Vector Spaces: Handwritten notes
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Vector Spaces (Handwritten notes)

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# Ring

def: A non-empty set $R$ is called ring if

i) $R$ is abelian group under multiplication. Addition.

ii) $R$ is semi-group under multiplication.

iii) Distributive law holds

\[ a(b+c) = ab + ac \]
\[ (a+b)c = ac + bc \]

Examples

i) \((\mathbb{Z}, +, \cdot)\) is a ring

where \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \)

ii) \((\mathbb{Q}, +, \cdot)\), where \( \mathbb{Q} \) is the set of rational numbers

iii) \((\mathbb{R}, +, \cdot)\), where \( \mathbb{R} \) is set of real numbers.

iv) \((\mathbb{Z}_n, +, \cdot)\), \( \mathbb{Z}_n \) residue classes of module \( n \).

# Field

def: A non-empty set \( F \) is called a field if

i) \( F \) is abelian group under addition.

ii) \( F \setminus \{0\} \) is abelian group under multiplication.

iii) Right distributive law holds in \( F \);

\( i.e \ a, b, c \in F \)

\[ (a+b)c = ac + bc \]

Examples

i) \((\mathbb{R}, +, \cdot)\) is a field.

ii) \((\mathbb{C}, +, \cdot)\) is a field.

iii) \((\mathbb{R}, +, \cdot)\) is a field.

iv) \((\mathbb{Z}, +, \cdot)\) is not a field

as \((\mathbb{Z} \setminus \{0\}, \cdot)\) is not group under multiplication.
# Vector Space

Define: let $V$ be a non-empty set and $F$ is field then $V$ is called vector space if

i) $V$ is abelian group under addition

ii) $a(v + w) = av + aw \ \forall \ a \in F, v, w \in V$

iii) $(a + b)v = av + bv \ \forall \ a, b \in F, v \in V$

iv) $a(bv) = (ab)v \ \forall \ a, b \in F, v \in V$

v) $1 \cdot v = v, 1 \in F$ and $v \in V$

i.e. 1 is identity under multiplication

**Example**

i) Let $V$ be a set of all polynomials of degree $\leq n$ then $V$ is vector space

$$V = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in F \ \forall \ i \leq n \in \mathbb{N}\}$$

addition is defined as

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i$$

and multiplication is defined as

$$r \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} ra_i x^i$$

$$= ra_0 + ra_1x + ra_2x^2 + \cdots + ranx^n$$

ii) Let $F$ is a field then the set

$$F^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in F, 1 \leq i \leq n\}$$

iii) The set $M_n$ of all $n \times n$ matrices with entries from a field $F$ is a vector space over $F$

iv) Every field is a vector space over itself.
Subspace:

Let $V$ be a vector space over $F$ and $W$ be its non-empty subset of $V$.

Then $W$ is a subspace of $V$ if $W$ itself is a vector space under operation induced (defined) in $V$.

**Theorem:**

A non-empty subspace subset $W$ of a vector space $V$ is a subspace of $V$ iff

1) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$
2) $\alpha \in F, w \in W \Rightarrow \alpha w \in W$.

**Proof:**

Let $W$ is subspace of vector field space $V$ then $W$ itself is a vectorspace i.e. $W$ is closed under addition and scalar multiplication.

Conversely, let $W$ is a subset satisfying condition (i) and (ii).

Then for $-1 \in F$ and $w_2 \in W$,

$-1 \cdot w_2 \in W$ by condition (iii).

$\Rightarrow -w_2 \in W$.

$\Rightarrow w_1 - w_2 \in W$

$\Rightarrow w_1 + (-w_2) \in W$ by condition (iv).

$\Rightarrow W$ is a subgroup under addition.

Since $W$ is a subset of $V$ and $V$ is abelian.

So $W$ is abelian.

Further condition II to $V$ of the definition are satisfied in $W$ as these are satisfied in $V$.

**Corollary:**

$W$ is non-empty subset of a vectorspace $V(F)$. Then $W$ is subspace of $V$ iff

$\alpha, \beta \in F, w_1, w_2 \in W \Rightarrow \alpha w_1 + \beta w_2 \in W$. 

[3]
Proof. Let \( W \) is a subspace of \( V(F) \), then \( W \) itself is a vector space. For a, b \( \in F \), \( w, w_1 \in W \),
\[
\Rightarrow aw_1 + bw_2 \in W.
\]
Conversely, let for a, b \( \in F \), \( w, w_1 \in W \).
\[
\Rightarrow aw + bw_1 \in W.
\]
Set \( a = b = 1 \)
then \( 1 \cdot w_1 + 1 \cdot w_2 \in W \)
\( \Rightarrow w_1 + w_2 \in W \).
also if \( b \cdot a \in F \)
For \( aw_1 + bw_2 \in W \)
\[
\Rightarrow aw_1 + a \cdot w_2 \in W.
\]
\[
\Rightarrow aw_1 \in W.
\]
\( \Rightarrow W \) is a subspace of \( V \).

# Definition (Linear Sum):
Let \( V \) be a vector space over \( F \) and \( W_1, W_2, \ldots, W_n \) be non-empty subset of \( V \) then their linear sum is defined as
\[
W_1 + W_2 + \cdots + W_n = \{ \sum_{i=1}^{n} a_i w_i \in W_1, a_i \in W_i \}
\]

Lemma. Let \( V \) be a vector space and \( W_1, W_2, \ldots, W_n \) be subspace. Prove that
\[
W = W_1 + W_2 + \cdots + W_n
\]
is also a subspace of \( V \).
**Lemma:**

$W_1, W_2, \ldots, W_n$ are subspaces of $V$ prove that $W = W_1 + W_2 + \ldots + W_n$ is a subspace of $V$.

**Proof:**

\[ c = c + 0 + 0 + \ldots + 0, \quad 0 \in W_0. \]

$\Rightarrow c \in W \Rightarrow W$ is non-empty.

Let $x, y \in W$, $a, b \in F$.

We have to show $ax + by \in W$.

\[ \Rightarrow x \in W \]

$\Rightarrow x = x_1 + x_2 + \ldots + x_n$ for $x_i \in W_i$.

$y = y_1 + y_2 + \ldots + y_n$ for $y_i \in W_i$.

Now,

\[ ax + by = a(x_1 + x_2 + \ldots + x_n) + b(y_1 + y_2 + \ldots + y_n). \]

\[ = ax_1 + ax_2 + \ldots + ax_n + by_1 + by_2 + \ldots + by_n. \]

As each $W_i$ is a subspace,

\[ \Rightarrow ax_i + by_i \in W_i, \quad i = 1, 2, \ldots, n. \]

So

\[ \sum_{i=1}^{n} ax_i + by_i \in \bigoplus_{i} W_i = W. \]

$\Rightarrow ax + by \in W$.

So $W$ is a subspace.

**Lemma:**

Let $V$ be a vector space and $W_i$ a family of subspaces of $V$. Then $NW_i$ is also a subspace of $V$.

**Proof:**

Let $v, w \in NW_i$.

Then $v, w \in W_i$ for each $i \in I$ and since each $W_i$ is a subspace so there must be $a, b \in F$ such that $av + bw \in W_i$ for each $i \in I$.

So $av + bw \in NW_i$. i.e. $NW_i$ is a subspace.
# Definition

Let $U$ and $V$ are two vector spaces over a field $F$, then $\mathcal{T}$ of $U$ into $V$ is called homomorphism if

- $\mathcal{T}(u_1 + u_2) = \mathcal{T}(u_1) + \mathcal{T}(u_2)$
- $\mathcal{T}(au) = a \mathcal{T}(u)$; $a \in F$.

# Definition

The kernel of homomorphism $\mathcal{T} : U \rightarrow V$ is defined as $\ker \mathcal{T} = \{ u \in U : \mathcal{T}(u) = 0 \}$.

(Question)

Prove that $\ker \mathcal{T}$ (kernel of homomorphism) is a subspace.

Solution. Let $u_1, u_2 \in \ker \mathcal{T}$.

$\Rightarrow \mathcal{T}(u_1) = 0, \mathcal{T}(u_2) = 0$.

Now, let $a, b \in F$

$\mathcal{T}(au_1 + bu_2) = \mathcal{T}(au_1) + \mathcal{T}(bu_2)$

$= a \mathcal{T}(u_1) + b \mathcal{T}(u_2)$

$= a \cdot 0 + b \cdot 0$

$= 0$.

$\Rightarrow au_1 + bu_2 \in \ker \mathcal{T}$.

So $\ker \mathcal{T}$ is subspace.

# Linear Combination

Let $V$ is a vector space.

Let $v_1, v_2, \ldots, v_n \in V$

$a_1, a_2, \ldots, a_n \in F$

then an element

$a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + a_n v_n$ is called linear combination.

The linear combination is trivial if each $a_i = 0$, and it is non-trivial if at least one of $a_i \neq 0$. 
Definition: \textbf{(Linear Span)}

Let $S$ be a subset of vector space $V$, then the set of all linear combinations of $S$ is called linear span, denoted by $<S>$ or $L(S)$ or $[S]$.

Theorem:

Prove that $<S>$ is a subspace of $V$ containing $S$. It is smallest subspace of $V$ containing $S$.

Proof:

Let $u, v \in <S>$

then $u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$

$v = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$

For $a, b \in F$ we have to prove $au + bv \in <S>$.

Now,

$au + bv = a(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n)$

$+ b(b_1 v_1 + b_2 v_2 + \ldots + b_n v_n)$

$= a_1 au_1 + a_2 au_2 + \ldots + a_n au_n$

$+ b_1 bv_1 + b_2 bv_2 + \ldots + b_n bv_n$

$\Rightarrow au + bv \in <S>$

$\Rightarrow <S>$ is a subspace.

Let $u \in S$

then $u = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n \in <S>$

i.e. $u \in <S>$

$\Rightarrow S \subseteq <S>$.

Let $W$ be any other subspace of $V$ containing $S$.

then $\exists a_i; u_i \in W$

$\Rightarrow W$ is subspace containing $S$

$\Rightarrow <S> \subseteq W$

i.e $<S>$ is smallest subspace containing $S$. 

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Definition (Finite Dimensional Vector Space)

A vector space $V$ is called finite dimensional if there is a subset $S$ of $V$ such that $\langle S \rangle = V$

Definition. (Linearly Dependent and Independent)

Let $V$ be a vector space, then the vectors $v_1, v_2, \ldots, v_n \in V$ are linearly dependent if $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ for some $a_1, a_2, \ldots, a_n \neq 0$.

If $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$

where each $a_i = 0$ then the vectors $v_1, v_2, \ldots, v_n$ are linearly independent.

Theorem.

Let $V$ be a vector space and consider a set of vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent then its subset is also independent.

ii) If $\{v_1, v_2, \ldots, v_n\}$ is dependent then $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ is also dependent.

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Lemma:

Let \( V(F) \) be a vector space and \( S = \{v_1, v_2, \ldots, v_n\} \) a set of vectors in \( V \). Then:

1) If \( S \) is independent, then any non-empty subset of \( S \) is also independent.

Proof:

Let \( \{v_1, v_2, \ldots, v_i\} \) be a subset of \( S \), \( 1 \leq i \leq n \).

Consider \( a_1v_1 + a_2v_2 + \cdots + a_iv_i = 0 \), \( a_i \in F \), then

\[
\ldots + a_iv_i + \cdots + 0_v + \cdots + 0_v = 0
\]

Since \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent,

\[
\Rightarrow \text{each } a_k = 0 \quad k = 1, 2, \ldots, i
\]

\[
\Rightarrow \{v_1, v_2, \ldots, v_i\} \text{ is L.I}
\]

Proof (ii)

If \( S \) is dependent, then

\( \{v_1, v_2, \ldots, v_i\} \) is also dependent.

i.e. \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) where all \( a_i \neq 0 \).

and then

\[
\ldots + a_iv_i + \cdots + a_nv_n + 0v = 0
\]

where all \( a_i \neq 0 \).

\( \Rightarrow \{v_1, v_2, \ldots, v_n\} \) is also dependent.

Theorem:

A set of non-zero vectors \( v_1, v_2, \ldots, v_n \in V \) is linearly dependent iff one of them is a linear combination of the other preceding vectors.

Proof:

\( \{v_1, v_2, \ldots, v_n\} \) is linearly dependent.

i.e. \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) where all \( a_i \)'s \neq 0.

For \( a_i \in F \)

Let \( a_k \) be the last non-coefficient of

\[
\ldots + a_kv_k + 2_{k+1}v_{k+1} + \cdots + a_nv_n
\]

where

\[
a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1} + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n
\]
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\[ a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \quad \Rightarrow \quad a_1 = a_2 = \ldots = a_n = 0 \]
\[ \Rightarrow \quad a_{k+1}v_{k+1} + \ldots + a_nv_n = 0 \]
\[ \Rightarrow \quad v_k = -\frac{1}{a_k} (a_1v_1 + a_2v_2 + \ldots + a_{k-1}v_{k-1}) \]

Conversely, let \( v_k \) is a linear combination of the preceding vectors

\[ v_1, v_2, v_3, \ldots, v_{k-1} \]

i.e.
\[ v_k = a_1v_1 + a_2v_2 + \ldots + a_{k-1}v_{k-1} \]
\[ \Rightarrow \quad a_1v_1 + a_2v_2 + \ldots + a_{k-1}v_{k-1} + (-1)v_k = 0 \]
\[ \Rightarrow \quad a_1v_1 + a_2v_2 + \ldots + a_{k-1}v_{k-1} + (-1)v_k + 0v_{k+1} + \ldots + 0v_n = 0 \]

then \( \{v_1, v_2, \ldots, v_n\} \) is Linearly Dependant

\[ \therefore \quad \text{at least one coefficient of } v_k \text{ is non-zero.} \]

# Basis of a Vector Space:

Let \( S \) be a subset of a vector space \( V(F) \)

then \( S \) is called basis for \( V \)

if i) \( S \) is linearly independent.

ii) \( S \) is spanning set of \( V \)

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# Theorem:

Any finite dimensional vector space contains a basis.

**Proof:**

Let \( \{v_1, v_2, \ldots, v_r\} \) be a spanning set of \( V \).

If \( \{v_1, v_2, \ldots, v_r\} \) is linearly independent then form a basis and there is nothing to prove.

Consider \( \{v_1, v_2, \ldots, v_r\} \) is linearly dependent then one of the vectors say \( v_r \) is a linear combination of the remaining \( \{v_1, v_2, \ldots, v_{r-1}\} \) we drop out this vector and obtain a set of \( r-1 \) vectors.

A vector linear combination of \( r \) vectors also a linear combination of \( r-1 \) vectors.

If this set \( \{v_1, v_2, \ldots, v_{r-1}\} \) is linearly independent then form a basis.

But if \( \{v_1, v_2, \ldots, v_{r-1}\} \) is dependent then the above process is continued. In this way, we can get a linear independent spanning set and hence a basis.

\[ \{v_1, v_2, \ldots, v_r\} \text{ is a basis.} \]

**Theorem:**

If \( v_1, v_2, \ldots, v_n \) is a basis of \( V(F) \) and if \( w_1, w_2, \ldots, w_m \in V \) are linearly independent then \( m \leq n \).

**Proof:**

Since \( v_1, v_2, \ldots, v_n \) is a basis of \( V \) so every element of \( V \) can be expressed as a linear combination of \( v_1, v_2, \ldots, v_n \).

In particular, \( w_m \in V \) is a linear combination of \( v_1, v_2, \ldots, v_n \).
\[ v_1, v_2, \ldots, v_n \] are dependent. Therefore, a proper subset \( \{ v_1, v_2, \ldots, v_r \} \), \( r \leq n-1 \) from a basis.

Similarly, \( \{ w_1, w_2, v_3, \ldots, v_n \} \) is dependent and its proper subset \( \{ w_1, w_2, v_3, \ldots, v_s \} \), \( s \leq n-2 \).

Repeating this procedure \((n-1)\) times, we get a basis \( \{ w_1, w_2, \ldots, v_1, v_2, \ldots, v_n \} \).

Since the vectors \( w_1 \) is not a k.c. of \( \{ w_2, w_3, \ldots, w_n \} \), \( t \geq 1 \),

\[ 1 \leq t \leq n-m+1 \]

\[ 1 \leq n-m+1 \]

\[ m \leq n \]
(Question: Show that the vectors

\[ v_1 = (1, 1, 1), \ v_2 = (1, 0, 1), \ v_3 = (0, 1, 1) \]

are linearly independent.

Solution:

Consider

\[ a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \]

\[ \Rightarrow a_1 (1, 1, 1) + a_2 (1, 0, 1) + a_3 (0, 1, 1) = 0 \]

\[ \Rightarrow (a_1, a_1 + a_2, a_1 + a_3) = (0, 0, 0) \]

\[ \Rightarrow a_1 + a_2 = 0 \quad (i) \]

\[ a_1 + a_3 = 0 \quad (ii) \]

\[ a_1 + a_2 + a_3 = 0 \quad (iii) \]

\[ \Rightarrow a_1 + a_2 = 0 \]

\[ \Rightarrow a_3 = 0 \quad \Rightarrow a_1 = 0, \ a_2 = 0. \]

Since \[ a_1 = a_2 = a_3 = 0 \]

\[ \Rightarrow \] the vectors are L . I.

(Question: Prove that the vectors

\[ v_1 = (3, 0, -3), \ v_2 = (-1, 1, 2), \ v_3 = (1, 2, -2) \]

\[ v_4 = (2, 1, 1) \]

are linearly dependent.

Solution:

Consider

\[ a v_1 + b v_2 + c v_3 + d v_4 = 0 \]

\[ \Rightarrow a (3, 0, -3) + b (-1, 1, 2) + c (1, 2, -2) + d (2, 1, 1) = 0 \]

\[ \Rightarrow (3a, -3a) + (-b, b, 2b) + (c, 2c, -2c) + (2d, d, d) = 0 \]

\[ \Rightarrow (3a - b + c + 2d, b + 2c + d, -3a + 2b - 2c + d) = 0 \]

\[ \Rightarrow 3a - b + c + 2d = 0 \]

\[ \Rightarrow b + 2c + d = 0 \]

\[ \Rightarrow -3a + 2b - 2c + d = 0 \]
Let \( d = 0 \) so \( c = \frac{1}{3} \) and \( d = 0 \).

\[
3a + b + 4c = 0 \quad (1)
\]

\[
b + 2c = 0
\]

\[-3a + 2b - 2c = 0\]

\[e = (1, 0, c, (1 - c))\]

\[e = (1, 0, \frac{1}{3}, (1 - \frac{1}{3}))
\]

\[(0, 0, 0) = (0, 0, 0, 0) + (0, 0, 0, 0)\]

\[
\sqrt{\text{Using } a = -2c, b = -2c, d = 0}
\]

into (1)

\[-2cV_1 - 2cV_2 + cV_3 + 0V_4 = 0\]

\[2V_1 + 2V_3 - V_3 + 0V_4 = 0\]

\[\implies V_1, V_2, V_3, V_4 \text{ are dependent}\]

\[\checkmark\]
# Definition (Quotient Space)

Let \( V \) be a vector space over a field \( F \)
and \( W \) be a subspace.

The set \( \frac{V}{W} \) of all left cosets along with two operations

\[
\begin{align*}
\text{(i)} & \quad (v_1 + W) + (v_2 + W) = v_1 + v_2 + W, \\
\text{(ii)} & \quad a(v + W) = av + W,
\end{align*}
\]

is called Quotient space.

# Lemma:

Let \( V \) be a vector space and \( W \) a subspace of \( V \) along with the operation

\[
\begin{align*}
\text{(i)} & \quad (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W, \\
\text{(ii)} & \quad \alpha (v + W) = \alpha v + W, \text{ is a subspace of vector space}
\end{align*}
\]

Proof:

\( \star \) It is easy to show that \( \frac{V}{W} \) is an

abelian group under addition with \( 0 + W = W \) as

identity

and \( -v + W \) as an inverse of \( v + W \in \frac{V}{W} \).

\( \star \) We see that scalar multiplication is

defined in \( \frac{V}{W} \).

\( \text{i.e. } v + W = v' + W \Rightarrow \alpha (v + W) = \alpha (v' + W) \]

Let \( v = v' + \omega \) for some \( \omega \in W \)

then \( \alpha (v + W) = \alpha \omega + W \)

\[
\begin{align*}
\alpha \omega & = \alpha (v' + w) + W, \\
& = \alpha v' + \alpha \omega + W, \\
& = \alpha v' + W, \quad \because \alpha \omega \in W. \\
& = \alpha (v' + W),
\end{align*}
\]

\( \star \) a Scalar multiplication is defined

Let \( v + W, v' + W \in \frac{V}{W}, \quad a \in F \)

\[
\begin{align*}
a((v + W) + (v' + W)) & = a(v + v' + W), \\
& = a(v + v') + W, \\
& = av + av' + W, \\
& = av + W + av' + W = a(v + W) + a(v' + W)
\end{align*}
\]
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\[
\begin{align*}
\dot{v} : (a+b)(v+w) &= (a+b)v + W \\
&= (av+ bv) + W \\
&= av + W + bv + W \\
&= a(v+W) + b(v+W).
\end{align*}
\]

\[
\begin{align*}
(a(b(v+w))) &= a(bv + w) \\
&= (ab)v + w \\
&= (ab)(v + w).
\end{align*}
\]

\[
\begin{align*}
1 \cdot (v + W) &= 1 \cdot v + W \\
&= v + W
\end{align*}
\]

Hence \( v/W \) is vector space.

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Theorem:

\[ V(F) \] is a finite dimensional vector space and if \( W \) is a subspace of \( V \). Then

i) \( W \) is finite dimensional and \( \dim W \leq \dim V \).

ii) \( \dim(V/W) = \dim V - \dim W \)

Proof:

Let \( \dim V = n \)

and let \( \{w_1, w_2, \ldots, w_m\} \) be a linearly independent set of vectors of \( W \).

Then \( m \leq n \)

If \( \{ w_1, w_2, \ldots, w_m, w_0 \} \) is linearly dependent, i.e., one of these vectors is a linear combination of the preceding vectors.

However, none of the vectors \( w_1, w_2, \ldots, w_m \) is a linear combination of the preceding vectors because the vectors \( w_1, w_2, \ldots, w_m \) are linearly independent.

So \( W \) can be written as a linear combination of \( w_1, w_2, \ldots, w_m \).

Since \( w \in W \) is an arbitrary element,

therefore \( \dim W = m \leq n \).

i.e., \( \dim W \leq \dim V \).

ii) Let \( \{w_1, w_2, \ldots, w_m\} \) be a basis of \( W \).

and \( \{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_k\} \) be a basis of \( V \).

We have to prove \( \{v_1 + W, v_2 + W, \ldots, v_k + W\} \) is a basis of \( V/W \).

Now,

\[ \alpha_1(v_1 + W) + \alpha_2(v_2 + W) + \cdots + \alpha_k(v_k + W) = 0 \]

\[ (\alpha_1 v_1 + W) + (\alpha_2 v_2 + W) + \cdots + (\alpha_k v_k + W) = 0 + W \]

Since \( W \) is identity of \( V/W \),

\[ \Rightarrow (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k) + W = 0 + W \]
Vector Spaces: Handwritten notes

\[ \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in W \quad \Rightarrow \quad \exists \mathbf{h} = \mathbf{v} \quad \exists \mathbf{a} \in H. \]

\[ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m. \]

\[ a_1 w_1, a_2 w_2, \ldots, a_m w_m \] is basis of \( W. \)

\[ p_1 w_1 + p_2 w_2 + \cdots + p_m w_m = \alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_n v_n = 0, \]

Since \( \{ w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_n \} \) is a basis of \( V. \)

\[ \Rightarrow p_1 = p_2 = \cdots = p_m = \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0. \]

\[ v + w, v_1 + w, v_2 + w, \ldots, v_n + w \] is linearly independent.

Set \( v + W \in V/W \) for \( v \in V. \)

Then \( v = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + b_1 v_1 + b_2 v_2 + \cdots + b_n v_n. \)

So \( v + W = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + b_1 v_1 + b_2 v_2 + \cdots + b_n v_n + W \)

\[ \Rightarrow v + W = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n + a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + W \]

\[ = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n + W \]

\[ \Rightarrow a_1 w_1 + a_2 w_2 + \cdots + a_m w_m + W = W \]

As \( a_1 w_1 + a_2 w_2 + \cdots + a_m w_m \in V/W, \)

\[ \Rightarrow (b_1 v_1 + W) + (b_2 v_2 + W) + \cdots + (b_n v_n + W). \] By def.

\[ = b_1 (v_1 + W) + b_2 (v_2 + W) + \cdots + b_n (v_n + W). \] By def.

\[ \Rightarrow v_1 + W, v_2 + W, \ldots, v_n + W \] generate \( V/W. \) And hence is a basis of \( V/W. \)

\[ \dim(V/W) = \mathcal{f} = (m + l) - m = \dim V - \dim W. \]
# Internal Direct Sum:

Let $U_1, U_2, \ldots, U_n$ be subspaces of a vector space $V$. For $v \in V$,

then if $v$ has one and only one expression of the form

$$v = u_1 + u_2 + \ldots + u_n$$

for $u_i \in U_i$,

then $V$ is called internal direct sum of subspaces $U_1, U_2, \ldots, U_n$.

# External Direct Sum:

Let $V_1, V_2, \ldots, V_n$ be vector spaces over a field $F$. Let $V$ be a vector space over field $F$.

Then $V$ is called external direct sum if

i) Two $n$-tuples $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ are equal iff $v_i = w_i$.

ii) $(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$

iii) $\alpha(v_1, v_2, \ldots, v_n) = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$

The external direct sum is denoted by $V_1 \oplus V_2 \oplus \cdots \oplus V_n$.

# Vector Space Homomorphism:

Let $V$ and $W$ be two vector spaces.

A mapping $T : V \rightarrow W$ is called homomorphism if

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$T(\alpha v) = \alpha T(v)$$

for $v_1, v_2 \in V$ and $\alpha \in F$.

# Theorem:

If a vector space $V$ is the internal direct sum of subspaces $U_1, U_2, \ldots, U_n$, then $V$ is isomorphic to the external direct sum of $U_1, U_2, \ldots, U_n$. 
**Proof.** Let \( v \in V \) where \( v = u_1 + u_2 + u_3 + \cdots + u_n \).

Define a mapping 
\[
T : V \rightarrow U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_n
\]
by 
\[
T(v) = T(u_1 + u_2 + \cdots + u_n) = (u_1, u_2, \cdots, u_n).
\]

1) **Mapping is well defined as \( v \in V \).**

\( v = u_1 + u_2 + \cdots + u_n \)
has one and only one representation.

(ii) \( T \) is onto because each 
\[
(u_1, u_2, u_3, \cdots, u_n) \in U_1 \oplus U_2 \oplus \cdots \oplus U_n
\]
is image of \( u_1 + u_2 + \cdots + u_n \in V \).

(iii) \( T \) is one-one

for 
\[
T(u_1 + u_2 + \cdots + u_n) = T(w_1 + w_2 + \cdots + w_n)
\]

\[
\Rightarrow (u_1, u_2, \cdots, u_n) = (w_1, w_2, \cdots, w_n)
\]

\[
\Rightarrow u_1 = w_1, u_2 = w_2, \cdots, u_n = w_n
\]

\[
\Rightarrow u_1 + u_2 + \cdots + u_n = w_1 + w_2 + \cdots + w_n.
\]

(iv) \( T(v + w) = T(u_1 + u_2 + u_3 + \cdots + u_n + w_1 + w_2 + \cdots + w_n) \)

\[
= T(u_1 + u_2 + u_3 + \cdots + u_n + w_1 + \cdots + w_n)
\]

\[
= (u_1 + w_1, u_2 + w_2, \cdots, u_n + w_n)
\]

\[
= (u_1, u_2, \cdots, u_n) + (w_1, w_2, \cdots, w_n)
\]

by def. of external direct sum.

\[
= T(v) + T(w).
\]

\( \lambda \in \mathbb{R} \) 
\[
T(\lambda v) = T(\lambda (u_1 + u_2 + \cdots + u_n)) = T(\lambda u_1 + \lambda u_2 + \cdots + \lambda u_n)
\]

\[
= \lambda (\lambda u_1, \lambda u_2, \cdots, \lambda u_n)
\]

\[
= \lambda \cdot T(v)
\]

Hence \( T \) is homomorphism. 

[20]
Theorem: If $A$ and $B$ are finite dimensional subspaces of a vector space $V(F)$, then $A + B$ is finite dimensional and $\dim(A + B) = \dim A + \dim B - \dim(\text{ANB})$.

Proof: Suppose $\{u_1, u_2, \ldots, u_r\}$ be a basis of $\text{ANB}$.

$\{u_1, u_2, \ldots, v_r, v_1, v_2, \ldots, v_m\}$ be a basis of $A$.

$\{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_n\}$ be a basis of $B$.

Then we have to prove that $\{u_1, u_2, \ldots, v_r, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\}$ is a basis of $A + B$.

Consider:

$$\alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_r u_r + \beta_1 v_1 + \ldots + \beta_m v_m + \gamma_1 w_1 + \ldots + \gamma_n w_n = 0$$

$\Rightarrow \alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_r u_r + \beta_1 v_1 + \ldots + \beta_m v_m = -\gamma_1 w_1 - \gamma_2 w_2 - \ldots - \gamma_n w_n$  \hspace{1cm} (1)

Since L.H.S of (1) is in $A$ so does R.H.S.

i.e $-\gamma_1 w_1 - \gamma_2 w_2 - \ldots - \gamma_n w_n \in A$.

Also $-\gamma_1 w_1 - \gamma_2 w_2 - \ldots - \gamma_n w_n \in B$.

$\therefore \gamma_1 w_1 + \gamma_2 w_2 + \ldots + \gamma_n w_n$ is a part of basis of $B$.

$\therefore -\gamma_1 w_1 - \gamma_2 w_2 - \ldots - \gamma_n w_n \in \text{ANB}$.

$\Rightarrow -\gamma_1 w_1 - \gamma_2 w_2 - \ldots - \gamma_n w_n = \delta_1 u_1 + \delta_2 u_2 + \ldots + \delta_r u_r$.

As $\{u_1, u_2, \ldots, u_r\}$ is a basis of $\text{ANB}$, $\delta_i \in F$.

$\Rightarrow \delta_1 u_1 + \delta_2 u_2 + \ldots + \delta_r u_r + \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_m v_m + \gamma_1 w_1 + \gamma_2 w_2 + \ldots + \gamma_n w_n = 0$.

Since $\{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_n\}$ is a basis of $B$ (L.1)

$\Rightarrow \delta_1 = \delta_2 = \ldots = \delta_r = \gamma_1 = \gamma_2 = \ldots = \gamma_n = 0$.

so that equation (1) becomes.
Vector Spaces: Handwritten notes

\[ \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_r u_r + p_1 v_1 + p_2 v_2 + \ldots + p_m v_m = 0 \]

But \( \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m\} \) is a basis of \( A \)

\[ \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_r = p_1 = p_2 = \ldots = p_m = 0 \]

i.e. each \( \alpha_i = p_i = c_i \alpha_i = 0 \).

Hence \( \{u_1, u_2, \ldots, u_r, v_1, \ldots, v_m, w_1, \ldots, w_n\} \) is linearly independent.

Let \( x + y \in A + B \) i.e. \( x \in A \) \& \( y \in B \).

As basis of \( A = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m\} \),

\[ x = a_1 u_1 + a_2 u_2 + \ldots + a_r u_r + b_1 v_1 + b_2 v_2 + \ldots + b_m v_m \]

Also basis of \( B = \{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_n\} \),

\[ y = a'_1 u_1 + a'_2 u_2 + \ldots + a'_r u_r + b'_1 w_1 + b'_2 w_2 + \ldots + b'_n w_n \]

By +ing,

\[ A + B = (a_1 + a'_1) u_1 + (a_2 + a'_2) u_2 + \ldots + (a_r + a'_r) u_r + b_1 v_1 + b_2 v_2 + \ldots + b_m v_m + b'_1 w_1 + \ldots + b'_n w_n \]

\[ \Rightarrow \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\} \text{ generates } A + B \]

and hence is a basis of \( A + B \).

\[ \Rightarrow A + B \text{ is a finite dimensional vector space \( \dim(A + B) = r + m + n \)} \]

\[ = (r + m) + (r + n) - r \]

\[ = \dim A + \dim B - \dim (A \cap B) \] proved
Theorem: Let $V$ and $W$ be vector spaces and $T$ be an isomorphism of $V$ onto $W$. Then $T$ maps a basis of $V$ onto a basis of $W$.

Proof:
$T: V \rightarrow W$ is isomorphism defined by $T(v) = w$.

Let $v_1, v_2, \ldots, v_n$ be a basis of $V$ then we have to prove $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis of $W$.

i) Consider

$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n) = 0 \quad \forall \alpha_i \in F$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = 0$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \in \ker T$

$\therefore T$ is isomorphism i.e one-one

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

$\therefore \{v_1, v_2, \ldots, v_n\}$ is basis of $V$.

$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Hence $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

ii) Let $w \in W$.

$T$ is onto there must be $v \in V$ such that $T(v) = w$.

Now $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \quad \forall \alpha_i \in F$.

$\therefore w = T(v)$

$= T(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n)$

$= T(a_1 v_1) + T(a_2 v_2) + \ldots + T(a_n v_n)$

$\therefore T$ is homo.

$\Rightarrow w = a_1 T(v_1) + a_2 T(v_2) + \ldots + a_n T(v_n)$.
Vector Spaces: Handwritten notes

...can be generated by 
\[ T(v_1), T(v_2), \ldots, T(v_n) \] 

Thus \[ T(v_1), T(v_2), \ldots, T(v_n) \] form a basis of \( W \).

The proof is complete.

**Theorem:**
Two finite-dimensional vector spaces are isomorphic iff they are of the same dimension.

**Proof:**
Let \( V \) and \( W \) are two vector spaces of same dimension \( n \) and \( \{v_1, v_2, \ldots, v_n\} \) be the basis of \( V \) and \( \{w_1, w_2, \ldots, w_n\} \) be the basis of \( W \).

Define a mapping 
\[ T: V \rightarrow W \] 
by \( T(v) = w \) for \( v \in V, w \in W \),

i.e. \( T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n \).

i) \( T \) is well defined.

For \( v, v' \in V \), if \( v = v' \),

\[ \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \alpha'_1 v_1 + \alpha'_2 v_2 + \cdots + \alpha'_n v_n \]

\[ \Rightarrow (\alpha_1 - \alpha'_1) v_1 + (\alpha_2 - \alpha'_2) v_2 + \cdots + (\alpha_n - \alpha'_n) v_n = 0 \]

Since \( \{v_1, v_2, \ldots, v_n\} \) is basis of \( V \),

\[ \alpha_1 - \alpha'_1 = 0 = \alpha_2 - \alpha'_2 = \cdots = \alpha_n - \alpha'_n \]

\[ \Rightarrow \alpha_1 = \alpha'_1, \alpha_2 = \alpha'_2, \ldots, \alpha_n = \alpha'_n \]

i.e. \( T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n \)

\[ = \alpha'_1 w_1 + \alpha'_2 w_2 + \cdots + \alpha'_n w_n = T(v') \]

ii) \( T \) is homomorphism

\[ T(v + v') = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + \alpha'_1 v_1 + \cdots + \alpha'_n v_n) \]

\[ = T((\alpha_1 + \alpha'_1) v_1 + (\alpha_2 + \alpha'_2) v_2 + \cdots + (\alpha_n + \alpha'_n) v_n) \]
\[
\begin{align*}
\text{and} \\
T(\alpha v) &= T(\alpha (\alpha_1 v_1 + \cdots + \alpha_n v_n)) \\
&= T(\alpha \alpha_1 v_1 + \cdots + \alpha_n v_n) \\
&= \alpha \alpha_1 w_1 + \cdots + \alpha_n w_n \\
&= \alpha (\alpha_1 w_1 + \cdots + \alpha_n w_n) \\
&= \alpha T(v) \\
\end{align*}
\]

(iii) \( T \) is one-one

\[ T(v) = T(v') \quad \text{for} \quad v, v' \in V. \]

\[ \Rightarrow \alpha_1 v_1 + \cdots + \alpha_n v_n = \alpha'_1 v'_1 + \cdots + \alpha'_n v'_n \]

\[ \Rightarrow T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = T(\alpha'_1 v'_1 + \cdots + \alpha'_n v'_n) \]

\[ \Rightarrow \alpha_1 w_1 + \cdots + \alpha_n w_n = \alpha'_1 w'_1 + \cdots + \alpha'_n w'_n \]

\[ \Rightarrow (\alpha_1 - \alpha'_1) w_1 + \cdots + (\alpha_n - \alpha'_n) w_n = 0. \]

\[ \Rightarrow \{w_1, w_2, \ldots, w_n\} \text{ is basis of } W. \]

\[ \Rightarrow \alpha_1 - \alpha'_1 = \alpha_2 - \alpha'_2 = \cdots = \alpha_n - \alpha'_n = 0. \]

\[ \Rightarrow \alpha_1 = \alpha'_1, \alpha_2 = \alpha'_2, \ldots, \alpha_n = \alpha'_n. \]

\[ \Rightarrow \alpha_1 v_1 + \cdots + \alpha_n v_n = \alpha'_1 v'_1 + \cdots + \alpha'_n v'_n. \]

\[ \Rightarrow v = v'. \]

(iv) \( T \) is onto as every element \( \alpha_1 w_1 + \cdots + \alpha_n w_n \in W \) is image of \( \alpha_1 v_1 + \cdots + \alpha_n v_n \in V \).

Conversely, let \( T: V \to W \) is isomorphism then we have to prove that the dimension of \( V \) and \( W \) are same.

Let \( \{v_1, v_2, \ldots, v_n\} \) be basis of \( V \) then we prove that \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) is a basis of \( W \).
**Vector Spaces: Handwritten notes**

**Set:** \( V \) and \( W \) are two vector spaces.

The set of all homomorphisms of \( V \) into \( W \) is denoted by \( \text{Hom}(V, W) \):

\[
\text{Hom}(V, W) = \{ T_1, T_2, \ldots, T_n \}
\]

where each \( T_i \) is a homomorphism.

**Theorem:**

Let \( V(F) \) and \( W(F) \) be two vector spaces introduce an operation in \( \text{Hom}(V, W) \) and prove that \( \text{Hom}(V, W) \) is a vector space under this operation.

**Proof:**

Let \( T_1, T_2 \in \text{Hom}(V, W) \), we define \((T_1 + T_2)(v) = T_1(v) + T_2(v)\) and \(\lambda T(v) = T(\lambda v)\).

To prove \( \text{Hom}(V, W) \) is a vector space we proceed as follows:

Let \( v_1, v_2 \in V \) and \( T_1, T_2 \in \text{Hom}(V, W) \);

\[
(T_1 + T_2)(v_1 + v_2) = T_1 (v_1 + v_2) + T_2 (v_1 + v_2)
\]

\[
= T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2)
\]

\[
= T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2)
\]

\[
= (T_1 + T_2)v_1 + (T_1 + T_2)v_2
\]

Also,

\[
(T_1 + T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v)
\]

\[
= \lambda T_1(v) + \lambda T_2(v)
\]

\[
\Rightarrow (T_1 + T_2)(\lambda v) = \lambda (T_1 + T_2)(v)
\]

\[
\Rightarrow T_1 + T_2 \in \text{Hom}(V, W)
\]

i.e \( \text{Hom}(V, W) \) is closed.

iii) Mapping \((T_1, T_2, \ldots, T_n)\) are associative in general.

Consider a mapping \(T_0\) which maps on
Element of $V$ into $0$ (zero) i.e.
\[ T_0(v) = 0 \]

Then $(T + T_0)v = T(v) + T_0(v) = T(v) + 0 = T(v)
\]

i.e \( T_0 + T = T \)

i.e \( T_0 \) is the identity of \( \text{Hom}(V, W) \),

also for \( T \in \text{Hom}(V, W) \)

so we have

\[ -T \in \text{Hom}(V, W) \text{ such that } \]

\[ (T + (-T))v = T(v) + (-1)T(v) = T(v) - T(v) = 0. \]

\[ = T_0(v) \]

\[ \implies \text{inverse exists}. \]

\[ \forall \ (T_1 + T_2)v = T_1(v) + T_2(v) = T_2(v) + T_1(v) = (T_2 + T_1)v \]

\[ \implies \text{Hom}(V, W) \text{ is an abelian group under } + \]

(ii)

\[ a(T_1 + T_2) = aT_1 + aT_2 \]

\[ a(T_1 + T_2)(v) = (T_1 + T_2)(av) = T_1(av) + T_2(av) = aT_1(v) + aT_2(v) \]

(iii)

\[ (a + b)T = aT + bT. \]

\[ (a + b)T(v) = T((a + b)v) = T(av + bv) = aT(v) + bT(v) \]

(iv) \[ a(b)T = (ab)T \]

\[ a(b)T(v) = aT((b)v) = T((a)b v) = T(abv) = abT(v) \]

\[ \text{p.t.o.} \]
1. \( T = T \)

\[ T(v) = T(1 \cdot v) = T(v) \]

As \( v \in V \) is a vector space.

Hence \( \text{Hom}(V, W) \) is a vector space.
Theorem: If \( V \) and \( W \) are of dimension \( m \) and \( n \) resp. then \( \text{Hom}(V, W) \) is of dimension \( mn \).

Proof:

Let \( \{ v_1, v_2, \ldots, v_m \} \) and \( \{ w_1, w_2, \ldots, w_n \} \) be basis of \( V \) and \( W \) respectively.

Define a mapping

\[
T_{ij}: V \rightarrow W \text{ defined by }
\]

\[
T_{ij}(v_k) = \begin{cases} 
\lambda_i w_j & \text{if } i = k \\
0 & \text{if } i \neq k
\end{cases}, \quad \lambda_{ij} \in F
\]

Let

\[
v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m
\]

\[
w = -\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m
\]

then

\[
T_{ij}(v + w) = T_{ij}\left( (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) + (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) \right)
\]

\[
= T_{ij}\left( (\lambda_1 + \lambda_1) v_1 + (\lambda_2 + \lambda_2) v_2 + \cdots + (\lambda_m + \lambda_m) v_m \right)
\]

\[
= (\lambda_1 + \lambda_2) w_j + \lambda_i w_j
\]

\[
= T_{ij}(v) + T_{ij}(w)
\]

And

\[
T_{ij}(\alpha v) = T_{ij}\left( \alpha (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) \right)
\]

\[
= T_{ij}\left( \alpha \lambda_1 v_1 + \alpha \lambda_2 v_2 + \cdots + \alpha \lambda_m v_m \right)
\]

\[
= \alpha \lambda_i w_j
\]

\[
= \alpha T_{ij}(v)
\]

Thus \( T_{ij} \) is homomorphism and \( T_{ij} \in \text{Hom}(V, W) \).

Now to prove \( \{ T_{11}, T_{12}, \ldots, T_{ij}, \ldots, T_{mn} \} \) is a basis.

Consider

\[
\alpha_{11} T_{11} + \alpha_{12} T_{12} + \cdots + \alpha_{ij} T_{ij} + \cdots + \alpha_{mn} T_{mn} = 0
\]

Now
Vector Spaces: Handwritten notes

\[
\left( \alpha_{11} T_{11} + \alpha_{12} T_{12} + \cdots + \alpha_{1n} T_{1n} \\
+ \alpha_{21} T_{21} + \alpha_{22} T_{22} + \cdots + \alpha_{2n} T_{2n} \\
\vdots \\
+ \alpha_{m1} T_{m1} + \alpha_{m2} T_{m2} + \cdots + \alpha_{mn} T_{mn} \right) v_1 = 0. \quad (1)
\]

\[
\Rightarrow \alpha_{11} T_{11} (v_1) + \alpha_{12} T_{12} (v_1) + \cdots + \alpha_{1n} T_{1n} (v_1) \\
+ \alpha_{21} T_{21} (v_1) + \alpha_{22} T_{22} (v_1) + \cdots + \alpha_{2n} T_{2n} (v_1) \\
+ \cdots \\
+ \alpha_{m1} T_{m1} (v_1) + \alpha_{m2} T_{m2} (v_1) + \cdots + \alpha_{mn} T_{mn} (v_1) = 0
\]

\[
\Rightarrow \alpha_{11} \lambda_1 w_1 + \alpha_{12} \lambda_2 w_2 + \cdots + \alpha_{1n} \lambda_n w_n \\
+ 0 + 0 + \cdots + 0 \\
+ 0 + 0 + \cdots + 0 = 0
\]

\[
\Rightarrow \alpha_{11} w_1 + \alpha_{12} w_2 + \cdots + \alpha_{1n} w_n = 0 \\
\text{and} \quad \{w_1, w_2, \ldots, w_n\} \text{ is basis of } \mathbb{V}
\]

\[
\Rightarrow \alpha_{11} = 0 = \alpha_{12} = \cdots = \alpha_{1n} = 0
\]

Similarly operating (ii) on \( v_k \) we have \( \alpha_{ij} = 0 \), \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \).

So the set \( \{T_{11}, T_{12}, \ldots, T_{21}, \ldots, T_{mn}\} \) is LI.

Now consider

\[
So = a_{11} T_{11} + a_{12} T_{12} + \cdots + a_{1n} T_{1n} \\
+ a_{21} T_{21} + a_{22} T_{22} + \cdots + a_{2n} T_{2n} \\
\vdots \\
+ a_{m1} T_{m1} + a_{m2} T_{m2} + \cdots + a_{mn} T_{mn}
\]

\[
So (v_1) = \left( a_{11} T_{11} + a_{12} T_{12} + \cdots + a_{1n} T_{1n} \\
+ a_{21} T_{21} + a_{22} T_{22} + \cdots + a_{2n} T_{2n} \\
\vdots \\
+ a_{m1} T_{m1} + a_{m2} T_{m2} + \cdots + a_{mn} T_{mn} \right) v_1
\]
Vector Spaces: Handwritten notes

\[ S_i(v_i) = a_{i1} T_{i1}(v_i) + a_{i2} T_{i2}(v_i) + \cdots + a_{in} T_{in}(v_i) \]
\[ + a_{21} T_{21}(v_i) + a_{22} T_{22}(v_i) + \cdots + a_{2n} T_{2n}(v_i) \]
\[ + \cdots + a_{m1} T_{m1}(v_i) + a_{m2} T_{m2}(v_i) + \cdots + a_{mn} T_{mn}(v_i) \]

Similarly,

\[ S_j(v_j) = a_{11} \lambda_1 w_1 + a_{12} \lambda_2 w_2 + a_{13} \lambda_3 w_3 + \cdots + a_{1n} \lambda_1 w_n \]
\[ + a_{21} \lambda_2 w_1 + a_{22} \lambda_2 w_2 + a_{23} \lambda_3 w_3 + \cdots + a_{2n} \lambda_2 w_n \]
\[ + \cdots + a_{m1} \lambda_1 w_1 + a_{m2} \lambda_2 w_2 + a_{m3} \lambda_3 w_3 + \cdots + a_{mn} \lambda_m w_n \]

\[ S_0(v_k) = a_{k1} \lambda_1 w_1 + a_{k2} \lambda_2 w_2 + a_{k3} \lambda_3 w_3 + \cdots + a_{kn} \lambda_k w_n \]

Let \( s \in \text{Hom}(V, W) \)

\[ \Rightarrow S(v_1), S(v_2), \ldots, S(v_k) \in W \]

So

\[ S(v_1) = a_{i1} w_1 + a_{i2} w_2 + \cdots + a_{in} w_n \]
\[ S(v_2) = a_{j1} w_1 + a_{j2} w_2 + \cdots + a_{jn} w_n \]
\[ S(v_k) = a_{k1} w_1 + a_{k2} w_2 + \cdots + a_{kn} w_n \]

i.e. \( s \in S_0 \) so \( s_0 \in \text{Hom}(V, W) \)

Thus \( S_{11}, S_{12}, \ldots, S_{mn} \) form a basis of \( \text{Hom}(V, W) \)

\[ \Rightarrow \dim(\text{Hom}(V, W)) = mn \]
Vector Spaces: Handwritten notes

# Definition (Dual Space):

Let \( V \) be a vector space over a field \( F \). Then \( \text{Hom}(V, F) \) is called the dual space and is denoted by \( V^* \) or \( \hat{V} \). Its elements are called linear functionals.

# Theorem:

If \( V \) is finite dimensional vector space over \( F \), then prove \( V \cong V^* \).

**Proof:**

Since \( \dim V = \dim V^* \), so consider \( \dim V = \dim V^* = m \).

Define a mapping \( T: V \to V^* \) by

\[
T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m
\]

i) \( T \) is a homomorphism.

\[
T(u + v) = T((\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) + (\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)) = T((\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \cdots + (\alpha_m + \beta_m) v_m) = (\alpha_1 + \beta_1) f_1 + (\alpha_2 + \beta_2) f_2 + \cdots + (\alpha_m + \beta_m) f_m = (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m) + (\beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m) = T(u) + T(v)
\]

and

\[
T(\alpha u) = T(\alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m)) = T(\alpha_1 \alpha v_1 + \alpha_2 \alpha v_2 + \cdots + \alpha_m \alpha v_m) = \alpha \alpha_1 f_1 + \alpha \alpha_2 f_2 + \cdots + \alpha \alpha_m f_m
\]

\[
= \alpha (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m) = \alpha T(u)
\]

ii) \( T \) is one-one.

If \( T(u) = T(v) \)

\[
\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = T(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m)
\]

\[
\Rightarrow \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m = \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_m f_m
\]

\[
\Rightarrow (\alpha_1 - \beta_1) f_1 + (\alpha_2 - \beta_2) f_2 + \cdots + (\alpha_m - \beta_m) f_m = 0
\]

\[
\Rightarrow (f_1, f_2, \ldots, f_m) \text{ is basis of } V^*
\]

\[
\Rightarrow \alpha_1 - \beta_1 = 0 = \alpha_2 - \beta_2 = \cdots = \alpha_m - \beta_m
\]
\[ \Rightarrow \alpha_1 = \beta_1, \ \alpha_2 = \beta_2, \ldots, \ \alpha_m = \beta_m \]
\[ \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_m v_m \]
\[ \Rightarrow v = v' \]

(iii) \( T \) is onto.

Since for \( \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_m f_m \in V^* \)
we have
\[ \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m \in V \]
such that
\[ T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m) = \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_m f_m \]
Thus \( T \) is onto.

and hence \( V \cong V^* \)

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Definition:
Let $T: V \rightarrow V'$ be a homomorphism of a vector space.
If $V_i(E)$ to a vector space $V_j(E)$ then $\ker T$ is called
null space denoted by $N(T)$.

The dimension of $N(T)$ is called nullity.

Theorem:
Let $T: V \rightarrow V'$ be a vector space homomorphism

then $\dim V_i = \dim N(T) + \dim R(T)$

Proof:
Let $\dim N(T) = m$
and $\dim (V_i) = n$

Let $\{v_1, v_2, \ldots, v_m\}$ be basis of $N(T) = \ker T$.

Since $N(T) = \ker T$ is a subspace of $V_i$

we can take basis of $V_i$

\[ \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\} \]

we have to prove:

$\{T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\}$ form basis of $R(T)$

Let $w \in R(T)$ then there is $v \in V_i$ such that

$T(v) = w$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \cdots + \alpha_n v_n) = w$

$\Rightarrow \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_m T(v_m) + \alpha_{m+1} T(v_{m+1}) + \cdots + \alpha_n T(v_n) = w$

$\Rightarrow \{v_1, v_2, \ldots, v_m\}$ \quad $\forall w \in N(T) = \ker T$

$\Rightarrow T(v_1) = 0, T(v_2) = 0, \ldots, T(v_m) = 0$

$\Rightarrow \alpha_{m+1} T(v_{m+1}) + \alpha_{m+2} T(v_{m+2}) + \cdots + \alpha_n T(v_n) = w$

$\Rightarrow T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)$ generates $R(T)$
Now consider

\[ b_{m+1} T(v_{m+1}) + b_{m+2} T(v_{m+2}) + \cdots + b_n T(v_n) = 0 \]

implies

\[ T(b_{m+1} v_{m+1} + b_{m+2} v_{m+2} + \cdots + b_n v_n) = 0 \]

\[ \Rightarrow b_{m+1} v_{m+1} + b_{m+2} v_{m+2} + \cdots + b_n v_n \in \ker T = N(T) \]

Since \( \{v_1, v_2, \ldots, v_m\} \) is a basis of \( N(T) \)

so \( \exists \delta_1, \delta_2, \ldots, \delta_m \in \mathbb{F} \) such that

\[ b_{m+1} v_{m+1} + b_{m+2} v_{m+2} + \cdots + b_n v_n = \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_m v_m \]

\[ \Rightarrow \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_m v_m - b_{m+1} v_{m+1} - b_{m+2} v_{m+2} - \cdots - b_n v_n = 0 \]

As \( \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\} \) is a basis of \( V \)

therefore \( \delta_1 = \delta_2 = \cdots = \delta_m = b_{m+1} = b_{m+2} = \cdots = b_n = 0 \)

i.e. \( b_{m+1} = b_{m+2} = \cdots = b_n = 0 \)

\[ \Rightarrow \{T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\} \] is L.I.

and hence form a basis of \( R(T) \)

so \( \dim R(T) = n - m \)

\[ = \dim V_1 - \dim N(T) \]

\[ \Rightarrow \dim V_1 = \dim N(T) + \dim R(T) \]

proved
Theorem

Let \( V \) be a vector space over \( F \) and \( \{ v_1, v_2, \ldots, v_n \} \) be a basis of \( V \). Let \( \phi_1, \phi_2, \ldots, \phi_n \in V^* = \text{Hom}(V, F) \) be linear functionals defined by

\[
\phi_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

Then \( \{ \phi_1, \phi_2, \ldots, \phi_n \} \) is a basis of \( V^* \).

Proof:

Let \( \phi \in V^* \) be taken

\[
\phi(v_1) = k_1, \quad \phi(v_2) = k_2, \ldots, \quad \phi(v_n) = k_n
\]

where \( k_1, k_2, \ldots, k_n \in F \)

Let

\[
\psi = k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n
\]

\[
\psi(v_i) = (k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n)(v_i)
\]

\[
= k_1 \phi_1(v_i) + k_2 \phi_2(v_i) + \cdots + k_n \phi_n(v_i)
\]

\[
= k_1(1) + k_2(0) + \cdots + k_n(0)
\]

\[
= k_i
\]

Also,

\[
\psi(v_2) = (k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n)(v_2)
\]

\[
= k_1 \phi_1(v_2) + k_2 \phi_2(v_2) + \cdots + k_n \phi_n(v_2)
\]

\[
= k_1(0) + k_2(1) + k_3(0) + \cdots + k_n(0)
\]

\[
= k_2
\]

\[
\Rightarrow \psi(v_2) = k_2 = \phi(v_2)
\]

\[
\Rightarrow \psi = \phi
\]

\[
\Rightarrow \phi = \psi = k_1 \phi_1 + k_2 \phi_2 + \cdots + k_n \phi_n
\]

So \( \{ \phi_1, \phi_2, \ldots, \phi_n \} \) spans \( V^* \).

To prove \( \{ \phi_1, \phi_2, \ldots, \phi_n \} \) is linearly independent:

Consider

\[
a_1 \phi_1 + a_2 \phi_2 + \cdots + a_n \phi_n = 0
\]

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then operating it on \( v_1 \)

\[(a_1 \Phi_1 + a_2 \Phi_2 + \cdots + a_n \Phi_n) v_1 = 0 \cdot v_1\]

\[\Rightarrow a_1 \Phi_1(v_1) + a_2 \Phi_2(v_1) + \cdots + a_n \Phi_n(v_1) = 0\]

\[\Rightarrow a_1(1) + a_2(0) + \cdots + a_n(0) = 0\]

\[\Rightarrow a_1 = 0\]

Similarly for \( i = 2, 3, \ldots, n \)

\[(a_1 \Phi_1 + a_2 \Phi_2 + \cdots + a_n \Phi_n) v_i = 0 \cdot v_i\]

\[\Rightarrow a_1 \Phi_1(v_i) + a_2 \Phi_2(v_i) + \cdots + a_i \Phi_i(v_i) + \cdots + a_n \Phi_n(v_i) = 0\]

\[\Rightarrow a_1(0) + a_2(0) + \cdots + a_i(1) + \cdots + a_n(0) = 0\]

\[\Rightarrow 0 + 0 + \cdots + a_i + \cdots + 0 = 0\]

\[\Rightarrow a_i = 0\]

\[\text{i.e. } a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_n = 0\]

Hence \( \{ \Phi_1, \Phi_2, \ldots, \Phi_n \} \) is L.I. and so is a basis of \( V^* \).
Example

Consider the basis of $\mathbb{R}^2$ as $v_1 = (2, 1)$ and $v_2 = (3, 1)$.

Find dual basis of $\{\phi_1, \phi_2\}$.

Solution.

$\phi_1(v_1) = 1, \quad \phi_1(v_2) = 0$

$\phi_2(v_1) = 0, \quad \phi_2(v_2) = 1$

Since $\phi_1, \phi_2$ are linear functionals, $\phi_1(x, y) = ax + by$ and $\phi_2(x, y) = cx + dy$

$\phi_1(v_1) = 1$ implies

$\phi_1(2, 1) = 1 \Rightarrow 2a + b = 1$  \hspace{1cm} (i)

$\phi_1(v_2) = 0$ implies

$\phi_1(3, 1) = 0 \Rightarrow 3a + b = 0$  \hspace{1cm} (ii)

By (i) and (ii)

$a = -1$ and $b = 3$

Now $\phi_2(v_1) = 0$

$\phi_2(2, 1) = 0 \Rightarrow 2c + d = 0$ \hspace{1cm} (iii)

and $\phi_2(v_2) = 1$

$\phi_2(3, 1) = 1 \Rightarrow 3c + d = 1$ \hspace{1cm} (iv)

Solving (iii) and (iv)

$c = 1$ and $d = -2$

Therefore $\phi_1 = -x + 3y$

$\phi_2 = x - 2y$

Example

Let a basis of $\mathbb{R}^3$ is $\{v_1, v_2, v_3\}$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Find dual basis $\phi_1, \phi_2$ and $\phi_3$ such that $\phi_i(v_j) = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}$

Do yourself as above.
Question

Let \( V = \{ a + bt : a, b \in \mathbb{R} \} \) be a vector space of polynomial of degree \( \leq 1 \).

Let \( \Phi_1, \Phi_2 : V \to \mathbb{R} \) be defined by

\[
\Phi_1(f(t)) = \int_0^1 f(t) dt \\
\Phi_2(f(t)) = \int_0^1 f(t) dt
\]

where \( \Phi_1, \Phi_2 \in V^* \) (dual space).

Find corresponding basis \( v_1, v_2 \) of \( V \).

Solution:

Let \( v_1 = a + bt \) and \( v_2 = a + bt + c + dt \).

By definition

\[
\Phi_1(v_1) = 1, \quad \Phi_1(v_2) = 0, \quad \Phi_2(v_1) = 0, \quad \Phi_2(v_2) = 1
\]

\[
\Phi_1(v_1) = 1 \quad \Rightarrow \quad \int_0^1 v_1 dt = 1 \quad \Rightarrow \quad \int_0^1 (a + bt) dt = 1
\]

\[
\Rightarrow \quad \int_0^1 a \, dt + \frac{bt^2}{2} \bigg|_0^1 = 1 \quad \Rightarrow \quad a + \frac{b}{2} = 1
\]

\[
\Rightarrow \quad 2a + b = 2 \quad \text{(i)}
\]

\[
\Phi_2(v_1) = 0 \quad \Rightarrow \quad \int_0^1 (a + bt) dt = 0 \quad \Rightarrow \quad \int_0^1 a \, dt + \frac{bt^2}{2} \bigg|_0^1 = 0
\]

\[
\Rightarrow \quad 2a + 2b = 0 \quad \Rightarrow \quad a + b = 0 \quad \text{(ii)}
\]

By (i) and (ii)

\[
a + b = 2 \quad \Rightarrow \quad a = 2 \quad \Rightarrow \quad b = -2
\]

Further \( \Phi_1(v_2) = 0 \)

\[
\Rightarrow \quad \int_0^1 v_2 dt = 0 \quad \Rightarrow \quad \int_0^1 (c + dt) dt = 0
\]
\[ \int_0^1 ct + \frac{d t^2}{2} \, dt = 0 \]

\[ \Rightarrow \quad c + \frac{d}{2} = 0 \quad \text{or} \quad 2c + d = 0 \quad (iii) \]

\[ \varphi_2(v_2) = 1 \]

\[ \Rightarrow \quad \int_0^2 v_2 \, dt = 1 \]

\[ \Rightarrow \quad \int_0^2 (c + dt) \, dt = 1 \quad \Rightarrow \quad \int_0^1 ct + \frac{d t^2}{2} \, dt = 1 \]

\[ \Rightarrow \quad 2c + 2d = 1 \quad (iv) \]

Subtracting (iii) from (iv):

\[ 2c + 2d = 1 \]

\[ 2c + d = 0 \]

\[ d = 1 \quad \Rightarrow \quad c = -\frac{1}{2} \]

Hence:

\[ v_1 = 2 - 2t \]

and \[ v_2 = -\frac{1}{2} + t \] are basis of \( V \) corresponding to dual basis \( V^* \)
# Eigen Value

Definition: let \( A \) be a \( n \times n \) square matrix, then \( \lambda \in \mathbb{F} \) is an eigenvalue of \( A \) if there exist a non-zero column vector \( v \in \mathbb{F}^n \) such that \( Av = \lambda v \).

Here \( v \) is an eigenvector corresponding to eigenvalue \( \lambda \).

# Exercise
Find eigenvalues and associated eigenvector of a matrix \( A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \).

Solution:

Let \( v = \begin{bmatrix} x \\ y \end{bmatrix} \). Then:

\[
Av = \lambda v
\]

\[
\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}
\]

\[
\begin{bmatrix} x + 2y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}
\]

\[
x + 2y = \lambda x
\]

\[
3x + 2y = \lambda y
\]

or

\[
\begin{align*}
(1 - \lambda)x + 2y &= 0 \\
3x + (2-\lambda)y &= 0
\end{align*}
\]

For non-trivial solution:

\[
\begin{vmatrix}
-\lambda & 2 \\
3 & 2-\lambda
\end{vmatrix} = 0
\]

\[
(1-\lambda)(2-\lambda) - 6 = 0
\]

\[
2 - \lambda - 2\lambda + \lambda^2 - 6 = 0
\]

\[
\lambda^2 - 3\lambda - 4 = 0
\]

\[
(\lambda - 4)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = 4, -1
\]
home $\lambda = 4, -1$ are eigen values. \\

$\lambda = -1$ in eq. ($\omega$) $\Rightarrow \sigma 2x + 2y = 0$ \\
$\sigma x - x + y = 0$ \\
for $y = -x$ \\
thus $v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ \\
i.e eigen vector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

and $\lambda = 4$ in eq. ($\omega$), $\Rightarrow -3x + 2y = 0$ \\
$\Rightarrow 2y = 3x$ \\
for $y = \frac{3}{2}x$ \\
thus $v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \frac{3}{2}x \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ \\
i.e eigen vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

# Note \\
$AV = \lambda v$ \\
$\Rightarrow AV - \lambda V = 0$ \\
$\Rightarrow (A - \lambda I)V = 0$ where I is identity \\
$\Rightarrow (A - \lambda I) = 0$ \\
and $AV = \lambda v$ \\
$A(kv) = k\lambda v$ \\
$= k(\lambda v)$ \\
$= \lambda(kv)$ \\
then $\lambda$ and k are eigen values for A.
# Eigen Value & Eigen Vector (Alternative)

**Definition:** Let \( T : V \rightarrow V \) be a linear operator then \( \lambda \in \mathbb{F} \) is called eigen value of \( T \) if there exist a non-zero vector \( v \) such that

\[
T(v) = \lambda v
\]

Here \( v \) is eigen vector.

Note that \( kv \) is also eigen vector for same eigen value \( \lambda \),

\[
T(kv) = kT(v) = k \lambda v = \lambda kv
\]

### Theorem

Let \( \lambda \) be an eigen value of an operator \( T : V \rightarrow V \). Let \( V_\lambda \) denotes set of all eigen vectors of \( T \) belonging to same eigen value \( \lambda \). The \( V_\lambda \) is a subspace of \( V \).

**Proof:**

Let \( \lambda \) be an eigen value of an operator.

Let \( v, w \in V_\lambda \).

Then \( T(v) = \lambda v \) and \( T(w) = \lambda w \).

Now \( T(av + bw) = T(av) + T(bw) \)

\[
= aT(v) + bT(w)
\]

\[
= a \lambda v + b \lambda w
\]

\[
= \lambda (av + bw)
\]

\( \Rightarrow av + bw \) is also an eigen vector for \( \lambda \).

Hence \( av + bw \in V_\lambda \)

\( \Rightarrow V_\lambda \) is a subspace.
**Theorem**

Let \( \{v_1, v_2, \ldots, v_n\} \) be non-zero eigen vectors of an operator \( T \) corresponding to distinct eigen values \( \lambda_1, \lambda_2, \ldots, \lambda_n \) respectively then \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent.

**Proof**

We prove the theorem by Mathematical Induction.

Let \( n = 1 \) so if \( a_1 v_1 = 0 \)

\[ \Rightarrow a = 0 \quad \text{as} \quad v_1 \neq 0 \]

so condition I is true.

Let the theorem is true for \( k = n-1 \)

i.e. \( v_1, v_2, \ldots, v_{n-1} \) are L. I. (linearly independent)

then \( a_1 v_1 + a_2 v_2 + \cdots + a_{n-1} v_{n-1} = 0 \)

\[ \Rightarrow a_1 = a_2 = \cdots = a_{n-1} = 0 \]

Consider

\[ b_1 v_1 + b_2 v_2 + \cdots + b_n v_n = 0 \quad \text{(i)} \]

\[ T(b_1 v_1 + b_2 v_2 + \cdots + b_n v_n) = 0 \]

\[ \Rightarrow T(b_1 v_1) + T(b_2 v_2) + \cdots + T(b_n v_n) = 0 \]

\[ \Rightarrow b_1 T(v_1) + b_2 T(v_2) + \cdots + b_n T(v_n) = 0 \]

\[ \Rightarrow b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \cdots + b_n \lambda_n v_n = 0 \]

or \( b_1 \lambda_1 v_1 + b_2 \lambda_2 v_2 + \cdots + b_{n-1} \lambda_{n-1} v_{n-1} + b_n \lambda_n v_n = 0 \) \quad \text{(ii)}

Multiplying eq (i) by \( \lambda_n \)

\[ \lambda_n b_1 v_1 + \lambda_n b_2 v_2 + \cdots + \lambda_n b_{n-1} v_{n-1} + \lambda_n b_n v_n = 0 \] \quad \text{(iii)}

Subtracting (iii) from (ii)
\[ b_1 (\lambda_1 - \lambda_n) v_1 + b_2 (\lambda_2 - \lambda_n) v_2 + \cdots + b_{n-1} (\lambda_{n-1} - \lambda_n) v_{n-1} = 0 \]

Since \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent,
\[ b_1 = a = b_2 = b_3 = \cdots = b_{n-1} = 0 \]

\[ \therefore \lambda_i - \lambda_n \neq 0 \quad i = 1, 2, \ldots, n-1 \]

because if \( \lambda_i = \lambda_n \),
\[ \Rightarrow \lambda_i = \lambda_n \quad \text{for } i = 1, 2, \ldots, n-1 \]

a contradiction as each \( \lambda_1, \lambda_2, \ldots, \lambda_n \)
are distinct.

Now from eq (ii)
\[ a + a + \cdots + a + b_n v_n = 0 \]
\[ \Rightarrow b_n v_n = 0 \]
\[ \Rightarrow b_n = 0 \quad \therefore v_n \neq 0 \]

Hence the vectors \( v_1, v_2, \ldots, v_n \)
are linearly independent.
Charateristic Polynomial/ Equation/ Matrix:

Let $A$ be an $n$ square matrix over $F$. Then $tI - A$ is called characteristic matrix and $\det(tI - A)$ is called characteristic polynomial.

And $\det(tI - A) = 0$ is called characteristic equation.

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$

$$tI - A = t \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1
\end{bmatrix} - \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$

$$= \begin{bmatrix}
t - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\
-a_{21} & t - a_{22} & -a_{23} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-a_{n1} & -a_{n2} & -a_{n3} & \cdots & t - a_{nn}
\end{bmatrix}$$

And $A(t) = \det(tI - A)$ is characteristic polynomial.

Also $A(t) = 0$ or $|tI - A| = 0$ is characteristic equation.

Exercise:

Find characteristic polynomial of

$$A = \begin{bmatrix}
1 & 3 & 0 \\
-2 & 2 & -1 \\
4 & 0 & -2
\end{bmatrix}$$

$$tI - A = t \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} - \begin{bmatrix}
1 & 3 & 0 \\
-2 & 2 & -1 \\
4 & 0 & -2
\end{bmatrix}$$
Vector Spaces: Handwritten notes

\[ \Delta_A(t) = \det(tI - A) \]

\[
\begin{pmatrix}
1 & -3 & 0 \\
2 & t-2 & 1 \\
-4 & 0 & t+2
\end{pmatrix}
\]

\[ t^3 - t^2 + 2t + 28 \] is characteristic polynomial.

Also, \[ \Delta_A(t) = 0 \]

\[ t^3 - t^2 + 2t + 28 = 0 \] is characteristic equation.

Note: Degree of eq. will be equal to the order of matrix.

Example:

\[ A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \]

\[ tI - A = t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} t-2 & -3 \\ -1 & t-5 \end{pmatrix} \]

\[ B(t) = \text{adj of } (tI - A) = \begin{pmatrix} t-5 & 3 \\ 1 & t-2 \end{pmatrix} \]

\[ = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 1 & -2 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t + \begin{pmatrix} -5 & 3 \\ 1 & -2 \end{pmatrix} = B_1t + B_0. \]

If \[ A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix} \]

Then, \[ B(t) = \text{adj of } (tI - A) \]

\[ = B_2t^2 + B_1t + B_0. \]
Calay Hamilton Theorem:

Every square matrix is zero of its characteristic polynomial.

OR Every square matrix satisfies its characteristic equation.

Proof:

Let $A$ be an $n$ square matrix and $\Delta_A(t) = |tI - A|$ be its characteristic polynomial, i.e. $\Delta_A(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1t + a_0$

Let $B(t)$ be adjoint of $tI - A$.

Since elements of $B(t)$ are cofactors of $tI - A$ and so are polynomial of degree not more than $n-1$, and we can write

$$B(t) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1t + B_0$$

where $B_i$ are square matrices of order $n$ over $F$.

Since by definition of adjoint of a matrix

$$(tI - A)B(t) = |tI - A|I$$

$$\Rightarrow (tI - A)(B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1t + B_0)$$

$$= (t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1t + a_0)I$$

Comparing the co-efficients:

Comparing $t^n \Rightarrow B_{n-1} = I$

Comparing $t^{n-1} \Rightarrow B_{n-2} - AB_{n-1} = a_{n-1}I$

Comparing $t^{n-2} \Rightarrow B_{n-3} - AB_{n-2} = a_{n-2}I$

\[\vdots\]

Comparing $t^1 \Rightarrow B_1I - AB_1 = a_1I$

Comparing $t^0 \Rightarrow -AB = a_0I$.

Multiplying above equations by first to last by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$ respectively, we have.
\[ A^n B_n I = A^n I \]
\[ A^{n-1} B_{n-2} I - A^n B_{n-1} I = a_{n-1} A^{n-1} I \]
\[ A^{n-2} B_{n-3} I - A^{n-1} B_{n-2} I = a_{n-2} A^{n-2} I \]

\[ AB_0 I, A^2 B_1 = a_1 A I \]
\[ AB_0 I = a_0 I \]

Adding both sides of the above equations,

\[ 0 = A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \cdots + a_1 A + a_0. \]

As required.
Vector Spaces: Handwritten notes

# Minimum Polynomial

A polynomial \( m(t) \) is called minimum polynomial if

i) \( m(t) \) divides \( \Delta(t) \)

ii) Each irreducible factor of \( \Delta(t) \) divides \( m(t) \)

iii) \( m(A) = 0 \).

**Question**

\[
A = \begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 2.4
\end{bmatrix}
\]

\[
 tI - A = \begin{bmatrix}
t-2 & -1 & 0 & 0 \\
0 & t-2 & 0 & 0 \\
0 & 0 & t-1 & -1 \\
0 & 0 & 2 & t-4
\end{bmatrix}
\]

\[
|tI - A| = \begin{bmatrix}
t-2 & -1 & 0 & 0 \\
0 & t-2 & 0 & 0 \\
0 & 0 & t-2 & -1 \\
0 & 0 & -2 & t-4
\end{bmatrix}
\]

Expanding by \( e_i \),

\[
\det(tI - A) = (t-2) \begin{vmatrix}
t-2 & 0 & 0 \\
0 & t-2 & -1 \\
0 & 0 & t-4
\end{vmatrix}
\]

\[
= (t-2)(t-2) \begin{vmatrix}
t-2 & -1 \\
2 & t-4
\end{vmatrix}
\]

\[
= (t-2)^3(t-3) = (t-2)^2(t-4) + 2
\]

\[
= (t^2 - 4t + 4)(t^2 - 6t + 8 + 2)
\]

\[
= t^4 - 10t^3 - 4t^2 + 40t + 4 (\text{after solving})
\]

is characteristic polynomial.
Possible minimum polynomials are

\[ i) \quad (t-2)(t-3) = f(t) \]

\[ ii) \quad (t-2)^2(t-3) = g(t) \]

\[ iii) \quad (t-2)^3(t-3) = h(t) \]

\[ f(A) = (A-2)(A-3) \]

\[ = (A-2I)(A-3I) \]

\[ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \neq 0 \]

\[ \Rightarrow f(t) \text{ is not minimum polynomial} \]

Now \( g(t) = (t-2)^2(t-3) \)

\[ \Rightarrow g(A) = (A-2)^2(A-3) \]

\[ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} = 0 \]

\[ \Rightarrow g(t) = (t-2)^2(t-3) \text{ is minimum polynomial} \]

\[ iii) \quad h(A) = (A-2)^3(A-3) \]

\[ \text{Do yourself} \]
Theorem: Prove that the minimum polynomial \( m(t) \) divides every polynomial which has \( A \) as a zero. In particular \( m(t) \) divides the characteristic polynomial \( \Delta(t) \) of \( A \).

Proof:

Let \( f(t) \) be a polynomial for which \( f(A) = 0 \). Then by division algorithm, there are polynomials \( q(t) \) and \( r(t) \) such that:

\[
f(t) = q(t) \cdot m(t) + r(t)
\]

Where \( r(t) \neq 0 \) or degree of \( r(t) \) is less than that of \( m(t) \).

From (i) \( f(A) = q(A) \cdot m(A) + r(A) \) by \( t = A \):

\[
0 = q(A) \cdot 0 + r(A) \Rightarrow r(A) = 0
\]

Then \( r(t) \) is a polynomial of degree less than that of \( m(t) \), which has \( A \) as a zero, which contradict the definition of \( m(t) \).

Hence \( r(t) = 0 \)

\[
f(t) = q(t) \cdot m(t)
\]

i.e \( m(t) \) divides \( f(t) \)

Also then \( m(t) \) divides \( \Delta(t) \)
Theorem

Let \( m(t) \) be the minimum polynomial of an \( n \times n \) square matrix \( A \). Then show that the characteristic polynomial of \( A \) divides \( (m(t))^n \).

Proof:

Let \( m(t) = t^n + c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_{n-1} t + c_n \).

Consider

\( B_0 = I \quad (i) \)

\( B_1 = A + c_1 I \quad (2) \)

\( B_2 = A^2 + c_1 A + c_2 I \quad (3) \)

\( B_3 = A^3 + c_1 A^2 + c_2 A + c_3 I \quad (4) \)

\( \vdots \)

\( B_{r-1} = A^{r-1} + c_1 A^{r-2} + \cdots + c_{r-1} I \quad (r) \)

Take

\( B(t) = t^{r-1} B_0 + t^{r-2} B_1 + t^{r-3} B_2 + \cdots + t B_{r-2} + B_{r-1} \)

Now

\( (tI - A) B(t) = (tI - A)(t^{r-1} B_0 + t^{r-2} B_1 + \cdots + t B_{r-2} + B_{r-1}) \)

\( = t^r B_0 I + t^{r-1} B_1 I + t^{r-2} B_2 I + \cdots + t B_{r-2} I + t B_{r-1} \)

\( - (t^{r-1} A B_0 + t^{r-2} A B_1 + \cdots + t A B_{r-2} + A B_{r-1}) \)

\( = t^r B_0 + t^{r-1} (B_1 - A B_0) + t^{r-2} (B_2 - A B_1) \)

\( + \cdots + t (B_{r-1} - A B_{r-2}) - A B_{r-1} \quad (a) \)

Now from eqs. (i) to (r) gives

\( B_1 - A B_0 = c_1 I \)

\( B_2 - A B_1 = c_2 I \)
\[ B_{r-1} = A B_{r-2} = c_{r-1} I \]

Also from \( r \)th equation,

\[
AB_{r-1} = A^r + c_1 A^{r-1} + \cdots + c_{r-1} A I = m(A) - c_r I
\]

\[ \Rightarrow AB_{r-1} = -c_r I \quad \therefore m(A) = 0. \]

Using all these values in eq. (a),

\[
(tI - A) \cdot B(t) = t^r I + t^{r-1} c_1 I + t^{r-2} c_2 I + \cdots + t c_{r-1} I + c_r I
\]

\[ = (t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + t c_{r-1} + c_r) I \]

Taking determinant on both sides,

\[
| (tI - A) \cdot B(t) | = | (t^r + t^{r-1} c_1 + t^{r-2} c_2 + \cdots + c_r) I | \]

\[ \Rightarrow | tI - A | | B(t) | = (t^{r-n} + c_1 t^{r-1} + c_2 t^{r-2} + \cdots + c_r)^n \]

\[ = (m(t))^n \]

\[ \Rightarrow | tI - A | \text{ divides } (m(t))^n \]

i.e. characteristic polynomial divides \((m(t))^n\)
## Similar Matrix

**def:** A matrix $B$ is similar to a matrix $A$ if there is a non-singular matrix $P$ such that $B = P^{-1}AP$ or $PB = AP$.

## Diagonalization of Matrix

**def:** A matrix $A$ is said to be diagonalizable if there is a matrix such that $B = P^{-1}AP$.

In this case, column of $P$ are eigen vectors of $A$ and diagonal elements of $B$ are corresponding eigen values of $A$.

**Question:** If $A = \begin{pmatrix} 4 & 2 \\ 8 & -1 \end{pmatrix}$ then diagonalize this matrix.

**Solution:**

To find eigen values

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 4 & -2 \\ -3 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 5, -2$$

i) $\lambda = 5$ then for eigen vectors $MX = 0$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 - 2x_2 \\ -3x_1 + 6x_2 \end{pmatrix} = 0$$

One of its solution is $x_2 = 1 \Rightarrow x_1 = 2$

eigen vector $(2, 1)^t$

ii) $\lambda = -2$

$$\Rightarrow MX = 0 \Rightarrow \begin{pmatrix} -6 & -2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$
Vector Spaces: Handwritten notes

\[ \Rightarrow \begin{align*}
-4x - 2y &= 0 \\
-3x - y &= 0
\end{align*} \]

\[ \Rightarrow \begin{align*}
x &= 1 \\
\Rightarrow y &= -3
\end{align*} \]

Eigen vector \( v = (1, -3)^t \)

\[ np = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \]

\[ |np| = -6 - 1 = -7 \]

\[ np^{-1} = \begin{pmatrix} 3/4 & 1/7 \\ -1/7 & -3/7 \end{pmatrix} \]

\[ np^{-1} np = \begin{pmatrix} 3/4 & 1/7 \\ -1/7 & -3/7 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \]

\[ = \begin{pmatrix} 3/4 & 1/7 \\ -1/7 & -3/7 \end{pmatrix} \begin{pmatrix} 10 & -2 \\ 3 & 6 \end{pmatrix} \]

\[ = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \]

is diagonal, where diagonal entries are eigenvalues of \( n \)

Question: Find \( A^{10} \) for \( n = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \)

\[ B = np^{-1} np \]

\[ np^{-1} np = A \]

\[ \Rightarrow A^{10} = (np^{-1} np)^{10} = np^{-1} np^{10} \]

\[ = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}^{10} \begin{pmatrix} 3/4 & 1/7 \\ -1/7 & -3/7 \end{pmatrix} \]

Simplify yourself.
Theorem:

Similar matrices $A$ and $\tilde{P}'AP$ have the same characteristic polynomial.

Proof:

Let $A$ and $B$ are similar matrices then $B = \tilde{P}'AP$

Using

$tI = \tilde{P}'tIP$

$|tI - B| = |tI - \tilde{P}'AP|$

$= |\tilde{P}'tIP - \tilde{P}'AP|$

$= |\tilde{P}'(tI - A)P|$

$= |\tilde{P}'| |tI - A| |P|$

$= |tI - A| |\tilde{P}'| |P|$

$= |tI - A|$

As required