

Vector Spaces: Handwritten notes

by

Atiq ur Rehman

<http://www.MathCity.org/atiq>

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Ring

def:- A non-empty set R is called ring if

- i) R is abelian group under ~~multiplication~~ addition.
- ii) R is semi-group under multiplication.
- iii) Distributive law holds

$$a(b+c) = a \cdot b + a \cdot c$$

$$(a+b)c = a \cdot c + b \cdot c$$

Examples

i) $(\mathbb{Z}, +, \cdot)$ is a ring

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$

ii) $(\mathbb{Q}, +, \cdot)$, where \mathbb{Q} is the set of rational numbers

iii) $(\mathbb{R}, +, \cdot)$, where \mathbb{R} is set of real numbers.

iv) $(\mathbb{Z}_n, +, \cdot)$, $\mathbb{Z}_n =$ residue classes of module n .

Field

def:- A non-empty set F is called a field if

- i) F is abelian group under addition.
- ii) $F - \{0\}$ is abelian group under multiplication.
- iii) Right distributive law holds in F .

i.e. $a, b, c \in F$

$$(a+b)c = ac + bc$$

Examples

i) $(\mathbb{R}, +, \cdot)$ is a field

ii) $(\mathbb{C}, +, \cdot)$ is a field

iii) $(\mathbb{Q}, +, \cdot)$ is a field

iv) $(\mathbb{Z}, +, \cdot)$ is not a field

as $(\mathbb{Z} - \{0\}, \cdot)$ is not group under multiplication.

Vector Space

def:- Let V be a non-empty set and F is field then V is called vector space if

i) V is abelian group under addition

ii) $a(v+w) = av + aw \quad \forall a \in F, v, w \in V$.

iii) $(a+b)v = av + bv \quad \forall a, b \in F, v \in V$.

iv) $a(bv) = (ab)v \quad \forall a, b \in F, v \in V$.

v) $1 \cdot v = v \cdot 1 = v$, $1 \in F$ and $v \in V$

i.e 1 is identity under multiplication

Example

i) Let V be a set of all polynomial of degree $\leq n$ then V is vector space.

$$V = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n \mid a_i \in F \quad \forall i \leq n \in \mathbb{N}\}$$

$$= \left\{ \sum_{i=0}^n a_i x^i \mid a_i \in F \quad \forall i \leq n \in \mathbb{N} \right\}$$

addition is defined as

$$\sum_{i=0}^n a_i x^i + \sum_{i=0}^n b_i x^i = \sum_{i=0}^n (a_i + b_i) x^i$$

and multiplication is defined as

$$\begin{aligned} r \sum_{i=0}^n a_i x^i &= \sum_{i=0}^n r a_i x^i \\ &= r a_0 + r a_1 x + r a_2 x^2 + \dots + r a_n x^n \end{aligned}$$

ii) Let F is a field then the set

$$F^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in F, 1 \leq i \leq n\}$$

iii) The set M_n of all $n \times n$ matrices with entries from a field F is a vector space over F .

iv) Every field is a vector space over itself.

Subspace:-

Let V be a vector space over F and W be its non-empty subset of V .

Then W is a subspace of V if W itself is vector space under operation induced (defined) in V .

* Theorem:-

A non-empty subset W of a vector space V is a subspace of V iff

$$i) w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$$

$$ii) \alpha \in F, w \in W \Rightarrow \alpha w \in W.$$

Proof:

Let W is subspace of vector space V .
then W itself is a vector space

i.e. W is closed under addition and scalar multiplication.

Conversely, let W is a subset, satisfying condition (i) and (ii).

Then for $-1 \in F$ and $w_2 \in W$.

$$\Rightarrow -1 \cdot w_2 \in W \quad \text{by condition (ii).}$$

$$\Rightarrow -w_2 \in W$$

$$\text{i.e. } w_1, -w_2 \in W$$

$$\Rightarrow w_1 + (-w_2) \in W \quad \text{by condition (i)}$$

$$\Rightarrow W \text{ is a subgroup under addition}$$

Since W is a subset of V and V is abelian.

So W is abelian.

Further condition II to V of the definition are satisfied in W as these are satisfied in V .

Corollary:-

W is non-empty subset of a vector space $V(F)$. Then W is subspace of V iff

$$a, b \in F, w_1, w_2 \in W \Rightarrow aw_1 + bw_2 \in W.$$

Proof Let W is a subspace of $V(F)$. Then W itself is a vector space.

i.e. for $a, b \in F, w_1, w_2 \in W$
 $\Rightarrow aw_1, bw_2 \in W$
 $\Rightarrow aw_1 + bw_2 \in W$. $\because W$ is closed under addition.

~~Con~~ Conversely,

Let for $a, b \in F, w_1, w_2 \in W$.

$\Rightarrow aw_1 + bw_2 \in W$.

Let $a = b = 1$

then $1 \cdot w_1 + 1 \cdot w_2 \in W$

i.e. $w_1 + w_2 \in W$.

also if $b = a \in F$

For $aw_1 + bw_2 \in W$

$\Rightarrow aw_1 + a \cdot w_2 \in W$

$\Rightarrow aw_1 \in W$.

$\Rightarrow W$ is a subspace of V .

Definition (Linear Sum)

Let V be a vector space over F and

W_1, W_2, \dots, W_n be non-empty subset of V .

then their linear sum is defined as

$$W_1 + W_2 + \dots + W_n = \{a_1 + a_2 + \dots + a_n, a_1 \in W_1, a_2 \in W_2, \dots, a_n \in W_n\}$$

Lemma:- Let V be a vector space and W_1, W_2, \dots, W_n be subspace, prove that

$$W = W_1 + W_2 + \dots + W_n$$

is also a subspace of V .

Lemma:

W_1, W_2, \dots, W_n are subspaces of V prove that
 $W = W_1 + W_2 + \dots + W_n$ is a subspace of V .

Proof:

$$0 = 0 + 0 + 0 + \dots + 0, \quad 0 \in W_i$$

$\Rightarrow 0 \in W \Rightarrow W$ is non-empty.

Let $x, y \in W, a, b \in F$

we have to show $ax + by \in W$.

$\because x \in W$

$$\Rightarrow x = x_1 + x_2 + \dots + x_n \text{ for } x_1 \in W_1, x_2 \in W_2, \dots, x_n \in W_n$$

$$y = y_1 + y_2 + \dots + y_n \text{ for } y_1 \in W_1, y_2 \in W_2, \dots, y_n \in W_n$$

Now

$$ax + by = a(x_1 + x_2 + \dots + x_n) + b(y_1 + y_2 + \dots + y_n)$$

$$= ax_1 + ax_2 + \dots + ax_n + by_1 + by_2 + \dots + by_n$$

$$= (ax_1 + by_1) + (ax_2 + by_2) + \dots + (ax_n + by_n)$$

As each W_i is a subspace

$$\Rightarrow ax_i + by_i \in W_i, \quad i = 1, 2, \dots, n$$

$$\text{So } \sum_{i=1}^n (ax_i + by_i) \in \sum_{i=1}^n W_i = W$$

$$\Rightarrow ax + by \in W$$

So W is a subspace.

Lemma:

Let V be a vector space and W_i a family of subspaces of V . Then $\bigcap W_i$ is also a subspace of V .

Proof

Let $v, w \in \bigcap W_i$

then $v, w \in W_i$ for each $i \in I$
 and since each W_i is a subspace

so there must be $a, b \in F$

such that $av + bw \in W_i$ for each $i \in I$.

so $av + bw \in \bigcap W_i$ i.e. $\bigcap W_i$ is a subspace.

Definition

Let U and V are two vector spaces over a field F .
then T of U into V is called homomorphism

$$\text{if } T(u_1 + u_2) = T(u_1) + T(u_2)$$

$$T(au) = aT(u) \quad ; \quad a \in F$$

Definition

The kernel of homomorphism $T: U \rightarrow V$, is defined as

$$\ker T = \{u \in U, T(u) = 0\}$$

Question.

Prove that $\ker T$ (ker. of homomorphism)
is a subspace.

Solution. Let $u_1, u_2 \in \ker T$

$$\Rightarrow T(u_1) = 0, \quad T(u_2) = 0$$

Now let $a, b \in F$

$$\begin{aligned} T(au_1 + bu_2) &= T(au_1) + T(bu_2) \\ &= aT(u_1) + bT(u_2) \\ &= a(0) + b(0) \\ &= 0 \end{aligned}$$

$$\Rightarrow au_1 + bu_2 \in \ker T$$

So $\ker T$ is subspace.

Linear Combination:-

Let V is a vector space

Let $v_1, v_2, \dots, v_n \in V$

$a_1, a_2, \dots, a_n \in F$

then an element

$a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n$ is called
linear combination.

The linear combination is trivial if each $a_i = 0$.

and it is non-trivial if at least one of $a_i \neq 0$

Definition: (Linear Span)

Let S be a subset of vector space V , then the set of all linear combination of S is called linear span denoted by $\langle S \rangle$ or $L(S)$ or $[S]$.

Theorem:-

Prove that $\langle S \rangle$ is a subspace of V containing S . It is smallest subspace of V containing S .

Proof:-

Let $u, v \in \langle S \rangle$

$$\text{then } u = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$$

$$v = b_1 v_1 + b_2 v_2 + \dots + b_n v_n$$

For $a, b \in F$ we have to prove $au + bv \in \langle S \rangle$.

Now

$$au + bv = a(a_1 u_1 + a_2 u_2 + \dots + a_n u_n)$$

$$+ b(b_1 v_1 + b_2 v_2 + \dots + b_n v_n)$$

$$= a a_1 u_1 + a a_2 u_2 + \dots + a a_n u_n$$

$$+ b b_1 v_1 + b b_2 v_2 + \dots + b b_n v_n$$

$$\Rightarrow au + bv \in \langle S \rangle$$

$\Rightarrow \langle S \rangle$ is a subspace.

Let $u_i \in S$

$$\text{then } u_i = 0u_1 + 0u_2 + \dots + 0u_{i-1} + 1 \cdot u_i + 0 \cdot u_{i+1} + \dots + 0 \cdot u_n \in \langle S \rangle$$

$$\text{i.e. } u_i \in \langle S \rangle$$

$$\Rightarrow S \subseteq \langle S \rangle$$

Let W be any other subspace of V containing S .

$$\text{then } \sum a_i u_i \in W$$

$\because W$ is subspace containing S .

$$\Rightarrow \langle S \rangle \subseteq W$$

i.e. $\langle S \rangle$ is smallest subspace containing S .

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Definition (Finite Dimensional Vector Space)

A vector space V is called finite dimensional if there is a subset S of V

such that $\langle S \rangle = V$.

Definition:- (Linear Dependent and Independent)

Let V be a vector space then the vectors

$v_1, v_2, v_3, \dots, v_n \in V$ are linearly dependent

if $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ and all $a_i \neq 0$.

If $a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n = 0$

where each $a_i = 0$ then the vectors

v_1, v_2, \dots, v_n are linearly independent.

Theorem:-

Let V be a vector space and consider a set of

vectors $\{v_1, v_2, \dots, v_n\}$ are linearly independent

then its subset is also independent.

ii) If $\{v_1, v_2, \dots, v_n\}$ is dependent then

$\{v_1, v_2, \dots, v_n, v\}$ is also dependent.

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Lemma:-

Let $V(F)$ be a vector space and $S = \{v_1, v_2, \dots, v_n\}$ a set of vectors in V . Then

i) If S is independent, then any non-empty subset of S is also independent

Proof:

Let $\{v_1, v_2, \dots, v_i\}$ be a subset of S , $1 \leq i < n$.

Consider $a_1 v_1 + a_2 v_2 + \dots + a_i v_i = 0$, $a_i \in F$

then

$$a_1 v_1 + a_2 v_2 + \dots + a_i v_i + 0 \cdot v_{i+1} + \dots + 0 \cdot v_n = 0$$

Since $\{v_1, v_2, \dots, v_n\}$ is Linearly Independent

\Rightarrow each $a_k = 0$, $k = 1, 2, \dots, n$

\therefore each $a_k = 0$, $k = 1, 2, \dots, i$

$\Rightarrow \{v_1, v_2, \dots, v_i\}$ is L.I.

2001
(ii)

If S is dependent then

$\{v_1, v_2, \dots, v_n\}$ is also dependent

i.e. $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ where all $a_i \neq 0$ and then

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n + 0v = 0$$

where all $a_i \neq 0$.

$\Rightarrow \{v_1, v_2, \dots, v_n, v\}$ is also dependent.

Theorem:-

A set of non-zero vectors $v_1, v_2, \dots, v_n \in V$ is linearly dependent iff one of them is a linear combination of the other / preceding vector.

Proof:

$\{v_1, v_2, \dots, v_n\}$ is linearly dependent

i.e. $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ where all a_i 's $\neq 0$ for $a_i \in F$.

Let a_k be the last ^{non-zero} coefficient of

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + \dots + a_k v_k + 0 + \dots + a_n v_n$$

$$= a_1 v_1 + a_2 v_2 + \dots + a_{k-1} v_{k-1} + a_k v_k + a_{k+1} v_{k+1} + \dots + a_n v_n$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0 \quad \because a_{k+1} = a_{k+2} = \dots = a_n = 0$$

$$\Rightarrow -a_k v_k - a_1 v_1 - a_2 v_2 - \dots - a_{k-1} v_{k-1} = 0$$

$$\Rightarrow v_k = -\frac{1}{a_k} (a_1 v_1 + a_2 v_2 + \dots + a_{k-1} v_{k-1})$$

Conversely, let v_k is a linear combination of the preceding vectors

$$v_1, v_2, v_3, \dots, v_{k-1}$$

$$\text{i.e. } v_k = a_1 v_1 + a_2 v_2 + \dots + a_{k-1} v_{k-1}$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \dots + a_{k-1} v_{k-1} + (-1) v_k = 0$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \dots + a_{k-1} v_{k-1} + (-1) v_k + 0 \cdot v_{k+1} + \dots + 0 v_n = 0$$

then $\{v_1, v_2, \dots, v_n\}$ is Linearly Dependent

\because at least one coefficient of v_k is non-zero,

Basis of a Vector Space :-

Let S be a subset of a vector space $V(F)$.

then S is called basis for V .

if i) S is linearly independent.

ii) S is spanning set of V .
generating.

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Theorem:

Any finite dimensional vector space contains a basis.

| Proof:

Let $\{v_1, v_2, \dots, v_n\}$ be a spanning set of V .

If $\{v_1, v_2, \dots, v_n\}$ is linearly independent then form a basis and there is nothing to prove.

Consider $\{v_1, v_2, \dots, v_n\}$ is linearly dependent then one of the vectors say v_r is a linear combination of the remaining $\{v_1, v_2, \dots, v_{r-1}\}$ we drop out this vector and obtain a set of ~~$n-1$~~ $r-1$ vectors.

A ~~vector~~ linear combination of r vectors also a linear combination of $r-1$ vectors.

If this set $\{v_1, v_2, \dots, v_{r-1}\}$ is linearly independent then form a basis.

But if $\{v_1, v_2, \dots, v_{r-1}\}$ is dependent then the above process is continued. In this way we can get a linear independent spanning set, and hence a basis.

$$\{v_1, v_2, \dots, v_n\} \text{ is n.s.r.}$$

Imp 2001, 11/02

Theorem:-

If v_1, v_2, \dots, v_n is a basis of $V(F)$ and if $w_1, w_2, \dots, w_m \in V$ are linearly independent then $m \leq n$.

| Proof:

Since v_1, v_2, \dots, v_n is a basis of V so every element of V can be expressed as a linear combination of v_1, v_2, \dots, v_n .

In particular $w_m \in V$ is a linear combination of v_1, v_2, \dots, v_m .

$w_m, v_1, v_2, \dots, v_n$ are dependent.

therefore, a proper subset $\{w_m, v_1, v_2, \dots, v_r\}$, $r \leq n-1$ form a basis

Similarly $\{w_{m-1}, w_m, v_1, v_2, \dots, v_r\}$ is dependent

and its proper subset

$$\{w_{m-1}, w_m, v_1, v_2, \dots, v_s\}, \quad s \leq n-2$$

Repeating this procedure $(m-1)$ times, we get a basis

$$w_1, w_2, \dots, w_{m-1}, w_m, v_1, v_2, \dots, v_k$$

$t \geq 1$ Since the vectors w_1

is not a l.c. of

$$w_1, w_2, \dots, w_n$$

$$\begin{cases} t \leq n - (m-1) \\ t \leq n - m + 1 \\ t \geq 1 \end{cases}$$

$$\Rightarrow 1 \leq t \leq n - m + 1$$

$$1 \leq n - m + 1$$

$$\Rightarrow 0 \leq n - m$$

$$\Rightarrow m \leq n$$

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Question: Show that the vectors

$$v_1 = (1, 1, 1), v_2 = (1, 0, 1), v_3 = (0, 1, 1)$$

are linearly independent.

Solution:-

$$\text{Consider } a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

$$\Rightarrow a_1(1, 1, 1) + a_2(1, 0, 1) + a_3(0, 1, 1) = 0$$

$$\Rightarrow (a_1, a_1, a_1) + (a_2, 0, a_2) + (0, a_3, a_3) = 0$$

$$\Rightarrow \cancel{a_1 + a_2}$$

$$(a_1 + a_2, a_1 + a_3, a_1 + a_2 + a_3) = (0, 0, 0)$$

$$\Rightarrow a_1 + a_2 = 0 \quad \text{--- (i)}$$

$$a_1 + a_3 = 0 \quad \text{--- (ii)}$$

$$a_1 + a_2 + a_3 = 0 \quad \text{--- (iii)}$$

$$\Rightarrow a_1 + a_2 + a_3 = 0$$

$$\underline{\underline{a_1 + a_2}}$$

$$a_3 = 0 \quad \Rightarrow a_1 = 0, a_2 = 0$$

Since $a_1 = a_2 = a_3 = 0$

\Rightarrow the vectors are L.I.

Question:- Prove that the vectors

$$v_1 = (3, 0, -3), v_2 = (-1, 1, 2), v_3 = (1, 2, -2)$$

$v_4 = (2, 1, 1)$ are linearly dependent.

Solution

Consider

$$a_1 v_1 + b v_2 + c v_3 + d v_4 = 0$$

$$\Rightarrow a(3, 0, -3) + b(-1, 1, 2) + c(1, 2, -2) + d(2, 1, 1) = 0$$

$$\Rightarrow (3a, 0, -3a) + (-b, b, 2b) + (c, 2c, -2c) + (2d, d, d) = 0$$

$$\Rightarrow (3a - b + c + 2d, b + 2c + d, -3a + 2b - 2c + d) = 0$$

$$3a - b + c + 2d = 0$$

$$b + 2c + d = 0$$

$$-3a + 2b - 2c + d = 0$$

let $d=0$ other stuff still valid? $\Rightarrow 3a + b + 4c = 0 \cdot e \cdot 1 = 0 \cdot (1, 1, 1) = 0$

$b + 3a = -4c$ ~~plugging in~~
 $-3a + 2b - 2c = 0$

$0 = 2v_1 + 2v_2 + v_3$
 $0 = (1, 0, 0) + (1, 0, 0) + (0, 1, 1)$
 $0 = (2, 0, 0) + (0, 1, 1)$
 $(0, 0, 0) = (2, 0, 0) + (0, 1, 1)$

✓ Using $a = -2c, b = -2c, d = 0$
 into (1)

$-2cv_1 - 2cv_2 + cv_3 + 0v_4 = 0$
 $2v_1 + 2v_2 - v_3 + 0v_4 = 0$

$\Rightarrow v_1, v_2, v_3, v_4$ are dependent

v_1, v_2 are a basis for \mathbb{R}^3 (check)

Definition: (Quotient Space)

Let V be a vector space over a field F and W be a subspace.

The set V/W of all left coset along with two operations

$$(i) (v_1 + W) + (v_2 + W) = v_1 + v_2 + W$$

$$(ii) a(v_1 + W) = av_1 + W$$

is called Quotient space.

~~2008~~

Lemma:-

Let V be a vector space and W a subspace of V along with the operation V/W

$$(i) (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

$$(ii) \alpha(v_1 + W) = \alpha v_1 + W \text{ is a subspace vector space}$$

Proof:-

a) It is easy to show that V/W is an abelian group under addition with $0 + W = W$ as its identity

and $-v + W$ as an inverse of $v + W \in V/W$.

b) We see that scalar multiplication is defined in V/W .

$$(\text{i.e. } v + W = v' + W \Rightarrow \alpha(v + W) = \alpha(v' + W))$$

Let $v = v' + w$ for some $w \in W$.

$$\text{then } \alpha(v + W) = \alpha v + W$$

$$= \alpha(v' + w) + W$$

$$= \alpha v' + \alpha w + W$$

$$= \alpha v' + W$$

$$\because \alpha w \in W$$

$$= \alpha(v' + W)$$

i.e. Scalar multiplication is defined.

Let $v + W, v' + W \in V/W, \alpha \in F$.

$$\alpha((v + W) + (v' + W)) = \alpha(v + v' + W)$$

$$= \alpha(v + v') + W$$

$$= \alpha v + \alpha v' + W$$

$$= \alpha v + W + \alpha v' + W$$

$$= \alpha(v + W) + \alpha(v' + W)$$

$$\begin{aligned}
 \text{(iv)} \quad (a+b)(v+w) &= (a+b)v + w \\
 &= (av + bv) + w \\
 &= av + w + bv + w \\
 &= a(v+w) + b(v+w)
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad a(b(v+w)) &= a(bv + w) \\
 &= (ab)v + w \\
 &= (ab)(v+w)
 \end{aligned}$$

$$\begin{aligned}
 \text{vi} \quad 1 \cdot (v+w) &= 1 \cdot v + w \\
 &= v + w
 \end{aligned}$$

Hence V/W is vector space.

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2. Theorem:-

$V(F)$ is a finite dimensional vector space and if W is a subspace of V . Then

i) W is finite dimensional and $\dim W \leq \dim V$.

ii) $\dim(V/W) = \dim V - \dim W$

Proof:

Let $\dim V = n$.

and let $\{w_1, w_2, \dots, w_m\}$ be linearly independent set of vectors of W .

then $m \leq n$

then the set $\{w_1, w_2, w_3, \dots, w_m, w\}$ is linearly dependent. i.e. one of these vectors is a linear combination of the preceding vectors.

however none of the vectors w_1, w_2, \dots, w_m is a linear combination of the preceding vectors because the vectors w_1, w_2, \dots, w_m are linearly independent.

so w can be written as a linear combination of w_1, w_2, \dots, w_m .

Since $w \in W$ is an arbitrary element therefore W is finite dimensional

and $\dim W = m \leq n$.

i.e. $\dim W \leq \dim V$.

ii) Let $\{w_1, w_2, \dots, w_m\}$ be a basis of W .

and $\{w_1, w_2, \dots, w_m, v_1, v_2, \dots, v_r\}$ be a basis of V .

we have to prove $\{v_1+W, v_2+W, \dots, v_r+W\}$ is a basis of V/W .

Now

$$\alpha_1(v_1+W) + \alpha_2(v_2+W) + \dots + \alpha_r(v_r+W) = 0$$

$$(\alpha_1 v_1 + W) + (\alpha_2 v_2 + W) + \dots + (\alpha_r v_r + W) = 0 + W$$

Since W is identity of V/W .

$$\Rightarrow (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r) + W = W$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_l v_l \in W \quad \left| \begin{array}{l} \because a+H = H \\ \Leftrightarrow a \in H \end{array} \right.$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_l v_l = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_m w_m$$

as $\{w_1, w_2, \dots, w_m\}$ is basis of W .

So

$$\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_m w_m - \alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_l v_l = 0$$

Since $\{w_1, w_2, \dots, w_m, v_1, v_2, \dots, v_l\}$ is a basis of V .

$$\Rightarrow \beta_1 = \beta_2 = \dots = \beta_m = \alpha_1 = \alpha_2 = \dots = \alpha_l = 0$$

i.e. $\{v_1 + W, v_2 + W, \dots, v_l + W\}$ is linearly independent.

Set $v + W \in V/W$ for $v \in V$.

$$\text{then } v = a_1 w_1 + a_2 w_2 + \dots + a_m w_m + b_1 v_1 + b_2 v_2 + \dots + b_l v_l$$

$$\text{So } v + W = a_1 w_1 + a_2 w_2 + \dots + a_m w_m + b_1 v_1 + b_2 v_2 + \dots + b_l v_l + W$$

$$\Rightarrow v + W = b_1 v_1 + b_2 v_2 + \dots + b_l v_l + a_1 w_1 + a_2 w_2 + \dots + a_m w_m + W \\ = b_1 v_1 + b_2 v_2 + \dots + b_l v_l + W$$

$$\because a_1 w_1 + a_2 w_2 + \dots + a_m w_m + W = W$$

$$\text{as } a_1 w_1 + a_2 w_2 + \dots + a_m w_m \in W$$

$$= (b_1 v_1 + W) + (b_2 v_2 + W) + \dots + (b_l v_l + W) \quad \text{by def.}$$

$$= b_1 (v_1 + W) + b_2 (v_2 + W) + \dots + b_l (v_l + W) \quad \text{by def.}$$

i.e. $\{v_1 + W, v_2 + W, \dots, v_l + W\}$ generate V/W
and hence is a basis of V/W .

$$\therefore \dim(V/W) = l$$

$$= (m + l) - m$$

$$= \dim V - \dim W$$

Internal Direct sum:-

def: Let U_1, U_2, \dots, U_n be subspace of a vector space V . For $v \in V$

then if v has one and only one expression of the form

$$v = u_1 + u_2 + \dots + u_n \quad \text{for } u_i \in U_i$$

then V is called internal direct sum of subspace U_1, U_2, \dots, U_n .

External Direct sum:-

def: Let V_1, V_2, \dots, V_n be vector spaces over a field F & V be a vector space over field F .

V be a vector space having n -ordered tuples (v_1, v_2, \dots, v_n) , $v_i \in V_i$. then V is called external direct sum if

i) Two n -tuples (v_1, v_2, \dots, v_n) and $(v'_1, v'_2, \dots, v'_n)$ are equal iff $v_i = v'_i$

$$\text{ii) } (v_1, v_2, \dots, v_n) + (v'_1, v'_2, \dots, v'_n) = (v_1 + v'_1, v_2 + v'_2, \dots, v_n + v'_n)$$

$$\text{iii) } \alpha(v_1, v_2, \dots, v_n) = (\alpha v_1, \alpha v_2, \dots, \alpha v_n)$$

external direct sum is denoted by

$$V_1 \oplus V_2 \oplus V_3 \oplus \dots \oplus V_n$$

Vector Space Homomorphism:-

Let V and W are two vector spaces.

A mapping $T: V \rightarrow W$ is called homomorphism if

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$T(\alpha v) = \alpha T(v) \quad \forall v_1, v_2 \in V \text{ \& } \alpha \in F$$

Theorem:-

If a vector space V is the internal direct sum of subspaces U_1, U_2, \dots, U_n then V is isomorphic to the external direct sum of U_1, U_2, \dots, U_n .

Proof Let $v \in V$ where $v = u_1 + u_2 + u_3 + \dots + u_n$

Define a mapping

$$T: V \rightarrow U_1 \oplus U_2 \oplus U_3 \oplus \dots \oplus U_n$$

$$\text{by } T(v) = T(u_1 + u_2 + \dots + u_n) \\ = (u_1, u_2, \dots, u_n)$$

i) Mapping is well defined as $v \in V$

$v = u_1 + u_2 + \dots + u_n$
has one and only one representation

(ii) T is onto because each

$$(u_1, u_2, u_3, \dots, u_n) \in U_1 \oplus U_2 \oplus \dots \oplus U_n$$

is image of $u_1 + u_2 + \dots + u_n \in V$.

(iii) T is one-one

$$\text{for } T(v) = T(w)$$

$$\Rightarrow T(u_1 + u_2 + \dots + u_n) = T(w_1 + w_2 + \dots + w_n)$$

\Rightarrow where $v_i, w_i \in U_i$

$$\Rightarrow (u_1, u_2, \dots, u_n) = (w_1, w_2, \dots, w_n)$$

$$\Rightarrow u_1 = w_1, u_2 = w_2, \dots, u_n = w_n$$

$$\Rightarrow u_1 + u_2 + \dots + u_n = w_1 + w_2 + \dots + w_n$$

$$\textcircled{a} \Rightarrow u = w$$

$$(iv) \quad T(v+w) = T(u_1 + u_2 + u_3 + \dots + u_n + w_1 + w_2 + \dots + w_n)$$

$$= T(u_1 + w_1 + u_2 + w_2 + \dots + u_n + w_n)$$

$$= (u_1 + w_1, u_2 + w_2, \dots, u_n + w_n)$$

$$= (u_1, u_2, \dots, u_n) + (w_1, w_2, \dots, w_n)$$

by def. of external direct sum

$$= T(v) + T(w)$$

$$\checkmark \quad T(\alpha v) = T(\alpha(u_1 + u_2 + \dots + u_n)) = T(\alpha u_1 + \alpha u_2 + \dots + \alpha u_n)$$

$$= T(\alpha u_1, \alpha u_2, \dots, \alpha u_n)$$

$$= \alpha(u_1, u_2, \dots, u_n)$$

$$= \alpha T(v)$$

hence T is homomorphism.

Theorem

If A and B are finite dimensional subspace of a vector space $V(F)$, then $A+B$ is finite dimensional and $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$.

Proof:

Suppose $\{u_1, u_2, \dots, u_r\}$ be a basis of $A \cap B$.

$\{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_m\}$ be a basis of A .

$\{u_1, u_2, \dots, u_r, w_1, w_2, \dots, w_n\}$ be a basis of B .

then we have to prove that

$\{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_n\}$
is a basis of $A+B$.

Consider

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r + \beta_1 v_1 + \dots + \beta_m v_m + \gamma_1 w_1 + \dots + \gamma_n w_n = 0$$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r + \beta_1 v_1 + \dots + \beta_m v_m = -\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n$$

Since L.H.S of (i) is in A so does R.H.S.

i.e. $-\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n \in A$

Also

$$-\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n \in B \quad \because w_1, w_2, \dots, w_n \text{ is a part of basis of } B.$$

$$\therefore -\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n \in A \cap B$$

$$\Rightarrow -\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n = \delta_1 u_1 + \delta_2 u_2 + \dots + \delta_r u_r$$

as $\{u_1, u_2, \dots, u_r\}$ is a basis of $A \cap B$

$$\forall \delta_i \in F.$$

$$\Rightarrow \delta_1 u_1 + \delta_2 u_2 + \dots + \delta_r u_r + \gamma_1 w_1 + \gamma_2 w_2 + \dots + \gamma_n w_n = 0$$

Since $\{u_1, u_2, \dots, u_r, w_1, w_2, \dots, w_n\}$ is a basis of B (L.I)

$$\Rightarrow \delta_1 = \delta_2 = \dots = \delta_r = \gamma_1 = \gamma_2 = \dots = \gamma_n = 0.$$

so that equation (i) becomes

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m = 0$$

But $\{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_m\}$ is a basis of A

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_r = \beta_1 = \beta_2 = \dots = \beta_m = 0$$

i.e. each $\alpha_i = \beta_i = \gamma_i = 0$

Hence $\{u_1, u_2, \dots, u_r, v_1, \dots, v_m, w_1, \dots, w_n\}$ is L.I

Let $x+y \in A+B$ i.e. $x \in A$ & $y \in B$

As basis of $A = \{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_m\}$

then $x = a_1 u_1 + a_2 u_2 + \dots + a_r u_r + b_1 v_1 + b_2 v_2 + \dots + b_m v_m$

Also basis of $B = \{u_1, u_2, \dots, u_r, w_1, w_2, \dots, w_n\}$

so

$$y = a'_1 u_1 + a'_2 u_2 + \dots + a'_r u_r + b'_1 w_1 + b'_2 w_2 + \dots + b'_n w_n$$

By +ing

$$A+B = (a_1 + a'_1)u_1 + (a_2 + a'_2)u_2 + \dots + (a_r + a'_r)u_r + b_1 v_1 + b_2 v_2 + \dots + b_m v_m + b'_1 w_1 + \dots + b'_n w_n$$

$$\therefore \{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_n\}$$

generates $A+B$

and hence is a basis of $A+B$

$\therefore A+B$ is a finite dimensional and

$$\dim(A+B) = r+m+n$$

$$= (r+m) + (r+n) - r$$

$$= \dim A + \dim B - \dim(A \cap B)$$

proved

Theorem: Let V and W be vector spaces

* If T is an isomorphism of V onto W .

Then T maps a basis of V onto a basis of W .

Proof:

$T: V \rightarrow W$ is isomorphism defined by

$$T(v) = w.$$

Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V .

then we have to prove

$\{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis of W .

i) Consider

$$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_n T(v_n) = 0, \quad \alpha_i \in F$$

$$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \dots + T(\alpha_n v_n) = 0 \quad \because T \text{ is homomorphism}$$

$$\therefore \alpha T(v) = T(\alpha v)$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = 0 \quad T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in \ker T$$

$\because T$ is isomorphism i.e. one-one

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

$\because \{v_1, v_2, \dots, v_n\}$ is basis of V .

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

Hence $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is linearly independent.

ii) Let $w \in W$

$\because T$ is onto there must be $v \in V$ such that

$$T(v) = w.$$

Now $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ for $a_i \in F$.

$$\therefore w = T(v)$$

$$= T(a_1 v_1 + a_2 v_2 + \dots + a_n v_n)$$

$$= T(a_1 v_1) + T(a_2 v_2) + \dots + T(a_n v_n) \quad \because T \text{ is homo.}$$

$$\Rightarrow w = a_1 T(v_1) + a_2 T(v_2) + \dots + a_n T(v_n)$$

i.e. w can be generated by $\{T(v_1), T(v_2), \dots, T(v_n)\}$.

Thus $\{T(v_1), T(v_2), \dots, T(v_n)\}$ form a basis of W .

The proof is complete.

2002

Theorem:-

Two finite dimensional vector space are isomorphic iff they are of the same dimension.

Proof:

Let V and W are two vector spaces of same dimension n and $\{v_1, v_2, \dots, v_n\}$ be the basis of V and $\{w_1, w_2, \dots, w_n\}$ be the basis of W .

Define a mapping

$T: V \rightarrow W$ by $T(v) = w$ for $v \in V, w \in W$.

i.e. $T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$.

i) T is well defined

For $v, v' \in V$, if $v = v'$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \alpha'_1 v_1 + \alpha'_2 v_2 + \dots + \alpha'_n v_n$$

$$\Rightarrow (\alpha_1 - \alpha'_1) v_1 + (\alpha_2 - \alpha'_2) v_2 + \dots + (\alpha_n - \alpha'_n) v_n = 0$$

Since $\{v_1, v_2, \dots, v_n\}$ is basis of V .

$$\therefore \alpha_1 - \alpha'_1 = 0 = \alpha_2 - \alpha'_2 = \dots = \alpha_n - \alpha'_n$$

$$\Rightarrow \alpha_1 = \alpha'_1, \alpha_2 = \alpha'_2, \dots, \alpha_n = \alpha'_n$$

$$\text{i.e. } T(\alpha v) = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$$

$$= \alpha'_1 w_1 + \alpha'_2 w_2 + \dots + \alpha'_n w_n$$

$$= T(v')$$

ii) T is homomorphism

$$T(v + v') = T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + \alpha'_1 v_1 + \dots + \alpha'_n v_n)$$

$$= T((\alpha_1 + \alpha'_1) v_1 + (\alpha_2 + \alpha'_2) v_2 + \dots + (\alpha_n + \alpha'_n) v_n)$$

$$\begin{aligned}
 &= (\alpha_1 + \alpha'_1)w_1 + (\alpha_2 + \alpha'_2)w_2 + \dots + (\alpha_n + \alpha'_n)w_n \\
 &= (\alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n) + (\alpha'_1 w_1 + \alpha'_2 w_2 + \dots + \alpha'_n w_n) \\
 &= T(v) + T(v')
 \end{aligned}$$

and

$$\begin{aligned}
 T(\alpha v) &= T(\alpha(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)) \\
 &= T(\alpha \alpha_1 v_1 + \alpha \alpha_2 v_2 + \dots + \alpha \alpha_n v_n) \\
 &= \alpha \alpha_1 w_1 + \alpha \alpha_2 w_2 + \dots + \alpha \alpha_n w_n \\
 &= \alpha(\alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n) \\
 &= \alpha T(v)
 \end{aligned}$$

iii) T is one-one

$$\text{Let } T(v) = T(v') \quad \text{for } v, v' \in V.$$

$$\Rightarrow \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = T(\alpha'_1 v_1 + \alpha'_2 v_2 + \dots + \alpha'_n v_n)$$

$$\Rightarrow \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n = \alpha'_1 w_1 + \alpha'_2 w_2 + \dots + \alpha'_n w_n$$

$$\Rightarrow (\alpha_1 - \alpha'_1)w_1 + (\alpha_2 - \alpha'_2)w_2 + \dots + (\alpha_n - \alpha'_n)w_n = 0$$

$\because \{w_1, w_2, \dots, w_n\}$ is basis of W .

$$\Rightarrow \alpha_1 - \alpha'_1 = \alpha_2 - \alpha'_2 = \dots = \alpha_n - \alpha'_n = 0$$

$$\Rightarrow \alpha_1 = \alpha'_1, \alpha_2 = \alpha'_2, \dots, \alpha_n = \alpha'_n$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \alpha'_1 v_1 + \alpha'_2 v_2 + \dots + \alpha'_n v_n$$

$$\Rightarrow v = v'$$

iv) T is onto as every element

$$\alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n \in W$$

is image of $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in V$

Conversely, let $T: V \rightarrow W$ is isomorphism

then we have to prove

Dimension of V and W are same

Let $\{v_1, v_2, \dots, v_n\}$ be basis of V . then we prove that

$\{T(v_1), T(v_2), T(v_3), \dots, T(v_n)\}$ is a basis of W .

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Vector Space Homomorphisms

Let V and W are two vector spaces

The set of all homomorphism of V into W is denoted by $\text{Hom}(V, W)$

$$\text{Hom}(V, W) = \{T_1, T_2, \dots, T_n\}$$

where each T_i is homomorphism

Theorem:

Let $V(F)$ & $W(F)$ be two vector spaces introduce an operation in $\text{Hom}(V, W)$ and prove that $\text{Hom}(V, W)$ is a vector space under this operation.

Proof:

Let $T_1, T_2 \in \text{Hom}(V, W)$

we define $(T_1 + T_2)(v) = T_1(v) + T_2(v)$

$$\& \quad \lambda T(v) = T(\lambda v)$$

to prove $\text{Hom}(V, W)$ is a vector space we proceed as follows:

Let $v_1, v_2 \in V$ & $T_1, T_2 \in \text{Hom}(V, W)$

Then

$$\begin{aligned} (T_1 + T_2)(v_1 + v_2) &= T_1(v_1 + v_2) + T_2(v_1 + v_2) \\ &= T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2) \\ &= T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2) \\ &= (T_1 + T_2)v_1 + (T_1 + T_2)v_2 \end{aligned}$$

Also

$$\begin{aligned} (T_1 + T_2)(\lambda v) &= T_1(\lambda v) + T_2(\lambda v) \\ &= \lambda T_1(v) + \lambda T_2(v) \end{aligned}$$

$$\Rightarrow (T_1 + T_2)(\lambda v) = \lambda (T_1 + T_2)(v)$$

$$\Rightarrow T_1 + T_2 \in \text{Hom}(V, W)$$

i.e $\text{Hom}(V, W)$ is closed

iii) Mapping (T_1, T_2, \dots, T_n) are associative in general

Consider a mapping T_0 which maps an

element of V into 0 (zero) i.e.

$$(v) T_0(v) = 0$$

$$\begin{aligned} \text{then } (T + T_0)v &= T(v) + T_0(v) \\ &= T(v) + 0 \\ &= T(v) \end{aligned}$$

$$\text{i.e. } T_0 + T = T$$

i.e. T_0 is the identity of $\text{Hom}(V, W)$
also for $T \in \text{Hom}(V, W)$

so we have

$$\begin{aligned} (-T) &\in \text{Hom}(V, W) \text{ such that} \\ (T + (-T))v &= T(v) + (-1)T(v) \\ &= T(v) - T(v) = 0 \\ &= T_0(v) \end{aligned}$$

\Rightarrow inverse exists.

$$\begin{aligned} \forall (T_1 + T_2)v &= T_1(v) + T_2(v) \\ &= T_2(v) + T_1(v) \\ &= (T_2 + T_1)v \end{aligned}$$

$\Rightarrow \text{Hom}(V, W)$ is an abelian group under '+'

(ii)

$$\begin{aligned} a(T_1 + T_2) &= aT_1 + aT_2 \\ a(T_1 + T_2)(v) &= (T_1 + T_2)(av) \\ &= T_1(av) + T_2(av) \\ &= aT_1(v) + aT_2(v) \end{aligned}$$

(iii)

$$\begin{aligned} (a+b)T &= aT + bT \\ (a+b)T(v) &= T((a+b)v) \\ &= T(av + bv) \\ &= aT(v) + bT(v) \end{aligned}$$

~~Hand~~ W
is a vector space

$$(iv) \quad a(b)T = (ab)T$$

$$\begin{aligned} a(b)T(v) &= aT(bv) = T((a)bv) = T(abv) \\ &= abT(v) \end{aligned}$$

P.T.O

$$v) \quad 1. \quad T \equiv T$$

$$\text{As } 1. \quad T(u) = T(1 \cdot u) = T(u)$$

As $V \in \mathcal{V}$ is a vector space

Hence $\text{Hom}(V, W)$ is a vector space.

Theorem:-

If V and W are of dimension m and n resp. then $\text{Hom}(V, W)$ is of dimension mn .

Proof:

Let $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ be basis of V and W respectively.

Define a mapping

$T_{ij} : V \rightarrow W$ defined by

$$T_{ij}(v_k) = \begin{cases} \lambda_i w_j & \text{if } i=k \\ 0 & \text{if } i \neq k \end{cases}, \lambda_{ij} \in F.$$

Let

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m$$

$$u = \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_m v_m$$

then

$$\begin{aligned} T_{ij}(u + v) &= T_{ij}(\mu_1 v_1 + \mu_2 v_2 + \dots + \mu_m v_m + (\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m)) \\ &= T_{ij}((\mu_1 + \lambda_1) v_1 + (\mu_2 + \lambda_2) v_2 + \dots + (\mu_m + \lambda_m) v_m) \\ &= (\mu_i + \lambda_i) w_j \\ &= \mu_i w_j + \lambda_i w_j \\ &= T_{ij}(u) + T_{ij}(v) \end{aligned}$$

And

$$\begin{aligned} T_{ij}(\alpha u) &= T_{ij}(\alpha(\mu_1 v_1 + \mu_2 v_2 + \dots + \mu_m v_m)) \\ &= T_{ij}(\alpha \mu_1 v_1 + \alpha \mu_2 v_2 + \dots + \alpha \mu_m v_m) \\ &= \alpha \mu_i w_j \\ &= \alpha T_{ij}(u) \end{aligned}$$

Thus T_{ij} is homomorphism and $T_{ij} \in \text{Hom}(V, W)$.

Now to prove $\{T_{11}, T_{12}, \dots, T_{ij}, \dots, T_{mn}\}$ is a basis.

Consider

$$\alpha_{11} T_{11} + \alpha_{12} T_{12} + \dots + \alpha_{ij} T_{ij} + \dots + \alpha_{mn} T_{mn} = 0$$

Now

$$\left(\alpha_{11}T_{11} + \alpha_{12}T_{12} + \dots + \alpha_{1n}T_{1n} + \alpha_{21}T_{21} + \alpha_{22}T_{22} + \dots + \alpha_{2n}T_{2n} + \dots + \alpha_{m1}T_{m1} + \alpha_{m2}T_{m2} + \dots + \alpha_{mn}T_{mn} \right) v_1 = 0 \quad (i)$$

$$\Rightarrow \alpha_{11}T_{11}(v_1) + \alpha_{12}T_{12}(v_1) + \dots + \alpha_{1n}T_{1n}(v_1) + \alpha_{21}T_{21}(v_1) + \alpha_{22}T_{22}(v_1) + \dots + \alpha_{2n}T_{2n}(v_1) + \dots + \alpha_{m1}T_{m1}(v_1) + \alpha_{m2}T_{m2}(v_1) + \dots + \alpha_{mn}T_{mn}(v_1) = 0$$

$$\Rightarrow \left. \begin{aligned} &\alpha_{11}\lambda_1 w_1 + \alpha_{12}\lambda_1 w_2 + \dots + \alpha_{1n}\lambda_1 w_n \\ &+ 0 + 0 + \dots + 0 \\ &+ \dots \\ &+ 0 + 0 + \dots + 0 = 0 \end{aligned} \right\} \begin{aligned} &T_{ij}(v_k) \\ &= \lambda_j \omega_j, i=k \\ &= 0, i \neq k \end{aligned}$$

$$\Rightarrow \alpha_{11}w_1 + \alpha_{12}w_2 + \dots + \alpha_{1n}w_n = 0 \quad \because \lambda_1 \neq 0$$

and $\{w_1, w_2, \dots, w_n\}$ is basis of W

$$\Rightarrow \alpha_{11} = 0 = \alpha_{12} = \alpha_{13} = \dots = \alpha_{1n}$$

Similarly operating (i) on v_2 we have

$$\alpha_{ij} = 0, \quad i=1, 2, \dots, m, \quad j=1, 2, \dots, n.$$

So the set $\{T_{11}, T_{12}, \dots, T_{ij}, \dots, T_{mn}\}$ is L.I.

Now consider

$$S_0 = \alpha_{11}T_{11} + \alpha_{12}T_{12} + \dots + \alpha_{1n}T_{1n} + \alpha_{21}T_{21} + \alpha_{22}T_{22} + \dots + \alpha_{2n}T_{2n} + \dots + \alpha_{m1}T_{m1} + \alpha_{m2}T_{m2} + \dots + \alpha_{mn}T_{mn}$$

So

$$S_0(v_1) = \left(\alpha_{11}T_{11} + \alpha_{12}T_{12} + \dots + \alpha_{1n}T_{1n} + \alpha_{21}T_{21} + \alpha_{22}T_{22} + \dots + \alpha_{2n}T_{2n} + \dots + \alpha_{m1}T_{m1} + \alpha_{m2}T_{m2} + \dots + \alpha_{mn}T_{mn} \right) v_1$$

$$\begin{aligned} \Rightarrow S_1(v_1) &= a_{11}T_{11}(v_1) + a_{12}T_{12}(v_1) + \dots + a_{1n}T_{1n}(v_1) \\ &+ a_{21}T_{21}(v_1) + a_{22}T_{22}(v_1) + \dots + a_{2n}T_{2n}(v_1) \\ &+ \dots \\ &+ a_{m1}T_{m1}(v_1) + a_{m2}T_{m2}(v_1) + \dots + a_{mn}T_{mn}(v_1) \\ &= a_{11}\lambda_1 w_1 + a_{12}\lambda_1 w_2 + a_{13}\lambda_1 w_3 + \dots + a_{1n}\lambda_1 w_n \end{aligned}$$

Similarly

$$S_1(v_2) = a_{21}\lambda_2 w_1 + a_{22}\lambda_2 w_2 + a_{23}\lambda_2 w_3 + \dots + a_{2n}\lambda_2 w_n$$

$$S_1(v_k) = a_{k1}\lambda_k w_1 + a_{k2}\lambda_k w_2 + a_{k3}\lambda_k w_3 + \dots + a_{kn}\lambda_k w_n$$

Let $S \in \text{Hom}(V, W)$

$$\Rightarrow S(v_1), S(v_2), \dots, S(v_k) \in W$$

so

$$S(v_1) = a_{11}w_1 + a_{12}w_2 + \dots + a_{1n}w_n$$

$$S(v_2) = a_{21}w_1 + a_{22}w_2 + \dots + a_{2n}w_n$$

$$S(v_k) = a_{k1}w_1 + a_{k2}w_2 + \dots + a_{kn}w_n$$

i.e. $S \in S_0$ so $S_0 \in \text{Hom}(V, W)$

Thus

$\{T_{11}, T_{12}, \dots, T_{ij}, \dots, T_{mn}\}$ form a basis of $\text{Hom}(V, W)$

$$\Rightarrow \dim(\text{Hom}(V, W)) = mn$$

Definition: (Dual Space):-

Let V be a vector space over a field F . Then $\text{Hom}(V, F)$ is called dual space and is denoted by V^* or \hat{V} . Its elements are called linear functionals.

Theorem:-

If V is finite dimensional vector space over F , then prove $V \cong V^*$.

Proof:

Since $\dim V = \dim V^*$
so consider $\dim V = \dim V^* = m$

Define a mapping $T: V \rightarrow V^*$ by

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_m f_m$$

i) T is homomorphism:-

$$\begin{aligned} T(v + v') &= T[(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m) + (\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m)] \\ &= T[(\alpha_1 + \beta_1)v_1 + (\alpha_2 + \beta_2)v_2 + \dots + (\alpha_m + \beta_m)v_m] \\ &= (\alpha_1 + \beta_1)f_1 + (\alpha_2 + \beta_2)f_2 + \dots + (\alpha_m + \beta_m)f_m \\ &= (\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_m f_m) + (\beta_1 f_1 + \beta_2 f_2 + \dots + \beta_m f_m) \\ &= T(v) + T(v') \end{aligned}$$

and

$$\begin{aligned} T(\alpha v) &= T(\alpha(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m)) \\ &= T(\alpha \alpha_1 v_1 + \alpha \alpha_2 v_2 + \dots + \alpha \alpha_m v_m) \\ &= \alpha \alpha_1 f_1 + \alpha \alpha_2 f_2 + \dots + \alpha \alpha_m f_m \\ &= \alpha(\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_m f_m) \\ &= \alpha T(v) \end{aligned}$$

ii) T is one-one

$$\text{if } T(v) = T(v')$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m) = T(\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m)$$

$$\Rightarrow \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_m f_m = \beta_1 f_1 + \beta_2 f_2 + \dots + \beta_m f_m$$

$$\Rightarrow (\alpha_1 - \beta_1)f_1 + (\alpha_2 - \beta_2)f_2 + \dots + (\alpha_m - \beta_m)f_m = 0$$

$\therefore \{f_1, f_2, \dots, f_m\}$ is basis of V^*

$$\therefore \alpha_1 - \beta_1 = 0 = \alpha_2 - \beta_2 = \dots = \alpha_m - \beta_m$$

$$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_m = \beta_m$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m$$

$$\Rightarrow v = v'$$

iii) T is onto

Since for $\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_m f_m \in V^*$
we have

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m \in V$$

such that

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_m f_m$$

Thus T is onto.

and hence $V \cong V^*$

Definition:

Let $T: V_1 \rightarrow V_2$ is homomorphism of a vector space $V_1(F)$ to a vector space $V_2(F)$ then $\ker T$ is called null space denoted by $N(T)$.

The dimension of $N(T)$ is called nullity.

* Theorem:

Let $T: V_1 \rightarrow V_2$ be a vector space homomorphism then $\dim V_1 = \dim N(T) + \dim R(T)$.

Proof:

Let $\dim N(T) = m$

and $\dim(V_1) = n$

Let $\{v_1, v_2, \dots, v_m\}$ be basis of $N(T) = \ker T$.

Since $N(T) = \ker T$ is a subspace of V_1

\therefore we can take basis of V_1

$\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$

we have to prove

$\{T(v_{m+1}), T(v_{m+2}), \dots, T(v_n)\}$ form basis of $R(T)$

Let $w \in R(T)$ then there is $v \in V_1$ such that

$$T(v) = w$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n) = w$$

$$\Rightarrow \alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_m T(v_m) + \alpha_{m+1} T(v_{m+1}) + \dots + \alpha_n T(v_n) = w$$

$$\because \{v_1, v_2, \dots, v_m\} \in N(T) = \ker T$$

$$\therefore T(v_1) = 0, T(v_2) = 0, \dots, T(v_m) = 0$$

$$\Rightarrow \alpha_{m+1} T(v_{m+1}) + \alpha_{m+2} T(v_{m+2}) + \dots + \alpha_n T(v_n) = w$$

$$\Rightarrow T(v_{m+1}), T(v_{m+2}), \dots, T(v_n) \text{ generates } R(T)$$

Now consider

$$\beta_{m+1} T(v_{m+1}) + \beta_{m+2} T(v_{m+2}) + \dots + \beta_n T(v_n) = 0$$

$$\Rightarrow T(\beta_{m+1} v_{m+1}) + T(\beta_{m+2} v_{m+2}) + \dots + T(\beta_n v_n) = 0$$

$\therefore T$ is homomorphism

$$\Rightarrow T(\beta_{m+1} v_{m+1} + \beta_{m+2} v_{m+2} + \dots + \beta_n v_n) = 0$$

$$\Rightarrow \beta_{m+1} v_{m+1} + \beta_{m+2} v_{m+2} + \dots + \beta_n v_n \in \text{Ker } T = N(T)$$

Since $\{v_1, v_2, \dots, v_m\}$ is basis of $N(T)$

so $\exists \delta_1, \delta_2, \dots, \delta_m \in F$ such that

$$\beta_{m+1} v_{m+1} + \beta_{m+2} v_{m+2} + \dots + \beta_n v_n = \delta_1 v_1 + \delta_2 v_2 + \dots + \delta_m v_m$$

$$\Rightarrow \delta_1 v_1 + \delta_2 v_2 + \dots + \delta_m v_m - \beta_{m+1} v_{m+1} - \beta_{m+2} v_{m+2} - \dots - \beta_n v_n = 0$$

As $\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ is basis of V

$$\text{therefore } \delta_1 = \delta_2 = \dots = \delta_m = \beta_{m+1} = \beta_{m+2} = \dots = \beta_n = 0$$

$$\text{i.e. } \beta_{m+1} = \beta_{m+2} = \dots = \beta_n = 0$$

$$\Rightarrow \{T(v_{m+1}), T(v_{m+2}), \dots, T(v_n)\} \text{ is L.I.}$$

and hence form a basis of $R(T)$.

$$\text{So } \dim R(T) = n - m \\ = \dim V - \dim N(T)$$

$$\Rightarrow \dim V = \dim N(T) + \dim R(T)$$

proved

Theorem

$\therefore V$ is a vector space over F and $\{v_1, v_2, \dots, v_n\}$ be a basis of V . Let $\varphi_1, \varphi_2, \dots, \varphi_n \in V^* = \text{Hom}(V, F)$ are linear functional defined by

$$\varphi_i(v_j) = \delta_{ij} = \begin{cases} 1 & ; i=j \\ 0 & ; i \neq j \end{cases}$$

Then $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is a basis of V^* .

Proof.

Let $\varphi \in V^*$ be taken

$$\varphi(v_1) = k_1, \quad \varphi(v_2) = k_2, \quad \dots, \quad \varphi(v_n) = k_n$$

where $k_1, k_2, \dots, k_n \in F$

Let

$$\psi = k_1\varphi_1 + k_2\varphi_2 + k_3\varphi_3 + \dots + k_n\varphi_n$$

$$\begin{aligned} \varphi(v_1) &= \psi(v_1) = (k_1\varphi_1 + k_2\varphi_2 + \dots + k_n\varphi_n)(v_1) \\ &= k_1\varphi_1(v_1) + k_2\varphi_2(v_1) + \dots + k_n\varphi_n(v_1) \\ &= k_1(1) + k_2(0) + \dots + k_n(0) \\ &= k_1 \end{aligned}$$

Also

$$\begin{aligned} \psi(v_2) &= (k_1\varphi_1 + k_2\varphi_2 + \dots + k_n\varphi_n)(v_2) \\ &= k_1\varphi_1(v_2) + k_2\varphi_2(v_2) + \dots + k_n\varphi_n(v_2) \\ &= k_1(0) + k_2(1) + k_3(0) + \dots + k_n(0) \\ &= k_2 \end{aligned}$$

$$\Rightarrow \psi(v_i) = k_i = \varphi(v_i)$$

$$\text{i.e. } \psi = \varphi$$

$$\Rightarrow \varphi = \psi = k_1\varphi_1 + k_2\varphi_2 + \dots + k_n\varphi_n$$

$$\text{So } \{\varphi_1, \varphi_2, \dots, \varphi_n\} \text{ span } V^*$$

To prove $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is a linearly independent.

Consider

$$a_1\varphi_1 + a_2\varphi_2 + \dots + a_n\varphi_n = 0$$

then operating it on v_1

$$(a_1\phi_1 + a_2\phi_2 + \dots + a_n\phi_n)v_1 = 0 \cdot v_1$$

$$\Rightarrow a_1\phi_1(v_1) + a_2\phi_2(v_1) + \dots + a_n\phi_n(v_1) = 0$$

$$\Rightarrow a_1(1) + a_2(0) + \dots + a_n(0) = 0$$

$$\Rightarrow a_1 = 0$$

Similarly for $i = 2, 3, \dots, n$

$$(a_1\phi_1 + a_2\phi_2 + \dots + a_n\phi_n)v_i = 0 \cdot v_i$$

$$\Rightarrow a_1\phi_1(v_i) + a_2\phi_2(v_i) + \dots + a_i\phi_i(v_i) + \dots + a_n\phi_n(v_i) = 0$$

$$\Rightarrow a_1(0) + a_2(0) + \dots + a_i(1) + \dots + a_n(0) = 0$$

$$\Rightarrow 0 + 0 + \dots + a_i + \dots + 0 = 0$$

$$\Rightarrow a_i = 0$$

$$\text{i.e. } a_1 = 0, a_2 = 0, a_3 = 0, \dots, a_n = 0$$

Hence $\{\phi_1, \phi_2, \dots, \phi_n\}$ is L.I and so is
a basis of V^*

* Example

Consider the basis of

$$\mathbb{R}^2 = \{v_1 = (2, 1), v_2 = (3, 1)\}$$

Find dual basis of $\{\phi_1, \phi_2\}$.

Solution.

$$\phi_1(v_1) = 1, \quad \phi_1(v_2) = 0$$

$$\phi_2(v_1) = 0, \quad \phi_2(v_2) = 1$$

Since ϕ_1, ϕ_2 are linear functional

$$\phi_1(x, y) = ax + by$$

and $\phi_2(x, y) = cx + dy$

$$\phi_1(v_1) = 1$$

$$\Rightarrow \phi_1(2, 1) = 1 \Rightarrow 2a + b = 1 \quad \text{--- (i)}$$

$$\phi_1(v_2) = 0$$

$$\Rightarrow \phi_1(3, 1) = 0 \Rightarrow 3a + b = 0 \quad \text{--- (ii)}$$

By (i) and (ii)

$$a = -1 \quad \text{and} \quad b = 3$$

Now $\phi_2(v_1) = 0$

$$\phi_2(2, 1) = 0 \Rightarrow 2c + d = 0 \quad \text{--- (iii)}$$

and $\phi_2(v_2) = 1$

$$\phi_2(3, 1) = 1 \Rightarrow 3c + d = 1 \quad \text{--- (iv)}$$

Solving (iii) and (iv)

$$c = 1 \quad \text{and} \quad d = -2$$

therefore $\phi_1 = -x + 3y$

$$\phi_2 = x - 2y$$

* Example

Let a basis of \mathbb{R}^3 is $\{v_1, v_2, v_3\}$

$$v_1 = \{1, -1, 3\}, \quad v_2 = \{0, 1, -1\}, \quad v_3 = \{0, 3, -2\}$$

Find dual basis ϕ_1, ϕ_2 and ϕ_3

such that $\phi_i(v_j) = \begin{cases} 1 & ; i=j \\ 0 & ; i \neq j \end{cases}$

Do yourself as above

* Question

Let $V = \{a + bt : a, b \in \mathbb{R}\}$ be a vector space of polynomial of degree ≤ 1 .

Let $\phi_1, \phi_2 : V \rightarrow \mathbb{R}$ be defined by

$$\phi_1(f(t)) = \int_0^1 f(t) dt$$

$$\phi_2(f(t)) = \int_0^2 f(t) dt$$

where $\phi_1, \phi_2 \in V^*$ (dual space).

Find corresponding basis v_1, v_2 of V .

Solution:

$$\text{let } v_1 = a + bt \text{ and } v_2 = \cancel{a+bt} c + dt$$

By definition

$$\phi_1(v_1) = 1, \quad \phi_1(v_2) = 0, \quad \phi_2(v_1) = 0, \quad \phi_2(v_2) = 1$$

$$\phi_1(v_1) = 1$$

$$\Rightarrow \int_0^1 v_1 dt = 1 \Rightarrow \int_0^1 (a + bt) dt = 1$$

$$\Rightarrow \left[at + \frac{bt^2}{2} \right]_0^1 = 1 \Rightarrow a + \frac{b}{2} = 1$$

$$\Rightarrow 2a + b = 2 \quad \text{--- (i)}$$

$$\phi_2(v_1) = 0$$

$$\int_0^2 (a + bt) dt = 0 \Rightarrow \left[at + \frac{bt^2}{2} \right]_0^2 = 0$$

$$\Rightarrow 2a + 2b = 0 \Rightarrow a + b = 0 \quad \text{--- (ii)}$$

By (i) and (ii)

$$2a + b = 2$$

$$\underline{a + b = 0}$$

$$\underline{a} = 2 \quad \Rightarrow \quad b = -2$$

Further $\phi_1(v_2) = 0$

$$\Rightarrow \int_0^1 v_2 dt = 0 \Rightarrow \int_0^1 (c + dt) dt = 0$$

$$\Rightarrow \left| ct + \frac{dt^2}{2} \right|_0^1 = 0$$

$$\Rightarrow c + \frac{d}{2} = 0 \quad \text{or} \quad 2c + d = 0 \quad \text{--- (iii)}$$

$$\Phi_2(v_2) = 1$$

$$\Rightarrow \int_0^2 v_2 dt = 1$$

$$\Rightarrow \int_0^2 (c + dt) dt = 1 \quad \Rightarrow \left| ct + \frac{dt^2}{2} \right|_0^2 = 1$$

$$\Rightarrow 2c + 2d = 1 \quad \Rightarrow \text{--- (iv)}$$

Subtracting (iii) from (iv)

$$2c + 2d = 1$$

$$\underline{-2c + d = 0}$$

$$d = 1$$

$$\Rightarrow c = -\frac{1}{2}$$

hence

$$v_1 = 2 - 2t$$

and $v_2 = -\frac{1}{2} + t$ are basis of V
corresponding to dual basis V^*

Eigen Value

def:- let 'A' be a n square matrix, then $\lambda \in F$ is eigen value of A if there exist a non-zero column vector $v \in F^n$ such that $A \cdot v = \lambda v$

here v is an eigen vector corresponding to eigen value λ :

Exercise

Find eigen values and associative eigen vector of a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Solution:-

$$\text{Let } v = \begin{bmatrix} x \\ y \end{bmatrix}^T = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore Av = \lambda v$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 2y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

$$\Rightarrow \begin{cases} x + 2y = \lambda x \\ 3x + 2y = \lambda y \end{cases}$$

$$\Rightarrow \begin{cases} x + 2y = \lambda x \\ 3x + 2y = \lambda y \end{cases}$$

$$\text{or } (1 - \lambda)x + 2y = 0 \quad \text{--- (1)}$$

$$3x + (2 - \lambda)y = 0 \quad \text{--- (2)}$$

For non-trivial solution

$$\begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 - \lambda)(2 - \lambda) - 6 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = 4, -1$$

hence $\lambda = 4, -1$ are eigen values

$$\lambda = -1 \text{ in eq (i)} \Rightarrow 2x + 2y = 0$$

$$\text{or } x + y = 0$$

$$y = -x$$

thus

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

i.e eigen vector is $[1 \ -1]^t$

$$\text{and } \lambda = 4 \text{ in eq (i)} \Rightarrow -3x + 2y = 0$$

$$\Rightarrow 2y = 3x$$

$$\text{or } y = \frac{3}{2}x$$

thus

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \frac{3}{2}x \end{pmatrix} = \frac{x}{2} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

i.e eigen vector is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}^t$

Note

$$Av = \lambda v$$

$$\Rightarrow Av - \lambda v = 0$$

$$\Rightarrow (A - \lambda I)v = 0 \quad \text{where } I \text{ is identity}$$

$$\Rightarrow |A - \lambda I| = 0$$

and

$$Av = \lambda v$$

$$A(kv) = k\lambda v = p(\lambda - 1) + r\lambda$$

$$= k(\lambda v)$$

$$= \lambda(kv)$$

then λ and k are eigen values for A .

Eigen Value & Eigen Vector (Alternative)

def:- Let $T: V \rightarrow V$ be a linear operator then $\lambda \in F$ is called eigen value of T if there exist a non-zero vector v such that

$$T(v) = \lambda v$$

here v is eigen vector.

Note that kv is also eigen vector for same eigen value λ .

$$T(kv) = kT(v)$$

$$= k\lambda v = \lambda kv$$

Theorem:-

Let λ be an eigen value of an operator $T: V \rightarrow V$. Let V_λ denotes set of all eigen vectors of T belonging to same eigen value λ . The V_λ is a subspace of V .

Proof:

~~let λ be an eigen value of an operator~~

let $v, w \in V_\lambda$.

then $T(v) = \lambda v$ and $T(w) = \lambda w$

$$\begin{aligned} \text{Now } T(av + bw) &= T(av) + T(bw) \\ &= aT(v) + bT(w) \\ &= a\lambda v + b\lambda w \\ &= \lambda(av + bw) \end{aligned}$$

$\Rightarrow av + bw$ is also an eigen vector for λ .

hence $av + bw \in V_\lambda$

$\Rightarrow V_\lambda$ is a subspace

Theorem

\therefore Let $\{v_1, v_2, \dots, v_n\}$ be non-zero eigen vectors of an operator T corresponding to distinct eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively then $\{v_1, v_2, \dots, v_n\}$ is linearly independent.

Proof:

We prove the theorem by Mathematical Induction.

Let $n=1$ so if $av_1 = 0$

$$\Rightarrow a = 0 \quad \text{as } v_1 \neq 0$$

so condition I is true.

Let the theorem is true for $k = n-1$

i.e. v_1, v_2, \dots, v_{n-1} are L.I (linearly independent)

then $a_1 v_1 + a_2 v_2 + \dots + a_{n-1} v_{n-1} = 0$

$$\Rightarrow a_1 = a_2 = \dots = a_{n-1} = 0$$

Consider

$$b_1 v_1 + b_2 v_2 + \dots + b_n v_n = 0 \quad \text{(i)}$$

$$T(b_1 v_1 + b_2 v_2 + \dots + b_n v_n) = 0$$

$$\Rightarrow T(b_1 v_1) + T(b_2 v_2) + \dots + T(b_n v_n) = 0$$

$$\Rightarrow b_1 T(v_1) + b_2 T(v_2) + \dots + b_n T(v_n) = 0$$

$$\Rightarrow b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \dots + b_n \lambda_n v_n = 0$$

or

$$b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \dots + b_{n-1} \lambda_{n-1} v_{n-1} + b_n \lambda_n v_n = 0$$

(ii)

Multiplying eq (i) by λ_n

$$\lambda_n b_1 v_1 + \lambda_n b_2 v_2 + \dots + \lambda_n b_{n-1} v_{n-1} + \lambda_n b_n v_n = 0$$

(iii)

Subtracting (iii) from (ii)

$$b_1(\lambda_1 - \lambda_n)v_1 + b_2(\lambda_2 - \lambda_n)v_2 + \dots + b_{n-1}(\lambda_{n-1} - \lambda_n)v_{n-1} = 0$$

Since $\{v_1, v_2, \dots, v_{n-1}\}$ is L.I

$$\Rightarrow b_1 = b_2 = \dots = b_{n-1} = 0$$

$$\because \lambda_i - \lambda_n \neq 0 \quad ; \quad i = 1, 2, \dots, n-1$$

because if $\lambda_i - \lambda_n = 0$

$$\Rightarrow \lambda_i = \lambda_n \quad \text{for } i = 1, 2, \dots, n-1$$

a contradiction as each $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct.

Now from eq (ii)

$$0 + 0 + \dots + 0 + b_n v_n = 0$$

$$\Rightarrow b_n v_n = 0$$

$$\Rightarrow b_n = 0 \quad \because v_n \neq 0$$

hence the vectors v_1, v_2, \dots, v_n are linearly independent.

$$v = (a_1 v_1) + \dots + (a_n v_n)$$

Characteristic Polynomial / Equation / Matrix :-

def:- Let A be a n square matrix over F .

then $tI - A$ is called characteristic matrix.

$|tI - A|$ is ^{called} characteristic polynomial.

and $|tI - A| = 0$ is called characteristic equation.

i.e

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$tI - A = t \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} t - a_{11} & -a_{12} & -a_{13} & \dots & -a_{1n} \\ -a_{21} & t - a_{22} & -a_{23} & \dots & -a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \dots & t - a_{nn} \end{bmatrix}$$

and $\Delta_A(t) = \det(tI - A)$ is characteristic polynomial.

Also $\Delta_A(t) = 0$ or $|tI - A| = 0$ is characteristic equation.

Exercise:

Find characteristic polynomial of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$tI - A = t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$= \begin{pmatrix} t-1 & -3 & 0 \\ 2 & t-2 & 1 \\ -4 & 0 & t+2 \end{pmatrix}$$

$$\Delta_A(t) = |tI - A|$$

$$= \begin{vmatrix} t-1 & -3 & 0 \\ 2 & t-2 & 1 \\ -4 & 0 & t+2 \end{vmatrix}$$

$$= t^3 - t^2 + 2t + 28 \text{ is characteristic polynomial.}$$

$$\text{Also } \Delta_A(t) = 0$$

$$\Rightarrow t^3 - t^2 + 2t + 28 = 0 \text{ is characteristic equation}$$

Note: Degree of eq. will be equal to the order of matrix.

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$tI - A = t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} t-2 & -3 \\ -1 & t-5 \end{bmatrix}$$

$$B(t) = \text{adj of } (tI - A) = \begin{bmatrix} t-5 & 3 \\ 1 & t-2 \end{bmatrix}$$

$$= \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} -5 & 3 \\ 1 & -2 \end{bmatrix} = B_1 t + B_0$$

$$\text{If } A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\text{Then } B(t) = \text{adj}(tI - A)$$

$$= B_2 t^2 + B_1 t + B_0$$

Cayley Hamilton Theorem:

∴ Every square matrix is zero of its characteristic polynomial.

OR Every square matrix satisfies its characteristic equation.

Proof:

Let A be n square matrix

and $\Delta_A(t) = |tI - A|$ be its characteristic polynomial.

i.e. $\Delta_A(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \dots + a_1t + a_0$

Let $B(t)$ is adjoint of $tI - A$.

Since elements of $B(t)$ are cofactors of $tI - A$ and so are polynomial of degree not more than $n-1$ and we can write

$$B(t) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \dots + B_1t + B_0$$

where B_i are square matrices of order n over F .

Since by definition of adjoint of a matrix

$$(tI - A)B(t) = |tI - A|I$$

$$\begin{aligned} \Rightarrow (tI - A)(B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \dots + B_1t + B_0) \\ = (t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \dots + a_1t + a_0)I \end{aligned}$$

Comparing the coefficients.

$$\text{Comparing } t^n \Rightarrow B_{n-1}I = I$$

$$\text{" } t^{n-1} \Rightarrow B_{n-2}I - AB_{n-1} = a_{n-1}I$$

$$\text{" } t^{n-2} \Rightarrow B_{n-3}I - AB_{n-2} = a_{n-2}I$$

$$\text{" } t^1 \Rightarrow B_0I - AB_1 = a_1I$$

$$\text{" } t^0 \Rightarrow -AB_0 = a_0I$$

Multiplying above equations by first to last by $A^n, A^{n-1}, A^{n-2}, \dots, A, I$ respectively.

we have.

$$A^n B_{n-1} I = A^n I$$

$$A^{n-1} B_{n-2} I - A^n B_{n-1} I = \alpha_{n-1} A^{n-1} I$$

$$A^{n-2} B_{n-3} I - A^{n-1} B_{n-2} I = \alpha_{n-2} A^{n-2} I$$

$$A B_0 I - A^2 B_1 = \alpha_1 A I$$

$$-A B_0 I = \alpha_0 I$$

Adding both sides of above equations

$$0 = A^n + \alpha_{n-1} A^{n-1} + \alpha_{n-2} A^{n-2} + \dots + \alpha_1 A + \alpha_0$$

As required.

Minimum Polynomial

A polynomial $m(t)$ is called minimum polynomial if

- i) $m(t)$ divides $\Delta(t)$
- ii) Each irreducible factor of $\Delta(t)$ divides $m(t)$
- iii) $m(A) = 0$.

Question:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$tI - A = \begin{bmatrix} t-2 & -1 & 0 & 0 \\ 0 & t-2 & 0 & 0 \\ 0 & 0 & t-1 & -1 \\ 0 & 0 & 2 & t-4 \end{bmatrix}$$

$$|tI - A| = \begin{vmatrix} t-2 & -1 & 0 & 0 \\ 0 & t-2 & 0 & 0 \\ 0 & 0 & t-1 & -1 \\ 0 & 0 & -2 & t-4 \end{vmatrix}$$

expanding by c_1

$$= (t-2) \begin{vmatrix} t-2 & 0 & 0 \\ 0 & t-2 & -1 \\ 0 & -2 & t-4 \end{vmatrix}$$

$$= (t-2)(t-2) \begin{vmatrix} t-2 & -1 \\ 2 & t-4 \end{vmatrix}$$

$$= (t-2)^3 (t-3) = (t-2)^2 ((t-2)(t-4) + 2)$$

$$= (t^2 - 4t + 4)(t^2 - 6t + 8 + 2)$$

$$= t^4 - 10t^3 - 4t^2 + 40t + 64 \quad (\text{after solving})$$

is characteristic polynomial.

Possible minimum polynomials are

$$i) (t-2)(t-3) = f(t)$$

$$ii) (t-2)^2(t-3) = g(t)$$

$$iii) (t-2)^3(t-3) = h(t)$$

$$f(A) = (A-2)(A-3)$$

$$= (A-2I)(A-3I)$$

$$= \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 2 & 0 & 0 & -2 & 1 \end{array} \right] \neq 0$$

$\Rightarrow f(t)$ is not minimum polynomial.

$$\text{Now } g(t) = (t-2)^2(t-3)$$

$$\Rightarrow g(A) = (A-2)^2(A-3)$$

$$= \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 2 & 0 & 0 & -2 & 1 \end{array} \right]$$

= solve = 0

$\Rightarrow g(t) = (t-2)^2(t-3)$ is minimum polynomial

$$iii) h(A) = (A-2)^3(A-3)$$

Do yourself

Theorem: -

∴ Prove that the minimum polynomial $m(t)$ divides every polynomial which has A as a zero.

In particular $m(t)$ divides the characteristic polynomial $\Delta(t)$ of A .

Proof:

Let $f(t)$ be a polynomial for which $f(A) = 0$. then by division algorithm, there are polynomial $q(t)$ and $r(t)$ such that

$$f(t) = q(t) \cdot m(t) + r(t) \quad \text{--- (i)}$$

where $r(t) = 0$ or degree of $r(t)$ is less than that of $m(t)$.

$$\text{From (i) } f(A) = q(A) \cdot m(A) + r(A) \quad \text{by } t = A.$$

$$\Rightarrow 0 = q(A) \times 0 + r(A)$$

$$\Rightarrow r(A) = 0$$

then $r(t)$ is a polynomial of degree less than that of $m(t)$, which has A as a zero.

which contradict the definition of $m(t)$.

$$\text{hence } r(t) = 0$$

$$\Rightarrow f(t) = q(t) \cdot m(t)$$

i.e. $m(t)$ divides $f(t)$

Also then $m(t)$ divides $\Delta(t)$

Theorem

\therefore Let $m(t)$ be the minimum polynomial of an n -square matrix A . Then show that characteristic polynomial of A divides $(m(t))^n$.

Proof.

$$\text{Let } m(t) = t^r + c_1 t^{r-1} + c_2 t^{r-2} + \dots + c_{r-1} t + c_r$$

Consider

$$B_0 = I \quad \text{--- (i)}$$

$$B_1 = A + c_1 I \quad \text{--- (2)}$$

$$B_2 = A^2 + c_1 A + c_2 I \quad \text{--- (3)}$$

$$B_3 = A^3 + c_1 A^2 + c_2 A + c_3 I \quad \text{--- (4)}$$

$$B_{r-1} = A^{r-1} + c_1 A^{r-2} + \dots + c_{r-1} I \quad \text{--- (r)}$$

Take

$$B(t) = t^{r-1} B_0 + t^{r-2} B_1 + t^{r-3} B_2 + \dots + t B_{r-2} + B_{r-1}$$

Now

$$(tI - A)B(t) = (tI - A)(t^{r-1} B_0 + t^{r-2} B_1 + \dots + t B_{r-2} + B_{r-1})$$

$$= t^r B_0 I + t^{r-1} B_1 I + t^{r-2} B_2 I + \dots + t^2 B_{r-2} I + t B_{r-1} I - (t^{r-1} A B_0 + t^{r-2} A B_1 + \dots + t A B_{r-2} + A B_{r-1})$$

$$= t^r B_0 + t^{r-1} (B_1 - A B_0) + t^{r-2} (B_2 - A B_1) + \dots + t (B_{r-1} - A B_{r-2}) - A B_{r-1}$$

--- (a)

Now from eqs (i) to (r) gives

$$B_1 - A B_0 = c_1 I$$

$$B_2 - A B_1 = c_2 I$$

$$B_{r-1} - AB_{r-2} = c_{r-1} I$$

Also from r th equation

$$\begin{aligned} AB_{r-1} &= A^r + c_1 A^{r-1} + \dots + c_{r-1} AI \\ &= A^r + c_1 A^{r-1} + \dots + c_{r-1} AI + c_r I - c_r I \\ &= m(A) - c_r I \end{aligned}$$

$$\Rightarrow AB_{r-1} = -c_r I \quad \therefore m(A) = 0$$

Using all these values in eq. (a)

$$(tI - A) \cdot B(t) = t^r I + t^{r-1} c_1 I + t^{r-2} c_2 I + \dots + t c_{r-1} I + c_r I$$

$$= (t^r + t^{r-1} c_1 + t^{r-2} c_2 + \dots + t c_{r-1} + c_r) I$$

taking determinant to both sides:

$$|(tI - A) B(t)| = |(t^r + t^{r-1} c_1 + t^{r-2} c_2 + \dots + t c_{r-1} + c_r) I|$$

$$\begin{aligned} \Rightarrow |tI - A| |B(t)| &= (t^r + c_1 t^{r-1} + c_2 t^{r-2} + \dots + c_r)^n \\ &= (m(t))^n \end{aligned}$$

$$\Rightarrow |tI - A| \text{ divides } (m(t))^n$$

ie characteristic polynomial divide $(m(t))^n$

Similar Matrix

def:- A matrix B is similar to a matrix A if there is non-singular matrix P such that

$$B = P^{-1}AP \quad \text{or} \quad PB = AP.$$

Diagonalization of Matrix:

def:- A matrix A is said to be diagonalizable if there is a matrix such that

$$B = P^{-1}AP$$

In this case column of P are eigen vectors of A and diagonal element of B are corresponding eigen values of A.

Question If $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$
then diagonalize this matrix

Solution:

To find eigen values

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 4 & -2 \\ -3 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 5, -2$$

i) $\lambda = 5$ then for eigen vectors

$$MX = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow x_1 - 2x_2 = 0$$

$$-3x_1 + 6x_2 = 0$$

One of its solution is $x_2 = 1 \Rightarrow x_1 = 2$

eigen vector $(2, 1)^t$

ii) $\lambda = -2$

$$\Rightarrow M \cdot X = 0 \Rightarrow \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow -6x - 2y = 0$$

$$-3x - y = 0$$

$$\Rightarrow \text{if } x=1 \Rightarrow y=-3$$

$$\text{eigen vector} = (1, -3)^t$$

Now

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

$$|P| = -6 - 1 = -7$$

$$P^{-1} = -\frac{1}{7} \begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{pmatrix}$$

Now

$$P^{-1}AP = \begin{pmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{pmatrix} \begin{pmatrix} 10 & -2 \\ 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$$

is diagonal where diagonal ~~val~~ elements are eigen values of A.

Question: Find A^{10} for $A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$

$$B = P^{-1}AP$$

$$PB\bar{P}^{-1} = A$$

$$\text{So } A^{10} = (PB\bar{P}^{-1})^{10}$$

$$= PB^{10}P^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^{10} \begin{pmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{pmatrix}$$

$$\therefore (PB\bar{P}^{-1})^2$$

$$= (PB\bar{P}^{-1})(PB\bar{P}^{-1})$$

$$= PB\bar{P}^{-1}PB\bar{P}^{-1}$$

$$= PBIB\bar{P}^{-1}$$

$$= PB^2\bar{P}^{-1}$$

Simplify yourself

Theorem:

Similar matrix A and $\bar{P}^{-1}AP$ have the same characteristic polynomial.

Proof.

Let A and B are similar matrices
then $B = \bar{P}^{-1}AP$

Using

$$tI = \bar{P}^{-1}tIP$$

$$|tI - B| = |tI - \bar{P}^{-1}AP|$$

$$= |\bar{P}^{-1}tIP - \bar{P}^{-1}AP|$$

$$= |\bar{P}^{-1}(tI - A)P|$$

$$= |\bar{P}^{-1}| |tI - A| |P|$$

$$= |tI - A| |\bar{P}^{-1}| |P|$$

$$= |tI - A|$$

As required