# Vector Spaces: Handwritten notes

by

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# Ring

def:- A non-empty set $R$ is called ring if

i) $R$ is abelian group under ~~multiplication~~ addition.

ii) $R$ is semi-group under multiplication.

iii) Distributive law holds

$$a(b+c) = a\cdot b + a\cdot c$$
$$(a+b)c = a\cdot c + b\cdot c$$

Examples

i) $(\mathbb{Z}, +, \cdot)$ is a ring
where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$

ii) $(\mathbb{Q}, +, \cdot)$, where $\mathbb{Q}$ is the set of rational numbers

iii) $(\mathbb{R}, +, \cdot)$, where $\mathbb{R}$ is set of real numbers.

iv) $(\mathbb{Z}_n, +, \cdot)$, $\mathbb{Z}_n$ = residue classes of module $n$.

# Field

def:- A non-empty set $F$ is called a field if

i) $F$ is abelian group under addition.

ii) $F - \{0\}$ is abelian group under multiplication.

iii) Right distributive law holds in $F$.

i.e $a, b, c \in F$

$$(a+b)c = ac + bc$$

Examples

i) $(\mathbb{R}, +, \cdot)$ is a field.

ii) $(\mathbb{C}, +, \cdot)$ is a field.

iii) $(\mathbb{Q}, +, \cdot)$ is a field.

iv) $(\mathbb{Z}, +, \cdot)$ is not a field
as $(\mathbb{Z} - \{0\}, \cdot)$ is not group under multiplication.

# Vector Space

def:- Let $V$ be a non-empty set and $F$ is field then $V$ is called vector space if

i) $V$ is abelian group under addition

ii) $a(v+w) = av + aw \quad \forall \; a \in F, \; v, w \in V$.

iii) $(a+b)v = av + bv \quad \forall \; a, b \in F, \; v \in V$.

iv) $a(bv) = (ab)v \quad \forall \; a, b \in F, \; v \in V$.

v) $1 \cdot v = v \cdot 1 = v \quad , \; 1 \in F$ and $v \in V$.

i.e $1$ is identity under multiplication

## Example

i) Let $V$ be a set of all polynomial of degree $\leq n$ then $V$ is vector space

$V = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in F \; \forall \; i \leq n \in \mathbb{N}\}$

$= \{\sum_{i=0}^{n} a_i x^i \mid a_i \in F \quad \forall \quad i \leq n \in \mathbb{N}\}$.

addition is defined as

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i$$

and multiplication is defined as

$$r \sum a_i x^i = \sum r a_i x^i$$
$$= r a_0 + r a_1 x + r a_2 x^2 + \cdots + r a_n x^n$$

ii) Let $F$ is a field then the set

$F^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in F, \; 1 \leq i \leq n\}$.

iii) The set $M_n$ of all $n \times n$ matrices with entries from a field $F$ is a vector space over $F$.

iv) Every field is a vector space over itself.

# Subspace:-

Let $V$ be a vector space over $F$ and $W$ be its non-empty subset of $V$.

Then $W$ is a subspace of $V$ if $W$ itself is vector space under operation induced (defined) in $V$.

# Theorem:-

A non-empty ~~subspace~~ subset $W$ of a vector space $V$ is a subspace of $V$ iff

i) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.

ii) $\alpha \in F, w \in W \Rightarrow \alpha w \in W$.

Proof:

Let $W$ is subspace of vector ~~field~~ space $V$.

then $W$ itself is a vectorspace

i.e $W$ is closed under addition and scalar multiplication.

Conversely, let $W$ is a subset satisfying condition (i) and (ii).

Then for $-1 \in F$ and $w_2 \in W$.

$\Rightarrow -1 \cdot w_2 \in W$ by condition (ii).

$\Rightarrow -w_2 \in W$.

i.e $w_1, -w_2 \in W$

$\Rightarrow w_1 + (-w_2) \in W$ by condition (i)

$\Rightarrow W$ is a subgroup under addition.

Since $W$ is a subset of $V$ and $V$ is abelian.

So $W$ is abelian.

Further condition II to V of the definition are satisfied in $W$ as these are satisfied in $V$.

# Corollary:-

$W$ is non-empty subset of a vectorspace $V(F)$. Then $W$ is subspace of $V$ iff

$a, b \in F, w_1, w_2 \in W \Rightarrow aw_1 + bw_1 \in W$.

Proof. Let $W$ is a subspace of $V(F)$. then $W$ itself is a vector space

i.e for $a, b \in F$, $w_1, w_2 \in W$

$\Rightarrow aw_1, bw_2 \in W$.

$\Rightarrow aw_1 + bw_2 \in W$. $\because W$ is closed under addition.

Conversely,

Let for $a, b \in F$, $w_1, w_2 \in W$.

$\Rightarrow aw_1 + bw_2 \in W$.

Let $a = b = 1$

then $1 \cdot w_1 + 1 \cdot w_2 \in W$

i.e $w_1 + w_2 \in W$.

also if $b = 0 \in F$

For $aw_1 + bw_2 \in W$

$\Rightarrow aw_1 + 0 \cdot w_2 \in W$

$\Rightarrow aw_1 \in W$.

$\Rightarrow W$ is a subspace of $V$

# Definition (Linear Sum):

Let $V$ be a vector space over $F$ and $W_1, W_2, \ldots, W_n$ be non-empty subset of $V$.
then their linear sum is defined as.

$W_1 + W_2 + \cdots + W_n = \{a_1 + a_2 + \cdots + a_n : a_1 \in W_1, a_2 \in W_2$ i.e $a_i \in W_i\}$

Lemma:- Let $V$ be a vector space and $W_1, W_2, \ldots, W_n$ be subspace. prove that

$W = W_1 + W_2 + \cdots + W_n$

is also a subspace of $V$.

**Lemma:-**

$W_1, W_2, \dots, W_n$ are subspaces of $V$. prove that $W = W_1 + W_2 + \dots + W_n$ is a subspace of $V$.

**Proof:-**

$0 = 0 + 0 + 0 + \dots + 0$ , $0 \in W_i$.

$\Rightarrow 0 \in W \Rightarrow W$ is non-empty.

Let $x, y \in W$ , $a, b \in F$

we have to show $ax + by \in W$.

$\because x \in W$

$\Rightarrow x = x_1 + x_2 + \dots + x_n$ for $x_1 \in W_1, x_2 \in W_2, \dots, x_n \in W_n$

$y = y_1 + y_2 + \dots + y_n$ for $y_1 \in W_1, y_2 \in W_2, \dots, y_n \in W_n$.

Now

$ax + by = a(x_1 + x_2 + \dots + x_n) + b(y_1 + y_2 + \dots + y_n)$.

$= ax_1 + ax_2 + \dots + ax_n + by_1 + by_2 + \dots + by_n$

$= (ax_1 + by_1) + (ax_2 + by_2) + \dots + (ax_n + by_n)$.

As each $W_i$ is a subspace

$\Rightarrow ax_i + by_i \in W_i$ , $i = 1, 2, \dots, n$.

So $\sum_{i=1}^{n} ax_i + by_i \in \sum W_i = W$.

$\Rightarrow ax + by \in W$

So $W$ is a subspace

**Lemma:-**

Let $V$ be a vector space and $W_i$ a family of subspaces of $V$. Then $\cap W_i$ is also a subspace of $V$.

**Proof**

Let $v, w \in \cap W_i$

then $v, w \in W_i$ for each $i \in I$

and since each $W_i$ is a subspace

so there must be $a, b \in F$

such that $av + bw \in W_i$ for each $i \in I$.

so $av + bw \in \cap W_i$ i.e $\cap W_i$ is a subspace.

# Definition

Let $U$ and $V$ are two vector spaces over a field $F$. then $\tau$ of $U$ into $V$ is called homomorphism

if $\tau(u_1+u_2) = \tau(u_1)+\tau(u_2)$

$\tau(au) = a\,\tau(u)$ ; $a \in F$.

# Definition.

The kernel of homomorphism $\tau : U \to V$, is defined as
$\ker\tau = \{u : u \in U, \tau(u) = 0\}$.

**Question.**

Prove that $\ker\tau$ (ker. of homomorphism) is a subspace.

Solution. Let $u_1, u_2 \in \ker\tau$

$\Rightarrow \tau(u_1) = 0$, $\tau(u_2) = 0$.

Now let $a, b \in F$

$$\begin{aligned}
\tau(au_1 + bu_2) &= \tau(au_1) + \tau(bu_2) \\
&= a\,\tau(u_1) + b\,\tau(u_2) \\
&= a(0) + b(0) \\
&= 0.
\end{aligned}$$

$\Rightarrow au_1 + bu_2 \in \ker\tau$.

So $\ker\tau$ is subspace.

# Linear Combination:-

Let $V$ is a vector space

Let $v_1, v_2, \ldots, v_n \in V$.

$a_1, a_2, \ldots, a_n \in F$

then an element

$a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n$ is called linear combination.

The linear combination is trivial if each $a_i = 0$.
and it is non-trivial if at least one of $a_i \neq 0$

# Definition: (Linear Span)

Let $S$ be a subset of vector space $V$, then the set of all linear combination of $S$ is called linear spam denoted by $<S>$ or $L(S)$ or $[S]$

# Theorem:-

Prove that $<S>$ is a subspace of $V$ containing $S$. It is smallest subspace of $V$ containing $S$.

Proof:-

Let $u, v \in <S>$

then $u = a_1u_1 + a_2u_2 + \cdots + a_nu_n$

$v = b_1v_1 + b_2v_2 + \cdots + b_nv_n$

For $a, b \in F$ we have to prove $au + bv \in <S>$.

Now

$$au + bv = a(a_1u_1 + a_2u_2 + \cdots + a_nu_n) + b(b_1v_1 + b_2v_2 + \cdots + b_nv_n)$$

$$= aa_1u_1 + aa_2v_2 + \cdots + aa_nu_n + bb_1v_1 + bb_2v_2 + \cdots + bb_nv_n$$

$\Rightarrow au + bv \in <S>$

$\Rightarrow <S>$ is a subspace.

Let $u_i \in S$

then $u_i = 0u_1 + 0u_2 + \cdots + 0u_{i-1} + 1 \cdot u_i + 0 \cdot u_{i+1} \cdots + 0 \cdot u_n \in <S>$

i.e $u_i \in <S>$

$\Rightarrow S \subseteq <S>$.

Let $W$ be any other subspace of $V$ containing $S$.

then $\sum a_i u_i \in W$.

$\because W$ is subspace containing $S$.

$\Rightarrow <S> \leq W$.

i.e $<S>$ is smallest subspace containing $S$.

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# Definition (Finite Dimensional Vector Space).

A vector space $V$ is called finite dimensional if there is a subset $S$ of $V$ such that $\langle S \rangle = V$.

# Definition:- (Linear Dependent and Independent)

Let $V$ be a vector space then the vectors $v_1, v_2, v_3, \ldots, v_n \in V$ are linearly dependent if $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ and all $a_i \neq 0$.

If $a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n = 0$ where each $a_i = 0$ then the vectors $v_1, v_2, \ldots, v_n$ are linearly independent.

**Theorem:-**

Let $V$ be a vector space and consider a set of vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent then its subset is also independent.

ii) If $\{v_1, v_2, \ldots, v_n\}$ is dependent then $\{v_1, v_2, \ldots, v_n, v\}$ is also dependent.

# Lemma:-

Let $V(F)$ be a vector space and $S=\{v_1, v_2, \dots, v_n\}$ a set of vectors in $V$. Then

i) If $S$ is independent, then any non-empty subset of $S$ is also independent.

Proof.

Let $\{v_1, v_2, \dots, v_i\}$ be a subset of $S$, $1 \le i < n$.
Consider $a_1v_1 + a_2v_2 + \dots + a_iv_i = 0$, $a_i \in F$
then

$$a_1v_1 + a_2v_2 + \dots + a_iv_i + 0.v_{i+1} + \dots + 0.v_n = 0$$

Since $\{v_1, v_2, \dots, v_n\}$ is Linearly Independent
$\Rightarrow$ each $a_k = 0$ ; $k = 1, 2, \dots, n$.
$\therefore$ each $a_k = 0$, $k = 1, 2, \dots, i$"
$\Rightarrow \{v_1, v_2, \dots, v_i\}$ is L.I.

2001 (ii)

If $S$ is dependent then
$\{v, v_1, v_2, \dots, v_n\}$ is also dependent.
i.e $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ where all $a_i \ne 0$.
and then

$$a_1v_1 + a_2v_2 + \dots + a_nv_n + 0v = 0$$

where all $a_i \ne 0$.

$\Rightarrow \{v_1, v_2, \dots, v_n, v\}$ is also dependent.

# Theorem:-

A set of non-zero vectors $v_1, v_2, \dots, v_n \in V$ is linearly dependent iff one of them is a linear combination of the other / preceding vector.

Proof:

$\{v_1, v_2, \dots, v_n\}$ is linearly dependent.
i.e $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ where all $a_i$'s $\ne 0$.
For $a_i \in F$

Let $a_k$ be the Last ~~non-~~ non-zero coeffients of

$\cancel{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + 0 + 0 + a_kv_k}$

$a_1v_1 + a_2v_2 + \dots + a_{k-1}v_{k-1} + a_kv_k + a_{k+1}v_{k+1} + \dots + a_nv_n$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0 \quad \because a_{k+1} = a_{k+2} = \cdots = a_n = 0$

$\Rightarrow -a_kv_k = a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}$

$\Rightarrow v_k = -\frac{1}{a_k}(a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1})$.

Conversely, let $v_k$ is a linear combination of the ~~prec~~ preceding vectors

$v_1, v_2, v_3, \ldots, v_{k-1}$

i.e $v_k = a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + (-1)v_k = 0$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + (-1)v_k + 0\cdot v_{k+1} + \cdots + 0v_n = 0$

then $\{v_1, v_2, \ldots, v_n\}$ is Linearly Dependant

$\because$ at least one co-efficient of $v_k$ is non-zero.

# Basis of a Vector Space:-

Let $S$ be a subset of a vector space $V(F)$. Then $S$ is called basis for $V$.

if i) $S$ is linearly independent.

ii) $S$ is spanning set of $V$.

generating.

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# Theorem:-

Any finite dimensional vector space contains a basis.

**Proof:**

Let $\{v_1, v_2, \ldots, v_r\}$ be a ~~spanning~~ spanning set of $V$.

If $\{v_1, v_2, \ldots, v_r\}$ is linearly independent then form a basis and there is nothing to prove.

Consider $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent then one of the vectors say $v_r$ is a linear combination of the remaining $\{v_1, v_2, \ldots, v_{r-1}\}$ we drop out this vector and obtain a set of ~~ve ...~~ $r-1$ vectors.

A ~~vector~~ linear combination of $r$ vectors also a linear combination of $r-1$ vectors.

If this set $\{v_1, v_2, \ldots, v_{r-1}\}$ is linearly independent then form a basis

But if $\{v_1, v_2, \ldots, v_{r-1}\}$ is dependent then the above process is continued. In this way we can get a linear independent spanning set, and hence a basis.

$\{v_1, v_2, \ldots, v_n\}$ $1 \le n \le r$.

Imp 2001, II/02

# Theorem:-

If $v_1, v_2, \ldots, v_n$ is a basis of $V(F)$ and if $w_1, w_2, \ldots, w_m \in V$ are linearly independent then $m \le n$.

**Proof:**

Since $v_1, v_2, \ldots, v_n$ is a basis of $V$ so every element of $V$ can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$.

In particular $w_m \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$,

$w_m, v_1, v_2, \ldots, v_n$ are dependent.
therefore, a proper subset $\{w_m, v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$, $r \le n-1$
form a basis
Similarly $\{w_{m-1}, w_m, v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ is dependent
and Its proper subset

$\{w_{m-1}, w_m, v_{j_1}, v_{j_2}, \ldots, v_{j_s}\}$, $s \le n-2$

Repeating this procedure $(m-1)$ times, we get a basis

$w_2, w_3, \ldots, w_{m-1}, w_m, v_{k_1}, v_{k_2}, \ldots, v_{k_t}$

$t \ge 1$ Since the vectors $w_1$ is not a L.C of $w_1, w_2, \ldots, w_n$

$t \le n-(m-1)$
$t \le n-m+1$
$t \ge 1$

$\Rightarrow \quad 1 \le t \le n-m+1$

$1 \le n-m+1$

$\Rightarrow \quad 0 \le n-m$

$\Rightarrow \quad m \le n$

Question: Show that the vectors

$v_1 = (1,1,1)$ , $v_2 = (1,0,1)$ , $v_3 = (0,1,1)$

are linearly independent.

Solution:-

Consider $a_1v_1 + a_2v_2 + a_3v_3 = 0$

$\Rightarrow a_1(1,1,1) + a_2(1,0,1) + a_3(0,1,1) = 0$

$\Rightarrow (a_1, a_1, a_1) + (a_2, 0, a_2) + (0, a_3, a_3) = 0$

$\Rightarrow$ ~~$(a_1+a_2)$~~

$(a_1+a_2 , a_1+a_3 , a_1+a_2+a_3) = (0,0,0)$

$\Rightarrow a_1+a_2 = 0$ ——— (i)

$a_1+a_3 = 0$ ——— (ii)

$a_1+a_2+a_3 = 0$ ——— (iii)

$\Rightarrow \cancel{a_1}+\cancel{a_2}+a_3 = 0$

$\underline{\cancel{a_1}+\cancel{a_2}}$

$a_3 = 0 \quad \Rightarrow a_1 = 0, \quad a_2 = 0.$

Since $a_1 = a_2 = a_3 = 0$

$\Rightarrow$ the vectors are L.I.

Question: Prove that the vectors

$v_1 = (3,0,-3)$ , $v_2 = (-1,1,2)$ , $v_3 = (1,2,-2)$

$v_4 = (2,1,1)$ are linearly dependent.

Solution

Consider

$av_1 + bv_2 + cv_3 + dv_4 = 0.$

$\Rightarrow a(3,0,-3) + b(-1,1,2) + c(1,2,-2) + d(2,1,1) = 0$

$\Rightarrow (3a, 0, -3a) + (-b, b, 2b) + (c, 2c, -2c) + (2d, d, d) = 0$

$\Rightarrow (3a-b+c+2d, \; b+2c+d, \; -3a+2b-2c+d) = 0$

$3a - b + c + 2d = 0$

$b + 2c + d = 0$

$-3a + 2b - 2c + d = 0$

Let $d=0$

$\Rightarrow 3a+b+4c=0$

b ~~+ 2c...~~

$-3a+2b-2c=0$

Using $a=-2c$, $b=-2c$, $d=0$

into (1)

$-2cv_1-2cv_2+cv_3+0v_4=0$

$2v_1+2v_3-v_3+0v_4=0$

$\Rightarrow v_1, v_2, v_3, v_4$ are dependent.

$\boxed{v_1, v_2, v_4 \text{ are L.I.}}$ check.

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# Definition: (Quotient Space)

Let $V$ be a vector space over a field $F$ and $W$ be a subspace.

The set $V/W$ of all left coset along with two operations

(i) $(v_1+W)+(v_2+W)=v_1+v_2+W$

(ii) $a(v_1+W)=av_1+W$

is called Quotient space.

2003

# Lemma:-

Let $V$ be a vector space and $W$ a subspace of $V$ then $V/W$ along with the operation

(i) $(v_1+W)+(v_2+W)=(v_1+v_2)+W$

(ii) $\alpha(v_1+W)=\alpha v_1+W$, is a ~~subspace~~ vector space

Proof:-

a) It is easy to show that $V/W$ is an abelian group under addition with $0+W=W$ as it identity and $-v+W$ as an inverse of $v+W\in V/W$.

b) We see that scalar multiplication is defined in $V/W$.

(i.e $v+W=v'+W \Rightarrow \alpha(v+W)=\alpha(v'+W)$)

Let $v=v'+w$ for some $w\in W$.

then $\alpha(v+W)=\alpha v+W$

$=\alpha(v'+w)+W$

$=\alpha v'+\alpha w+W$

$=\alpha v'+W \quad \because \alpha w\in W.$

$=\alpha(v'+W).$

i.e Scalar multiplication is defined.

Let $v+W,\ v'+W\in V/W$, $a\in F$.

$a((v+W)+(v'+W))=a(v+v'+W)$

$=a(v+v')+W$

$=av+av'+W$

$=av+W+av'+W$

$=a(v+W)+a(v'+W)$

(iv) $(a+b)(v+W) = (a+b)v + W$
$= (av + bv) + W$
$= av + W + bv + W$
$= a(v+W) + b(v+W)$.

(v) $a(b(v+W)) = a(bv + W)$
$= (ab)v + W$
$= (ab)(v + W)$

(vi) $1 \cdot (v+W) = 1 \cdot v + W$
$= v + W$

Hence $V/W$ is vector space.

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Theorem:-

$V(F)$ is a finite dimensional vector space and if $W$ is a subspace of $V$. Then

i) $W$ is finite dimensional and $\dim W \leq \dim V$.

ii) $\dim(V/W) = \dim V - \dim W$

Proof:

Let $\dim V = n$.

and let $\{w_1, w_2, \dots, w_m\}$ be linearly independent set of vectors of $W$,

then $m \leq n$

then the set $\{w_1, w_2, w_3, \dots, w_m, w\}$ is linearly dependent.

i.e one of these vectors in a linear combination of the preceeding vector.

however none of the vectors $w_1, w_2, \dots, w_m$ is a linear combination of the preceeding vectors. because the vectors $w_1, w_2, \dots, w_m$ are linearly independent.

so $w$ can be written as a linear combination of $w_1, w_2, \dots, w_m$.

Since $w \in W$ is an arbitrary element therefore $W$ is finite dimensional

and $\dim W = m \leq n$.

i.e $\dim W \leq \dim V$.

ii) Let $\{w_1, w_2, \dots, w_m\}$ be a basis of $W$.

and $\{w_1, w_2, \dots, w_m, v_1, v_2, \dots, v_\ell\}$ be a basis of $V$.

we have to prove $\{v_1+W, v_2+W, \dots, v_\ell+W\}$ is a basis of $V/W$.

Now

$$\alpha_1(v_1+W) + \alpha_2(v_2+W) + \dots + \alpha_\ell(v_\ell+W) = 0.$$

$$(\alpha_1 v_1 + W) + (\alpha_2 v_2 + W) + \dots + (\alpha_\ell v_\ell + W) = 0 + W$$

Since $W$ is identity of $V/W$.

$$\Rightarrow (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_\ell v_\ell) + W = W.$$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_l v_l \in W$ $\qquad$ $\because a+H = H \Leftrightarrow a \in H$.

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_l v_l = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m$

as $\{w_1, w_2, \ldots, w_m\}$ is basis of $W$.

So

$\beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m - \alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_l v_l = 0$.

Since $\{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_l\}$ is a basis of $V$.

$\Rightarrow \beta_1 = \beta_2 = \cdots = \beta_m = \alpha_1 = \alpha_2 = \cdots = \alpha_l = 0$.

i.e. $\{v_1 + W, v_2 + W, \ldots, v_l + W\}$ is linearly independent.

Set $v + W \in V/W$ for $v \in V$.

then $v = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + b_1 v_1 + b_2 v_2 + \cdots + b_l v_l$.

So $v + W = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + b_1 v_1 + b_2 v_2 + \cdots + b_l v_l + W$

$\Rightarrow v + W = b_1 v_1 + b_2 v_2 + \cdots + b_l v_l + a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + W$

$= b_1 v_1 + b_2 v_2 + \cdots + b_l v_l + W$

$\because a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + W = W$

as $a_1 w_1 + a_2 w_2 + \cdots + a_m w_m \in W$.

$= (b_1 v_1 + W) + (b_2 v_2 + W) + \cdots + (b_l v_l + W)$ by def.

$= b_1 (v_1 + W) + b_2 (v_2 + W) + \cdots + b_l (v_l + W)$. by def.

i.e. $\{v_1 + W, v_2 + W, \ldots, v_l + W\}$ generate $V/W$

and hence is a basis of $V/W$.

$\therefore \dim(V/W) = l$

$= (m + l) - m$

$= \dim V - \dim W$

# Internal Direct sum:-
def. Let $U_1, U_2, \ldots, U_n$ be subspace of a vector space $V$. For $v \in V$.
then if $v$ has one and only one expression of the form
$$v = u_1 + u_2 + \ldots + u_n \quad \text{for } u_i \in U_i$$
then $V$ is called internal direct sum of subspace $U_1, U_2, \ldots, U_n$.

# External Direct Sum:-
def. Let $V_1, V_2, \ldots, V_n$ be vector spaces over a field $F$ & $V$ be a vector space over field $F$.
$V$ be a vector space having n-ordered tuples $(v_1, v_2, \ldots, v_n)$, $v_i \in V_i$. then $V$ is called external direct sum if
i) Two n-tuples $(v_1, v_2, \ldots, v_n)$ and $(v_1', v_2', \ldots, v_n')$ are equal iff $v_i = v_i'$
ii) $(v_1, v_2, \ldots, v_n) + (v_1', v_2', \ldots, v_n')$
$$= (v_1 + v_1', v_2 + v_2', \ldots, v_n + v_n')$$
iii) $\alpha(v_1, v_2, \ldots, v_n) = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$.
external direct sum is denoted by.
$$V_1 \oplus V_2 \oplus V_3 \oplus \ldots \oplus V_n$$

# Vector Space Homomorphism:-
Let $V$ and $W$ are two vector spaces.
A mapping $T: V \to W$ is called homomorphism if
$$T(v_1 + v_2) = T(v_1) + T(v_2)$$
$$T(\alpha v) = \alpha T(v) \quad \forall\ v_1, v_2 \in V \ \&\ \alpha \in F.$$

# Theorem:-
If a vector space $V$ is the internal direct sum of subspaces $U_1, U_2, \ldots, U_n$ then $V$ is isomorphic to the external direct sum of $U_1, U_2, \ldots, U_n$.

**Proof** Let $v \in V$ where $v = u_1 + u_2 + u_3 + \cdots + u_n$

Define a mapping

$T : V \to U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_n$

by $T(v) = T(u_1 + u_2 + \cdots + u_n)$

$= (u_1, u_2, \cdots, u_n)$.

1) Mapping is well defined as $v \in V$.

$v = u_1 + u_2 + \cdots + u_n$

has one and only one representation

(ii) $T$ is onto because each

$(u_1, u_2, u_3, \cdots, u_n) \in U_1 \oplus U_2 \oplus \cdots \oplus U_n$

is image of $u_1 + u_2 + \cdots + u_n \in V$.

iii) $T$ is one one

for $T(v) = T(w)$

$\Rightarrow T(u_1 + u_2 + \cdots + u_n) = T(w_1 + w_2 + \cdots + w_n)$.

where $v_i, w_i \in U_i$

$\Rightarrow (u_1, u_2, \cdots, u_n) = (w_1, w_2, \cdots, w_n)$

$\Rightarrow u_1 = w_1,\ u_2 = w_2, \cdots, u_n = w_n$

$\Rightarrow u_1 + u_2 + \cdots + u_n = w_1 + w_2 + \cdots + w_n$.

$\Rightarrow u = w$.

(iv) $T(v + w) = T(u_1 + u_2 + u_3 + \cdots + u_n + w_1 + w_2 + \cdots + w_n)$

$= T(u_1 + w_1 + u_2 + w_2 + \cdots + u_n + w_n)$

$= (u_1 + w_1, u_2 + w_2, \cdots, u_n + w_n)$.

$= (u_1, u_2, \cdots, u_n) + (w_1, w_2, \cdots, w_n)$

by def. of external direct sum,

$= T(v) + T(w)$

& $T(\alpha v) = T(\alpha(u_1 + u_2 + \cdots + u_n))$

$= T(\alpha u_1 + \alpha u_2 + \cdots + \alpha u_n)$

$= (\alpha u_1, \alpha u_2, \cdots, \alpha u_n)$

$= \alpha(u_1, u_2, \cdots, u_n)$

$= \alpha T(v)$ hence $T$ is homomorphism.

# Theorem:

If $A$ and $B$ are finite dimensional subspace of a vector space $V(F)$, then $A+B$ is finite dimensional and $\dim(A+B) = \dim A + \dim B - \dim(A\cap B)$.

**Proof:**

Suppose $\{u_1, u_2, \dots, u_r\}$ be a basis of $A\cap B$

$\{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_m\}$ be a basis of $A$.

$\{u_1, u_2, \dots, u_r, w_1, w_2, \dots, w_n\}$ be a basis of $B$.

then we have to prove that

$\{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_n\}$

is a basis of $A+B$.

Consider

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r + \beta_1 v_1 + \dots + \beta_m v_m + \gamma_1 w_1 + \dots + \gamma_n w_n = 0$$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r + \beta_1 v_1 + \dots + \beta_m v_m = -\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n \quad \text{--- (i)}$$

Since L.H.S of (i) is in $A$ so does R.H.S.

i.e $-\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n \in A$.

Also

$-\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n \in B$ $\quad \because w_1, w_2, \dots, w_n$ is a part of basis of $B$.

$\therefore -\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n \in A\cap B$.

$\Rightarrow -\gamma_1 w_1 - \gamma_2 w_2 - \dots - \gamma_n w_n = \delta_1 u_1 + \delta_2 u_2 + \dots + \delta_r u_r$

as $\{u_1, u_2, \dots, u_r\}$ is a basis of $A\cap B$

& $\delta_i \in F$.

$$\Rightarrow \delta_1 u_1 + \delta_2 u_2 + \dots + \delta_r u_r + \gamma_1 w_1 + \gamma_2 w_2 + \dots + \gamma_n w_n = 0.$$

Since $\{u_1, u_2, \dots, u_r, w_1, w_2, \dots, w_n\}$ is a basis of $B$ (L.I)

$\Rightarrow \delta_1 = \delta_2 = \dots = \delta_r = \gamma_1 = \gamma_2 = \dots = \gamma_n = 0$.

so that equation (i) becomes.

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m = 0$.

But $\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m\}$ is a basis of A.

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_r = \beta_1 = \beta_2 = \cdots = \beta_m = 0$.

i.e each $\alpha_i = \beta_i = \gamma_i = 0$.

Hence $\{u_1, u_2, \ldots, u_r, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is L.I

Let $x+y \in A+B$ i.e $x \in A$ & $y \in B$.

As basis of $A = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m\}$,

then

$x = a_1 u_1 + a_2 u_2 + \cdots + a_r u_r + b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$

Also basis of $B = \{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_n\}$.

so

$y = a_1' u_1 + a_2' u_2 + \cdots + a_r' u_r + b_1' w_1 + b_2' w_2 + \cdots + b_n' w_n$

By ting

$A+B = (a_1+a_1')u_1 + (a_2+a_2')u_2 + \cdots + (a_r+a_r')u_r + b_1 v_1 + b_2 v_2 + \cdots + b_m v_m + b_1' w_1 + \cdots + b_n' w_n$

$\therefore \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m\}$ generates A+B

and hence is a basis of A+B.

$\therefore$ A+B is a finite dimensional and

$$\begin{aligned}\dim(A+B) &= r+m+n \\ &= (r+m)+(r+n)-r \\ &= \dim A + \dim B - \dim(A\cap B).\end{aligned}$$

proved

# Theorem:- Let V and W be vector spaces
If T is an isomorphism of V onto W.
Then T mapsps a basis of V onto a basis of W.

Proof:-
$T: V \to W$ is isomorphism defined by
$$T(v) = w.$$
Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V.
then we have to prove
$\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis of W.

i) Consider
$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = 0$ , $\alpha_i \in F$.
$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \cdots + T(\alpha_n v_n) = 0$ $\because$ T is homomorphism $\therefore \alpha T(v) = T(\alpha v)$
$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0$. $T(v_1 + v_2) = T(v_1) + T(v_2)$
$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in \ker T$
$\because$ T is isomorphism i.e one-one
$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$
$\because \{v_1, v_2, \ldots, v_n\}$ is basis of V.
$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
Hence $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

ii) Let $w \in W$
$\because$ T is onto there must be $v \in V$ such that
$$T(v) = w.$$
Now $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ for $a_i \in F$.
$\therefore w = T(\underline{v})$
$= T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$.
$= T(a_1 v_1) + T(a_2 v_2) + \cdots + T(a_n v_n)$ , $\because$ T is homo.
$\Rightarrow w = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$.

i.e $w$ can be generated by
$\{T(v_1), T(v_2), \ldots, T(v_n)\}$.

Thus $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ form a basis of $W$.

The proof is complete.

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# Theorem:-

Two finite dimensional vector space are isomorphic iff they are of the same dimension.

Proof:-

Let $V$ and $W$ are two vector spaces of same dimension $n$ and $\{v_1, v_2, \ldots, v_n\}$ be the basis of $V$ and $\{w_1, w_2, \ldots, w_n\}$ be the basis of $W$.

Define a mapping

$T : V \to W$ by $T(v) = w$ for $v \in V$, $w \in W$.

i.e $T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$.

i) $T$ is well defined

For $v, v' \in V$, if $v = v'$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \alpha_1' v_1 + \alpha_2' v_2 + \cdots + \alpha_n' v_n.$$

$$\Rightarrow (\alpha_1 - \alpha_1') v_1 + (\alpha_2 - \alpha_2') v_2 + \cdots + (\alpha_n - \alpha_n') v_n = 0.$$

Since $\{v_1, v_2, \ldots, v_n\}$ is basis of $V$.

$$\therefore \alpha_1 - \alpha_1' = 0 = \alpha_2 - \alpha_2' = \cdots = \alpha_n - \alpha_n'$$

$$\Rightarrow \alpha_1 = \alpha_1', \alpha_2 = \alpha_2', \ldots, \alpha_n = \alpha_n'$$

i.e

$$\begin{aligned} T(\alpha v) &= \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n \\ &= \alpha_1' w_1 + \alpha_2' w_2 + \cdots + \alpha_n' w_n \\ &= T(v'). \end{aligned}$$

ii) $T$ is homomorphism

$$\begin{aligned} T(v + v') &= T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + \alpha_1' v_1 + \cdots + \alpha_n' v_n) \\ &= T((\alpha_1 + \alpha_1') v_1 + (\alpha_2 + \alpha_2') v_2 + \cdots + (\alpha_n + \alpha_n') v_n) \end{aligned}$$

$= (\alpha_1+\alpha_1')w_1 + (\alpha_2+\alpha_2')w_2 + \cdots + (\alpha_n+\alpha_n')w_n$
$= (\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n) + (\alpha_1' w_1 + \alpha_2' w_2 + \cdots + \alpha_n' w_n)$
$= T(v) + T(v')$

and

$T(\alpha v) = T(\alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n))$
$= T(\alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \cdots + \alpha\alpha_n v_n)$
$= \alpha\alpha_1 w_1 + \alpha\alpha_2 w_2 + \cdots + \alpha\alpha_n w_n$
$= \alpha(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n)$
$= \alpha T(v)$.

iii) $T$ is one-one

Let $T(v) = T(v')$ for $v, v' \in V$.

$\Rightarrow \cancel{\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n}$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = T(\alpha_1' v_1 + \alpha_2' v_2 + \cdots + \alpha_n' v_n)$.

$\Rightarrow \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n = \alpha_1' w_1 + \alpha_2' w_2 + \cdots + \alpha_n' w_n$.

$\Rightarrow (\alpha_1-\alpha_1')w_1 + (\alpha_2-\alpha_2')w_2 + \cdots + (\alpha_n-\alpha_n')w_n = 0$.

$\because \{w_1, w_2, \ldots, w_n\}$ is basis of $W$.

$\Rightarrow \alpha_1-\alpha_1' = \alpha_2-\alpha_2' = \cdots = \alpha_n-\alpha_n' = 0$.

$\Rightarrow \alpha_1 = \alpha_1', \ \alpha_2 = \alpha_2', \ \ldots, \ \alpha_n = \alpha_n'$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \alpha_1' v_1 + \alpha_2' v_2 + \cdots + \alpha_n' v_n$

$\Rightarrow v = v'$.

iv) $T$ is onto as every element

$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n \in W$

is image of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in V$

Conversely, let $T: V \to W$ is isomorphism then we have to prove:

Dimension of $V$ and $W$ are same.

Let $\{v_1, v_2, \ldots, v_n\}$ be basis of $V$. then we prove that

$\{T(v_1), T(v_2), T(v_3), \ldots, T(v_n)\}$ is a basis of $W$.

See on page 3 $\boxed{V-24}$

# Vector Space Homomorphism:

Let $V$ and $W$ are two vector spaces. The set of all homomorphism of $V$ into $W$ is denoted by $Hom(V, W)$

$$Hom(V, W) = \{T_1, T_2, \dots, T_n\}$$

where each $T_i$ is homomorphism.

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# Theorem:

Let $V(F)$ & $W(F)$ be two vector spaces introduce an operation in $Hom(V, W)$ and prove that $Hom(V, W)$ is a vector space under this operation.

Proof:

Let $T_1, T_2 \in Hom(V, W)$.

we define $(T_1 + T_2)(v) = T_1(v) + T_2(v)$

& $\lambda T(v) = T(\lambda v)$

to prove $Hom(V, W)$ is a vector space we proceed as follows:

Let $v_1, v_2 \in V$ & $T_1, T_2 \in Hom(V, W)$.

then

$$(T_1 + T_2)(v_1 + v_2) = T_1(v_1 + v_2) + T_2(v_1 + v_2)$$

$$= T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2)$$

$$= T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2)$$

$$= (T_1 + T_2)v_1 + (T_1 + T_2)v_2$$

Also

$$(T_1 + T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v)$$

$$= \lambda T_1(v) + \lambda T_2(v)$$

$$\Rightarrow (T_1 + T_2)(\lambda v) = \lambda (T_1 + T_2)(v)$$

$$\Rightarrow T_1 + T_2 \in Hom(V, W).$$

i.e $Hom(V, W)$ is closed.

iii) Mapping $(T_1, T_2, \dots, T_n)$ are associative in general

Consider a mapping $T_0$ which mapps an

element of $V$ into $0$ (zero) i.e

$$T_0(v) = 0$$

then $(T+T_0)v = T(v) + T_0(v)$

$$= T(v) + 0$$

$$= T(v)$$

i.e $T_0 + T = T$

i.e $T_0$ is the identity of $\text{Hom}(V, W)$,

also for $T \in \text{Hom}(V, W)$

so we have

$-T \in \text{Hom}(V, W)$ such that

$$[T+(-T)](v) = T(v) + (-1)T(v)$$

$$= T(v) - T(v) = 0.$$

$$= T_0(v)$$

$\Rightarrow$ inverse exists.

& $(T_1+T_2)v = T_1(v) + T_2(v)$

$$= T_2(v) + T_1(v)$$

$$= (T_2+T_1)(v)$$

$\Rightarrow$ $\text{Hom}(V, W)$ is an abelian group under '+'

(ii)

$$a(T_1+T_2) = aT_1 + aT_2$$

$$a(T_1+T_2)(v) = (T_1+T_2)(av)$$

$$= T_1(av) + T_2(av)$$

$$= aT_1(v) + aT_2(v)$$

(iii) $(a+b)T = aT + bT$

$$(a+b)T(v) = T((a+b)v)$$

$$= T\{av + bv\}$$

$$= aT(v) + bT(v)$$

~~Hence~~ $V/W$ is ~~a~~ vector space

(iv) $a(b)T = (ab)T$

$$a(b)T(v) = aT\{(b)v\} = T\{(a)bv\} = T\{(ab)v\}$$

$$= abT(v)$$

P.T.O

v) $1 . T = T$

As $1 . T(v) = T(1 \cdot v) = T(v)$

As $v \in V$ is a vector space

Hence $Hom(V, W)$ is a vector space.

# Theorem:-

If $V$ and $W$ are of dimension $m$ and $n$ resp. then $Hom(V, W)$ is of dimension $mn$.

Proof:

Let $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ be basis of $V$ and $W$ respectively.

Define a mapping

$T_{ij} : V \to W$ defined by

$$T_{ij}(v_k) = \begin{cases} \lambda_i w_j & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}, \quad \lambda_{ij} \in F$$

Let.

$$\underline{v} = \lambda_1 v_1 + \lambda_2 v_2 + \cdots\cdots + \lambda_m v_m$$

$$\underline{u} = \mu_1 v_1 + \mu_2 v_2 + \cdots\cdots + \mu_m v_m$$

then

$$\begin{aligned} T_{ij}(\underline{u} + \underline{v}) &= T_{ij}\big[(\mu_1 v_1 + \mu_2 v_2 + \cdots\cdots + \mu_m v_m) \\ &\qquad + (\lambda_1 v_1 + \lambda_2 v_2 + \cdots\cdots + \lambda_m v_m)\big] \\ &= T_{ij}\big[(\mu_1 + \lambda_1) v_1 + (\mu_2 + \lambda_2) v_2 + \cdots \\ &\qquad \cdots + (\mu_m + \lambda_m) v_m\big] \\ &= (\mu_i + \lambda_i) w_j \\ &= \mu_i w_j + \lambda_i w_j \\ &= T_{ij}(\underline{u}) + T_{ij}(\underline{v}) \end{aligned}$$

And

$$\begin{aligned} T_{ij}(\alpha \underline{u}) &= T_{ij}\big(\alpha(\mu_1 v_1 + \mu_2 v_2 + \cdots\cdots + \mu_m v_m)\big) \\ &= T_{ij}(\alpha\mu_1 v_1 + \alpha\mu_2 v_2 + \cdots\cdots + \alpha\mu_m v_m) \\ &= \alpha \mu_i w_j \\ &= \alpha T_{ij}(\underline{u}) \end{aligned}$$

Thus $T_{ij}$ is homomorphism and $T_{ij} \in Hom(V, W)$.

Now to prove $\{T_{11}, T_{12}, \ldots, T_{ij}, \ldots, T_{mn}\}$ is a basis

Consider

$$\alpha_{11}T_{11} + \alpha_{12}T_{12} + \cdots\cdots + \alpha_{ij}T_{ij} + \cdots\cdots \alpha_{mn}T_{mn} = 0$$

Now

P.T.O

$$(\alpha_{11}T_{11}+\alpha_{12}T_{12}+\cdots\cdots+\alpha_{1n}T_{1n}$$
$$+\alpha_{21}T_{21}+\alpha_{22}T_{22}+\cdots\cdots+\alpha_{2n}T_{2n}$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\alpha_{m1}T_{m1}+\alpha_{m2}T_{m2}+\cdots\cdots\alpha_{mn}T_{mn})\,v_1=0 \quad \text{--- (i)}$$

$$\Rightarrow \alpha_{11}T_{11}(v_1)+\alpha_{12}T_{12}(v_1)+\cdots\cdots+\alpha_{1n}T_{1n}(v_1)$$
$$+\alpha_{21}T_{21}(v_1)+\alpha_{21}T_{21}(v_1)+\cdots\cdots+\alpha_{2n}T_{2n}(v_1)$$
$$+\cdots\cdots\cdots\cdots\cdots$$
$$+\alpha_{m1}T_{m1}(v_1)\,\alpha_{m1}T(v_1)+\cdots\cdots+\alpha_{mn}T_{mn}(v)=0$$

$$\Rightarrow \alpha_{11}\lambda_1\omega_1+\alpha_{12}\lambda_1\omega_2+\cdots\cdots+\alpha_{1n}\lambda_1\omega_n$$
$$+0+0+\cdots\cdots+0$$
$$+\cdots\cdots\cdots\cdots$$
$$+0+0+\cdots\cdots+0=0$$

$\because T_{ij}(v_k) = \lambda_j\omega_j,\ i=k$
$= 0,\ i\neq k$

$$\Rightarrow \alpha_{11}\omega_1+\alpha_{12}\omega_2+\cdots\cdots+\alpha_{1n}\omega_n=0 \qquad \because \lambda_1\neq 0$$

and $\{\omega_1,\omega_2,\cdots,\omega_n\}$ is basis of $W$

$$\Rightarrow \alpha_{11}=0=\alpha_{12}=\alpha_{13}=\cdots\cdots=\alpha_{1n}$$

Similarly operating (i) on $v_k$ we have

$$\alpha_{ij}=0 \quad,\ i=1,2,\cdots,m,\ \ j=1,2,\cdots,n.$$

So the set $\{T_{11},T_{12},\cdots,T_{2j},\cdots,T_{mn}\}$ is L.I.

Now consider

$$S_0=a_{11}T_{11}+a_{12}T_{12}+\cdots\cdots+a_{1n}T_{1n}$$
$$+a_{21}T_{21}+a_{22}T_{22}+\cdots\cdots+a_{2n}T_{2n}$$
$$+\cdots\cdots\cdots\cdots$$
$$+a_{m1}T_{m1}+a_{m2}T_{m2}+\cdots\cdots+a_{mn}T_{mn}$$

So

$$S_0(v_1)=(a_{11}T_{11}+a_{12}T_{12}+\cdots\cdots+a_{1n}T_{1n}$$
$$+a_{21}T_{21}+a_{22}T_{22}+\cdots\cdots+a_{2n}T_{2n}$$
$$+\cdots\cdots\cdots\cdots$$
$$+a_{m1}T_{m1}+a_{m2}T_{m2}+\cdots\cdots+a_{mn}T_{mn})\,v_1$$

$$\Rightarrow S_0(v_1) = a_{11}T_{11}(v_1) + a_{12}T_{12}(v_1) + \cdots + a_{1n}T_{1n}(v_1)$$
$$+ a_{21}T_{21}(v_1) + a_{22}T_{22}(v_1) + \cdots + a_{2n}T_{2n}(v_1)$$
$$+ \cdots$$
$$+ a_{m1}T_{m1}(v_1) + a_{m2}T_{m2}(v_2) + \cdots + a_{mn}T_{mn}(v_1)$$
$$= a_{11}\lambda_1 w_1 + a_{12}\lambda_2 w_2 + a_{13}\lambda_1 w_3 + \cdots + a_{1n}\lambda_1 w_n$$

Similarly

$$S_0(v_2) = a_{21}\lambda_2 w_1 + a_{22}\lambda_2 w_2 + a_{23}\lambda_2 w_3 + \cdots + a_{2n}\lambda_2 w_n$$

$$\cdots$$

$$S_0(v_k) = a_{k1}\lambda_k w_1 + a_{k2}\lambda_k w_2 + a_{k3}\lambda_k w_3 + \cdots + a_{kn}\lambda_k w_n$$

Let $S \in Hom(V, W)$

$$\Rightarrow S(v_1), S(v_2), \ldots, S(v_k) \in W$$

so

$$S(v_1) = a_{11}w_1 + a_{12}w_2 + \cdots + a_{1n}w_n$$
$$S(v_2) = a_{21}w_1 + a_{22}w_2 + \cdots + a_{2n}w_n$$

$$\cdots$$

$$S(v_k) = a_{k1}w_1 + a_{k2}w_2 + \cdots + a_{kn}w_n$$

i.e $S \simeq S_0$ so $S_0 \in Hom(V, W)$

Thus

$\{T_{11}, T_{12}, \ldots, T_{ij}, \ldots, T_{mn}\}$ form a basis of $Hom(V, W)$

$$\Rightarrow \dim(Hom(V, W)) = mn$$

# Definition (Dual Space):-

Let $V$ be a vector space over a field $F$. Then $Hom(V, F)$ is called dual space and is denoted by $V^*$ or $\hat{V}$. Its element are called linear functional.

# Theorem:-

If $V$ is finite dimensional vector space over $F$. then prove $V \cong V^*$.

**Proof.**

Since $\dim V = \dim V^*$

so consider $\dim V = \dim V^* = m$

Define a mapping $T : V \to V^*$ by

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m$$

i) $T$ is homomorphism

$$\begin{aligned} T(v+v') &= T[(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) + (\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)] \\ &= T((\alpha_1+\beta_1)v_1 + (\alpha_2+\beta_2)v_2 + \cdots + (\alpha_m+\beta_m)v_m) \\ &= (\alpha_1+\beta_1)f_1 + (\alpha_2+\beta_2)f_2 + \cdots + (\alpha_m+\beta_m)f_m \\ &= (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m) + (\beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m) \\ &= T(v) + T(v') \end{aligned}$$

and

$$\begin{aligned} T(\alpha v) &= T(\alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m)) \\ &= T(\alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \cdots + \alpha\alpha_m v_m) \\ &= \alpha\alpha_1 f_1 + \alpha\alpha_2 f_2 + \cdots + \alpha\alpha_m f_m \\ &= \alpha[\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m] \\ &= \alpha\, T(v) \end{aligned}$$

ii) $T$ is one-one

if $T(v) = T(v')$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = T(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)$

$\Rightarrow \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m = \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m$

$\Rightarrow (\alpha_1 - \beta_1) f_1 + (\alpha_2 - \beta_2) f_2 + \cdots + (\alpha_m - \beta_m) f_m = 0$

$\because \{f_1, f_2, \ldots, f_m\}$ is basis of $V^*$

$\therefore \alpha_1 - \beta_1 = 0 = \alpha_2 - \beta_2 = \cdots = \alpha_m - \beta_m$

$\Rightarrow \alpha_1 = \beta_1, \ \alpha_2 = \beta_2, \ \cdots, \ \alpha_m = \beta_m$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m$

$\Rightarrow v = v'$

iii) T is onto

Since for $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m \in V^*$

we have

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \in V$$

such that

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m$$

Thus T is onto.

~~and~~ hence $V \cong V^*$

# Definition:

Let $T: V_1 \to V_2$ is homomorphism of a vector space $V_1(F)$ to a vector space $V_2(F)$ then $\ker T$ is called null space denoted by $N(T)$.

The dimension of $N(T)$ is called nullity.

# Theorem:

Let $T: V_1 \to V_2$ be a vector space homomorphism then $\dim V_1 = \dim N(T) + \dim R(T)$.

**Proof:**

Let $\dim N(T) = m$

and $\dim(V_1) = n$

Let $\{v_1, v_2, \ldots, v_m\}$ be basis of $N(T) = \ker T$

Since $N(T) = \ker T$ is a subspace of $V_1$

$\because$ we can take basis of $V_1$

$$\{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$

we have to prove

$$\{T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\} \text{ form basis of } R(T)$$

Let $w \in R(T)$ then there is $v \in V_1$ such that

$$T(v) = w$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \cdots + \alpha_m v_m) = w$$

$$\Rightarrow \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_m T(v_m) + \alpha_{m+1} T(v_{m+1}) + \cdots + \alpha_m T(v_m) = w$$

$\because \{v_1, v_2, \ldots, v_m\} \in N(T) = \ker T$

$\therefore T(v_1) = 0, T(v_2) = 0, \ldots, T(v_m) = 0$

$$\Rightarrow \alpha_{m+1} T(v_{m+1}) + \alpha_{m+2} T(v_{m+2}) + \cdots + \alpha_n T(v_n) = w$$

$\Rightarrow T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)$ generates $R(T)$

Now consider

$\beta_{m+1}T(v_{m+1}) + \beta_{m+2}T(v_{m+2}) + \cdots + \beta_n T(v_n) = 0$

$\Rightarrow T(\beta_{m+1}v_{m+1}) + T(\beta_{m+2}v_{m+2}) + \cdots + T(\beta_n v_n) = 0$

$\because T$ is homomorphism

$\Rightarrow T(\beta_{m+1}v_{m+1} + \beta_{m+2}v_{m+2} + \cdots + \beta_n v_n) = 0$

$\Rightarrow \beta_{m+1}v_{m+1} + \beta_{m+2}v_{m+2} + \cdots + \beta_n v_n \in \operatorname{Ker} T = N(T)$

Since $\{v_1, v_2, \ldots, v_m\}$ is basis of $N(T)$

so $\exists\ \delta_1, \delta_2, \ldots, \delta_m \in F$ such that

$\beta_{m+1}v_{m+1} + \beta_{m+2}v_{m+2} + \cdots + \beta_n v_n = \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_m v_m$

$\Rightarrow \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_m v_m - \beta_{m+1}v_{m+1} - \beta_{m+2}v_{m+2} - \cdots - \beta_n v_n = 0$

As $\{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ is basis of $V_1$

therefore $\delta_1 = \delta_2 = \cdots = \delta_m = \beta_{m+1} = \beta_{m+2} = \cdots = \beta_n = 0$

i.e $\beta_{m+1} = \beta_{m+2} = \cdots = \beta_n = 0$

$\Rightarrow \{T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\}$ is L.I

and hence form a basis of $R(T)$.

So $\dim R(T) = n - m$

$= \dim V_1 - \dim N(T)$

$\Rightarrow \dim V_1 = \dim N(T) + \dim R(T)$

proved

# Theorem

-: $V$ is a vector space over $F$ and $\{v_1, v_2, \dots, v_n\}$ be a basis of $V$. Let $\varphi_1, \varphi_2, \dots, \varphi_n \in V^* = Hom(V, F)$ are linear functional defined by

$$\varphi_i(v_j) = \delta_{ij} = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}$$

then $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is a basis of $V^*$.

Proof.

Let $\varphi \in V^*$ be taken

$\varphi(v_1) = k_1$, $\varphi(v_2) = k_2, \dots, \varphi(v_n) = k_n$

where $k_1, k_2, \dots, k_n \in F$

Let

$\psi = k_1\varphi_1 + k_2\varphi_2 + k_3\varphi_3 + \dots + k_n\varphi_n$

$$\begin{aligned} \psi(v_1) &= (k_1\varphi_1 + k_2\varphi_2 + \dots + k_n\varphi_n)v_1 \\ &= k_1\varphi_1(v_1) + k_2\varphi_2(v_2) + \dots + k_n\varphi_n(v_1) \\ &= k_1(1) + k_2(0) + \dots + k_n(0) \\ &= k_1 \end{aligned}$$

Also

$$\begin{aligned} \psi(v_2) &= (k_1\varphi_1 + k_2\varphi_2 + \dots + k_n\varphi_n)v_2 \\ &= k_1\varphi_1(v_2) + k_2\varphi_2(v_2) + \dots + k_n\varphi_n(v_2) \\ &= k_1(0) + k_2(1) + k_3(0) + \dots + k_n(0) \\ &= k_2 \end{aligned}$$

$\Rightarrow \psi(v_i) = k_i = \varphi(v_i)$

i.e $\psi = \varphi$

$\Rightarrow \varphi = \psi = k_1\varphi_1 + k_2\varphi_2 + \dots + k_n\varphi_n$

So $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ span $V^*$.

To prove $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is a linearly independent:

Consider

$a_1\varphi_1 + a_2\varphi_2 + \dots + a_n\varphi_n = 0$

then operating it on $v_1$

$$(a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n)v_1 = 0 \cdot v_1$$

$\Rightarrow a_1\phi_1(v_1) + a_2\phi_2(v_1) + \cdots + a_n\phi_n(v_1) = 0$

$\Rightarrow a_1(1) + a_2(0) + \cdots + a_n(0) = 0$

$\Rightarrow a_1 = 0$

Similarly for $i = 2, 3, \ldots, n$

$$(a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n)v_i = 0 \cdot v_i$$

$\Rightarrow a_1\phi_1(v_i) + a_2\phi_2(v_i) + \cdots + a_i\phi_i(v_i) + \cdots + a_n\phi_n(v_i) = 0$

$\Rightarrow a_1(0) + a_2(0) + \cdots + a_i(1) + \cdots + a_n(0) = 0$

$\Rightarrow 0 + 0 + \cdots + a_i + \cdots + 0 = 0$

$\Rightarrow a_i = 0$

i.e $a_1 = 0,\ a_2 = 0,\ a_3 = 0, \ldots, a_n = 0$

Hence $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is L.I and so is a basis of $V^*$

# Example

Consider the basis of

$R^2 = \{v_1 = (2,1),\ v_2 = (3,1)\}$

Find dual basis of $\{\phi_1, \phi_2\}$.

Solution.

$\phi_1(v_1) = 1 \quad , \quad \phi_1(v_2) = 0$

$\phi_2(v_1) = 0 \quad , \quad \phi_2(v_2) = 1$

Since $\phi_1, \phi_2$ are linear functional

$\therefore \quad \phi_1(x,y) = ax + by$

and $\phi_2(x,y) = cx + dy$

$\phi_1(v_1) = 1$

$\Rightarrow \phi_1(2,1) = 1 \Rightarrow 2a + b = 1$ —— (i)

$\phi_1(v_2) = 0$

$\Rightarrow \phi_1(3,1) = 0 \Rightarrow 3a + b = 0$ —— (ii)

By (i) and (ii)

$a = -1$ and $b = 3$

Now $\phi_2(v_1) = 0$

$\phi_2(2,1) = 0 \Rightarrow 2c + d = 0$ —— (iii)

and $\phi_2(v_2) = 1$

$\phi_2(3,1) = 1 \Rightarrow 3c + d = 1$ —— (iv)

Solving (iii) and (iv)

$c = 1$ and $d = -2$

therefore $\phi_1 = -x + 3y$

$\phi_2 = x - 2y$

# Example

Let a basis of $\mathbb{R}^3$ is $\{v_1, v_2, v_3\}$

$v_1 = \{1,-1,3\},\ v_2 = \{0,1,-1\},\ v_3 = \{0,3,-2\}$

Find dual basis $\phi_1, \phi_2$ and $\phi_3$

such that $\phi_i(v_j) = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}$

Do youself as above

# Question

Let $V = \{a+bt : a, b \in \mathbb{R}\}$ be a vector space of polynomial of degree $\leq 1$.

Let $\phi_1, \phi_2 : V \to \mathbb{R}$ be defined by

$$\phi_1(f(t)) = \int_0^1 f(t)\,dt$$

$$\phi_2(f(t)) = \int_0^2 f(t)\,dt$$

where $\phi_1, \phi_2 \in V^*$ (dual space).

Find corresponding basis $v_1, v_2$ of $V$.

Solution:

let $v_1 = a+bt$ and $v_2 = \cancel{a+bt}\ c+dt$

By definition

$$\phi_1(v_1) = 1, \quad \phi_1(v_2) = 0, \quad \phi_2(v_1) = 0, \quad \phi_2(v_2) = 1$$

$$\phi_1(v_1) = 1$$

$$\Rightarrow \int_0^1 v_1\,dt = 1 \Rightarrow \int_0^1 (a+bt)\,dt = 1$$

$$\Rightarrow \left| at + \frac{bt^2}{2} \right|_0^1 = 1 \Rightarrow a + \frac{b}{2} = 1$$

$$\Rightarrow 2a + b = 2 \quad \text{——— (i)}$$

$$\phi_2(v_1) = 0$$

$$\int_0^2 (a+bt)\,dt = 0 \Rightarrow \left| at + \frac{bt^2}{2} \right|_0^2 = 0$$

$$\Rightarrow 2a + 2b = 0 \Rightarrow a + b = 0 \quad \text{——— (ii)}$$

By (i) and (ii)

$$\begin{array}{r} 2a + b = 2 \\ \underline{\underset{-}{a} \underset{-}{+} b = \underset{-}{0}} \\ a \quad = 2 \end{array} \Rightarrow b = -2$$

Further $\phi_1(v_2) = 0$

$$\Rightarrow \int_0^1 v_2\,dt = 0 \Rightarrow \int_0^1 (c+dt)\,dt = 0$$

$$\Rightarrow \left| ct + \frac{dt^2}{2} \right|_0^1 = 0$$

$$\Rightarrow c + \frac{d}{2} = 0 \quad \text{or} \quad 2c + d = 0 \quad \text{—— (iii)}$$

$$\phi_2(v_2) = 1$$

$$\Rightarrow \int_0^2 v_2 \, dt = 1$$

$$\Rightarrow \int_0^2 (c + dt)\,dt = 1 \Rightarrow \left| ct + \frac{dt^2}{2} \right|_0^2 = 1$$

$$\Rightarrow 2c + 2d = 1 \Rightarrow \text{—— (iv)}$$

Subtracting (iii) from (iv)

$$\begin{array}{r} \cancel{2c} + 2d = 1 \\ \underline{-\cancel{2c} \pm d = -0} \\ d = 1 \end{array} \Rightarrow c = -\frac{1}{2}$$

Hence

$v_1 = 2 - 2t$

and $v_2 = -\frac{1}{2} + t$ are basis of $V$ corresponding to dual basis $V^*$

# Eigen Value

def:- Let 'A' be a n square matrix, then $\lambda \in F$ is eigen value of A if there exist a non-zero column vector $v \in F^n$ such that $A \cdot v = \lambda v$

here v is an eigen vector corresponding to eigen value $\lambda$.

# Exercise

Find eigen values and associative eigen vector of a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Solution:-

Let $v = [x \ \ y]^t = \begin{bmatrix} x \\ y \end{bmatrix}$

$\because \quad Av = \lambda v$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 2y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

$$\Rightarrow \quad x + 2y = \lambda x$$
$$3x + 2y = \lambda y$$

or
$$(1-\lambda)x + 2y = 0 \quad \text{——— (1)}$$
$$3x + (2-\lambda)y = 0 \quad \text{——— (2)}$$

For non-trivial solution

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$$
$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$
$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$\Rightarrow (\lambda - 4)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = 4, -1$$

hence $\lambda = 4, -1$ are eigen values.

$\lambda = -1$ in eq (i) $\Rightarrow 2x + 2y = 0$

or $x + y = 0$

or $y = -x$

thus

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

i.e eigen vector is $[1 \ -1]^t$

and $\lambda = 4$ in eq (i) $\Rightarrow -3x + 2y = 0$

$\Rightarrow 2y = 3x$

or $y = \frac{3}{2}x$

thus

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{2}x \end{bmatrix} = \frac{x}{2}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

i.e eigen vector is $[2 \ 3]^t$

# Note

$Av = \lambda v$

$\Rightarrow Av - \lambda v = 0$

$\Rightarrow (A - \lambda I)v = 0$ where $I$ is identity

$\Rightarrow |A - \lambda I| = 0$

and

$Av = \lambda v$

$A(kv) = k\lambda v$

$= k(\lambda v)$

$= \lambda(kv)$

then $\lambda$ and $k$ are eigen values for $A$.

# Eigen Value & Eigen Vector (~~Alternative~~)

def:- Let $T: V \to V$ be a linear operator then $\lambda \in F$ is called eigen value of $T$ if there exist a non-zero vector $v$ such that

$$T(v) = \lambda v$$

here $v$ is eigen vector.

Note that $kv$ is also eigen vector for same eigen value $\lambda$.

$$T(kv) = kT(v)$$
$$= k\lambda v = \lambda kv.$$

# Theorem:-

Let $\lambda$ be an eigen value of an operator $T: V \to V$. Let $V_\lambda$ denotes set of all eigen vectors of $T$ belonging to same eigen value $\lambda$. The $V_\lambda$ is a subspace of $V$.

Proof:

~~Let $\lambda$ be an eigen value of an operator~~

Let $v, w \in V_\lambda$.

then $T(v) = \lambda v$ and $T(w) = \lambda w$

Now $T(av + bw) = T(av) + T(bw)$
$$= aT(v) + bT(w)$$
$$= a\lambda v + b\lambda w$$
$$= \lambda(av + bw)$$

$\Rightarrow$ $av + bw$ is also an eigen vector for $\lambda$.

hence $av + bw \in V_\lambda$

$\Rightarrow$ $V_\lambda$ is a subspace

# Theorem

:- Let $\{v_1, v_2, \ldots, v_n\}$ be non-zero eigen vectors of an operator $T$ corresponding to distinct eigen value's $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively then $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

**Proof:**

We prove the theorem by Mathematical Induction.

Let $n=1$ so if $a v_1 = 0$

$$\Rightarrow a = 0 \quad \text{as } v_1 \neq 0$$

so Condition I is true.

Let the theorem is true for $k = n-1$

i.e $v_1, v_2, \ldots, v_{n-1}$ are L.I (linearly independent)

then $a_1 v_1 + a_2 v_2 + \cdots + a_{n-1} v_{n-1} = 0$

$$\Rightarrow a_1 = a_2 = \cdots = a_{n-1} = 0$$

Consider

$$b_1 v_1 + b_2 v_2 + \cdots + b_n v_n = 0 \quad \text{------ (i)}$$

$$T(b_1 v_1 + b_2 v_2 + \cdots + b_n v_n) = 0$$

$$\Rightarrow T(b_1 v_1) + T(b_2 v_2) + \cdots + T(b_n v_n) = 0$$

$$\Rightarrow b_1 T(v_1) + b_2 T(v_2) + \cdots + b_n T(v_n) = 0$$

$$\Rightarrow b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \cdots + b_n \lambda_n v_n = 0$$

or

$$b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \cdots + b_{n-1} \lambda_{n-1} v_{n-1} + b_n \lambda_n v_n = 0 \quad \text{------ (ii)}$$

Multiplying eq (i) by $\lambda_n$

$$\lambda_n b_1 v_1 + \lambda_n b_2 v_2 + \cdots + \lambda_n b_{n-1} v_{n-1} + \lambda_n b_n v_n = 0 \quad \text{------ (iii)}$$

Subtracting (iii) from (ii)

$b_1(\lambda_1-\lambda_n)v_1 + b_2(\lambda_2-\lambda_n)v_2 + \cdots + b_{n-1}(\lambda_{n-1}-\lambda_n)v_{n-1} = 0$

Since $\{v_1, v_2, \cdots, v_{n-1}\}$ is L.I

$\Rightarrow b_1 = 0 = b_2 = b_3 = \cdots = b_{n-1}$

$\because \lambda_i - \lambda_n \neq 0 \quad ; \quad i = 1, 2, \cdots, n-1$

because if $\lambda_i - \lambda_n = 0$

$\Rightarrow \lambda_i = \lambda_n$ for $i = 1, 2, \cdots, n-1$

a contradiction as each $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct.

Now from eq (ii)

$0 + 0 + \cdots + 0 + b_n v_n = 0$

$\Rightarrow b_n v_n = 0$

$\Rightarrow b_n = 0 \quad \because v_n \neq 0$

Hence the vectors $v_1, v_2, \cdots, v_n$ are Linearly independent

# Characteristic Polynomial / Equation / Matrix :-

def:- Let $A$ be a $n$ square matrix over $F$.
then $tI - A$ is called characteristic matrix.
$|tI - A|$ is called characteristic polynomial.
and $|tI - A| = 0$ is called characteristic equation.

i.e

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$tI - A = t\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} t-a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & -a_{23} & & -a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & t-a_{nn} \end{bmatrix}$$

and $\Delta_A(t) = \det(tI - A)$ is characteristic polynomial.

Also $\Delta_A(t) = 0$ or $|tI - A| = 0$ is characteristic equation.

Exercise:

Find characteristic polynomial of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$tI - A = t\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} t-1 & -3 & 0 \\ 2 & t-2 & 1 \\ -4 & 0 & t+2 \end{bmatrix}$$

$$\Delta_A(t) = |tI - A|$$

$$= \begin{vmatrix} t-1 & -3 & 0 \\ 2 & t-2 & 1 \\ -4 & 0 & t+2 \end{vmatrix}$$

$= t^3 - t^2 + 2t + 28$ is characteristic polynomial.

Also $\Delta_A(t) = 0$

$\Rightarrow t^3 - t^2 + 2t + 28 = 0$ is characteristic equation.

**Note:** Degree of eq. will be equal to the order of matrix.

# Example:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$tI - A = t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} t-2 & -3 \\ -1 & t-5 \end{bmatrix}$$

$$B(t) = \text{adj of } (tI - A) = \begin{bmatrix} t-5 & 3 \\ 1 & t-2 \end{bmatrix}$$

$$= \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} -5 & 3 \\ 1 & -2 \end{bmatrix} = B_1 t + B_0$$

If $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

then $B(t) = \text{adj}(tI - A)$

$$= B_2 t^2 + B_1 t + B_0$$

# Calay Hamilton Theorem:

:- Every square matrix is zero of its characteristic polynomial.

OR Every square matrix satisfies its characteristic equation.

**Proof:**

Let $A$ be n square metrix

and $\Delta_A(t) = |tI - A|$ be its characteristic polynomial.

i.e $\Delta_A(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0$

Let $B(t)$ is adjoint of $tI - A$.

Since elements of $B(t)$ are cofactors of $tI - A$ and so are polynomial of degree not more than $n-1$.

and we can write

$$B(t) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1 t + B_0$$

where $B_i$ are square matrices of order $n$ over $F$.

Since by definition of adjoint of a matrix

$$(tI - A)\,B(t) = |tI - A|\,I$$

$$\Rightarrow (tI - A)(B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1 t + B_0)$$
$$= (t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0)\,I$$

Comparing the co-effients.

Comparing $t^n \Rightarrow B_{n-1}I = I$

" $t^{n-1} \Rightarrow B_{n-2}I - AB_{n-1} = a_{n-1}I$

" $t^{n-2} \Rightarrow B_{n-3}I - AB_{n-2} = a_{n-2}I$

.....................................

.....................................

" $t^1 \Rightarrow B_0 I - AB_1 = a_1 I$

" $t^0 \Rightarrow -AB = a_0 I$

Multiplying above equations by first to last by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$ respectively.

we have.

$$A^n B_{n-1} I = A^n I$$

$$A^{n-1} B_{n-2} I - A^n B_{n-1} I = a_{n-1} A^{n-1} I$$

$$A^{n-2} B_{n-3} I - A^{n-1} B_{n-2} I = a_{n-2} A^{n-2} I$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A B_0 I - A^2 B_1 = a_1 A I$$

$$-A B_0 I = a_0 I$$

Adding both sides of above equations

$$0 = A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \cdots + a_1 A + a_0$$

As required.

# Minimum Polynomial

A polynomial $m(t)$ is called minimum polynomial if

i) $m(t)$ divides $\Delta(t)$

ii) Each irreducible factor of $\Delta(t)$ divides $m(t)$

iii) $m(A) = 0$.

Question.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$tI - A = \begin{bmatrix} t-2 & -1 & 0 & 0 \\ 0 & t-2 & 0 & 0 \\ 0 & 0 & t-1 & -1 \\ 0 & 0 & 2 & t-4 \end{bmatrix}$$

$$|tI - A| = \begin{vmatrix} t-2 & -1 & 0 & 0 \\ 0 & t-2 & 0 & 0 \\ 0 & 0 & t-2 & -1 \\ 0 & 0 & -2 & t-4 \end{vmatrix}$$

expanding by $c_1$

$$= (t-2) \begin{vmatrix} t-2 & 0 & 0 \\ 0 & t-2 & -1 \\ 0 & -2 & t-4 \end{vmatrix}$$

$$= (t-2)(t-2) \begin{vmatrix} t-2 & -1 \\ 2 & t-4 \end{vmatrix}$$

$$= (t-2)^3(t-3) = (t-2)^2((t-2)(t-4)+2)$$

$$= (t^2-4t+4)(t^2-6t+8+2)$$

$$= t^4 - 10t^3 - 4t^2 + 40t + 64 \quad \text{(after solving)}$$

is characteristic polynomial.

Possible minimum polynomial are

i) $(t-2)(t-3) = f(t)$

ii) $(t-2)^2(t-3) = g(t)$

iii) $(t-2)^3(t-3) = h(t)$.

$$f(A) = (A-2)(A-3)$$
$$= (A-2I)(A-3I)$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \neq 0$$

$\Rightarrow f(t)$ is not minimum polynomial.

Now $g(t) = (t-2)^2(t-3)$

$$\Rightarrow g(A) = (A-2)^2(A-3)$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$
$$= \text{solve} = 0$$

$\Rightarrow g(t) = (t-2)^2(t-3)$ is minimum polynomial

iii) $h(A) = (A-2)^2(A-3)$

Do yourself

# Theorem:-

:- Prove that the minimum polynomial $m(t)$ divides every polynomial which has $A$ as a zero.

In particular $m(t)$ divides the characteristic polynomial $\Delta(t)$ of $A$.

**Proof:**

Let $f(t)$ be a polynomial for which $f(A)=0$. then by division algorithm, there are polynomial $q(t)$ and $r(t)$ such that

$$f(t) = q(t).m(t) + r(t) \quad \text{---} \quad (i)$$

where $r(t)=0$ or degree of $r(t)$ is less then that of $m(t)$.

From (i) $\quad f(A) = q(A)\, m(A) + r(A) \quad$ by $t = A$.

$$\Rightarrow \quad 0 = q(A) \times 0 + r(A)$$

$$\Rightarrow \quad r(A) = 0$$

then $r(t)$ is a polynomial of degree less than that of $m(t)$, which has $A$ as a zero which contradict the definition of $m(t)$.

hence $r(t) = 0$

$$\Rightarrow \quad f(t) = q(t).m(t)$$

i.e $m(t)$ divides $f(t)$

Also then $m(t)$ divides $\Delta(t)$

# Theorem

:- Let $m(t)$ be the minimum polynomial of an $n$-square matrix $A$. Then show that characteristic polynomial of $A$ divides $(m(t))^n$.

Proof.

Let $m(t) = t^r + c_1 t^{r-1} + c_2 t^{r-2} + \cdots + c_{r-1} t + c_r$

Consider

$$B_0 = I \text{ ——— (1)}$$
$$B_1 = A + c_1 I \text{ ——— (2)}$$
$$B_2 = A^2 + c_1 A + c_2 I \text{ ——— (3)}$$
$$B_3 = A^3 + c_1 A^2 + c_2 A + c_3 I \text{ ——— (4)}$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$B_{r-1} = A^{r-1} + c_1 A^{r-2} + \cdots + c_{r-1} I \text{ ——— (r)}.$$

Take

$$B(t) = t^{r-1} B_0 + t^{r-2} B_1 + t^{r-3} B_2 + \cdots + t B_{r-2} + B_{r-1}$$

Now

$$(tI - A)B(t) = (tI - A)(t^{r-1} B_0 + t^{r-2} B_1 + \cdots \cdots + t B_{r-2} + B_{r-1})$$

$$= t^r B_0 I + t^{r-1} B_1 I + t^{r-2} B_2 I + \cdots + t^2 B_{r-2} I + t B_{r-1} I - (t^{r-1} A B_0 + t^{r-2} A B_1 + \cdots \cdots + t A B_{r-2} + A B_{r-1})$$

$$= t^r B_0 + t^{r-1}(B_1 - A B_0) + t^{r-2}(B_2 - A B_1) + \cdots + t(B_{r-1} - A B_{r-2}) - A B_{r-1}$$

——— (a)

Now from eqs (1) to (r) gives

$$B_1 - A B_0 = c_1 I$$
$$B_2 - A B_1 = c_2 I$$
$$\cdots\cdots\cdots$$

$\therefore \quad B_{r-1} - AB_{r-2} = c_{r-1} I$

Also from rth equation

$$AB_{r-1} = A^r + c_1 A^{r-1} + \cdots + c_{r-1} AI$$
$$= A^r + c_1 A^{r-1} + \cdots + c_{r-1} AI + c_r I - c_r I$$
$$= m(A) - c_r I$$

$\Rightarrow \quad AB_{r-1} = -c_r I \qquad \because m(A) = 0$

Using all these values in eq (8)

$$(tI - A)\cdot B(t) = t^r I + t^{r-1} c_1 I + t^{r-2} c_2 I + \cdots + t c_{r-1} I + c_r I$$

$$= (t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + t c_{r-1} + c_r) I$$

taking determinant to both sides.

$$|(tI - A)\, B(t)| = |(t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + c_r) I|$$

$$\Rightarrow |tI - A|\,|B(t)| = (t^r + c_1 t^{r-1} + c_2 t^{r-2} + \cdots + c_r)^n$$
$$= (m(t))^n$$

$\Rightarrow |tI - A|$ divides $(m(t))^n$.

i.e characteristic polynomial divide $(m(t))^n$

# Similar Matrix

def:- A matrix B is similar to a matrix A if there is non-singular matrix P such that
$$B = P^{-1}AP \quad \text{or} \quad PB = AP.$$

# Diagonalization of Matrix:-

def:- A matrix A is said to be diagonalizable if there is a matrix such that
$$B = P^{-1}AP$$
In this case column of P are eigen vectors of A and diagonal element of B are corresponding eigen values of A.

Question If $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ then diagonalize this matrix

Solution:

To find eigen values
$$|\lambda I - A| = 0$$
$$\Rightarrow \begin{vmatrix} \lambda - 4 & -2 \\ -3 & \lambda + 1 \end{vmatrix} = 0$$
$$\Rightarrow \lambda = 5, -2$$

i) $\lambda = 5$ then for eigen vectors
$$MX = 0$$
$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\Rightarrow x_1 - 2x_2 = 0$$
$$-3x_1 + 6x_2 = 0$$
One of its solution is $x_2 = 1 \Rightarrow x_1 = 2$

eigen vector $(2, 1)^t$

ii) $\lambda = -2$
$$\Rightarrow MX = 0 \Rightarrow \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow -6x - 2y = 0$
$-3x - y = 0$

$\Rightarrow$ if $x = 1 \Rightarrow y = -3$

eigen vector $= (1, -3)^t$

Now

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

$$|P| = -6 - 1 = -7$$

$$P^{-1} = -\frac{1}{7}\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{bmatrix}$$

Now

$$P^{-1}AP = \begin{pmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{pmatrix}\begin{pmatrix} 10 & -2 \\ 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$$

is diagonal where diagonal ~~val~~ elements are eigen values of $A$.

Question: Find $A^{10}$ for $A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$

$B = P^{-1}AP$

$PBP^{-1} = A$

So $A^{10} = (PBP^{-1})^{10}$

$= PB^{10}P^{-1}$

$$= \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}^{10}\begin{bmatrix} 3/7 & 1/7 \\ 1/7 & -2/7 \end{bmatrix}$$

$\because (PBP^{-1})^2$
$= (PBP^{-1})(PBP^{-1})$
$= PBP^{-1}PBP^{-1}$
$= PBIBP^{-1}$
$= PB^2P^{-1}$

Simplify yourself

# Theorem:

Similar matrix $A$ and $P^{-1}AP$ have the same characteristic polynomial.

Proof:

Let $A$ and $B$ are similar matrices then $B = P^{-1}AP$

Using

$$tI = P^{-1}tIP$$

$$|tI - B| = |tI - P^{-1}AP|$$

$$= |P^{-1}tIP - P^{-1}AP|$$

$$= |P^{-1}(tI - A)P|$$

$$= |P^{-1}|\,|tI - A|\,|P|$$

$$= |tI - A|\,|P^{-1}|\,|P|$$

$$= |tI - A|$$

As required