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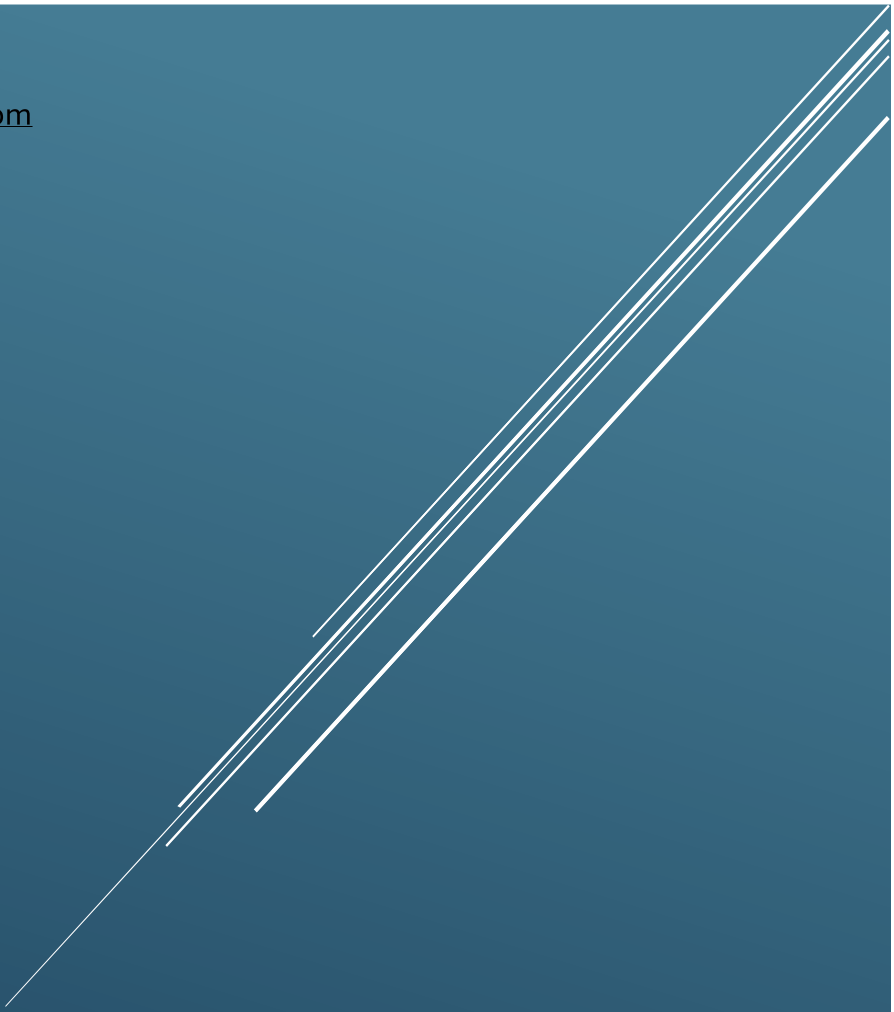

ADVANCE FUNCTIONAL ANALYSIS

Waseem Akram

Dedicated to

My Dearest Mam

Miss. Arsla Afzal

&

My Parents

Collected and compose by Waseem Akram

Lectures(2025)

Miss.Arsla Afzal (Lecturer)

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Book;

Introductory functional Analysis with Applications

(Erwin Kreyszig)

Collected and compose by Waseem Akram

Fundamental Theorem for normed and banach space

Partially Order Set:

A partially order set is a set M on which there is a defined partial ordering ,i.e a bianry relation written as $\leq$ and satisfies the following condition

1. Reflexivity: $a \leq a$ for every $a \in M$.

2.Anti-symmetry: If $a \leq b$ and $b \leq a$ then $a = b$.

3. Transitive: If $a \leq b$ and $b \leq c$ then $a \leq c$.

Comparable:

Two elements a and b are said to be comparable if $a \leq b$ and $b \leq a$ or both.

Totally Order Set:

A totally order set is partial order set such that every two elements of the set is comaparable.

Maximal elements:

In partially ordered set $(P, \leq)$ an element $m \in P$ is called a maximal element if there is no other element $x \in P$ such that $m \leq x$ and $m \neq x$.

Example#

Set of real number is totally order and has no maximal element.

Zorn lemma:

Let $M \neq \phi$ be a partially ordered set .Suppose that every chain $C \subset M$ has an upper bound .Then M hasat least one maximal element .

Hahn-Banach space Theorem:

Let X be a real vector space and sublinear functional P defined on X.Morever,Let f be a linear functional defined on subspace Z of X $(Z \subset X)$ s.t

$$f(x) \leq p(x) \quad \forall x \in Z$$

Then there is a linear extension f' of f from Z to X s.t

$$f'(x) \leq P(x) \qquad \forall x \in X$$

Proof: Let E be the set of all linear extension g of f i.e

$$E = \{g | f \leq g\} \qquad \text{such that } g(x) \leq P(x) \forall x$$

$E \neq \phi$ (at ;least $f \in E$) Define partial ordering on E , $g \leq h$ (h is extension of g) . By defination

$$g(x) = h(x) \qquad \forall x \in D(g), D(g) \subseteq D(h)$$

1. Reflexvie: $(E, \leq)$ is reflexive because $g \leq g$ (every functional is extension of itself) .

2. Anti-symmetric : $(E, \leq)$ is anti-symmetric because if $g \leq h$ and $h \leq g$ then $g = h$.

3. Transitive: $(E, \leq)$ is transitive because if $g \leq h$ and $h \leq i$ then $g \leq i$.

Hence, $(E, \leq)$ is partial ordering .Consider a chain $C \subset E$ and define $g^{\hat{}}$ on C .

$$\hat{g}(x) \leq g(x) \qquad g \in C \ \forall x \in D(g)$$

Clearly, $D(\hat{g}) = \bigcup_{g \in C} D(g)$. Further, if $\hat{g}$ is related with any $g \in C$ .we may write

$$g \leq \hat{g} \text{ i.e } \hat{g}(x) = g(x) \ \forall x \in g \text{ and } D(g) \subset D(\hat{g})$$

Hence, $\hat{g}$ is upper bound of C .Since, C was arbitary So, every chain of E has an upper bound .By zorn's lemma E must have maximal element say f' and by defination of E .

$$\begin{aligned} f'(x) &\leq P(x) \cdots\cdots\cdots (1) \\ \forall x &\in D(f'(x)) \end{aligned}$$

Next we have to show that $D(f'(x)) = X$.

Suppose on contray ,let $D(f') \neq X$.Let $y \in x|D(f')$ clearly, $y_1 \neq 0$ as $0 \in D(f')$ .Consider a subspace y_1 spanned by Y_1 and $D(f')$ i.e $x \in Y_1$

$$\begin{aligned} x &= y + \alpha y_1 \cdots\cdots (2) \\ y &\in D(f') \text{ and } \alpha \text{ is scalar} \end{aligned}$$

Define linear functional g on Y_1

$$g_1(x) = g_1(y + \alpha y_1) = f'(y) + \alpha C \cdots\cdots\cdot (3)$$

where c is constant if $\alpha = 0$ then $g_1(x) = g_1(y) = f'(y)$ or $g_1(x) = f'(y)$ $x \in D(g_1)$ .Clearly, g_1 ios extenion of $f\ '$.Also from (1)

$$\begin{aligned} g_1(x) &\leq P(x) \cdots\cdots (4) \\ \forall x &\in D(y) \end{aligned}$$

Let

$$\begin{aligned} y, Z &\in D(f') \\ f'(y) - f'(Z) &= f'(y - Z) \because f \text{ is linear} \\ &\leq P(y - Z) \text{ from } (1) \\ &= P(y + y_1 - y_1 - Z) \\ &\leq P(y + y_1) + P(-y_1 - Z) \text{ By triangular inequlity} \end{aligned}$$

OR

$$-P\left(-y_1-Z\right)-f'\left(Z\right)\leq P\left(y+y_1\right)-f'\left(y\right)$$

Taking supremum over left side $m_\circ$ and infimum over right side m_1 i.e $m_\circ \leq m_1$.Consider a C such that $m_\circ \leq C \leq m_1$. we may write above inequality as

$$-P\left(-y_1-Z\right)-f'\left(Z\right)\leq C\leq P\left(y+y_1\right)-f'\left(y\right)$$

or

$$\begin{aligned} -P\left(-y_1-Z\right)-f'\left(Z\right) &\leq C\cdots\cdots(5) \\ C &\leq P\left(y+y_1\right)-f'\left(y\right)\cdots\cdots(6) \end{aligned}$$

For $\alpha<0$ put $Z=y$ in (5)

$$-P\left(-y_1-\frac{1}{\alpha}y\right)-f'\left(\frac{1}{\alpha}y\right)\leq C$$

multiplying with $-\alpha>0$

$$\begin{aligned} \alpha P\left(-y-\frac{1}{\alpha}y\right)+\_\alpha f'\left(\frac{1}{\alpha}y\right) &\leq \alpha C \\ \alpha C+f'\left(y\right) &\leq -\alpha P\left(-y_1-\frac{1}{\alpha}y\right) \\ &\leq P\left(\alpha y_1+y\right) \end{aligned}$$

from (2) and (3)

$$g_1\left(x\right)\leq P\left(x\right) \qquad \forall x\in D\left(g\right)$$

if $\alpha=0$ from (4) we already have

$$g_1\left(x\right)\leq P\left(x\right)\ \ \forall x\in D\left(g_1\right)$$

if $\alpha>0$ replace $y=\frac{1}{\alpha}y$ in (6)

$$C\leq P\left(\frac{1}{\alpha}y+y_1\right)-f'\left(\frac{1}{\alpha}y\right)$$

Multiplying with α

$$\begin{aligned} \alpha C &\leq \alpha P\left(\frac{1}{\alpha}y+y_1\right)-f'\left(\frac{1}{\alpha}y\right) \\ &\leq P\left(y+\alpha y_1\right)-f'\left(y\right) \end{aligned}$$

P and f' are linear

$$f'\left(y\right)+\alpha C\leq P\left(y+\alpha y_1\right)$$

from (2) and (3)

$$g_1\left(x\right)\leq P\left(x\right) \qquad \forall x\in D\left(g_1\right)$$

Hence, for any chose of α

$$g_1(x) \leq P(x) \qquad \forall x \in D(g_1)$$

$g_1 \in E$ and g_1 being the extension of f' which contraduct the maximality of f'

$$\Rightarrow D(f') \neq X$$

which is wrong

$$\Longrightarrow D(f') = X$$

Generalized Hahn-Banach Theorem: Let X be a complex or real vector space .A functional P is defined on X which is subadditive i.e

$$P(x+y) \leq P(x) + P(y) \;\; \forall x \in X$$

and satisfies

$$P(\alpha x) = |\alpha| P(x) \qquad \forall x \in X$$

Furthermore, there isa subspace Z of X on which a linear functional f is defined and it satisfies $|f(x)| \leq P(x) \quad \forall x \in X$ .Then there is linesr extension f' of f from Z to X s.t

$$|f'(x)| \leq P(x) \qquad \forall x \in X$$

Proof:

(a) **Let** X is real then we write

$$\begin{aligned} f(x) &\leq P(x) \cdots\cdots (1) \qquad \forall x \in X \\ -f'(x) &= f'(-x) \leq P(-x) = |-1| P(x) = P(x) \qquad \text{Using (1) and above condition} \\ -f'(x) &\leq P(x) \;\; \cdots\cdots\cdots (2) \qquad \forall x \in X \end{aligned}$$

(b) Let X is complex then Z is alsdo complex So, that f is complex valued fucntion i.e

$$f(x) = f_1(x) + i f_2(x) \qquad \forall x \in Z$$

where f_1 and f_2 are real valued w.r.t Z_r and X_r (Real spaces)

$$f_1(x) \leq |f(x)|$$

$\because$ Real part of complex number never exceeds absoulte value and

$$\begin{aligned} f_1(x) &\leq |f(x)| \leq P(x) \qquad \forall x \in Z \\ &\Rightarrow f_1(x) \leq P(x) \qquad \forall x \in Z_r \end{aligned}$$

Since, Z_r is real .So, by Hahn Banach theorem is linear f_1 w.r.tX_r s.t

$$f_1'(x) \leq P(x) \cdots\cdots (3) \qquad \forall x \in X_r$$

Next

$$\begin{aligned} if\left(x\right) &= i\left[f_1\left(x\right)+if_2\left(x\right)\right]=f_1\left(ix\right)+f_2\left(ix\right) \\ if_1\left(x\right)-f_2\left(x\right) &= f_1\left(ix\right)+if_2\left(ix\right) \end{aligned}$$

Comparing real parts at both sides

$$-f_2\left(x\right)=f_1\left(ix\right)-if_1\left(ix\right) \qquad \forall x\in Z$$

Consider an extension f' of f from Z to X s.t

$$f'\left(x\right)=f_1'\left(x\right)-if_1'\left(ix\right) \qquad \forall x\in X$$

$\because$ f' is linear

$$\begin{aligned} f(\left(a+ib\right)x) &= f_1'\left(\left(a+ib\right)x\right)-if_1'\left(\left(a+ib\right)x\right) \\ &= af_1'\left(x\right)+ibf_1'\left(x\right)-i\left[af_1'\left(x\right)+ibf_1'\left(ix\right)\right] \end{aligned}$$

Using (4) and f is linear

$$\begin{aligned} &= \left(a+ib\right)f_1'\left(x\right)-i\left(a+ib\right)f_1'\left(ix\right) \\ &= \left(a+ib\right)f_1'\left(x\right) \end{aligned}$$

Hence, f is linear .

Since, $f'\left(x\right)$ is complex number . we may it write it as

$$\begin{aligned} f'\left(x\right) &= \left|f'\left(x\right)\right|e^{i\theta} \\ &\Longrightarrow \left|f'\left(x\right)\right|=f'\left(x\right)e^{-i\theta} \\ &= f'\left(e^{-i\theta}x\right) \end{aligned}$$

$\because$ f is linear

$$\left|f'\left(x\right)\right|=f'\left(e^{-i\theta}x\right)=f_1'\left(e^{-i\theta}x\right)-if_1'\left(e^{-i\theta}x\right)$$

Since, L.H.S is real .So, by comparing the real part .

$$\begin{aligned} \left|f'\left(x\right)\right| &= f_1'\left(e^{-i\theta}x\right) \\ &\Rightarrow \left|f'\left(x\right)\right|\leq P\left(e^{-i\theta}x\right) \end{aligned}$$

Using (4)

$$\begin{aligned} &\Rightarrow \left|f'\left(x\right)\right|=\left|e^{-i\theta}\right|P\left(x\right) \\ &= P\left(x\right) \qquad \forall x\in X \end{aligned}$$

Hence,

$$\left|f'\left(x\right)\right|\leq P\left(x\right) \qquad \forall x\in X$$

Hence, proved.

Hahn Banach Theorem (Normalized Space) :

Let X be a normed space .A bounded linear functional f defined on subspace Z of X .Then there is functional f' on X .|So, that

$$||f'||_x = ||f||_z$$

where

$$\begin{aligned} ||f'||_x &= \sup_{x\in X \ \ , \ ||X||=1} |f'(x)| \\ ||f||_z &= \sup_{x\in z \ , \ ||z||=1} |f(x)| \end{aligned}$$

Proof :

Since f is bounded linear functional defined on Z .By defination norm of f.

$$\begin{aligned} ||f||_z &= \sup_{x\in z \ \ ,||x||\neq 0} \frac{|f(x)|}{||x||} \\ &\Longrightarrow \frac{|f(x)|}{||x||} \le ||f||_z \qquad \forall x \in z \\ &\Longrightarrow f(x) \le ||f||_z \, ||x|| \qquad \forall x \in z \end{aligned}$$

Let $||f||_z \, ||x|| = P(x)$ So, that we can write

$$|f(x)| \le P(x) \qquad \forall x \in z$$

Morever,

$$\begin{aligned} P(x+y) &= ||f||_z \, ||x+y|| \le ||f||_z \left(||x|| + ||y||\right) \\ &= ||f||_z \, ||x|| + ||f||_z \, ||y|| \\ &= P(x) + P(y) \end{aligned}$$

i.e

$$P(x+y) \le P(x) + P(y)$$

$\because P$ is subaddtive and

$$\begin{aligned} P(\alpha x) &= ||f||_z \, ||\alpha x|| = |\alpha| \, ||f||_z \, ||x|| \\ &= P(\alpha x) = |\alpha| \, P(x) \end{aligned}$$

Hence, by using hahn banach theorem(*Generalized*) there is an extrension f' of f s.t

$$\begin{aligned} |f'(x)| &\le P(x) \qquad \forall x \in X \\ &\Rightarrow |f'(x)| \le ||f||_z \, ||x|| \\ &\Rightarrow \frac{|f'(x)|}{||x||} \le ||f||_z \qquad \forall x \in X \end{aligned}$$

Taking supremum over all values of X with $||x|| \neq 0$

$$\begin{aligned} \sup_{x \in X, ||x|| \neq 0} \frac{|f'(x)|}{||x||} &\leq ||f||_z \\ &\Rightarrow ||f'||_x \leq ||f||_z \end{aligned}$$

Since, f' is extension of f so $||f||$ never exceeds $||f'||$.

Hence, $||f'||_x = ||f||_z$

Self Adjoint Operator:

Let $T : X \longrightarrow Y$ be bounded linear operater ,where X and Y are normned spaces .Then the adjoint operator $T^{\times} : Y' \longrightarrow X'$ of T is defined as

$$f(x) = \left(T^{\times} g\right)(x) = g(Tx)$$

where X' and Y' are dual spaces of X and Y are respectively,

Theorem:

$T^{\times}$ is bounded linear and satisfies

$$\left|\left|T^{\times}\right|\right| = ||T||$$

Proof: $T^{\times}$ is linear

$$\begin{aligned} T^{\times}(\alpha g_1 + \beta g_2)(x) &= (\alpha g_1 + \beta g_2)(Tx) \\ &= \alpha g_1(Tx) + \beta g_2(Tx) \qquad \because T \text{ is linear} \\ &= \alpha\left(T^{\times} g_1\right)(x) + \beta\left(T^{\times} g_2\right)(x) \\ &\Longrightarrow T^{\times} \text{ is linear} \end{aligned}$$

Now,we prove $T^{\times}$ is bounded

$$\begin{aligned} f(x) &= \left(T^{\times} g\right)(x) \\ &\Longrightarrow f = T^{\times} g \cdots (1) \\ ||f|| &= \left|\left|T^{\times} g\right|\right| \cdots (2) \\ f(x) &= g(Tx) \cdots (A) \\ |f(x)| &= |g(Tx)| \leq ||g|| \, ||T|| \, ||x|| \\ \frac{|f(x)|}{||x||} &\leq ||g|| \, ||T|| \qquad \forall x \in X \end{aligned}$$

Taking the supremum over all values of X with $||X|| \neq 0$

$$\begin{aligned} \sup_{x \in X, ||x|| \neq 0} \frac{|f(x)|}{||x||} &\leq ||g|| \, ||T|| \\ ||f|| &\leq ||g|| \, ||T|| \qquad \text{by defination of norm} \end{aligned}$$

from (2)

$$\begin{aligned} \left|\left|T^{\times} g\right|\right| &\leq ||g|| \, ||T|| \\ \frac{||T^{\times} g||}{||g||} &\leq ||T|| \qquad \forall g \in Y' \end{aligned}$$

Taking supremum of over all values of Y' with $||g|| \neq 0$

$$\begin{aligned} \sup_{g \in Y', ||g|| \neq 0} \frac{||T^{\times} g||}{||g||} &\leq ||T|| \\ \left|\left|T^{\times}\right|\right| &\leq ||T|| \cdots \cdots (3) \left(\text{ by def of norm of } T^{\times} \right) \end{aligned}$$

This shows $||T^{\times}||$ is bounded . As we know ,for a normed space Y' for any $x_{\circ} \in X$ there is $g_{\circ} \in Y'$ s.t

$$||g_{\circ}|| = 1$$

and $g_{\circ}\left(Tx_{\circ}\right) = ||Tx_{\circ}||$

$$\begin{aligned} ||Tx_{\circ}|| &= g_{\circ}\left(Tx_{\circ}\right) \\ &= f\left(x_{\circ}\right) \end{aligned}$$

from (A)

$$\begin{aligned} ||Tx_{\circ}|| &= \left|f_{\circ}\left(x_{\circ}\right)\right| \leq ||f_{\circ}|| \, ||x_{\circ}|| \\ \frac{||Tx_{\circ}||}{||x_{\circ}||} &\leq ||f_{\circ}|| = \left|\left|T^{\times} g_{\circ}\right|\right| \end{aligned}$$

from (2)

$$\frac{||Tx_{\circ}||}{||x_{\circ}||} \leq \left|\left|T^{\times}\right|\right| ||g_{\circ}|| \qquad \because ||g_{\circ}|| = 1$$

Taking the supremum over all values $x_{\circ} \in X$ with $||x|| \neq 0$

$$\begin{aligned} \sup_{x_{\circ} \in X, ||x_{\circ}||} \frac{||Tx_{\circ}||}{||x_{\circ}||} &\leq \left|\left|T^{\times}\right|\right| (1) \\ ||T|| &\leq \left|\left|T^{\times}\right|\right| \cdots \cdot (4) \end{aligned}$$

combing (3)and (4)

$$||T|| = \left|\left|T^{\times}\right|\right|$$

Hence, proved

Rare OR Nowhere dense :

A subset M of metric space X is said to be rare in X if its closure $\overline{M}$ has no interior points.

Meager Or The first Category:

A subset M of metric space X is said to be meager in X if M is the union of countably many sets each of which is rare in X. A set which is not meager is called non-meager or second category.

Theorem (Baires Category Theorem) :

If a metric space is comlete then X is non-meager in it self i.e

$$X \neq \bigcup_{i=1}^{k} M_k$$

where each M_k rare.

Proof:

Supose on the contary .Let X be meager .

$\Longrightarrow$ $X = \bigcup\limits_{i=1}^{k} M_k$ where each M_k is rare. .Since, M_1 is rare $\overline{M_1}$ does not contain any open set i.e $\overline{M_1} \neq X$ (as X is open) .

$\Longrightarrow$ $\overline{M_1} \neq \phi$ and is open .Let $P_1 \in \overline{M_1}^c$ and choose $\in_1 < \frac{1}{2}$ the open ball

$$B_1\left(P_1, \in_1\right) \subset \overline{M_1}^c \quad \left(\in_1 < \frac{1}{2}\right)$$

By assumpition M_2 is rare .So, $\overline{M_2}$ does not contain any open set

$$\begin{array}{rcl} B_1 & \subset & \overline{M_2} \\ & \Longrightarrow & B_1 \cap \overline{M_2}^c \neq \phi \end{array}$$

for $P_2 \in \left(B_1 \cap \overline{M_2}^c\right)$. consider an open ball

$$\begin{array}{lll} B_2\left(P_2, \in_2\right) & \subset & B\left(P_1, \frac{1}{2} \in_1\right) \subset B_1\left(P_1, \in_1\right) \qquad \left(\in_2 \leq \frac{1}{2} \in_1\right) \Rightarrow \left(\in_2 < \frac{1}{2} . \frac{1}{2}\right) \Rightarrow \in_2 < 2^{-2} \\ B_3\left(P_3, \in_3\right) & \subset & B\left(P_2, \frac{1}{2} \in_2\right) \subset B_2\left(P_2, \in_2\right) \qquad \in_3 < 2^{-3} \end{array}$$

By induction

$$B_{k+1} \subset B\left(P_k, \frac{1}{2} \in_k\right) \subset B_k\left(P_k, \in_k\right) \qquad \in_k < 2^{-k}$$

$\left(P_k\right)$ sequence of centers is cauchy sequence and is convergent ($P_k \longrightarrow P$ (say) when $k \longrightarrow \infty$)as X is complete space because for m, n with $n > m$

$$\begin{array}{lll} B_n\left(P_n, \in_n\right) & \subset & B\left(P_m, \frac{1}{2} \in_m\right) \subset B_m\left(P_m, \in_m\right) \qquad \left(\in_m < 2^{-m}\right) \\ d\left(P_m, P_n\right) & < & \frac{1}{2} \in_m \qquad P_k \text{ is cauchy} \end{array}$$

Now ,

$$d\left(P_m, P\right) \leq d\left(P_m, P_n\right) + d\left(P_n, P\right)$$

when $n \longrightarrow \infty, \left(d\left(P_n, P\right) = 0\right)$ and $d\left(P_m, P\right) < \frac{1}{2} \in_m$

$$\begin{array}{ll} \Longrightarrow & d\left(P_m, P\right) < \frac{1}{2} \in_m +0 = \frac{1}{2} \in_m \\ \Longrightarrow & P \in B_m \subset \overline{M_m}^c \\ \Longrightarrow & P \notin \bigcup\limits_{m=1}^{k} M_m = X \\ \Longrightarrow & P \notin X \end{array}$$

which is contraduction .

Hence, our supposition is wrong

$\implies$ X is non-meager.

Theorem(Uniform Boundness) :

Let (T_n) be a sequence of bounded linear operaters.$T_n : X \longrightarrow Y$ where X is banach space and Y is normed space .The sequence $(||T_n||)$ is bounded that is

$$||T_n x|| \leq C_x \qquad \forall x \in X \qquad \because C_x \text{ is real number} \qquad n = 1, 2, 3 \cdots \cdot$$

Then the sequence of norms $(||T_n||)$ is bounded that is

$$||T_n|| \leq C_x \quad C \text{ is constant} \qquad n = 1, 2, 3 \cdot \cdot$$

Proof: Given that

$$||T_n x|| \leq C_x \qquad \forall \;\; x \in X \;\; n = 1, 2, 3 \cdot \cdot \cdot$$

Let $A_k \subseteq X$ s.t

$$A_k = \{x \in X, ||T_n x|| \leq k\}$$

Let (x_i) be a convergent sequence of A_ki.e$(x_i \longrightarrow x)$ .Since $x_i \in A_k$ $||T_n x_i|| \leq k$ (by defination of A_k) .Since T_n is sequence of bounded linear operator

$\implies$ T_n is continous

$$\begin{aligned} &\implies \quad ||T_n x|| \leq k \\ &\implies \quad x \in A_k \end{aligned}$$

Hence, A_k is closed .or $A_k = \overline{A_k}$ .as we write

$$X = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \overline{A_k}$$

and since X is banach (*complete*) space ,so by baire's category theorem at least one $A_{k_\circ}$ contains an open ball (say) $B_\circ$.

$$B_\circ (x_\circ, r) \subset A_{k_\circ}$$

Let $x \in X$ and set Z

$$\begin{aligned} Z \quad &= \quad x_\circ + \gamma x \qquad \text{where} \quad \gamma = \frac{r}{2\,||x||} \\ &\Rightarrow \quad Z - x_\circ = \gamma x \cdot \cdot \cdot \cdot \cdot (1) \\ &\Rightarrow \quad ||Z - x_\circ|| = \gamma\,||x|| = \frac{r}{2\,||x||}\,||x|| = \frac{r}{2} \\ &\implies \quad ||z - x_\circ|| < r \\ &\Rightarrow \quad Z \in B_\circ (x_\circ, r) \subset A_{k_\circ} \\ &\implies \quad Z \in A_{k_\circ} \\ &\implies \quad ||T_n Z|| \leq k_\circ \qquad (\text{ by defination of } A_{k_\circ}) \\ \text{also } x_\circ \quad &\in \quad A_{k_\circ} \\ &\Rightarrow \quad ||T_n x_\circ|| \leq k_\circ \end{aligned}$$

from(1)

$$x = Z - x_\circ$$

Applying T_n.

$$\begin{aligned} T_n x &= \frac{1}{\gamma} T_n (Z - x_\circ) \\ ||T_n x|| &= \frac{1}{\gamma} \, ||T_n \, (Z - x_\circ)|| \\ &\leq \frac{1}{\gamma} \, (||T_n Z|| + ||T_n x_\circ||) \\ &\leq \frac{2 \, ||x||}{r} \, (k_\circ + k_\circ) \\ \frac{||T_n x||}{||x||} &\leq \frac{4k_\circ}{r} \end{aligned}$$

Taking supremum over all $x \in X$ with $||x|| \neq 0$

$$\begin{aligned} \sup_{x \in X, ||x|| \neq 0} \frac{||T_n x||}{||x||} &\leq \frac{4k_\circ}{r} = \vec{C} \, (say) \\ ||T_n|| &\leq C \end{aligned}$$

Hence, proved.

Reflexive Spaces:

A space X is said to be algebrically reflexive if the canonical mapping $C : X \longrightarrow X''$ is surjective .Here, X'' is the second algebric dual space of X and the mapping C is defined by $x \longrightarrow g_x$ where

$$g_x \, (f) = f \, (x) \qquad\qquad x \in X'$$

Lemma (Norm of g$_x$) :

For every fixed x in a normed space X .The functional g_x is a bounded linear on X'.So that the $g_x \in X''$ and has the norm

$$||g_x|| = ||x||$$

Strong Convergence :

A sequence (x_n) in a normed space X is said to be strongly convergent if there is an $x \in X$ such that

$$\lim_{n \longrightarrow \infty} ||x_n - x|| = 0$$

and it is written as

$$\lim_{n \longrightarrow \infty} x_n = x$$

Weak Convergence:

A sequence (x_n) in normed space X is said to be weaky convergent if there is an $x \in X$ such that for evrey $f \in X'$

$$\lim_{n \longrightarrow \infty} f(x_n) = f(x)$$

and it is written as

$$x_n \longrightarrow x \quad (weakly)$$

Lemma:

Let (x_n) be weakly convergent sequence in a normed space X, say $x_n \longrightarrow x\,(weakly)$ then

(1) The weak limit x of (x_n) is unique.

(2) Every subsequence of (x_n) converges weakly to x .

(3)The sequence $(||x_n||)$ is bounded.

Proof:

(1) Given that

$$\begin{aligned} x_n &\longrightarrow x\,(\text{weakly}) \\ &\Longrightarrow \lim_{n \longrightarrow \infty} f(x_n) = f(x) \end{aligned}$$

To prove lim of x_n is unique.

Let x and y be the limit of sequence .i.e

$$\lim_{n \longrightarrow \infty} f(x_n) = f(x) \qquad \text{and} \quad \lim_{n \longrightarrow \infty} f(x_n) = f(y)$$

$f(x_n)$ is sequence of real no.s . So, it has unique limit .

$$\begin{aligned} &\Longrightarrow \quad f(x) = f(y) \\ &\Longrightarrow \quad f(x) - f(y) = 0 \\ &\Longrightarrow \quad f(x - y) = 0 \quad \because f \text{ is linear} \\ &\Longrightarrow \quad x - y = 0 \\ &\Longrightarrow \quad x = y \end{aligned}$$

(2) To prove every subsequence of x_n converges weakly to x.Let

$$\begin{aligned} x_n &\longrightarrow x\,(weakly) \\ &\Longrightarrow f(x_n) \longrightarrow f(x) \end{aligned}$$

$\Longrightarrow$ $f(x_n)$ is convergent sequence of real numbers.So, every subsequence of $f(x_n)$ is also convergent

$\Longrightarrow$ Every subsequence of $f(x_n)$ converges weakly.

(3) Let $x_n \longrightarrow x\,(weakly)$.

$\Longrightarrow$ $f(x_n)$ is convergent sequence of real numbers .

$\Longrightarrow$ $f(x_n)$ is bounded

$$\Longrightarrow \ |f(x_n)| < C_f \, (C_f \ \text{ is constant } \ \text{ where } f \in X')$$

Now, using canonical mapping

$$\begin{aligned} C \quad &: \quad X \longrightarrow X'' \\ g_n(f) \quad &= \quad f(x_n) \quad \forall f \in X' \\ &\Longrightarrow \quad |g_n(f)| = |f(x_n)| \leq C_f, f \in X' \\ &\Longrightarrow \quad g_n(f) \ \text{ is bounded.} \end{aligned}$$

Also we know that X' is banach space .So, by uniform boundness theorem

$$||g_n|| \leq (f \in X')$$

that is $(||g_n||)$ is bounded.Now, Since g_n is bounded linear operator on X' with

$$||g_n|| = ||x_n|| \leq C$$

$\Longrightarrow$ $||x_n||$ is bounded .

Theorem:

Let (x_n) be a sequence in normed space X Then.

(1) Strong convergence implies convergence.

(2) If $\dim X < \infty$,then weakly convergence implies strong convergence.

Proof:

(1) Given that $x_n \longrightarrow x$ that is $\lim\limits_{n \longrightarrow \infty} ||x_n - x|| = 0$

$$\begin{aligned} |f(x_n) - f(x)| \quad &= \quad |f(x_n - x)| \quad (\because f \text{ is linear}) \\ &\leq \quad ||f|| \, ||x_n - x|| \qquad (\because |f(x)| \leq ||f|| \, ||x||) \end{aligned}$$

when $n \longrightarrow \infty$

$$\begin{aligned} \lim_{n \longrightarrow \infty} |f(x_n) - f(x)| \quad &\leq \quad ||f|| \lim_{n \longrightarrow \infty} ||x_n - x|| \\ \lim_{n \longrightarrow \infty} |f(x_n) - f(x)| \quad &\leq \quad 0 \\ &\Longrightarrow \quad x_n \longrightarrow x \ \ (weakly) \end{aligned}$$

(2) Given that $x_n \longrightarrow x \, (weakly)$ that is $\lim\limits_{n \longrightarrow \infty} |f(x_n) - f(x)| = 0$ and consider $\dim X = k \, (finite\ number)$ and $\{e_1, e_2, \cdots \cdot e_k\}$ be the set of basis of X .Then

$$x_n = \alpha_1^{(n)} e_1 + \alpha_2^{(n)} e_2 + \cdots + \alpha_k^{(n)} e_k = \sum_{i=1}^{k} \alpha_i^{(n)} e_i \qquad \text{where } \ (n) \ \text{ is super subscripit}$$

and

$$x = \{\alpha_1 e_1 + \alpha_2 e_2, + \cdots \cdot + \alpha_k e_k\} = \sum_{i=1}^{k} \alpha_i e_i$$

consider $\{f_1, f_2, \cdots f_k\}$ set of basis of X' s.t

$$\begin{aligned}
f_j(e_j) &= 1, f_j(e_m) = 0 \, (j \neq m) \\
f_j(x_n) &= \alpha_j^{(n)}, f_j(x) = \alpha_j \\
\because & \lim_{n \longrightarrow \infty} |f_j(x_n) - f_j(x)| = 0 \\
\Longrightarrow & \lim_{n \longrightarrow \infty} \left| \alpha_j^{(n)} - \alpha_j \right| = 0
\end{aligned}$$

Now,

$$\begin{aligned}
||x_n - x|| &= \left|\left| \sum_{i=1}^{k} \alpha_j^{(n)} e_j - \sum_{i=1}^{k} \alpha_j e_j \right|\right| \\
&= \left|\left| \sum_{i=1}^{k} \left(\alpha_j^{(n)} - \alpha_j \right) e_j \right|\right| \\
&\leq \sum_{i=1}^{k} \left| \alpha_j^{(n)} - \alpha_j \right| ||e_i||
\end{aligned}$$

when $n \longrightarrow \infty$

$$\begin{aligned}
\lim_{n \longrightarrow \infty} ||x_n - x|| &= \sum_{i=1}^{k} \lim_{n \longrightarrow \infty} \left| \alpha_j^{(n)} - \alpha_j \right| ||e_j|| \\
\lim_{n \longrightarrow \infty} ||x_n - x|| &= 0 \qquad \because \lim_{n \longrightarrow \infty} \left| \alpha_j^{(n)} - \alpha_j \right| = 0 \\
&\Longrightarrow x_n \longrightarrow x
\end{aligned}$$

Theorem:

$B(x_\circ, r) = x_\circ + rB(0;1)$

Proof:

Let

$$\begin{aligned}
x &\in B(x_\circ, r) \\
&\Longrightarrow |x - x_\circ| < r \, (r > 0) \\
&\Longrightarrow \frac{|x - x_\circ|}{r} < 1
\end{aligned}$$

Let

$$\begin{aligned}
Z &= \frac{x - x_\circ}{r} \\
|Z| &= \frac{|x - x_\circ|}{1} < 1 \, (\because r > 0) \\
|Z| &< 0 \Longrightarrow Z \in B(0;1) \\
Z &= \frac{x - x_\circ}{r} \Rightarrow x = x_\circ + rZ \in x_\circ + rB(0;1)
\end{aligned}$$

$\because x$ was arbitrary

$$\Longrightarrow B(x_\circ; r) \subseteq x_\circ + rB(0;1) \cdots\cdots (1)$$

Let

$$\begin{aligned} x &\in x_\circ + rB(0;1) \\ &\Longrightarrow x = x_\circ + rZ \qquad\qquad Z \in B(0;1) \text{ i.e } |Z| < 1 \\ x - x_\circ &= rZ \\ |x - x_\circ| &= r|Z| < r(1) \qquad\quad \because |Z| < 1 \\ |x - x_\circ| &< r \quad OR\ x \in B(x_\circ, r) \\ &\Longrightarrow x_\circ + rB(0;1) \subseteq B(x_\circ; r) \cdots\cdots (2) \end{aligned}$$

Combining (1) and (2)

$$x_\circ + rB(0,1) = B(x_\circ; r)$$

oR we can write

$$B(0;1) = \frac{1}{r}[B(x_\circ; r) - x_\circ]$$

Remarks:

(1) A mapping $T : X \longrightarrow Y$ is continous iff $T^{-1} : Y \longrightarrow X$ is open mapping.

(2) Let $T : X \longrightarrow Y$ be linear operators if T^{-1}exist ,T^{-1} is linear.

(3) Let $T : X \longrightarrow Y$ be linear operator T ios continous iff T is bounded .

(4) $B(0,1) = \frac{1}{r}(B(x;r) - x_\circ)$

(5) A bounded linear operator from a banach space X to banach space Y has property the image $T(B_\circ)$ of open unit ball $B_\circ(0,1)$ contains an open ball centered at 0.

Open Mapping Theorem OR Bounded Inverse Theorem:

A bounded linear operater T from a banach space X to banach space Y is open mapping if T is bijective ,T^{-1} is continous and Thus,bounded .

Proof:

Let $A \subset X$ (open in X) .For any $x \in A$,$y = Tx \in T(A) \subset Y$ and since A is open for $x \in A$ there is an open ball contained in A centered at x.

$\Longrightarrow$ $A - x$ contained open ball centre at 0.

$\Longrightarrow$ $\frac{1}{r}(A - x)$ contains ,$B(0;1)$, where r is radius of ball.Hence,

$$T\left(\frac{1}{r}(A - x) = \frac{1}{r}(T(A) - Tx)\right)$$

$$= \frac{1}{r}(T(A) - y)$$

contains an open ball centered at 0. and so, does $((TA) - y)$.

$\Longrightarrow$ $T(A)$ contains open ball centered at y. Since $y \in T(A)$ was arbitrary

$\Longrightarrow T(A)$ is open set .

Hence, T is open mapping

Further, if T is bijective so there exist $T^{-1}$and is linear and also T^{-1} is open mapping

$\Longrightarrow T^{-1}$ is continous

$\Longrightarrow T^{-1}$ is bounded.

Close Linear Operator:

Let X, Y be normed space .A linear operator $T : X \longrightarrow Y$ is said to be close if its graph

$G(T) = \{(x, y) \,|x \in D(T), y = Tx\}$ is close in $X \times Y$ where $X \times Y$ is usual two operations are defined $(on\ X \times Y)$ as

$(i)\ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$(ii)\ \ \alpha(x, y) = (\alpha x, \alpha y)$

and norm on $X \times Y$ is $||(x, y)|| = ||x|| + ||y||$

Note:

A subset M of cmplete space X is itself complete iff M is close in X .

Close Grapgh Theorem :

Let X and Y are banach spaces $T : D(T) \longrightarrow Y$ be a closed linear operator $(D(T) \subset X)$ if $D(T)$ is close in X then$T$ is bounded.

Proof: Given X and Y are banach spaces cauchy seqs in both spaces are convergent i.e if $(x_n) \in X$ is cauchy ,$x_n \longrightarrow x\ OR \lim\limits_{n\longrightarrow\infty} ||x_n - x|| < \frac{\epsilon}{2}$ and if $(y_n) \in Y$ is cauchy $y_n \longrightarrow y\ OR\ \lim\limits_{n\longrightarrow\infty} ||y_n - y|| \,\frac{\epsilon}{2}\,(where\ \epsilon > 0)$ .Let $(Z_n) \in X \times Y$ be a cauchy sequence where

$$\begin{aligned}
Z_n &= (x_n, y_n); z = (x, y)\,.Then \lim_{n\longrightarrow\infty} ||Z_n - Z|| = \lim_{n\longrightarrow\infty} ||(x_n, y_n) - (x, y)|| = \lim_{n\longrightarrow\infty} ||x_n - x, y_n - y|| \\
&= \lim_{n\longrightarrow\infty} ||x_n - x|| + \lim_{n\to\infty} ||y_n - y|| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\Longrightarrow \lim_{n\longrightarrow\infty} ||Z_n\_Z|| < \epsilon \\
&\Longrightarrow Z_n \longrightarrow Z
\end{aligned}$$

OR (Z_n) is convergent in $X \times Y$. Since,(Z_n) was arbitrary .So, $X \times Y$ is complete . Now, $G(T)$ is close subset of $X \times Y$ (complete) and $D(T)$ is close in X (complete) .

Consider bijective mapping

$$P : G(T) \longrightarrow D(T) \quad \text{s.t } P(x, y) = x$$

Let $(x_1, y_1), (x_2, y_2) \in G(T)$

$$\begin{aligned}
P(\alpha(x_1, y_1) + \beta(x_2, y_2)) &= P(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \\
&= \alpha x_1 + \beta x_2 \\
&= \alpha P(x_1, y_1) + \beta P(x_2, y_2)
\end{aligned}$$

$\Longrightarrow$ P is linear.

Also

$$||P(x,y)|| = ||x|| \leq ||x|| + ||Tx|| = ||x, Tx||$$

P is bounded and since P is bijective .So, $P^{-1} : D(T) \longrightarrow G(T)$ exists by bounded inverese theorem .

P^{-1} is bounded i.e

$$\left|\left|P^{-1}(x)\right|\right| = ||x, Tx|| \leq b\,(say)$$

$$\begin{aligned} ||Tx|| &\leq ||x|| + ||Tx|| = ||x, Tx|| \leq b \\ &\Longrightarrow ||Tx|| \leq b \end{aligned}$$

$\Longrightarrow$ T is bounded .

Banach Fixed Point Theorem:

Fixed Point:

A fixed point of mapping $T : X \longrightarrow Y$ of set X in to itself is an $x \in X$ which is mapped onto it self that is

$$Tx = x$$

the image Tx concide with x.

Contraction:

Let $X = (X, d)$ be a metric space .A mapping $T : X \longrightarrow X$ is called a contraction on X if there is a positive real number $0 < \alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y) \qquad 0 < \alpha < 1$$

Example#

Let $x = \{x \in \mathbb{R} | x \geq 1\} \subset \mathbb{R}$ and let the mapping $T : X \longrightarrow X$ be defined by $Tx = \frac{x}{2} + x^{-1}$.Show that T is contraction and find the samllest α.

Solution:

For contraction $d(Tx, Ty) \leq \alpha d(x, y)$ $(o < \alpha < 1)$

Given

$$Tx = \frac{x}{2} + x^{-1} \;\; OR \;\; Ty = \frac{y}{2} + y^{-1}$$

$$\begin{aligned} d(Tx,Ty) &= |Ty-Tx| \\ &= \left|\left(\frac{y}{2}+y^{-1}\right)-\left(\frac{x}{2}+x^{-1}\right)\right| \\ &= \left|\frac{1}{2}(y-x)+\left(\frac{1}{y}-\frac{1}{x}\right)\right| \\ &= \left|\frac{1}{2}(y-x)-\frac{y-x}{xy}\right| \\ &= \left|\left(\frac{1}{2}-\frac{1}{xy}\right)(y-x)\right| \\ &= \left|\frac{xy-2}{2xy}\right||y-x| \\ &= \left|\frac{xy-2}{2xy}\right|d(x,y) \end{aligned}$$

For contraction $0<\alpha<1$ so, $\alpha=\left|\frac{xy-2}{2xy}\right|<1\ \ (say)$

$$\Longrightarrow\ d(Tx,Ty)<\alpha d(x,y)$$

Hence, T is comtraction .

For least value of α .Let $\alpha>0$

$$\begin{aligned} &\Longrightarrow\quad \frac{xy-2}{2xy}=0 \\ &\Longrightarrow\quad xy>2 \end{aligned}$$

Fixed Point :

A fixed pooint of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself (is kept fixed by T) that is

$$Tx = x$$

The image of Tx is coincide with x.♡

Example#

$f(x) = x^3$ put $x = 0$ we get $f(0) = 0$ is a fixed point.

Contraction:

Let $X = (X, d)$ be a metric space .A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y) \spadesuit$$

Babach fixed point Theorem(Contraction Theorem)$^{\clubsuit}$:

Cosider a metric space $X = (X, d)$ where $X \neq \emptyset$.Suppose that X is complete and let $T : X \to X$ be a contraction on X .Then T has precisely one fixed point .

Proof: *Consider a sequence* (x_n) in X defined as

$$x_\circ, x_1 = Tx_\circ, x_2 = Tx_1 = T^2x_\circ, x_3 = Tx_2 = T^3x_\circ \cdots x_n = Tx_{n-1} = T^nx_\circ$$

This is the sequence of images of $X_\circ$ under repeated application of T.

$$\begin{aligned} d(x_m, x_{m+1}) &= d(Tx_{m-1}, Tx_m) \\ &\leq \alpha d(x_{m-1}, x_m) \\ &\leq \alpha d(Tx_{m-2}, Tx_{m-1}) \qquad \because T \text{ is contraction} \\ &\leq \alpha^2 d(Tx_{m-2}, Tx_{m-1}) \\ &\vdots \\ d(x_m, x_{m-1}) &\leq \alpha^m d(x_\circ, x_1) \end{aligned}$$

Now, using triangular inequality for $n > m$

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \alpha^m d(x_\circ, x_1) + \alpha^{m+1} d(x_\circ, x_1) + \cdots + \alpha^{n-1} d(x_\circ, x_1) \\ &= \alpha^m \left(1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-m-1}\right) d(x_\circ, x_1) \end{aligned}$$

On right side of the inequality .we have geometric series

$$d(x_m, x_n) \leq \alpha^m \frac{1 - \alpha^{n-m}}{1 - \alpha} d(x_m, x_n)$$

Since $\alpha < 1$ and $d(x_\circ, x_1)$ is fixed we can make the term on right side as small as we need by taking an sufficiently large .Hence,$d(x_m, x_n) \to 0$ or (x_n) is cauchy sequence and is convergent as X is complete.i.e $d(x_n, x) \to 0$ when $n \to \infty$

$$\begin{aligned} d(x, Tx) &\leq d(x, x_n) + d(x_n, Tx) \\ &= d(x_n, x) + d(Tx_{n-1}, Tx) \\ &\leq d(x_n, x) + \alpha(x_{n-1}, x) \end{aligned}$$

when $n \to \infty$, $d(x_n, x) \to 0$ also $d(x_{n-1}, x) \to 0$ .So, that

$$\begin{aligned} d(x, Tx) &= 0 \\ &\implies x = Tx \end{aligned}$$

i.e x is fixed point .Let $\overline{x}$ be another fixed point

$$\begin{aligned} T\overline{x} &= \overline{x} \\ d(x, \overline{x}) &= d(Tx, T\overline{x}) \leq \alpha d(x, \overline{x}) \\ d(x, \overline{x}) - \alpha d(x, \overline{x}) &\leq 0 \end{aligned}$$

as $1 - \alpha \neq 0 \, (\because \alpha < 1)$

$$\begin{aligned} &\implies d(x, \overline{x}) = 0 \\ &\implies x = \overline{x} \end{aligned}$$

OR x is the only fixed point.■

Spectral theory in finite dimensional Normed space :

Definations(Eigen values,Eigen vectors,Eigen spaces ,spectrum Resolvent set of Matrix):

An eigen value of square matarix $A = (\alpha_{jk})$ is a number λ such that $Ax = \lambda x$ has solution $x \neq 0$ .This x is called an eigen vector of A corresponding to that egien value λ.the eigenvectors correspondng to that eigen value λ and zero vector form a vector subspace of X which is called eigenspaces corresponding to that eigen value λ.The $\sigma(A)$ of all eigen values of A is called spectrum of A.Its complement $\rho(A) = \mathbb{C} - \sigma(A)$ in the complex plane is called the resolvent of A .♣

Theorem(Eigen values of an operater)$^{\heartsuit\heartsuit}$:

All matrices representing a given linear operater $T : X \to X$ on a finite dimensional Normed space X relative to various bases for X have the same eigen values.

proof:

Let X be a $n-$dim space and $T : X \to X$ be operater .Consider two basis set for X .

$$e = (e_1, e_2, \cdots e_n) \,;\, \bar{e} = (\bar{e}_1, \bar{e}_2, \cdots \cdot \bar{e}_n)$$

by defination of basis each $\bar{e}$ can be represented as linear combination of e and conversly that is

$$\begin{aligned} \bar{e} &= eC \qquad \textit{where } C \text{ is non singular matrix} \\ &\implies (\bar{e})^T = C^T e^T \end{aligned}$$

Since each $x \in X$ can be represented as

$$x = ex_1 = \sum_{i=1}^{n} \xi_i e_i \text{ and } x = \bar{e}x_2 = \sum_{i=1}^{n} \xi_i \bar{e}_i \text{ where } x_1 = \sum_{i=1}^{n} e_i$$

using (1) and above equation $x_2 = \sum_{i=1}^{n} \bar{e}_i$

$$\begin{aligned} ex_1 &= \bar{e}x_2 = eCx_2 \\ &\Longrightarrow x_1 = Cx_2 \end{aligned}$$

Similarly, for $Tx = y \in X$ we have $y_1 = Cy_2$.Now, let T_1 and T_2 denote the operaters w.r.t e and $\bar{e}$ ($both$ different) $y_1 = Tx_1$ and $y_2 = T_2x_2$

$$\begin{aligned} CT_2x_2 &= Cy_2 = y_1 = T_1x_1 = T_1Cx_2 \\ &\Longrightarrow CT_2x = T_1Cx_2 \\ &\Longrightarrow CT_2 = T_1C \\ &\Longrightarrow C^{-1}CT_2 = C^{-1}CT_1C \\ &\Longrightarrow T_2 = C^{-1}T_1C \end{aligned}$$

Now,

$$\begin{aligned} \det\left(T_2 - \lambda I\right) &= \det\left(C^{-1}T_1C - \lambda\left(C^{-1}IC\right)\right) \\ &= \det\left(C^{-1}\left(T_1 - \lambda I\right)C\right) \\ &= \det C^{-1} \det\left(T_1 - \lambda I\right)\det C \\ &= \det\left(T_1 - \lambda I\right).\det C^{-1}.\det C \\ &= \det\left(T_1 - \lambda I\right) \qquad \because \det C^{-1}.\det\left(C\right) = 1 \end{aligned}$$

showing that characteristic equation w.r.t both operaters are same hence gives same values of λ (eigen value) ■

Definations(Regular value,Resolvent set,Spectrum)$^{\heartsuit\clubsuit}$:

Let $X \neq \{0\}$ be complex normed space and $T : D(T) \to X$ a linear operater with $D(T) \subset X$ .A regular value λ of T is complex number such that

(**R$_1$**) $R_\lambda(T)$ exists,

(**R$_2$**) $R_\lambda(T)$ is bounded,

(**R$_3$**) $R_\lambda(T)$ is define on set which is dense in X .

The resolvent set $\rho(T)$ of T is the of all regualr value of λ of T. Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in the complex plane $\mathbb{C}$ is called **Spectrum** of T and a $\lambda \in \sigma(T)$ is called a **spectral value of** T.♣

Point Spectrum:

The point spectrum or discrete spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exists .A $\lambda \in \sigma_p(T)$ is called an eigen value of T .♣

Continous Spectrum:

The continous spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ exists and (R_3) holds but (R_2) deos not hold that is $R_\lambda(T)$ is unbounded.♣

Residual Spectrum:

The residual spectrum $\sigma_r(T)$ is trhe set such $R_\lambda(T)$ exists (and maybe bounded or not) butr does not satisfies (R_3) .♣

Spectral properties of Bounded Linear Operaters:

Theorem :

Let $T \in B(X,X)$ where x is banach space if $||T|| < 1$ Then $(I-T)^{-1}$ exists and bounded on whole space and

$$(I-T)^{-1} = I + T + T^2 + T^3 + \cdots$$

Proof:

Consider sequence $\left(\left|\left|T^j\right|\right|\right)$ (sequence of norm of operaters) $\therefore$ $\left|\left|T^j\right|\right| \leq ||T||^j$ .The series $\sum ||T||^j$ is absolutely convergent because $||T|| < 1$.

$\Longrightarrow ||T||^j$ is convergent as $B(X,X)$ is complete ($\because X$ is complete)

.

$\Longrightarrow ||T||$ is bounded .Now, we have to prove $(I-T)^{-1} = I+T+T^2+T^3+\cdots$ for this Suppose

$$\begin{aligned}
S &= I + T + T^2 + T^3 + \cdots \cdot + T^n \\
(I-T)S &= S(I-T) \\
(I-T)\left(I + T + T^2 + T^3 + \cdots + T^n\right) &= \left(I + T + T^2 + T^3 + \cdots + T^n\right)(I-T) \\
&= I - T + T - T^2 + T^3 - T^4 + \cdot \cdot + T^n - T^{n+1} \\
&= I - T^{n+1}
\end{aligned}$$

When $n \to \infty$,$T^{n+1} \to 0$ $\because ||T|| < 1$

$$\begin{aligned}
&\Longrightarrow \lim_{n\to\infty} (I-T)S = \lim_{n\to\infty} S(I-T) \\
&= \lim_{n\to\infty} I - T^{n+1}\ OR \\
(I-T)\left(I + T + T^2 + T^3 + \cdots\right) &= I - 0 = I \\
&\Longrightarrow I + T + T^2 + T^3 + \cdots \cdot = (I-T)^{-1} .\blacksquare
\end{aligned}$$

Theorem:

The resolvent $\rho(T)$ of a bounded linear operater T on a complex banach space X is open .Hence the spectrum$\sigma(T)$ is closed.

Proof:

If $\rho(T) = \Phi$ then it is open .Let $\rho(T) \neq \phi$ for some $\lambda_\circ \in \rho(T)$ and any $\lambda \in (1)$

$$\begin{aligned} T-\lambda I &= T-\lambda_\circ I+\lambda_\circ I-\lambda I \\ &= (T-\lambda_\circ I)-(\lambda-\lambda_\circ)I \\ &= (T-\lambda_\circ I)\left[I-(\lambda-\lambda_\circ)(T-\lambda_\circ I)^{-1}\right] \\ T_\lambda &= T_{\lambda_\circ}\left[I-(\lambda-\lambda_\circ)R_{\lambda_\circ}\right]\ OR \\ T_\lambda &= T_{\lambda_\circ}V \text{ where } V=I-(\lambda-\lambda_\circ)R_{\lambda_\circ} \end{aligned}$$

Since $\lambda_\circ \in \rho(T)$ and T is bounded .Hence,$R_{\lambda_\circ}$ exists and is defined on $B(X,X)$ i.e $R_{\lambda_\circ}=T_{\lambda_\circ}^{-1}\in B(X,X)$

$\Longrightarrow V^{-1}$ exists and

$$\begin{aligned} V^{-1} &= \sum_{j=1}^{n}\left[I-(\lambda-\lambda_\circ)R_{\lambda_\circ}\right]^j \\ &= \sum(\lambda-\lambda_\circ)^j R_{\lambda_\circ} \end{aligned}$$

for $||\lambda-\lambda_\circ||<1$

$$\Longrightarrow |\lambda-\lambda_\circ|<\frac{1}{||R_{\lambda_\circ}||}\cdots(2)$$

Since $T_{\lambda_\circ}^{-1}$ is exists and bounded also V^{-1} exists .

$$\begin{aligned} \Longrightarrow&\ \ T_\lambda^{-1}=(T_{\lambda_\circ}V)^{-1} \\ =&\ \ V^{-1}T_{\lambda_\circ}^{-1} \end{aligned}$$

exists and is bounded i.e

$$T_\lambda^{-1}\in B(X,X)\left(as\ T_{\lambda_\circ}^{-1}\in B(X,X)\right)$$

is regular value of T.i.e $\lambda_\circ\rho(T)$ .Moreever (2) implies $\lambda_\circ$ belongs to neighbourhood of $\lambda\in\rho(T)$ $\because \lambda_\circ\in\rho(T)$ was arbitary .

$\Longrightarrow$Each elemnent of $\rho(T)$ has its own neighbourhood OR $\rho(T)$ is open set $\sigma(T)$ being complement of $\rho(T)$ is then closed.■

Resolvent Theorem(Resolvent):

For X and T every $\lambda_\circ\in\rho(T)$ the resolvent $R_\lambda(T)$ has the representation

$$R_\lambda=\sum_{j=1}^{\infty}(\lambda-\lambda_\circ)^j R_\lambda^{j+1}$$

Proof:

we khow that

Since $\lambda_\circ\in\rho(T)$ and T is bounded

$$\Longrightarrow R_{\lambda_\circ}=T_{\lambda_\circ}^{-1}$$

Put value of $T_{\lambda_\circ}^{-1} = R_{\lambda_\circ}$

$$\Longrightarrow R_\lambda = V^{-1} R_{\lambda_\circ}$$

V has an inverse

$$\begin{aligned} V^{-1} &= \sum_{j=1}^{\infty} [(\lambda - \lambda_\circ) \, R_{\lambda_\circ}]^j \\ V^{-1} &= \sum_{j=1}^{\infty} (\lambda - \lambda_\circ)^j \, R_{\lambda_\circ}^j \end{aligned}$$

(2) becomes

$$\begin{aligned} R_\lambda &= \sum_{j=1}^{\infty} (\lambda - \lambda_\circ)^j \, R_{\lambda_\circ}^j R_{\lambda_\circ}^j \\ &= \sum_{j=1}^{\infty} (\lambda - \lambda_\circ)^j \, R_{\lambda_\circ}^{j+1} \end{aligned}$$

Now, for

$$||(\lambda - \lambda_\circ) \, R_{\lambda_\circ}|| < 1$$

Since the series being absolutely convergent for every λ in the open disk

$$|\lambda - \lambda_\circ| < \frac{1}{||R_{\lambda_\circ}||}$$

In the complex plane .This disk is a subset of $\rho(T)$.■

Theorem(Spectrum):

The spectrum $\sigma(T)$ of bounded linear operator $T : X \to Y$ on a complex banach space X is compact and lies in the disk given by $|\lambda| \leq ||T||$ .Hence,the resolvent set $\rho(T)$ of$T$is not empty.

,**Proof:**

Let $\lambda \neq 0$ and $\kappa = \frac{1}{\lambda}$.Since resolvent can be written as

$$\begin{aligned} R_\lambda &= (T - \lambda I)^{-1} \\ &= \left[\lambda \left(T.\frac{1}{\lambda} - I\right)\right]^{-1} \\ &= \left[-\lambda \left(I - \frac{1}{\lambda} T\right)\right]^{-1} \\ &= -\frac{1}{\lambda} (I - \kappa T)^j \\ &= -\frac{1}{\lambda} \sum_{j=1}^{\infty} (\kappa T) \end{aligned}$$

where the series converges for all λ such that

$$\begin{aligned} \left|\left|\frac{1}{\lambda}T\right|\right| &= \frac{||T||}{|\lambda|} < 1 \quad i.e\,|\lambda| > ||T|| \\ &\Longrightarrow \lambda \in \rho(T) \end{aligned}$$

Hence, the spectrum $\sigma(T)$ lies in the disk $|\lambda| < ||T||$.

$\Longrightarrow \sigma(T)$ is bounded.Furthermore,$\sigma(T)$ is closed by spectrum closed theorem.Hence, $\sigma(T)$ is compact.■

Theorem: **Let** $T \in B(X, X)$, where X is a banach space and $\lambda, \mu \in \rho(T)$. Then show that

(a) $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda$

(b) R_λCommutes with any $S \in B(X, X)$ which commutes with T

(c) $R_\mu R_{\ \lambda} = R_\lambda R_\mu$

Proof:

(a) The range of T_λ is all of X. Hence, $I = T_\lambda R_\lambda$ where I is identity operater on X.Also $I = R_\mu T_\mu$ Consequently,

$$\begin{aligned} R_\mu - R_\lambda &= R_\mu (T_\lambda R_\lambda) - (R_\mu T_\mu) R_\mu \\ &= R_\mu (T_\lambda - T_\mu) R_\lambda \\ &= R_\mu [(T - \lambda I - (T - \mu I))] R_\lambda \\ &= (\mu - \lambda) R_\mu R_\lambda \end{aligned}$$

(b) By assumption $ST = TS$ hence, $ST_\lambda = T_\lambda S$ now using $I = T_\lambda R_\lambda = R_\lambda T_\lambda$. we obtain

$$\begin{aligned} R_\lambda S &= R_\lambda S T_\lambda R_\lambda \\ &= R_\lambda T_\lambda S R_\lambda \\ R_\lambda S &= S R_\lambda \end{aligned}$$

(c) we khow that R_μ (Commutes with T by (b)) .Hence, R_λ commutes with R_μ by (b).

Question#1

If a linear operater T defined on an infinite dimensional normed space then it can have spectral values which are not eigen values proved\disprove .

Solution:

Operater with a soectral value which is not an eigen value *.on the hilbert sequence spcae* $X = l^2$. we define linear operater

$$T : l^2 \to l^2 \text{ by } (x'_1, x'_2, x'_3 \cdot\cdot) \to (0, x'_1, x'_2, \cdot\cdot) \cdots (1)$$

where $x = x'_i \in l^2$. The operater T is called right shift operater .T is bounded

$$\begin{aligned} ||T|| &= 1 \\ ||Tx||^2 &= \sum_{i=1}^{\infty} |x'_i| = ||x||^2 \end{aligned}$$

$R_\circ(T) = T^{-1} : Tx \to x$ exists left shift opearter $(x_1', x_2', \cdot\cdot) \to (x_1', x_2', \cdots)$. But $R_\circ(T)$ does not satisfies (1) because (1) show that $T(x)$ is not dense in $x.T(x)$ is subspace of Y consisting of all $Y = Y_i'$ with $Y_i' = 0$.By defination $\lambda = 0$ is spectral value of T.Furthermore , $\lambda = 0$ is not eigen value .from (1)

$$Tx = 0 \qquad x = 0$$

Hence, zero vector is not an eigen vector.

Application of banach fixed point theorem:

Question:1

Solve the differential equation by fixed point method

$$\frac{dx}{dt} = (x+t)\,t \qquad\qquad x(0) = 0,\ 0 \le t \le 2$$

Solution:

Given that $\quad x \in [0,2] \quad \frac{dx}{dt} = (x+t)\,t$ If $x(0) = 0 \quad 0 \le t \le 2$ then

$$\begin{aligned}
f_x(t) &= \beta + \int_\infty^t f(x(0), u)\,du \\
&\implies f_x(t) = 0 + \int_\infty^t (x+u)\,u du \\
&= 0 + \int_0^t \left(xu + u^2\right) du \\
f_x(t) &= \int_0^t x(u)\,u du + u^2 du \\
|f_x(t) - f_y(t)| &= \left|\int_0^t (x(u)\,u du) + u^2 du - \int_0^t y(u)\,u du - u^2 du\right| \\
&= \left|\int_0^t ((x(u))\,u - y(u))du\right| \\
&\le |x(u) - y(u)| \int_0^t u du \\
\max_{t\in[0,2]} |f_x(t) - f_y(t)| &\le \max_{\mu\in(0,2)} |x(u) - y(u)| \int_0^t u du \\
d(f_x, f_y) &\le d(x,y)\left[\frac{u^2}{2}\right]_0^t \\
d(f_x, f_y) &\le \frac{t^2}{2} d(x,y)
\end{aligned}$$

$\implies d = \frac{4}{2} = 2 > 1$ So, by banach fixed point theorem $x_\circ$ is contraduction .

So, exists as unique fixed point $x(t) = fx^*(t)$ ■