Vector Spaces: Handwritten notes
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# Ring

defn: A non-empty set $R$ is called ring if
i) $R$ is abelian group under multiplication .addition
ii) $R$ is semi-group under multiplication.
iii) Distributive law holds,
\[ a(b + c) = ab + ac \]
\[ (a + b)c = ac + bc \]

Examples
i) $(\mathbb{Z}, +, \cdot)$ is a ring
where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$
ii) $(\mathbb{Q}, +, \cdot)$, where $\mathbb{Q}$ is the set of rational numbers.
iii) $(\mathbb{R}, +, \cdot)$, where $\mathbb{R}$ is set of real numbers.
iv) $(\mathbb{Z}_n, +, \cdot)$, $\mathbb{Z}_n$ = residue classes of module $n$.

# Field

defn: A non-empty set $F$ is called a field if
i) $F$ is abelian group under addition
ii) $F-\{0\}$ is abelian group under multiplication.
iii) Right distributive law holds in $F$.
\[ a(b + c) = ab + ac \]
\[ (a + b)c = ac + bc \]

Examples
i) $(\mathbb{R}, +, \cdot)$ is a field.
ii) $(\mathbb{C}, +, \cdot)$ is a field.
iii) $(\mathbb{Q}, +, \cdot)$ is a field.
iv) $(\mathbb{Z}, +, \cdot)$ is not a field
as $(\mathbb{Z}-\{0\}, \cdot)$ is not group under multiplication.
Vector Space

definition: let $V$ be a non-empty set and $F$ is a field, then $V$ is called a vector space if:

i) $V$ is an abelian group under addition.

ii) $a(v + w) = av + aw \quad \forall \ a, v, w \in V$

iii) $(a + b)v = av + bv \quad \forall \ a, b \in F, \ v \in V$

iv) $a(bv) = (ab)v \quad \forall \ a, b \in F, \ v \in V$

v) $1 \cdot v = v, 1 = v, 1 \in F$ and $v \in V$

i.e. $1$ is identity under multiplication.

Example:

i) Let $V$ be a set of all polynomials of degree $\leq n$ then $V$ is vector space.

$$V = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \mid a_i \in F, \forall i \leq n \in N \}$$

Addition is defined as:

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i$$

and multiplication is defined as:

$$1 \cdot \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (a_i x^i)$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

ii) Let $F$ be a field then the set

$$F^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in F, 1 \leq i \leq n \}$$

iii) The set $M_{n \times n}$ of all $n \times n$ matrices with entries from a field $F$ is a vector space over $F$.

iv) Every field is a vector space over itself.
Subspace:
Let $V$ be a vector space, over $F$ and $W$ be its non-empty subset of $V$.
Then $W$ is a subspace of $V$ if $W$ itself is a vector space under operation induced (defined) in $V$.

Theorem:
A non-empty subspace subset $W$ of a vector space $V$ is a subspace of $V$ iff:
\[ \text{i) } w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W, \]
\[ \text{ii) } a \in F, w \in W \Rightarrow aw \in W. \]

Proof:
Let $W$ is subspace of vector field space $V$.
then $W$ itself is a vector space i.e. $W$ is closed under addition and scalar multiplication.

Conversely, let $W$ is a subset satisfying:
condition (i) and (ii)

Then for $-1 \in F$ and $w_2 \in W$,
\[ \Rightarrow -w, \in W \text{ by condition (ii)}. \]
\[ \Rightarrow -w_2 \in W. \]
\[ \Rightarrow w_1 - w_2 \in W \]
\[ \Rightarrow w_1 + (-w_2) \in W \text{ by condition (i)}. \]
\[ \Rightarrow W \text{ is a subgroup under addition}. \]
Since $W$ is a subset of $V$ and $V$ is abelian.
So $W$ is abelian.

Further condition II to $V$, of the definition are satisfied in $W$ as these are satisfied in $V$.

Corollary:
$W$ is non-empty subset of a vector space $V(F)$. Then $W$ is subspace of $V$ iff:
\[ a, b \in F, w_1, w_2 \in W \Rightarrow aw_1 + bw_2 \in W. \]
Proof. Let \( W \) be a subspace of \( V(F) \), then \( W \) itself is a vector space.

For \( a, b \in F \), \( w, w' \in W \),
\[
aw, bw' \in W.
\]

\( W \) is closed under addition.

Conversely,

Let \( a, b \in F \), \( w, w' \in W \).
\[
aw + bw' \in W.
\]

Set \( a = b = 1 \),
\[
1w + 1w' \in W.
\]
\( w + w' \in W \).

Also, if \( b \in F \),
\[
aw, bw' \in W.
\]
\( aw + bw' \in W \).
\( aw + bw' \in W \).

\( W \) is a subspace of \( V \).

Definition (Linear Sum).

Let \( V \) be a vector space over \( F \) and \( W_1, W_2, \ldots, W_n \) be non-empty subsets of \( V \). Then their linear sum \( S \) is defined as:

\[
S = \{ \sum_{i=1}^{n} a_i w_i \mid a_i \in F, w_i \in W_i \}.
\]

Lema. Let \( V \) be a vector space and \( W, W_1, \ldots, W_n \) be subspaces. Prove that:

\[
W = W_1 + W_2 + \ldots + W_n
\]

is also a subspace of \( V \).
Lemma:

$W_1, W_2, \ldots, W_n$ are subspaces of $V$ prove that

$W = W_1 + W_2 + \ldots + W_n$ is a subspace of $V$.

Proof:

$c = c + 0 + \ldots + 0, \quad 0 \in W_2$.

$\Rightarrow c \in W \Rightarrow W$ is non-empty.

Let $x, y \in W, \quad a, b \in F$

we have to show $ax + by \in W$.

$x \in W$,

$\Rightarrow x = x_1 + x_2 + \ldots + x_n$ for $x_1 \in W_1, x_2 \in W_2, \ldots, x_n \in W_n$.

$y = y_1 + y_2 + \ldots + y_n$ for $y_1 \in W_1, y_2 \in W_2, \ldots, y_n \in W_n$.

Now

$ax + by = a(x_1 + x_2 + \ldots + x_n) + b(y_1 + y_2 + \ldots + y_n)$

$= ax_1 + ax_2 + \ldots + ax_n + by_1 + by_2 + \ldots + by_n$

$= (ax_1 + by_1) + (ax_2 + by_2) + \ldots + (ax_n + by_n)$.

As each $W_i$ is a subspace

$\Rightarrow ax_i + by_i \in W_i, \quad i = 1, 2, \ldots, n$.

$\Rightarrow \sum_{i=1}^{n} ax_i + by_i \in \bigoplus_{i=1}^{n} W_i = W$.

$\Rightarrow ax + by \in W$.

So $W$ is a subspace.

Lemma:

Let $V$ be a vector space and $W_i$ a family of subspaces of $V$. Then $\bigcap W_i$ is also a subspace of $V$.

Proof:

Let $v, w \in \bigcap W_i$.

then $v, w \in W_i$ for each $i \in I$

and since each $W_i$ is a subspace

so there must be $a, b \in F$

such that $av + bw \in W_i$ for each $i \in I$.

so $av + bw \in \bigcap W_i$, i.e. $\bigcap W_i$ is a subspace.
**Definition**

Let \( U \) and \( V \) are two vector spaces over a field \( F \). Then \( \mathcal{T} \) of \( U \) into \( V \) is called homomorphism if 
\[
\mathcal{T}(u_1 + u_2) = \mathcal{T}(u_1) + \mathcal{T}(u_2)
\]
\[
\mathcal{T}(au) = a \mathcal{T}(u) ; \quad a \in F.
\]

**Definition**

The kernel of homomorphism \( \mathcal{T} : U \rightarrow V \) is defined as 
\[
\ker \mathcal{T} = \{ u \in U ; \mathcal{T}(u) = 0 \}.
\]

(Question)

Prove that \( \ker \mathcal{T} \) (kernel of homomorphism) is a subspace.

**Solution**

Let \( u, u_1 \in \ker \mathcal{T} \).

\[
\mathcal{T}(u) = 0 ; \quad \mathcal{T}(u_1) = 0.
\]

Now, let \( a, b \in F \).

\[
\mathcal{T}(au_1 + bu_2) = \mathcal{T}(au_1) + \mathcal{T}(bu_2)
\]
\[
= a \mathcal{T}(u_1) + b \mathcal{T}(u_2)
\]
\[
= a(0) + b(0)
\]
\[
= 0.
\]

\[
\Rightarrow au_1 + bu_2 \in \ker \mathcal{T}.
\]

So, \( \ker \mathcal{T} \) is subspace.

**Linear Combination**

Let \( V \) is a vector space. Let \( v_1, v_2, \ldots, v_n \in V \).

\( a_1, a_2, \ldots, a_n \in F \).

Then an element
\[
a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n
\]
is called linear combination.

The linear combination is trivial if each \( a_i = 0 \).

And it is non-trivial if at least one of \( a_i \neq 0 \).
# Definition: (Linear Span)

Let \( S \) be a subset of vector space \( V \), then the set of all linear combinations of \( S \) is called linear span, denoted by \( \langle S \rangle \) or \( L(S) \) or \( [S] \).

# Theorem

Prove that \( \langle S \rangle \) is a subspace of \( V \), containing \( S \). It is smallest subspace of \( V \) containing \( S \).

Proof:

Let \( u, v \in \langle S \rangle \).

- then \( u = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n \)
  \( v = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n \).

For \( a, b \in F \) we have to prove \( au + bv \in \langle S \rangle \).

Now,

\[
au + bv = a(a_1 u_1 + a_2 u_2 + \ldots + a_n u_n) + b(b_1 v_1 + b_2 v_2 + \ldots + b_n v_n)
\]

\[
= a_1 au_1 + a_2 au_2 + \ldots + a_n au_n + b_1 bv_1 + b_2 bv_2 + \ldots + b_n bv_n
\]

\( \Rightarrow au + bv \in \langle S \rangle \)

\( \Rightarrow \langle S \rangle \) is a subspace.

Let \( u \in S \).

- then \( u = \sum_{i=1}^{n} c_i u_i \in \langle S \rangle \)

\( \Rightarrow u \in \langle S \rangle \).

- Let \( W \) be any other subspace of \( V \) containing \( S \).

\( \Rightarrow \exists a_1 u_1, a_2 u_2, \ldots \in W \)

\( \Rightarrow W \) is subspace containing \( S \).

\( \Rightarrow \langle S \rangle \subseteq W \).

i.e \( \langle S \rangle \) is smallest subspace containing \( S \).
Definition (Finite Dimensional Vector Space)
A vector space \( V \) is called finite dimensional if there is a subset \( S \) of \( V \) such that \( < S > = V \).

Definition (Linearly Dependent and Independent)
Let \( V \) be a vector space. Then the vectors \( v_1, v_2, \ldots, v_n \) in \( V \) are linearly dependent if \( a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \) and not all \( a_i \) are zero.

If \( a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \) where each \( a_i = 0 \) then the vectors \( v_1, v_2, \ldots, v_n \) are linearly independent.

Theorem:
Let \( V \) be a vector space and consider a set of vectors \( \{v_1, v_2, \ldots, v_n\} \) are linearly independent, then its subset is also independent.

ii) If \( \{v_1, v_2, \ldots, v_n\} \) is dependent then \( \{v_1, v_2, \ldots, v_n, v_{n+1}\} \) is also dependent.

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# Lemma:

Let \( V, (F) \) be a vector space and \( S = \{v_1, v_2, \ldots, v_n\} \) a set of vectors in \( V \). Then:

i) If \( S \) is independent, then any non-empty subset of \( S \) is also independent.

Proof:

Let \( \{v_1, v_2, \ldots, v_i\} \) be a subset of \( S \), \( 1 \leq i \leq n \). Consider:

\[
a_1v_1 + a_2v_2 + \cdots + a_iv_i = 0, \quad a_i \in F
\]

then

\[
a_1v_1 + a_2v_2 + \cdots + a_iv_i = 0v_1 + 0v_2 + \cdots + 0v_i = 0
\]

Since \( \{v_1, v_2, \ldots, v_i\} \) is linearly independent

\[
\Rightarrow \text{each } a_i = 0; \quad k = 1, 2, \ldots, i
\]

\[
\Rightarrow \{v_1, v_2, \ldots, v_i\} \text{ is L.I.}
\]

Proof (ii)

If \( S \) is dependent, then

\[
\{v_1, v_2, \ldots, v_n\} \text{ is also dependent.}
\]

i.e. \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) where all \( a_i \neq 0 \).

and then

\[
a_1v_1 + a_2v_2 + \cdots + a_nv_n + 0v = 0
\]

where all \( a_i \neq 0 \).

\[
\Rightarrow \{v_1, v_2, \ldots, v_n, v_k\} \text{ is also dependent}
\]

# Theorem:

A set of non-zero vectors \( v_1, v_2, \ldots, v_n \in V \) is linearly dependent iff one of them is a linear combination of the other preceding vectors.

Proof:

\( \{v_1, v_2, \ldots, v_n\} \) is linearly dependent.

i.e. \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) where all \( a_i \)'s \( \neq 0 \).

for \( a_i \in F \)

Let \( a_k \) be the last non-coefficient of

\[
a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n
\]

\[
a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n
\]
\[ a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \]
\[ \Rightarrow a_{k+1} = a_{k+2} = \cdots = a_n = 0 \]

\[ \Rightarrow -a_{k-1}v_k = a_1v_1 + a_2v_2 + \cdots + a_{k-2}v_{k-2} \]
\[ \Rightarrow v_k = -\frac{1}{a_{k-1}} (a_1v_1 + a_2v_2 + \cdots + a_{k-2}v_{k-2}) \]

Conversely, let \( v_k \) is a linear combination of the preceding vectors:

\[ v_1, v_2, \ldots, v_{k-1} \]

\[ v_k = a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} \]
\[ \Rightarrow a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + (-1)v_k = 0 \]
\[ \Rightarrow a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + 0v_k + \cdots + 0v_n = 0 \]

then \( \{v_1, v_2, \ldots, v_n\} \) is Linearly Dependent.

\[ \Rightarrow \text{at least one coefficient of } v_k \text{ is non-zero.} \]

# Basis of a Vector Space:

Let \( S \) be a subset of a vector space \( V(F) \), then \( S \) is called basis for \( V \) if

i) \( S \) is linearly independent.

ii) \( S \) is spanning set of \( V \).
Theorem:

Any finite dimensional vector space contains a basis.

Proof:

Let \( \{v_1, v_2, \ldots, v_r\} \) be a spanning set of \( V \).

If \( \{v_1, v_2, \ldots, v_r\} \) is linearly independent then it forms a basis, and there is nothing to prove.

Consider \( \{v_1, v_2, \ldots, v_r\} \) is linearly dependent then one of the vectors, say \( v_r \), is a linear combination of the remaining \( \{v_1, v_2, \ldots, v_{r-1}\} \).

We drop out this vector and obtain a set of \( r-1 \) vectors.

A vector linear combination of \( r \) vectors is also a linear combination of \( r-1 \) vectors.

If this set \( \{v_1, v_2, \ldots, v_{r-1}\} \) is linearly independent then it forms a basis.

But if \( \{v_1, v_2, \ldots, v_{r-1}\} \) is dependent then the above process is continued. In this way we can get a linear independent spanning set.

and hence a basis.

\[ \{v_1, v_2, \ldots, v_r\} \text{ is a basis.} \]

Theorem:

If \( v_1, v_2, \ldots, v_r \) is a basis of \( V(F) \) and if \( w_1, w_2, \ldots, w_m \subset V \) are linearly independent then \( m \leq r \).

Proof:

Since \( v_1, v_2, \ldots, v_r \) is a basis of \( V \) so every element of \( V \) can be expressed as a linear combination of \( v_1, v_2, \ldots, v_r \).

In particular \( w_m \in V \) is a linear combination of \( v_1, v_2, \ldots, v_r \).

[11]
\[ v_1, v_2, \ldots, v_n \text{ are dependent.} \]

Therefore, a proper subset \( \{v_1, v_2, \ldots, v_r\}, r \leq n-1 \) from a basis.

Similarly, \( \{v_s, v_{s+1}, \ldots, v_n\}, s \leq n-2 \) is dependent and the proper subset

\[ \{v_1, v_2, \ldots, v_s, v_{s+1}, \ldots, v_n\}, s \leq n-2 \]

Repeating this procedure \((n-1)\) times, we get a basis.

\[ \{v_1, v_2, \ldots, v_t\}, t \geq 1 \text{ since the vectors } v_t \text{ is not a linear combination} \]

\[ (1 \leq t \leq n-m+1) \]

\[ 1 \leq n-m+1 \]

\[ 0 \leq n-m \]

\[ m \leq n \]

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Question. Show that the vectors
\[ v_1 = (1, 1, 1), \ v_2 = (1, 0, 1), \ v_3 = (0, 1, 1) \]
are linearly independent.

Solution:
Consider \( a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \)
\[ \Rightarrow a_1 (1, 1, 1) + a_2 (1, 0, 1) + a_3 (0, 1, 1) = 0. \]
\[ \Rightarrow (a_1, a_1, a_1) + (a_2, 0, a_2) + (0, a_3, a_3) = 0. \]
\[ \Rightarrow (a_1 + a_2, a_1 + a_3, a_2 + a_3) = (0, 0, 0). \]
\[ \Rightarrow a_1 + a_2 = 0 \quad (i) \]
\[ a_1 + a_3 = 0 \quad (ii) \]
\[ a_2 + a_3 = 0 \quad (iii) \]
\[ \Rightarrow \frac{a_1}{a_2} = \frac{a_3}{a_2} \]
\[ a_3 = 0 \quad \Rightarrow a_1 = 0, \ a_2 = 0. \]

Since \( a_1 = a_2 = a_3 = 0 \)
\[ \Rightarrow \text{the vectors are L.I.} \]

Question. Prove that the vectors
\[ v_1 = (3, 0, -3), \ v_2 = (-1, 1, 2), \ v_3 = (1, 2, -2) \]
\[ v_4 = (2, 1, 1) \]
are linearly dependent.

Solution:
Consider \( a_1 v_1 + b v_2 + c v_3 + d v_4 = 0 \)
\[ \Rightarrow a_1 (3, 0, -3) + b (-1, 1, 2) + c (1, 2, -2) + d (2, 1, 1) = 0 \]
\[ \Rightarrow (3a, 0, -3a) + (-b, b, 2b) + (c, 2c, -2c) + (2d, d, d) = 0 \]
\[ \Rightarrow (3a - b + c + 2d, b + 2c + d, -3a + 2b - 2c + d) = 0 \]
\[ 3a - b + c + 2d = 0 \]
\[ b + 2c + d = 0 \]
\[ -3a + 2b - 2c + d = 0 \]
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Let \( d = 0 \) and \( \alpha = \), then

\[
3a + b + 4c = 0 \quad (1)
\]

\[
b + d = 0
\]

\[-3a + 2b - 2c = 0
\]

Substituting \( a = -2c \), \( b = -2c \), \( d = 0 \)

into (1)

\[
-2cV_1 - 2cV_2 + cV_3 + 0V_4 = 0
\]

\[
2V_1 + 2V_3 - V_3 + 0V_4 = 0
\]

\[
\Rightarrow V_1, V_2, V_3, V_4 \text{ are dependent.}
\]

\[
\text{Check:}
\]

Since \( V_1, V_2, V_3, V_4 \) are dependent, they do not span a unique solution.
**Definition:** (Quotient Space)

Let $V$ be a vector space over a field $F$, and $W$ be a subspace of $V$. The set $V/W$ of all left cosets along with two operations

- $\text{i)} \quad (v_1 + W) + (v_2 + W) = v_1 + v_2 + W$
- $\text{ii)} \quad a(v + W) = av + W$

is called Quotient space.

**Theorem:**

Let $V$ be a vector space and $W$ a subspace of $V$ along with the operations

- $\text{i)} \quad (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$
- $\text{ii)} \quad a(v + W) = av + W$, is a subspace vector space

**Proof:**

1. It is easy to show that $V/W$ is an abelian group under addition with $0 + W = W$ as identity, and $-v + W$ is an inverse of $v + W \in V/W$.
2. We can see that scalar multiplication is defined in $V/W$.

(i.e. $v + w = v' + w \Rightarrow a(v + w) = a(v' + w)$)

Let $v = v' + w$ for some $w \in W$.

then $a(v + W) = a(\alpha v + w)$

$\text{=} a(v' + w) + W$

$\text{=} a\alpha v' + aw + W$

$\text{=} a\alpha v' + W$ \quad \because aw \in W$

$\text{=} a(v' + W)$

i.e. Scalar multiplication is defined.

Let $v + W, v' + W \in V/W$, $a \in F$

$a((v + W) + (v' + W)) = a(v + v' + W)$

$= a(v + v') + W$

$= av + av' + W$

$= av + W + av' + W$

$= a(v + W) + a(v' + W)$
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\[ (a + b)(v + w) = (a + b)v + (a + b)w = av + bw + \frac{a+b}{2} (v+w) = a(v+w) + b(v+w) \]

\[ a \left( b(v + w) \right) = a (bv + vw) = (ab)v + vw = (ab)(v + w) \]

\[ 1 \cdot (v + w) = \frac{1}{1} v + \frac{1}{1} w = v + w \]

Hence \( v/w \) is vector space.

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Theorem: \( V(F) \) is a finite dimensional vector space and if \( W \) is a subspace of \( V \). Then

i) \( W \) is finite dimensional and \( \dim W \leq \dim V \).

ii) \( \dim (V/W) = \dim V - \dim W \)

Proof:

Let \( \dim V = n \)

and \( \{w_1, w_2, \ldots, w_m\} \) be a linearly independent set of vectors of \( W \).

Then \( m \leq n \)

Therefore, the set \( \{w_1, w_2, \ldots, w_m, w_{m+1}\} \) is linearly dependent.

i.e., one of these vectors in a linear combination of the preceding vector.

However, none of the vectors \( w_1, w_2, \ldots, w_m \) is a linear combination of the preceding vectors.

because the vectors \( w_1, w_2, \ldots, w_m \) are linearly independent.

so \( w \) can be written as a linear combination of \( w_1, w_2, \ldots, w_m \)

Since \( w \in W \) is an arbitrary element.

therefore, \( \dim W \) is finite dimensional

and \( \dim W = m \leq n \)

i.e., \( \dim W \leq \dim V \).

ii) Let \( \{v_1, v_2, \ldots, v_m\} \) be a basis of \( V \)

and \( \{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_k\} \) be a basis of \( V/W \)

we have to prove \( \{v_1 + W, v_2 + W, \ldots, v_k + W\} \) is a basis of \( V/W \)

Now

\[ \alpha_1 (v_1 + W) + \alpha_2 (v_2 + W) + \cdots + \alpha_k (v_k + W) = 0 \]

\[ (\alpha_1 v_1 + W) + (\alpha_2 v_2 + W) + \cdots + (\alpha_k v_k + W) = 0 + W \]

since \( W \) is ideality

\[ (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k) + W = 0 + W \].
Vector Spaces: Handwritten notes

\[ a_1v_1 + a_2v_2 + \cdots + a_lv_l \in W \]

\[ a_1v_1 + a_2v_2 + \cdots + a_lv_l = \beta_1w_1 + \beta_2w_2 + \cdots + \beta_mw_m \]

So \[ a_1v_1, a_2v_2, \ldots, a_lv_l \] is a basis of \( W \).

So \[ \beta_1 = \beta_2 = \cdots = \beta_m = 0 \]

\[ v_1, w_1, v_2 + w_2, \ldots, v_l + w_l \] is linearly independent.

Let \( v + w \in V/W \) for \( v \in V \)

Then \[ v = b_1v_1 + b_2v_2 + \cdots + b_lv_l \]

So \[ v + w = b_1v_1 + b_2v_2 + \cdots + b_lv_l + w \]

\[ = b_1(v_1 + w) + b_2(v_2 + w) + \cdots + b_l(v_l + w) \]

And hence is a basis of \( V/W \).

\[ \dim(V/W) = \dim(V) - \dim(W) = (m + l) - m = \dim V - \dim W \]
Internal Direct Sum:

Let \( U_1, U_2, \ldots, U_n \) be subspaces of a vector space \( V \). For \( v \in V \),

then if \( v \) has one and only one expression of the form

\[ v = u_1 + u_2 + \ldots + u_n \]
for \( u_i \in U_i \),

then \( V \) is called internal direct sum of subspaces \( U_1, U_2, \ldots, U_n \).

External Direct Sum:

Let \( V_1, V_2, \ldots, V_n \) be vector spaces over a field \( F \). \( V \) be a vector space over field \( F \).

\( V \) be a vector space having \( n \)-ordered tuples \((v_1, v_2, \ldots, v_n)\), \( v_i \in V_i \), then \( V \) is called external direct sum if

i) Two \( n \)-tuples \((v_1, v_2, \ldots, v_n)\) and \((v'_1, v'_2, \ldots, v'_n)\) are equal iff \( v_i = v'_i \.

ii) \((v_1, v_2, \ldots, v_n) + (v'_1, v'_2, \ldots, v'_n) = (v_1 + v'_1, v_2 + v'_2, \ldots, v_n + v'_n)\).

iii) \( \alpha (v_1, v_2, \ldots, v_n) = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n) \).

External direct sum is denoted by \( V_1 \oplus V_2 \oplus V_3 \oplus \ldots \oplus V_n \).

Vector Space Homomorphism:

Let \( V \) and \( W \) are two vector spaces.

A mapping \( T: V \to W \) is called homomorphism if

\[ T(v_1 + v_2) = T(v_1) + T(v_2) \]
and \( T(\alpha v) = \alpha T(v) \) for \( v_1, v_2 \in V \) and \( \alpha \in F \).

Theorem:

If a vector space \( V \) is the internal direct sum of subspaces \( U_1, U_2, \ldots, U_n \), then \( V \) is isomorphic to the external direct sum of \( U_1, U_2, \ldots, U_n \).
Proof. Let \( v \in \mathcal{V} \) where \( v = u_1 + u_2 + \cdots + u_n \).

Define a mapping

\[ T : \mathcal{V} \rightarrow U_1 \oplus U_2 \oplus \cdots \oplus U_n \]

by \[ T(v) = T(u_1 + u_2 + \cdots + u_n) = (u_1, u_2, \ldots, u_n). \]

1) Mapping is well defined as \( v \in \mathcal{V} \).

\( v = u_1 + u_2 + \cdots + u_n \)

has one and only one representation.

(ii) \( T \) is onto because each

\[ (u_1, u_2, \ldots, u_n) \in U_1 \oplus U_2 \oplus \cdots \oplus U_n \]

is image of \( u_1 + u_2 + \cdots + u_n \in \mathcal{V} \).

(iii) \( T \) is one-one.

\[ T(v) = T(w) \]

\( \Rightarrow T(u_1 + u_2 + \cdots + u_n) = T(w_1 + w_2 + \cdots + w_n) \)

\[ \Rightarrow (u_1, u_2, \ldots, u_n) = (w_1, w_2, \ldots, w_n) \]

where \( u_i, w_i \in U_i \).

\( \Rightarrow u_1 = w_1, u_2 = w_2, \ldots, u_n = w_n \)

\( \Rightarrow u_1 + u_2 + \cdots + u_n = w_1 + w_2 + \cdots + w_n \)

(\( \oplus \)) \( \Rightarrow u = w \)

(iv) \( T(u + w) = T(u_1 + u_2 + \cdots + u_n + w_1 + w_2 + \cdots + w_n) \)

\[ = T(u_1 + w_1 + u_2 + w_2 + \cdots + u_n + w_n) \]

\[ = (u_1 + w_1, u_2 + w_2, \ldots, u_n + w_n) \]

\[ = (u_1, u_2, \ldots, u_n) + (w_1, w_2, \ldots, w_n) \]

by def. of external direct sum.

\[ = T(u) + T(w) \]

\[ \lambda T(\alpha v) = T(\alpha(u_1 + u_2 + \cdots + u_n)) = T(\alpha u_1 + \alpha u_2 + \cdots + \alpha u_n) \]

\[ = \alpha T(u_1, u_2, \ldots, u_n) \]

\[ = \alpha T(v) \]

Hence \( T \) is homomorphism.

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Theorem

If $A$ and $B$ are finite-dimensional subspaces of a vector space $V(F)$, then $A + B$ is finite-dimensional and $\dim(A + B) = \dim A + \dim B - \dim(AB)$.

Proof:

Suppose $\{u_1, u_2, \ldots, u_k\}$ be a basis of $AB$,
$\{u_{k+1}, u_{k+2}, \ldots, u_{m} \}$ be a basis of $A$,
$\{v_1, v_2, \ldots, v_{n} \}$ be a basis of $B$.

Then we have to prove that
$\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{n} \}$ is a basis of $A + B$.

Consider

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_n v_n = 0$,

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_n v_n = 0, \ldots, 0, \ldots, -\gamma_n v_n$.

Since L.H.S of (1) is in $A$, so does R.H.S, i.e.

$-\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n \in A$.

Also,

$-\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n \in B$.

Thus $\gamma_1 w_1, \gamma_2 w_2, \ldots, \gamma_n w_n$ is a part of a basis of $B$.

Hence $-\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n \in AB$.

$\Rightarrow -\gamma_1 w_1 - \gamma_2 w_2 - \cdots - \gamma_n w_n = \delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_k u_k$.

as $\{u_1, u_2, \ldots, u_k\}$ is a basis of $AB$, $\delta_i \in F$.

$\Rightarrow \delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_k u_k + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n = 0$.

Since $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_n\}$ is a basis of $B$, (L.H.S)

$\Rightarrow \delta_1 = \delta_2 = \cdots = \delta_k = \gamma_1 = \gamma_2 = \cdots = \gamma_n = 0$.

so that equation (1) becomes
Vector Spaces: Handwritten notes

\[ \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_r v_r + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_m v_m = 0 \]

But \( \{ u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m \} \) is a basis of \( A \).

\[ \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_r = \beta_1 = \beta_2 = \ldots = \beta_m = 0 \]

i.e., each \( \alpha_i = \beta_i = 0 \).

Hence \( \{ u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m \} \) is linearly independent.

Let \( x + y \in A + B \) i.e., \( x \in A, y \in B \).

As basis of \( A = \{ u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m \} \),

\[ x = a_1 u_1 + a_2 u_2 + \ldots + a_r u_r + b_1 v_1 + b_2 v_2 + \ldots + b_m v_m \]

Also basis of \( B = \{ w_1, w_2, \ldots, w_1, w_2, \ldots, w_m \} \)

\[ y = a'_1 u_1 + a'_2 u_2 + \ldots + a'_r u_r + b'_1 w_1 + b'_2 w_2 + \ldots + b'_m w_m \]

By adding,

\[ A + B = (a_1 + a'_1) u_1 + (a_2 + a'_2) u_2 + \ldots + (a_r + a'_r) u_r + b_1 v_1 + b_2 v_2 + \ldots + b_m v_m \]

\[ + b'_1 w_1 + b'_2 w_2 + \ldots + b'_m w_m \]

\[ \therefore \{ u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m \} \]

generates \( A + B \) and hence is a basis of \( A + B \).

\( \therefore \) \( A + B \) is a finite dimensional and

\[ \dim (A + B) = r + m + n \]

\[ = (r + m) + (r + n) - r \]

\[ = \dim A + \dim B - \dim (A \cap B) \]

proved.
Theorem: Let $V$ and $W$ be vector spaces. If $T$ is an isomorphism of $V$ onto $W$, then $T$ maps a basis of $V$ onto a basis of $W$.

Proof:

$T: V \rightarrow W$ is isomorphism defined by $T(v) = w$.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $V$, then we have to prove $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis of $W$.

i) Consider

$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = 0 \quad \forall \alpha_i \in F$.

$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \cdots + T(\alpha_n v_n) = 0$.

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0$.

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in \ker T$.

$\therefore \; T$ is isomorphism one-one.

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Hence $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

ii) Let $w \in W$.

$T$ is onto, there must be $v \in V$ such that $T(v) = w$.

Now $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ for $a_i \in F$.

$\therefore \; w = T(v)$

$= T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$

$= T(a_1 v_1) + T(a_2 v_2) + \cdots + T(a_n v_n)$.

$\Rightarrow \; w = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$.
Vector Spaces: Handwritten notes

i.e. \( \mathbf{w} \) can be generated by \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \).

Thus \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) form a basis of \( \mathbf{W} \).

The proof is complete.

---

**Theorem:**

Two finite dimensional vector spaces are isomorphic iff they are of the same dimension.

**Proof:**

Let \( \mathbf{V} \) and \( \mathbf{W} \) are two vector spaces of same dimension \( n \) and \( \{v_1, v_2, \ldots, v_n\} \) be the basis of \( \mathbf{V} \) and \( \{w_1, w_2, \ldots, w_n\} \) be the basis of \( \mathbf{W} \).

Define a mapping \( T: \mathbf{V} \to \mathbf{W} \) by \( T(v) = w \) for \( v \in \mathbf{V}, w \in \mathbf{W} \).

i.e. \( T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \).

i) \( T \) is well defined

For \( v, v' \in \mathbf{V} \), if \( v = v' \),

\[ \Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n. \]

\[ \Rightarrow \quad (\alpha_1 - \alpha_1') v_1 + (\alpha_2 - \alpha_2') v_2 + \ldots + (\alpha_n - \alpha_n') v_n = 0. \]

Since \( \{v_1, v_2, \ldots, v_n\} \) is basis of \( \mathbf{V} \),

\[ \Rightarrow \quad \alpha_1 - \alpha_1' = 0 = \alpha_2 - \alpha_2' = \ldots = \alpha_n - \alpha_n'. \]

\[ \Rightarrow \quad \alpha_1 = \alpha_1', \alpha_2 = \alpha_2', \ldots, \alpha_n = \alpha_n'. \]

i.e. \( T(\alpha v) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \)

\[ = \alpha_1' w_1 + \alpha_2' w_2 + \ldots + \alpha_n' w_n \]

\[ = T(v'). \]

ii) \( T \) is homomorphism

\[ T(v + v') = T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha_1' v_1 + \ldots + \alpha_n' v_n) \]

\[ = T((\alpha_1 + \alpha_1') v_1 + (\alpha_2 + \alpha_2') v_2 + \ldots + (\alpha_n + \alpha_n') v_n) \]

[24]
\[ (\alpha_1, \alpha_2, \ldots, \alpha_n) \mapsto \sigma(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\sigma_1, \sigma_2, \ldots, \sigma_n) \]

and

\[ T(v) = T(v') \quad \text{for} \quad v, v' \in V. \]

Thus, \( T \) is one-to-one.

I. \( T \) is onto as every element
\[ w = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in W \]

is image of \( \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \in V \).

Conversely, let \( T: V \to W \) is isomorphism then we have to prove \( \dim V = \dim W \).

Dimension of \( V \) and \( W \) are same.

Let \( \{v_1, v_2, \ldots, v_n\} \) be basis of \( V \), then we prove that \( \{T(v_1), T(v_2), T(v_3), \ldots, T(v_n)\} \) is a basis of \( W \).

See page 25 [v=24]
Vector Spaces: Handwritten notes

Set $V$ and $W$ are two vector spaces.

The set of all homomorphism of $V$ into $W$ is denoted by $\text{Hom}(V, W)$.

$\text{Hom}(V, W) = \{ T_1, T_2, \ldots, T_n \}$

where each $T_i$ is a homomorphism.

Let $V(F)$ and $W(F)$ be two vector spaces.

introduce an operation in $\text{Hom}(V, W)$ and prove that $\text{Hom}(V, W)$ is a vector space under this operation.

Proof:

Let $T_1, T_2 \in \text{Hom}(V, W)$.

we define $(T_1 + T_2)(v) = T_1(v) + T_2(v)$

and $\lambda T(v) = T(\lambda v)$

to prove $\text{Hom}(V, W)$ is a vector space we proceed as follows:

Let $v_1, v_2 \in V$ such that $T_1, T_2 \in \text{Hom}(V, W)$.

Then

$(T_1 + T_2)(v_1 + v_2) = T_1(v_1 + v_2) + T_2(v_1 + v_2) = T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2) = (T_1 + T_2)(v_1) + (T_1 + T_2)(v_2)$

Also

$(T_1 + T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v) = \lambda T_1(v) + \lambda T_2(v)$

$\Rightarrow (T_1 + T_2)(\lambda v) = \lambda (T_1 + T_2)(v)$

$\Rightarrow T_1 + T_2 \in \text{Hom}(V, W)$

i.e $\text{Hom}(V, W)$ is closed.

iii). Mapping $T_1, T_2, \ldots, T_n$ are associative in general.
element of \( V \) into \( \mathbb{R} \) (zero) \( i.e. \):

\[
T_o(v) = 0
\]

\[
(T + T_o)v = T(v) + T_o(v) = T(v) + 0 = T(v)
\]

i.e. \( T_o + T = T \)

\( T_o \) is the identity of \( \text{Hom} (V, W) \),

also for \( T \in \text{Hom} (V, W) \),

so we have

\[
-T \in \text{Hom} (V, W) \text{ such that } \]

\[
(T + (-T))(v) = T(v) + (-1)T(v) = T(v) - T(v) = 0
\]

\[
= T_o(v)
\]

\( \Rightarrow \) inverse exist.

\[
(V, T_1 + T_2) = T_1(v) + T_2(v)
\]

\[
= T_2(v) + T_1(v)
\]

\( \Rightarrow \) \( \text{Hom} (V, W) \) is an abelian group under +.

(ii)

\[
a(T_1 + T_2) = aT_1 + aT_2
\]

\[
a(T_1 + T_2)(v) = (T_1 + T_2)(av) = T_1(av) + T_2(auv) = aT_1(v) + aT_2(v)
\]

(iii)

\[
(a + b)T = aT + bT
\]

\[
(a + b)T(v) = T((a + b)v) = T(au + bv) = aT(v) + bT(v)
\]

(iv)

\[
a(b)T = (ab)T
\]

\[
a(b)T(v) = aT((b)v) = T((a)b v) = T(ab v) = abT(v)
\]

p.t.o
Vector Spaces: Handwritten notes

1. \( T = T \)

As \( T(v) = T(1 \cdot v) = T(v) \)

As \( v \in V \) is a vector space

Hence \( \text{Hom}(V, W) \) is a vector space.
Theorem: If \( V \) and \( W \) are of dimension \( m \) and \( n \) respectively, then \( \text{Hom}(V, W) \) is of dimension \( mn \).

Proof:

Let \( \{v_1, v_2, \ldots, v_m\} \) and \( \{w_1, w_2, \ldots, w_n\} \) be basis of \( V \) and \( W \) respectively.

Define a mapping \( T_{ij} : V \to W \) defined by

\[
T_{ij}(v_k) = \begin{cases} 
\lambda_i w_j & \text{if } i = k \\
0 & \text{if } i \neq k
\end{cases}, \quad \lambda_{ij} \in F.
\]

Let:

\[
v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m
\]

\[
w = -\mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n
\]

then

\[
T_{ij}(v + w) = T_{ij} \left( (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) + (-\mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n) \right)
\]

\[
= T_{ij} \left( (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) \right) + T_{ij} \left( (-\mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n) \right)
\]

\[
= \left( \lambda_1 + \mu_1 \right) w_j + \lambda_2 w_j + \cdots + \lambda_m w_j
\]

And

\[
T_{ij}(\alpha v) = T_{ij} \left( \alpha (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) \right)
\]

\[
= \alpha \left( \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \right)
\]

\[
= \alpha T_{ij}(v)
\]

Thus, \( T_{ij} \) is homomorphism and \( T_{ij} \in \text{Hom}(V, W) \).

Now to prove \( \{T_{11}, T_{12}, \ldots, T_{mn}\} \) is a basis.

Consider

\[
\alpha_{11} T_{11} + \alpha_{12} T_{12} + \cdots + \alpha_{ij} T_{ij} + \cdots + \alpha_{mn} T_{mn} = 0
\]

Now

\[\square\]
Vector Spaces: Handwritten notes

\[ (\alpha_{11} T_{11} + \alpha_{12} T_{12} + \cdots + \alpha_{1n} T_{1n} + \cdots + \alpha_{m1} T_{m1} + \alpha_{m2} T_{m2} + \cdots + \alpha_{mn} T_{mn}) v_i = 0 \quad (i) \]

\[ \Rightarrow \alpha_{11} T_{11} (v_i) + \alpha_{12} T_{12} (v_i) + \cdots + \alpha_{1n} T_{1n} (v_i) + \alpha_{21} T_{21} (v_i) + \alpha_{22} T_{22} (v_i) + \cdots + \alpha_{2n} T_{2n} (v_i) + \cdots + \alpha_{mn} T_{mn} (v_i) = 0 \]

\[ \Rightarrow \alpha_{11} \lambda_1 v_1 + \alpha_{12} \lambda_2 v_2 + \cdots + \alpha_{1n} \lambda_n v_n = 0 \]

\[ \Rightarrow \alpha_{11} = 0 = \alpha_{12} = \alpha_{13} = \cdots = \alpha_{1n} \]

Similarly operating \( w \) on \( v_i \), we have:

\[ \alpha_{ij} = 0 \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, m \]

so the set \( \{ T_{11}, T_{12}, \ldots, T_{1n}, T_{21}, T_{22}, \ldots, T_{mn} \} \) is linearly independent.

Now consider:

\[ s_0 = a_{11} T_{11} + a_{12} T_{12} + \cdots + a_{1n} T_{1n} + a_{21} T_{21} + a_{22} T_{22} + \cdots + a_{2n} T_{2n} + \cdots + a_{mn} T_{mn} \]

So:

\[ s_0 (v_i) = (a_{11} T_{11} + a_{12} T_{12} + \cdots + a_{1n} T_{1n} + a_{21} T_{21} + a_{22} T_{22} + \cdots + a_{2n} T_{2n} + \cdots + a_{mn} T_{mn}) v_i \]
\[ s_i(v_j) = a_{11}T_{1i}(v_j) + a_{12}T_{12}(v_j) + \cdots + a_{1n}T_{1n}(v_j) + \cdots + a_{21}T_{2i}(v_j) + a_{22}T_{22}(v_j) + \cdots + a_{2n}T_{2n}(v_j) + \cdots + a_{m1}T_{m1}(v_j) + a_{m2}T_{m2}(v_j) + \cdots + a_{mn}T_{mn}(v_j). \]

Similarly,
\[ s_i(v_k) = a_{11}l_{1k} \omega_1 + a_{12}l_{1k} \omega_2 + a_{13}l_{1k} \omega_3 + \cdots + a_{1n}l_{1k} \omega_n \]
\[ s_i(v_k) = a_{21}l_{2k} \omega_1 + a_{22}l_{2k} \omega_2 + a_{23}l_{2k} \omega_3 + \cdots + a_{2n}l_{2k} \omega_n \]
\[ s_i(v_k) = a_{31}l_{3k} \omega_1 + a_{32}l_{3k} \omega_2 + a_{33}l_{3k} \omega_3 + \cdots + a_{3n}l_{3k} \omega_n \]
\[ s_i(v_k) = a_{41}l_{4k} \omega_1 + a_{42}l_{4k} \omega_2 + a_{43}l_{4k} \omega_3 + \cdots + a_{4n}l_{4k} \omega_n \]
\[ \vdots \]
\[ s_i(v_k) = a_{m1}l_{mk} \omega_1 + a_{m2}l_{mk} \omega_2 + a_{m3}l_{mk} \omega_3 + \cdots + a_{mn}l_{mk} \omega_n \]

Let \( s \in \text{Hom}(V_W) \)
\[ s(v_1), s(v_2), \ldots, s(v_n) \in W \]
so
\[ s(v_i) = a_{11} \omega_1 + a_{12} \omega_2 + \cdots + a_{1n} \omega_n \]
\[ s(v_2) = a_{21} \omega_1 + a_{22} \omega_2 + \cdots + a_{2n} \omega_n \]
\[ s(v_3) = a_{31} \omega_1 + a_{32} \omega_2 + \cdots + a_{3n} \omega_n \]
\[ s(v_k) = a_{k1} \omega_1 + a_{k2} \omega_2 + \cdots + a_{kn} \omega_n \]
\[ \vdots \]
\[ s(v_k) = a_{mk} \omega_1 + a_{mk} \omega_2 + \cdots + a_{mn} \omega_n \]

Thus, \( s \) \( s_i \) \( s_k \), so \( s \in \text{Hom}(V_W) \)

Thus, \( \{ T_{ij}, T_{j1}, \ldots, T_{jn} \} \) form a basis of \( \text{Hom}(V_W) \)
\[ \Rightarrow \dim(\text{Hom}(V_W)) = mn \]


**Definition (Dual Space):**

Let $V$ be a vector space over a field $F$. Then $\text{Hom}(V, F)$ is called the dual space and is denoted by $V^*$ or $\hat{V}$. Its elements are called linear functionals.

**Theorem:**

If $V$ is a finite dimensional vector space over $F$, then prove $V \cong V^*$.

**Proof:**

Since $\dim V = \dim V^*$, so consider $\dim V = \dim V^* = m$.

Define a mapping $T: V \to V^*$ by

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m$$

i) $T$ is a homomorphism.

$$T(v + v') = T[(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) + (\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)]$$

$$= T[(\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \cdots + (\alpha_m + \beta_m) v_m]$$

$$= (\alpha_1 + \beta_1) f_1 + (\alpha_2 + \beta_2) f_2 + \cdots + (\alpha_m + \beta_m) f_m$$

$$= (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m) + (\beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m)$$

$$= T(v) + T(v')$$

and

$$T(\alpha v) = T(\alpha (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m))$$

$$= T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m)$$

$$= \alpha \alpha_1 f_1 + \alpha \alpha_2 f_2 + \cdots + \alpha \alpha_m f_m$$

$$= \alpha [\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m]$$

$$= \alpha T(v)$$

ii) $T$ is one-one.

$$\Rightarrow T(v) = T(v')$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = T(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)$$

$$\Rightarrow \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m = \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m$$

$$\Rightarrow (\alpha_1 - \beta_1) f_1 + (\alpha_2 - \beta_2) f_2 + \cdots + (\alpha_m - \beta_m) f_m = 0$$

$$\Rightarrow \{f_1, f_2, \ldots, f_m\} \text{ is basis of } V^*$$

$$\Rightarrow \alpha_1 - \beta_1 = 0 = \alpha_2 - \beta_2 = \cdots = \alpha_m - \beta_m$$

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\[ \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m \]

\[ \Rightarrow v = v \]

iii) \( T \) is onto.

Since for \( \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m \in V^* \) we have

\[ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \in V \]

such that

\[ T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m \]

Thus \( T \) is onto.

and hence \( V \cong V^* \).
**Definition.** Let $T : V \rightarrow V'$ be a homomorphism of a vector space $V$, to a vector space $V'$, then $\ker T$ is called the kernel of $T$. The dimension of $\ker T$ is called the nullity of $T$.

**Theorem.** Let $T : V \rightarrow V'$ be a vector space homomorphism. Then $\dim V = \dim \ker T + \dim \text{Im } T$.

**Proof.** Let $\dim \ker T = m$ and $\dim V = n$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $\ker T$.

Since $\ker T$ is a subspace of $V$, we can take a basis of $V$.

$\{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$.

We have to prove:

$\{T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\}$ form a basis of $\text{Im } T$.

Let $w \in \text{Im } T$ then there is $v \in V$ such that $T(v) = w$.

$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \cdots + \alpha_n v_n) = w$.

$\Rightarrow \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_m T(v_m) + \alpha_{m+1} T(v_{m+1}) + \cdots + \alpha_n T(v_n) = w$.

$\Rightarrow \alpha_1 v_1, \alpha_2 v_2, \ldots, v_n \in \ker T$.

$\Rightarrow T(\alpha_1 v_1) = 0, T(\alpha_2 v_2) = 0, \ldots, T(v_n) = 0$.

$\Rightarrow \alpha_1 T(v_{m+1}) + \alpha_2 T(v_{m+2}) + \cdots + \alpha_n T(v_n) = w$.

$\Rightarrow T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)$ generate $\text{Im } T$. 

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Now consider

\[ \beta_{m+1} T(v_{m+1}) + \beta_{m+2} T(v_{m+2}) + \ldots + \beta_n T(v_n) = 0 \]

\[ \Rightarrow T(\beta_{m+1} v_{m+1}) + T(\beta_{m+2} v_{m+2}) + \ldots + T(\beta_n v_n) = 0 \]

\[ \Rightarrow T(\beta_{m+1} v_{m+1} + \beta_{m+2} v_{m+2} + \ldots + \beta_n v_n) = 0 \]

So \( T \) is homomorphism.

\[ \Rightarrow \beta_{m+1} v_{m+1} + \beta_{m+2} v_{m+2} + \ldots + \beta_n v_n \in \ker T = N(T) \]

Since \( \{v_1, v_2, \ldots, v_m\} \) is basis of \( N(T) \),

so \( \exists \delta_1, \delta_2, \ldots, \delta_m \in \mathbb{F} \) such that

\[ \beta_{m+1} v_{m+1} + \beta_{m+2} v_{m+2} + \ldots + \beta_n v_n = \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_m v_m \]

\[ \Rightarrow \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_m v_m - \beta_{m+1} v_{m+1} - \beta_{m+2} v_{m+2} - \ldots - \beta_n v_n = 0 \]

As \( \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\} \) is basis of \( V \),

therefore \( \delta_1 = \delta_2 = \ldots = \delta_m = \beta_{m+1} = \beta_{m+2} = \ldots = \beta_n = 0 \)

i.e. \( \beta_{m+1} = \beta_{m+2} = \ldots = \beta_n = 0 \)

\[ \Rightarrow \{T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\} \text{ is L.I.} \]

and hence form a basis of \( R(T) \)

so \( \dim R(T) = n - m \)

\[ = \dim V_1 - \dim N(T) \]

\[ \Rightarrow \dim V_1 = \dim N(T) + \dim R(T) \]

proved
Theorem

Let $V$ be a vector space over $F$ and \( \{v_1, v_2, \ldots, v_n\} \) be a basis of $V$. Let 
\[ \phi_1, \phi_2, \ldots, \phi_n \in V^* = \text{Hom}(V, F) \]
are linear functionals defined by
\[ \phi_i(v_j) = \delta_{ij} = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases} \]

Then \( \{\phi_1, \phi_2, \ldots, \phi_n\} \) is a basis of $V^*$.

Proof

Let $\phi \in V^*$ be taken
\[ \phi(v_1) = k_1, \quad \phi(v_2) = k_2, \ldots, \quad \phi(v_n) = k_n \]
where $k_1, k_2, \ldots, k_n \in F$

Let
\[ \psi = k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n \]
\[ \psi(v) = \psi(v_1) = (k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n)(v) \]
\[ = k_1 \phi_1(v) + k_2 \phi_2(v) + \cdots + k_n \phi_n(v) \]
\[ = k_1 (v) + k_2 (v) + \cdots + k_n (v) = k_1 \]

Also
\[ \psi(v_2) = (k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n)(v_2) \]
\[ = k_1 \phi_1(v_2) + k_2 \phi_2(v_2) + \cdots + k_n \phi_n(v_2) \]
\[ = k_1 (v_2) + k_2 (v_2) + \cdots + k_n (v_2) = k_2 \]

\[ \Rightarrow \psi(v_2) = k_2 = \phi(v_2) \]
\[ \therefore \psi = \phi \]

\[ \Rightarrow \phi = \psi = k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n \]
so \( \{\phi_1, \phi_2, \ldots, \phi_n\} \) span $V^*$.

To prove \( \{\phi_1, \phi_2, \ldots, \phi_n\} \) is a linearly independent.

Consider
\[ a_1 \phi_1 + a_2 \phi_2 + \cdots + a_n \phi_n = 0 \]
then operating it on $v_i$,

$$(a_1 \Phi_1 + a_2 \Phi_2 + \cdots + a_n \Phi_n) v_i = 0 \cdot v_i$$

$\Rightarrow a_1 \Phi_1 (v_i) + a_2 \Phi_2 (v_i) + \cdots + a_n \Phi_n (v_i) = 0$

$\Rightarrow a_1 (1) + a_2 (0) + \cdots + a_n (0) = 0$

$\Rightarrow a_1 = 0$

Similarly for $i = 2, 3, \cdots, n$

$$(a_1 \Phi_1 + a_2 \Phi_2 + \cdots + a_n \Phi_n) v_i = 0 \cdot v_i'$$

$\Rightarrow a_1 \Phi_1 (v_i) + a_2 \Phi_2 (v_i) + \cdots + a_n \Phi_n (v_i) = 0$

$\Rightarrow a_1 (0) + a_2 (0) + \cdots + a_i (1) + \cdots + a_n (0) = 0$

$\Rightarrow 0 + a + \cdots + a_i + \cdots + 0 = 0$

$\Rightarrow a_i = 0$

$x \in \{a_1 = 0, a_2 = 0, a_3 = 0, \cdots, a_n = 0\}$

Hence $\{\Phi_1, \Phi_2, \cdots, \Phi_n\}$ is LI and so is a basis of $V^*$.
Vector Spaces: Handwritten notes

\[ \Phi_1(v_1) = 1 \quad \Phi_1(v_2) = 0 \]
\[ \Phi_2(v_1) = 0 \quad \Phi_2(v_2) = 1 \]

Since \( \Phi_1, \Phi_2 \) are linear functional

\[ \Phi_1(x, y) = 3x + by \]
and \( \Phi_2(x, y) = cx + dy \)

\[ \Phi_1(v_1) = 1 \]
\[ \Rightarrow \Phi_1(2, 1) = 1 \quad \Rightarrow 2a + b = 1 \quad \text{(i)} \]
\[ \Phi_1(v_2) = 0 \]
\[ \Rightarrow \Phi_1(3, 1) = 0 \quad \Rightarrow 3a + b = 0 \quad \text{(ii)} \]

By (i) and (ii)

\[ a = -1 \quad \text{and} \quad b = 3 \]

Now \( \Phi_2(v_1) = 0 \)

\[ \Phi_2(2, 1) = 0 \quad \Rightarrow 2c + d = 0 \quad \text{(iii)} \]
and \( \Phi_2(v_2) = 1 \)

\[ \Phi_2(3, 1) = 1 \quad \Rightarrow 3c + d = 1 \quad \text{(iv)} \]

Solving (iii) and (iv)

\[ c = 1 \quad \text{and} \quad d = -2 \]

Therefore

\[ \Phi_1 = -x + 3y \]
\[ \Phi_2 = 2x - 2y \]

**Example**

Let a basis of \( \mathbb{R}^3 \) is \( \{v_1, v_2, v_3\} \)

\[ v_1 = \{1, -1, 3\}, \quad v_2 = \{0, 1, -1\}, \quad v_3 = \{0, 3, -2\} \]

Find dual basis \( \Phi_1, \Phi_2, \text{ and } \Phi_3 \)

such that \( \Phi_i(v_j) = \delta_{ij} \) \( i, j = 1, 2, 3 \)

Do yourself as above.
Question

Let \( V = \{a + bt : a, b \in \mathbb{R}\} \) be a vector space of polynomials of degree < 1.

Let \( \Phi_1, \Phi_2 : V \to \mathbb{R} \) be defined by

\[
\Phi_1(f(t)) = \int f(t) \, dt
\]

\[
\Phi_2(f(t)) = \int f(t) \, dt
\]

where \( \Phi_1, \Phi_2 \in V^* \) (dual space).

Find corresponding basis \( v_1, v_2 \) of \( V \).

Solution:

Let \( v_1 = a + bt \) and \( v_2 = a + bt + c + dt \).

By definition,

\[
\Phi_1(v_1) = 1, \quad \Phi_1(v_2) = 0, \quad \Phi_2(v_1) = 0, \quad \Phi_2(v_2) = 1
\]

\[
\Phi_1(v_1) = 1
\]

\[
\int v_1 \, dt = 1 \quad \Rightarrow \quad \int (a + bt) \, dt = 1
\]

\[
\int \left[ b t + \frac{b t^2}{2} \right]_0^1 = 1 \quad \Rightarrow \quad \frac{a}{2} + \frac{b}{2} = 1
\]

\[
\Rightarrow \quad 2a + b = 2 \quad \text{(i)}
\]

\[
\Phi_2(v_1) = 0
\]

\[
\int (a + bt) \, dt = 0 \quad \Rightarrow \quad \int \left[ b t + \frac{b t^2}{2} \right]_0^1 = 0
\]

\[
\Rightarrow \quad 2a + 2b = 0 \quad \Rightarrow \quad a + b = 0 \quad \text{(ii)}
\]

By (i) and (ii)

\[
2a + b = 2
\]

\[
a + b = 0
\]

\[
\frac{a}{2} = 2 \quad \Rightarrow \quad b = -2
\]

Further \( \Phi_1(v_2) = 0 \)

\[
\Rightarrow \quad \int v_2 \, dt = 0 \quad \Rightarrow \quad \int (c + dt) \, dt = 0
\]

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\[
\Rightarrow \left[ e t + \frac{d t^2}{2} \right]_0^1 = 0
\]
\[
\Rightarrow e + \frac{d}{2} = 0 \quad \text{or} \quad 2e + d = 0 \quad \text{--- (iii)}
\]
\[
\Phi_2(v_2) = 1
\]
\[
\Rightarrow \int_0^1 v_2 \, dt = 1
\]
\[
\Rightarrow \int_0^1 (e + dt) \, dt = 1 \quad \Rightarrow \left[ e t + \frac{d t^2}{2} \right]_0^1 = 1
\]
\[
\Rightarrow 2e + 2d = 1 \quad \Rightarrow \quad \text{--- (iv)}
\]
Subtracting (iii) from (iv),
\[
2e + 2d = 1
\]
\[
-2e + d = 0
\]

Hence,
\[
ed = 1 \quad \Rightarrow \quad e = -\frac{1}{2}
\]

\[
v_1 = 2 - 2t
\]
and
\[
v_2 = -\frac{1}{2} + t
\]
are basis of \( V \) corresponding to dual basis \( V^* \)
**Eigen Value**

Def.: Let \( A \) be a \( n \) square matrix, then \( \lambda \in \mathbb{F} \) is an eigenvalue of \( A \) if there exist a non-zero column vector \( v \in \mathbb{F}^n \) such that \( Av = \lambda v \).

Here \( v \) is an eigenvector corresponding to eigenvalue \( \lambda \).

**Exercise**

Find eigenvalues and associated eigenvector of a matrix \( A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \).

**Solution**

Let \( v = \begin{bmatrix} x \\ y \end{bmatrix} \)

\[ Av = \lambda v \]

\[ \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \]

\[ \Rightarrow \begin{bmatrix} x + 2y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \]

\[ \Rightarrow \begin{cases} x + 2y = \lambda x \\ 3x + 2y = \lambda y \end{cases} \]

or

\[ (1 - \lambda)x + 2y = 0 \tag{1} \]

\[ 3x + (2 - \lambda)y = 0 \tag{2} \]

For non-trivial solution

\[ \begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = 0 \]

\[ \Rightarrow (1 - \lambda)(2 - \lambda) - 6 = 0 \]

\[ \Rightarrow 2 - 2\lambda - \lambda^2 - 6 = 0 \]

\[ \Rightarrow \lambda^2 - 3\lambda - 4 = 0 \]

\[ \Rightarrow (\lambda - 4)(\lambda + 1) = 0 \]

\[ \Rightarrow \lambda = 4, -1 \]
Vector Spaces: Handwritten notes

\[ \lambda = 4, -1 \] are eigenvalues.

\[ \lambda = -1 \] in eq. (ii) \[ \Rightarrow 2x + 2y = 0 \]
\[ x, y = 0 \]
\[ \forall x, y \in \mathbb{R} \]

Thus,
\[ v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

i.e., eigen vector is \[ \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \]

and \[ \lambda = 4 \] in eq. (ii) \[ \Rightarrow -3x + 2y = 0 \]
\[ 2y = 3x \]
\[ \Rightarrow y = \frac{3}{2} x \]

Thus,
\[ v = \begin{bmatrix} x \\ \frac{3}{2} x \end{bmatrix} = \frac{x}{2} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

i.e., eigen vector is \[ \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T \]

\# Note

\[ A\mathbf{v} = \lambda \mathbf{v} \]
\[ \Rightarrow A\mathbf{v} - \lambda \mathbf{v} = 0 \]
\[ \Rightarrow (A - \lambda I)\mathbf{v} = 0 \]
where \( I \) is identity

\[ \Rightarrow |A - \lambda I| = 0 \]

and
\[ A\mathbf{v} = \lambda \mathbf{v} \]
\[ A(\mathbf{k}\mathbf{v}) = k\lambda \mathbf{v} = \mathbf{v}(\lambda \mathbf{k}) + x\mathbf{e} \]
\[ = k(\lambda \mathbf{v}) \]

Then \( \lambda \) and \( k \) are eigenvalues for \( A \).
* Eigen Value & Eigen Vector (Alternative).

Let \( T : V \to V \) be a linear operator. Then \( \lambda \in F \) is called eigen value of \( T \) if there exist a non-zero vector \( v \) such that

\[
T(v) = \lambda v
\]

Here \( v \) is eigen vector.

Note that \( kv \) is also eigen vector for same eigen value \( \lambda \).

\[
T(kv) = kT(v) = k\lambda v = \lambda kv
\]

**Theorem:**

Let \( \lambda \) be an eigen value of an operator \( T : V \to V \). Let \( V_\lambda \) denotes set of all eigen vectors of \( T \) belonging to same eigen value \( \lambda \). The \( V_\lambda \) is a subspace of \( V \).

**Proof:**

Let \( \lambda \) be an eigen value of an operator.

Let \( v, w \in V_\lambda \).

Then \( T(v) = \lambda v \) and \( T(w) = \lambda w \).

Now \( T(av + bw) = T(\lambda v + \lambda w) = \lambda T(v) + \lambda T(w) \)

\[
= a\lambda v + b\lambda w = \lambda (av + bw)
\]

\( av + bw \) is also an eigen vector for \( \lambda \).

\( \therefore \) \( av + bw \in V_\lambda \)

\( \therefore V_\lambda \) is a subspace.
Theorem: Let \( \{v_1, v_2, \ldots, v_n\} \) be non-zero eigen vectors of an operator \( T \) corresponding to distinct eigen values \( \lambda_1, \lambda_2, \ldots, \lambda_n \) respectively then \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent.

Proof: We prove the theorem by Mathematical Induction.

Let \( n=1 \) so if \( \alpha v_1 = 0 \)

\[ \Rightarrow \alpha = 0 \quad \text{as} \quad v_1 \neq 0 \]

so condition I is true.

Let the theorem is true for \( k=n-1 \)

i.e. \( v_1, v_2, \ldots, v_{n-1} \) are L.I. (linearly independent)

then \( \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{n-1} v_{n-1} = 0 \)

\[ \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0 \]

Consider \( b_1 v_1 + b_2 v_2 + \cdots + b_n v_n = 0 \) (i)

\[ T(b_1 v_1 + b_2 v_2 + \cdots + b_n v_n) = 0 \]

\[ \Rightarrow T(b_1 v_1) + T(b_2 v_2) + \cdots + T(b_n v_n) = 0 \]

\[ \Rightarrow b_1 T(v_1) + b_2 T(v_2) + \cdots + b_n T(v_n) = 0 \]

\[ \Rightarrow b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \cdots + b_n \lambda_n v_n = 0 \]

or

\[ b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \cdots + b_{n-1} \lambda_{n-1} v_{n-1} + b_n \lambda_n v_n = 0 \] (ii)

Multiplying eq (i) by \( \lambda_n \nabla \)

\[ \lambda_n b_1 v_1 + \lambda_n b_2 v_2 + \cdots + \lambda_n b_{n-1} v_{n-1} + \lambda_n b_n v_n = 0 \] (iii)

Subtracting (iii) from (ii)
\[ b_1(\lambda_1 - \lambda_n)v_1 + b_2(\lambda_2 - \lambda_n)v_2 + \cdots + b_n(\lambda_n - \lambda_n)v_n = 0. \]

Since \( \lambda_1, \lambda_2, \ldots, \lambda_n \) is linearly independent,
\[ \Rightarrow b_1 = a = b_2 = b_3 = \cdots = b_n. \]

\[ \therefore \lambda_i - \lambda_n \neq 0 \quad \forall i = 1, 2, \ldots, n-1 \]

because if \( \lambda_i = \lambda_n \) then \( b_i = \lambda_i \) \( \Rightarrow \) \( \lambda_i = \lambda_n \) \( \forall i = 1, 2, \ldots, n-1 \),
which is a contradiction since each \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are distinct.

Now from eq. (ii)
\[ 0 + 0 + \cdots + 0 + b_nv_n = 0 \]
\[ \Rightarrow b_nv_n = 0. \]

\[ \therefore b_n = 0. \quad \forall v_n \neq 0 \]

Hence the vectors \( v_1, v_2, \ldots, v_n \) are linearly independent.
Characteristic Polynomial / Equation / Matrix:

Let $A$ be a $n \times n$ square matrix over $F$. Then $tI - A$ is called characteristic matrix, $|tI - A|$ is called characteristic polynomial, and $|tI - A| = 0$ is called characteristic equation.

$$A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$

$$tI - A = t \begin{bmatrix}
    1 & 0 & \cdots & 0 \\
    0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1
\end{bmatrix} - \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} = \begin{bmatrix}
    t - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\
    -a_{21} & t - a_{22} & -a_{23} & \cdots & -a_{2n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    -a_{n1} & -a_{n2} & -a_{n3} & \cdots & t - a_{nn}
\end{bmatrix}$$

And $\Delta_A(t) = \det(tI - A)$ is characteristic polynomial.

Also $\Delta_A(t) = 0$ or $|tI - A| = 0$ is characteristic equation.

Exercise:

Find characteristic polynomial of

$$A = \begin{bmatrix}
    1 & 3 & 0 \\
    -2 & 2 & -1 \\
    4 & 0 & -2
\end{bmatrix}$$

$$tI - A = t \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix} - \begin{bmatrix}
    1 & 3 & 0 \\
    -2 & 2 & -1 \\
    4 & 0 & -2
\end{bmatrix}$$
\[
\begin{pmatrix}
1 & -1 & -3 & 0 \\
2 & 1 & -2 & 1 \\
-4 & 0 & 1 & 2
\end{pmatrix}
\]

\[\Delta_{A(t)} = |tI - A| \]

\[\begin{pmatrix}
t-1 & -3 & 0 \\
2 & t-2 & 1 \\
-4 & 0 & t+2
\end{pmatrix}
\]

\[t^3 - t^2 + 2t + 28 \text{ is characteristic polynomial}
\]

Also \[\Delta_{A(t)} = 0\]

\[t^3 - t^2 + 2t + 28 = 0 \text{ is characteristic equation}
\]

Note: Degree of eq. will be equal to the order of matrix.

**Example:**

\[A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}
\]

\[tI - A = t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} t-2 & -3 \\ -1 & t-5 \end{pmatrix}
\]

\[B(t) = \text{adj of } (tI - A) = \begin{pmatrix} t-5 & 3 \\ 1 & t-2 \end{pmatrix}
\]

\[= \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 1 & -2 \end{pmatrix} = B_1 + B_2
\]

If \[A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix}
\]

Then \[B(t) = \text{adj} (tI - A) = B_2t^2 + B_1t + B_0\]
Calay Hamilton Theorem: Every square matrix is zero of its characteristic polynomial. OR Every square matrix satisfies its characteristic equation.

Proof: Let \( A \) be a square matrix and \( \Delta_A(t) = |tI - A| \) be its characteristic polynomial, i.e. \( \Delta_A(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1t + a_0 \)

Let \( B(t) \) is adjoint of \( tI - A \).

Since elements of \( B(t) \) are cofactors of \( tI - A \) and so are polynomial of degree not more than \( n-1 \), and we can write

\[
B(t) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1t + B_0
\]

where \( B_i \) are square matrices of order \( n \) over \( F \).

Since by definition of adjoint of a matrix

\[
(tI - A)B(t) = |tI - A|I
\]

\[
\Rightarrow (tI - A)(B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1t + B_0) = (t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1t + a_0)I
\]

Comparing the co-efficients.

Comparing \( t^n \) \[ B_{n-1}I = I \]

\( t^{n-1} \) \[ B_{n-2}I - AB_{n-1} = a_{n-1}I \]

\( t^{n-2} \) \[ B_{n-3}I - AB_{n-2} = a_{n-2}I \]

\( t^1 \) \[ B_1I - AB_0 = a_1I \]

\( t^0 \) \[ AB = a_0I \]

Multiplying above equations by first to last by \( A^n, A^{n-1}, A^{n-2}, \ldots, A, I \), respectively, we have.
\[ A^n B_n I = A^n I \]
\[ A^{n-1} B_{n-2} I - A^n B_{n-1} I = a_{n-1} A^{n-1} I \]
\[ A^{n-2} B_{n-3} I - A^{n-1} B_{n-2} I = a_{n-2} A^{n-2} I \]

\[ A B_0 I - A^2 B_1 = a_1 A I \]
\[ A B_0 I = a_1 I \]

Adding both sides of above equations

\[ 0 = A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \ldots + a_1 A + a_0. \]

As required.
# Minimum Polynomial

A polynomial \( m(t) \) is called minimum polynomial if

i) \( m(t) \) divides \( \Delta(t) \)

ii) Each irreducible factor of \( \Delta(t) \) divides \( m(t) \)

iii) \( m(A) = 0 \)

Question

\[
A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}
\]

\[
(tI - A) = \begin{bmatrix} t - 2 & -1 & 0 & 0 \\ 0 & t - 2 & 0 & 0 \\ 0 & 0 & t - 1 & -1 \\ 0 & 0 & 2 & t - 4 \end{bmatrix}
\]

\[
|tI - A| = \begin{bmatrix} t - 2 & -1 & 0 & 0 \\ 0 & t - 2 & 0 & 0 \\ 0 & 0 & t - 2 & -1 \\ 0 & 0 & -2 & t - 4 \end{bmatrix}
\]

Expanding by \( c_1 \),

\[
= (t - 2) \begin{vmatrix} t - 2 & 0 & 0 \\ 0 & t - 2 & 1 \\ 0 & -2 & t - 4 \end{vmatrix}
\]

\[
= (t - 2)(t - 2) \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}
\]

\[
= (t - 2)^3 (t - 3) = (t - 2)^2 (t - 4) + 2
\]

\[
= (t^2 - 4t + 4) (t^2 - 6t + 8 + 2)
\]

\[
= t^4 - 10t^3 + 4t^2 + 48t + 64 \quad \text{(after solving)}
\]

is characteristic polynomial.
Possible minimum polynomial are:

\[ \begin{align*}
\text{i)} & \quad (t-2)(t-3) = f(t) \\
\text{ii)} & \quad (t-2)^2(t-3) = g(t) \\
\text{iii)} & \quad (t-2)^3(t-3) = h(t).
\end{align*} \]

\[ f(A) = (A-2)(A-3) \]
\[ = (A-2I)(A-3I) \]

\[
\begin{bmatrix}
  1 & 1 & 0 & 0 & 0 \\
  0 & 1 & -1 & 0 & 0 \\
  0 & 0 & -1 & 1 & 0 \\
  0 & 0 & -2 & 2 & 0 \\
  0 & 0 & -2 & 2 & 1
\end{bmatrix} \neq 0
\]

\[ \Rightarrow f(t) \text{ is not minimum polynomial.} \]

Now \[ g(t) = (t-2)^2(t-3) \]

\[ g(A) = (A-2)^2(A-3) \]

\[
\begin{bmatrix}
  1 & 1 & 0 & 0 & 0 \\
  0 & 1 & -1 & 0 & 0 \\
  0 & 0 & -1 & 1 & 0 \\
  0 & 0 & -2 & 2 & 0 \\
  0 & 0 & -2 & 2 & 1
\end{bmatrix} = 0
\]

\[ \Rightarrow g(t) = (t-2)^2(t-3) \text{ is minimum polynomial.} \]

\[ h(A) = (A-2)^3(A-3) \]

Do yourself
Theorem:

Prove that the minimum polynomial $m(t)$ divides every polynomial which has $A$ as a zero.

In particular $m(t)$ divides the characteristic polynomial $\Delta(t)$ of $A$.

Proof:

Let $f(t)$ be a polynomial for which $f(A) = 0$ then by division algorithm, there are polynomials $q(t)$ and $r(t)$ such that

$$f(t) = q(t) \cdot m(t) + r(t)$$  (i)

where $r(t) = 0$ or degree of $r(t)$ is less than that of $m(t)$.

From (i) $f(A) = q(A) \cdot m(A) + r(A)$ by $t = A$

$\Rightarrow 0 = q(A) \cdot 0 + r(A)$

$\Rightarrow r(A) = 0$

Then $r(t)$ is a polynomial of degree less than that of $m(t)$, which has $A$ as a zero which contradict the definition of $m(t)$.

Hence $r(t) = 0$

$\Rightarrow f(t) = g(t) \cdot m(t)$

i.e. $m(t)$ divides $f(t)$

Also then $m(t)$ divides $\Delta(t)$
Theorem

Let \( m(t) \) be the minimum polynomial of an \( n \times n \) square matrix \( A \). Then show that characteristic polynomial of \( A \) divides \( (m(t))^n \).

Proof:

Let \( m(t) = t^r + c_1 t^{r-1} + c_2 t^{r-2} + \cdots + c_r t + c_r \).

Consider

\[ B_0 = I \quad (i) \]
\[ B_1 = A + c_1 I \quad (ii) \]

\[ \vdots \]
\[ B_r = \cdots \]

Take

\[ B(t) = t^{r-1} B_0 + t^{r-2} B_1 + t^{r-3} B_2 + \cdots + t B_{r-2} + B_r \]

Now,

\[ (tI - A) B(t) = (tI - A)(t^{r-1} B_0 + t^{r-2} B_1 + \cdots + t B_{r-2} + B_r) \]

\[ = t^r B_0 I + t^{r-1} B_1 I + t^{r-2} B_2 I + \cdots + t^2 B_{r-2} I \]
\[ + t B_{r-1} I - (t^{r-1} A B_0 + t^{r-2} A B_1 + \cdots + t A B_{r-2} + A B_{r-1}) \]

\[ = t^r B_0 + t^{r-1}(B_1 - A B_0) + t^{r-2}(B_2 - A B_1) \]
\[ + \cdots + t(B_{r-1} - A B_{r-2}) - A B_{r-1} \]

Now from eqs. (i) to (r) give

\[ B_r = A B_0 = c_1 I \]
\[ B_2 = A B_1 = c_2 I \]
\[ B_{r+1} = A B_{r+2} = c_{r-1} I \]

Also from the equation:

\[ A B_{r-1} = A^r + c_1 A^{r-1} + \cdots + c_{r-1} A I \]

\[ = A^r + c_1 A^{r-1} + \cdots + c_{r-1} A I + c_{r-1} I - c_r I \]

\[ \Rightarrow A B_{r+1} = -c_r I \quad \therefore m(A) = \varnothing. \]

Using all these values in eq. (3):

\[(tI - A) \cdot B(t) = t^r I + t^{r-1} c_1 I + t^{r-2} c_2 I + \cdots + t c_{r-1} I + c_r I \]

\[ = (t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + t c_{r-1} + c_r) I \]

Taking determinant to both sides:

\[ |(tI - A) \cdot B(t)| = |(t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + c_r) I| \]

\[ \Rightarrow |tI - A| \cdot |B(t)| = (t^r + c_1 t^{r-1} + c_2 t^{r-2} + \cdots + c_r)^n \]

\[ = (m(t))^n \]

\[ \Rightarrow |tI - A| \text{ divides } (m(t))^n. \]

i.e. characteristic polynomial divides \((m(t))^n\).
# Similar Matrix

**Definition:** A matrix $B$ is similar to a matrix $A$ if there is a non-singular matrix $P$ such that $B = P^{-1}AP$ or $PB = AP$.

# Diagonalization of Matrix

**Definition:** A matrix $A$ is said to be diagonalizable if there is a matrix such that $B = P^{-1}AP$.

In this case, column of $P$ are eigen vectors of $A$ and diagonal element of $B$ are corresponding eigen values of $A$.

**Question:** If $A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ then diagonalize this matrix.

**Solution:**

To find eigen values $[\lambda I - A] = 0$

$$\begin{vmatrix} \lambda - 4 & -2 \\ -3 & \lambda + 1 \end{vmatrix} = 0$$

$\Rightarrow \lambda = 5, -2$.

i) $\lambda = 5$, then for eigen vectors $MX = 0$

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow x_1 - 2x_2 = 0$

$-3x_1 + 6x_2 = 0$

One of its solution is $x_1 = 1 \Rightarrow x_1 = 2$

eigen vector $(2, 1)^t$.

ii) $\lambda = -2$

$$MX = 0 \Rightarrow \begin{bmatrix} -6 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
\[ -2x - 2y = 0 \]
\[ -3x - y = 0 \]

If \( x = 1 \) then \( y = -3 \).

Eigenvector: \((1, -3)^t\).

Now:
\[
P = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}
\]

\[
|P| = -6 - 1 = -7
\]

\[
P^{-1} = -\frac{1}{7} \begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{3}{7} \end{bmatrix}
\]

Now:
\[
P^{-1}AP = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{3}{7} \end{bmatrix}
\]

\[
= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}
\]

is diagonal where diagonal is the diagonal are eigenvalues of \( A \).

**Question:** Find \( A^{10} \) for \( A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \)

\[
PBP^{-1} = A
\]

\[
PBP^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{3}{7} \end{bmatrix}
\]

\[
= \frac{3}{7} \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}
\]

\[
= \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}
\]

\[
\text{Simplify yourself.}
\]
# Theorem:

Similar matrices $A$ and $P'AP$ have the same characteristic polynomial.

Proof:
The matrices $A$ and $B$ are similar matrices, then $B = P'AP$.

Using $tI = P'tIP$,

$$|tI - B| = |tI - P'AP|$$

$$= |P'tIP - P'AP|$$

$$= |tI - A|$$

As required.