order. \therefore if

$$a_{i_1} \leq a_{i_2}$$

$$\Rightarrow f(a_{i_1}) \leq f(a_{i_2})$$

$$\Rightarrow b_1 \leq b_2$$

$\Rightarrow f$ is similarity mapping. Hence

All the finite totally ordered set with same number of elements are similar to each other.

(78)

Principle of Mathematical Induction.

Let S be a subset of the set \mathbb{N} of natural numbers with following two properties

i) $1 \in S$

ii) If $n \in S$ then $n+1 \in S$

Then $S = \mathbb{N}$ i.e. S is actually the set of ~~all~~ countable numbers.

*

INITIAL SEGMENT :-

Let A be a well ordered set the initial segment $S(a)$ of an element $a \in A$ consist of all elements in A which strictly precede a . In other words

$$S(a) = \{x : x \in A \wedge x < a\}$$

e.g. $\mathbb{N} \rightarrow$ well ordered set

$$S(9) = \{1, 2, 3, \dots, 8\}$$

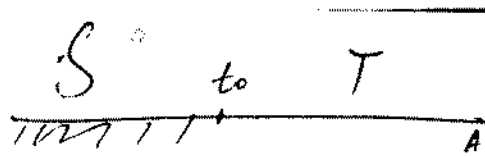
Theorem

Let S be a subset of well ordered set A with the following properties

i) $a_0 \in S$ where a_0 is the 1st element

ii) $S(a) \subseteq S \Rightarrow a \in S \quad \forall a \in A$

Then $S = A$.



$$N = \{1, 2, 3, \dots\}$$

$$1 \in S$$

$$S(1) = \{1\} \subseteq S \Rightarrow 2 \in S$$

$$S(2) = \{1, 2\} \subseteq S \Rightarrow 3 \in S$$

(79)

Proof ~

On contrary Suppose $S \neq A$. i.e. $A \setminus S = T$ is non-empty. Since A is well ordered so T has 1st element say (t_0) . each element $x \in S(t_0)$, strictly proceed to. and therefore x belongs to T , as t_0 is first element of T . so $x \in S$. Hence $S(t_0) \subseteq S \Rightarrow t_0 \in S$ by (ii) hypothesis. which is contradiction to the fact that $t_0 \in T$. Hence our assumption is wrong i.e. $S \neq A$ is not true. In other word $S = A$.

Thm. Theorems of $S(A)$ denote the collection of initial segments of elements in well ordered set A and let $S(A)$ be ordered by set inclusion. Then A is similar to $S(A)$. i.e. $A \approx S(A)$.

Proof ~

let $f: A \rightarrow S(A)$ defined by.

$$f(a) = S(a) \quad \forall a \in A.$$

By definition 'f' is onto.

Suppose $x \neq y$, $x, y \in A$.

\Rightarrow one of them ^{we} say

$$x < y$$

$$\Rightarrow x \in S(y).$$

$$\text{but } x \notin S(x)$$

(80)

$$\Rightarrow S(x) \neq S(y)$$

$\Rightarrow f$ is one-one.

Now we show that f preserve the order i.e. if $x < y \Leftrightarrow S(x) \subset S(y)$
Now of $x < y$ also suppose $a \in S(x)$ i.e. segment of x

$$\Rightarrow a < x < y.$$

$$\Rightarrow a < y$$

$$\Rightarrow a \in S(y)$$

$$\Rightarrow S(x) \subset S(y).$$

Now suppose that

$$x \neq y$$

Then

$$\Rightarrow y \notin S(x) \quad y < x \Rightarrow y \in S(x)$$

But by definition of initial segment

Hence

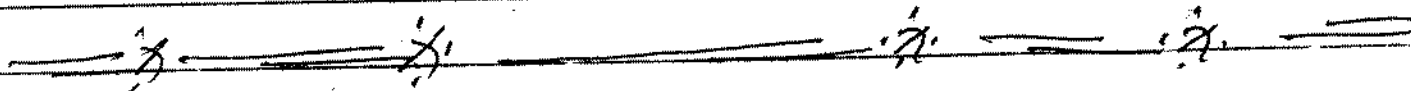
$$S(x) \not\subset S(y).$$

In other words $x < y \Leftrightarrow S(x) \subset S(y)$.

$\Rightarrow f$ is similarity mapping

Hence

$$A \cong S(A).$$



Sol:

Theorem Let A be well ordered set and B be subset of A . Also $f: A \rightarrow B$ is similarity mapping of A onto B . Then for every $a \in A$ $a \leq f(a)$.

Proof - let

$$D = \{x \mid x \in A \wedge f(x) < x\}$$

There are two cases to be discussed

Case-I

If $D = \emptyset$ Then the result is trivial.

Case II. If $D \neq \emptyset$. Then since A is well order so D has 1st element say (d_0) .

Now $d_0 \in D$

$$\Rightarrow f(d_0) < d_0$$

$$\Rightarrow f[f(d_0)] < f(d_0)$$

$$\Rightarrow f(d_0) \in D \quad \because D = \{x \mid x \in A \text{ and } f(x) < x\}$$

Now as $d_0 \in D$ and $f(d_0) \in D$

$$\text{also } f(d_0) < d_0$$

which is contradiction as 'd₀' is the 1st element of D . Hence our assumption is wrong and $D = \emptyset$ and result is true

i.e

$$a \leq f(a).$$

• ————— • ————— • ————— • ————— •

(82)

(A) Similar (B) Similar (A) \Rightarrow $f: A \rightarrow B$ (Similar)

*

Theorem Let A and B be similar well-ordered sets. Then there exist only one similarity mapping of A into B .

Proof Let

$f: A \rightarrow B$ and $g: A \rightarrow B$ be two similarity mappings. That is $f \neq g$.

Then \exists an element $x \in A$ such that $f(x) \neq g(x)$

where $f(x), g(x) \in B$.

\Rightarrow Either $f(x) < g(x)$ or $g(x) < f(x)$

Say $f(x) < g(x)$

\because $g: A \rightarrow B$ is similarity mapping

Then

$g^{-1}: B \rightarrow A$ is also similarity mapping.

Furthermore

$g^{-1} \circ f: A \rightarrow A$ is similarity mapping

as composition mapping of two similarity mappings is also similarity mapping but

$f(x) < g(x)$ and $f(x), g(x) \in B$.

$\Rightarrow g^{-1}(f(x)) < g^{-1}(g(x))$

$\Rightarrow g^{-1}(f(x)) < g^{-1}g(x) = x$

$\Rightarrow g^{-1}(f(x)) < x$

$\Rightarrow g^{-1} \circ f(x) < x$

which is contradiction to the theorem

"If A is well order set & $B \subset A$ & $f: A \rightarrow B$ is similarity mapping then $\forall a \in A$ $a < f(a)$ "

Hence our assumption that $f \neq g$ is wrong. Hence similarity mapping is unique. //

NOTE - IDENTITY relation is
 $P.O.R \Leftrightarrow E.R.$

(83)

Theorem A well order set can not be similar to any initial segment.

Proof - let A be well ordered set and $S(a)$ be one of its initial segment.

Suppose $f: A \rightarrow S(a)$ be similarity mapping then

$$f(a) \in S(a)$$

$$\Rightarrow f(a) \prec a \text{ By def of initial segment.}$$

Which is contradiction to The Theorem

"If A is well ordered set and $B \subset A$ and $f: A \rightarrow B$ be a similarity mapping then $\forall a \in A \quad a \prec f(a)$ ".

Hence there is no similarity mapping b/w a well order set and its initial segment.

* Def
Theorem

Two different initial segments of well order set cannot be similar.

Proof - let $S(a)$ and $S(b)$ be different

initial segments i.e $a \neq b$

Then either $a \prec b$ or $b \prec a$

Suppose

$$a \prec b \Rightarrow a \in S(b)$$

$$\text{let } x \in S(a)$$

$$\Rightarrow x \prec a$$

$$\Rightarrow x \prec a \prec b$$

$$\Rightarrow x \prec b$$

$$\Rightarrow x \in S(b).$$

$$\Rightarrow S(a) \subset S(b).$$

$$N = \{1, 2, 3, \dots\}$$

$$S(1) = \{1, 2, 3, \dots\}$$

$$S(10) = \{1, 2, 3, \dots, 9\}$$

$$\text{Since } \# S(1) \neq \# S(10)$$

Then there can never be bijec.

$$\Rightarrow S(a) \subseteq S(b)$$

Hence $S(b)$ is well ordered being a subset of well ordered set and $S(a)$ of its initial segment. Therefore can't be similar to its initial segment.

As by Theorem "A well ordered set can't be similar to its initial segment."

Theorem 2 Let A and B be two well ordered set and let an initial segment $S(a)$ of A is similar to an initial segment $S(b)$ of B . Then $S(a)$ is similar to a unique initial segment $S(b')$ of B .

Proof

Let $a \in A$ and $b, b' \in B$ also

Suppose that

$$S(a) \approx S(b) \quad \text{Then } S(b) \approx S(a) \text{ --- (1)}$$

and

$$S(a) \approx S(b') \quad \text{--- (2)}$$

Then from (1) & (2) by transitive property

$$S(b) \approx S(b')$$

Since $b, b' \in B$. Then $S(b) \approx S(b')$ is contradiction because by the Theorem

or any two different segment of a well ordered set is not similar.

Hence $S(a)$ is similar to a unique initial segment $S(b)$ of well order set B .

Imply:

Theorem: Let A be well order set and S be subset of A with the following properties

if $a < b$, $b \in S$ then $a \in S$

Then

$S = A$ or S is an initial segment of A .

Proof w

Suppose $S \neq A$. Then

$$A \setminus S = T \neq \emptyset$$

Also 'T' has first element being the subset of well order set A .

say $a_0 \in T$ is first element of T , where $a_0 \notin S$. we show that

$$S = S(a_0)$$

Suppose $x \in S(a_0)$

$$\Rightarrow x < a_0$$

$$\Rightarrow x \notin A \setminus S = T$$

$$\Rightarrow x \in S$$

$\therefore a_0$ is 1st element of T .

$$\Rightarrow S(a_0) \subseteq S \text{ --- (1)}$$

$$A = \{1, 2, 3, 4, \dots\}$$

$$1 < 2, 2 \in S$$

$$\Rightarrow 1 \in S$$

$$S = \{1, 2\}$$

(86)

Now - Suppose $y \notin S(a_0)$
Then

also Suppose $a_0 \prec y$
 $y \in S$ and $a_0 \not\leq y$

$\Rightarrow a_0 \in S$ (By given property)

which is contradiction as a_0 is 1st element of T . Hence our assumption that $y \in S$ is wrong. we have ~~$\exists y \in S(a_0)$~~

$y \notin S(a_0)$

$\Rightarrow y \notin S$

$\Rightarrow S \subseteq S(a_0) - \text{②}$

From ① & ②

$S = S(a_0)$.

Now we are to show that

$A = S$

Suppose that $A \neq S$ i.e. $A \setminus S = T$
Since A is well order Therefore $T \subset A$ has 1st element a_0 (say). each element

$x \in S(a_0)$

$\Rightarrow x \prec a_0$

$\Rightarrow x \notin T$ $\because a_0$ is 1st element

$\Rightarrow x \in S$

$\Rightarrow S(a_0) \subseteq S$

$\Rightarrow a_0 \in S$

which is contradiction

NOTE $f: A \rightarrow B$ where A, B are finite
of f is 1-1. Then it is bijective.

(87)

to the fact that $a_0 \in T$ is the
1st element of T . Hence our assumption
is wrong.

$$\text{So } A = \emptyset$$

Theorem

Let A and B be two well ordered
set such that an initial segment $S(a)$ of A
is similar to initial segment $S(b)$ of B .
Then each initial segment of $S(a)$ is similar
to an initial segment of $S(b)$. i.e

$$a' \prec a \Rightarrow S(a') \approx S(b') \text{ where } b' \prec b \text{ where } b' \text{ is image of } a'.$$

Proof

Let $f: S(a) \rightarrow S(b)$ be
similarity mapping where $S(a)$ & $S(b)$
are initial segment of A and B are
respectively.

$$\text{Let } f(a') = b'$$

Note that f restricted to $S(a')$ is
1-1 and preserve the order.

Hence

$$S(a') \approx f(S(a')) \text{ i.e}$$

$$S(a') \approx f(S(a')) \text{ --- (1)}$$

Furthermore since f is similarity
mapping and of $a^* \prec a'$

$$\Rightarrow f(a^*) \prec f(a')$$

(88)

$$\Rightarrow f(a^*) \leq f(a') = b'$$

i.e.

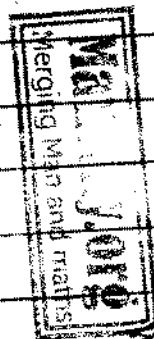
$$f(a^*) \leq b'$$

$$a^* \in S(a) \Leftrightarrow f(a^*) \in f(S(a)) = S(b')$$

$$\Rightarrow f(S(a)) = S(b') \quad \text{--- (2)}$$

From (1) & (2)

$$S(a) \cong S(b')$$



* Theorem Let A & B be well ordered sets
 Let S and T are the subsets of
 A and B respectively such that

$$S = \{x : x \in A \quad S(x) \cong S(y) ; y \in B\}$$

$$T = \{y : y \in B \quad S(y) \cong S(x) ; x \in A\}$$

Then

$$S \cong T.$$

Proof Let $x \in S$, then by the theorem
 "Let A, B be well ordered and
 initial segment of a i.e. $S(a)$ is similar
 to initial segment of b i.e. $S(b)$. Then
 $S(a)$ is similar to unique $S(b)$. i.e.
 $S(a) \cong S(b)$."

Then $S(x)$ is similar to a unique $S(y)$
 of the set B .

Thus to each $x \in S$ There correspond to unique $y \in T$ such that

$$S(x) \cong S(y) \text{ and vice versa.}$$

Hence

$f: S \rightarrow T$ defined by

$$f(x) = y \text{ if } S(x) \cong S(y).$$

$\Rightarrow f(x)$ is one-one & onto hence bijective.

now let $x, x' \in S$, such that

$x' \prec x$ Then we are to show that $f(x') \prec f(x)$

$$\Rightarrow y' \prec y$$

let $g: S(x) \rightarrow S(y)$ be similarity mapping of $S(x)$ into $S(f(x)) = S(y)$.

By Theorem.

"let A & B be well ordered set $S(a)$ and $S(b)$ be initial segment of a and b respectively such that $f: S(a) \rightarrow S(b)$ is similarity mapping on $S(b)$ - where $a' \preceq a \Rightarrow f(a') \preceq f(a)$ and $b' \preceq b$.

Then

$g|_{S(x')}$ is similarity mapping of $S(x')$ onto $S(g(x'))$ of B . but there exist only one similarity mapping of $S(x')$ into B . Then consequently

$$g(x') = f(x') = y'$$

$$\text{Since } g(x') \in S(y) \Rightarrow y' \in S(y)$$

Hence f is similarity mapping. $\Rightarrow y' \prec y$
 \square

(90)

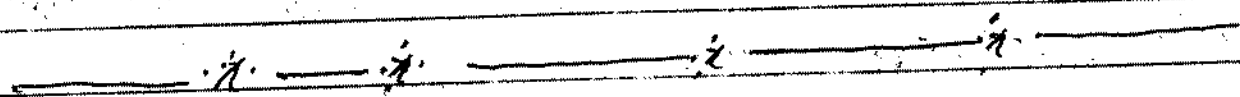
Immediate Successor & Immediate Predecessor:-

If $a, b \in A$ where A is order set then b is said to be immediate successor

if $a < b$ such that there exist no element $c \in A$ such that $a < c < b$. Then it is denoted by

$$a \triangleleft b$$

where a is immediate predecessor.



Theorem Every element in a well order set has a unique immediate successor except the last element.

Proof Let $a \in A$ and $M(a)$ be the set of all those elements of A which strictly succeeds a . If a is not the last element then $M(a) \neq \emptyset$

Since set A is well order then $M(a)$ has 1st element say b . We claim that b is an immediate successor of a . Otherwise there is an element $c \in A$ such that

$$a < c < b \text{ then}$$

$$c \in M(a) \because a < c.$$

and this contradicts the fact that b is the 1st element of $M(a)$. Hence b is immediate successor.

We claim that b is the unique immediate successor of a otherwise there is another immediate successor of a say $d \in M(a)$.

and since 'b' is the 1st element of $M(a)$
 so a bad.

This contradicts the assumption 'd'
 is the immediate successor of $M(a)$. Thus
 1st element 'b' is the unique immediate
 successor of 'a'.

BOOK.

Theorem Every element in well ordered
 set has a unique immediate predecessor
 except the ~~last~~^{1st} element.

Proof let $a \in A$ and $S(a)$ be the set of all
 those element of A which strictly
 precede 'a'. If 'a' is not the 1st element
 then $S(a) \neq \emptyset$.

~~set~~ Since set A well ordered, then $S(a)$ has
~~last~~ element say 'b'. we claim that 'b' is an
 immediate predecessor of 'a'. otherwise there
 exist an element $c \in A$ such that

$$b \prec c \prec a.$$

$$\Rightarrow c \in S(a).$$

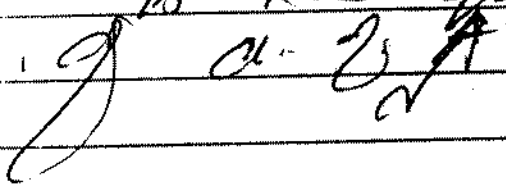
and This contradicts the fact that 'b'
 is last element of $S(a)$ bss $b \prec c$. Hence
 'b' is immediate predecessor.

we claim that 'b' is the unique
 predecessor of 'a' otherwise there is another
 immediate predecessor of 'a' say $d \in S(a)$
 and since 'b' is the ~~first~~^{last} element of $S(a)$
 so $b \prec d \prec a$ bad αa

This contradicts that assumption that
 'd' is the immediate predecessor of

(92)

$S(a)$. Thus last element b is the unique immediate predecessor



* Shorter and Longer Sets

Suppose 'A' &

'B' are well ordered set and suppose A is similar to an initial segment of B. Then A is said to be shorter than B and B is said to be longer than A.

i.e

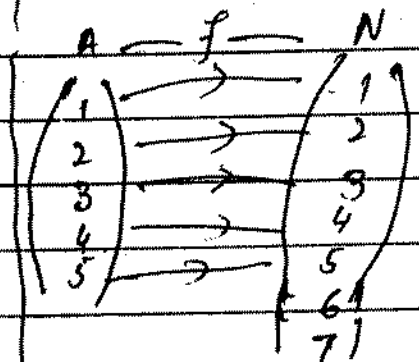
$$A \sim S(b) \subset B$$

Then

A is shorter than B

and

B is longer than A.



$$A \sim S(b)$$

where $f(x) = x$.

Then

A is shorter &

N is longer.

Theorem:

let \mathcal{A} be a collection of pairwise non-similar well ordered sets, Then there exist a set A_0 in \mathcal{A} such that A_0 is shorter than every other set in \mathcal{A} .

Proof: let B be the set in \mathcal{A} defined

$$B = \{x : x \in \mathcal{A}, x \text{ is shorter than } B\}$$

If $B = \emptyset$ Then B satisfies the requirements of Theorem i.e. B is shortest.

$$\mathcal{A} = \{A_1, A_2, A_3, A_4, \dots\}$$

$$B = \{x : x \in \mathcal{A}, x \subseteq S(b) \text{ for some } b \in B\}$$

$$B' = \{S(b) : \forall b \in B\}$$

But if $B \neq \emptyset$. If we show that

B has shortest set A_0 Then

considering the way B , we defined A_0 will also be the shortest set \mathcal{A} .

Since every set $A \in B$ is shorter than B so each A is similar to $\text{initial segment } S(b)$ of B . i.e. $A \sim S(b)$.

let B' be the collection of all those initial segments of B , each of which is similar to a set of B . Then B' contains an initial segment $S(a_0)$ which is shorter than every other initial segment of B . since B is well ordered. Consequently the set $A_0 \in B$ which is similar to $S(a_0)$ is shorter than to any set of B .

Therefore A_0 is the shortest set in \mathcal{A} and A_0 satisfying the requirement of the Theorem.

Ordinal Numbers

Consider a collection L of well ordered sets. Each set A in L is assigned a symbol in such a way that any two well-ordered sets A and B are assigned the symbol α if A and B are similar. This symbol is called the ordinal number of A and B , and we will write

$$\alpha = \text{ord}(A) = \text{ord}(B) \text{ iff } A \sim B \text{ and similar}$$

Transfinite Ordinal numbers

The ordinal numbers of each of the well order set $\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \dots$ is denoted by $0, 1, 2, 3, \dots$ respectively and are called finite ordinal numbers, all other ordinal numbers are called transfinite Ordinal numbers. The ordinal number of set of natural number is denoted by $\omega = \text{ord}(\mathbb{N})$. ' ω ' is also called 1st limit ordinal.

— * — * — * — * —

* Inequality And Ordinal Numbers:

An equality relation is defined by the ordinal numbers as follows

let ' λ ' and ' μ ' be ordinal numbers and let A and B be well-ordered sets such that

$$\lambda = \text{ord}(A) \quad \& \quad \mu = \text{ord}(B)$$

Then

$\lambda < \mu$ if A is shorter than B , or A is similar to an initial segment of B .

* Ordinal addition:

An operation of addition is defined for ordinal number as follows

let ' λ ' and ' μ ' be ordinal numbers and let A & B disjoint sets such that

$$\lambda = \text{ord}(A) \quad \text{and} \quad \mu = \text{ord}(B)$$

Then

$$\lambda + \mu = \text{ord}[A; B]$$

(96)

// Ordinal Multiplication :-

An operation on multiplication is defined for ordinal numbers as follows

let α and β be ordinal numbers
and let A and B well order sets
such that

$$\alpha = \text{ord}(A) \text{ and } \beta = \text{ord}(B).$$

Then

$$\alpha\beta = \text{ord}(A \times B).$$

Ex let $A = \{a, b\}$ and $B = \mathbb{P} = \omega$ be well order sets then

$$\text{ord}(A) = 2, \text{ord}(\mathbb{P}) = \omega \quad \text{Then}$$

$$A \times B = \{a, b\} \times \mathbb{P}.$$

$$A \times B = \{(a, 1), (a, 2), (a, 3), \dots, (b, 1), (b, 2), (b, 3), \dots\}$$

Then $\text{ord}(A \times B) = \text{ord} \left\{ \begin{array}{l} (a, 1), (a, 2), \dots \\ (b, 1), (b, 2), \dots \end{array} \right\}$

Now as

$$\begin{aligned} \left\{ (a, 1), (a, 2), (a, 3), \dots \right\} &\cong \mathbb{N} \\ \left\{ (b, 1), (b, 2), (b, 3), \dots \right\} &\cong \mathbb{N} \end{aligned}$$

$$\text{ord}(A \times B) = \omega * \omega$$

$$= \underline{\underline{\omega^2}}.$$

Structure of ordinal Number:-

we now write down many of the ordinal numbers according to their numbers (structures)
 First we come to finite ordinal which are $0, 1, 2, 3, \dots$

Then comes to 1st limit ordinal ω and its successors are $\omega, \omega+1, \omega+2, \omega+3, \dots$
 Hence next comes to second limit ordinal number ω_2 and its successors are

$\omega_2, \omega_2+1, \omega_2+2, \omega_2+3, \dots$. The next limit ordinal number is ω_3 . we proceed as follows

$\omega_3, \omega_3+1, \omega_3+2, \dots$

$\omega_4, \omega_4+1, \omega_4+2, \dots$

\dots

\dots

\dots

$\omega \cdot \omega = (\omega)^2 = (\omega)^2, (\omega)^2+1, \omega^2+2, \dots$

$(\omega)^3, \dots, (\omega)^4, \dots, (\omega)^\omega, \dots$

$\dots, (\omega^\omega)^1, \dots, (\omega^\omega)^2, \dots, (\omega^\omega)^\omega, \dots$

$\dots, ((\omega^\omega)^\omega)^\omega, \dots$

After all these ordinal we have the ordinal ϵ_0 we can continue

$\epsilon_0, \epsilon_0+1, \epsilon_0+2, \epsilon_0+3, \dots$

we note each of the ordinal numbers we have enumerated is still the ordinal number of countable set.

b/c union of countable set is countable.

(98)

Theorem

// let $\mu = \text{ord}(A)$ let $\mu < \lambda$
 Then there is a unique initial segment $S(a)$
 of A . such that
 $\mu = \text{ord}(S(a))$.

Proof let $\mu = \text{ord}(B)$
 and since $\mu < \lambda = \text{ord}(A)$
 and $\mu < \lambda$ then B is similar
 to A . i.e. B is similar to an initial
 segment $S(a)$ of A . Therefore $\mu = \text{ord}(S(a))$.
 Since two different initial segments of
 a well order set cannot be similar. Therefore
 $S(a)$ is the only initial segment whose
 ordinal number is μ . i.e.

$$\mu = \text{ord}(S(a)).$$



* gmp

Theorem

// let $S(\lambda)$ be the set of ordinals
 less than the ordinal λ Then

$$\lambda = \text{ord}(S(\lambda))$$

$$S(9) = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$\text{ord } S(9) = 9$$

$$S(A) = \{S(a) : a \in A\}$$

Proof

let $\lambda = \text{ord}(A)$

and let $S(A)$ denote the collection
 of all initial segments of A . ordered by
 set inclusion.

Since $A \sim S(A)$

So $\text{ord } S(A) = \lambda$.

Next we show that

$$S(\lambda) \cong S(A).$$

Let

$$\mu \in S(\lambda).$$

Then

$$\mu < \lambda$$

Then there is unique initial segment $S(\alpha)$ of A such that

$$\mu = \text{ord}(S(\alpha)) \quad \text{by the previous theorem}$$

Now we defined a function

$$f: S(\lambda) \longrightarrow S(A).$$

By

$$f(\mu) = S(\alpha) \quad \text{if } \mu = \text{ord}(S(\alpha)).$$

So f is clearly 1-1. Furthermore we show that f is onto.

Suppose $S(b) \in S(A)$. Then $S(b)$ is shorter than A . \therefore therefore ordinal number of $S(b) = \gamma < \text{ord}(A) = \lambda$
 $\gamma < \lambda$.

$\Rightarrow \exists \mu \in S(\lambda)$. $\therefore S(\lambda)$ is the set of ordinal numbers which are less than λ .

$$f(\mu) = S(b) \quad \text{where } \mu = \text{ord}(S(b)) < \lambda.$$

$\Rightarrow f$ is onto.

Hence f is bijective.

Now we will prove that f preserves order. For that suppose $\mu, \gamma \in S(\lambda)$ such that $\mu < \gamma$

(100)

where $\mu = \text{ord}(S(a))$

$\nu = \text{ord}(S(b))$

ie $f(\mu) = S(a)$

and

$f(\nu) = S(b)$.

Since $\mu < \nu$ Therefore

$S(a)$ is an initial segment of $S(b)$. Hence $S(a)$ is proper subset of $S(b)$. In other words under the ordering of $S(A)$,

$S(a) < S(b)$.

$\therefore S(a) \subset S(b)$

$\Rightarrow f(\mu) < f(\nu) \Rightarrow f$ preserves the order

$\Rightarrow f$ is similarity mapping

ie

$S(N) \cong S(A)$

$\Rightarrow \text{ord}(S(N)) = \text{ord}(S(A))$

$= n$

Hence

$n = \text{ord}(S(N))$

//

~~$\alpha < \beta \Rightarrow \alpha \in \alpha \subset \beta$~~

* ORDINARY DIFFERENTIAL EQUATION:- OF HIGHER ORDER HYPER GEOMETRIC D.E's

In mathematics the hypergeometric D.E is a 2nd order linear ordinary D.E whose solutions are given by the hypergeometric series:

* Hypergeometric Series

A hypergeometric series is a power series in which the ratios of successive coefficients is a rational function. The series if convergent will define a hypergeometric function.

Hypergeometric function generalize many special functions including the Bessel function, The incomplete Gamma Function, The ERROR Function.

Symbolically hypergeometric function can be written as

$$F(a, b, c; z) = F \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right]$$

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

where $(a)_n$ read as a factorial function of order n or generalize factorial function or rising function.

$$(a)_n = a(a+1)(a+2)(a+3) \dots + (a+n-1)$$

(102)

now we take $a=e$ and $b=1$
we get

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(e)_n (1)_n}{(c)_n n!} z^n$$

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

* GAMMA FUNCTION:

The GAMMA Function represented by the capitalized Greek letter " Γ " which is an extension of factorial function to the real and complex numbers z with positive (+ve) real part. ~~It is~~ ~~defined~~ ~~as~~

NOTATION $\Gamma(z) = \text{Gamma } z$ It is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

Properties

$$\Gamma(1) = 1$$

using

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

Put $z=1$

$$\begin{aligned} \Gamma(1) &= \int_0^{\infty} e^{-t} t^{1-1} dt. \\ &= \int_0^{\infty} e^{-t} dt. \end{aligned}$$

$$\begin{aligned}
 \Gamma(1) &= \lim_{h \rightarrow \infty} \int_0^h e^{-t} dt \\
 &= \lim_{h \rightarrow \infty} \left[-e^{-t} \right]_0^h \\
 &= \lim_{h \rightarrow \infty} \left[-\frac{1}{e^h} + \frac{1}{e^0} \right] \\
 &= \lim_{h \rightarrow \infty} \left[-\frac{1}{e^{\infty}} + 1 \right] \\
 &= \lim_{h \rightarrow \infty} [-0 + 1] \\
 &= \lim_{h \rightarrow \infty} (1)
 \end{aligned}$$

Hence $\Gamma(1) = 1$ Ans.

ii)

$$\Gamma(2+1) = \int_0^{\infty} t^{2+1-1} e^{-t} dt$$

$$\Gamma(2+1) = \int_0^{\infty} e^{-t} \cdot t^{2+1-1} dt$$

$$= \int_0^{\infty} e^{-t} \cdot t^2 dt$$

$$= \lim_{h \rightarrow \infty} \int_0^h e^{-t} t^2 dt$$

$$= \lim_{h \rightarrow \infty} \left[\frac{t^2 e^{-t}}{-1} - \int \frac{e^{-t}}{-1} \cdot 2t dt \right]$$

$$= \lim_{h \rightarrow \infty} \left[\frac{t^2 e^{-t}}{-1} - \int \frac{e^{-t}}{-1} \cdot 2t dt \right]$$

(104)

$$= 0 + \lim_{h \rightarrow \infty} z \int_0^h e^{-t} t^{z-1} dt.$$

$$= z \lim_{h \rightarrow \infty} \int_0^h e^{-t} t^{z-1} dt.$$

$$= z \int_0^{\infty} e^{-t} t^{z-1} dt.$$

Hence

$$\Gamma(z+1) = z \Gamma(z) \quad \text{prove } \Gamma(z+1) = z!$$

Since

$$\Gamma(z+1) = z \cdot \Gamma(z)$$

$$= z \cdot (z-1) \Gamma(z-1)$$

$$= z \cdot (z-1) (z-2) \Gamma(z-2)$$

$$= z \cdot (z-1) (z-2) (z-3) \Gamma(z-3)$$

$$\vdots$$

$$z \cdot (z-1) (z-2) (z-3) \dots \cdot 1 \Gamma(1)$$

$$= z \cdot (z-1) (z-2) (z-3) \dots \cdot 1$$

$$\Gamma(z+1) = z!$$

✓ Show that $0! = 1$.

we know that

$$n! = n(n-1)!$$

$$\Rightarrow (n-1)! = \frac{n!}{n}$$

put $n=1$

$$0! = \frac{1!}{1} = \frac{1}{1} = 1$$

$$\boxed{0! = 1}$$

//

Beta function:

Notation: $B(x, y)$

It is defined as.

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \text{ where } x, y \in \mathbb{Z}.$$

* Relation b/w Gamma & Beta function.

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{\Gamma(y) \Gamma(x)}{\Gamma(y+x)}$$

✓ $B(x, y) = B(y, x)$
i.e. Beta function is symmetric.

Some useful result:

i) Binomial Theorem

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n$$

$$\text{ii) } \Gamma(\alpha+n) = \alpha(\alpha+n-1) \Gamma(\alpha+n-1)$$

$$= (\alpha+n-1)(\alpha+n-2) \Gamma(\alpha+n-2)$$

$$= (\alpha+n-1)(\alpha+n-2) \dots \alpha \Gamma(\alpha)$$

$$\Gamma(\alpha+n) = (\alpha)_n \Gamma(\alpha) \Rightarrow \Gamma(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

(106)

Question

of $|z| < 1$ & $\text{Re}(r) > \text{Re}(\beta) > 0$

Then

$$F(\alpha, \beta, r; z) = \frac{\Gamma(r)}{\Gamma(\beta)\Gamma(r-\beta)} \int_0^1 t^{\beta-1} (1-t)^{r-\beta-1} (1-tz)^{-\alpha} dt$$

Ans: since

$$F(\alpha, \beta, r; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\Gamma)_n n!} z^n \quad \text{--- (1)}$$

take

$$\frac{(\beta)_n}{(\Gamma)_n} = \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \cdot \frac{\Gamma(r)}{\Gamma(r+n)} \quad \because (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(r)}{\Gamma(\beta)\Gamma(r-\beta)} \cdot \frac{\Gamma(\beta+n)\Gamma(r-\beta)}{\Gamma(r+n)} \quad \left| \begin{array}{l} x = \beta+n \\ y = r-\beta \\ x+y = r+n \end{array} \right.$$

$$= \frac{\Gamma(r)}{\Gamma(\beta)\Gamma(r-\beta)} \cdot B(\beta+n, r-\beta) \quad \left. \begin{array}{l} \text{By defn} \\ \text{of} \\ \text{Beta} \\ \text{function} \end{array} \right\}$$

$$\frac{(\beta)_n}{(\Gamma)_n} = \frac{\Gamma(r)}{\Gamma(\beta)\Gamma(r-\beta)} \int_0^1 t^{\beta+n-1} (1-t)^{r-\beta-1} dt$$

using the above value in eq (1)

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 (t)^{\beta+n-1} (1-t)^{\gamma-\beta-1} dt \frac{(\alpha)_n z^n}{n!}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 (t)^{\beta-1} (1-t)^{\gamma-\beta-1} dt \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{n!}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (tz)^n}{n!} dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt$$

$$\therefore (1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{n!}$$

Hence.

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt$$

~~=====~~

(108)

Q11 $\operatorname{Re}(c-a-b) > 0$ & $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$
Then

$$F(a, b, c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}$$

Soln Since

$$F(a, b, c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} (1)^n$$

$$F(a, b, c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} \quad \text{--- (1)}$$

take

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)}$$

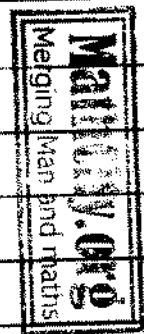
$$= \frac{\Gamma(c) \Gamma(b+n) \Gamma(c-b)}{\Gamma(b) \Gamma(c-b) \Gamma(c+n)}$$

$$= \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} B(b+n, c-b) \quad \because B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b+n-1} (1-t)^{c-b-1} dt$$

using in eqn (1)

$$F(a, b, c; 1) = \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b+n-1} (1-t)^{c-b-1} \frac{(a)_n}{n!} dt$$



$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \sum_{n=0}^{\infty} (t)^{b-1} \cdot t^n (1-t)^{c-b-1} \cdot \frac{(c)_n}{n!} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(c)_n}{n!} t^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t)^{-a} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-a-b-1} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot B(b, c-a-b)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b) \cdot \Gamma(c-a-b)}{\Gamma(b+c-a-b)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b) \cdot \Gamma(c-b-a)}{\Gamma(c-a)}$$

$$= \frac{\Gamma(c) \Gamma(c-b-a)}{\Gamma(c-b) \Gamma(c-a)}$$

Result //

(111)

$$Q_0 \quad F(-n, b, c; 1) = \frac{(c-b)_n}{(c)_n}$$

for

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n (b)_n}{(c)_n n!} (1)^n$$

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n (b)_n}{(c)_n n!} \quad \text{--- (1)}$$

take

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(c+n)}{\Gamma(c)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)}$$

$$= \frac{\Gamma(c) \cdot \Gamma(b+n) \cdot \Gamma(c-b)}{\Gamma(b) \cdot \Gamma(c-b) \cdot \Gamma(c+n)}$$

$$= \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} B(b+n, c-b) \quad \because B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{(b+n-1)} (1-t)^{(c-b-1)} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} t^n dt$$

using in (1)

(112)

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n}{n!} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} t^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(-n)_n}{n!} t^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t)^{-n} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b+n-1} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot B(b, c-b+n) \text{ by def of Beta fun}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b) \cdot \Gamma(c-b+n)}{\Gamma(b+c-b+n)} \dots B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{\Gamma(c)}{\Gamma(c-b)} \cdot \frac{\Gamma(c-b+n)}{\Gamma(c+n)}$$

$$= \frac{(c-b)_n}{(c)_n}$$

Hence

$$F(-n, b, c; 1) = \frac{(c-b)_n}{(c)_n} //$$

$$(a+1-c)_n = (a+1-c)(a+1-c+1)(a+1-c+2) \dots + (a+1-c+(n-1))$$

$$= (a+1-c)(a+2-c)(a+3-c) \dots (a+n-c), \quad (113)$$

Ans

10. $F(-n, a+n, c; 1) = \frac{(-1)^n (a+1-c)_n}{(c)_n}$

So
As

$$F(a, b, c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}$$

Then

$$F(-n, a+n, c; 1) = \frac{\Gamma(c) \Gamma(c+n-a-n)}{\Gamma(c+n) \Gamma(c-a-n)}$$

$$= \frac{\Gamma(c) \Gamma(c-a)}{\Gamma(c+n) \Gamma(c-a-n)}$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{(c-a-1) \Gamma(c-a-1)}{\Gamma(c-a-n)}$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{(c-a-1)(c-a-2) \dots (c-a-n) \Gamma(c-a-n)}{\Gamma(c-a-n)}$$

$$= \frac{1}{(c)_n} \cdot (-1)^n (a+1-c)(a+2-c) \dots (a+n-c)$$

$$= \frac{(-1)^n (a+1-c)_n}{(c)_n} = \frac{(-1)^n (a+1-c)_n}{(c)_n}$$

$$(114) \quad (a)_1 = a, \quad (a)_2 = a(a+1).$$

*

$$\underline{\text{Q.11}} \quad \frac{d}{dx} F(a, b, c; x) = \frac{ab}{c} F(a+1, b+1, c+1; x)$$

Solⁿ

we know that,

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

$$= 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{c(c+1)2!} x^2$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!} x^3 + \dots$$

Differentiating both sides

$$\frac{d}{dx} F(a, b, c; x) = \frac{ab}{c} + \frac{2ab(a+1)(b+1)}{c(c+1)2!} x$$

$$+ \frac{3ab(a+1)(a+2)(b+1)(b+2)}{c(c+1)(c+2)3!} x^2 + \dots$$

$$= \frac{ab}{c} \left[1 + \frac{(a+1)(b+1)}{(c+1)} x + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)2!} x^2 \right]$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n n!} x^n + \dots$$

$$= \frac{ab}{c} F(a+1, b+1, c+1; x)$$

Hence

$$\frac{d}{dx} F(a, b, c; x) = \frac{ab}{c} F(a+1, b+1, c+1; x)$$

Q11 Show That

$$\frac{d^n}{dx^n} \left[x^{a+n-1} F(a, b, c; x) \right] = (a)_n x^{a-1} F(a+n, b, c; x)$$

let

$$f(x) = x^{a+n-1} F(a, b, c; x)$$

$$= x^{a+n-1} \left[\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \right]$$

$$= x^{a+n-1} \left[1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{c(c+1)2!} x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!} x^3 + \dots \right]$$

$$f(x) = x^{a+n-1} + \frac{ab}{c} x^{a+n} + \frac{a(a+1)b(b+1)}{c(c+1)2!} x^{a+n+1} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!} x^{a+n+2} + \dots$$

$$\frac{d}{dx} f(x) = (a+n-1) x^{a+n-2} + \frac{ab(a+n)}{c} x^{a+n-1} + \frac{a(a+1)b(b+1)(a+n+1)}{c(c+1)2!} x^{a+n} + \dots$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)(a+n+2)}{c(c+1)(c+2)3!} x^{a+n+1} + \dots$$

(116)

$$\frac{d^2}{dx^2} [g(x)] = \left[\frac{(a+n-1)(a+n-2)x^{a+n-3}}{e} + \frac{ab(a+n)(a+n-1)x^{a+n-2}}{e} \right. \\ \left. + \frac{a(a+1)b(b+1)(a+n+1)(a+n)x^{a+n-1}}{e(c+1)2!} \right. \\ \left. + \frac{a(a+1)(a+2)b(b+1)(b+2)(a+n+2)(a+n+1)x^{a+n}}{e(c+1)(c+2)} \right]$$

Generally:

$$\frac{d^n}{dx^n} [g(x)] = \left[\frac{(a+n-1)(a+n-2)(a+n-3)\dots(a+n-(n-1))x^{a-1}}{e} + \frac{ab(a+n)(a+n-1)(a+n-2)\dots(a+1)x^a}{e} \right. \\ \left. + \frac{a(a+1)b(b+1)(a+n+1)(a+n)(a+n-1)(a+n-2)\dots(a+1)x^{a+1}}{e(c+1)2!} \right]$$

$$\frac{d^n}{dx^n} [g(x)] = \left[a(a+1)(a+2)\dots(a+(n-1))x^{a-1} + \frac{a(a+1)(a+2)\dots(a+n-1)(a+n)b x^a}{e} \right. \\ \left. + \frac{a(a+1)(a+2)\dots(a+3)\dots(a+n-1)(a+n)(a+n+1)b(b+1)x^{a+1}}{e(c+1)2!} \right]$$

2

$$\frac{d^n [g(x)]}{dx^n} = (a)_n x^{a-1} + (a)_n \frac{(a+n)x^a}{e} + (a)_n \frac{(a+n)(a+n+1)b(b+1)}{c(c+1)2!} x^{a+1}$$

$$= (a)_n x^{a-1} \left[1 + \frac{(a+n)b}{e} x + \frac{(a+n)(a+n+1)b(b+1)}{c(c+1)2!} x^2 + \dots \right]$$

$$= (a)_n x^{a-1} \left[\sum_{n=0}^{\infty} \frac{(a+n)_n (b)_n}{(c)_n n!} x^n \right]$$

$$\therefore (a)_n x^{a-1} F(a, b, c; x)$$

Hence

$$\frac{d^n}{dx^n} \left[x^{a+n-1} F(a, b, c; x) \right] = (a)_n x^{a-1} F(a+n, b, c; x)$$

~~$$\dots$$~~

(118) Results Prove that.

$$i) B(x+1, y) = \frac{x}{x+y} B(x, y).$$

$$ii) B(x, y+1) = \frac{y}{x+y} B(x, y).$$

Proof :-

$$B(x+1, y) = \frac{\Gamma(x+1) \Gamma(y)}{\Gamma(x+y+1)}.$$

$$= \frac{x \Gamma(x) \Gamma(y)}{(x+y) \Gamma(x+y)} \quad \because \Gamma(z+1) = z \Gamma(z)$$

$$= \frac{x}{x+y} \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}.$$

$$B(x+1, y) = \frac{x}{x+y} B(x, y)$$

$$ii) B(x, y+1) = \frac{y}{x+y} B(x, y).$$

L.H.S

$$B(x, y+1) = \frac{\Gamma(x) \Gamma(y+1)}{\Gamma(x+y+1)}.$$

$$= \frac{\Gamma(x) \cdot y \Gamma(y)}{(x+y) \Gamma(x+y)}$$

$$B(x, y+1) = \frac{y}{x+y} \cdot \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$B(x, y+1) = \frac{y}{x+y} B(x, y).$$

Prove that.

$$\Gamma(z) = 2 \int_0^{\infty} e^{-t^2} t^{2z-1} dt.$$

Proof

$$\text{Since } \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt. \quad \text{--- (1)}$$

$$\text{Put } t = u^2 \Rightarrow dt = 2u du.$$

when $t \rightarrow 0$ then $u \rightarrow 0$

when $t \rightarrow \infty$ then $u \rightarrow \infty$

(1) \Rightarrow

$$\Gamma(z) = \int_0^{\infty} e^{-u^2} (u^2)^{z-1} (2u du).$$

$$= 2 \int_0^{\infty} e^{-u^2} u^{2z-2} \cdot u du.$$

$$= 2 \int_0^{\infty} e^{-u^2} u^{2z-1} du.$$

$$= 2 \int_0^{\infty} e^{-t} t^{z-1} dt \quad \left(\because u \text{ is dummy variable} \right)$$

(120)

Theorem:
$$\int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta d\theta$$
$$= \frac{\Gamma(x) \Gamma(y)}{2\Gamma(x+y)}$$

Proof: Consider

$$I = \iint_R e^{-t^2} \cdot e^{-u^2} \cdot t^{2x-1} \cdot u^{2y-1} dt du$$

where R is the region of the tu -plane consisting of first quadrant only.

$$R: 0 \leq t < \infty, 0 \leq u < \infty$$

$$I = \int_0^{\infty} \int_0^{\infty} e^{-t^2} \cdot e^{-u^2} \cdot t^{2x-1} \cdot u^{2y-1} dt du$$

$$I = \int_0^{\infty} \left(\int_0^{\infty} e^{-t^2} \cdot t^{2x-1} dt \right) e^{-u^2} u^{2y-1} du$$

$$I = \int_0^{\infty} \frac{1}{2} \Gamma(x) e^{-u^2} u^{2y-1} du \quad \because \int_0^{\infty} e^{-t^2} t^{2x-1} dt = \frac{1}{2} \Gamma(x)$$
$$= \frac{1}{2} \Gamma(x) \cdot \frac{1}{2} \Gamma(y) \quad \text{--- (1)}$$

Now we find I in polar coordinates

put

$$x = r \cos \theta, \quad y = r \sin \theta.$$

$$dA = r dr d\theta.$$

where

$$R: 0 \leq r < \infty \text{ and } 0 \leq \theta < \pi/2.$$

$$I = \int_0^{\infty} \int_0^{\pi/2} e^{-r^2 \cos^2 \theta} \cdot e^{-r^2 \sin^2 \theta} \cdot (r \cos \theta)^{2x-1} (r \sin \theta)^{2y-1} r dr d\theta$$

$$I = \int_0^{\infty} \int_0^{\pi/2} e^{-r^2 (\cos^2 \theta + \sin^2 \theta)} \cdot r^{2x+y-1} \cos^{2x-1} \theta \sin^{2y-1} \theta dr d\theta$$

$$I = \int_0^{\infty} \int_0^{\pi/2} e^{-r^2} r^{2(x+y)-1} \cos^{2x-1} \theta \sin^{2y-1} \theta dr d\theta$$

$$I = \int_0^{\pi/2} \frac{1}{2} \Gamma(x+y) \cos^{2x-1} \theta \sin^{2y-1} \theta d\theta \cdot \int_0^{\infty} e^{-r^2} r^{2(x+y)-1} dr$$

$$I = \frac{1}{2} \Gamma(x+y) \int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta d\theta \quad \text{--- (1)}$$

Comparing eqn (1) & (2) we get

$$\frac{1}{2} \Gamma(x+y) \int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta d\theta = \frac{1}{2} \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

$$\int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta d\theta = \frac{\Gamma(x) \Gamma(y)}{2 \Gamma(x+y)}$$

(122)

Theorem:

Imp Q: $\Gamma(1/2) = \sqrt{\pi}$

Proof: Since $\int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}$

put $x = y = 1/2$

Then $\int_0^{\pi/2} \cos^0 \theta \cdot \sin^0 \theta d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

2) $\int_0^{\pi/2} (1) (1) d\theta = \frac{\Gamma(1/2)\Gamma(1/2)}{2\Gamma(1)} \because x^0 = 1$

2) $\int_0^{\pi/2} d\theta = \frac{(\Gamma(1/2))^2}{2}$

2) $\theta \Big|_0^{\pi/2} = \frac{(\Gamma(1/2))^2}{2}$

$\Rightarrow \frac{\pi}{2} = \frac{(\Gamma(1/2))^2}{2}$

$\pi = (\Gamma(1/2))^2$

$\Rightarrow \sqrt{\pi} = \Gamma(1/2)$

interchanging

$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$

T. from stai notes
Assignment =

(123)

Prove That

$$\Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Proof: Since we know That

$$\int_0^{\pi/2} \cos^{2n-1} \theta \sin^{2m-1} \theta d\theta = \frac{\Gamma(x) \Gamma(y)}{2 \Gamma(x+y)}$$

put $y = 1-x$

$$\int_0^{\pi/2} \cos^{2n-1} \theta \sin^{2(1-x)-1} \theta d\theta = \frac{\Gamma(x) \Gamma(1-x)}{2 \Gamma(x+(1-x))}$$

$$\int_0^{\pi/2} \cos^{2n-1} \theta \sin^{1-2x} \theta d\theta = \frac{\Gamma(x) \Gamma(1-x)}{2 \Gamma(1)}$$

$$\int_0^{\pi/2} \cos^{2n-1} \theta \sin^{1-2x} \theta d\theta = \frac{\Gamma(x) \Gamma(1-x)}{2}$$

$$\Rightarrow 2 \int_0^{\pi/2} \cos^{2n-1} \theta \sin^{1-2x} \theta d\theta = \Gamma(x) \Gamma(1-x)$$

Available at <http://www.MathCity.org/msc/notes>

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✓
 Prove that $\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

Proof $\Gamma(x) = 2 \int_0^{\infty} e^{-t^2} t^{2x-1} dt$.

Put $x = 1/2$

$$\Gamma(1/2) = 2 \int_0^{\infty} e^{-t^2} t^{2(1/2)-1} dt.$$

$$\Gamma(1/2) = 2 \int_0^{\infty} e^{-t^2} dt.$$

$$\sqrt{\pi} = 2 \int_0^{\infty} e^{-t^2} dt.$$

$$\frac{\sqrt{\pi}}{2} = \int_0^{\infty} e^{-t^2} dt$$

Hence $\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

Prove that

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

Proof \Rightarrow Since

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Put $t = \cos^2 \theta \Rightarrow dt = -2 \cos \theta \sin \theta d\theta$

when $t \rightarrow 0$ then $\theta \rightarrow \pi/2$

when $t \rightarrow 1$ then $\theta \rightarrow 0$

(126)

$$B(x, y) = \int_{\pi/2}^0 (\cos^2 \theta)^{x-1} (1 - \cos^2 \theta)^{y-1} (-2 \cos \theta \sin \theta) d\theta.$$

$$= 2 \int_0^{\pi/2} \cos^{2x-2} \theta \cdot \sin^{2y-2} \theta \cdot \cos \theta \sin \theta d\theta$$

$$= 2 \int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta d\theta.$$

$$= \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}$$

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

Express $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$ in terms of Gamma funⁿ or Beta funcⁿ and simplify if possible.

Proof:- $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta = \int_0^{\pi/2} \left(\frac{\sin \theta}{\cos \theta} \right)^{1/2} d\theta.$

$$= \int_0^{\pi/2} \cos^{-1/2} \theta \sin^{1/2} \theta d\theta. \quad \text{--- (1)}$$

Since $\int_0^{\pi/2} \cos^{2\alpha-1} \theta \sin^{2\beta-1} \theta d\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}$ --- (2)

put $2\alpha-1 = -1/2 \Rightarrow 2\alpha = 1/2 \Rightarrow \alpha = 1/4$
and

$2\beta-1 = 1/2 \Rightarrow 2\beta = 3/2 \Rightarrow \beta = 3/4$

putting value of α & β in eqn (2)

$$\int_0^{\pi/2} \cos^{-1/2} \theta \sin^{1/2} \theta d\theta = \frac{\Gamma(1/4) \cdot \Gamma(3/4)}{2\Gamma(1/4+3/4)}$$

$$= \frac{\Gamma(1/4) \cdot \Gamma(1-1/4)}{2\Gamma(1)}$$

$$= \frac{\Gamma(1/4) \cdot \Gamma(3/4)}{2}$$

$$= \frac{\pi}{2 \operatorname{Sin} \pi (1/4)} \quad \because \frac{\Gamma(x) \cdot \Gamma(1-x)}{\operatorname{Sin} \pi x}$$

$$= \frac{\pi}{2 \operatorname{Sin} \pi/4}$$

Hence $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta = \frac{\pi}{\sqrt{2}} // \quad = \frac{\pi}{2 \cdot \frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}}$

(128)

Bessel Function:-

Notation: $J_n(x)$

It is defined as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n} m! \Gamma(m+n+1)}$$

is called Bessel function of index 'n' and we obtain the Bessel function by solving Bessel differential equation of order 'n',

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

where 'n' is non-negative integer.

* Show that

$$J_{-n}(x) = (-1)^n J_n(x).$$

Proof:- we know

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n} m! \Gamma(m+n+1)}$$

put $n = -n$.

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-n}}{2^{2m-n} m! \Gamma(m-n+1)}$$



$\Gamma(m-n+1)$ diverges if

$$m-n+1 \leq 0$$

$$\Rightarrow m \leq n-1$$

$$\left[\begin{aligned} \sqrt{(z+1)} &= (z) \sqrt{z} \\ \sqrt{(z+2)} &= (z+1) z \sqrt{z} \end{aligned} \right]$$

$$2! = 2 \quad (129)$$

for this value

$$J_{-n}(x) = 0$$

so $m \geq n$.

~~Since~~

$$\Rightarrow J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-n}}{2^{2m-n} (m)! \Gamma(m-n+1)}$$

let

$$k = m - n \Rightarrow m = k + n$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n} (x)^{2(k+n)-n}}{2^{2(k+n)-n} (k+n)! \Gamma(k+n-n+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+n} (x)^{2k+2n-n}}{2^{2k+2n-n} (k+n)! \Gamma(k+1)}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^n (x)^{2k+n}}{2^{2k+n} (k+n)(k+(n-1)) \dots (k+1) k! \Gamma(k+1)}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+n}}{2^{2k+n} k! \Gamma(k+n+1)}$$

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

required result.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sqrt{x} = (x-1)^{1/2} = (x-1)(2-1)$$

(130)

Show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

Ans: As $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+n}}{2^{2k+n} \cdot k! \Gamma(k+n+1)}$

Put $n = 1/2$:

$$J_{1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+1/2}}{2^{2k+1/2} \cdot k! \Gamma(k+1/2+1)}$$

$$J_{1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+1/2}}{2^{2k+1/2} \cdot k! \Gamma(k+3/2)}$$

$$= \frac{x^{1/2}}{2^{1/2} \Gamma(3/2)} + \frac{(-1)^1 (x)^{5/2}}{2^{5/2} \Gamma(5/2)} + \frac{(-1)^2 (x)^{9/2}}{2^{9/2} \cdot 2! \Gamma(7/2)} + \dots$$

$$J_{1/2}(x) = \frac{(x/2)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2! \Gamma(7/2)} - \frac{(x/2)^{11/2}}{3! \Gamma(9/2)} + \dots$$

$$J_{1/2}(x) = (x/2)^{1/2} \left[\frac{1}{2 \Gamma(3/2)} - \frac{(x/2)^2}{\frac{3}{2} \cdot \frac{1}{2} \Gamma(5/2)} + \frac{(x/2)^4}{2! \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \Gamma(7/2)} - \dots \right]$$

(B1)

$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\frac{1}{2}\Gamma(1/2)} \left[1 - \frac{x^2}{2^2 \cdot 3/2} + \frac{x^4}{2^4 \cdot 2! \cdot \frac{5}{2} \cdot \frac{3}{2}} - \frac{x^6}{2^6 \cdot 3! \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}} + \dots \right]$$

$$= \frac{x^{1/2}}{\frac{1}{2} \cdot 2^{1/2} \sqrt{\pi}} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right]$$

$$= \frac{2^{1/2} \cdot x^{1/2}}{x \sqrt{\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x.$$

Hence $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

4
3!
2!
1/2

(132)

Show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Sol:- As by the definition Bessel function

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n} m! \Gamma(m+n+1)}$$

Put

$$n = -1/2$$

$$J_{(-1/2)}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-1/2}}{2^{2m-1/2} m! \Gamma(m-1/2+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-1/2}}{2^{2m-1/2} m! \Gamma(m+1/2)}$$

$$= \frac{x^{-1/2}}{2^{-1/2} \Gamma(1/2)} + \frac{(-1)^1 (x)^{3/2}}{2^{3/2} (1!) \Gamma(3/2)} + \frac{(-1)^2 (x)^{7/2}}{2^{7/2} (2!) \Gamma(5/2)} + \frac{(-1)^3 (x)^{11/2}}{2^{11/2} (3!) \Gamma(7/2)} + \dots$$

$$= \frac{x^{-1/2}}{2^{-1/2} \Gamma(1/2)} \left[\frac{1}{\Gamma(1/2)} - \frac{x^2}{2^2 \cdot \frac{1}{2} \Gamma(1/2)} + \frac{x^4}{2^4 \cdot \frac{1}{2} \cdot \frac{3}{2} \Gamma(1/2)} - \dots \right]$$

$$= \frac{1}{2} \cdot \frac{1}{x^{1/2} \Gamma(1/2)} \left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cdot \frac{1}{\sqrt{\pi}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x.$$

Hence $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$

V-imp

* Differential Recurrence Relation of Bessel function:-

$$\text{As } J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m! \Gamma(m+n+1)} \quad \text{--- (1)}$$

Multiplying (x^n) on both sides

$$x^n J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m! \Gamma(m+n+1)}$$

$$x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2n}}{2^{2m+n} m! \Gamma(m+n+1)}$$

On Differentiating

$$x^n J_n'(x) + nx^{n-1} J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot 2(2m+n) x^{2m+2n-1}}{2^{2m+n} m! \Gamma(m+n)}$$

Show $\frac{d}{dx} (x^n J_n(x)) = x^n J_{n-1}(x) \quad / \quad \frac{d}{dx} x^2 J_5(x) = x^2 J_4(x)$

(134)

$$x^n J_n'(x) + n x^{n-1} J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+2n-1}}{2^{2m+n-1} \cdot m! \Gamma(m+n)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n-1} \cdot (x)^n}{2^{2m+n-1} \cdot m! \Gamma(m+n)}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n-1}}{2^{2m+n-1} \cdot m! \Gamma(m+n-1+1)}$$

$$= x^n J_{n-1}(x)$$

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

$$J_n'(x) = J_{n-1}(x) - n x^{-1} J_n(x) \quad \text{--- (2)}$$

Now Multiplying x^{-n} on both sides of (1)

$$x^{-n} J_n(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n}}{2^{2m+n} \cdot m! \Gamma(m+n+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m}}{2^{2m+n} \cdot m! \Gamma(m+n+1)}$$

on Differentiating

$$x^{-n} J_n'(x) - n x^{-n-1} J_n(x)$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m \cdot 2^m \cdot x^{2m-1}}{2^{2m+n} \cdot m! \cdot \sqrt{(m+n+1)}}$$

put

$$m-1 = k \Rightarrow m = k+1$$

$$x^{-n} J_n'(x) - n x^{-n-1} J_n(x)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot 2^{k+1} \cdot x^{2k+1}}{2^{2k+n+2} \cdot (k+1)! \cdot \sqrt{(k+n+2)}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (-1) \cdot (k+1) \cdot x^{2k+1}}{2^{2k+n+1} \cdot (k+1)k! \cdot \sqrt{(k+n+2)}}$$

$$= - \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+n+1} \cdot x^{-n}}{2^{2k+n+1} \cdot k! \cdot \sqrt{(k+n+1+1)}} \quad \checkmark$$

$$= - x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+n+1}}{2^{2k+n+1} \cdot k! \cdot \sqrt{(k+n+1+1)}} \quad \left. \begin{array}{l} \text{Use} \\ n = n+1 \end{array} \right\}$$

$$+ x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = - x^{-n} J_{n+1}(x)$$

Available at

(i) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$ when multiply by x^n & differentiate.

(ii) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$ when multiply by x^n & diff.

(iii) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

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(iv) $J_n(x) = \frac{x}{2n} [J_{n-1}(x) + J_{n+1}(x)]$

$$\Rightarrow x J_n'(x) - n x J_n'(x) = -J_{n+1}(x)$$

$$J_n'(x) = n x J_n'(x) - J_{n+1}(x) \quad \text{--- (3)}$$

adding (2) & (3) we have.

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

This relation is called differential recurrence relation.

now subtracting (3) from (2)

$$0 = J_{n-1}(x) - 2n x J_n'(x) + J_{n+1}(x)$$

$$2n x J_n'(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$2n J_n(x) = x [J_{n-1}(x) + J_{n+1}(x)]$$

$$J_n(x) = \frac{x}{2n} [J_{n-1}(x) + J_{n+1}(x)]$$

This relation is called pure recurrence or non-differential relation.

~~.....~~

Spherical Bessel Function:

General representation of spherical Bessel function is

$$J_{n+1/2}(z) = A(z) \sin z + B(z) \cos z$$

where $A(z)$ and $B(z)$ are polynomial in z .

* Properties of Bessel function:-

$$1) J_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left[\frac{\sin z - z \cos z}{z} \right]$$

Or we know

$$J_n(z) = \frac{z}{2n} \left[J_{n-1}(z) + J_{n+1}(z) \right]$$

$$\text{Put } n = 1/2.$$

$$J_{1/2}(z) = \frac{z}{2(1/2)} \left[J_{1/2-1}(z) + J_{1/2+1}(z) \right]$$

$$J_{1/2}(z) = z \left[J_{-1/2}(z) + J_{3/2}(z) \right]$$

$$= z J_{-1/2}(z) + z J_{3/2}(z)$$

$$z J_{3/2}(z) = J_{1/2}(z) - z J_{-1/2}(z)$$

$$J_{3/2}(z) = z^{-1} \left[J_{1/2}(z) - z J_{-1/2}(z) \right]$$

(138)

$$J_{3/2}(z) = z^{-1} [J_{1/2}(z) - z J_{-1/2}(z)]$$

$$= z^{-1} \left[\sqrt{\frac{2}{\pi z}} \sin z - z \sqrt{\frac{2}{\pi z}} \cos z \right]$$

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left[\frac{\sin z - z \cos z}{z} \right]$$

Assignment

$$J_{-3/2}, J_{5/2}, J_{-5/2}$$

(9) Show that $\int x^n J_{n-1}(x) dx$

$$\int x^n J_{n-1}(x) dx = x^n J_n(x) + C$$

$$= \int \frac{d}{dx} (x^n J_n(x)) dx$$

$$= x^n J_n(x) + C$$

$$\left. \begin{aligned} \frac{d}{dx} (x^{-n} J_n(x)) \\ (189) \end{aligned} \right\}$$

$$b) \int x^{-n} J_{n+1}(x) dx.$$

$$= - \int \frac{d}{dx} (x^{-n} J_n(x)) dx.$$

$$= - x^{-n} J_n(x) + C.$$

e.g. $\int x^{-3} J_4(x) dx.$

$$= - x^{-3} J_3(x) + C.$$

* Find $J_{-3/2}(x).$

we know that

$$J_n(x) = \frac{x}{2n} [J_{n-1}(x) + J_{n+1}(x)]$$

put $n = -1/2.$

$$J_{-1/2}(x) = \frac{x}{2(-1/2)} [J_{-1/2-1}(x) + J_{-1/2+1}(x)]$$

$$J_{-1/2}(x) = -x [J_{-3/2}(x) + J_{1/2}(x)] \quad ?$$

$$J_{-1/2}(x) = -x J_{-3/2}(x) - x J_{1/2}(x).$$

$$x J_{-3/2}(x) = J_{-1/2}(x) - x J_{1/2}(x).$$

$$J_{-3/2}(x) = \frac{1}{x} [J_{-1/2}(x) - x J_{1/2}(x)]$$

(140)

*

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$$J_{-3/2}(x) = \frac{1}{x} \left[J_{-1/2}(x) - x J_{1/2}(x) \right]$$

$$J_{-3/2}(x) = \frac{1}{x} \left[\sqrt{\frac{2}{\pi x}} \cos x - x \left[\sqrt{\frac{2}{\pi x}} \left(\frac{\sin x - x \cos x}{x} \right) \right] \right]$$

$$= \frac{1}{x} \sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} (\sin x + x \cos x) \sqrt{\frac{2}{\pi x}}$$

$$= \left(\frac{1}{x} + x \right) \sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(x \cos x + \frac{1}{x} \cos x - \sin x \right)$$

Find $J_{5/2}(x)$?

we know that

$$J_n(x) = \frac{x}{2n} \left[J_{n-1}(x) + J_{n+1}(x) \right]$$

Put $n = 3/2$.

$$J_{3/2}(x) = \frac{x}{2(3/2)} \left[J_{1/2}(x) + J_{5/2}(x) \right]$$

$$J_{3/2}(x) = \frac{x}{3} J_{1/2}(x) + \frac{x}{3} J_{5/2}(x)$$

$$J_{5/2}(x) = \frac{3}{x} \left[J_{3/2}(x) - \frac{x}{3} J_{1/2}(x) \right]$$

$$= \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$$

$$J_{5/2}(x) = \frac{3}{x} \cdot \left[\sqrt{\frac{2}{\pi x}} \left(\frac{\sin x - x \cos x}{x} \right) \right] - \sqrt{\frac{2}{\pi x}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{3(\sin x - x \cos x)}{x^2} - \sin x \right]$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3 \sin x - 3x \cos x - x^2 \sin x}{x^2} \right]$$

*) Friend

$J_{-5/2}(x)$?

we know that

$$J_n(x) = \frac{x}{2n} \left[J_{n-1}(x) + J_{n+1}(x) \right]$$

$$\text{put } n = -3/2$$

$$J_{-3/2}(x) = \frac{x}{2(-3/2)} \left[J_{-5/2}(x) + J_{-1/2}(x) \right]$$

$$J_{-3/2}(x) = -\frac{x}{3} \left[J_{-5/2}(x) + J_{-1/2}(x) \right]$$

$$J_{-5/2}(x) = \frac{3}{x} \left[-\frac{x}{3} J_{-1/2} - J_{-3/2}(x) \right]$$

$$J_{-5/2}(x) = \left[-J_{-1/2} - \frac{3}{x} J_{-3/2}(x) \right]$$

$$= -\sqrt{\frac{2}{\pi x}} \cos x - \frac{3}{x} \left[\sqrt{\frac{2}{\pi x}} \left(x \cos x + \frac{1}{\pi} \cos x - \sin x \right) \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[-\cos x - \frac{3}{x} \left(x \cos x + \frac{1}{\pi} \cos x - \sin x \right) \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[-\cos x - 3 \cos x - \frac{3 \cos x}{x} + \frac{3 \sin x}{x} \right] \text{ (Ans)}$$

* i) $\int z^n J_{n-1}(z) dz = z^n J_n(z)$ } power is greater than order of Bessel functions
 ii) $\int z^n J_{n+1}(z) dz = -z^n J_n(z)$

(142)

Q11 Evaluate $\int x^3 J_3(x) dx$.

$$= \int x^5 \cdot x^{-2} J_3(x) dx$$

$$= x^5 \cdot \int x^{-2} J_3(x) dx - \int (5x^4) (-x^{-2} J_2(x)) dx$$

$$= x^5 \cdot (-x^{-2} J_2(x)) + 5 \int x^2 J_2(x) dx$$

$$= -x^3 J_2(x) + 5 \int x^1 \cdot x^2 J_2(x) dx$$

$$= -x^3 J_2(x) + 5 \left[x^3 \int x^{-1} J_2(x) dx - \int 3x^2 (-x^{-1} J_1(x)) dx \right]$$

$$= -x^3 J_2(x) + 5 \left[-x^3 \cdot x^{-1} J_1(x) + 3 \int x^2 \cdot x^{-1} J_1(x) dx \right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15 \int x J_1(x) dx$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15 \left[\int x \cdot x^{-1} J_1(x) dx \right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15 \left[x \int x^{-1} J_1(x) dx - \int (-x^{-1} J_0(x)) dx \right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15x (-x^{-1} J_0(x)) + 15 \int x^{-1} J_0(x) dx$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) - 15x J_0(x) + 15 \int J_0(x) dx$$

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Q11. a) $\int x^4 J_1(x) dx.$

i) $\frac{d}{dx} x^n J_n(x)$
 $= x^n J_{n-1}(x).$

b) $\int x^4 J_4(x) dx.$

ii) $\int x^n J_{n-1}(x)$
 $= x^n J_n(x)$

c) $\int x^3 J_2(x) dx.$

iii) $\int x^n J_{n+1}(x)$
 $= x^n J_n(x).$

sol:

a) $\int x^4 J_1(x) dx.$

$$= \int x^2 x^2 J_1(x) dx$$

$$= x^2 \int x^2 J_1(x) dx - \int 2x \cdot x^2 J_2(x) dx$$

$$= x^2 \cdot x^2 J_2(x) - 2 \int x^3 J_2(x) dx$$

~~$$= x^4 J_2(x) - 2 \int x \cdot x^2 J_2(x) dx$$~~

~~$$= x^4 J_2(x) - 2 \int x \cdot x^2 J_2(x) dx = \int x^4 J_2(x) dx$$~~

~~$$= x^4 J_2(x) - 2 \int x^3 J_2(x) dx = \int x^3 x^{-1} J_2(x) dx$$~~

~~$$= x^4 J_2(x) - 2 \int x^3 J_2(x) dx + 2 \int x^3 (-x^{-1} J_1(x)) + \int 3x^2 x^{-1} J_1(x) dx$$~~

~~$$= x^4 J_2(x) - 2x^3 J_2(x) + 2x^2 J_1(x) + 6 \int x J_1(x) dx$$~~

~~$$= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) + 6 \int x \cdot x^0 J_1(x) dx$$~~

~~$$= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) + 6 \int x (-x^0 J_0(x))$$~~

~~$$+ \int (-x^0 J_0(x)) dx$$~~

~~$$= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) - 6 J_0(x) - 6 \int J_0(x) dx$$~~

Ans

(144) Q. no

$$b) \int x^4 J_4(x) dx.$$

$$= \int x^7 \cdot x^{-3} J_4(x) dx.$$

$$= x^7 (-x^{-3} J_3(x)) + 7 \int x^6 \cdot x^{-3} J_3(x) dx.$$

$$= -x^4 J_3(x) + 7 \int x^3 J_3(x) dx.$$

$$= -x^4 J_3(x) + 7 \int x^5 x^{-2} J_3(x) dx.$$

$$= -x^4 J_3(x) + 7 \left[x^5 (-x^{-2} J_2(x)) + 5 \int x^4 x^{-2} J_2(x) dx \right]$$

$$= -x^4 J_3(x) - 7x^3 J_2(x) + 35 \int x^2 J_2(x) dx$$

$$= -x^4 J_3(x) - 7x^3 J_2(x) + 35 \left[\int x^2 x^{-1} J_2(x) dx \right]$$

$$= -x^4 J_3(x) - 7x^3 J_2(x) + 35 \left[-x^3 \cdot x^{-1} J_1(x) + 3 \int x^2 x^{-1} J_1(x) dx \right]$$

$$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105 \int x J_1(x) dx$$

$$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105 \left[x \cdot x^{-1} J_0(x) dx \right]$$

$$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105 \left[-x \cdot x^{-1} J_0(x) + \int J_0(x) dx \right]$$

$$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) - 105x J_0(x) + \int J_0(x) dx$$

//

$$i) \int x^n J_{n-1}(x) dx = x^n J_n(x) \quad (145)$$

$$ii) \int x^{-n} J_{n+1} dx = -x^{-n} J_n$$

c)

olve

$$\int x^3 J_2(x) dx.$$

$$= x^3 J_3(x).$$

$$a) \int x^4 J_1(x) dx.$$

$$= \int x^2 \cdot x^2 J_1(x) dx.$$

$$= x^2 \int x^2 J_1(x) - \int 2x \cdot \int x^2 J_1(x) dx.$$

$$= x^2 \int x^2 J_1(x) - 2 \int x \cdot x^2 J_2(x) dx.$$

$$= x^4 J_2(x) - 2 \int x^3 J_2(x)$$

$$= x^4 J_2(x) - 2 x^3 J_3(x)$$

1146)

These notes are available on at

<http://www.mathcity.org/msc/notes>

MathCity.org

$$\Gamma(3) = 2 \Gamma(2) = 2 \cdot 1 \cdot \Gamma(1) = 2 \cdot 1$$

Q11 ^{imp} Show that (147)

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots$$

Soln

we know

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+n}}{2^{2k+n} \cdot k! \cdot \Gamma(k+n+1)}$$

Put $n=1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{2^{2k+1} \cdot k! \cdot \Gamma(k+2)}$$

$$= \frac{x}{2 \cdot 0! \Gamma(2)} + \frac{(-1)^1 x^3}{2^3 \cdot 1! \Gamma(3)} + \frac{(-1)^2 x^5}{2^5 \cdot 2! \Gamma(4)}$$

$$+ \frac{(-1)^3 x^7}{2^7 \cdot 3! \Gamma(5)} + \dots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \cdot 2} + \frac{x^5}{2^6 \cdot 3 \cdot 2} - \frac{x^7}{2^7 \cdot 3! \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \dots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots$$

which is required result.

Q.E.D.

$$F_m \rightarrow \underline{y^{m-1}}$$

(148)

* Hypergeometric function:-

we define the general hypergeometric function with 'm' upper parameters and 'n' lower parameters written in

by

$${}_m F_n (a_1, a_2, a_3, \dots, a_m; b_1, b_2, b_3, \dots, b_n; x)$$

$${}_m F_n = \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r (a_3)_r \dots (a_m)_r}{(b_1)_r (b_2)_r (b_3)_r \dots (b_n)_r} \frac{x^r}{r!}$$

The notation ${}_m F_n \left[\begin{matrix} a_1, a_2, a_3, \dots, a_m \\ b_1, b_2, b_3, \dots, b_n \end{matrix} ; x \right]$

is also often used.

now we discussed two cases

if

$$m = n = 1$$

Then the general hypergeometric fn will be called the "confluent" hypergeometric function. This confluent hypergeometric function.

${}_1 F_1 (a; b; x)$ is often denoted by

$$M(a; b; x).$$

if $m = 2, n = 1$. Then fn is hypergeometric ${}_2 F_1 (a, b; c; x)$

Some special cases of ${}_m F_n$.

i) when there are no upper and lower parameters,

$${}_0 F_0 (; ; +x) = e^{+x}$$

As

$${}_0 F_0 (; ; x) = \sum_{k=0}^{\infty} \frac{(x)^k}{k!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= e^x \quad \text{--- (1)}$$

Also

$${}_0 F_0 (; ; -x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$$

$$= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots$$

$$= e^{-x} \quad \text{--- (2)}$$

Combining (1) & (2)

$${}_0 F_0 (; ; \pm x) = e^{\pm x}$$

~~—————~~

$$\begin{aligned} (1/2)_2 &= 1/2(1/2+1) \\ &= 3/6 \cdot (3/2+1) \\ &= 3/2(3/2+1) \end{aligned}$$

$$\begin{aligned} (1/2)_n &= 4(1/2)_1(1/2)_2 \dots (1/2)_n \\ (1/2)_1 &= 1/2+0 = 1/2 \end{aligned}$$

(150) //

$$2) \quad {}_1F_0(a; ; x) = (1-x)^{-a}$$

$$\begin{aligned} \text{Proof} \quad {}_1F_0(a; ; x) &= \sum_{k=0}^{\infty} \frac{(a)_k}{k!} x^k \\ &= (1-x)^{-a} \end{aligned}$$

$$3) \quad x \cdot {}_2F_1(1/2, 1/2; 3/2; x)$$

$$= x + \frac{x^2}{2 \cdot 3} + \frac{3x^3}{3 \cdot 5} + \dots$$

4) //

$$\begin{aligned} x \cdot {}_2F_1(1, 1; 2; -x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\ &= \log(1+x) \end{aligned}$$

3) L.H.S.

sol:

$$x \cdot {}_2F_1(1/2, 1/2; 3/2; x)$$

$$= x \cdot \left[\sum_{k=0}^{\infty} \frac{(1/2)_k (1/2)_k}{(3/2)_k k!} x^k \right]$$

$$= x \left[\frac{(1/2)_0 (1/2)_0}{(3/2)_0 0!} x^0 + \frac{(1/2)_1 (1/2)_1}{(3/2)_1 1!} x + \frac{(1/2)_2 (1/2)_2}{(3/2)_2 2!} x^2 + \dots \right]$$

~~$$= x \left[1 + \frac{1}{6} x + \frac{3/4 \cdot 3/4}{2 \cdot 2} x^2 + \dots \right]$$~~

$$= x \left[1 + \frac{1}{6} x + \frac{3}{3 \cdot 5} x^2 + \dots \right]$$

$$= x + \frac{x^2}{2 \cdot 3} + \frac{3}{3 \cdot 5} x^3 + \dots \quad \checkmark$$

$$(1)_2 = (1+0)(1+1) = 2$$

$$(2)_2 = (2)(2+1) = 6$$

(151)

Q11

$${}_2F_1(1, 1; 2; -x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

L.H.S

$${}_2F_1(1, 1; 2; -x)$$

$$= x \sum_{n=0}^{\infty} \frac{(1)_2 (1)_2}{(2)_2 n!} (-x)^n$$

$$= x \left[1 + \frac{(1)_1 (1)_1}{(2)_1 1!} (-x)^1 + \frac{(1)_2 (1)_2}{(2)_2 2!} (-x)^2 + \dots \right]$$

$$= x \left[1 + \frac{1}{2} (-x)^1 + \frac{2 \cdot 2}{6 \cdot 2!} (-x)^2 + \dots \right]$$

$$= x \left[1 - \frac{1}{2} x + \frac{4}{12} x^2 + \dots \right]$$

$$= x \left[1 - \frac{1}{2!} x + \frac{x^2}{3} + \dots \right]$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$= \log(1+x) \quad \text{Ans}$$

(152)

$$\begin{aligned} (1/2)_2 &= \frac{1}{2} (1/2 + 1) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4} \\ (3/2)_2 &= \frac{3}{2} (3/2 + 1) = \frac{3}{2} \cdot \frac{5}{2} = \frac{15}{4} \end{aligned}$$

$$5) \quad x^2 F_1\left(\frac{1}{2}; \frac{3}{2}; x^2\right)$$

$$= x^2 \sum_{k=0}^{\infty} \frac{(1/2)_k (1)_k}{(3/2)_k k!} (x^2)^k$$

$$= x^2 \left[1 + \frac{(1/2)_1 (1)_1}{(3/2)_1 1!} (x^2)^1 + \frac{(1/2)_2 (1)_2}{(3/2)_2 2!} (x^2)^2 + \dots \right]$$

(1/2)(1/2)

$$= x^2 \left[1 + \frac{1/2}{3/2} x^2 + \frac{(3/4)(2)}{15/4 \cdot 2!} x^4 + \dots \right]$$

$$= x^2 \left[1 + \frac{1}{3} x^2 + \frac{1}{5} x^4 + \dots \right]$$

$$= x^2 + \frac{1}{3} x^4 + \frac{1}{5} x^6 + \dots$$

$$5) x {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; x^2\right)$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

$$= \tanh^{-1} x.$$

$$6) x {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right) = \tan^{-1} x.$$

$$7) x {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = \sin^{-1} x$$

$$8) x {}_0F_1\left(\frac{3}{2}; +\frac{x^2}{4}\right) = \sin x.$$

$$9) x {}_0F_1\left(\frac{3}{2}; -\frac{x^2}{4}\right) = \cos x.$$

Objective
Problem

$$J_0'(x) = -J_1(x).$$

Sol: we know that

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x).$$

Put $n=0$

$$J_0'(x) = -J_1(x) + \frac{0}{x} J_0(x)$$

$$\boxed{J_0'(x) = -J_1(x)}$$

(154)

Show that

$$\frac{d}{dx} (\alpha J_1(x)) = \alpha J_0(x)$$

Pr: we know that

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} \cdot k! \Gamma(k+n+1)}$$

Put $n=1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{2^{2k+1} \cdot k! \Gamma(k+2)}$$

Multiplying by α .

$$\alpha J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+2}}{2^{2k+1} \cdot k! \Gamma(k+2)}$$

Differentiating on both sides

$$\frac{d}{dx} \alpha J_1(x) = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+2}}{2^{2k+1} \cdot k! \Gamma(k+2)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2(k+1) x^{2k+1}}{2^{2k+1} \cdot k! \Gamma(k+2)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2(k+1) x^{2k+1}}{2^{k+1} \cdot k! \cdot (2k+1) \Gamma(k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{2^{2k} \cdot k! \cdot (k+1)}$$

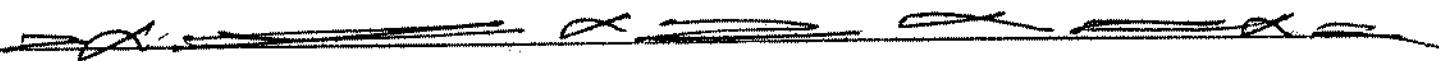
$$= x \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{2^{2k} \cdot k! \cdot (k+1)}$$

$$= x \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+0}}{2^{2k+0} \cdot k! \cdot (k+1+0)}$$

$$= x J_0(x)$$

Hence

$$\frac{d}{dx} (x J_1(x)) = x J_0(x)$$



Q: Find $J_3(x)$ in terms of $J_2(x)$ and $J_1(x)$.

Sol: we know that

$$2n J_n(x) = x (J_{n-1}(x) + J_{n+1}(x)) \quad \text{--- (A)}$$

- put $n=2$

$$2(2) J_2(x) = x (J_1(x) + J_3(x))$$

$$x J_3(x) = 4 J_2(x) - x J_1(x) \quad \text{--- (C)}$$

(156)

Again putting $n=1$ in (4)

$$2(1) J_1(x) = x (J_0(x) + J_2(x))$$

$$x J_2(x) = 2 J_1(x) - x J_0(x)$$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

using this value in equation (1)

$$x J_3(x) = 4 \left(\frac{2}{x} J_1(x) - J_0(x) \right) - x J_2(x)$$

$$J_3(x) = \frac{4}{x} \left(\frac{2}{x} J_1(x) - J_0(x) \right) - J_2(x)$$

Q11 Show that

$$J_n''(x) = \frac{1}{2} \left(J_{n-1}(x) - 2 J_n(x) + J_{n+1}(x) \right)$$

we know that

$$J_n'(x) = n x^{-1} J_n(x) - J_{n+1}(x)$$

Differentiating we have.

Q11

Show $J_n''(x) = \frac{1}{2} [J_{n-1}(x) - 2J_n(x) + J_{n+1}(x)]$

As we know that

$$2 J_n'(x) = J_{n-1}(x) + J_{n+1}(x).$$

Differentiating we have

$$2 J_n''(x) = J_{n-1}'(x) + J_{n+1}'(x).$$

$$2 J_n''(x) = \frac{1}{2} [J_{n-2}(x) + J_n(x)] + \frac{1}{2} [J_n(x) - J_{n+2}(x)]$$

$$= \frac{1}{2} [J_{n-2} + 2J_n(x) - J_{n+2}]$$

$$J_n''(x) = \frac{1}{4} [J_{n-2} + 2J_n(x) - J_{n+2}] \quad \downarrow \quad (A)$$

(1.58)

$$y_1(x) = \frac{c_1}{x} - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{3!} x^3 - \frac{(n-2)(n-4)(n+1)(n+3)(n+5)}{4!} x^4 + \dots$$
$$y_2(x) = \frac{c_2}{x - n}$$

y

t

c

c

c

* Legendre Polynomial: (159)

The equation

$$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + n(n+1)y = 0$$

or

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is called the Legendre equation.

By the method of series solution (Frobenius solution).

we get two independent solutions of above equation of the form

$$y_1(x) = c_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \frac{(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \dots \right]$$

&

$$y_2(x) = c_1 \left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \dots \right]$$

If n is even then $y_1(x)$ (1st eqn) get terminate and we get $y_1(x)$ as a polynomial of degree n . while 2nd series is an infinite series.

now if n is odd then 2nd eqn get terminate and we get $y_2(x)$ a polynomial of degree n and 1st series an infinite series. These polynomials are known as Legendre polynomials.

The coefficients of such a Legendre polynomial are found from the recurrence relation

$$c_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} c_j$$

(160)

Initial Condition for Legendre polynomial

$$P_n(1) = 1 \quad \forall n \in \mathbb{N}.$$

of $n = 2$

$$P_2(x) = C_0 \left[1 - \frac{2(2+1)}{2!} x^2 \right]$$

$$= C_0 \left[1 - \frac{2(3)}{2} x^2 \right]$$

$$P_2(x) = C_0 [1 - 3x^2]$$

As $P_2(1) = 1$ Therefore

$$1 = C_0 [1 - 3(1)^2]$$

$$1 = C_0 (1 - 3)$$

$$1 = -2C_0 \Rightarrow C_0 = -\frac{1}{2}.$$

So required Legendre polynomial is

$$P_2(x) = -\frac{1}{2} [1 - 3x^2]$$

$$P_2(x) = -\frac{1}{2} (1 - 3x^2)$$

Ans. for $n=3$

$$P_3(x) = c_1 \left[x - \frac{(3-1)(3+2)}{3!} x^3 \right]$$

$$P_3(x) = c_1 \left[x - \frac{2(5)}{3!} x^3 \right]$$

$$P_3(x) = c_1 \left[x - \frac{5}{3} x^3 \right]$$

$$P_3(1) = c_1 \left[1 - \frac{5}{3} \right]$$

as $P_3(1) = 1$

Therefore

$$1 = c_1 \left(-\frac{2}{3} \right) \Rightarrow c_1 = -\frac{3}{2}$$

Hence

$$P_3(x) = -\frac{3}{2} \left[x - \frac{5}{3} x^3 \right]$$

Find the Legendre polynomial for $n=4$

$$P_4(x) = c_0 \left[1 - \frac{4(4+1)}{2!} x^2 + \frac{4(4-2)(4+1)(4+3)}{4!} x^4 \right]$$

$$P_4(x) = c_0 \left[1 - 10x^2 + \frac{35}{3} x^4 \right]$$

$$P_4(1) = c_0 \left[1 - 10 + \frac{35}{3} \right]$$

$$P_4(1) = c_0 \left[\frac{8}{3} \right]$$

$$1 = c_0 \left(\frac{8}{3} \right) \Rightarrow c_0 = \frac{3}{8}$$

$$P_4(x) = \frac{3}{8} \left[1 - 10x^2 + \frac{35}{3} x^4 \right]$$

(16.2)

$f(x)$ $P_4(x)$

Ans

$$P_4(x) = C_0 + C_2 x^2 + C_4 x^4$$

As we know

Jhāī

$$C_n = \frac{(2n)!}{2^n (n!)^2}$$

for $n=4$

$$C_4 = \frac{8!}{2^4 (4!)^2}$$

$$C_4 = \frac{1 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{\cancel{16} \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{4} \cdot 3 \cdot 2 \cdot 1}$$

$$C_4 = \frac{35}{8}$$

Now

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} \cdot C_j$$

for $j=2$

$$C_4 = \frac{(2-4)(2+4+1)}{(2+1)(2+2)} C_2$$

$$\frac{35}{8} = \frac{(-2)(7)}{(3)(4)} C_2$$

$$\frac{35}{8} = -\frac{7}{6} C_2 \Rightarrow C_2 = -\frac{6}{7} \times \frac{35}{8}$$

$$= -\frac{35}{4}$$

(165)

put $x=0$

$$C_2 = \frac{(0-4)(0+4+1)}{(0+1)(0+2)} C_0$$

$$-\frac{15}{4} = -\frac{10}{2} C_0$$

$$C_0 = -\frac{15}{4} \times \frac{1}{10} = +\frac{3}{8}$$

$$C_0 = \frac{3}{8}$$

So $P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$

$$P_5(x)$$

Ans

$$P_5(x) = C_1x + C_3x^3 + C_5x^5$$

Now

$$C_5 = \frac{(10!)}{2^5 \cdot (5!)^2}$$

$$C_5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{32 \cdot 8 \cdot 4 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$C_5 = \frac{63}{8}$$

(164)

Now we know that

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} C_j$$

for $j=3$,

$$C_5 = \frac{(3-5)(3+5+1)}{(3+1)(3+2)} C_3$$

$$C_5 = \frac{(-2)(9)}{(4)(5)} C_3$$

$$\frac{63}{8} = \frac{(-2)(9)}{(4)(5)} C_3$$

$$C_3 = -\frac{63}{8} \cdot \frac{1}{(2)(9)}$$

$$C_3 = -\frac{35}{4}$$

for $j=1$

$$C_3 = \frac{(1-5)(1+5+1)}{(1+1)(1+2)} C_1$$

$$-\frac{35}{4} = \frac{(-4)(7)}{(2)(3)} C_1$$

$$C_1 = +\frac{35}{4} \cdot \frac{5}{(2)(7)} = \frac{15}{8}$$

(165)

So

$$f_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

 $f_8(x)$

So the Legendre eqn is

$$f_8(x) = C_0 + C_2 x^2 + C_4 x^4 + C_6 x^6 + C_8 x^8$$

$$C_8 = \frac{(16!)}{2^8 (8!) (8!)}$$

$$C_8 = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$C_8 = \frac{7!5}{32}$$

Now we know that

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} C_j$$

Put $j=6$

$$C_8 = \frac{(-2)(15)}{(7)(8)} C_6$$

(166)

$$C_6 = \frac{\overset{48}{\cancel{715}} \cdot \overset{1}{\cancel{7}} \cdot \overset{1}{\cancel{18}}}{\underset{4}{\cancel{33}} \cdot \underset{3}{\cancel{2}} \cdot \underset{3}{\cancel{15}}} = \frac{143 \times 7}{4 \times 2 \times 3}$$

$$C_6 = \frac{\cancel{2275}}{\cancel{24}}$$

$$C_6 = \frac{1001}{24}$$

put $v = 4$

$$C_6 = \frac{(4-8)(4+8+1)}{(4+1)(4+2)} C_4$$

$$\frac{1001}{24} = \frac{(4)(13)}{(5)(6)} C_4$$

$$C_4 = \frac{1001}{24} \cdot \frac{(5)(6)}{(4)(13)}$$

$$C_4 = \frac{5005}{208}$$

(167)

put $j = 2$

$$C_4 = \frac{(2-8)(2+8+1)}{(2+1)(2+2)} C_2$$

$$\frac{5005}{208} = \frac{(-6)(11)}{(3)(4)} C_2$$

$$C_2 = -\frac{5005}{208} \frac{(3)(4)}{(6)(11)}$$

$$C_2 = -\frac{5005}{1144}$$

Now
put $j = 0$

$$C_2 = \frac{(-8)(9)}{(1)(2)} C_0$$

$$C_0 = -\frac{5005}{1144} \frac{(-1)(2)}{(8)(9)}$$

$$C_0 = \frac{+5005}{41184}$$

Now

$$P_8(x) = \frac{5005}{41184} - \frac{5005}{1144} x^2 + \frac{5005}{208} x^4 - \frac{1001}{24} x^6 + \frac{715}{32} x^8 //$$

(168) Generating function :-

Suppose we have a function

$$F(x, t) = \sum_{m=0}^{\infty} f_m(x) t^m$$

we called $F(x, t)$ a generating function for a set of functions $\{f_m(x)\}$ provided the above series converges. For e.g.

Consider

$$\frac{1}{1-xt} = (1-xt)^{-1}$$

$$= 1 + xt + x^2t^2 + x^3t^3 + \dots$$

$$= \sum_{m=0}^{\infty} x^m t^m$$

\Rightarrow $F(x, t)$ generate a function a set of function $\{x^m\}$

Solⁿ.

Ans. Show the function is generating fn. (109)

$$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} \text{ gives } \sum_{n=0}^{\infty} P_n(x)t^n.$$

i.e. This function generates Legendre polynomial $P_n(x)$.

$$\text{Sol: } \frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-1/2}$$

$$= (1-(2xt-t^2))^{-1/2}$$

$$= (1-d)^{-1/2} \text{ where } d = 2xt-t^2.$$

$$= 1 + (-1/2)(-d) + \frac{(-1/2)(-1/2-1)(-d)^2}{2!} + \dots$$

$$= 1 + d/2 + \frac{3/4}{2!} d^2 + \dots$$

$$= 1 + d/2 + \frac{(3/4)d^2}{2} + \dots$$

$$= 1 + \frac{(2xt-t^2)}{2} + \frac{3}{8} (2xt-t^2)^2 + \dots$$

$$= 1 + xt - \frac{1}{2}t^2 + \frac{3}{8} (4x^2t^2 + t^4 - 4xt^3) + \dots$$

$$= x^0t^0 + xt + \left(-\frac{1}{2} + \frac{3}{2}x^2\right)t^2 + \dots$$

$$= P_0(x)t^0 + P_1(x)t + \frac{1}{2}(3x^2-1)t^2 + \dots$$

$$(f(x) = x) \text{ and } (1) = 1$$

(170)

$$= p_0(x)t^0 + p_1(x)t + p_2(x)t^2 + \dots$$

$$F(x,t) = \sum_{n=0}^{\infty} p_n(x)t^n$$

Hence, given function is
generating function

• Rodrigues's Formula.

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

This formula is used to find
the Legendre polynomial of different
degree.

* Show that $p_n(x)$ is $\begin{cases} \text{even if } n \text{ is even} \\ \text{odd if } n \text{ is odd} \end{cases}$

Proof of n is even.

Then

$$P_n(x) = C_0 + C_2 x^2 + C_4 x^4 + \dots + C_n x^n$$

Put $x = -x$

$$P_n(-x) = C_0 + C_2(-x)^2 + C_4(-x)^4 + \dots + C_n(-x)^n$$

$$= C_0 + C_2x^2 + C_4x^4 + \dots + C_nx^n$$

$$P_n(-x) = P_n(x)$$

$\Rightarrow P_n(x)$ is even.
If n is odd Then.

$$P_n(x) = C_1 + C_3x^3 + C_5x^5 + \dots + C_nx^n$$

Put $x = -x$.

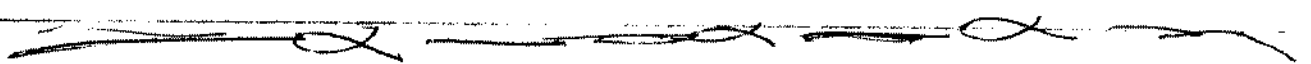
$$P_n(-x) = C_1 + C_3(-x)^3 + C_5(-x)^5 + \dots + C_n(-x)^n$$

$$= C_1 - C_3x^3 - C_5x^5 + \dots - C_nx^n$$

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(-x) = -P_n(x) \quad \because (-1)^n = -1$$

as n is odd



(172)

~~9/11~~

Show that

$$F(x, t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Then $P_n(x)$ is even if n is even
and

$P_n(x)$ is odd if n is odd.

Now as

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Change 'x' by $-x$, gives

$$\frac{1}{\sqrt{1+2xt+t^2}} = \sum_{n=0}^{\infty} P_n(-x) t^n$$

Now change 't' by $-t$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(-x) (-t)^n$$

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} P_n(-x) (-1)^n t^n$$

$$P_0(x) t^0 + P_1(x) t^1 + P_2(x) t^2 + \dots$$

$$= P_0(-x) (-1)^0 t^0 + P_1(-x) (-1)^1 t^1 + P_2(-x) (-1)^2 t^2 + \dots$$

on comparing the coefficients of " x^n " we have

$$P_n(x) = P_n(-x) (-1)^n$$

now if n is even

$$\text{then } P_n(x) = P_n(-x) \quad \because (-1)^n = 1 \text{ as } n \text{ is even.}$$

$\Rightarrow P_n(x)$ is even.

now if n is odd

$$P_n(x) = P_n(-x) (-1)^n$$

$$\Rightarrow P_n(x) = -P_n(-x) \quad \because (-1)^n = -1 \text{ as } n \text{ is odd.}$$

$\Rightarrow P_n(x)$ is odd.

* Q11. Show that

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

Q11:

$$\text{Since } P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Put $n = n+1$

$$P_{n+1} = \frac{1}{2^{n+1} (n+1)!} \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^{n+1}$$

$$P_{n+1}' = \frac{1}{2^{n+1} (n+1)!} \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^{n+1}$$

(174)

$$= \frac{1}{2^{n+1} \cdot (n+1)!} \frac{d^n}{dx^n} \left[\frac{d}{dx} (x^2-1)^{n+1} \right]$$

$$= \frac{1}{2^{n+1} \cdot (n+1)!} \frac{d^n}{dx^n} \left[\frac{d}{dx} (n+1)(x^2-1)^n 2x \right]$$

$$= \frac{2(n+1)}{2^{n+1} \cdot (n+1)!} \frac{d^n}{dx^n} \left[n(x^2-1)^{n-1} (x)(2x) + (x^2-1)^n (1) \right]$$

$$= \frac{2(n+1)}{2^{n+1} \cdot (n+1)!} \frac{d^n}{dx^n} \left[2x^2 n (x^2-1)^{n-1} + (x^2-1)^n \right]$$

$$= \frac{2}{2^{n+1} \cdot n!} \frac{d^n}{dx^n} \left[2x^2 n (x^2-1)^{n-1} + (x^2-1)^n \right]$$

$$= \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} \left[2x^2 n (x^2-1)^{n-1} \right]$$

$$+ \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P'_{n+1}(x) = \frac{2n}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2 (x^2-1)^{n-1}) + P_n(x).$$

↓

Now By Reducing formula

$$P_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^{n-1}$$

By taking derivative

$$P'_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dx^n} (x^2-1)^{n-1}$$

Now Subtracting (2) from (1) (2)
we get.

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{2n}{2^n n!} \frac{d^n}{dx^n} [x^2(x^2-1)^{n-1}]$$

$$+ P_n(x)$$

$$- \frac{1}{2^n (n-1)!} \frac{d^n}{dx^n} (x^2-1)^{n-1}$$

$$= \frac{2n}{2^n n!} \frac{d^n}{dx^n} [x^2(x^2-1)^{n-1}] - \frac{1}{2^n (n-1)!} \frac{d^n}{dx^n} (x^2-1)^{n-1}$$

$$+ P_n(x)$$

$$= \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dx^n} \left[\frac{2n x^2 (x^2-1)^{n-1}}{2n} - \frac{(x^2-1)^{n-1}}{1} \right] + P_n(x)$$

$$= \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dx^n} [(x^2-1)^{n-1} \cdot (x^2-1)] + P_n(x)$$

(176)

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{1}{2^{n-1} \cdot (n-1)!} \frac{d^n}{dx^n} (x^2-1)^n + P'_n(x)$$

$$= \frac{2n}{2^n (n!)} \frac{d^n}{dx^n} (x^2-1)^n + P'_n(x)$$

$$= 2n P_n(x) + P'_n(x)$$

$$= (2n+1) P_n(x) \checkmark$$

i.e

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x)$$

$$(2n+1) P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

for $n=1, 2, 3$

So: we know that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Differentiating w.r.t to t

(177)

$$(-1/2) (1-2xt+t^2)^{-3/2} (-2x+2t)$$

$$= \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$\frac{(x-t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$\frac{x-t}{(1-2xt+t^2)^{1/2} (1-2xt+t^2)} = \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$(x-t) \cdot \frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$(x-t) \sum_{n=0}^{\infty} P_n(x) t^n = (1-2xt+t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$\begin{aligned} x \sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} &= \sum_{n=0}^{\infty} P_n(x) \cdot n \cdot t^{n-1} \\ &\quad - 2x \sum_{n=0}^{\infty} P_n(x) \cdot n t^n \\ &\quad + \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n+1} \end{aligned}$$

P.P.O

(178)

$$\Rightarrow \sum_{n=0}^{\infty} P_n(\alpha) t^n + 2n\alpha \sum_{n=0}^{\infty} P_n(\alpha) t^n$$

$$= \sum_{n=0}^{\infty} P_n(\alpha) \cdot n \cdot t^{n-1} + \sum_{n=0}^{\infty} P_n(\alpha) \cdot n t^{n+1} + \sum_{n=0}^{\infty} P_n(\alpha) t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} (1+2n)\alpha P_n(\alpha) t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (1+2n)\alpha P_n(\alpha) t^n = \sum_{n=0}^{\infty} P_n(\alpha) \cdot n t^{n-1} + \sum_{n=0}^{\infty} P_n(\alpha) \cdot n t^{n+1} + \sum_{n=0}^{\infty} P_n(\alpha) t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} (2n+1)\alpha P_n(\alpha) t^n = \sum_{n=0}^{\infty} P_n(\alpha) \cdot n t^{n-1} + \sum_{n=0}^{\infty} (n+1)P_n(\alpha) t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} (2n+1)\alpha P_n(\alpha) t^n = \sum_{n=1}^{\infty} P_n(\alpha) n t^{n-1} + \sum_{n=0}^{\infty} (n+1)P_n(\alpha) t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} (2n+1)x P_n(x) t^n$$

$$= \sum_{n=0}^{\infty} P_{n+1}(x)(n+1)t^n$$

By taking
 $n = n+1$
 $n = n-1$

$$+ \sum_{n=1}^{\infty} n P_{n-1}(x) t^n$$

Now on comparing coefficients of t^n
 for $n = 1$ and onward we have

$$\Rightarrow (2n+1)x P_n(x)$$

$$= P_{n+1}(x)(n+1) + n P_{n-1}(x)$$

Hence

$$(2n+1)x P_n(x) = P_{n+1}(x)(n+1) + n P_{n-1}(x).$$

————— \cdot ————— \cdot ————— \cdot —————

11 (180)

Theorem:

* The hypergeometric function is solution of D.E

$$z(1-z) \frac{d^2y}{dz^2} + \{ \gamma - (\alpha + \beta + 1)z \} \frac{dy}{dz} - \alpha\beta y = 0 \quad (1)$$

which is called Hypergeometric differential equation α, β, γ are constants.

Proof:

we take a trial solution

$$y = \sum_{k=0}^{\infty} C_k z^{k+\gamma} \quad (2)$$

$$\Rightarrow \frac{dy}{dz} = \sum_{k=0}^{\infty} C_k (k+\gamma) z^{k+\gamma-1}$$

$$\Rightarrow \frac{d^2y}{dz^2} = \sum_{k=0}^{\infty} C_k (k+\gamma)(k+\gamma-1) z^{k+\gamma-2}$$

Using in eqn (1)

$$z(1-z) \left(\sum_{k=0}^{\infty} C_k (k+\gamma)(k+\gamma-1) z^{k+\gamma-2} \right)$$

$$+ \{ \gamma - (\alpha + \beta + 1)z \} \sum_{k=0}^{\infty} C_k (k+\gamma) z^{k+\gamma-1}$$

$$- \alpha\beta \sum_{k=0}^{\infty} C_k z^{k+\gamma} = 0$$

→

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+\gamma) (k+\gamma-1) z^{k+\gamma-1}$$

$$- \sum_{k=0}^{\infty} C_k (k+\gamma) (k+\gamma-1) z^{k+\gamma}$$

$$+ \gamma \sum_{k=0}^{\infty} C_k (k+\gamma) z^{k+\gamma-1}$$

$$- (\alpha + \beta + 1) \gamma \sum_{k=0}^{\infty} C_k (k+\gamma) z^{k+\gamma-1}$$

$$- \alpha \beta \sum_{k=0}^{\infty} C_k z^{k+\gamma} = 0$$

On comparing Least power of z i.e. $z^{\gamma-1}$ by putting $k=0$. we get

$$C_0 \gamma (\gamma-1) + \gamma C_0 (\gamma) = 0$$

$$C_0 \gamma (\gamma-1 + \gamma) = 0$$

$$\Rightarrow C_0 \neq 0 \text{ so } \gamma (\gamma-1 + \gamma) = 0$$

$$\Rightarrow \gamma = 0 \text{ or } \gamma = 1 - \gamma$$

Now comparing coefficient of $z^{k+\gamma-1}$.

$$C_k (k+\gamma) (k+\gamma-1) - C_{k-1} (k+\gamma-1) (k+\gamma-2)$$

$$+ \gamma C_k (k+\gamma) - (\alpha + \beta + 1) C_{k-1} (k+\gamma-1)$$

$$- \alpha \beta C_{k-1} = 0 \rightarrow \textcircled{3}$$

(182)

r

Case-I

when $\gamma = 0$

so

$$C_n [n(n-1) + \gamma n] \\ = C_{n-1} [(n-1)(n-2) \\ + (\alpha + \beta + 1)(n-1) \\ + \alpha\beta]$$

$$C_n = \frac{C_{n-1} [(n-1)(n-2) + (\alpha + \beta + 1)(n-1) + \alpha\beta]}{n(n-1) + r n}$$

$$C_n = \frac{C_{n-1}^{(n-1)} [(n + \alpha - 1 + \beta) + \alpha\beta]}{n [n-1 + r]}$$

$$C_n = \frac{C_{n-1} [(n-1)(n + \alpha - 1) + \beta(n-1) + \alpha\beta]}{n [n-1 + r]}$$

$$C_n = \frac{C_{n-1} [(n-1)(n + \alpha - 1) + \beta(n + \alpha - 1)]}{n (n-1 + r)}$$

$$C_n = C_{n-1} \left[\frac{(n + \alpha - 1)(n + \beta - 1)}{n (n + r - 1)} \right]$$

NOTE 1st term of Geometric series is always 1. i.e. $C_0 = 1$

(183)

for $n = n-1$

$$C_{n-1} = C_{n-2} \left[\frac{(n-2+\alpha)(n-2+\beta)}{(n-1)(n-2+\gamma)} \right]$$

$$C_2 = C_1 \left[\frac{(1+\alpha)(1+\beta)}{(2)(1+\gamma)} \right]$$

and

$$C_1 = C_0 \left[\frac{\alpha \cdot \beta}{1 \cdot \gamma} \right]$$

$$C_1 = C_0 \left(\frac{\alpha \beta}{\gamma} \right)$$

$$C_1 = \frac{\alpha \beta}{\gamma}$$

$$C_2 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{2! \gamma(\gamma+1)}$$

$$C_3 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3! \gamma(\gamma+1)(\gamma+2)}$$

Hence, Choose $C_0 = 1$
using back substitution we get

$$C_n = \frac{(n-1+\alpha)(n-2+\alpha)\dots(n-(n-1)+\alpha)\alpha \cdot (n-1+\beta)(n-2+\beta)\dots(n-(n-1)+\beta)\beta}{n \cdot (n-1)(n-2)\dots 3 \cdot 2 \cdot 1 \cdot (n-1+\gamma)(n-2+\gamma)\dots(n-(n-1)+\gamma)\gamma}$$

$$C_n = \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} \checkmark$$

(184)

using in solution we have

$$y = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k! (\gamma)_k} z^{k+1} \quad \because \gamma \neq 0$$

Case II

when $\gamma = 1 - r$
using in equation (3) we see

$$C_k [(k+1-r)(k-r) + r(k+1-r)] \\ = C_{k-1} [(k-r)(k-1-r) + (\alpha+\beta+1)(k-r) + \alpha\beta] = 0$$

$$C_k = C_{k-1} [(k-r)(k-1-r) + (\alpha+1)(k-r) + \beta(k-r) + \alpha\beta]$$

$$(k+1-r)(k-r) + r(k+1-r)$$

$$C_k = C_{k-1} [(k-r)[k-1-r+k-r+\alpha+1] + \beta(k-r+\alpha)]$$

$$(k+1-r)(k-r+r)$$

$$C_k = C_{k-1} \left[\frac{(k-r+\alpha)(k-r+\beta)}{(k+1-r)(k)} \right] \quad (18)$$

put

$$k = k-1$$

$$C_{k-1} = C_{k-2} \left[\frac{(k-1-r+\alpha)(k-1-r+\beta)}{(k-1+1-r)(k-1)} \right]$$

$$C_2 = C_1 \left[\frac{(2-r+\alpha)(2-r+\beta)}{2(2-r+1)} \right]$$

$$C_1 = C_0 \left[\frac{(1-r+\alpha)(1-r+\beta)}{1 \cdot (1-r+1) = (2-r)} \right]$$

Now choose $C_0 = 1$

$$C_k = \frac{(k-r+\alpha)(k-1-r+\alpha)(k-2-r+\alpha) \cdots (1-r+\alpha)}{(k-r+\beta)(k-1-r+\beta)(k-2-r+\beta) \cdots (1-r+\beta)} \cdot \frac{k(k-1)(k-2) \cdots 3 \cdot 2 \cdot 1}{(k+1-r)(k-r)(k-1-r) \cdots (3-r)(2-r)}$$

$$C_k = \frac{(1-r+\alpha)_k \cdot (1-r+\beta)_k}{k! \cdot (2-r)_k}$$

using in solution

$$y = \sum_{k=0}^{\infty} \frac{(1-r+\alpha)_k \cdot (1-r+\beta)_k}{(2-r)_k} x^{k+r}$$

$\because r = r$

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Bessel Functions:-

Bessel's Differential Equation.

The differential equation

Available at
www.mathcity.org

$$x^2 y'' + x y' + (x^2 - n^2) y = 0, \quad n \geq 0$$

The solution of Bessel's D.E is called Bessel function.

* Find the solution of Bessel's D.E

$$x^2 y'' + x y' + (x^2 - n^2) y = 0, \quad n \geq 0$$

Proof :-

$$\text{Let } y = \sum_{k=0}^{\infty} C_k x^{k+\gamma} \text{ be a}$$

Series solution of Bessel's D.E. we may write

$$y = \sum_{k=-\infty}^{\infty} C_k x^{k+\gamma} \quad \text{where } C_k = 0 \text{ for } k < 0$$

$$\Rightarrow y' = \sum_{k=0}^{\infty} C_k (k+\gamma) x^{k+\gamma-1}$$

$$y'' = \sum_{k=0}^{\infty} C_k (k+\gamma)(k+\gamma-1) x^{k+\gamma-2}$$

using in eqn (1) we get

$$\begin{aligned} & \chi^2 \sum_{k=0}^{\infty} C_k (k+\delta)(k+\delta-1) \chi^{k+\delta-2} \\ & + \chi \sum_{k=0}^{\infty} C_k (k+\delta) \chi^{k+\delta-1} \\ & + (\chi^2 - \eta^2) \sum_{k=0}^{\infty} C_k \chi^{k+\delta} = 0 \end{aligned}$$

$$\begin{aligned} & \chi^2 \sum_{k=0}^{\infty} C_k (k+\delta)(k+\delta-1) \chi^{k+\delta-2} \\ & + \chi \sum_{k=0}^{\infty} C_k (k+\delta) \chi^{k+\delta-1} \\ & + \chi^2 \sum_{k=0}^{\infty} C_k \chi^{k+\delta} - \eta^2 \sum_{k=0}^{\infty} C_k \chi^{k+\delta} = 0 \end{aligned}$$

$$\begin{aligned} & \sum_{k=0}^{\infty} C_k (k+\delta)(k+\delta-1) \chi^{k+\delta} + \sum_{k=0}^{\infty} C_k (k+\delta) \chi^{k+\delta} \\ & + \sum_{k=0}^{\infty} C_k \chi^{k+\delta+2} - \eta^2 \sum_{k=0}^{\infty} C_k \chi^{k+\delta} = 0 \end{aligned}$$

$k \geq 0$

Comparing coefficient of $\chi^{k+\delta}$ we get

$$C_k (k+\delta)(k+\delta-1) + C_k (k+\delta)$$

$$+ C_{k-2} - \eta^2 C_k = 0$$

$$C_k (k+\delta)(k+\delta-1+1) + C_{k-2} - \eta^2 C_k = 0$$

$$C_k (k+\delta)(k+\delta) + C_{k-2} - \eta^2 C_k = 0$$

(188)

$$\Rightarrow C_k [(k+\gamma)^2 - n^2] = -C_{k-2} \quad (2)$$

This is known as recurrence relation or identical relation.

Put $k=0$ in eqn (2)

$$C_0 [(0+\gamma)^2 - n^2] = -C_{-2}$$

$$C_0 [\gamma^2 - n^2] = 0 \quad \because C_{-2} = 0 \quad \forall k < 0$$

Either $C_0 = 0$ or $\gamma^2 - n^2 = 0$

$k=2$ in eqn (2)

$$C_2 [(2+\gamma)^2 - n^2] = -C_0$$

$$k=4$$

$$C_4 [(4+\gamma)^2 - n^2] = -C_2$$

So all C_k for even value of k depends upon C_0 . To get a non-trivial solution $C_0 \neq 0$

$$\text{So } \gamma^2 - n^2 = 0$$

$$\gamma^2 = n^2$$

$$\gamma = \pm n$$

from eq (2) we get

$$C_k = \frac{(-1)^{k-2} C_{k-2}}{(k+\gamma)^2 - \eta^2}$$

(3)

$$C_k = \frac{(-1)^{k-2} C_{k-2}}{(k+\gamma+\eta)(k+\gamma-\eta)}$$

$$C_k = \frac{(-1)^{k-2} C_{k-2}}{(k+\gamma+\eta)(k+\gamma-\eta)}$$

Now for

$$k=1$$

$$C_1 = \frac{(-1)^{C_{-1}}}{(1+\gamma+\eta)(1+\gamma-\eta)}$$

$$C_1 = 0 \quad \because C_{-1} = 0$$

$$C_3 = \frac{(-1)^{C_1}}{(3+\gamma+\eta)(3+\gamma-\eta)}$$

$$C_3 = 0 \quad \because C_1 = 0$$

so

$$C_1 = C_3 = C_5 = C_7 = \dots = 0$$

* Case I

when $\lambda = \eta$ put in eqn (3)

$$C_k = \frac{(-1)^{k-2} C_{k-2}}{k(2\eta+k)}$$

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$$C_{k-2} = \frac{(-1) C_{k-4}}{(k-2)(2n+k-2)}$$

$$C_4 = \frac{(-1) C_2}{(4)(2n+4)}$$

$$C_2 = \frac{(-1) C_0}{(2)(2n+2)}$$

let

$$C_0 = \frac{1}{2^n (n+1)}$$

$$C_2 = \frac{(-1)}{(2)(2n+2)} \cdot \frac{1}{2^n (n+1)}$$

$$= \frac{(-1)}{2^{n+1} \cdot 2! (n+1) (n+1)} = \frac{(-1)}{2^{n+2} \cdot 1! (n+2)}$$

$$C_2 = \frac{(-1)}{2^{n+1} \cdot 2! \sqrt{(n+2)}} \quad \because \sqrt{(2+1)} = 2 \cdot \sqrt{(2)}$$

Now

$$C_4 = \frac{(-1) C_2}{(4)(2n+4)}$$

$$C_4 = \frac{(-1)^2}{4(2n+4) \cdot 2^{n+1} \cdot 2! \cdot (n+2)}$$

$$C_4 = \frac{(-1)^2}{2^3 (2n+2^2) \cdot 2^{n+1} \cdot 2! \cdot (n+2)}$$

$$C_4 = \frac{(-1)^2}{2^{n+4} (n+2) \cdot 2! \cdot (n+2)}$$

$$C_4 = \frac{(-1)^2}{2^{n+4} (2!) \cdot (n+2+1)}$$

$$C_{k=2m} = \frac{(-1)^{2m/2=m}}{2^{n+2m} (m!) \cdot (n+m+1)}$$

using these values of C_k and δ in the solution

$$y = \sum_{m=0}^{\infty} \frac{(-1)^m 2^{2m+n}}{2^{2m+n} \cdot m! \cdot (m+n+1)} = I_n(x)$$

(192)

Case-II

when $x = -n$ put in
eqn (3)

$$C_k = \frac{(-1) C_{k-2}}{(k-n)^2 - n^2}$$

$$C_k = \frac{(-1) C_{k-2}}{k^2 + n^2 - 2kn + n^2} = \frac{(-1) C_{k-2}}{k^2 - 2kn}$$

$$C_k = \frac{(-1) C_{k-2}}{k(k-2n)}$$

$$C_{k-2} = \frac{(-1) C_{k-4}}{(k-2)(k-2-2n)}$$

$$C_4 = \frac{(-1) C_2}{(4)(4-2n)}$$

$$C_2 = \frac{(-1) C_0}{(2)(2-2n)}$$

let $C_0 = \frac{1}{2^n (-n+1)}$ $\because n = -n$

so

$$C_2 = \frac{(-1)}{2^2(1-n)} \cdot \frac{1}{2^n \sqrt{(-n+1)}}$$

$$C_2 = \frac{(-1)}{2^{2-n} (1-n) \sqrt{(-n+1)}}$$

$$C_2 = \frac{(-1)}{2^{2-n} \sqrt{(-n)}} \quad \therefore \frac{(1-n) \sqrt{(-n+1)}}{= \sqrt{(-n)}}$$

$$C_4 = \frac{(-1)^2 C_2}{4(4-4n)}$$

$$C_4 = \frac{(-1)^2}{4(4-4n) \cdot 2^{2-n} \sqrt{(-n+0)}}$$

$$C_4 = \frac{(-1)^2}{2^3(2-n) \cdot 2^{2-n} \sqrt{(-n+0)}}$$

$$C_4 = \frac{(-1)^2}{2^{4-n} \cdot 2! (2-n) \sqrt{(-n+0)}}$$

$$C_4 = \frac{(-1)^2}{2^{4-n} \cdot 2! (-n+2) \sqrt{(-n+0)}}$$

(194)

* Find the solution of the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{①}$$

Ans

we assume a solution of eqn ① as

$$y = \sum_{k=0}^{\infty} C_k x^{k+\beta} \quad \text{and } C_k = 0 \text{ for } k < 0$$

\Rightarrow

$$y' = \sum_{k=0}^{\infty} C_k (k+\beta) x^{k+\beta-1}$$

$$y'' = \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1) x^{k+\beta-2}$$

Using values in equation ① we get

$$(1-x^2) \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1) x^{k+\beta-2}$$

$$- 2x \sum_{k=0}^{\infty} C_k (k+\beta) x^{k+\beta-1}$$

$$+ n(n+1) \sum_{k=0}^{\infty} C_k x^{k+\beta} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1) x^{k+\beta-2} - \sum_{k=0}^{\infty} 2C_k (k+\beta) x^{k+\beta-1} + n(n+1) \sum_{k=0}^{\infty} C_k x^{k+\beta} = 0$$

Comparing coefficient of $x^{k+\beta}$, we get

(195)

$$C_{k+2} (k+\beta+2) (k+1+\beta) - C_k (k+\beta) (k+\beta-1) - 2 C_k (k+\beta) + n(n+1) C_k = 0$$

$$C_{k+2} (k+\beta+2) (k+\beta+1) + C_k \left[n(n+1) - (k+\beta) (k+\beta-1) - 2(k+\beta) \right] = 0$$

$$C_{k+2} (k+\beta+2) (k+\beta+1) + C_k \left[n(n+1) - (k+\beta) (k+\beta-1+2) \right] = 0$$

$$C_{k+2} (k+\beta+2) (k+\beta+1) + C_k \left[n(n+1) - (k+\beta) (k+\beta+1) \right] = 0$$

This is known as identical eqn. or recurrence equation.

↓
③

Put $k = -2$.

$$C_0 \beta(\beta-1) + C_{-2} \left[n(n+1) - (-2+\beta) (-2+\beta+1) \right] = 0$$

$$C_0 \beta(\beta-1) = 0 \quad \therefore C_{-2} = 0$$

Assume $C_0 \neq 0$

Then either $\beta = 0$ or $\beta - 1 = 0 \Rightarrow \beta = 1$

Case-I put $\beta = 0$ in eqn (3) we get

$$C_{k+2} (k+2) (k+1) + C_k \left[n(n+1) - k(k+1) \right] = 0$$

$$C_{k+2} = \frac{k(k+1) - n(n+1)}{(k+1)(k+2)} C_k$$

(196)

Put $k=0$

$$C_2 = \frac{-n(n+1)}{1 \cdot 2} C_0$$

$$C_2 = \frac{-n(n+1)}{2!} C_0$$

Put $k=1$

$$C_3 = \frac{1(2) - n(n+1)}{2 \cdot 3} \cdot C_1$$

$$C_3 = \frac{2 - n^2 - n}{2 \cdot 3} C_1$$

$$C_3 = \frac{2 - 2n + n - n^2}{2 \cdot 3} C_1$$

$$C_3 = \frac{2(1-n) + n(1-n)}{2 \cdot 3} C_1$$

$$C_3 = \frac{(2+n)(1-n)}{3!} C_1$$

Put $k=2$

$$C_4 = \frac{2 \cdot 3 - n(n+1)}{3 \cdot 4} C_2$$

$$= \frac{6 - n^2 - n}{3 \cdot 4} C_2$$

$$= \frac{-n^2 - n + 6}{3 \cdot 4} C_2$$

$$= - \frac{(n^2 + n - 6)}{3 \cdot 4} C_2$$

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$$C_4 = - \frac{(3+n)(n-2)}{4 \cdot 3} C_2.$$

$$C_4 = - \frac{(3+n)(n-2)}{4 \cdot 3} \cdot \left(\frac{-n(n+1)}{2!} C_0 \right)$$

$$C_4 = \frac{n(n-2)(n+1)(n+3)}{24 = 4!}$$

Put $k=3$

$$C_5 = \frac{3 \cdot 4 - n(n+1)}{4 \cdot 5} C_3 = \frac{12 - n^2 - n}{4 \cdot 5} C_3$$

$$C_5 = - \frac{n^2 - n + 12}{4 \cdot 5} C_3 = - \frac{(n^2 + n - 12)}{4 \cdot 5} C_3$$

$$C_5 = - \frac{(4+n)(n-3)}{5 \cdot 4} C_3.$$

$$C_5 = - \frac{(n+4)(n-3)}{5 \cdot 4} - \frac{(n+2)(n-1)}{3!} C_1$$

$$C_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} C_1.$$

using all values in equation (2) we get

$$y = C_0 \left[1 - \frac{n(n+1)}{3!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 \right]$$

$$+ C_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 \right]$$

which is the

required solution of (1)

This is a solution with two arbitrary constants so not considered. $\beta=1$ //

CH # 01 Lebesgue Measure

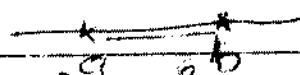
Introduction

Interval: An interval is the set of real numbers e.g. $I = [a, b] = \{x : a \leq x \leq b\}$

The length of an interval is equal to the difference of the end points of the interval e.g. $I = [a, b]$

$$l(I) = b - a$$

The intervals $[a, b]$ and (a, b) have same length.



* SET FUNCTION:

A function f is said to be a set function if its domain is a collection of sets and M is a collection of sets then the function

as $f: M \rightarrow \{-1, 1\}$ defined

$$f(A) = \begin{cases} -1 & \text{if } \text{ord} A = \infty \\ 1 & \text{if } \text{ord} A < \infty \end{cases}$$

* Non-negative Function

A set function is said to be non-negative function if its range is non-negative extended real number.

e.g. if we define a function

$f: M \longrightarrow [0, \infty]$ as

$$f(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$$

Additivity of a Set Function:

A set function

$f: M \longrightarrow [0, \infty]$ is said to be additive if for $A, B \in M$

$$f(A \cup B) = f(A) + f(B).$$

* Finitely Additive Set Function:

The set function

$f: M \longrightarrow [0, \infty]$ is said to be finitely additive set function if

$$f\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n f(A_i).$$

* Countable Additive Set Function:

A) 'M' is a collection of sets and

$f: M \longrightarrow [0, \infty]$ is said to be countable additive or sigma additive if for $A_1, A_2, A_3, \dots, A_n, \dots \in M$

NOTE Every additive fn is sub additive
but converse not true.

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also $A_1 \cup A_2 \cup A_3 \dots A_n \dots \in M$

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} f(A_i)$$

* Sub additivity of set function.
A set function $f: M \rightarrow [0, \infty]$

is said to be sub additive set function if $A, B \in M$ also $A \cup B \in M$ such that

$$f(A \cup B) \leq f(A) + f(B).$$

* Finite sub additive of set function:
A set function $f: M \rightarrow [0, \infty]$

is said to be finite sub additive set function if $f\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n f(A_i)$

* Countable sub additive of set function:

A set function $f: M \rightarrow [0, \infty]$

is said to be countably sub additive set function if for $A_1, A_2, A_3, \dots, A_n, \dots \in M$
also $A_1 \cup A_2 \cup A_3 \dots \cup A_n \dots \in M$

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} f(A_i)$$

201)

* Pre-Measure:- A set function

$f: M \rightarrow [0, \infty]$ is said to be
pre-measure if $f(\varnothing) = 0$

e.g.

Let M be the collection of all
open interval and

$f: M \rightarrow [0, \infty]$ defined as

$$f(I) = l(I).$$

let $\bigcup a_n \in M$

clearly

$$\bigcup a_n = \varnothing$$

$$\Rightarrow f(\varnothing) = f(\bigcup a_n) \\ = l(\bigcup a_n)$$

Hence f is $\stackrel{=0}{\text{pre measure}}$.

Outer Measure:

A set function $f: 2^X \rightarrow [0, \infty]$ is said to be outer measure if

i) $f(\emptyset) = 0$

ii) $\forall A_1, A_2, \dots, A_n, \dots \in 2^X$

$\& A \subseteq \bigcup A_i \quad \text{or} \quad \mu^*(E) \leq \mu^*(E_i)$

$f(A) \leq \sum f(A_i)$ (monotonicity) $\forall E_i \subseteq E$

* Monoton set function:

$f: M \rightarrow [0, \infty]$ A set function is said to be monoton if for

$A \subseteq B$

$f(A) \leq f(B)$.

Remark:

- i) Every outer measure is pre-measure but converse may not true.
 ii) An outer measure is a monoton sub-additivity.

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Ex: A set function

$f: 2^X \rightarrow [0, \infty]$ defined as

$$f(A) = \text{Ord}(A).$$

Sol: - Now first if

i) $A = \emptyset$.

$$\text{ord}(A) = \text{ord}(\emptyset)$$

$$= 0$$

i.e

$$f(A) = 0 \text{ for } A = \emptyset.$$

ii) Let $A_1, A_2, A_3, \dots, A_i, \dots \in 2^X$

\varnothing

$$A \subseteq \cup A_i$$

Then clearly

$$\text{ord}(A) \leq \text{ord}(\cup A_i)$$

$$\Rightarrow f(A) \leq f(\cup A_i) = \sum f(A_i)$$

$$\Rightarrow f(A) \leq \sum f(A_i) \quad \because \text{Sub-additivity property}$$

which shows that f

is an outer measure.

~~$2^X \supseteq A \supseteq \emptyset \supseteq \emptyset \supseteq \emptyset \supseteq \emptyset$~~

Ex.

A set function

 $f: 2^X \rightarrow [0, 1]$ defined as

$$f(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \end{cases}$$

Sol.

Now first if $A = \emptyset$ Then by definition of this function

$$f(A) = f(\emptyset) = 0$$

ii) Now Let $A_1, A_2, A_3, \dots, A_i, \dots \in 2^X$ and

$$A \subseteq \cup A_i$$

Now if $A = \emptyset$ Then $f(A) = 0$ also if $\cup A_i$ is improperly contained in A Then

$$f(\cup A_i) = 0 \quad \text{--- (1)}$$

But if A is properly contained in $\cup A_i$ i.e. There must exist at least one non-empty A_i such that

$$f(A) \leq \sum f(A_i) \quad \text{--- (2)}$$

Combining (1) & (2) we get

$$f(A) \leq \sum f(A_i)$$

Now if $A \neq \emptyset$ Then $f(A) = 1$ and $A \subseteq \cup A_i$

(205)

Then There must exist at least one non-empty set A_i such that

$$f(A) \leq \sum f(A_i)$$

Hence f is outer measure.

$$f(A) = \begin{cases} 0 & \text{if } y \notin A = \emptyset \\ 1 & \text{if } y \in A \neq \emptyset \end{cases}$$

Assignment

of $\mu: 2^X \rightarrow [0, \infty]$
defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is uncountable} \end{cases}$$

Sol: - i) if $A = \emptyset$ \Rightarrow A is finite and countable

$$\mu(A) = \mu(\emptyset) = 0$$

$$\mu(A) = 0$$

ii) Now Let $A_1, A_2, A_3, \dots, A_i, \dots$
and

$$A \subseteq \cup A_i$$

if A is countable and each A_i is countable so $\cup A_i$ is also countable. Then $f(A) = 0$ $\text{---} \textcircled{1}$

and

A is ~~uncountable~~

Then ~~at~~ at least one non-empty set A_i is non-countable.

Then

$$f(A) < \sum f(A_i) \quad \text{--- (2)}$$

from

(1) & (2) we get.

$$f(A) \leq \sum f(A_i)$$

Now if

A is uncountable then

There is at least one non-empty set A_i which is uncountable
Hence

$$f(A) \leq \sum f(A_i)$$

$\Rightarrow f$ is outermeasure.

$\dots \dots \dots$

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~~Def in paper~~

Sum of Two Outer Measures.

Let f and g be two outer measures from $\mathcal{Q} \rightarrow [0, \infty]$. Then the

function $(f+g): \mathcal{Q} \rightarrow [0, \infty]$

defined as

$$(f+g)A = f(A) + g(A)$$

Pr: Now if $A = \emptyset$.
Then

$$(f+g)\emptyset = f(\emptyset) + g(\emptyset)$$

$$= 0 + 0$$

$\therefore f, g$ are outer measures.

$$(f+g)\emptyset = 0$$

(ii) Let $A_1, A_2, A_3, \dots, A_i, \dots \in \mathcal{Q}$

and $A \subseteq \cup A_i$

$$\text{Then } (f+g)A = f(A) + g(A)$$

$$\leq \sum f(A_i) + \sum g(A_i)$$

Since both are outer measures

$$= \sum (f(A_i) + g(A_i))$$

$$\geq \sum (f+g)A_i$$

$$\Rightarrow (f+g)A \leq \sum (f+g)A_i$$

\Rightarrow " $f+g$ " is outer measure.

* Scalar Multiplication of an outer measure; \rightarrow outer measure

let f be an outer measure defined as

$f: \mathcal{G}^X \rightarrow [0, \infty]$ & c be a non-negative real number

Define

$$cf: \mathcal{G}^X \rightarrow [0, \infty]$$

$$\text{as } cf(A) = c f(A)$$

to check 'cf' is outer measure

i)

$$A = \emptyset.$$

$$cf(A) = cf(\emptyset) = c(0) = 0 \because f \text{ is outer measure}$$

$$\Rightarrow c f(\emptyset) = 0$$

\Rightarrow cf is pre-measure

ii) let $A_1, A_2, A_3, \dots, A_i, \dots \in \mathcal{G}^X$.

such that

$$A \subseteq \cup A_i$$

$$cf(A) = c f(A)$$

$$\leq c \sum f(A_i)$$

$$= \sum cf(A_i)$$

$$\Rightarrow cf(A) \leq \sum cf(A_i)$$

hence $cf: \mathcal{G}^X \rightarrow [0, \infty]$ is outer measure.

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✓ * Difference of two outer measure needs not be an outer measure.

Sol:

Let $f: 2^X \rightarrow [0, \infty]$ and

$g: 2^X \rightarrow [0, \infty]$ are two outer measure defined as

$$f(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \end{cases}$$

and

$$g(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is uncountable} \end{cases}$$

Then $g-f: 2^X \rightarrow [0, \infty]$ defined

by

$$(g-f)(A) = g(A) - f(A)$$

Now if $A = \emptyset$.
Then

$$(\mathcal{G} - \mathcal{F})(\emptyset) = \mathcal{G}(\emptyset) - \mathcal{F}(\emptyset)$$

$$= 0 - 0$$

$$= 0$$

$\therefore \mathcal{G}$ & \mathcal{F} are outer measure

Now

if A is Countable but $A \neq \emptyset$.

$$(\mathcal{G} - \mathcal{F})(A) = \mathcal{G}(A) - \mathcal{F}(A)$$

$$= 0 - 1$$

$(\mathcal{G} - \mathcal{F})(A) = -1$ i.e. $(\mathcal{G} - \mathcal{F})(A)$ not belongs $[0, \infty]$

Hence $(\mathcal{G} - \mathcal{F})$ is not an outer measure.

Remark: Difference of two outer measure is ~~not~~ measure but may or may not outer measure.

— $\mathcal{G} - \mathcal{F}$ — $\mathcal{G} - \mathcal{F}$ — $\mathcal{G} - \mathcal{F}$ — $\mathcal{G} - \mathcal{F}$ — $\mathcal{G} - \mathcal{F}$ —

(211)

* Lebesgue Outer measure:
of a set of real numbers,

Let A be set of real numbers
and let $\{I_n\}$ be a countable collection
of open intervals covering A
i.e.

$$A \subseteq \bigcup I_n$$

and let $\sum l(I_n)$ be the sum of
lengths of all these intervals.
we define Lebesgue outer measure
of A denoted by

$m^*(A)$ by infimum of all
numbers $\sum l(I_n)$
specifically

$$m^*(A) = \inf \left\{ \sum l(I_n) / A \subseteq \bigcup I_n \right\}$$

————— · · ————— · · ————— · · ————— · · —————

$$E = (a, b) = b - a$$

$$E + \gamma = (a + \gamma, b + \gamma) = b - a$$

length of interval does not change.

(212)

* Translation of a set:

Let E be subset of \mathbb{R} and $\gamma \in \mathbb{R}$ Then

$E + \gamma = \{x + \gamma : x \in E\}$ is called translation of the set E .

* ~~Proof~~ Outer measure is translationally invariant i.e.

$E \subset \mathbb{R}, \gamma \in \mathbb{R}$ Then

$$m^*(E + \gamma) = m^*(E).$$

Proof: Let $\{I_n\}$ be any countable cover of E i.e. $E \subseteq \cup I_n$ Then

$\{I_n + \gamma\}$ will be countable open cover of $E + \gamma$. Now $I = (a, b) \in I_n$ Then $I + \gamma = (a + \gamma, b + \gamma) \in I_n + \gamma$

$$\begin{aligned} l(I + \gamma) &= l(a + \gamma, b + \gamma) \\ &= b + \gamma - (a + \gamma) \\ &= b + \gamma - a - \gamma \end{aligned}$$

$$= b - a$$

$$l(I + \gamma) = l(I)$$

$$\Rightarrow \sum l(I_n + \gamma) = \sum l(I_n)$$

$$\Rightarrow m^*(E + \gamma) \leq m^*(E) \quad \text{--- (1)}$$

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(2/3)

$(E+\delta) - \delta = E$
i.e. translation of $E+\delta$ by the real number $(-\delta)$ is E so by the same argument

$$m^*(E) \leq m^*(E+\delta) \quad \text{--- (2)}$$

From (1) & (2)

$$m^*(E+\delta) = m^*(E)$$

\Rightarrow outer measure is translationally invariant.

100 ——— \cdot \cdot \cdot \cdot \cdot ——— 0

Theorem:- (Countable Subadditivity)

Let $\{A_n\}$ be countable collection of real numbers. Then

$$m^*(\cup A_n) \leq \sum m^*(A_n)$$

Proof:- Case I:- If one of the set $\{A_n\}$ has outer measure ∞ . Then

$$m^*(\cup A_n) = \sum m^*(A_n)$$

Case II

if all A_n have finite outer measure. Then there is a countable collection $\{I_n\}$ of open intervals such that for any $\epsilon > 0$

$$\sum \sum \frac{1}{2^n} \rightarrow 1$$

$$\Rightarrow \sum \epsilon = \epsilon(1) = \epsilon$$

9. series eq: to 1.

(214)

$$\sum_i l(I_{n,i}) \leq m^*(A_n) + \frac{\epsilon}{2^n}$$

Then $\{I_{n,i}\}_{n,i}$ is countable

collection of open intervals as
each $\{I_{n,i}\}_i$ is countable

$$\cup A_n \subseteq \cup \{I_{n,i}\}_{n,i}$$

$$\Rightarrow m^* \cup A_n \leq \sum_{n,i} l(I_{n,i})$$

$$\Rightarrow m^* \cup A_n \leq \sum_n \sum_i l(I_{n,i})$$

$$\leq \sum_n \left(m^*(A_n) + \frac{\epsilon}{2^n} \right)$$

∵ eq (1)

$$\Rightarrow m^* \cup A_n \leq \sum_n m^*(A_n) + \sum_n \frac{\epsilon}{2^n}$$

$$\Rightarrow m^* \cup A_n \leq \sum_n m^*(A_n) + \epsilon$$

Since ϵ is an arbitrary

$$\Rightarrow m^* \cup A_n \leq \sum_n m^* A_n$$

as $\epsilon \rightarrow 0$

————— α ————— α ————— α —————

(215)

Problem: If $m^*(A) = 0$ Then

$$m^*(A \cup B) = m^*(B).$$

Prove:

Since ~~$A \subseteq A \cup B$~~ $B \subseteq A \cup B$

So

$$\Rightarrow m^*(B) \leq m^*(A \cup B) \quad \text{--- (1)}$$

Now By Theorem

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

$$\Rightarrow m^*(A \cup B) \leq m^*(B) \quad \text{--- (2)} \quad \because m^*(A) = 0$$

From (1) & (2) we get (2)

$$m^*(A \cup B) = m^*(B)$$

Monotone property:-

of any two sets of real numbers
s.t. $A, B \in \mathcal{E}^*$ & $A \subseteq B$ Then.

$$m^*(A) \leq m^*(B).$$

Proof let $\{I_n\}$ be countable collection
of open intervals which covers B.
i.e.

$$B \subseteq \bigcup I_n$$

$$m^*(B) = \inf \left\{ \sum l(I_n) \mid B \subseteq \bigcup I_n \right\}$$

$$\Rightarrow m^*(B) = \inf \left\{ \sum \ell(I_n) \mid B \subseteq \bigcup I_n \right\} \geq \inf \left\{ \sum \ell(I_n) \mid A \subseteq \bigcup I_n \right\}$$

$$= m^*(A)$$

$$\Rightarrow m^*(A) \leq m^*(B).$$

V-Domp.
Theorem

Outer measure of an interval is its length.

Proof in

we begin with case in which we have a finite closed interval say $I = [a, b]$, for any $\epsilon > 0$ the closed interval $[a, b]$ is contained in

$$(a - \epsilon, b + \epsilon) \text{ i.e. } [a, b] \subseteq (a - \epsilon, a + \epsilon)$$

Then

$$m^*[a, b] \leq \ell(a - \epsilon, b + \epsilon) = b + \epsilon - (a - \epsilon) = b - a + 2\epsilon$$

$$\Rightarrow m^*[a, b] \leq b - a + 2\epsilon$$

as $\epsilon \rightarrow 0$ since ϵ is arbitrary

4

$$m^*[a, b] \leq b - a \quad \text{--- (1)}$$

So we only have to prove

$$b - a \leq m^*[a, b]$$

or

$$m^*[a, b] \geq b - a.$$

but it is equivalent to show that

$$\sum \ell(I_n) \geq b - a \quad \text{--- (2)}$$

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$\{I_n\}$

$a_1, a_2, \dots, b_1, \dots$

a, b

For any countable cover $\{I_n\}$
By Heine-Borel Theorem

$\{I_n\}$ is reducible to finite subcover (1) for so it is sufficient to prove (2) for a finite subcover.

Now

$a \in \cup I_n$. Then \exists an open interval $(a_1, b_1) \in \{I_n\}^N$ of which contains a . This is (a_1, b_1)

$b_1 \leq b$ as $b_1 \notin (a_1, b_1)$

Then there is an open interval $(a_2, b_2) \in \{I_n\}^N$ s.t.

in the collection $\{I_n\}$

$b_1 \in (a_2, b_2)$. $\Rightarrow a_2 < b_1 < b_2$
Proceeding in the same way, we reach on an open interval

(a_k, b_k) s.t.

(Process in the same way we obtain a sequence $(a_1, b_1), (a_2, b_2), \dots$)

$a_k < b < b_k$ i.e. $b \in (a_k, b_k)$.

It is because $\{I_n\}$ finite subcover. Also in our process we have each

a_i since $\{I_n\}^N$ is a finite collection

from the collection that are

$a_i < b_{i-1} < b_i$

$a_i < b_i$

as our process must stop

as $\cup I_n \supset \cup (a_i, b_i)$

$$\Rightarrow \sum_{n=1}^m l(I_n) \geq \sum_{i=1}^k (b_i - a_i)$$

since

$$= (b_k - a_k) + (b_{k-1} - a_{k-1})$$

$\cup_{i=1}^k (a_i, b_i)$

$$+ \dots + (b_2 - a_2) + (b_1 - a_1) \quad m \geq k$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2})$$

$$\geq b_k - a_1 \geq b - a$$

and $b_k = a_1 > b - a$
 as $b_k > b$ and $a_1 < a$ so $b_k = a_1 > b - a$ (218)

$\Rightarrow \sum_{n=1}^m l(I_n) \geq b - a$ — (3)

From (1), (2) & (3)

$m^*[a, b] \geq b - a$ — (4)

From (1) & (4) we have

$m^*[a, b] = b - a.$

Case II ✓

Now let 'I' be any arbitrary finite interval. Then for $\epsilon > 0$ there is a closed interval 'J' s.t.

$J \subseteq I$

and $l(J) \geq l(I) - \epsilon$

Then

$l(I) - \epsilon \leq l(J) = m^*(J) \leq m^*(I)$

$\Rightarrow l(I) - \epsilon \leq m^*(I) \leq l(I).$

As ϵ is an arbitrary and approaching to zero therefore so

$l(I) \leq m^*(I) \leq l(I)$

$\Rightarrow m^*(I) = l(I)$

Case III

let 'I' be an infinite interval. Then given any positive real number Δ we have a closed interval

(2.19)

$$m^*(A) = m^* \left\{ \sum l(I_n) : A \subseteq \bigcup I_n \right\}$$

$$J \subset I$$

Set

$$l(J) = \Delta$$

$$\Rightarrow m^*(I) \supset m^*(J) = l(J) = \Delta$$

$$\Rightarrow m^*(I) \supset \Delta$$

$$\Rightarrow m^*(I) = \infty \quad \because \Delta \text{ is any}$$

$$\Rightarrow m^*(I) = l(I) \quad \text{Positive Real numbers.}$$

$$\text{i.e. } l(I) = \infty$$

* Lebesgue Measure of set of Real Numbers.

Let E be a set of real numbers
we say E is Lebesgue measurable
if for any set A of real numbers
we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \bar{E})$$

Since we always have

$$\begin{aligned} m^*(A) &= m^*(A \cap R) \\ &= m^*(A \cap (E \cup \bar{E})) \\ &= m^*(A \cap E \cup A \cap \bar{E}) \end{aligned}$$

$$\leq m^*(A \cap E) + m^*(A \cap \bar{E})$$

\therefore By countable
sub-additivity

So we only have to prove

Lemma $m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$.
 If E_1 and E_2 are measurable.
 Then $E_1 \cup E_2$ is also measurable.

Proof: Let A be any set of real numbers. Then we have to show that

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim(E_1 \cup E_2))$$

Now

$$\begin{aligned} m^*(A \cap (E_1 \cup E_2)) &= m^*((A \cap E_1) \cup (A \cap E_2)) \\ &\leq m^*(A \cap E_1) + m^*(A \cap E_2) \\ &= m^*(A \cap E_1) + m^*(A \cap (E_2 \cap \tilde{E}_1)) \end{aligned}$$

Adding $m^*(A \cap \sim(E_1 \cup E_2))$ on both sides of inequality we have

$$\begin{aligned} m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim(E_1 \cup E_2)) &\leq m^*(A \cap E_1) + m^*(A \cap (E_2 \cap \tilde{E}_1)) \\ &\quad + m^*(A \cap (\tilde{E}_1 \cap \tilde{E}_2)) \end{aligned}$$

$$= m^*(A \cap E_1) + m^*((A \cap \tilde{E}_1) \cap E_2) + m^*((A \cap \tilde{E}_1) \cap \tilde{E}_2)$$

$$\leq m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1)$$

$\therefore E_2$ is measurable.

$$\begin{aligned} \Rightarrow m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim(E_1 \cup E_2)) &\leq m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1) \\ &\leq m^*(A) \quad \because E_1 \text{ is measurable} \end{aligned}$$

$$\Rightarrow m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim(E_1 \cup E_2)) \quad \text{--- (1)}$$

NOTE If E_1 is measurable then E_1^c is measurable.

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Generally.

$$m^*(A) \leq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \quad \text{②}$$

From ① & ② we have.

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

$\Rightarrow E_1 \cup E_2$ is measurable.

Theorem

Let E_1 & E_2 are two measurable sets. Then $E_1 \cap E_2$ is also measurable.

Proof: Let X be any set of real numbers. we have to prove that.

$$m^*(X) \geq m^*(X \cap (E_1 \cap E_2)) + m^*(X \cap (E_1 \cap E_2)^c)$$

By De Morgan's Law.

$$\begin{aligned} X \cap (E_1 \cap E_2)^c &= X \cap (\bar{E}_1 \cup \bar{E}_2) \\ &= (X \cap \bar{E}_1) \cup (X \cap \bar{E}_2) \end{aligned}$$

$$\Rightarrow m^*(X \cap (E_1 \cap E_2)^c) = m^*((X \cap \bar{E}_1) \cup (X \cap \bar{E}_2))$$

$$\leq m^*(X \cap \bar{E}_1) + m^*(X \cap \bar{E}_2)$$

adding $m^*(X \cap (E_1 \cap E_2))$ on both sides of inequality we get.

$$m^*(x \cap (E_1 \cap E_2)) + m^*(x \cap (E_1 \cap E_2^c))$$

$$\leq m^*(x \cap \tilde{E}_1) + m^*(x \cap \tilde{E}_2) + m^*(x \cap (E_1 \cap E_2))$$

$$\leq m^*(x \cap E_1 \cap E_2) + m^*(x \cap \tilde{E}_1)$$

$$+ m^*(x \cap (E_1 \cap E_2^c))$$

$$= m^*(x \cap E_1 \cap E_2) + m^*(x \cap \tilde{E}_1) + m^*(x \cap E_1 \cap E_2^c)$$

$$\leq m^*(x \cap E_1) + m^*(x \cap \tilde{E}_1)$$

$\therefore E_2$ is measurable

$$\leq m^*(x) \quad \therefore E_1 \text{ is measurable}$$

$$m^*(x \cap (E_1 \cap E_2)) + m^*(x \cap (E_1 \cap E_2^c)) \leq m^*(x) \quad \downarrow$$

①

Generally,

$$m^*(x \cap (E_1 \cap E_2)) + m^*(x \cap (E_1 \cap E_2^c))$$

$$\geq m^*(x) \quad \downarrow$$

From ① & ② we get

②

$$m^*(x) = m^*(x \cap (E_1 \cap E_2)) + m^*(x \cap (E_1 \cap E_2^c))$$

$\Rightarrow E_1 \cap E_2$ is measurable

α

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Qⁿ If A and B are measurable
Then $A - B$ is measurable.

Proof For any set Given that
 A & B are measurable. Also

$$A - B = A \cap B^c$$

Since B is measurable, so
 B^c is also measurable as
Complement of measurable set is
also measurable.

Also we know that intersection
of two measurable set is also
measurable. so $A \cap B^c$ is measurable
 $\Rightarrow A - B$ is measurable.

* Boolean Algebra or
Simply algebra of sets:-

A collection ' α ' of subsets of
Real numbers is called Algebra of sets

if

- i) $A \in \alpha \Rightarrow \bar{A} \in \alpha$.

- ii) $A, B \in \alpha \Rightarrow A \cup B \in \alpha$.

* Sigma Algebra (σ Algebra):-
An algebra of sets is called
 σ -algebra if every union of
Countable collection of members of
' α ' belongs to ' α '.

Result: The collection of measurable sets is an algebra of sets.

Proof: Let \mathcal{A} be the collection of measurable sets. Then for any $A \in \mathcal{A}$.

$\Rightarrow \bar{A}$ is also measurable as complement of measurable set is also measurable.

$\Rightarrow \bar{A} \in \mathcal{A}$.

Secondly let $A, B \in \mathcal{A}$.

i.e. A & B are measurable.

$\Rightarrow A \cup B$ is measurable.

$\Rightarrow A \cup B \in \mathcal{A}$.

Hence

The collection of measurable sets is an algebra of sets. \parallel

Lemma:

Let A be any set of real numbers and $E_1, E_2, E_3, \dots, E_n$ be the collection of disjoint measurable sets. Then

$$m^*(A \cap \bigcup_{i=1}^n E_i) = \sum_{i=1}^n m^*(A \cap E_i)$$

Proof: we prove the lemma by induction on 'n'.

for $i=1$

$$m^*(A \cap E_1) = \sum_{i=1}^1 m^*(A \cap E_i)$$

so it is true.

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Suppose the result is true
 for $i = n-1$ i.e.

$$m^* \left(A \cap \bigcup_{i=1}^{n-1} E_i \right) = \sum_{i=1}^{n-1} m^* (A \cap E_i)$$

Now

E_i 's are disjoint,
 $(A \cap \bigcup_{i=1}^n E_i)$ is any subset of real
 numbers and E_n is measurable

Then

$$\begin{aligned} m^* \left(A \cap \bigcup_{i=1}^n E_i \right) &= m^* \left(A \cap \bigcup_{i=1}^n E_i \cap E_n \right) \\ &\quad + m^* \left(A \cap \bigcup_{i=1}^n E_i \cap \bar{E}_n \right) \\ &= m^* (A \cap E_n) + m^* \left(A \cap \bigcup_{i=1}^{n-1} E_i \right) \\ &= m^* (A \cap E_n) + \sum_{i=1}^{n-1} m^* (A \cap E_i) \quad \text{By Hypothesis} \\ &= \sum_{i=1}^n m^* (A \cap E_i) \end{aligned}$$

i.e. The result is true for $i = n$.

Hence

$$m^* \left(A \cap \bigcup_{i=1}^n E_i \right) = \sum_{i=1}^n m^* (A \cap E_i)$$

Now

for $i = 1, 2, 3, \dots$

$$\begin{aligned} (A \cap \bigcup_{i=1}^{\infty} E_i) &\supset (A \cap \bigcup_{i=1}^n E_i) \\ \Rightarrow m^* \left(A \cap \bigcup_{i=1}^n E_i \right) &\leq m^* \left(A \cap \bigcup_{i=1}^{\infty} E_i \right) \end{aligned}$$

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$$\Rightarrow \sum_{i=1}^n m^*(A \cap E_i) \leq m^*(A \cap \bigcup_{i=1}^n E_i)$$

Since R.H.S of inequality is independent of n . Therefore

$$\sum_{i=1}^{\infty} m^*(A \cap E_i) \leq m^*(A \cap \bigcup_{i=1}^{\infty} E_i)$$

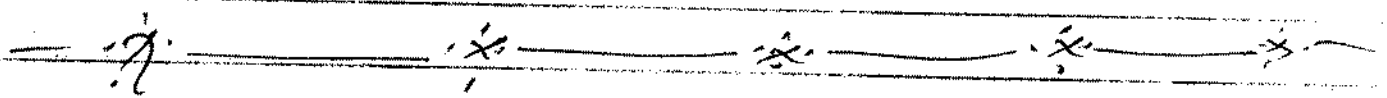
also by countable sub additivity ①

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

From ① & ② we have: ②

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Hence The proof is.



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~~W-2007~~

Theorem:-

The collection M of measurable sets is σ -Algebra.

Proof

i) As if $A \in M$ then $\bar{A} \in M$
Since complement of measurable set is again measurable.

ii) If $A, B \in M$ then $A \cup B \in M$.
As union of measurable set is again measurable.

iii) we have to prove union of countable collection of measurable sets is also measurable.

Let A be any set of real number and E can be expressed as

$$E = \bigcup_{i=1}^{\infty} E_i$$

where E_i 's are pairwise disjoint measurable set.

our requirement is to show

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \bar{E})$$

let

$$F_n = \bigcup_{i=1}^n E_i$$

Then F_n is measurable \because union of finite measurable set measurable.

now

$$F_n \subset E$$

$$\Rightarrow F_n \supset E$$

Since F_n is measurable.

Therefore

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap \tilde{F}_n)$$

$$m^*(A) \geq m^*(A \cap F_n) + m^*(A \cap \tilde{E})$$

$$\because \tilde{F}_n \supset \tilde{E}$$

$$\Rightarrow m^*(A) \geq m^*(A \cap \bigcup_{i=1}^n E_i) + m^*(A \cap \tilde{E})$$

$$m^*(A) \geq \sum_{i=1}^n m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

Since L.H.S is independent of 'n'

$$\Rightarrow m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

$$\Rightarrow m^*(A) \geq m^*(A \cap \bigcup_{i=1}^{\infty} E_i) + m^*(A \cap \tilde{E})$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E}) \quad \downarrow$$

Generally

$$= m^*(A) \leq m^*(A \cap E) + m^*(A \cap \tilde{E}) \quad \downarrow$$

From (1) & (2) we get

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$$

$\Rightarrow E$ is measurable.

Hence from (i), (ii) and (iii) The collection \mathcal{M} of measurable set \mathcal{S} algebra.

Borel Subsets of Real line:-

Let β be the intersection of all the σ -Algebra of subsets R containing every ^{open} subset of R . The members of β are called Borel subsets of R .

The members of the ^{smallest} σ -Algebra of subsets of R are called Borel subsets of R .

Lemma

The open interval (a, ∞) is measurable

Proof:-

Let A be any set of real numbers then $A_1 = A \cap (a, \infty)$, $A_2 = A \cap (-\infty, a]$
So we must have to prove

$$m^*(A) \geq m^*(A_1) + m^*(A_2)$$

Now

if $m^*(A) = \infty$

Then there is nothing to prove.

But if

$$m^*(A) < \infty$$

Then for $\epsilon > 0$ \exists countable collection $\{I_n\}$ ^{of open interval} such that

$$\sum \ell(I_n) \leq m^*(A) + \epsilon \rightarrow (1)$$

$$\text{let } I_n' = I_n \cap (a, \infty)$$

&

$$I_n'' = I_n \cap (-\infty, a]$$

Then

I_n' and I_n'' are intervals and

$$I_n = I_n' \cup I_n''$$

$$\Rightarrow \sum \ell(I_n) = \sum \ell(I_n') + \sum \ell(I_n'')$$

Now

$$A_1 \subset \cup I_n' \quad \& \quad A_2 \subset \cup I_n''$$

$$\Rightarrow m^*(A_1) \leq \sum \ell(I_n') = m^*(I_n') \quad (2)$$

&

$$m^*(A_2) \leq \sum \ell(I_n'') = m^*(I_n'') \quad (3)$$

From (1)

$$m^*(A) + \epsilon \geq \sum \ell(I_n)$$

$$= \sum \ell(I_n') + \sum \ell(I_n'')$$

$$m^*(A) \geq \sum \ell(I_n') + \sum \ell(I_n'') \quad ; \epsilon \rightarrow 0$$

$$\geq m^*(A_1) + m^*(A_2)$$

$$\Rightarrow m^*(A) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a])$$

Generally

$$m^*(A) \leq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]) \quad (4)$$

From (4) & (5) we get

$$m^*(A) = m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]) \quad (5)$$

Hence the open interval (a, ∞) is measurable.

(23)

* imp

Every Borel set is measurable.

Proof:-

By the definition of Borel set it is sufficient to prove that every open set is measurable and since every open set is union of countable collection of disjoint open interval so it is sufficient to prove that every open interval (a, b) is measurable we have (a, ∞) is measurable.

$\Rightarrow N(a, \infty)$ is also measurable.

as $N(a, \infty) = (-\infty, a]$ is measurable for every $a \in \mathbb{R}$

now

$$(-\infty, b) = \bigcup_{n \rightarrow \infty} (-\infty, b - \frac{1}{n}] \text{ where}$$

$\Rightarrow (-\infty, b)$ is measurable since countable union of measurable set is measurable

now

$$(a, b) = (a, \infty) \cap (-\infty, b)$$

$\Rightarrow (a, b)$ is measurable being the intersection of two measurable set is again measurable. Hence our result is proved

ie Every Borel set is measurable.

Every countable Borel set is of measure zero.

* Lebesgue Measure of Measurable set:-

If E is measurable then its Lebesgue measure is denoted by $m(E)$ will be taken as its outer measure.

* Let $\langle E_i \rangle$ be sequence of measurable sets then $m \cup E_i \leq \sum m E_i$
if E_i 's are disjoint then
 $m \cup E_i = \sum m E_i$

Proof:- we have

$$m^* \cup E_i \leq \sum m^* E_i$$

Since E_i 's are measurable therefore

$$m^* \cup E_i = m \cup E_i$$

$$\Rightarrow m \cup E_i \leq \sum m E_i \quad \text{--- (1)}$$

Now if $\langle E_i \rangle$ is finite sequence of mutually disjoint measurable then we have

$$m^* (A \cap \bigcup_{i=1}^n E_i) \leq \sum_{i=1}^n m^* (A \cap E_i)$$

Now

$$A = R$$

$$m^* (R \cap \bigcup_{i=1}^n E_i) \leq \sum_{i=1}^n m^* (R \cap E_i)$$

$$m^* (\bigcup_{i=1}^n E_i) \leq \sum_{i=1}^n m^* (E_i)$$

As E_i 's are measurable and disjoint

(233)

$$\Rightarrow m \bigcup_{i=1}^n E_i = \sum_{i=1}^n m E_i$$

Now

$$\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^n E_i$$

$$\Rightarrow \bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^{\infty} E_i$$

$$\Rightarrow m \left(\bigcup_{i=1}^n E_i \right) \leq m \bigcup_{i=1}^{\infty} E_i$$

$$\begin{aligned} \because (a, b) &\subset [a, b] \\ m^*(a, b) &= b - a \\ \therefore m^*[a, b] &= b - a \end{aligned}$$

$$\Rightarrow \sum_{i=1}^n m E_i \leq m \bigcup_{i=1}^{\infty} E_i$$

$$\therefore m \bigcup_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} m E_i$$

Since R.H.S is independent of n

$$\Rightarrow \sum_{i=1}^{\infty} m E_i \leq m \bigcup_{i=1}^{\infty} E_i \quad \text{--- (2)}$$

also by countable subadditivity

$$m \bigcup_{i=1}^{\infty} E_i \leq \sum_{i=1}^{\infty} m E_i \quad \text{--- (3)}$$

From (2) & (3) we get

$$m \bigcup_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} m E_i \quad //$$

* Proposition:

Let $\{E_n\}$ be a decreasing sequence of measurable sets i.e.

$$E_1 \supset E_2 \supset E_3 \supset E_4 \supset \dots$$

and let mE_1 is finite. Then

$$m \bigcap_{i=1}^{\infty} E_i = \lim_{n \rightarrow \infty} mE_n$$

Proof:

$$\text{Let } E = \bigcap_{i=1}^{\infty} E_i \quad \& \quad \text{let } F_i = E_i - E_{i+1}$$

Then

$$F_1 = E_1 - E_2, F_2 = E_2 - E_3, \dots$$

i.e.

F_i 's are mutually disjoint sets.

$$m \cup F_i = \sum m F_i$$

$$m \cup F_i = \sum m (E_i - E_{i+1}) \longrightarrow \textcircled{1}$$

~~$$m(E_1 - E) = \sum m$$~~

$$\cup F_i = (E_1 - E_2) \cup (E_2 - E_3) \cup (E_3 - E_4) \cup \dots$$

$$= E_1 - E \quad (\text{where } E = \bigcap E_i)$$

$$m(\cup F_i) = m(E_1 - E) \longrightarrow \textcircled{2}$$

From $\textcircled{1}$ & $\textcircled{2}$ we have

$$m(E_1 - E) = \sum m(E_i - E_{i+1})$$

(235)

$$E_1 = E \cup (E_1 - E)$$



$$m E_1 - m E = \sum m (E_i - E_{i+1})$$

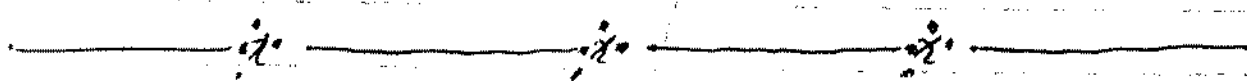
$$= m(E_1 - E_2) + m(E_2 - E_3) + \dots$$

$$\Rightarrow m E_1 - m E = \cancel{m E_1} - \cancel{m E_2} + \cancel{m E_2} - \cancel{m E_3} + \cancel{m E_3} - \dots$$

$$m E = \lim_{n \rightarrow \infty} m E_n$$

$$m E = \lim_{n \rightarrow \infty} m E_n$$

$$\boxed{m \bigcap_{i=1}^{\infty} E_i = \lim_{n \rightarrow \infty} m E_n} \checkmark$$



Ass:

Show that condition $m(E_i) < \infty$ is necessary in the above proposition.

All these notes are available on

<http://www.MathCity.org>

Notes for FSc, BSc, MSc, MPhil and PhD can be downloaded from MathCity.org

Other notes for MSc Mathematics are also Available.

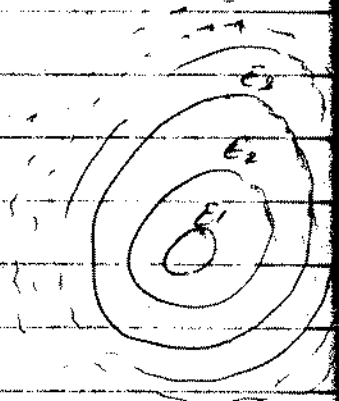
(237)

$$\underbrace{\left(\left(\left(E_3 \ E_2 \ E_1 \right) \right) \right)}$$

Proposition: \checkmark Let $\langle E_n \rangle$ be an infinite increasing sequence of measurable set. i.e.

Then $E_1 \subset E_2 \subset E_3 \subset \dots$

$$m \bigcup_{i=1}^{\infty} E_i = \lim_{n \rightarrow \infty} m E_n.$$



Proof:

Then $\exists m E_n = \infty$ for some n

$$m \bigcup_{i=1}^{\infty} E_i = \infty$$

\nexists

$$\lim_{n \rightarrow \infty} m E_n = \infty$$

$$\Rightarrow m \bigcup_{i=1}^{\infty} E_i = \lim_{n \rightarrow \infty} m E_n.$$

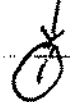
Consider the case

$$m(E_n) < \infty \text{ for all } n \in \mathbb{N}.$$

$$= E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2)$$

the R.H.S are mutually disjoint

$$= m E_1 + m(E_2 - E_1) + m(E_3 - E_2) + \dots$$



$$E_{i+1} = E_i \cup (E_{i+1} \cap \bar{E}_i)$$

$$\begin{aligned} m(E_{i+1}) &= mE_i + m(E_{i+1} \cap \bar{E}_i) \\ &= mE_i + m(E_{i+1} - E_i) \end{aligned}$$

$$\Rightarrow m(E_{i+1} - E_i) = mE_{i+1} - mE_i$$

$$\textcircled{1} \Rightarrow$$

$$\begin{aligned} m\left(\bigcup_{i=1}^{\infty} E_i\right) &= mE_1 + mE_2 - mE_1 + mE_3 - mE_2 \\ &\quad + mE_4 - mE_3 + \dots \end{aligned}$$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} m(E_n)$$

Imp. Problem:

If E is measurable then each translation of E is also measurable.

Solution: Let E be any sub set of real numbers. Then for $y \in \mathbb{R}$, we have to prove

$$m^*(A) = m^*(A \cap E + y) + m^*(A \cap \bar{E} + y)$$

we have from set theory

$$(A - y) \cap E + y = A \cap E + y$$

$$(A - y) \cap \bar{E} + y = A \cap \bar{E} + y$$

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$$f: [0, \infty] \rightarrow [0, \infty]$$

Now E is mble and $A - \gamma$ is the subset of real numbers.
Then

$$m^*(A - \gamma) = m^*(A - \gamma \cap E) + m^*(A - \gamma \cap E^c)$$

(2)

Since outer measure is translationaly invariant

$$m^*[(A - \gamma) \cap E] + \gamma$$

$$= m^*(A - \gamma \cap E)$$

$$m^*[(A - \gamma \cap E^c) + \gamma] = m^*(A - \gamma \cap E^c)$$

\therefore invariant property.

also

$$m^*(A - \gamma) = m^*(A).$$

eq (2) \Rightarrow

$$m^*(A - \gamma) = m^*[(A - \gamma) \cap E] + m^*[(A - \gamma) \cap E^c]$$

$$m^*(A) = m^*(A \cap E + \gamma) + m^*(A \cap E^c + \gamma)$$

$\Rightarrow E + \gamma$ is mble. each
Hence each translation of E is measurable
// // // // //

* Measurable Function:

Let f be an extended real valued function whose domain is measurable then the following statements are equivalent.

i) The set $\{x : f(x) > \alpha\}$ is mble

ii) The set $\{x : f(x) \geq \alpha\}$ is mble

iii) The set $\{x : f(x) < \alpha\}$ is mble

iv) The set $\{x : f(x) \leq \alpha\}$ is mble. where ' α ' is any real number.

Proposition

The above four statements are equivalent.

Proof:- (i) \Leftrightarrow (iv)

From (iv) $\{x : f(x) \leq \alpha\}$

$= D - \{x : f(x) > \alpha\}$ where D

The R.H.S is measurable being the difference of two measurable set. There L.H.S is also measurable. i.e

$\{x : f(x) \leq \alpha\}$ is mble.

Now

$\{x : f(x) > \alpha\} = D - \{x : f(x) \leq \alpha\}$

as R.H.S is mble being difference of mble set is mble. so L.H.S is also mble.

Available at
www.mathcity.org

(241) $\bigcup_{i=1}^{\infty} I_i$ is mble $\Rightarrow (\bigcup_{i=1}^{\infty} I_i)^c$ is mble.
 ~~$\alpha-1 \quad \alpha-\frac{1}{2} \quad \dots \quad \alpha$~~
 ~~$\{x: f(x) > \alpha\}$~~

now

(i) \Leftrightarrow (ii)

$$\{x: f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\} \quad (ii)$$

R.H.S is mble being countable intersection of mble set. Therefore L.H.S is also mble.

now

$$\{x: f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x: f(x) \geq \alpha + \frac{1}{n}\right\} \quad (iii)$$

R.H.S is mble being countable union of mble set. Therefore L.H.S is mble.

(ii) \Leftrightarrow (iii)

Consider $\{x: f(x) < \alpha\}$

$$= D - \{x: f(x) \geq \alpha\}$$

Since D and $\{x: f(x) \geq \alpha\}$ is measurable therefore R.H.S is measurable.

$\Rightarrow \{x: f(x) < \alpha\}$ is mble.

now

$$\{x: f(x) \geq \alpha\} = D - \{x: f(x) < \alpha\}$$

L.H.S is mble being the difference

~~day 3 day 2 of 1 2~~

(242)

of two measurable sets.
Hence L.H.S is measurable.

(iii) \Leftrightarrow (iv)

$$\{x: f(x) \leq \alpha\} = \bigcup_{n=1}^{\infty} \{x: f(x) < \alpha - \frac{1}{n}\}$$

R.H.S measurable being the countable union of measurable sets is measurable.
Hence R.H.S is measurable.

Now

$$\{x: f(x) < \alpha\} = \bigcap_{n=1}^{\infty} \{x: f(x) < \alpha + \frac{1}{n}\}$$

R.H.S is measurable being the countable union of measurable sets is measurable.

$i \Leftrightarrow (i) \Leftrightarrow (iii) \Leftrightarrow (iv)$

————— α ————— α —————

(243)

f

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Proposition 2.

The four statements are equivalent to

$$\{x: f(x) = \alpha\} \text{ --- (V)}$$

Proof :- (ii) & (iv) \implies (v)

Case-I.

$$\{x: f(x) = \alpha\} = \{x: f(x) \geq \alpha\} \cap \{x: f(x) \leq \alpha\}$$

which exist if α is finite

Case-II

when $\alpha = \infty$ then

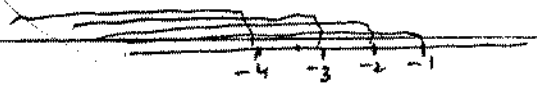


$$\{x: f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x: f(x) \geq n\}$$

The R.H.S is measurable being a countable intersection of measurable sets ~~for R.H.S~~
~~R.H.S~~ so L.H.S is measurable.

when

$$\alpha = -\infty$$



$$\{x: f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x: f(x) \leq -n\}$$

R.H.S is measurable being a countable intersection of measurable set is measurable
so L.H.S is measurable.

(245)

NOTE:
f is said to be measurable if the following four statements are equivalent.

- i) $\{x: f(x) > \alpha\}$ is measurable
 - ii) $\{x: f(x) \geq \alpha\}$ is measurable
 - iii) $\{x: f(x) < \alpha\}$ is measurable
 - iv) $\{x: f(x) \leq \alpha\}$ is measurable
- \Downarrow
- $\{x: f(x) = \alpha\}$

$$c + f(x) = c + f(x) \quad (c + f, cf)$$

(246)

Proposition

Let c be any constant and f be a measurable function then

$'c+f'$ and $'cf'$ are also measurable.

Proof ~

Let α be any real number, then

$$\{x : c + f(x) > \alpha\} = \{x : f(x) > \alpha - c\}$$

R.H.S is measurable since f is measurable
 ∴ L.H.S is measurable. i.e. $'c+f'$ is measurable.

Now

$$\{x : cf(x) > \alpha\} = \begin{cases} \{x : f(x) > \frac{\alpha}{c}\} & \text{if } c > 0 \\ \{x : f(x) < \frac{\alpha}{c}\} & \text{if } c < 0 \end{cases}$$

In both cases

R.H.S is f measurable. so L.H.S measurable

i.e. $'cf'$ is measurable.

$$\begin{aligned} (c+f)(x) &= c + f(x) > \alpha \\ &\implies f(x) > \alpha - c \end{aligned}$$

$$cf(x) = c f(x)$$

(247)



Proposition: Let f and g are measurable functions Then

$f+g$, $f-g$ and $f \cdot g$ is measurable. $(f+g) \cdot x = f \cdot x + g \cdot x$

Proof: Let α be any real number and $f(x) + g(x) < \alpha$

$\Rightarrow f(x) < \alpha - g(x)$ (two real numbers) (as)

$\Rightarrow \exists$ a rational number γ such that $f(x) < \gamma < \alpha - g(x)$

$$\left\{ x : f(x) + g(x) < \alpha \right\}$$

$$= \bigcup_{\gamma \in \mathbb{Q}} \left\{ x : f(x) < \gamma \right\} \cap \left\{ x : g(x) < \alpha - \gamma \right\}$$

Since f and g measurable functions and \mathbb{Q} is countable. Therefore R.H.S being countable union of measurable is measurable i.e. R.H.S is measurable

\Rightarrow L.H.S is measurable

$\Rightarrow f+g$ is measurable

(272) $C \subseteq \mathbb{R}$
 $f: C \rightarrow \mathbb{R}$
 $f(x) = \sqrt{x}$
 $x \in C$
 (248)

Now as f is measurable.
 $\Rightarrow C \subseteq \mathbb{R}$ is also measurable.
 In particular if $C = \mathbb{R}$

$\Rightarrow -f$ is measurable

$\Rightarrow f + (-f) = f - f$ is also measurable.

Now for any real number α

Case I if $\alpha > 0$

$$\{x: f^2(x) < \alpha\} = \{x: f(x) < \sqrt{\alpha}\} \cup \{x: f(x) < -\sqrt{\alpha}\}$$

$$= \{x: f(x) < \sqrt{\alpha}\} \cup \{x: f(x) > -\sqrt{\alpha}\}$$

if $\alpha \leq 0$

as R.H.S is measurable being the union of measurable set. so L.H.S is measurable.

Now $f(x) > -\alpha = x$

$\alpha < 0$

$$\{x: f^2(x) > \alpha\} = \text{Domain of } f$$

i.e. D

where D is the domain of which is measurable. so $\{x: f^2(x) > \alpha\}$ is measurable.

Hence f^2 is measurable for all α .

$$\text{Now } fg = \frac{1}{2} [(f+g)^2 - (f-g)^2]$$

Since f & g are measurable therefore f^2 and g^2 and $f+g$ are also

(249)

$$f \vee g(x) = \max\{f(x), g(x)\}$$

measurable. also cf' is measurable. Therefore R.H.S is measurable. Hence L.H.S is measurable.

so $f \cdot g$ is measurable.

Proposition: Let f & g be measurable function. Then

$$(i) f \vee g \quad (ii) f \wedge g$$

Let α be any real number

$$\psi \quad f \vee g = \max\{f(x), g(x)\}$$

$$\Rightarrow \{x: (f \vee g)(x) > \alpha\}$$

$$= \{x: f(x) > \alpha\} \cup \{x: g(x) > \alpha\}$$

Since R.H.S is measurable being union of two measurable set. So L.H.S is measurable therefore

$$\{x: (f \vee g)(x) > \alpha\} \text{ is measurable}$$

ii) Let α be any real number

$$\psi \quad f \wedge g = \min\{f(x), g(x)\}$$

$$\{x: (f \wedge g)(x) > \alpha\} = \{x: f(x) > \alpha\} \cap \{x: g(x) > \alpha\}$$

R.H.S measurable. being intersection

mp

Almost Everywhere:

A property is said to be "Almost everywhere" if the set of points where does not hold is a set of measure zero.

Proposition

If f is measurable function and $f = g$ (almost everywhere) then g is measurable.

Proof: Let D be the domain of both function f and g and let E be the set where $f \neq g$. Then by hypothesis $m^* E = 0$.

Let α be any real number. Then the set

$$\{x \in D \mid g(x) > \alpha\} = \{x \in D \mid f(x) > \alpha\} \cup \{x \in E \mid g(x) > \alpha\}$$

Now the 1st set on R.H.S is measurable by hypothesis i.e. f is measurable and the 2nd set being the subset of E of measure zero is also measurable. Hence R.H.S is measurable being the union of two measurable sets therefore L.H.S is also measurable.

$$(25) \quad \begin{cases} \{x : f(x) < \alpha \\ \{x : f(x) > \alpha \\ \alpha > f'(a) \end{cases} \quad \begin{matrix} f: [0, \infty) \rightarrow (0, \infty) \\ \{0, 1\} \end{matrix}$$

* The characteristic function χ_A of A is measurable iff A is measurable

Proof: Suppose that χ_A is measurable. Then the set

$$\{x \mid \chi_A(x) > 0\} = A \quad \text{By definition of characteristic function}$$

Hence A is measurable.

Conversely suppose that A is measurable.

$$\{x \mid \chi_A > 0\} = \begin{cases} A & \text{if } 0 < \alpha < 1 \\ \mathbb{R} & \alpha < 0 \\ \emptyset & \alpha \geq 1 \end{cases}$$

Now A is measurable by assumption and \mathbb{R} and \emptyset are measurable. Hence χ_A is measurable.

* Let f be measurable function. Then for any interval I , $f^{-1}(I)$ is measurable.

Proof: Let $I = (a, b) = (a, \infty) \cap (-\infty, b)$

$$\begin{aligned} f^{-1}(I) &= f^{-1}[(a, \infty) \cap (-\infty, b)] \\ &= f^{-1}(a, \infty) \cap f^{-1}(-\infty, b) \\ &= \{x \mid f(x) > a\} \cap \{x \mid f(x) < b\} \end{aligned}$$

Both the sets on R.H.S are measurable. Hence

$f^{-1}(I)$ is measurable

$$\pm f(x) > \alpha.$$

$$f(x) > \alpha, f(x) < -\alpha \quad (252)$$

A \checkmark * $|f|$ is mble if f is mble.

Proof let α be any real number
Then $\alpha = 0$

$$\{x : |f(x)| > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : f(x) < -\alpha\}$$

R.H.S mble being the union of two mble sets hence $|f|$ is mble.

if $\alpha < 0$.

$$\{x : |f(x)| > \alpha\} = D$$

R.H.S mble hence $|f|$ is mble.

if $\alpha > 0$

$$\{x : |f(x)| > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : f(x) < -\alpha\}$$

Again $|f|$ is mble

$\frac{1}{f}$ is mble if f is mble:

$$\{x : \frac{1}{f(x)} > \alpha\} = D - \{x : f(x) \geq \alpha'\} \quad \alpha' = \frac{1}{\alpha}$$

$$= \{x : f(x) < \alpha'\}$$

Since R.H.S mble therefore R.H.S is mble. That

$\frac{1}{f}$ is mble.

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Problem: Let D and E be mble sets and f be real value function with domain $(D \cup E)$. Then f is mble iff its restriction to D and E are mble.

Proof Let α be any real value and f be mble function.

$$\{x \mid f|_D(x) > \alpha\} = \{x \in D \cup E \mid f(x) > \alpha\} \cap D$$

Since 1st set on R.H.S is mble as f is mble and D is measurable by hypothesis hence restriction of $f|_D$ is measurable. Similarly $f|_E$ is mble.

Conversely let $f|_D$ and $f|_E$ are mble. Then for any real number α .

$$\begin{aligned} \{x \mid f(x) > \alpha \mid x \in D \cup E\} \\ = \{x \mid f|_D(x) > \alpha\} \cup \{x \mid f|_E(x) > \alpha\} \end{aligned}$$

Since R.H.S measurable therefore L.H.S is measurable. Hence

f is measurable of Domain

$D \cup E$.

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Problem Let D is a mble set and f be a function with Domain D Then f is mble \iff the function g defined by

$$f(x) = g(x) \text{ if } x \in D \text{ and}$$

$$g(x) = 0 \text{ if } x \notin D \text{ is measurable.}$$

Proof Let α be any real number and f is measurable. we will have to prove that g is measurable. Let E be the domain of g : Then E is mble $\forall \alpha > 0$

$$\{x : x \in E \mid g(x) > \alpha\} = \{x \in D \mid f(x) > \alpha\}$$

Since the R.H.S is mble therefore L.H.S is mble

Now if $\alpha < 0$

$$\{x : x \in E \mid g(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup (E - D)$$

Again R.H.S mble therefore L.H.S mble.

Hence g is mble.

Conversely Let g is measurable then the set

$$\{x \in D \mid f(x) > \alpha\} = \{x \in E \mid g(x) > \alpha\} \cap D$$

R.H.S measurable. Therefore

f is mble.

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2nd

*) If $f: E \rightarrow \mathbb{R}$ is continuous and E is mble then f^{-1} is also mble.

Proof Let α be any real number then we have to show that

$$B = \{x \in E \mid f(x) > \alpha\} \text{ is mble.}$$

Let

x be an arbitrary point of B . Then $f(x) > \alpha$ and f is continuous at x . So there is open interval U_x containing x such that $f(y) > \alpha \forall y \in U_x$.

Then

$$E \cap U_x \subseteq B$$

Let

$$U = \bigcup_{x \in B} (U_x)$$

$$E \cap U = E \cap \bigcup_{x \in B} (U_x)$$

$$= \bigcup_{x \in B} (E \cap U_x)$$

Since $E \cap U_x \subseteq B$ for every $x \in B$,

$$\Rightarrow E \cap U = \bigcup_{x \in B} (E \cap U_x) \subseteq B \quad \text{--- (1)}$$

$$\text{But } B \subseteq E \cap U \quad \text{--- (2)}$$

From (1) & (2) we have

$$B = E \cap U \quad \checkmark$$

U is open being union of open interval and hence measurable. Since E is also mble therefore $E \cap U$ is mble Hence

$B = \{x : f(x) > \alpha\}$ is mble.

NOTE: If f and g are boreal mble then then $g \circ f$ is also boreal mble.

Boreal Measurable:

A function is called Boreal measurable if for every real number α the set

$\{x \mid f(x) > \alpha\}$ is boreal set.

Theorem If f is mble and B a Boreal set then $f^{-1}(B)$ is mble. (see).

Proof:- By definition of boreal set we know $f^{-1}(B)$ is mble for an open set B .

Now every open set is a union of countable collection of open interval then

$$\begin{aligned} B &= \cup I_i \\ f^{-1}(B) &= f^{-1}(\cup I_i) = \cup f^{-1}(I_i) \\ &= \cup f^{-1}(a_i, b_i) \\ &= \cup (f^{-1}(-\infty, b) \cap (a_i, \infty)) \end{aligned}$$

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$$f^{-1}(B) = \{ f^{-1}(-\infty, b) \cap f^{-1}(a_i, \infty) \}$$
$$= \{ x : f(x) < b \} \cap \{ x : f(x) > a_i \}$$

Since R.H.S mble being the ~~and~~
Intersection of two mble sets. So R.H.S
i.e

$f^{-1}(B)$ is mble.

* // If f is Borel mble and B is
Borel set Then $f^{-1}(B)$ is also
Borel set.

Proof :: By definition of Borel set it
is sufficient to prove that
 $f^{-1}(B)$ is a Borel set for any open set

\therefore An open set the countable union
of disjoint open interval we have

$$B = \cup I_i$$

$$f^{-1}(B) = f^{-1}(\cup I_i)$$

$$= \cup f^{-1}(I_i)$$

Since set of all Borel sets is
a sigma algebra

So

$$I = (a, b) = (a, \infty) \cap (-\infty, b)$$

$$f^{-1}(I) = f^{-1}(a, \infty) \cap f^{-1}(-\infty, b)$$

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$$= \{x \mid f(x) > a\} \cap \{x \mid f(x) < b\} \quad (1)$$

NOW

$$\begin{aligned} \{x \mid f(x) < b\} &= \{x \mid f(x) \geq b\}^c \\ &= \left[\bigcap_{n=1}^{\infty} \{x \mid f(x) > b - 1/n\} \right]^c \end{aligned}$$

Since intersection of Borel set is Borel set so by definition of Borel set 2nd set on R.H.S of (1) is Borel set being the complement of countable intersection of Borel sets. Hence $f^{-1}(I)$ is Borel set.

* Let f be bounded function and E be measurable set of finite measure then simple function ϕ and ψ

$$\inf \int \psi dx = \sup \int \phi dx$$

f is measurable.

Proof Let f is bounded by M and suppose that f is measurable then the sets

$$E_k = \left\{ x : \frac{kM}{n} \geq f(x) \geq \frac{(k-1)M}{n} \right\}, -n \leq k \leq n$$

are measurable disjoint and have union E .

Thus

$$\sum_{k=-n}^n m E_k = m E$$

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The simple function defined by

$$\psi_n(x) = \frac{M}{n} \sum_{k=-n}^n k \chi_{E_k}(x).$$

and

$$\phi_n(x) = \frac{M}{n} \sum_{k=1}^n (k-1) \chi_{E_k}(x).$$

satisfy

$$\phi_n(x) \leq f(x) \leq \psi_n(x)$$

Thus

$$\inf_E \int \phi_n(x) \leq \int f(x) dx = \frac{M}{n} \sum_{k=-n}^n k m E_k.$$

and

$$\sup_E \int \psi_n(x) dx \geq \int \phi_n(x) dx = \frac{M}{n} \sum_{k=1}^n (k-1) m E_k.$$

$$0 \leq \inf_E \int \psi_n(x) dx - \sup_E \int \phi_n(x) dx \leq \frac{M}{n} \sum_{k=-n}^n m E_k = \frac{M}{n} m A.$$

Since n is arbitrary we have

$$\inf_E \int \psi(x) dx - \sup_E \int \phi(x) dx = 0$$

$$\inf_E \int \psi(x) dx = \sup_E \int \phi(x) dx.$$

and the condition is sufficient

Problem:

Let D and E be measurable sets and f be real value function on $D \cup E$. Then f is measurable

Conversely Suppose that

$$\inf \int \psi(x) dx = \sup \int \phi(x) dx.$$

Then given n , there are simple functions

$$\phi_n(x) \text{ and } \psi_n(x)$$

$$\phi_n(x) \leq f(x) \leq \psi_n(x).$$

and

$$\int \psi_n(x) dx - \int \phi_n(x) dx < \frac{1}{n}$$

Then the functions

$\psi^* = \inf \psi_n$ and $\phi^* = \sup \phi_n$ are measurable by hypothesis and

$$\phi^*(x) \leq f(x) \leq \psi^*(x)$$

Now the set

$$\Delta = \{x; \phi^*(x) < \psi^*(x)\}$$

is the union of the sets

$$\Delta_V = \{x; \phi^*(x) < \psi^*(x) - 1/V\}$$

But each of the sets contained in the set

$$A_V = \{x; \phi_n(x) < \psi_n(x) - 1/V\} \text{ and}$$

this latter set has measure less than $1/n$

since n is arbitrary $m \Delta_V = 0$ and

so $m \Delta = 0$ Thus $\phi^* = \psi^*$ except on a set of measure zero. Thus f is measurable.