These are the handwritten notes. These notes are lecture delivered by Mr. Tahir Mehmood.

1. Metric space ........................................1
2. Minkowski’s inequality .............................5
3. Open set .............................................7
4. Closed ball ..........................................9
5. Closed set .........................................10
6. Bounded set ........................................11
7. Limit point ..........................................13
8. Closure of a set ....................................14
9. Convergence in metric space and complete metric space ...........................................18
10. Cauchy sequence ....................................19
11. Bounded sequence ..................................20
12. Nested interval property or Cantor’s intersection theorem ...........................................26
13. Continuous function ................................28
14. Topological spaces ..................................38
15. Metric topology, cofinite topology ..........39
16. Open set .............................................41
17. Closed set ..........................................43
18. Closure of a set ....................................44
19. Neighbourhood ......................................48
20. Interior point, exterior point .....................49
21. Boundary point .....................................50
22. Limit point (with respect to topology) ......52
23. Isolated point .......................................62
24. Dense ...............................................63
25. Separable set; Countable set .....................64
26. Base of topology ....................................65
27. Neighbourhood base or local base or base at a point ...............................................71
28. Open cover; Lindelof space .....................73
29. Lindelof theorem ...................................74
30. Relative topology, subspace .....................77
31. Separation axioms; $T_0$-space .................85
32. $T_1$-space ........................................87
33. Subbase; Generation of topologies ..........92
34. $T_2$-space ..........................................93
35. Continuous function (with respect to topologies) .........................................................95
36. Product topology ....................................98
37. Convergence of sequence in topological spaces .........................................................101
38. Regular space .......................................109
39. Completely regular space .........................111
40. Compactness in topological spaces ..........125
41. Homeomorphism ....................................134
42. Countably compact space .........................141
43. Bolzano Weierstrass property ................145
44. Lebesgue number; Big set; Lebesgue covery lemma .....................................................147
45. $\epsilon$-net; Totally bounded ......................149
46. Connected spaces; Disconnected ..........157
47. Component .........................................170
48. Totally disconnected ................................173
49. Separated ..........................................180
50. Normed space ......................................186
51. Uniformly continuous ............................189
52. Closed unit ball; Convex set ...................190
53. Vector space ........................................191
54. Linear combination; Spanning set; Linearly independent ..........................................192
55. Linearly dependent ................................193
56. Linearly independent lemma .................194
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>57. Finite dimensional; Subspace</td>
<td>67. Direct sum</td>
</tr>
<tr>
<td>58. Equivalent norms</td>
<td>68. Orthogonal set; Orthonormal set</td>
</tr>
<tr>
<td>59. Banach space</td>
<td>69. Bessel’s inequality</td>
</tr>
<tr>
<td>60. Reiz Lemma</td>
<td>70. Total orthonormal sets (definition); Parseval’s equality</td>
</tr>
<tr>
<td>61. Hilbert spaces; Inner product spaces</td>
<td>71. Linear Operator; The Kernel or Null space of a linear operator; Continuous linear operator</td>
</tr>
<tr>
<td>63. Cauchy Schewarz inequality</td>
<td>73. Norm of a bounded linear operator</td>
</tr>
<tr>
<td>64. Appalonius identity</td>
<td>74. Linear functionals</td>
</tr>
<tr>
<td>65. Hilbert space; Pythagorian theorem</td>
<td></td>
</tr>
<tr>
<td>66. Minimizing vector</td>
<td></td>
</tr>
</tbody>
</table>

Available at [www.MathCity.org/msc/notes/](http://www.MathCity.org/msc/notes/)

If you have any question, ask at [www.facebook.com/MathCity.org](http://www.facebook.com/MathCity.org)

*MathCity.org* is a non-profit organization, working to promote mathematics in Pakistan. If you have anything (notes, model paper, old paper etc.) to share with other peoples, you can send us to publish on MathCity.org.

For more information visit: [www.MathCity.org/participate/](http://www.MathCity.org/participate/)
**Definition:**

Let \( X \) be a non-empty set. A function \( d : X \times X \to \mathbb{R} \) is said to be metric on \( X \) if, for all \( x, y, z \in X \), it satisfies the following axioms:

- \( M_1): \quad d(x, y) \geq 0 \)
- \( M_2): \quad d(x, y) = d(y, x) \)
- \( M_3): \quad d(x, y) = 0 \iff x = y \)
- \( M_4): \quad d(x, z) \leq d(x, y) + d(y, z) \)

**Example 1:**

Let \( X = \mathbb{R} \) then \( d(x, y) = |x - y| \).

**Solution:**

i) As \( |x - y| \geq 0 \)

\[ d(x, y) \geq 0 \]

ii) \( d(x, y) = |x - y| = |y - x| = d(y, x) \)

iii) \( d(x, y) = 0 \iff |x - y| = 0 \iff x = y \)

iv) \( d(x, z) = |x - z| = |x - y + y - z| \\
    \leq |x - y| + |y - z| \)

\[ d(x, z) \leq d(x, y) + d(y, z) \]

Then \((X, d)\) is a metric space.
EXAMPLE 2: For set \( R^2 = \mathbb{R}(x_1, x_2, y_1, y_2) \),
\[
d((x_3, x_2), (y_1, y_2)) = |x_3 - y_1| + |x_2 - y_2|
\]

SOLUTION:

i) As \( |x_1 - y_1| + |x_2 - y_2| \geq 0 \),
\[
d((x_1, x_2), (y_1, y_2)) \geq 0.
\]

ii) \( d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1 + x_2 - y_2| = d((y_1, y_2), (x_1, x_2)) \)

iii) \( d((x_1, x_2), (y_1, y_2)) = 0 \)
\[
\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0.
\]
\[
\Leftrightarrow |x_1 - y_1| = 0, \quad |x_2 - y_2| = 0.
\]
\[
\Leftrightarrow x_1 = y_1, \quad x_2 = y_2.
\]
\[
\Leftrightarrow (x_1, x_2) = (y_1, y_2).
\]

iv) \( d((x_1, x_2), (z_1, z_2)) = |x_1 - z_1| + |x_2 - z_2| = |x_1 - y_1 + y_1 - z_1 + x_2 - y_2 + y_2 - z_2| = |(x_1 - y_1 + y_1 - z_1) + (x_2 - y_2 + y_2 - z_2)| = (x_1 - y_1 + x_2 - y_2) + (y_1 - z_1 + y_2 - z_2) \)

Available at www.mathcity.org
\begin{align*}
&d((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (y_2, y_2)) + d((y_2, y_2), (z_2, z_2)) \\
&\text{As all the axioms of the metric are satisfied, so } (X, d) \text{ is a metric space.}
\end{align*}

**Example 3:**
Let \((X, d_1)\) and \((X, d_2)\) be two metric spaces defined as:
\[
\frac{1}{2} \sum_{i=1}^{n} d_i(x_i, y_i)
\]
Is \(d_1\) metric on \(X\)?

**Example 4:**
Let \(X = \mathbb{C}\) (complex no.) and \(d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}\) be defined as:
\[
d(z_1, z_2) = |z_1 - z_2|
\]

**Solution:**
\[
i) \quad d(z_1, z_2) = |z_1 - z_2| \geq 0 \\
ii) \quad d(z_1, z_2) = |z_1 - z_2| = \frac{1}{2} \sum_{i=1}^{n} |z_i - z_i| = d(z_2, z_1)
\]
\[
iii) \quad d(z_1, z_2) = 0 \iff |z_1 - z_2| = 0 \iff z_1 = z_2
\]
(iv) Let $x_1, x_2, x_3 \in C$.
\[
d(x_1, x_3) = |x_1 - x_3|
\]
\[
= |x_1 - x_2 + x_2 - x_3|
\]
\[
\leq |x_1 - x_2| + |x_2 - x_3|
\]
\[
\therefore d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)
\]
\[
\Rightarrow (X, d) \text{ is a metric space.}
\]

**Example 5: (Discrete Metric Space).**

Let $X$ be a non-empty set and $d: X \times X \rightarrow \mathbb{R}$ defined as:
\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y
\end{cases}
\]

Then show that $d$ is a metric on $X$.

**Solution:**

i) $d(x, y) = 0 \iff x = y$

and $d(x, y) = 1 \iff x \neq y$

\[
\Rightarrow d(y, y) = 0
\]

ii) $d(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y
\end{cases}
\]

\[
d(y, y) = \begin{cases} 
0 & \text{if } y = x \\
1 & \text{if } y \neq x
\end{cases}
\]

\[
= d(y, x).
\]
iii) \( d(x, y) = 0 \) if \( x = y \) (By definition)

iv) \( d(x, z) \leq d(x, y) + d(y, z) \)

Case I: \( x \neq y \neq z \)

\[ d(x, y) = 1, \quad d(y, z) = 1, \quad d(x, z) = 1 \]

\[ \frac{1}{2} + \frac{1}{2} + 1 \]

\[ \Rightarrow d(x, z) \leq d(x, y) + d(y, z) \to 0 \]

Case II: \( x = y = z \)

\[ d(x, y) = 1, \quad d(y, z) = 0, \quad d(x, z) = 1 \]

\[ \frac{1}{2} = \frac{1}{2} \cdot 0 \]

\[ d(x, z) = d(x, y) + d(y, z) \to 2 \]

Case III: \( x = y \neq z \)

\[ d(x, y) = 0, \quad d(y, z) = 0, \quad d(x, z) = 0 \]

\[ 0 = 0 + 0 \]

\[ d(x, z) = d(x, y) + d(y, z) \to 3 \]

From 1, 2, and 3, we conclude

\[ d(x, z) \leq d(x, y) + d(y, z) \]

Thus, \( d \) is a metric on \( X \).
**Minkowski's Inequality:**

\[
\sum_{i=1}^{\infty} d(x_i - y_i)^p \leq \sum_{i=1}^{\infty} (x_i - y_i)^p + \sum_{i=1}^{\infty} (y_i)^p.
\]

**Open Ball:** (Def)

Let \((X, d)\) be a metric space and \(x \in X\). Then, for \(r \in \mathbb{R}^+\), we define an open ball with center at \(x\) and radius \(r\) as a set consisting of all those points of \(X\) whose distance from \(x\) is less than \(r\). Mathematically, it is defined and denoted as:

\[B(x, r) = \{y \in X : d(x, y) < r\}.
\]

**Example:**

An open ball in a usual metric is an open interval.

**Solution:**

Let \((X, d)\) be the usual metric. Let us consider the open ball

\[S(x, r) = \{y \in X \cap \mathbb{R} : d(x, a) < r\}.
\]

\[= \{y \in \mathbb{R} : |x - a| < r\}.
\]

\[= \{x \in \mathbb{R} : a - r < x < a + r\}.
\]

\[= \{a - r < x < a + r\}.
\]

Which is an open interval.
**Open Set (Def):**

Let \((X,d)\) be the metric space and \(A \subseteq X\). Then, \(A\) is said to be an open set if, for each \(x \in A\), there exist some open ball \(S_y(x)\) such that \(\forall y \in S_y(x) \subseteq A\).

**Theorem:**

Prove that an open ball is an open set.

**Proof:** Let us consider,

\[ S_x(x_0) = \{ x \in X : d(x,x_0) < r \} \]

Let \(x \in S_x(x_0) \Rightarrow d(x,x_0) < r \).

\[ r > d(x,x_0) \]

\[ \Rightarrow r - d(x,x_0) > 0 \]

Put \(r_1 = r - d(x,x_0) > 0\).

Consider \(S_{r_1}(x_0) = \{ y \in X : d(y,x_0) < r_1 \} \).

We now show that \(S_{r_1}(x) \subseteq S_x(x_0)\).

Let \(y \in S_{r_1}(x_0) \Rightarrow d(y,x_0) < r_1 = r - d(x,x_0)\).

\[ d(y,x) + d(x,x_0) < r_1 = r - d(x,x_0) \]

Now, \(d(y,x) \leq d(y,x_0) + d(x,x_0) \leq r_1 + d(x,x_0) = \frac{r}{2} \)

\[ \Rightarrow d(y,x) < \frac{r}{2} \]

\[ y \in S_x(x_0) \]

\[ \Rightarrow S_{r_1}(x) \subseteq S_x(x_0) \]

Hence, \(S_x(x_0)\) is open set.
THEOREM: Let \((X,d)\) be a metric space, then:

i) \(\emptyset\) and \(X\) are open sets.

ii) Union of any number of open sets is open.

iii) Intersection of finite number of open sets is open.

**Proof:**

i) To prove: \(\emptyset\) is open.

For this, we have to prove for each \(x \in \emptyset\), there exists an open ball \(B(x,r)\) such that \(x \in B(x,r) = \emptyset\). But since \(\emptyset\) contains no element, so automatically, it is proved that \(\emptyset\) is open.

Next to prove, \(X\) is open. Let \(x \in X\) and let \(y \in \mathbb{R}^+\), define \(B(x,y) = \{y \in X : d(y,x) < y\}\). Hence, \(X\) is open.

ii) To prove: Union of any number of open sets is open.

Let \(\{U_i : i \in I\}\) be collection of open sets.

To prove: \(U = \bigcup_{i \in I} U_i\) is open.

Let \(x \in \bigcup_{i \in I} U_i\), then \(x \in U_i\) for some \(i \in I\).

As \(x \in U_i\) and \(U_i\) is open, then by definition of open set there exist an open ball \(B(x,r)\) such that...
\( x \in B(x, r) \subseteq U_k. \)
\[ \Rightarrow x \in B(x, r) = U_{k_1}. \]
\[ \Rightarrow U_{k_1} \text{ is open set.} \]

iii) Let \( U_1, U_2, \ldots, U_n \) be the finite collection of open sets.

To prove: \( \bigcap_{i=1}^{n} U_i \) is open.

Let \( x \in \bigcap_{i=1}^{n} U_i \Rightarrow x \in U_i \) for each \( 1 \leq i \leq n \)

\[ \Rightarrow \text{There exist open ball } B(r_2, x_i) \text{ such that } \]
\[ x \in B(r_2, x_i) \subseteq U_i \Rightarrow U_i \text{ is open set.} \]

Let \( r = \min \{ r_1, r_2, \ldots, r_n \} \)

Then for each \( i, 1 \leq i \leq n \)

\[ B(x, r) \subseteq B(x, r_2, x_i). \]
\[ \Rightarrow \text{for each } i, 1 \leq i \leq n \]

\[ x \in B(r, x_i) \subseteq B(x, r_2, x_i) \subseteq U_i. \]
\[ \Rightarrow x \in B(r, x) \subseteq U_i \text{ for each } i \]
\[ \Rightarrow x \in B(r, x) \subseteq \bigcap_{i=1}^{n} U_i. \]

Hence, \( \bigcap_{i=1}^{n} U_i \) is an open set.

**Closed Ball** (Def).

Let \((X, d)\) be a metric space, then for \( x \in X \) and \( r \in \mathbb{R}^+ \), we denote and defined closed ball as:

\[ B(x, r) = \{ y \in X : d(x, y) \leq r \}. \]
CLOSED SET. (DEF.).

Let $(X,d)$ be a metric space and $A \subseteq X$. Then, $A$ is said to be closed if and only if $A'$ is open.

REMARK.

We have just proved that intersection of a finite number of open sets is open. This is not valid for the case of intersection of infinite number of open sets.

For example,

Let $(\mathbb{R}, d)$ be the usual metric and $I_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) : n \in \mathbb{N}$ be an infinite collection of open sets. Then, $\bigcap_{n=1}^{\infty} I_n = \left(0, 1\right] \neq \emptyset$ is not an open set.

DISTANCE OF A POINT FROM A SET. (DEF.).

Let $(X,d)$ be a metric space with $A \subseteq X$ and $a \in X$. Then, the distance of $a$ from $A$ is denoted and defined as:

$$d(a, A) = \inf \{d(a,y) : y \in A\}$$

DIAMETER OF A SET. (DEF.).

Let $(X,d)$ be a metric space and $A \subseteq X$. Then, diameter of $A$ is denoted and defined as:

$$d(A) = \delta(A) = \sup \{d(x,y) : x, y \in A\}$$
**Bounded Set (Def)**

Let $(X,d)$ be the metric space and $A \subseteq X$. Then, $A$ is said to be bounded if and only if the diameter of $A$ is finite.

**Remark:**
Diameter of an empty set is considered as $-\infty$.

**Theorem:**
Diameter of a closed ball is less than or equal to two times of its radius.

**Proof:**
Let $A = \overline{B}(x_0, r) = \{ y \in X : d(x_0, y) \leq r \}$ be a closed ball in metric space $(X,d)$.

Now, $\delta(A) = \operatorname{sup}d(x,y) : x,y \in A^3 \leq 2r$.

**Theorem:**
Prove that the union of two bounded sets is bounded.

**Proof:**
Let $A$ and $B$ be two bounded sets in metric space $(X,d)$.

To prove: $A \cup B$ is bounded.

Since $A$ and $B$ are bounded,
so $\delta(A)$ and $\delta(B)$ are finite.

Let $x,y \in A \cup B$. 

Then, there arises the three cases:

Case I: If \( x, y \in A \),
\[
\text{Then } d(x, y) \leq s(A),
\]
\[
\Rightarrow \sup_{x, y \in A} d(x, y) \leq s(A),
\]
\[
\Rightarrow s(A) \subseteq s(A).
\]

Since \( s(A) \) is finite,
\( s(A) \) is also finite.

Hence, \( A \cup B \) is bounded.

Case II: If \( x, y \in B \),
\[
\text{Then by the similar argument as in Case I, } A \cup B \text{ is again bounded.}
\]

Case III: If \( x \in A \) and \( y \in B \),
\[
\text{Let } a \in A \text{ and } b \in B.
\]
\[
\text{Now } d(x, y) \leq d(x, a) + d(a, y),
\]
\[
\leq d(x, a) + d(a, b) + d(b, y),
\]
\[
\sup_{a \in A} d(x, a) \leq \sup_{a \in A} d(a, b) + \sup_{b \in B} d(b, y),
\]
\[
\Rightarrow s(A) \subseteq s(A) + d(a, b) + s(B).
\]

Since, R.H.S is finite,
\( L.H.S \) is also finite.

Hence, \( A \cup B \) is bounded.
LIMIT POINT (DEF):
Let \((X, d)\) be a metric space and \(A \subseteq X\). Then an element \(x \in X\) is said to be limit point of \(A\) if, and only if, for every open ball \(B(x, \varepsilon)\):
\[
B(x, \varepsilon) \cap A \setminus \{x\} \neq \emptyset
\]
In other words, each open ball \(B(x, \varepsilon)\) contains a point of \(A\) different from \(x\).

The set of all limit points of \(A\) is said to be a derived set and is denoted by \(d(A)\) or \(\text{D}(A)\) or \(Ad\) or \(A_d\).

EXAMPLE: Let \((\mathbb{R}, d)\) be the usual metric and \(A = \{2, 3\}, B = \{1, 2, 3, 4\}\)

SOLUTION:
Consider \(A = \{2, 3\}\).
Let \(x \in A\), i.e., \(x \leq 3\).
Then obviously for any \(\varepsilon > 0\) however small, \(B(x, \varepsilon) \cap A \setminus \{x\} \neq \emptyset\).
So \(x\) is then limit point of \(A\).
Let \(x = 2\), then again for any \(\varepsilon > 0\) however small, \(B(x, \varepsilon) \cap A \setminus \{x\} \neq \emptyset\).
\(\Rightarrow\) \(x = 2\) is limit point of \(A\).
If \(x < 2\) or \(x > 3\), then \(x\) is not limit point of \(A\).
\(\Rightarrow\) if \(x = 1.2\), then \(B(x, 0.1) \cap A \setminus \{x\} = \emptyset\) \(\Rightarrow\) \(D(A) = \{2, 3\}\).
Also \(\text{D}(B) = \emptyset\) because \(B\) is finite set.
DEF: Closure of a Set:
Let \((X,d)\) be a metric space and \(A \subseteq X\). Then, the closure of \(A\) is denoted and defined as:
\[
\bar{A} = \overline{\text{cl}(A)}
\]

THEOREM:
Let \((X,d)\) be the metric space and \(A \subseteq X\), then:

i) \(\bar{A}\) is the intersection of all the closed supersets of \(A\).

ii) \(\bar{A}\) is the smallest closed superset of \(A\).

iii) \(\bar{A}\) is closed.

Proof:
Let us define:
\[
\mathcal{F} = \{F : F \text{ is closed set of } A\}
\]
To prove: \(\bar{A} = \cap \mathcal{F}\).

Let \(x \in A \Rightarrow x \in \overline{\text{cl}(A)}\)
\[
\Rightarrow \{x \in A \text{ or } x \in \overline{\text{cl}(A)}\}
\]

If \(x \in A \Rightarrow x \notin \overline{\text{cl}(A)}\)
If \(x \notin A \Rightarrow x \in \overline{\text{cl}(A)}\) and we have to prove:

Suppose \(x \notin \overline{\text{cl}(A)}\), then \(F \neq \emptyset\) for some \(F \in \mathcal{F}\).
\[
\Rightarrow x \in F^c
\]
As \(F\) is closed \(\Rightarrow F^c\) is open.

Then, by the definition of a closed set, there exist some open ball \(B(x,r)\) such that:

\[
B(x,r) \cap \overline{\text{cl}(A)} = \emptyset
\]
\[ x \in B(x, \delta) \subseteq F' \]

Now as \( F \cap F' = \emptyset \), and \( B(x, \delta) \subseteq F' \) and \( A \subseteq F \),

\[
\Rightarrow B(x, \delta) \cap A = \emptyset.
\]

\[
\Rightarrow B(x, \delta) \cap \overline{A} = \emptyset.
\]

\[
\Rightarrow x \text{ is not a limit point of } A.
\]

\[
\Rightarrow x \notin D(A).
\]

Which is a contradiction:

\[
\Rightarrow x \in D(A).
\]

So, our supposition is wrong.

Hence, \( x \in \overline{A} \Rightarrow A \subseteq \overline{A} \Rightarrow \emptyset \).

Now let \( x \in \overline{A} \).

To prove: \( x \in A = \text{Au}(A) \).

If \( x \in A \), then \( x \in \text{Au}(A) \Rightarrow x \in A \).

If \( x \notin A \), then to prove \( x \in D(A) \).

Suppose \( x \notin D(A) \).

\[
\Rightarrow x \text{ is not the limit point of } A.
\]

Then, there exists some open ball \( B(x, \delta) \)

such that \( B(x, \delta) \cap A - \overline{x} = \emptyset \).

\[
\Rightarrow B(x, \delta) \cap \overline{A} = \emptyset.
\]

\[
\Rightarrow A = (B(x, \delta)).
\]

Since \( B(x, \delta) \) is an open set,

so, \( (B(x, \delta))' \) is a closed set.

\[
\Rightarrow (B(x, \delta))' \text{ is the closed super set of } A.
\]

With \( x \notin B(x, \delta) \), \( x \notin \overline{A} \).

Which is a contradiction:

So, our supposition is wrong.

And hence, \( x \in D(A) \Rightarrow x \in \text{Au}(A) \).

\[
\Rightarrow x \in A \Rightarrow \overline{A} \subseteq A \Rightarrow \emptyset.
\]

\[ \]
ii) To prove: $A$ is the smallest closed superset of $A$. 

Since $A$ is the intersection of all closed subsets of $A$.

As intersection of any number of closed sets is closed, so $A$ is the closed superset of $A$.

Now, we prove $A$ is the smallest such set, let $B$ be another closed superset of $A$.

$B \in Y \Rightarrow \forall Y \subset B \Rightarrow A \in B \Rightarrow A = B$.

Hence $A$ is the smallest closed superset of $A$.

iii) Since $A$ is the intersection of closed sets. Hence $A$ is closed.

Theorem: Let $(X, d)$ be a metric space and $A \subseteq X$. Then, $A$ is open if and only if $A$ is the union of open balls/ spheres.

Proof: Let $(X, d)$ be a metric space and $A \subseteq X$. To prove: $A$ is the union of open spheres.

Let $x$ be an arbitrary point of $A$. Let $x$ be an arbitrary point of $A$. Let
As \( A \) is open, so there exists some open ball \( S_{r_1}(x) \) such that:

\[
\forall x \in S_{r_1}(x) \subseteq A.
\]

Then, \( \bigcup_{x \in A} S_{r_1}(x) = A \).

So, \( \bigcup_{x \in A} S_{r_1}(x) = A \).

Conversely, suppose that \( A \) is the union of open spheres.

To prove: \( A \) is open.

Since each open sphere is an open set in a metric space and the union of any number of open sets is also open.

So, \( A \), being the union of open sets, is open.

**Theorem:**

Let \((X, d)\) be a metric space and \( A = X \). Then a point \( x \in X \) and \( x \notin A \) is called limit point of \( A \) if every open ball containing \( x \) contains infinite number of points of \( A \).

**Proof:**

Suppose \( B(x, 1) \) containing \( x \) contains finite number of points of \( A \), then:

\[
B(x, 1) \cap A = \{ x_1, x_2, \ldots, x_n \}.
\]
\( x \notin A \) \Rightarrow \forall i \in \mathbb{N}, \exists x_i \mid d(x, x_i) = \frac{1}{i} \Rightarrow d(x, x_i) > 0 \quad \forall i = 1, 2, \ldots, n. \\
\Rightarrow d(x, x_i) = h_i. \\
\text{Put } d(x, x_i) = h_i. \\
\text{and let } h^* = \min h_1, h_2, \ldots, h_n. \\
\Rightarrow h^* < h_i \forall i = 1, 2, \ldots, n. \\
\text{Now, consider, } \\
B(x, h^*) = \{ y \in X : d(x, y) < h^* \}. \\
\text{For } y \in B(x, h^*) \Rightarrow d(x, y) < h^* \Rightarrow h_i \leq d(x, y) < h^* \Rightarrow d(x, y) \leq h_i = d(x, x_i). \\
\Rightarrow y \neq x_i \quad i = 1, 2, \ldots, n. \Rightarrow B(x, h^*) \text{ does not contain any point of } A \text{ different from } x. \\
\Rightarrow x \text{ is not the limit point of } A. \\
\text{Which is a contradiction.} \\
\Rightarrow \text{our supposition is wrong.} \\
\Rightarrow \text{And hence, every open ball containing } x \text{ contains infinite number of points of } A. \\

**Convergence in Metric Spaces and Complete Metric Spaces**

**Definition:**

Let \( (x_n) \) be a metric space and \( x_n \) be a sequence in \( X \). Then \( x_n \) converges to a point \( x \) [written as \( x_n \to x \) or \( x_n \xrightarrow{d} x \)] if for every \( \varepsilon > 0 \) there exists a positive integer
\[ d(x_n, x) \leq \varepsilon \] whenever \( n > n_0 \) or in other words:
\[
\lim_{n \to \infty} d(x_n, x) = 0
\]

Then, we write \( \lim_{n \to \infty} x_n = x \).

Then, \( x \) is called the limit point of \( \{x_n\} \).

**Cauchy Sequence (Def.):**

Let \( (X, d) \) be a metric space and \( \{x_n\} \) be a sequence in \( X \). Then, \( \{x_n\} \) is said to be a Cauchy sequence if, for every \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that:
\[
d(x_m, x_n) \leq \varepsilon \text{ whenever } m, n > n_0
\]

**Theorem:**

Prove that every convergent sequence is Cauchy.

**Proof:**

Let \( \{x_n\} \) be a convergent sequence in a metric space \( (X, d) \).

To prove \( \{x_n\} \) is Cauchy in \( X \),

Since \( \{x_n\} \) is convergent in \( X \), to 0,

\[
\text{there exist some } x \in X \text{ such that } x_n \to x.
\]

Then, for every \( \varepsilon > 0 \), there exist some positive integer \( n_0 \) such that:
\[
d(x_n, x) < \frac{\varepsilon}{2} \text{ whenever } n > n_0
\]
Now consider \( m, n \neq \infty \).

Then, \( d(x_m, x) \leq \varepsilon /2 \) and \( d(x_m, x_n) \leq \varepsilon /2 \).

Now, \( d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \).

\[ \Rightarrow \frac{d(x_m, x_n)}{2} \leq \varepsilon /2 + \varepsilon /2 \]

\[ \Rightarrow d(x_m, x_n) \leq \varepsilon (m, n \neq \infty) \]

Hence, sequence is Cauchy in \( X \).

**Remark.**

Converse of the above theorem is not true in general, i.e., there may be a sequence which is Cauchy in \( X \), but is not convergent in \( X \).

E.g., let \( d = ||x-y|| \) and define \( d: X \times X \rightarrow R \) by \( d(x, y) = |x - y| \). Then, \((X, d)\) is metric.

Notice: Consider \( x_n = \frac{1}{n} \).

Then, \( x_n \) is Cauchy in \( X \).

But \( x_n \) is not convergent in \( X \).

\[ \frac{1}{n} \rightarrow 0 \notin X \]

**Bounded Sequence.** (Def.)

A sequence \( x_n \) in metric space \((X, d)\) is said to be bounded if there exists some positive real number \( \lambda \), however large, such that:

\[ d(x_m, x_n) < \lambda \text{ for all } m, n \]

Here, \( x_\infty \) is some fixed element.

Available at

www.mathcity.org
THEOREM:

Let \((X, d)\) be a metric space. Then,

i) Every convergent sequence is bounded and the limit of the convergent sequence is unique.

ii) If \(x_n \to x\) and \(y_n \to y\). Then,
\[
d(x_n, y_n) \to d(x, y).
\]

Proof:

i) a) Since \(x_n\) is convergent, so say, \(x_n \to x \in X\). Then, for every \(\varepsilon > 0\), there exist some positive integer \(n_0\) such that:
\[
d(x_n, x) < \varepsilon \quad \text{whenever} \quad n \geq n_0.
\]

Let \(n = \max \{d(x_{n_1}, x), d(x_{n_2}, x), \ldots, d(x_{n_{-1}}, x), 3\}.

Then, \(d(x_n, x) \leq n + 3\) \(\forall n \geq 1\)
\[
\Rightarrow d(x_n, x) \leq n \quad \forall n \geq 1
\]
\[
\Rightarrow (x_n) \text{ is bounded.}
\]

b) Now, we prove the uniqueness of the limit. On the contrary, say \(x_n \to x\) and \(x_n \to y\).

Then, \(\lim_{n \to \infty} d(x_n, x) = 0\)
\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \text{and} \quad \lim_{n \to \infty} d(x_n, y) = 0.
\]
Now, \(d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0 + 0\)
\[
\Rightarrow d(x, y) \leq 0
\]
\[
\Rightarrow d(x, y) = 0 \iff x = y
\]
Hence, limit of a convergent sequence is unique.

ii) Given \( x_n \to x \) and \( y_n \to y \).
To prove: \( d(x_n, y_n) \to d(x, y) \).

Since, \( x_n \to x \) and \( y_n \to y \).

So, \( \lim_{n \to \infty} d(x_n, y_n) = 0 \) and \( \lim_{n \to \infty} d(x_n, y) = 0 \).

Now, \( |d(x_n, y_n) - d(x, y)| = |d(x_n, y_n) - d(x, y_n) + d(x, y_n) - d(x, y)| \)
\( \geq |d(x_n, y_n) - d(x, y)| \leq |d(x_n, x) + d(y_n, y) + d(x, y)| \to 0 \)
\( \Rightarrow d(x_n, y_n) \to d(x, y) \)

**V-Ind**

**Theorem:**
Let \((X, d)\) be a metric space and \(M \subseteq X\),
then:

i) \( x \in \overline{M} \) if and only if there exist a sequence \( x_n \) in \( M \) such that \( x_n \to x \).

ii) \( M \) is closed if and only if the situation above holds.

Then, \( x \in \overline{M} \).

**Proof:**

i) Suppose \( x \in \overline{M} \Rightarrow x \in \operatorname{cl}(M) \).
\( \Rightarrow x \in M \) or \( x \in \operatorname{cl}(M) \).

If \( x \in M \), then we have the constant sequence \((d(x, x, x, \ldots)) \to x \).
23

If \( x \in M \), then \( x \) is a limit point of \( M \).

\[ \forall \epsilon > 0, \exists y \in M \text{ such that } d(x, y) < \epsilon. \]

Conversely, suppose that there is a sequence \( \{x_n\} \) in \( M \) and a point \( p \) in \( \mathbb{R} \) such that:\n
\[ d(x_n, p) \to 0 \text{ as } n \to \infty. \]

Then \( \{x_n\} \) is a sequence in \( M \) such that \( p \) is a limit point of \( M \).

Let us consider \( N \) to be closed and \( B(x, \epsilon) \) by (ii).

To prove: \( x \in M \).

We wish to consider \( M \) such that \( x \in M \).

\[ \forall \epsilon > 0, \exists y \in M \text{ such that } d(x, y) < \epsilon. \]

\[ \Rightarrow x \in M. \]

\[ \Rightarrow d(x_n, y) < \epsilon \to 0 \text{ as } n \to \infty. \]

This is a sequence \( \{x_n\} \) in \( M \) and a point \( B(A, \epsilon) \).

\[ \Rightarrow d(x_n, p) \to 0 \text{ as } n \to \infty. \]
Since $M$ is closed \& $M = \overline{M}$ \\
\[ \Rightarrow x \in M \]

Conversely, let $\{x_n\}$ be a sequence in $M$, such that $x_n \to x$ and $x \notin M$.

To prove: $M$ is closed.
\[ M = \overline{M} \Rightarrow \]

Let $x \in \overline{M}$, then by part (i) there exist a sequence $\{x_n\}$ in $M$, such that $x_n \to x$.

Then, by given $x \in \overline{M}$, \\
\[ \Rightarrow x \in M \Rightarrow \]

(i) and \[ \Rightarrow M = \overline{M} \]

\[ \Rightarrow M \text{ is closed.} \]

**Theorem:** If a convergent sequence in a metric space has infinitely many distinct points, then its limit is a limit point of the set of the points of the sequence.

**Proof:** Let $(X, d)$ be a metric space and $\{x_n\}$ be a convergent sequence in $X$ with infinitely many distinct points. Let $A$ be the set of points of the sequence and $x_n \to x$.

To prove:
$x$ is the limit point of $A$.

Let us consider any open ball $B(A,r)$. Then, obviously by given condition $B(x, r)$ contains the infinite many distinct points of $A$.

$\Rightarrow x \in \overline{A}$.

**Complete Metric Space:** (Def).

Let $(X, d)$ be a metric space then $X$ is said to be complete metric space if and only if every cauchy sequence in $X$ converges to a point in $X$.

**Theorem:**

A subspace $M$ of a complete metric space $X$ is complete if and only if $M$ is closed in $X$.

**Proof:**

Suppose $M$ is complete.

To prove: $M$ is closed in $X$.

Let $x_n \in M$ be a sequence in $M$ such that $x_n \rightarrow x$.

As every convergent sequence is Cauchy.

So, $x_n \rightarrow x$ in $M$.

As $M$ is complete, so, $x \in M$.

But $x_n \rightarrow x$, so $x \in M$. So, $M$ is closed. (As $M$ is closed if and only if the limit of sequence $x_n$ in $M$ such that $x_n \rightarrow x$. Then $x \in M$).
Conversely, suppose $M$ is closed in $X$.
To prove: $M$ is complete.

Let $\{a_n\}$ be a Cauchy sequence in $M$.
Since, $\{a_n\}$ is a sequence in $M$ and $M = X$.
So $\{a_n\}$ is also a Cauchy sequence in $X$.
As $X$ is complete, so $\lim_{n \to \infty} a_n = x \in X$.

Hence $\{a_n\}$ is a Cauchy sequence in $M$ and $a_n \to x$.
But, $M$ is closed.
Then, by above theorem 1, i.e. $M$.

$\Rightarrow M$ is complete.

**Nested Sequence (Def.)**

Let $(X,d)$ be a metric space and $\{A_n\}$ be a sequence of non-empty subsets of $X$. Then, this sequence is called nested sequence if:

1. $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1}$ for all $n$.
2. $\lim_{n \to \infty} A_n = \emptyset$ when $n \to \infty$.

**Nested Interval Property or Cantor’s Intersection Theorem.**

**Statement:**

Let $(X,d)$ be a complete metric space and $\{F_n\}$ be a decreasing sequence of closed subsets of $X$, such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Then, $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.
Proof:

First, we show that \( \bigcap_{n=1}^{\infty} F_n \neq \emptyset \).

As \( F_n \) is non-empty for all \( n \),

so, let \( x_n \in F_n \) and \( x_m \in F_m \).

Put \( m = \max(m,n) \).

Then, as \( \{F_n\} \) is decreasing so \( F_n = F_m \):

\[ x_m \leq F_m = F_n \]

\[ x_m = F_n \]

Thus, both \( x_m \) and \( x_n \) are in \( F_n \).

\[ d(x_m, x_n) \leq S(F_n) \to 0 \text{ when } n \to \infty \]

\[ d(x_m, x_n) \to 0 \text{ when } n \to \infty \]

\( \{x_n\} \) is a Cauchy sequence in \( X \).

But \( X \) is complete.

As \( X \) is complete, \( \{x_n\} \) converges to a point in \( X \), say \( x_n \to x \in X \).

Now, there arises two cases:

Case I:

If the sequence \( \{x_n\} \) contain finite number of distinct points, then \( x \) is a point which repeats infinitely many times. Then, there exists a positive integer \( n \) such that:

\[ x \in F_{n} \quad \forall n \geq n \]

But as \( \{F_n\} \) is decreasing:

\[ x \in F_n \quad \forall n \]

\[ x \in \bigcap_{n=1}^{\infty} F_n \]

\[ \bigcap_{n=1}^{\infty} F_n \neq \emptyset \]

Case II:
If \( \mathcal{F} \) contains infinite many distinct points, then, \( \exists \) \( x \) is the limit point of the set \( \mathcal{F} \).

Then, using the facts that \( \mathcal{F}_n \) is decreasing and its elements are closed sets, we have:

\[
\forall n, \quad \bigcap_{n=1}^{\infty} \mathcal{F}_n \neq \emptyset.
\]

Now we show that \( \bigcap_{n=1}^{\infty} \mathcal{F}_n \) contains exactly one point.

Consider \( x, y \in \bigcap_{n=1}^{\infty} \mathcal{F}_n \).

\[
\Rightarrow x, y \in \mathcal{F}_n \quad \forall n.
\]

\[
\Rightarrow d(x, y) = s(\mathcal{F}_n) \to 0 \quad \text{when} \quad n \to \infty.
\]

\[
\Rightarrow d(x, y) \to 0 \quad \Rightarrow x = y.
\]

Hence, \( \bigcap_{n=1}^{\infty} \mathcal{F}_n \) contains exactly one point.

**Continuous Function:**

Let \( (X, d_X) \) and \( (Y, d_Y) \) be two metric spaces then a function \( f : X \to Y \) is said to be continuous at a point \( x_0 \in X \), if for every \( \varepsilon > 0 \) there must exist some \( \delta > 0 \) such that:

\[
d_Y(f(x_0), f(x)) < \varepsilon \quad \text{whenever} \quad d_X(x_0, x) < \delta.
\]
THEOREM:

Let \((X, d)\) be a metric space, and \(f: X \to \mathbb{R}\) is defined as \(f(x) = d(x, z)\), where \(z\) is a fixed point of \(X\), then show that \(f\) is continuous on \(X\).

Proof:

Let \(y \in X\) and \(\varepsilon > 0\). Choose \(\delta = \varepsilon\) such that \(d(x, y) < \delta\).

Now, if \(|f(x) - f(y)| = |d(x, z) - d(y, z)| \leq d(x, y) < \delta = \varepsilon\),

\[|f(x) - f(y)| < \varepsilon.

Hence, \(f\) is continuous at \(y\). Since \(y\) is an arbitrary point of \(X\), \(f\) is continuous on \(X\).

Imp. THEOREM:

Let \((X, d_{X})\) and \((Y, d_{Y})\) be two metric spaces and \(f: X \to Y\) is a function then \(f\) is said to be continuous at \(x_0 \in X\) if and only if there exist a sequence \(\{x_n\}\) in \(X\) such that \(f(x_n) \to f(x_0)\) when \(x_n \to x_0\).

Proof:

Suppose \(f: X \to Y\) is continuous at \(x_0 \in X\).

And let \(\{x_n\}\) be a sequence in \(X\) such that \(x_n \to x_0\).

To prove: \(f(x_n) \to f(x_0)\).
Since $f$ is continuous at $x_0$, for every $\varepsilon > 0$, there exist a $\delta > 0$ such that:

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{whenever} \quad d(x, x_0) < \delta \Rightarrow \Box$$

Now as $x_n \to x_0$, $\delta$, for $\varepsilon > 0$, there exist some positive integer $n_0$ such that:

$$d(x_n, x_0) < \delta \quad \text{whenever} \quad n > n_0$$

So by $\Box$, whenever $n > n_0$:

$$d_Y(f(x_n), f(x_0)) < \varepsilon \Rightarrow f(x_n) \to f(x_0)$$

Conversely, suppose $x_n \to x_0$, then $f(x_n) \to f(x_0)$ to prove $f$ is continuous at $x_0$.

Suppose on the contrary, $f$ is discontinuous at $x_0$.

Then for every $\varepsilon > 0$, there exist $\delta > 0$ such that:

$$d(x, x_0) < \delta \text{ but } d_Y(f(x), f(x_0)) > \varepsilon$$

Let $\delta = \frac{1}{n}$.

Then there exist $x_n \in X$ such that:

$$d(x_n, x_0) < \frac{1}{n} \text{ but } d_Y(f(x_n), f(x_0)) > \varepsilon$$

Now as $d(x_n, x_0) < \frac{1}{n}$

So when $n \to \infty$, then $d(x_n, x_0) \to 0$

$$\Rightarrow \quad x_n \to x_0$$

Then, by the hypothesis $f(x_n) \to f(x_0)$. 
But as \( \lim_{x \to a} f(x_n) = \lim_{x \to a} f(x) \),
\[ f(a_n) \neq f(a) \]
Which is a contradiction.
So our supposition is wrong.
Hence, \( f \) is continuous at \( a \).

**Theorem:**
Prove that every singleton set in a metric space is closed.

**Proof:**
Let \( F = \{ x \} \) be a singleton subset of a metric space \( X \).

Let \( y \in F^c \Rightarrow y \notin F \)
\[ \Rightarrow d(x, y) \neq 0 \]
Let \( d(x, y) = r \)
then, \( x \neq b \) if \( b(y) \neq F^c \)
\[ \Rightarrow F^c \text{ is open} \]
\[ \Rightarrow F \text{ is closed} \]

**Theorem:**
Prove that every finite set in a metric space is closed.

**Proof:**
Let \( F \) be a finite subset of a metric space \( X \), then we have to show that \( F \) is closed in \( X \).
Suppose on the contrary that \( F \) is not closed in \( X \), then there must exist at least one limit point \( x \) of \( F \) which is not in \( F \). Let that limit point of \( F \) be \( x \), then each open ball centered at \( x \) must contain infinite number of points of \( F \). But \( F \) is finite.

So, this condition cannot be satisfied. This shows that \( x \) is not a limit point of \( F \).

Which is a contradiction to our assumption that \( x \) is not a limit point of \( F \).

So, \( F \) is closed.

**Theorem:**

Let \((X,d)\) be a metric space and \( A \subseteq X \). Then, \( A \) is closed if and only if \( \overline{D(A)} = A \).

**Proof:**

Suppose \( A \) is closed.

To prove: \( \overline{D(A)} = A \).

Since \( A \) is closed, so \( A' \) is open.

Let \( x \in \overline{D(A)} \).

Then, \( x \) is the limit point of \( A \).

Then, for each open ball \( B(x, \varepsilon) \),

\[ B(x, \varepsilon) \cap A - x \neq \emptyset \]

Now if \( x \in A \), then there is nothing to prove.

Suppose \( x \notin A \Rightarrow A - x \neq A \).

\[ \Rightarrow B(x, \varepsilon) \cap A - x \neq \emptyset \]

\[ \Rightarrow B(x, \varepsilon) \cap A \neq \emptyset \]

\[ \Rightarrow B(x, \varepsilon) \neq A' \]
\[ A' \text{ is not open set} \]
\[
\Rightarrow \ A \text{ is not closed.}
\]
Which is a contradiction.

So our supposition is wrong.

Hence \( x \in A \Rightarrow D(x) = A \).

Conversely, let us suppose \( D(A) = A \).

To prove: \( A \) is closed set. For this, we prove

\( A' \) is open.

Let \( x \in A' \Rightarrow x \notin A \Rightarrow x \notin D(A) \Rightarrow D(x) = A \).

\[ x \text{ is not the limit point of } \ A. \]

Then there exist an open ball \( B(x, r) \) such that:

\[ B(x, r) \cap \overline{A} = \emptyset \]

\[ \Rightarrow B(x, r) \cap A = \emptyset \]

\[ \Rightarrow B(x, r) = A' \]

\[ \Rightarrow x \in B(x, r) = A' \Rightarrow A' \text{ is open} \]

\[ \Rightarrow A \text{ is closed} \]

**Theorem:**

Prove that closed ball in usual metric space is closed.

**Proof:**

Let \( \mathbb{R}^d \) be the usual metric space.

Let us consider the closed ball

\[ S_r(a) = \{ x : x \in \mathbb{R}^d \land d(x, a) \leq r \} \]

\[ = \{ x : x \in \mathbb{R}^d \land |a - x| \leq r \} \]

\[ = \{ x : x \in \mathbb{R}^d \land a - r \leq x \leq a + r \} \]

Which is closed interval.
THEOREM:
Prove that in metric space \((X, d)\), closed ball is a closed set.

Proof:
Let us consider,

\[ S_r[x_0] = \{ x \in X : d(x, x_0) \leq r \} \]

Then we have to show that closed ball \( S_r[x_0] \) is a closed set.

Let \( x \in S_r[x_0] \):

\[ x \notin S_r[x_0] \]

So, \( d(x, x_0) > r \) \( \Rightarrow \) \( d(x, x_0) - r > 0 \)

Let \( x_1 = d(x, x_0) - r \) \( \rightarrow \theta \)

Consider \( S_{x_1}[x] = \{ y \in X : d(y, x) \leq x_1 \} \). We now show that \( S_{x_1}[x] \subseteq S_r[x_0] \).

Let \( y \in S_{x_1}[x] \) \( \Rightarrow \) \( d(y, x) \leq x_1 \)

\[ d(x, x_0) \leq d(x, y) + d(y, x_0) \]

\[ d(x, x_0) \leq x_1 + d(y, x_0) \]

\[ d(y, x_0) > d(x, x_0) - x_1 \] \( \rightarrow \theta \)

\[ d(y, x_0) > d(x, x_0) - d(x, x_0) + x \] (Using \( \theta \))

\[ d(y, x_0) > x \]

So, \( y \notin S_r[x_0] \)

\[ \Rightarrow y \in S_{x_1}[x] \]

\[ \Rightarrow S_{x_1}[x] \subseteq S_r[x_0] \]

\[ \Rightarrow S_r[x_0] \text{ is open} \]
\[
\Rightarrow S_Y[\bar{a}] \text{ is closed.}
\]

Hence, closed balls are closed sets.

**Theorem:**

Prove that in metric space \((X,d)\):

i) \(\emptyset\) and \(X\) are closed sets.

ii) Intersection of any number of closed sets is closed.

iii) Union of finite number of closed sets is closed.

**Proof:**

i) As \(\emptyset\) is pen. So \(\emptyset'\) is closed.

But \(\emptyset' = X \Rightarrow X\) is closed.

As \(X\) is pen, so \(X'\) is closed.

As \(X' = \emptyset' \Rightarrow \emptyset\) is closed.

ii) Let \(\mathcal{A}_x : x \in I_2 \) be any collection of closed sets.

\(\Rightarrow \bigcap_{x \in I_2} \mathcal{A}_x \) is the collection of pen sets in \(X \Rightarrow \bigcup_{x \in I_2} \mathcal{A}_x \) is pen set in \(X\).

\(\Rightarrow (\bigcup_{x \in I_2} \mathcal{A}_x)' \) is closed.

As, \( (\bigcup_{x \in I_2} \mathcal{A}_x)' = \bigcap_{x \in I_2} (\mathcal{A}_x)' = \bigcap_{x \in I_2} \mathcal{A}_x \)

\(\Rightarrow \bigcap_{x \in I_2} \mathcal{A}_x \) is closed set.

iii) Let \(\mathcal{A}_x, \mathcal{A}_x, \ldots, \mathcal{A}_x\) be a finite collection of closed sets.

\(\Rightarrow \bigcap_{x \in I_3} \mathcal{A}_x \) is the collection of pen sets. As intersection of finite number
Theorem:  
\[ C(X,\mathbb{R}) \text{ is the closed.} \]

Proof:  
To prove \[ C(X,\mathbb{R}) = C(X,\mathbb{R}) \]

\[ \forall f : C(X,\mathbb{R}) \subseteq C(X,\mathbb{R}) \implies (\forall) \]  

Now let \( f \in C(X,\mathbb{R}) \). If \( f \in C(X,\mathbb{R}) \) then there is nothing to prove.

If \( f \notin C(X,\mathbb{R}) \) then \( f \) is a limit point of \( C(X,\mathbb{R}) \).

Then for each open ball \( B(f, \varepsilon/3) \):

\[ B(f, \varepsilon/3) \cap C(X,\mathbb{R}) \neq \emptyset \]

\[ \implies B(f, \varepsilon/3) \cap C(X,\mathbb{R}) \neq \emptyset \]

\[ \implies f_1 \in B(f, \varepsilon/3) \cap C(X,\mathbb{R}) \]

\[ \implies f_1 \in B(f, \varepsilon/3) \text{ and } f_1 \in C(X,\mathbb{R}) \]

As \( f_1 \in B(f, \varepsilon/3) \) and \( d(f, f_1) = \inf_{x \in \mathbb{R}} |f(x) - f_1(x)| < \varepsilon/3 \)
\[ \Rightarrow |f_i(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in \mathbb{R}. \]

As \( f_i \in C(X, \mathbb{R}) \), so \( f_i \) is continuous at \( a_0 \in X \). Then, for \( \varepsilon > 0 \) there exist \( S > 0 \) such that:

\[ |f_i(x) - f_i(a_0)| < \frac{\varepsilon}{3} \text{ whenever } d(x, a_0) < S. \]

Now,

\[ |f(x) - f(a_0)| = |f(x) - f_i(x) + f_i(x) - f_i(a_0) - f_i(a_0) + f(a_0) - f(a_0)| \]

\[ \leq |f(x) - f_i(x)| + |f_i(x) - f_i(a_0)| + |f_i(a_0) - f(a_0)| + |f(a_0) - f(a_0)|. \]

\[ \Rightarrow |f(x) - f(a_0)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \]

\[ \Rightarrow f \text{ is continuous on } X. \]

\[ \Rightarrow f \in C(X, \mathbb{R}). \]

\[ \Rightarrow \overline{C(X, \mathbb{R})} \subseteq \overline{C(X, \mathbb{R})} - (*) \]

\( 0 \) and \( 2 \) \( \Rightarrow \overline{C(X, \mathbb{R})} = \overline{C(X, \mathbb{R})}. \)

\[ \Rightarrow C(X, \mathbb{R}) \text{ is closed.} \]
Topological Spaces

Topology is the generalization of the Metric Space. The word Topology is composed of two words.

- **Top** means twisting instruments.
- **Logy** a Latin word means Analysis.

**Definition 1: Topological spaces**

Suppose that $X$ be a non-empty set and $\tau$ be the collection of subsets of $X$, then $\tau$ is called a topology on $X$ if the following axioms are satisfied.

1. $\phi$ and $X$ are in $\tau$.
2. The union of the elements of any sub collection of $\tau$ is in $\tau$.
3. The intersection of the elements of any finite sub collection of $\tau$ is in $\tau$.

We call the set $X$ together with topology $\tau$ is a topological space and denote it $(X, \tau)$.

The subset $A$ of $X$ is an open subset of $X$ if $A \in \tau$, so we can say that a topological space together with its subsets are all open, such that $X$ and $\phi$ are both open and also the infinite union and finite intersection of open sets is also open.

**Examples:**

1. Let $X = \{a, b, c\}$, and consider the collection
   
   $$\tau = \{X, \phi, \{a\}, \{b, c\}\}$$
   
   - $X$ and $\phi$ belongs to $\tau$.
   - The union of any sub collection of $\tau$ belongs to $\tau$.
   - The intersection of finite sub collection of $\tau$ belongs to $\tau$.

   All the three axioms are satisfied, hence $\tau$ is a topology on $X$.

2. Let $X$ be any set, and $P(X)$ called the power set of $X$ consisting of all subsets of $X$ is a topology on $X$. It is called discrete topology.

3. The collection consisting of the set $X$ and empty set only is also a topology on $X$, it is called indiscrete topology or trivial topology.
METRIC TOPOLOGY: (Def).
Let \((X,d)\) be a metric space and 
\(T = \{U \subseteq X: U \text{ is open in } (X,d)\}\). Then \(T\) is 
said to be topology on \(X\) and is called 
metric topology.

EXAMPLE:
Let \(X = \{1, 2, 3\}\), then make all the possible 
topologies on \(X\).

COFINITE TOPOLOGY: (Def).
Let \(X\) be a non-empty set and 
\(\mathcal{F}_c\) be the collection containing \(\emptyset\) and all 
finite subsets of \(X\) whose complements are finite. Then \(\mathcal{F}_c\) is a topology on \(X\). 
it is called cofinite topology.

SOLUTION:
1) \(\emptyset \in \mathcal{F}_c\) (Given).
Also, as \(X' = \emptyset\), which is finite.
So \(X \in \mathcal{F}_c\).

2) Let \(\mathcal{Y}\) be the collection of elements \(y\).
\(\mathcal{F}_c\). To prove: \(U \in \mathcal{F}_c\).
Here arises two cases:

Case I: \(\emptyset \notin \mathcal{Y}\).
Now \((U,\mathcal{Y}) = (U \cup F') = F' \cap \mathcal{F}_c\).
Now here for each \( F \subseteq \Gamma_c \):

\[
\Rightarrow F' \text{ is finite.} \\
\Rightarrow \bigcap F' \text{ is finite.} \\
\Rightarrow (U \upharpoonright)' \text{ is finite.} \\
\Rightarrow U \in \Gamma_c.
\]

Case II. If \( \phi \in \Gamma \),

Then, we have \( \chi \in \Gamma \), such that:

\[
\chi = \chi - \phi \exists \phi.
\]

Then, \( \phi \in \chi \).

Then, by Case I; \( U \chi \in \Gamma_c \).

As \( \chi U = \phi(U \chi) = U \chi \in \Gamma_c \),

\( U \in \Gamma_c \).

iii) Let \( \alpha = F_1, F_2, \ldots, F_n \) be a finite collection of elements of \( \Gamma_c \).

To prove: \( \bigcap_{i=1}^{n} F_i \in \Gamma_c \).

Here arises two cases:

Case I: If \( \phi \notin \alpha \),

Then \( (\bigcap_{i=1}^{n} F_i)' = (\bigcap_{i=1}^{n} F_i)' = \bigcup_{i=1}^{n} F_i' \).

As \( F_i \in \alpha \Rightarrow \phi \notin \Gamma_i \) for each \( i, 1 \leq i \leq n \).
\( \mathcal{F}_i \) is finite.

As finite union of finite sets is finite,

so \( \bigcup_i \mathcal{F}_i \) is finite.

\[ \Rightarrow (\mathcal{N})' \text{ is finite} \Rightarrow m \in \mathcal{F}_c \]

**Case II:** If \( \phi \in \mathcal{N} \).

Then, we can find \( \beta = \alpha - \epsilon / 3 \).

Then \( \phi \subseteq \beta \). Then by Case I: \( \mathcal{N} \subseteq \mathcal{F}_c \).

\[ \forall \alpha, \mathcal{N} \alpha = \phi \cap (\mathcal{N} \beta) \]

\[ = \phi \subseteq \mathcal{F}_c \]

Hence, \( \mathcal{F}_c \) is a topology on \( X \).

**Remarks.**

(i) If \( X \) is finite, then \( \mathcal{F}_0 = \mathcal{F}_c \).

(ii) \( \mathcal{F}_0 \) is a singleton set, then \( \mathcal{F}_0 = \mathcal{F}_c \).

**Open Sets (Def.).**

Let \((X, \mathcal{F})\) be a topological space and \( A \subseteq X \). Then \( A \) is said to be open set, if \( A \in \mathcal{F} \).

**Example:**

If \( X = \{1, 2, 3, 4\} \),

\[ \mathcal{F} = \{\emptyset, X, \{1\}, \{3\}, \{2, 3\}, \{1, 2, 3\}\} \]

is a topology on \( X \).

Then \( \{1, 2, 3\} \) is open set in \((X, \mathcal{F})\).

But \( \{1\}, \{3\}, \{2, 3\}, \{1, 3\} \) are not open sets in \((X, \mathcal{F})\).
THEOREM:
Let \((X, T)\) be a topological space. Then:

i) \(\emptyset\) and \(X\) are open sets.

ii) Union of any number of open sets is open.

iii) Intersection of finite number of open sets is open.

PROOF:

i) \(A \in T \Rightarrow \emptyset \in T\) \(\Rightarrow \emptyset\) is open.

\[A \in T \Rightarrow X \in T\]

\(\Rightarrow X\) is open.

ii) Let \(\mathcal{A}_x : x \in I_3\) be a collection of any number of open sets.

To prove: \(\bigcup \mathcal{A}_x\) is open.

\[A_x \in T\]

\(\Rightarrow A_x \in T\)

\(\Rightarrow \mathcal{A}_x \in T\) \(\because T\) is topology on \(X\).

\(\Rightarrow \bigcup \mathcal{A}_x\) is open.

iii) Let \(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\) be a finite collection of open sets.

To prove: \(\bigcap_{i=1}^{n} A_i\) is open.

\[A_i \in T \Rightarrow A_i \in T\]

\(\Rightarrow \bigcap_{i=1}^{n} A_i \in T\) \(\because T\) is topology on \(X\).

\(\Rightarrow \bigcap_{i=1}^{n} A_i\) is open.

Hence proved.
**Closed Set:** (Def)

Let $(X,T)$ be a topological space and $A \subseteq X$, then $A$ is said to be closed if and only if $A'$ is open.

**Theorem:**

Let $(X,T)$ be a topological space, then:

i) $\emptyset$ and $X$ are closed sets.

ii) Intersection of any number of closed sets is closed.

iii) Union of a finite number of closed sets is closed.

**Proof:**

i) As $\emptyset$ is open, so $\emptyset'$ is closed.

As $\emptyset' = X \Rightarrow X$ is closed.

As $X$ is open, so $X'$ is closed.

As $X' = \emptyset \Rightarrow \emptyset$ is closed.

ii) Let $\{A_i : i \in I\}$ be any collection of closed sets.

Then $\{A_i : i \in I\}$ is the collection of open sets.

$\Rightarrow \bigcup_{i \in I} A_i$ is an open set in $X$.

$\Rightarrow (\bigcup_{i \in I} A_i)'$ is closed.

As $(\bigcup_{i \in I} A_i)' = \bigcap_{i \in I} (A_i)'$, and

$$\bigcap_{i \in I} (A_i) = \bigcup_{i \in I} A_i$$

$\Rightarrow \bigcap_{i \in I} A_i$ is closed set.
iii) Let $A_1, A_2, \ldots, A_n$ be a finite collection of closed sets.

So, $\bigcap_{i=1}^n A_i$ is the finite collection of open sets.

As intersection of finite number of open sets is open, so

$\bigcap_{i=1}^n A_i$ is open.

$\Rightarrow (\bigcap_{i=1}^n A_i)^c$ is closed.

$\Rightarrow \bigcup_{i=1}^n A_i$ is closed.

**Closure of A Set:** (Def)

Let $(X,T)$ be a topological space and $A \subseteq X$. Then, closure of $A$, denoted by $\bar{A}$ is the intersection of all the closed supersets of $A$.

**Example:**

Let $X = \{a, b, c, d\}$, $\emptyset = \emptyset$, $X$, $\{b, c\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$.

$A = \{b, c\}$ then find $\bar{A}$.

**Solution:**

Closed sets are: $X$, $\emptyset$, $\{b, c\}$, $\{b, c, d\}$, $\{a, b, c, d\}$,

Closed supersets of $A$ are: $X$, $\{b, c\}$, $\{b, c, d\}$.

$\bar{A} = X \cap \{b, c, d\} = \{b, c, d\}$.  

Theorem:
Let \((X, \mathcal{T})\) be a topological space. \(A, B \subseteq X\).

Then:
1. \(\overline{A}\) is closed.
2. \(A\) is closed iff \(A = \overline{A}\).
3. \(\emptyset = \emptyset\) and \(X = X\).
4. If \(A \subseteq B\), then \(\overline{A} \subseteq \overline{B}\).
5. \(A \cup B = \overline{A} \cup \overline{B}\).
6. \(A \cap B = \overline{A} \cap \overline{B}\). Also show by example \(\overline{A \cap B} = \overline{A} \cap \overline{B}\).

Proof:
1. \(\overline{A}\) is closed.
   - As \(\overline{A}\) is the intersection of closed sets, (which are also supersets of \(A\)) and intersection of closed sets is closed, so \(\overline{A}\) is closed.

2. \(A\) is closed iff \(A = \overline{A}\).
   - Suppose \(A\) is closed.
     - To prove: \(A = \overline{A}\).
       - As \(\overline{A}\) is the intersection of all the closed supersets of \(A\),
         - \(A \subseteq \overline{A}\) \(\Rightarrow 1\).

       - Now let \(x \in \overline{A}\).
         - Then, \(x\) belongs to each closed superset.
         - \(\exists\ \overline{A}_x\) \(\subseteq A\).
         - As \(A \subseteq \overline{A}_x\) and \(\overline{A}_x\) is closed,
           - \(A\) is the closed superset of \(A\).
So, if \( A \in \mathbb{R} \) then \( \overline{A} \in \mathbb{R} \) \( \Rightarrow \) ii).

i) \( \overline{A} = A \).

ii) \( \overline{\emptyset} = \emptyset \) and \( X = X \).

iii) \( \overline{\emptyset} = \emptyset \) and \( X = X \).

iv) \( \overline{A} \in \mathbb{R} \).

To prove: \( A = B \).

As \( A \subseteq B \) and \( B \subseteq B \),

\( \Rightarrow \) \( A = B \).

Also, as \( B \) is closed, \( B \) is the closed

superset of \( A \).
But \( \overline{A} \) is the smallest closed superset of \( A \):

\[
\overline{A} \subseteq B.
\]

\[
\overline{A} \subseteq \overline{B}.
\]

vi) \( \overline{A \cap B} = \overline{A} \cap \overline{B} \)

\[
\overline{A \cap B} \subseteq \overline{A} \quad \text{and} \quad \overline{A \cap B} \subseteq \overline{B}.
\]

\[
\overline{A} \subseteq \overline{A \cap B} \quad \text{and} \quad \overline{A \cap B} \subseteq \overline{B}.
\]
EXAMPLE:

Show that $A \cap B = \overline{A \cap B}$

Let $A = \{1, 3\}$, $B = \{2, 3\}$,
$X = \{1, 2, 3, 4\}$, $\phi = \{\phi, X\}$

Then, $\overline{A} = X$, $\overline{B} = X$
$\overline{A \cap B} = X \cap X = X \rightarrow (1)$

As $A \cap B = \phi$
And $A \cap B = \phi$
$\phi \rightarrow (2)$

(1) and (2) $\Rightarrow A \cap B = \overline{A \cap B}$

Hence Proved.

NEIGHBOURHOOD (DEF.):

Let $(X, \tau)$ be a topological space
and $x \in X$. Then, a subset $N$ of $X$ is said to be neighbourhood of $x$, if there exists some open set $U$ in $X$ such that $x \in U \subseteq N$.

EXAMPLE:

Let $X = \{1, 2, 3, 4, 5\}$, $\tau = \{\phi, X, \{1, 3\}, \{4, 5\}, \{1, 4, 5\}, \{2\}, \{3\}, \{1, 2, 3\}, \{4, 2, 3\}, \{5, 2, 3\}, N = \{1, 2, 4, 5\}$

SOLUTION:

As $1 \in \{1, 3\} \subseteq N$, $1 \in N$
$N$ is not the neighbourhood of $2$.
As $3$ does not belong to $N$.
$N$ is not the neighbourhood of $3$.
As $4 \in \{1, 4, 5\} \subseteq N$, $4 \in N$
$N$ is the neighbourhood of $1$ and $4$. 
REMARK:

i) If \( N \) is an open set, then, as for each \( x \in N \):
   \[ x \in N \Rightarrow N = N \]  
   So it means \( N \) is the neighbourhood of each of its element. It is called
   \( N \) neighbourhood.

ii) If \( x \in X \), then the set of all neighbourhoods
    of \( x \) is denoted by \( N_x \) and is called
    \( N_x \) neighbourhood system. It is i.e., \( N_x \) is
    always non-negative empty because
    \( X \) is always in \( N_x \).

INTERIOR POINT: (DEF).

Let \((X, \mathcal{T})\) be a topological space

and \( A \subseteq X \). Then, an element \( x \in A \) is said?

1. to be an interior point of \( A \) if there
    exists some open set \( U \) in \( X \) such that
    \( x \in U \subseteq A \). In other words, \( x \) is an interior
    point of \( A \) if and only if \( A \) is \( x \) neighbourhood
    \( x \).

OR:

\( x \) is an interior point of \( A \) if the

there exist an open set \( U \) containing \( x \) such

that \( U \cap A \neq \emptyset \).

EXTERIOR POINT: (DEF).

Let \((X, \mathcal{T})\) be a topological space

and \( A \subseteq X \). Then \( x \in X \) is said to be an

exterior point of \( A \) if \( x \) is an exterior

point of \( A \) i.e., \( x \) is said to be exterior

point of \( A \) if there exist some open set...
A subset of \( \mathbb{R} \) is open if it contains all its limit points.

\[ \forall x \in A, \quad \exists r > 0 \text{ such that } \{ y \in \mathbb{R} : |x-y| < r \} \subseteq A \]

Let \( U \) be a non-empty open set.

**Boundary Point:** (Def.)

Let \( (X, \mathcal{T}) \) be a topological space and \( A \subseteq X \). Then, \( x \in X \) is said to be a boundary point of \( A \) if \( x \) is neither an interior point of \( A \), nor an exterior point of \( A \). In other words, \( x \in X \) is said to be a boundary point of \( A \) if for every open set \( U \) containing \( x \),

\[ U \cap A \neq \emptyset \quad \text{and} \quad U \cap A^c \neq \emptyset \]

**Remark:**

\( A^o \) or \( int(A) \) denotes the set of all interior points of \( A \), \( Ext(A) \) denotes the set of all exterior points of \( A \), \( b(A) \) or \( Fr(A) \) denotes the set of all boundary points of \( A \).

**Example:**

Let \( X = \{ x \in \mathbb{R} : x < 5 \} \)

\[ A = \{ 1, 3, 4, 5 \} \]

and \( A^c = \{ 6, 7, 8, 9 \} \). Then find \( A^o, Ext(A), b(A) \).

**Solution:**

If \( A = \{ 4, 5 \} \) then \( A^o = (1, 3) \cup (6, 9) \).
1. $e \in b(A)$
2. $e \notin \text{Int}(A)$
3. $e \in \text{Int}(A)$
4. $e \in b(A)$

\[ \text{Int}(A) = E_1, E_2 \]
\[ \text{Ext}(A) = \emptyset \]
\[ b(A) = E_1 \cup E_2 \]

i) \[ \text{Int}(A) \cup \text{Ext}(A) \cup b(A) = X \]

ii) \[ \text{Int}(A) \cap \text{Ext}(A) = \emptyset \]
\[ \text{Int}(A) \cap b(A) = \emptyset \]
\[ b(A) \cap \text{Ext}(A) = \emptyset \]

**Theorem:**
Let $(X, T)$ be a topological space and $A \subseteq X$.
Then, $\overline{A} = \text{Ext}(X \setminus \text{Int}(A))$, and each open neighborhood of $A$ intersects $\overline{A}$ or $A = \text{Int}(X)$. For each open set $U$ containing $A$, $U \cap \overline{A} = \emptyset$.

**Proof:**

Let $B = \text{Ext}(X \setminus \text{Int}(A))$, and each open set $U$ containing $A$, $U \cap \overline{A} = \emptyset$.

To show: $\overline{A} = B$.

Let $x \in A$. To show: $x \in B$.
Suppose $x \notin B$. Then there exist some open set $U$ containing $x$ such that $U \cap A = \emptyset$.

$\Rightarrow A \subseteq U'$

As $U$ is open, $U'$ is closed.

$\Rightarrow U'$ is the closed superset of $A$. 

As \( x \in U \), so \( x \notin U' \)

\[ \Rightarrow x \notin A \quad (\because A \text{ is the intersection of all the closed subsets of } A) \]

Which is a contradiction.

\[ \Rightarrow x \notin A \]

So our supposition is wrong.

Hence, \( x \in B \Rightarrow A \subseteq B \Rightarrow \Box \)

Now let \( x \in B \). To prove: \( x \notin A \).

Suppose \( x \notin A \). Then there exists a closed subset \( F \) of \( A \) such that \( x \notin F \).

\[ \Rightarrow x \in F = U \]

Since, \( F \) is closed, \( x \in F = U \) is false.

with \( x \in U \). As \( A = F \Rightarrow A \cap F = \emptyset \Rightarrow A \cup U = \emptyset \).

\[ \Rightarrow \text{There is an open set } U \text{ containing } x \text{ such that } U \cap A = \emptyset \Rightarrow x \notin B \]

Which is a contradiction.

\[ \Rightarrow x \notin B \]

So, our supposition is wrong.

Hence, \( x \notin A \Rightarrow B \subseteq A \Rightarrow \Box \).

\[ \Box \text{ and } \Box \Rightarrow B = A \]

**LIMIT POINT**: (DEF).

Let \((X, T)\) be a topological space and \( A \subseteq X \). Then a point \( x \in X \) is said to be a limit point of \( A \) if for every open set \( U \)
containing \( A \), \( \cap \cup A \neq \emptyset \), i.e., each open set \( U \) containing \( A \) contains at least one point of \( A \) different from \( x \).

The set of all limit points of \( A \) is denoted by \( D(A) \) or \( D(A) \), \( A' \), or \( A^* \), and is called derived set of \( A \).

**Example:**
Let \( X = \{1, 2, 3, 4, 5\} \)
\( F = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)
\( A = \{1, 2, 3, 4\} \)
Find the limit points of \( A \).

**Solution:**
\( x = 1 \) is not the limit point of \( A \)
because for each set \( 1 \), \( 2, 3 \cup A \neq \emptyset \),

\( x = 2 \), open sets containing \( 2 = \{2\} \)
\( X = \{2, 3, 4\} \)
\( 2 \cap A - E 2 \neq \emptyset \),
\( 2, 3 \cap A - E 2 \neq \emptyset \),
\( 2, 3 \cap A - E 2 \neq \emptyset \),

\( x = 2 \) is the limit point of \( A \).

\( x = 3 \) is not the limit point of \( A \)
because for each set \( 3, 4 \cap A - E 2 = \emptyset \),

\( x = 4 \), open sets containing \( 4 = \{4\} \)
\( X = \{2, 3, 4\} \)
\( 2 \cap A - E 4 \neq \emptyset \),

\( 4 \) is the limit point of \( A \).

\( \Rightarrow D(A) = \{2, 3, 4\} \).
REMARK:
For any topological space \((X, T)\) and \(A \subseteq X\), if \(\forall x \in A\) is an open set, then \(x\) is not the limit point of \(A\) because:
\[\exists x \in A \Rightarrow \exists U \subseteq X - \exists x \subseteq U = \emptyset\]

THEOREM:
Let \((X, T)\) be a topological space and \(A \subseteq X\). Then,

1) \(A^o\) is the union of all open sets that are contained in \(A\), or, \(A^o\) is the union of all open subsets of \(A\).

Proof:
Let \(x = \exists V \subseteq V \text{ is an open subset of } A \subseteq \).
To prove: \(A^o = U \subseteq V \).

Let \(x \in U \subseteq V \).
To prove: \(x \in A^o\).

Now, \(x \in U \subseteq V \Rightarrow x \in V \text{ for some } V \subseteq X\).

\[\Rightarrow x \in V \subseteq A \quad \text{(By definition of } A\text{)}\]
\[\Rightarrow x \in A^o\]
\[\Rightarrow U \subseteq V \subseteq A^o \Rightarrow 1\]

Now, let \(x \in A^o\), then there exist some open set \(V \subseteq X\) such that \(x \in V \subseteq A\).

\[\Rightarrow x \in U \subseteq V \subseteq X\]
\[\Rightarrow A^o \subseteq U \subseteq V \Rightarrow 0\]
ii). $A^o$ is the largest open subset of $A$.

Proof: To prove, $A^o$ is the largest open subset of $A$.

Since $A^o$ is the union of all open subsets of $A$, and union of any number of open sets is open and union of subsets is again subset of that set.

So $A^o$ is an open subset of $A$.

Now, let $D$ be another open subset of $A$. Then, by part i, $D \subseteq A^o$.

$\Rightarrow \ D \subseteq U^o$.

$\Rightarrow D \subseteq A^o$ (as $A^o \subseteq U^o$).

$\Rightarrow A^o$ is the largest open subset of $A$.

iii). $A^o$ is an open set. $A^o \subseteq A$.

Proof: Based above in (ii).

iv). $A$ is open if and only if $A = A^o$.

Proof: Let $V = \{V \subseteq X : V$ is open subset of $A^o\}$. Then, $A^o \subseteq U^o$. Let $A$ be an open set. To prove $A = A^o$.

As $A$ is open and $A \subseteq A$

$\Rightarrow A^o \subseteq A$.
\[ A = U \xrightarrow{\phi} A \subseteq A^0 \xrightarrow{\phi} A^0 = A^0 \]

But \( A^0 = A \), so \( \phi \)

And hence \( A = A^0 \).

Conversely, suppose \( A = A^0 \).

To prove: \( A = A^0 \).

As we have already proved that \( A^0 \) is an open set and here as \( A = A^0 \).

So, \( A \) is open.

v) \( (A^0)^0 = A^0 \).

Proof: As \( A^0 \) is an open set (Proved above), so by part (iv) \( (A^0)^0 = A^0 \).

vi) \( \phi^0 = \phi \), \( X^0 = X \).

Proof: As \( \phi \) and \( X \) are open sets, so \( \phi^0 = \phi \) and \( X^0 = X \).

vii) If \( A = B \), then \( A^0 = B^0 \).

Proof: Given \( A = B \). To prove \( A^0 = B^0 \).

Let \( x \in A^0 \), then there exists an open set \( U \) such that \( x \in U \subseteq A^0 \).

\( \Rightarrow x \in U \subseteq A = B \)

\( \Rightarrow x \in U \subseteq B \)

\( \Rightarrow x \in B^0 \)

\( \Rightarrow A^0 = B^0 \).
Viii) \((AnB)^o = A^o \cap B^o\).

Proof: As \(AnB \subseteq A\) and \(AnB \subseteq B\),
\[
(AnB)^o = A^o \cap B^o.
\]
\[
(AnB)^o = A^o \cap B^o \Rightarrow (AnB)^o = A^o \cap B^o.
\]

Now as \(A^o \cap B^o \subseteq A\) and \(A^o \cap B^o \subseteq B\),
\[
A^o \cap B^o = AnB.
\]
\[
A^o \cap B^o = (AnB)^o \Rightarrow \quad (AnB)^o = A^o \cap B^o.
\]

\[
A^o \cap B^o \text{ is an open subset of } AnB.
\]
But \((AnB)^o\) is the largest open subset of \(AnB\).
\[
A^o \cap B^o = (AnB)^o \Rightarrow \quad (AnB)^o = A^o \cap B^o.
\]

ix) \((A\cup B)^o = (A^o \cup B^o)\). Give an example to show:
\[
(A\cup B)^o \neq A^o \cup B^o.
\]

Proof: As \(A \subseteq A\cup B\) and \(B \subseteq A\cup B\),
\[
A^o \subseteq (A\cup B)^o \quad \text{and} \quad B^o \subseteq (A\cup B)^o.
\]
\[
A^o \cup B^o = (A\cup B)^o.
\]

Example:
\[
\begin{align*}
&\text{Let } X = \{a, b, c, d\}; A = \{a, c\}, B = \{b, c, d\}; \\
&\mathcal{P}(X) = \emptyset, X, \{a, c\}, \{b, c, d\}, \{a, b, c, d\}.
\end{align*}
\]

Solution:
\[
A \cup B = \{a, b, c, d\} = X,
\]
\[
(A \cup B)^o = X^o = X
\]
\[
A^o = \{a, c\}, B^o = \{b, c, d\},
\]
\[
A^o \cup B^o = \{a, b, c, d\} = (A \cup B)^o
\]

Hence, proved.
THEOREM:
Let \((X, \tau)\) be a topological space and \(A\) be a closed subset of \(X\). Then, \(A\) is the disjoint union of \(A^o\) and \(\text{b}(A)\).

PROOF:
To prove:
1) \(A = A^o \cup \text{b}(A)\)
2) \(A^o \cap \text{b}(A) = \emptyset\)

\(i)\) Let \(x \in A\). To prove: \(x \in A^o \cup \text{b}(A)\).
If \(x \in A^o\), then \(x \in A^o \cup \text{b}(A)\) and there is nothing to prove.
If \(x \notin A^o\), then to prove \(x \in \text{b}(A)\).
Suppose \(x \notin \text{b}(A)\), then \(x \notin \text{b}(A)\)
\[ \text{Either } x \notin A^o \text{ or } x \in \text{Ext}(A) \]
But \(x \notin A^o\), so \(x \in \text{Ext}(A)\). Then, there exist some open set \(U\) such that \(x \in U \subseteq A^c\).
\[ \Rightarrow x \in A' \]
Which is a contradiction.
\[ \Rightarrow x \in A \]
So, our supposition is wrong.
Hence, \(x \in \text{b}(A)\).
\[ \Rightarrow x \notin A^o \cup \text{b}(A) \Rightarrow A = A^o \cup \text{b}(A) \Rightarrow \Box \]

Now, let \(x \in A^o \cup \text{b}(A)\). To prove, \(x \notin A\).
\[ \Rightarrow x \notin A^o \text{ or } x \notin \text{b}(A) \text{ (Given)} \]
If \(x \notin A^o\) then \(x \notin A^o \subseteq A\), so \(x \notin A\) and there is nothing to prove.
If \(x \notin A^o\), then \(x \in \text{b}(A)\) and to prove \(x \notin A\).
Suppose \( x \notin A \Rightarrow x \in A' \).

Since \( A' \) is closed, so \( A' \) is open.

\[ \Rightarrow x \in A' \subseteq A' \]

\[ \Rightarrow x \in \text{Ext}(A) \Rightarrow x \notin b(A) . \]

Which is a contradiction.

\[ x \in b(A) \]

So, our supposition is wrong.

Hence, \( x \notin A \Rightarrow A^0 \cup b(A) \subseteq A \Rightarrow \emptyset \).

\( \emptyset \) and \( \emptyset \) \( \Rightarrow A = A^0 \cup b(A) . \)

i) To prove: \( A^0 \cap b(A) = \emptyset \).

Let \( A^0 \cap b(A) \neq \emptyset \). Then, there exist an element \( x \in A^0 \cap b(A) \).

\[ \Rightarrow x \in A^0 \text{ and } x \in b(A) \]

If \( x \in A^0 \), then there exist an \( \emptyset \) set \( U \) containing \( x \) such that \( x \in U \subseteq A \Rightarrow UNX' = \phi \).

\[ \Rightarrow x \notin b(A) \ (\because UNX' = \phi) \]

Which is a contradiction.

\[ x \notin b(A) \]

So, our supposition is wrong.

Hence, \( A^0 \cap b(A) = \emptyset \).

Proved.

**Theorem:**

Let \((X, T)\) be a topological space and \( A \subseteq X \). Then:

i) \( A = A^0 \cup b(A) . \)

ii) \( A \) is closed iff \( b(A) \subseteq A \).
Proof:

1. \( A = \text{Au}(A) \)

   Let \( x \in \text{Au}(A) \). To prove: \( x \in \text{Au}(A) \).

   If \( x \in A \), then \( x \in \text{Au}(A) \). Then, there is nothing to prove.

   If \( x \notin A \), then, to prove \( x \in \text{D}(A) \).

   Suppose \( x \notin \text{D}(A) \).

   \[ \iff \text{it is not the limit point of } A. \]

   Then, there exist at least one open set \( U \) containing \( A \) such that \( \text{Un}A \cap U \neq \emptyset \).

   \[ \Rightarrow \text{Un}A = \emptyset \Rightarrow A = U' \]

   As \( x \in U \Rightarrow x \notin U' \).

   As \( U \) is open, \( U' \) is closed.

   \[ \Rightarrow U' \text{ is the closed superset of } A. \]

   As \( x \in U \) and \( x \notin U' \),

   \[ \Rightarrow x \notin A \text{ (since } A \text{ is the intersection of all closed supersets of } A) \]

   Which is a contradiction.

   So, our supposition is wrong.

   Hence, \( x \in \text{D}(A) \Rightarrow x \in \text{Au}(A) \).

   \[ \Rightarrow A = \text{Au}(A) \rightarrow \Box. \]

   Now, let \( x \notin \text{Au}(A) \). To prove: \( x \notin A \).

   As \( x \notin \text{Au}(A) \),

   \[ \Rightarrow x \notin A \text{ or } x \notin \text{D}(A) \.]
If $x \in A \Rightarrow x \in \overline{A}$ (\(= A \subseteq \overline{A}\)).

If $x \notin A$, then $x \not\in D(A)$. To prove: $x \in A$.

Suppose, $x \notin A$.

Then, there exist at least one closed superset $F$ of $A$ such that $x \notin F \Rightarrow x \in F$.

Now, $A \subseteq F \Rightarrow A \cap F = \emptyset$

$\Rightarrow F \cap A = \emptyset$

$\Rightarrow F \cap A \cap D(A) = \emptyset$ (\(= x \notin A\)).

$\Rightarrow x \notin D(A)$.

Which is a contradiction.

So, our supposition is wrong.

Hence, $x \in A \Rightarrow A \cup D(A) = A \Rightarrow (1)$

(1) and (2) $\Rightarrow A = A \cup D(A)$.

1. $A$ is closed iff $D(A) \subseteq A$.

Suppose $A$ is closed.

To prove: $D(A) \subseteq A$.

Let $x \in D(A)$.

$\Rightarrow x \in A \cup D(A)$.

$\Rightarrow x \in A \Rightarrow A$.

As $A$ is closed, $A = A$.

$\Rightarrow x \in A$.

$\Rightarrow D(A) \subseteq A$.

Conversely, suppose that $D(A) \subseteq A$.
To prove: $A$ is closed.

For this we have to show that $A = \overline{A}$.

As $A \subseteq \overline{A}$ \(\Rightarrow 1\)

Let $x \in \overline{A}$

\(\Rightarrow x \in A \cup \text{cl}(A)\)

If $x \in A$, then there is nothing to prove.

If $x \in \text{cl}(A)$, then $x \in A$.

\(\Rightarrow x \in A \Rightarrow 2\)

\(1\) and \(2\) \(\Rightarrow A = \overline{A} \Rightarrow A\) is closed.

**Isolated Point: (Def)**

Let $(X, \tau)$ be a topological space and $A = X$. Then a point $x \in A$ is said to be an isolated point of $A$ if $x$ is not the limit point of $A$, i.e., there exist an open set $U$ containing $x$ such that $\overline{U} \cap X = \{x\}$. The set of all isolated points of $A$ is denoted by $A^*$.

**Theorem:**

Let $(X, \tau)$ be a topological space. Then, any closed subset $A$ of $X$ is the disjoint union of $A^*$ and $\text{cl}(A)$.

**Proof:** It is obvious $A^* \cap \text{cl}(A) = \emptyset$.

Now, we prove that $A = A^* \cup \text{cl}(A)$.

Let $x \in A$; to prove $x \in A^* \cup \text{cl}(A)$.

If $x \in A^*$, then $x \in A^* \cup \text{cl}(A)$ and there is nothing to prove.
If $x \notin A^*$, then to have $x \in D(A)$.
Suppose, $x \notin D(A)$.

Then, $x \notin A^*$.

Which is a contradiction.

\[ x \notin A^* \]

So, our supposition is wrong.

Hence, $x \notin D(A) \Rightarrow x \in A^* \cup D(A)$.

\[ \Rightarrow A = A^* \cup D(A) \rightarrow (1) \]

Conversely, let $x \in A^* \cup D(A)$. To prove, $x \in A$.
As, $x \in A^* \cup D(A) \Rightarrow x \in A^*$ or $x \in D(A)$.

If $x \in A^*$, then $x \in A$ (by definition) and there is nothing to prove.

If $x \notin A^*$, then $x \in D(A)$. To prove, $x \in A$.

Suppose, $x \notin A$, then $x \in A^*$.

Since, $A$ is closed. So, $A^*$ is open. And $x \in A^*$.

Now, $A \cap A^* = \emptyset$.

\[ \Rightarrow A \cap A^* = \emptyset = x \notin D(A) \]

Which is a contradiction.

\[ x \in D(A) \]

So, our supposition is wrong.

Hence, $x \in A \Rightarrow x \in A^* \cup D(A) \Rightarrow A = (2)$.

$(1)$ and $(2) \Rightarrow A = A^* \cup D(A)$.

Hence proved.

**Dense:** (Def.)

Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then, $A$ is called Dense (everywhere dense) in $X$ if $\overline{A} = X$. 
EXAMPLE:
Let $X = \{1, 2, 3, 4, 5\}$.
$A = \{4\}$, $A \subset X$.
Then show that $A$ and $B$ are dense in $X$.

SOLUTION: Let $A = \{1, 3, 4, 5\}$, $B = \{1, 4, 5\}$.
Closed sets are $\emptyset$, $\{1\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{3, 4, 5\}$, $\{1, 3, 4, 5\}$.
Closed subsets of $A$ are $\emptyset$, $A = \{1, 3, 4, 5\}$.
Closed subsets of $B$ are $\emptyset$, $B = \{1, 4, 5\}$, $\{1, 3, 4, 5\}$.
This shows that $A$ is dense in $X$ and $B$ is not dense in $X$.

SEPARABLE: (DEF): Let $(X, \mathcal{A})$ be a topological space. Then it is said to be separable if it has a countable dense set.

COUNTABLE (DEF): Any set $A$ is said to be countable if:

i) It is finite.
ii) It has one-to-one correspondence with the set of natural numbers.

$\{1, 2, 3, \ldots, 10, 3, 2, 1, 4, 3, 2, 3\}$, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are countable.

EXAMPLE:
If: i) $X$ is itself countable then for any topology
$\mathcal{T}$, $(X, \mathcal{T})$ is separable ($X$ is itself
dense in $X$).

ii) Let $X = \mathbb{R}$ (Uncountable set) and $\mathcal{T} = \mathcal{T}_0 = \mathcal{P}(\mathbb{R})$.
Then $(X, \mathcal{T})$ is not separable.

Solution:

Let $X = \mathbb{R}$ and $\mathcal{T} = \mathcal{P}(\mathbb{R})$.
Then $\mathbb{R}$ is the only subset of $\mathbb{R}$ such that
with respect to this $\mathcal{T}$, $\mathbb{R} = \mathbb{R}$. Because for
any proper subset $A$, $\emptyset \subseteq A \subseteq \mathbb{R}$.

Suppose $A$ is closed.

Then $\overline{A} = A$ (i.e., $A$ is closed).

$\Rightarrow \overline{A} \subseteq A$.

$\Rightarrow A = \mathbb{R}$ (i.e., $A = \mathbb{R}$).

$\Rightarrow \mathbb{R}$ is the only dense set in $\mathbb{R}$.
Since $\mathbb{R}$ is uncountable.
Then $(\mathbb{R}, \mathcal{T})$ is not separable.

Base: (Def).
Let $(X, \mathcal{T})$ be a topological space. Then
a collection $B$ of subsets of $X$ is said to be
base for $X$ if:

i) $B \subseteq \mathcal{T}$.

ii) For every $U \in \mathcal{T}$, there is a subfamily
$\mathcal{Y}$ of $B$ such that $U = \bigcup \mathcal{Y}$.
(The second
condition can also be stated as every
$U \in \mathcal{T}$ can be expressed as the union of
some members of $B$).
EXAMPLE: Let \( X = \{1, 2, 3, 4, 5\} \),
\( F = \emptyset, X, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{5\}, \{4, 5\} \)
\( B = \{\emptyset, \{1, 2\}, \{3\}, \{3, 4\}\} \), show that \( B \) is a base.

\( \phi \) \( X \).

SOLUTION: As both conditions of \( B \) as a base are satisfied, \( B \) is a base for \( X \).

REMARK: It is not always necessary that \( \emptyset \in B \) because if \( \emptyset \notin B \), then for every \( \emptyset \notin B \), we consider the empty subfamily \( \{\}\) of \( X \) such that \( \emptyset \in U \).

THEOREM: Let \((X, T)\) be a topological space and
\( B \) be a family of some subsets of \( X \), then
\( B \) is base for \( X \) if and only if

i) \( B = T \)

ii) for every \( U \in T \) and \( \emptyset \subseteq U \), there is \( V \subseteq B \) such that \( \emptyset \subseteq V \subseteq U \).

Proof: Let \( B \) be the base for \( X \). Then,

1) \( B = T \)
2) For every open set \( U \in T \), there is a subfamily \( \{\} \) of \( B \) such that \( U = \cup \).

Now, by (1) and (2), we have to prove

i) and (ii).

i) is same as (i). Now we prove (ii).

Let \( U \in T \) and \( \emptyset \subseteq U \), then by (2), there is a subfamily \( \{\} \) of \( B \) such that \( U = \cup \).

Now, \( \emptyset \subseteq U \Rightarrow \emptyset \subseteq V \Rightarrow \emptyset \subseteq V \) for some \( V \).

\( \Rightarrow \emptyset \subseteq V \subseteq U \Rightarrow V \subseteq B \).
\[ \forall V \in \mathcal{V} \Rightarrow V \subseteq U \Rightarrow U \subseteq \bigcup V = U \Rightarrow \# \cdot V = U \]

Hence, we have \( V \in \mathcal{B} \) such that \( \# \cdot V = U \).

Conversely, let \( \mathcal{U} \) and \( \mathcal{V} \) be \( \mathcal{B} \) such that \( \# \cdot \mathcal{U} = \# \cdot \mathcal{V} = U \).

To prove \( \# \cdot \mathcal{B} \) is true for \( \mathcal{V} \). For this, we have to prove:

(1) \( \mathcal{B} \subseteq \mathcal{F} \).

(2) For every \( U \in \mathcal{F} \), there is a subfamily \( \gamma \subseteq \mathcal{B} \) such that \( U = \# \cdot \gamma \).

(4) is same as (2). To prove (4), let \( U \in \mathcal{F} \) if \( U = \emptyset \). Then, we can consider the empty subfamily \( \gamma \subseteq \mathcal{B} \) such that \( U = \# \cdot \emptyset \).

But if \( U \neq \emptyset \), then, let \( x \in U \). Then, by given ii) there is \( V_x \in \mathcal{B} \) such that \( x \in V_x \subseteq U \).

Then, consider \( \gamma = \{ V_x \in \mathcal{B} : d \cdot V_x \subseteq U \} \).

Then, \( x \in V_x \subseteq U \Rightarrow \exists V_x : V_x \subseteq U \subseteq \gamma \).

\[ \Rightarrow U \subseteq \# \cdot \gamma \subseteq U \subseteq U \Rightarrow \# \cdot \gamma = U \Rightarrow \exists \gamma \subseteq \mathcal{B} \text{ such that } \forall U \in \mathcal{F} \text{ and } \# \cdot \gamma = U \Rightarrow \mathcal{B} \text{ is true for } \mathcal{V} \text{.}

**Theorem:**

Let \( \mathcal{F} \) be a non-empty set. A family \( \mathcal{B} \) of subsets of \( \mathcal{F} \) is true for some topology \( \mathcal{F} \) on \( \mathcal{F} \) if and only if:

i) \( \mathcal{F} \subseteq \mathcal{B} \).

ii) For \( B_1, B_2 \in \mathcal{B} \) and \( d \cdot \mathcal{B}_1 \mathcal{B}_2 \), then there exist \( B_3 \in \mathcal{B} \) such that \( d \cdot B_3 \subseteq B_1 \mathcal{B}_2 \).
Proof: Given \( B \) is base for some topology \( T \) on \( X \).
To prove condition (ii) and (iii) described in the theorem.

i) Let \( x \in X \) (note that \( x \in T \)).
Then there exist some \( B_1 \in B \) such that \( x \in B_1 \subseteq X \).
\[
\Rightarrow \exists x_1 \in B_1 = x \\
\Rightarrow \exists x_1 \in \bigcup_{B_1} B_1 = X \\
\Rightarrow \exists x \in x \in B_1 \subseteq X \\
\Rightarrow \exists x \in \bigcup_{B_1} B_1 = X \\
\Rightarrow X = \bigcup_{B_1} B_1 \\
\]
So, condition (i) is proved.

ii) Now let \( B_1, B_2 \in B \) and \( x \in B_1 B_2 \).
Since \( B \) is base for \( T \), \( B \in T \).
\[
\Rightarrow B_1, B_2 \subseteq T \\
\Rightarrow B_1 B_2 \subseteq T \\
\]
Since \( T \) is topology, \( B_1 B_2 \subseteq T \).
Hence, we have \( B_1 B_2 \subseteq T, \forall x \in B_1 B_2 \).
and \( B \) is base for \( T \).
So there exists \( B_3 \in B \) such that
\[
\Rightarrow x \in B_3 \subseteq B_1 B_2 \\
\Rightarrow \exists x \in B_3 \\
\Rightarrow \exists x \in \bigcup_{B_3} B_3 = X \\
\Rightarrow \exists x \in \bigcup_{B_3} B_3 = X \\
\]
So condition (ii) is proved.

Conversely, let the collection \( B \) of subsets of \( X \) satisfies the two conditions described in the statement of the theorem.

We have to prove that \( B \) is base for some topology \( T \) on \( X \).
Let \( T \) be the collection of all possible
unions of all the subfamily \( U \) of \( B \). Now we prove that \( F \) is topology on \( X \) and \( B \) is the base for this topology.

1) Let \( X \) be the empty subfamily \( F \). Then \( U = \emptyset \). Then by definition of \( F \), \( U = \emptyset \in F \). Now, by given condition (i), there is a subfamily \( B : x \in B_x \) such that \( x \in U \).

Then, by the construction of \( F \),

\[ \bigcup_{x \in B_x} x \in F \] i.e. \( x \in F \).

2) Union of any number of elements of \( F \) is in \( F \). Let \( \{U_x : x \in I\} \) be a collection of elements of \( F \). To prove: \( \bigcup_{x \in I} U_x \in F \).

Since for each \( x \in I \), \( U_x \in F \), so, by the construction of \( F \), there exists some subfamily \( B_x : x \in I \) such that \( U_x = \bigcup_{x \in B_x} x \in F \).

\[ \bigcup_{x \in I} U_x = \bigcup_{x \in B_x} x \in F \] (by construction \( F \)).

\[ \bigcup_{x \in I} U_x \in F \]

3) Intersection of finite number of elements of \( F \) is in \( F \).
Let $U_1, U_2 \in \mathcal{F}$. To prove: $U_1 \cup U_2 \in \mathcal{F}$.

As $U_1 \in \mathcal{F}$, so there exists a subfamily

$B_a : a \in I \text{ s.t. } U_1 = \bigcap_{a \in I} B_a$

As $U_2 \in \mathcal{F}$, so there exists a subfamily

$B_b : b \in I \text{ s.t. } U_2 = \bigcap_{b \in I} B_b$

$U_1 \cup U_2 = (\bigcap_{a \in I} B_a) \cap (\bigcap_{b \in I} B_b)$

$= \bigcap_{a \in I} (B_a \cap B_b)$ (Distributive Law)

Here arises two cases:

a) If $B_a \cap B_b = \emptyset \ \forall a, b$, then

$\Rightarrow U(B_a \cap B_b) = \emptyset$

$\Rightarrow U(\bigcap_{a \in I} B_a) = \emptyset$

$\Rightarrow U_1 \cup U_2 = \emptyset$ (\emptyset in \mathcal{F})

b) If $B_a \cap B_b \neq \emptyset \Rightarrow \exists x \in B_a \cap B_b$.

Then, by given condition, there exists some $B_{x} \in B$ such that $x \in B_x \approx B_a \cap B_b$.

$\Rightarrow B_x = B_a \cap B_b$

Thus, $U \cup U_2 = U(B_x) \in \mathcal{F}$

$\Rightarrow U_1 \cup U_2 \in \mathcal{F}$

$\Rightarrow \mathcal{F}$ is topology for $X$. 
Further, by the construction $B_1^2$, $B$ is base for $X$.

**Neighbourhood Base or Local Base at A Point**: (Def)

Let $(X, \tau)$ be a topological space and $x \in X$. Then, a collection $B_x$ of subsets of $X$ is said to be a local base for $X$ at $x$ if:

1. $B_x \subseteq \tau$.
2. For every $U \in \tau$ such that $x \in U$, there exists some $V \in B_x$ such that $x \in V \subseteq U$.

**Example**: Let $X = \{1, 2, 3, 4, 5, X, Y, Z, 433\}$.

**Solution**: $x = 1$, $B_1 = \{3, 4, 133\}$.

Then $B_1$ is the local base at $x = 1$.

**First Countable Space**: (Def)

A topological space $(X, \tau)$ is said to be first countable space if it has a countable local base at each of its points.

**Second Countable Space**: (Def)

A topological space $(X, \tau)$ is said to be second countable space if it satisfies the 2nd axiom of countability if it has a countable base.

**Examples**:

1. For any set $X$ with any topology $\tau$ on $X$, if $X$ itself is countable, then $(X, \tau)$ is both...
first countable as well as 2nd countable.

2) If \( X \) is uncountable and \( S = \mathbb{R} \times \mathbb{R} \), then \((X, S)\) is both 1st and 2nd countable.

3) If \( X = \mathbb{R} \) (i.e., uncountable) and \( S = \mathbb{R} \cdot \), then space \((X, S)\) is first countable but it is not the 2nd countable space because the base for \((X, S)\) with minimum number of elements is \( B = \{ \rho(x, y) \mid x, y \in X^2 \} \). \( x \in X^2 \) is also uncountable.

**Theorem:** Prove that every second countable space is first countable.

**Proof:** Let \((X, S)\) be a 2nd countable space.

To prove: \( X \) is first countable.

For this, let \( \forall x \) be an arbitrary point of \( X \). We have to prove that this \( X \) has a countable local base for this \( x \).

Since \( X \) is second countable space, so \( X \) has a countable base.

Let \( B = \{ B_n \}_{n \geq 1} \) be a countable base for \( X \).

Put \( B_0 = \{ B_n \} \mid x \in B_n \text{ and } B_0 \in B \} \).

Then, \( B_0 \subseteq B \).

\( \Rightarrow \) \( B_0 \) is countable (since \( B \) is countable).

Further, also \( B_0 \subseteq F \subseteq B_0 \subseteq F \).

Now, let \( U \subseteq F \) such that \( x \in U \). Then as \( B \) is base so there exist \( V \subseteq F \) such that \( x \in V \subseteq U \). Hence, \( B_0 \) is the local base at \( x \).

\( \Rightarrow \) \( X \) is first countable space.
**Remark:** Converse of the above theorem is not true in general, i.e., a first countable space need not be second countable necessarily. E.g. If *X* is an uncountable set and *T* is discrete topology on *X*, then (*X, *T*) is a first countable space because for every *x* ∈ *X* we have countable base *B*ₚ = { {*x*} : *x* ∈ *X* }, but (*X, *T*) is not second countable space because the smallest base for *X* that can be considered is *B* = { {x} : *x* ∈ *X* }, which is not countable.

**Open Cover:** (Def.)

Let (*X, *T*) be a topological space, then a collection *{Uₖ : k ∈ I}*, of open sets in *X* is said to be an open cover for *X* if *X* = ∪ₖ∈I Uₖ.

Since, for each *k* ∈ I, *Uₖ* ⊆ *X*,

so ∪ₖ∈I Uₖ ⊆ *X*. Hence *X* ∈ ∪ₖ∈I Uₖ is an open cover for *X*.

For *X*, then ∪ₖ∈I Uₖ = *X*.

**Example:** Let *X* = {1, 2, 3, 4, 5, 6, 7},

*T* = {∅, *X*, {1}, {2, 3}, {4, 5}, {4, 5, 6}, {4, 5, 6, 7}, {1, 2, 3}, {1, 4, 5}, {1, 2, 3, 4, 5, 6, 7}}.

**Solution:**

*Y* = {1, 5, 6} ∈ *T*.

As *U*₉ = *X*, so *Y* is an open cover for *X*.

As *β* = {1, 5, 6, 7} ∈ *T*,

Also *U*₉ = *X*. Then *β* is an open subcover for *X*.

Every open cover has at least one open subcover which is an open cover itself.

**Lindelöf Space:** (Def.)

A topological space (*X, *T*) is said to
KINDELF: THEOREM

Let $X$ be a second countable space.

If a non-empty set $S$ is separable as a union of countable sets, then $S$ is separable.

Proof:

Let $S_1$ be a collection of open sets in $X$ such that $S = \bigcup S_1$, then $S$ can be separated as a countable union of open sets.

Choose a countable subset $S_2$ of $S_1$.

For each subset $G_i$ of $S_2$, we can choose a countable subset $G_i'$ of $S_2$ such that $G_i' \subseteq G_i$. This is because $S_2$ is a countable set.

Now let $S_3 = \bigcup G_i'$.

Since $S_3$ is a countable union of sets, it is separable.

Now let $S_4 = \bigcup S_2$.

Since $S_4$ is a countable union of sets, it is separable.

Now let $S_5 = \bigcup S_1$.

Since $S_5$ is a countable union of sets, it is separable.

Therefore, $S$ is separable as a countable union of open sets.
This way is countable and its union is $\mathcal{G}$.

**Theorem:** Every separable metric space is second countable.

**Proof:** Let $(X,d)$ be a separable metric space. To prove $X$ is second countable, we need to show that $X$ has a countable dense subset. Let $A$ be a countable dense subset of $X$. Then $A$ is a countable subset of $X$ and $A = X$.

Put $\mathcal{G} = \{B(a, r): a \in A \text{ and } r \in \mathbb{Q}\}$, then it is clear that $\mathcal{G}$ is countable.

Now, we prove that $\mathcal{G}$ is base for $X$.

Clearly, $\emptyset \in \mathcal{G}$.

Further, let $G$ be an open set in $X$, and $y \in G$. Since $G$ is an open set, there exist an open ball $B(y, r)$ such that $y \in B(y, r) \subseteq G$.

Now consider $B(y, r/2)$ as the concentric open ball of $B(y, r)$, then as $A$ is dense in $X$, $B(y, r/2) \cap A \neq \emptyset$.

Let $a \in B(y, r/2) \cap A$.

$\Rightarrow a \in B(y, r/2)$ and $a \in A$.

Let $a \in A$ such that $1/3 < 2r < 2/3$.

We know the claim that $B(a, r) = B(y, r)$.

Let $x \in B(a, r)$, then $d(x, a) < r$.

Now as, $d(x, y) \leq d(x, a) + d(a, y)$

$\leq 2r/3 + 2/3 (\because a \in B(y, r/2) \Rightarrow d(a, y) < r/2)$

$\leq 2r/3 + 2/3 = 2r$. 


Theorem: Let $X$ be an uncountable set with a complete topology. Then $X$ is neither first countable nor second countable.

Proof: Suppose $X$ is first countable. Then, for each $x \in X$, there exists a countable local base for $x$. Let $B = \{B_n\}$ be a countable local base at $x$. Since $B = F_c$ (by definition of local base), so for each $B_n \in B \Rightarrow B_n \in F_c \Rightarrow \forall n, B_n$ is finite, $\Rightarrow \forall n, B_n$ is countable.

Let $G = \bigcup B_n \Rightarrow G$ is countable. As $G$ is countable and $X$ is uncountable (given), so $G = X \setminus G$ is uncountable.

Then, there exists $y \in G'$ such that $x \neq y$.

As $y \in G'$, so $y \in G \Rightarrow y \notin \bigcup B_n$.

$\Rightarrow y \in (\bigcup B_n) ^c \Rightarrow \exists n \in \mathbb{N}, y \notin B_n \Rightarrow y \in B_n$.

Put, $U = X \setminus \{y\}$, Then $U = \{x \mid x \neq y\} \Rightarrow U \in F_c$.

Hence, $U$ is an open set and $x \in U$.

($U = X \setminus \{y\}$ and $x \notin U$).

and $U$ is a local base at $x$. Then, there exist
$B_\in B$, such that $x \in B = U$.

Now as, $y \notin U$ and $B_n \subseteq U \Rightarrow y \notin B_n$.

Which is a contradiction, $y \in B_n \forall n$.

So our supposition is wrong.

Hence, $X$ is not 1st Countable.

Now, suppose $X$ is 2nd countable. Then, by a well known theorem $X$ is first countable.

Which is a contradiction because $X$ is not 1st countable (proved above).

So our supposition is wrong.

Hence, $X$ is not 2nd countable.

PROVED

**Relative Topology**

**SUBSPACE**: (DEF).

Let $(X, \tau)$ be a topological space and $Y$ be a non empty subset of $X$. Then, the collection $\tau_Y = \{ U \cap Y : U \in \tau \}$ is a topology on $Y$ and is called subspace $\tau$ of $(X, \tau)$.

**EXAMPLE**:

Let $X = \{ 1, 2, 3, 4, 5 \}$.

$A = \{ 1 \}, \{ 1, 2 \}, \{ 1, 2, 3 \}, \{ 1, 2, 3, 4 \}, \{ 1, 2, 3, 4, 5 \}$.

Then $(X, \tau)$ be a topological space.

Let $Y = \{ 2, 3, 4 \}$.

Then $\tau_Y = \{ 2 \}, \{ 2, 3 \}, \{ 2, 3, 4 \}, \{ 2, 3, 4, 5 \}$.

**THEOREM**: Every subspace of first countable space is first countable.

**PROOF**: Let $X$ be 1st countable space and $Y$ be a subspace of $X$. 
To prove: \( Y \) is first countable.

Let \( x, y \in Y \). To prove: there exists a countable local base at \( x \) for \( Y \).

Now, as \( x, y \in Y \) and \( Y = X \cdot 0 \), \( x, y \in X \) and \( X \) is first countable. So, \( X \) has a countable local base at \( x \) for \( X \). Let \( B_x \) be the countable local base at \( x \) for \( X \).

Put \( B^*_x = \{ V 
Y : V \in B_x \} \).

Then, \( B^*_x \) is countable (\( B_x \) is countable).

Now, let \( U_Y \) be an open set in \( Y \) and \( x \in U_Y \). Since \( U_Y \) is an open set in \( Y \) and \( Y \) is a subspace of \( X \), then there exist an open set \( U_X \) in \( X \) such that \( U_Y = U_X \cap Y \).

As \( x \in U_Y \Rightarrow x \in U_X \cap Y \).

\( \Rightarrow x \in U_X \) and \( x \in Y \).

Now, as \( x \in U_X \), \( U_X \) is open in \( X \) and \( B_x \) is a local base at \( x \) for \( X \). Then there exists \( V \in B_x \) such that \( x \in V \subseteq U_X \).

As \( x \in V \Rightarrow x \in V \cap Y \subseteq U_X \cap Y = U_Y \).

\( \Rightarrow x \in V \cap Y \subseteq U_Y \). Where \( V \in B^*_x \).

\( \Rightarrow B^*_x \) is local base at \( x \) for \( Y \).

Hence, \( Y \) is first countable.

**Theorem:** Every closed subspace of a Lindelöf space is Lindelöf.

**Proof:** Let \( X \) be a Lindelöf space and \( Y \) be a closed subspace of \( X \).

To prove: \( Y \) is Lindelöf.

Let \( \mathcal{U}_X : x \in I \) be an open cover for \( X \).
\[ y = \bigcup_{i \in I} y \cap y. \] Since for each \( a \in I \), \( U_a \) is an open set in \( Y \), so for each \( a \in I \), there exists an open set \( U_a \) in \( X \) such that \( U_a = V_a \cap Y \).

Further, as \( Y \) is closed in \( X \), so \( Y \) is open in \( X \).

Now as \( U_a = V_a \cap Y \Rightarrow U_a = V_a \) for each \( a \in I \).

\[ Y \cup_{a \in I} Y \cap U_a \Rightarrow \bigcup_{a \in I} Y \cap U_a \Rightarrow \bigcup_{a \in I} Y \cap U_a = \left( \bigcup_{a \in I} Y \right) \cap U_a. \]

\[ \Rightarrow \bigcup_{a \in I} Y \cap U_a = \bigcup_{a \in I} Y \cap U_a \Rightarrow \bigcup_{a \in I} Y \cap U_a = \bigcup_{a \in I} Y \cap U_a. \]

\[ \Rightarrow Y = \bigcup_{a \in I} Y \cap U_a \Rightarrow Y = \bigcup_{a \in I} Y \cap U_a. \]

Since \( X \) is Lindelöf, so this open cover for \( X \) has a countable subcover. Let \( \{ U_n \} \) be a countable subcover for \( X \).

\[ \Rightarrow Y = \bigcup_{n} U_n \cap Y \Rightarrow Y = \bigcup_{n} U_n \cap Y. \]

\[ \Rightarrow Y = \bigcup_{n} U_n \cap Y \Rightarrow Y = \bigcup_{n} U_n \cap Y. \]

\[ \Rightarrow Y = \bigcup_{n} U_n \cap Y \Rightarrow Y = \bigcup_{n} U_n \cap Y. \]

\[ \Rightarrow \text{ is a countable subcover for } Y. \]

Thus, \( Y \) is Lindelöf.

**Theorem:** Every second countable space is Lindelöf.

**Proof:** Let \( X \) be a second countable space.

To prove: \( X \) is Lindelöf.
As $X$ is 2nd countable, so $X$ has a countable base. Let $B = \{B_n\}$ be a countable base for $X$ and let $\tau = \{U_i: i \in I\}$ be an open cover for $X$. Then $X = U_i$.

Let $x \in X \Rightarrow x \in U_i \Rightarrow x \in U_i$ for some $i$. If $x \in \bigcup_{i \in I} U_i$, then there exists some $B_n \in B$ such that $x \in B_n = U_k$.

Since for some $i \in I$, $x \in U_i$ and $B = \{B_n\}$ is a base for $X$, then there exists some $B_n \in B$ such that $x \in B_n = U_k$.

$\Rightarrow \exists \, x \in B_n = U_k$.

$\Rightarrow U_k \ni x \subseteq \bigcup_{i \in I} U_i$.

Now, for each $B_n \in B$, we can find a $U_k \subseteq U_k$ such that $B_n = U_k$ and $x \in U_k$. As $E_i$ is countable, so $E_i$ is also countable.

$\Rightarrow \exists \, U_k \ni x$ is a countable subcover for $X$.

Hence, $X$ is Lindelöf.

**Theorem:** Let $(X, \tau)$ be a topological space and $A \subseteq X$, then $A$ has empty boundary if and only if $A$ is both open and closed.

**Proof:** Suppose $A$ is both open and closed.

To prove: $\text{bd}(A) = \emptyset$.

Since $A$ is open, then $A = A^o$.

Since $A$ is also closed, $A^c$ is open.

$\Rightarrow (A^c)^o = A^c \Rightarrow A^c = \text{Ex}(A)$ ($x$ is an exterior point of $A$ if $x$ is an interior point of $A^c$).
\[ A \cup \text{Ext}(A) = X \quad (\because \quad A \cup A^0 = X) \]

Now, \( b(A) = X \implies \exists A^0 \cup \text{Ext}(A)^3 \)

[Equation]

\[ = X - X = \emptyset \]

Conversely, suppose \( b(A) = \emptyset \)

To prove: \( A \) is both open and closed.

Since \( b(A) = \emptyset \), so \( X = A^0 \cup \text{Ext}(A) \).

Now, let \( x \in A \implies x \in X \implies A = X \).

[Table]

<table>
<thead>
<tr>
<th>\text{Now to show}</th>
<th>A \text{ is closed}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( x \in A^0 \implies x \notin \text{Ext}(A) )</td>
<td>( x \in A \equiv \text{Int}(A) )</td>
</tr>
<tr>
<td>( \implies x \notin \text{Ext}(A) \implies A \subseteq \text{Ext}(A) )</td>
<td>( \text{Ext}(A) \subseteq A \equiv \text{Int}(A) )</td>
</tr>
<tr>
<td>( \implies A^0 \cup \text{Ext}(A) = A )</td>
<td>( A ) is closed</td>
</tr>
<tr>
<td>( \implies A ) is closed</td>
<td></td>
</tr>
</tbody>
</table>

[Theorem]: Prove that every second countable space is separable.

[Proof]:

Hence, proved.
Let $X$ be a second countable space.

To prove: $X$ is separable.

As $X$ is 2nd countable, so $X$ has countable base. Let $B = \{B_n\}^\infty_{n=1}$ be a countable base for $X$.

Put $A = \{x \in X : x_n \in B_n \}$.

then $A \subseteq X \supseteq \emptyset$.

Now let $x \in X$. Further let $U \subseteq X$ such that $x \in U$. Then exists $B_n \in B$ such that $x \in B_n \cup U$ ($\cdot$ $B$ is base for $X$).

Hence, $B_n \cup U \in A$ and $B_n \cup A \neq \emptyset$.

$\Rightarrow UNA \neq \emptyset$.

$\Rightarrow$ For each $x \in X$ containing $x$, $UNA \neq \emptyset$.

$\Rightarrow$ For each $x \in A$, $x \in \text{cl}(A)$

$\Rightarrow$ $X = A \supseteq \emptyset$.

$\Rightarrow A = X$.

Further, $A$ is countable.

So, $A$ is countable.

$\Rightarrow A$ is countable dense in $X$.

$\Rightarrow X$ is separable.

Theorem: Every subspace of a second countable space is second countable.

Proof: Let $X$ be a second countable space and $Y$ be a subspace of $X$.

To prove: $Y$ is second countable.

As $X$ is second countable so $X$ has a countable base $B = \{B_n\}^\infty_{n=1}$.

Let $B^n = \{B_n \cap Y : B_n \in B \}^\infty_{n=1}$.

$\Rightarrow Y$ is second countable.
As \( B \) is countable, so \( B^* \) is also countable.

Now let \( U \) be open in \( Y \). Then, there exist an \( Y \) such that \( U = Y \).

Let \( x \in Y \) such that \( x \in U \).

As \( x \in Y \subseteq X \Rightarrow x \in X \).

Now \( x \in U = Y \).

\( x \in V \) and \( x \in Y \).

As \( x \in V \) and \( B \) is base for \( X \).

So, there exist \( B_n \in B \) such that \( x \in B_n \subseteq V \).

\[ x \in B_n \cap Y \subseteq V \cap Y \]

\[ x \in B_n \cap Y \subseteq U \quad \text{and} \quad B_n \cap Y \subseteq B^* \]

\( B^* \) is base for \( Y \).

\( Y \) is second countable.

Hence Proved.

These Notes are the Lectures delivered by Tahir Mahmood

Available at:

www.mathcity.org
Theorem: Let $S$ be a non-empty collection of subsets of $X$ with $\cup S = X$. Then $S$ is sub-base for some topology on $X$.

Proof: Let $B$ be the collection of all possible finite intersections of members (sub-families) of $S$, i.e.

$$B = \{ B_1 \cap B_2 : B_1, B_2 \in S, 1 \leq n \}.$$ 

Then $X = \cup S \subseteq UB \subseteq X$.

Further let $B_1, B_2 \in B$.

$x \in B_1 \cap B_2$. Since $B_1$ and $B_2$ are finite intersections of sub-families of $S$, so $B_1 \cap B_2$ is finite intersection of some sub-family of $S$.

$x \in B_1 \cap B_2 \Rightarrow x \in B_1 \cap B_2 \in B$. Then $x \in B_2 \subseteq B_1 \cap B_2$

Then, by a theorem of base, $B$ is base for some topology on $X$. 
**Separation Axiom**

**To-Space:** (Def)

A topological space \((X, T)\) is said to be a **To-space** if for each \(x \in X\) such that \(x \notin U\) and \(y \notin U\) or there exists an open set \(V\) such that \(x \in V\) and \(y \notin V\).

**Examples:**
1. Let \(X = \{1, 2, 3\}\) and \(T = \{\emptyset, X, \{1, 3\}, \{2\}\}\). Then \((X, T)\) is a To-space.
2. \((X, T_0)\) is also a To-space.
3. \((X, T_f)\) is not a To-space.

**Theorem:** Every subspace of a To-space is To.

**Proof:** Let \((X, T)\) be a To-space and \(Y\) be a subspace of \(X\). To prove: \(Y\) is To.

Let \(x \in Y \setminus \overline{Y}\). Let \(x \in Y\) such that \(x \notin \overline{Y}\).

Since \(x \in Y\), and \(Y \subseteq X\), \(x \in X\).

Since \(X\) is a To-space, so either there exists an open set \(U\) such that \(x \in U\) and \(y \notin U\) or there exists an open set \(V\) such that \(y \in V\) and \(x \notin V\).

Without any loss of generality, suppose there exists an open set \(U\) such that \(x \in U\) and \(y \notin U\).

Now as \(U\) is open in \(X\), so \(U \cap Y\) is open in \(Y\).

As \(x \in U\) and \(x \in Y\) \(\Rightarrow x \in U \cap Y\).

As \(x \notin U\) \(\Rightarrow y \notin U\).

Hence, we have found open set \(U\) in \(Y\).
such that \( a \in U \) and \( y \notin U \),

\[ \Rightarrow y \text{ is } T_0 \text{ space.} \]

**Theorem:** A topological space \((X, \tau)\) is \( T_0 \) if for every \( a, b \in X \) such that \( a \neq b \) then \( \exists A \neq B \) such that

**Proof:** Suppose \( X \) is \( T_0 \)-space and \( a, b \in X \) such that \( a \neq b \). To prove \( \exists A \neq B \) such that

Since \( X \) is \( T_0 \) and \( a, b \in X \) with \( a \neq b \),

so say, there exists an open set \( U \) in \( X \) such that \( a \in U \) and \( b \notin U \).

As \( a \in U \) and \( b \notin U \), \( U \) is an open set containing \( \mathbf{a} \) such that \( \bigcup \mathbf{b} X = X \)

\[ \Rightarrow a \notin \mathbf{b} \] \( (A=\mathbf{a}, \text{ for each open } \)

\[ \mathbf{b} \]

\[ \Rightarrow \exists A \neq B \]

Further as \( \mathbf{b} \) \( \leq \mathbf{a} \)

\[ \Rightarrow \mathbf{a} \mathbf{b} \neq \mathbf{b} \]

\[ \Rightarrow \mathbf{a} \mathbf{b} \neq \mathbf{b} \]

Conversely, suppose that for every \( a, b \in X \) such that \( a \neq b \),

\[ \Rightarrow \exists A \neq B \] \( \text{ such that } \mathbf{a} \mathbf{b} \neq \mathbf{b} \]

To prove \( X \) is \( T_0 \)-space.

Suppose on the contrary that \( X \) is not \( T_0 \)-space.

Then there is a pair \( a, b \in X \) such that

\[ a \neq b. \] and for every open set \( U \) containing \( a \), also contain \( b \) and for every open set \( V \) containing \( b \) contains \( a \).

Hence, now for every open set \( U \) containing...

\[ a \cup b \cup \phi \Rightarrow a \in \mathcal{E}(\mathcal{E}) \Rightarrow \mathcal{E}(\mathcal{E}) = \mathcal{E}(\mathcal{E}) \]
\[ \Rightarrow \mathcal{E}(\mathcal{E}) \leq \mathcal{E}(\mathcal{E}) \ (\because \mathcal{A} \subseteq \mathcal{A}) \]
\[ \Rightarrow \mathcal{E}(\mathcal{E}) = \mathcal{E}(\mathcal{E}) \ (\because \mathcal{A} = \mathcal{A}) \]

Similarly, \( \mathcal{E}(\mathcal{E}) \leq \mathcal{E}(\mathcal{E}) \)
\[ \Rightarrow \mathcal{E}(\mathcal{E}) = \mathcal{E}(\mathcal{E}) \]

Which is a contradiction.

\[ \Rightarrow \mathcal{E}(\mathcal{E}) \neq \mathcal{E}(\mathcal{E}) \]

So our supposition is wrong.

Hence, \( X \) is \( T_1 \) space.

\textbf{Ti-space: (DEF)}

A topological space \( (X, \mathcal{T}) \) is said to be a \( T_i \)-space if, for every \( x, y \in X \) such that \( x \neq y \), there exists two open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( \bar{U} \cap \bar{V} = \emptyset \).

\textbf{EXAMPLE:} Let \( X = \mathcal{E}(\mathcal{E}) \)
\[ \mathcal{T} = \mathcal{E}(\mathcal{E}), \mathcal{E}(\mathcal{E}) \subset \mathcal{E}(\mathcal{E}) \]

\textbf{THEOREM:}

Every subspace \( Y \) of a \( T_i \)-space is \( T_i \).

\textbf{PROOF:}

Let \( (X, \mathcal{T}) \) be a \( T_i \)-space and \( Y \) be a

subspace of \( X \). To prove: \( Y \) is \( T_i \).

Let \( x, y \in Y \) such that \( x \neq y \).

Now, \( x, y \in Y \) and \( Y \subseteq X \), so \( x, y \in X \).

As \( X \) is \( T_i \) space, so there exists two open sets \( U \) and \( V \) in \( X \) such that \( x \in U \), \( y \in V \) and \( \bar{U} \cap \bar{V} = \emptyset \).

Now as \( U \) and \( V \) are open sets in \( X \), \( U \cup V \) is an open set in \( X \).

As \( x \in U \) and \( y \in V \) \( \Rightarrow x \in U \cup V \) and as \( U \cup V = \bar{U} \cup \bar{V} \).

As \( y \in V \) and \( y \in V \) \( \Rightarrow y \in U \cup V \) and as \( U \cup V = \bar{U} \cup \bar{V} \).
Hence, we have found two open sets \( U \) and \( V \) in \( Y \) such that:
\[ \forall x \in U, \forall y \in V \text{ and } y \in V, x \neq V. \]
Hence \( Y \) is Ti-space.

**THEOREM:** Prove that every Ti-space is To-space.

**Proof:** Let \( X \) be Ti-space. To prove: \( X \) is To-space.

Let \( x \in X \), such that \( x \neq y \).

Since \( X \) is Ti, so there exists two open sets \( U \) and \( V \) in \( X \) such that \( x \in U \), \( y \notin U \) and \( y \in V \), \( x \notin V \).

Since here, we have an open set \( U \) in \( X \) with \( x \in U \), \( y \notin U \Rightarrow X \) is also Ti-space.

**Remark:** Convex of the above theorem is not true in general, i.e., a Ti-space need not necessarily be Ti-space.

**Example:** Let \( X = \mathbb{R}, 0 = \{0\}, 0 = \mathbb{R} \) except \( 0 \).

Hence \( X \) is To-space. But it is not Ti-space.

**Theorem:** A topological space \((X, T)\) is Ti-space iff each singleton subset of \( X \) is closed in \( X \).

**Proof:** Let us suppose \( X \) is Ti-space.

To prove: Each singleton subset of \( X \) is closed.

Let \( x \in X \) be the singleton subset of \( X \).

To prove: \( x \) is closed.

For this, we have \( x \notin x \) is open in \( X \).

Let \( y \in x \Rightarrow y \notin x \) \( x \notin x \), \( y \neq y \).

Hence, we have no \( y \in X \) such that \( x \neq y \) and \( x \).
is $T_{3}$-space. So there exists two open sets $U_{x}$ and $V_{y}$ in $X$ such that $x \in U_{x}$, $y \notin U_{x}$ and $y \in V_{y}$.

Now as $V_{y} \subseteq X$ and $\emptyset \neq V_{y} \Rightarrow V_{y} \subseteq X - E_{x} Y_{x}$.
As $y \in V_{y} \subseteq X - E_{x} Y_{x} \Rightarrow E_{x} Y_{y} \subseteq V_{y} = E_{x} Y_{x}$.

\[ \therefore U \cdot E_{x} Y_{x} \subseteq U \cdot V_{y} \subseteq E_{x} Y_{x}. \]
\[ \forall x \in E_{x} Y_{x}, \forall y \in E_{x} Y_{x}. \]
\[ \therefore E_{x} Y_{x} \subseteq U \cdot V_{y} \subseteq E_{x} Y_{x}. \]
\[ \therefore E_{x} Y_{x} \subseteq U \cdot V_{y}. \]
\[ \therefore E_{x} Y_{x} \subseteq E_{x} Y_{x}. \]

Since $V_{y}$ is open set and union of any number of open sets is open, so $U \cdot V_{y}$ is open.
\[ \therefore E_{x} Y_{x} \text{ is open } \Rightarrow E_{x} Y_{x} \text{ is closed.} \]

Conversely, suppose each singleton subset $y \in X$ is closed. To prove, $X$ is $T_{3}$-space.

For this, let $x, y \in X$ such that $x \neq y$.
Then by supposition, $E_{x} Y_{x}$ and $E_{y} Y_{y}$ are closed.
\[ \therefore E_{x} Y_{x} \text{ and } E_{y} Y_{y} \text{ are open}. \]
\[ \text{Let } U = E_{y} Y_{y} \text{ and } V = E_{x} Y_{x}. \]
\[ \text{Then } x \in U, y \notin U, y \in V, x \notin V. \]
\[ \therefore x \in X \text{ is } T_{3}-\text{space}. \]

**Theorem:** Every finite $T_{3}$-space is discrete.

**Proof:** Let $X$ be a $T_{3}$-space. To prove $X$ is discrete.
For this, we will have to show that each subset of $X$ is closed. Let $A \subseteq X$. 

If \( A = \emptyset \), then \( A \) is closed.
If \( A \neq \emptyset \), then \( A \) contains some elements.
Since \( X \) is finite, so \( A \) is also finite.
\[
\text{Let } A = \{ x_1, x_2, \ldots, x_n \} \implies A = \bigcup_{i=1}^{n} x_i.
\]

Since \( X \) is \( T_1 \), so each singleton subset of \( X \) is closed \( \Rightarrow \) for each \( i, j \leq n, x_i \notin x_j \) is closed.
Since union of a finite number of closed sets is closed, \( \text{so } A = \bigcup_{i=1}^{n} x_i \text{ is closed} \Rightarrow A \text{ is closed.} \)
\[
\implies X \text{ is discrete.}
\]

**Theorem:** A topological space \((X, T)\) is \( T_1 \).
Space \( Y \), each subset of \( X \) is the intersection of all its open supersets.

**Proof:** Let \( X \) be a \( T_1 \)-space and \( A \subseteq X \).
To prove: \( A \) is the intersection of all its open supersets.

Let \( Y \subseteq X \) such that \( Y \cap A \neq \emptyset \implies Y \subseteq A \).
Now, as \( X \) is \( T_1 \)-space, so each singleton subset of \( X \) is closed \( \Rightarrow \) \((Y \cap x) \text{ is closed} \).
\[
\implies \bigcap_{x \in A} x = A = \bigcap_{x \in A} x \subseteq Y \cap A.
\]

Let \( x \in A \implies x \in \bigcup_{x \in A} x \implies x \in \bigcup_{x \in A} x \subseteq Y \cap A \).
Now, we prove \( A = \bigcap_{x \in A} x \subseteq Y \cap A \).

Let \( x \in A \implies x \in Y \implies A \subseteq Y \cap A \).
\[
\implies \bigcap_{x \in A} x = A \subseteq Y \cap A \implies \bigcap_{x \in A} x \subseteq Y \cap A \implies a \text{ is a subset of } Y \cap A \]
Now, let $x \in \bigcup_{y \in A} Y \Rightarrow x \in A$, for all $y \in A$.

$\Rightarrow x \in A \land \forall y \in A \Rightarrow x \in A \land y \in A$.

$\Rightarrow x \in A \Rightarrow x \in A \Rightarrow \bigcup_{y \in A} Y \subseteq A \Rightarrow 0$.

Therefore, $A = \bigcup_{y \in A} Y$.

Hence, $A$ is the intersection of its supersets.

Conversely, suppose that in topological space $(X, T)$ each subset of $X$ is the intersection of its ten supersets. To prove: $X$ is $T_i$.

Suppose $X$ is not $T_i$.

Then, there exists $x \in X$ such that $x \in A$.

And either each open set containing $x$ also contains $y$ or each open set containing $y$ also contains $x$.

Say, each open set containing $x$ also contains $y$.

So, by given condition $Y \subseteq X$ is the intersection of its ten supersets.

$\Rightarrow Y \subseteq X$ is open.

Which is a contradiction. $y \not\in x$.

So, our supposition is wrong.

Hence, $X$ is $T_i$.

**Theorem:** Let $X$ be $T_i$ space and $A \subseteq X$ and $x \in X$ is the limit point of $A$, then every open set containing $x$ contains infinite number of distinct points of $A$.

**Proof:** Suppose given is not true, i.e. each open set containing $x$ contains only finite number of distinct points of $A$. This is a contradiction. Therefore, the given statement is true.
containing \( k \) does not contain infinite number of distinct points of \( A \). Then, there exists an open set \( U \) containing \( k \) which contains finite number of distinct points \( B \) of \( A \). i.e.

\[
U \cap A = \{x_1, x_2, \ldots, x_n \} = B
\]

As \( X \) is \( T_1 \) space and \( B \) is finite subset of \( X \), \( B \) is closed.

\( \Rightarrow B' \) is open as \( x \in B \) so \( x \in B' \).

\( \Rightarrow B' \) is an open set containing \( x \).

Also \( B' \cap A = \emptyset \Rightarrow B' \cap A = \emptyset \Rightarrow x \notin D(A) \)

\( \Rightarrow \) which is a contradiction. \( \Rightarrow x \in D(A) \).

So our supposition is wrong.

And hence, each open set in \( X \) containing \( k \) contains infinite number of distinct points of \( A \).

**Subbase (Def):**

If \((X,T)\) is a topological space. Then a subcollection \( S \) of \( T \) is called subbase for \( T \) if and only if finite intersection of members of \( S \) form a base for \( T \).

**Example:** Let \( X = \{a, b, c, d, e, f\} \)

\( T = \{\emptyset, X, \{a, b, c\}, \{a, b, c, d, e, f\}, \{a, d, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}\} \)

and \( S = \{a, b, c, d, e, f\} \)

Then \( S \) is subbase for \( T \).

Because \( B = \{a, b, c, d, e, f\}, \emptyset, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}. \)

**Generation of Topologies:**

Let \( X \) be a non-empty set and \( S \).
be the collection of some subsets of $X$ including $X$ itself. Then, $S$ is the subbase for some topology on $X$, i.e.,

- $S$ All possible finite intersections of elements of $S$
- $E$ All possible unions of elements of $E$

This procedure is called generation of topologies.

**Remark:** $X$ may not be included in $S$ because in some cases, we take an empty subcollection $E$ of $S$ such that $\mathcal{M} = X$.

For example, let $X = \{1, 2, 3, 4, 5\}$

$$S = \{\{1, 2, 3\}, \{3, 4, 5\}\}$$

$$E = \{\{1, 2, 3\}, \{3, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4, 5\}, \emptyset, X\}$$

$T$ is a $T_2$ space. (Def.)

A $(X, T)$ is called a topological space, then it is said to be $T_2$-space or Hausdorff space if for every $x, y \in X$ such that $x \neq y$, then there exist two open sets $U$ and $V$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

**Example:** Let $X = \{1, 2, 3\}$, $T = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

$$\Rightarrow (X, T) \text{ is a } T_2\text{ space.}$$

**Theorem:** Every $T_2$ space is $T_2$ space.

**Proof:**

Let $X$ be $T_2$ space. To prove: $X$ is $T_2$-space.

Let $x, y \in X$ such that $x \neq y$.

As $X$ is $T_2$, so there exist two open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. 


Now as \( x \in U \) and \( U \cup V = \emptyset \Rightarrow x \not\in V \),

As \( y \in V \) and \( U \cup V = \emptyset \Rightarrow y \not\in U \).

\( \Rightarrow X \) is Ti-space.

**Remark:** Converse of the above theorem is not true in general i.e., a Ti-space is not necessarily a Tα-space.

**Example:** Let \( X = \mathbb{N} \) and \( \mathcal{F} = \mathcal{P}(\mathbb{N}) \).

Now, for every \( x \in X \) such that \( x \neq y \). We have:

\( x \notin \{x\} \cup \{x\} \) and \( y \notin \{y\} \cup \{y\} \).

\( x \notin \{x\} \cup \{x\} \) and \( y \notin \{y\} \cup \{y\} \) are open in \( X \).

Hence, \( X \) is Ti-space.

But \( X \) is not Tα-space.

Because, on the contrary if we suppose that \( X \) is Tα-space, then there exists two open sets \( U \) and \( V \) such that \( x \notin U \), \( y \in V \) and \( U \cup V = \emptyset \).

Now \( U \cup V = \emptyset \Rightarrow (U \cup V)' = \emptyset \).

\( \Rightarrow U' \cup V' = X \Rightarrow U' \cup V' = \mathbb{N} \).

Now as \( (x, \mathcal{F}) \) is cofinite and \( U, V \)

are open in \( X \).

\( \Rightarrow U' \) and \( V' \) are finite.

\( \Rightarrow U' \cup V' \) is finite \( \Rightarrow X = \mathbb{N} \) is finite.

Which is a contradiction.

So our supposition is wrong.

Hence, \( X \) is not Tα-space.

**Theorem:** Every subspace \( Y \) of a Ti-space is Tα-space.

**Proof:** Let \( X \) be a Ti-space and \( Y \) be a subspace of \( X \). To prove: \( Y \) is Tα.
Let $Y \subseteq X$ such that $x \not\in Y$.

As $x \not\in Y$ and $x \in X$, there exist two open sets $U$ and $V$ in $X$ such that $x \in U$, $x \in V$, and $U \cap V = \emptyset$.

Put $U_1 = U \cap Y$ and $V_1 = V \cap Y$.

Then, $U_1$ and $V_1$ are open in $Y$.

As $x \in U_1$, $x \in V_1 \Rightarrow x \in U_1 \cap V_1 = U_1 \cap V_1$.

As $x \in V_1$, $x \in U_1 \cap V_1 \Rightarrow x \in V_1$.

Now, $U_1 \cap V_1 = U_1 \cap V_1 = \emptyset$.

Hence, $Y$ is $T_2$.

**Theorem:** In $T_2$-space, no finite subset has an limit point.

**Proof:**

Let $X$ be a $T_2$-space and $A$ be a finite subset of $X$. Suppose $x \in D(A)$.

Then, each open set $U$ containing $x$ contains an infinite number of distinct points of $A$.

Which is a contradiction.

As it is finite.

So our supposition is wrong.

Hence, in $T_2$-space, a finite set has no limit point.

**Continuous Function:** (Def.)

Let $(X, T_1)$ and $(Y, T_2)$ be the two topological spaces. Then a function $f : X \rightarrow Y$ is said to be continuous at a point $x \in X$ if, for every open set $V$ in $Y$ containing $f(x)$, there exist an open set $U$ in $X$ containing $x$ such that $f(U) \subseteq V$.

$f$ is said to be continuous on $X$ if, it is continuous at each point on $X$. 
**Example:** Let \( X = \{a, b, c\} \), \( Y = \{1, 2, 3\} \), \( f_1 = \{a, b, 1\} \), \( f_2 = \{a, b, 2, 3\} \).

We define a function \( f : X \to Y \) as:

- \( x = a \), \( f(a) = 1 \)
- \( y = 2, 3 \), \( \forall y \in Y \), \( f(y) \in V \)
- \( U = X \), \( a \in X \)
- \( f(U) = f(X) = Y \leq V \)
- \( V = \{2, 3\} \), \( f(a) \in V \), \( U = \{a, b\} \), \( a \in U \)
- \( f(U) = \{2, 3\} \in V \).

Hence, \( f \) is continuous at \( a \). Similarly, we can check \( f \) is continuous at \( x = b, c \).

**Therefore, \( f \) is continuous on \( X \).**

**Theorem:** Let \((X, T_X)\) and \((Y, T_Y)\) be two topological spaces. A function \( f : X \to Y \) is said to be continuous if inverse image of each open set is open.

**Proof:** Suppose \( f \) is continuous on \( X \).

To prove: Inverse image of each open set in \( Y \) is open in \( X \). Let \( G \) be an open set in \( Y \). To prove: \( f^{-1}(G) \) is open in \( X \).

If \( G = \emptyset \), then \( f^{-1}(G) = \emptyset \), which is open in \( X \).

And there is nothing to prove.

If \( G \neq \emptyset \), then \( f^{-1}(G) + \emptyset ) \subseteq f^{-1}(G) \Rightarrow f^{-1}(G) \).

As \( f \) is continuous on \( X \), \( f \) is continuous at \( x \). Then, by the definition of continuity, there exist an open set \( U_x \) in \( X \) such that \( x \in U_x \) and \( f(U_x) \subseteq G \).

\[ \Rightarrow x \in U_x \text{ and } U_x \subseteq f^{-1}(G) \]
\[ \Rightarrow \text{Let } \mathcal{U} \subseteq f^{-1}(G) \]
\[ \Rightarrow \exists \mathcal{U} \subseteq \mathcal{U} \subseteq f^{-1}(G) \]
\[ \Rightarrow u, \exists \mathcal{U} \subseteq \mathcal{U} \subseteq f^{-1}(G) \]
\[ \text{def } f^{-1}(G) \]
\[ \Rightarrow f^{-1}(G) \subseteq \mathcal{U} \subseteq f^{-1}(G) \]
\[ \text{def } f^{-1}(G) \]
\[ \Rightarrow f^{-1}(G) = \mathcal{U} \cup \mathcal{U} = f^{-1}(G) \]

Since \( \mathcal{U} \) is open in \( X \) and union of open sets is open.

\[ \Rightarrow f^{-1}(G) \text{ is open in } X. \]

Conversely, suppose inverse image of each open set in \( Y \) is open in \( X. \)

To prove: \( f \) is continuous on \( X. \)

Let \( x \in X \), to show \( f \) is continuous at \( x. \)

Let \( V \) be an open set in \( Y \) containing \( f(x) \)

\[ \text{i.e. } f(x) \in V \Rightarrow \exists \mathcal{U} : f^{-1}(V) = \mathcal{U} \]

Since \( f^{-1}(V) \) is open in \( X \), and \( V \) is open in \( Y \).

So \( U = f^{-1}(V) \) is open in \( X \) and \( x \in U. \)

Now, as \( V = f^{-1}(V) \Rightarrow f(U) \subseteq V. \)

\[ \Rightarrow f \text{ is continuous at } x \in X. \]

Since, \( x \) is arbitrary, \( f \) is continuous at each point of \( X. \) Hence, \( f \) is continuous on \( X. \)

**THEOREM:** Every metric space is \( T_0 \)-space.

**Proof:** Let \( (X, d) \) be the metric space.

To prove: \( X \) is \( T_0 \)-space.

Let \( x, y \in X \) such that \( x \neq y. \)

\[ d(x, y) > 0 \]

Let \( d(x, y) = \varepsilon > 0. \)
Now consider \( U = B(x, \frac{\varepsilon}{2}) \) and \( V = B(y, \frac{\varepsilon}{2}) \).

\[ \Rightarrow \] \( U \) and \( V \) are open sets in \( X \) (open balls are open sets).

and \( x \in U, y \in V \).

Now to show \( U \cap V \neq \emptyset \).

Suppose, on the contrary that \( U \cap V = \emptyset \).

\[ \Rightarrow \] \( x \notin U \) and \( y \notin V \).

\[ \Rightarrow \] \( d(x, y) > \frac{\varepsilon}{2} \) and \( d(x, y) > \frac{\varepsilon}{2} \).

Now \( d(x, y) < d(x, y) + d(x, y) \).

\[ \Rightarrow \] \( x < y \).

A contradiction.

So our supposition is wrong.

Hence \( U \cap V \neq \emptyset \).

\[ \Rightarrow \] \( x \in \mathbb{R} \) is \( T_2 \) space.

**Product Topology, (Def).**

Let \( (X, \mathcal{T}_X), (Y, \mathcal{T}_Y) \) be two topological spaces and \( X \times Y \) be the cartesian product of \( X \) and \( Y \). Define a subset \( \mathcal{U} \times \mathcal{V} \) of \( X \times Y \) to be open in \( X \times Y \) if \( U \in \mathcal{T}_X \) and \( V \in \mathcal{T}_Y \). Then the class \( \mathfrak{B} \) of all subsets \( \mathcal{U} \times \mathcal{V} \) of \( X \times Y \) is the base for the topology \( \mathcal{T} \) on \( X \times Y \), called the product topology on \( X \times Y \).

**Example:** Let \( X = \{1, 2, 3\} \), \( Y = \{a, b, c, d\} \), \( X \times Y = \{1a, 1b, 1c, 1d, 2a, 2b, 2c, 2d, 3a, 3b, 3c, 3d\} \), \( \mathcal{T}_X = \{\emptyset, X\} \), \( \mathcal{T}_Y = \{\emptyset, Y\} \), \( \mathfrak{B} = \{1a, 1b, 1c, 1d, 2a, 2b, 2c, 2d, 3a, 3b, 3c, 3d\} \).
\[ T = \{ U \subseteq X : X \text{ is a subfamily of } \mathcal{B} \} \]

**THEOREM:** The following statements about the topological space are equivalent:
\[ i) \text{ } X \text{ is } T_2 \text{ space. } \]
\[ ii) \text{ } \text{The diagonal } D = \{(x, x) : x \in X\} \text{ is closed in } X \times X. \]

**PROOF:** 
\[ i) \implies ii)\text{ } \] i.e., we assume that \( X \) is a \( T_2 \) space and prove that \( D \) is closed in \( X \times X \).

For this, we have that \( D' \) is open in \( X \times X \).

Let \( (x, y) \in D' \implies x \neq y \).
Here, we have \( x, y \in X \) such that \( x \neq y \) and \( X \) is a \( T_2 \) space. So, there exists two open sets \( U_x \) and \( U_y \) such that \( x \in U_x \) and \( y \in U_y \) and \( U_x \cap U_y = \emptyset \).

Now, let \( (U, V) \in U \times V \).
\[ \implies (U, V) \cap \emptyset = \emptyset \implies U \cap V = \emptyset \implies (U, V) \in D' \]
\[ \implies U_x \times V_y \subseteq D' \]
\[ \implies (x, y) \in D' \]
\[ \implies D' \subseteq U_x \times V_y \subseteq D' \]
\[ \implies D' = U_x \times V_y \]

Now, as \( U_x \) and \( V_y \) are open in \( X \) so, \( U_x \times V_y \) is open in \( X \times X \) and \( \bigcup \) any union of any number of open sets is open. So \( D' \) is open.
\[ \implies D \text{ is closed.} \]
ii) \( \Rightarrow \) i) i.e., here we assume that \( D \) is closed in \( XXX \) and we have to prove that \( X \) is T₃-space.

Let \( x, y \in X \) such that \( x \neq y \) and \( (x, y) \in D \).

Now as \( D \) is an open set in \( XXX \), so, there exists an open set \( U \times V \) in \( XXX \) such that \( (x, y) \in U \times V \).

\[ \exists \ z \in U \text{ and } y \in V. \]

Now to prove \( U \times V = \emptyset \),

Suppose on the contrary, \( U \times V \neq \emptyset \).

Let \( z \in U \times V \) and \( z = (z_1, z_2) \).

\[ \Rightarrow (z_1, z_2) \in U \times V. \]

\[ \Rightarrow (z_1, z_2) \in D \] (\( : \, U \times V \subseteq D \))

\[ \Rightarrow z_1 = z \]

Which is a contradiction.

\[ \Rightarrow U \times V = \emptyset. \]

So our supposition is wrong.

Hence, \( U \times V = \emptyset \).

\[ \Rightarrow X \text{ is T}_3 \text{-space}. \]

**Theorem:** Let \( (X, \tau) \) and \( (Y, \tau') \) be two topological spaces. Then a function \( f : X \to Y \) is said to be continuous on \( X \) if and only if inverse image of each closed set is closed.

**Proof:**

Suppose \( f \) is continuous and \( A \) is closed in \( Y \).

\[ \Rightarrow f^{-1}(A) \text{ is open in } X. \]

\[ \Rightarrow f^{-1}(A) \text{ is closed in } X. \]

Now, \[ (f^{-1}(A))' = [f^{-1}(Y - A)]' = (f^{-1}(Y)) - f^{-1}(A) = [X - f^{-1}(A)]' = f^{-1}(A) \].
$\Rightarrow f^{-1}(A)$ is closed in $X$.

Conversely, suppose inverse image of each closed set in $Y$ is closed in $X$.
Let $A$ be any closed in $Y$.
$\Rightarrow f^{-1}(A)$ is closed in $X$.
$\Rightarrow f^{-1}(A)$ is open in $X$.
$\Rightarrow f$ is continuous.

**Convergence** (Def):
Let $(X, \mathcal{T})$ be a topological space. Then a sequence $\{x_n\}$ in $X$ is said to converge to a point $x \in X$ if for every open set $U$ containing $x$, there is a natural number $n_0$ such that $x_n \in U$, for all $n \geq n_0$.

**Theorem**: Let $X$ be a $T_2$-space. Then, any sequence in $X$ can converge to atmost one point.
i.e., in $T_2$-space, limit of the sequence is unique.

**Proof**: Suppose $\{x_n\}$ is a sequence in $X$ and $x_n \to x$ and $x_n \to y$ and suppose $x \neq y$.
As $X$ is $T_2$-space so, then there exists two open sets $U$ and $V$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.
Now as $x_n \to x$, so there exists some positive integer $n_0$, such that $x_n \in U \forall n \geq n_0$.
Also as $x_n \to y$, so there exist some
positive integer $n$ such that $\forall x \in V$, $\forall u > n$.

Let $m = \max \{m_0, m\}$.

Then $\forall x \in m$, $\forall u \in U$ and $x \in V$.

$\Rightarrow \, \, U \cap V \neq \emptyset$

$\Rightarrow \, \, U \neq \emptyset$

$\Rightarrow \, \, \text{Which is a contradiction.}$

$\Rightarrow \, \, U \cap V = \emptyset$

So, our supposition is wrong.

Hence $x = y$.

Hence, limit of the sequence is unique.

**Theorem:** Let $(X, Y)$ be a topological space and $Y$ be a $T_2$-space and $f : X \rightarrow Y$ is a continuous function, then the graph $G = \{(x, y) : y = f(x)\}$ is closed in $X \times Y$.

**Proof:** We prove $G'$ is open in $X \times Y$.

Let $(a, y) \in G \Rightarrow y = f(a)$.

As $Y$ is a $T_2$-space, $y \neq f(a)$ and $Y$ is $T_2$, so there exists two open sets $V$ and $W$ in $Y$ such that $y \notin V$, $f(a) \notin W$, and $V \cap W = \emptyset$.

Let $U = f^{-1}(W)$.

As $V$ is open in $Y$ and $f$ is continuous, so inverse image $f^{-1}(V) = U$ is open in $X$.

$\Rightarrow \, \, f(U) = f(f^{-1}(W)) = W$.

Thus $U \subseteq V$.

$\Rightarrow \, \, y \in V \Rightarrow y \in V \Rightarrow (a, y) \in U \times V$

Hence, $(a, y) \in U \times V \subseteq G'$.

But $U \times V$ is open in $X \times Y$.

$\Rightarrow \, \, G'$ is open in $X \times Y$.

$\Rightarrow \, \, G$ is closed in $X \times Y$. 

THEOREM: Let $X$ be a topological space and $Y$ be a $T_2$-space, and let $f, g : X \to Y$ be two continuous functions, then prove that $A = \{ x \in X : f(x) = g(x) \}$ is closed in $X$.

PROOF: We prove $A$ is open in $X$.

Let $a \in A \Rightarrow f(a) = g(a)$.

As $a \in X$, $f(a), g(a) \in Y$ and $f(a) = g(a)$ and $Y$ is $T_2$, so there exist two open sets $V$ and $W$ in $Y$ such that 

$f(a) \in V, g(a) \in W$, and $V \cap W = \emptyset$.

As $f$ and $g$ are continuous, so $f^{-1}(V)$ and $g^{-1}(W)$ are open in $X$.

As $f(a) \in V$ \Rightarrow $a \in f^{-1}(V)$

$g(a) \in W$ \Rightarrow $a \in g^{-1}(W)$

$\Rightarrow a \in f^{-1}(V) \cap g^{-1}(W) = A$.

As $f^{-1}(V) \cap g^{-1}(W)$ is open in $X$,

$\Rightarrow A$ is open in $X$.

\[ \Rightarrow A $ is closed in $X$.

THEOREM: Let $f : X \to Y$ and $g : X \to Y$ be two continuous functions from a topological space $X$ to a $T_2$ space $Y$ and $f(x) = g(x)$ for all $x \in D$, where $D$ is dense in $X$. Then $f(x) = g(x), \forall x \in X$.

PROOF: If $D = X$, then the theorem is trivially proved.

If $D \neq X$, then $D \cap O = \emptyset$, then there is no $x \in X$ such that $x \in D$. To prove: $f(x) = g(x)$.

Suppose $f(x) \neq g(x)$. As $f(x), g(x) \in Y$,

$f(x) \neq g(x)$ and $Y$ is a $T_2$-space. Then there
exists two open sets U and V such that $f(x) \in U$, $g(x) \in V$ and $UNV = \emptyset$.

Put $U_1 = f^{-1}(U)$ and $V_1 = g^{-1}(V)$.

Since U and V are open in Y and f, g are continuous functions so then $U_1$ and $V_1$ are open in X.

Further $f(x) \in U_1$ and $g(x) \in V_1$.

$\Rightarrow \exists e f^{-1}(U)$ and $\exists e g^{-1}(V)$.

$\Rightarrow \exists e U_1$ and $\exists e V_1$ $\Rightarrow \exists e UNV$.

Now as $D$ is dense in X, $2D = X$.

$\Rightarrow \exists e D$.

$\Rightarrow \exists (UNV)D = \emptyset$.

Let $d = (UNV)D$.

$\Rightarrow d \in U_1$, $d \in V_1$ and $d \in D$.

$\Rightarrow d \in f^{-1}(U)$ and $d \in g^{-1}(V)$ and $d \in D$.

Now $d \in D$ $\Rightarrow f(d) = g(d)$ ($\forall x \in D$, $f(x) = g(x)$).

Also, $d \in f^{-1}(U)$ and $d \in g^{-1}(V)$.

$\Rightarrow f(d) \in U$ and $g(d) \in V$.

$\Rightarrow f(d) = g(d)$ ($\forall x \in D$, $f(x) = g(x)$).

Which is a contradiction.

So, our supposition is wrong.

Hence, $f(x) = g(x)$, $\forall x \in X$.

**Theorem:** A topological space X is T$_{1}$-space if for any two distinct points a, b in X, there are closed sets C and C such that $a \in C$, $b \in C$ and $C \cap C = \emptyset$.

**Proof:**

Suppose X is T$_{1}$-space.

Available at

www.mathcity.org
As \( a, b \in X \) and \( a \neq b \). As \( X \) is \( T_{\infty} \)-space, \( \exists \) two open sets \( U \) and \( V \) such that:

- \( a \in U \) and \( b \in V \) and \( U \cap V = \emptyset \).

Put \( C_1 = V \) and \( C_2 = U \).

As \( U \) and \( V \) are open, \( \exists \) \( a \notin C_1 \) and \( b \notin C_2 \).

are closed in \( X \).

As \( a \in U \) and \( U \cap V = \emptyset \), \( a \notin V \).

be \( V \) and \( \emptyset \neq U \), \( a \notin C_1 \).

Now, \( a \in U \), \( \emptyset \neq U \), \( a \notin C_2 \).

\( a \notin V \), \( \emptyset \neq V \), \( a \notin C_1 \).

\( b \notin V \), \( \emptyset \neq V \), \( b \notin C_1 \).

Further, \( \emptyset \neq V \), \( \emptyset \neq U \), \( \emptyset \neq C_1 \).

Conversely, suppose \( C_1 \cup C_2 = X \).

To prove: \( X \) is \( T_{\infty} \).

Let \( U = C_2 \) and \( V = C_1 \).

As \( C_1 \) and \( C_2 \) are closed.

So \( U \) and \( V \) are open in \( X \).

As \( a \notin C_2 \Rightarrow a \notin C_1 \Rightarrow a \notin U \).

\( b \notin C_1 \Rightarrow b \notin C_2 \Rightarrow b \notin V \).

Now, \( U \cap V = \emptyset \), \( C_1 \cap C_2 = \emptyset \).

\( \Rightarrow X \) is \( T_{\infty} \)-space.

\[ \text{THEOREM: A topological space } X \text{ is } T_{\infty} \text{ space iff for every point } a \in X, \exists a \in X \text{ such that } \forall \in X, \exists a \in X \text{ where} \]

\[ \text{ each } C_a \text{ is a closed set containing an open set } U \text{ such that } a \in U \text{.} \]

\[ \text{Proof: Suppose } X \text{ is } T_{\infty} \text{-space.} \]
To prove: \( i a \mathfrak{J} = \mathfrak{n} C_x \) for all \( x \in U \).

Let \( b \in X \) such that \( a \neq b \). Then there exists two open sets \( U \) and \( V \) such that

\[ a \in U, \ b \in \mathfrak{V} \text{ and } \mathfrak{U} \cap \mathfrak{V} = \emptyset. \]

Put \( V = C_x \). As \( V \) is open, \( x \in V \) is closed.

Now as \( \mathfrak{U} \cap \mathfrak{V} = \emptyset \), so \( U \cap V = \emptyset \).

\[ b \in V \quad \Rightarrow \quad b \in \mathfrak{V} \subset \mathfrak{n} C_x. \]

Now as for every point \( b \in X \), distinct from \( a \), we have a closed set \( C_x \) such that \( a \in C_x \) and \( b \notin C_x \).

\[ \forall a \in U \subseteq C_x. \quad \forall b \in \mathfrak{V} \ni b \notin C_x. \]

\[ \Rightarrow \forall a \in U \subseteq C_x. \quad \forall b \in \mathfrak{V} \ni b \notin C_x. \]

Conversely, suppose in a topological space \( X \), for every point \( a \in X \), \( i a \mathfrak{J} = \mathfrak{n} C_x \), where \( C_x \) is a closed set containing an open set \( U \) such that \( a \in U \).

To prove: \( X \) is \( T_a \)-space.

Let \( b \in X \) such that \( a \neq b \).

\[ \Rightarrow b \notin \mathfrak{n} C_x \]

\[ \Rightarrow b \in C_x \text{ for some } x. \]

Put \( V = C_x \Rightarrow b \in V \).

\[ \Rightarrow a \in U \text{ and } b \in V. \]

Now as \( U \subseteq C_x \),

\[ \Rightarrow U \cap C_x = \emptyset \Rightarrow U \cap V = \emptyset. \]

\[ \Rightarrow X \text{ is } T_a \text{-space.} \]
Theorem: A 1st countable space $X$ is $T_{\infty}$-space if and only if every convergent sequence has a unique limit.

Proof: Suppose $X$ is 1st countable space which is $T_{\infty}$. To prove: Every convergent sequence has unique limit.

Suppose on the contrary that $x_0 \neq y$ and $x_n \to x_0$ and $x_n \neq y$.

Since $X$ is $T_{\infty}$, there exist two open sets $U$ and $V$ such that $x_0 \in U$, $y \in V$ and $U \cap V = \emptyset$.

Now $x_n \to x_0 \in U$, so there exist $n_1$ such that $x_n \in U \, \forall \, n > n_1$.

As $x_n \to y \in V$, so there exist $n_2$ such that $x_n \in V \, \forall \, n > n_2$.

Put $n_0 = \max \{n_1, n_2\}$.

$\Rightarrow x_n \in U \, \forall \, n > n_0$.

$\Rightarrow x_n \in V \, \forall \, n > n_0$.

$\Rightarrow U \cap V \neq \emptyset \Rightarrow \emptyset$, contradiction.

Thus our assumption is wrong.

Hence $x = y$.

$\Rightarrow$ each sequence has unique limit.

Conversely, suppose in a first countable space $X$, every convergent sequence has a unique limit. To prove: $X$ is $T_{\infty}$-space.

Let $a, b \in X$ such that $a \neq b$.
To prove: \( X \) is \( T_{\alpha} \)-space.

We suppose \( X \) is not \( T_{\alpha} \)-space.

Then, every open set containing \( a \) has a non-empty intersection with every open set which contains \( b \).

Let \( \mathcal{U}_n \) and \( \mathcal{V}_n \) be countable nested bases at \( a \) and \( b \) respectively.

Then \( \bigcap U_n \cap \bigcap V_n \neq \emptyset \)

\[ \Rightarrow \text{and} \ U_n \cap V_n \neq \emptyset. \]

Then \( a_n \to a \) and \( b_n \to b \).

Which is a contradiction.

Every convergent sequence in \( X \) has unique limit.

So our supposition is wrong.

Hence, \( X \) is \( T_{\alpha} \)-space.

**Theorem:**

Every \( T_{\alpha} \)-space is \( T_{\beta} \)-space.
**Regular Space (Def).**
A topological space \((X, T)\) is said to be regular if for every \(x \in X\) and for any closed subset \(A \subseteq X\) with \(x \notin A\), there exists two open sets \(U\) and \(V\) such that \(x \in U\), \(A \subseteq V\), and \(U \cap V = \emptyset\).

**Example:** Let \(X = \{a, b, 3\}\),
\[ T = \{\emptyset, X, \{a, 3\}, \{b, 3\}\}, \]
then \((X, T)\) is regular.

**Theorem:** The following statements about a topological space are equivalent:

1. \(X\) is regular.
2. For any point \(x\) in \(X\) and \(x \notin U\), there is an open set \(V\) containing \(x\) such that \(x \in V \subseteq U\).
3. Each element of \(X\) has a local base containing closed sets.

**Proof:** 1 \(\Rightarrow\) 2: i.e., here it is given that \(X\) is regular and to prove 2.

Let \(U\) be an open set in \(X\) with \(x \in U\). To prove: There exists an open set \(V\) in \(X\) containing \(x\) such that \(x \in V \subseteq U\).

Now, as \(x \in U\) and \(U\) is open set,
\[ \exists \emptyset \subseteq U \quad \text{and} \quad U \text{ is closed}. \]
Then, by the definition of regular...
space, there exist two open sets \( V \) and \( V_i \) such that \( \delta e V \), \( U' \subseteq V_i \), and \( V_i \cap V = \emptyset \).

Now, \( U' \subseteq V_i \Rightarrow V' \subseteq U \).
Also, \( V_i \cap V = \emptyset \Rightarrow V \subseteq V_i \).

\( \Rightarrow \delta e V \subseteq V' \subseteq U \).

Now, as \( V_i \) is an open set, so \( V_i \) is a closed set. So \( V_i \) is a closed super set of \( V \).
But \( V \) is the smallest closed super set of \( V \Rightarrow \delta e V \subseteq V \subseteq V_i \subseteq U \).

\( \Rightarrow \delta e V \subseteq U \).

2) \( \Rightarrow \) 3): Let \( \delta e X \). To prove \( X \) has a local base containing closed sets.

Let \( U \) be an open set such that \( \delta e U \).
Then, by condition \( 2 \), there exist an open set \( V \) such that \( \delta e V \subseteq U \).

This shows that a local base at \( \delta \) contains sets of the form \( V \) which is, of course, a closed set.

3) \( \Rightarrow \) 2): Let \( \delta e X \) and \( A \) be closed subset of \( X \) such that \( \delta e A \).

\( \Rightarrow \delta e A \).
Further, as \( A \) is closed,
so, \( A \) is open set. Then by \( 3 \), there
is a closed set. \( B \) in the local base at \( x \) such that \( x \in B \subseteq A \). Now \( B \subseteq A \Rightarrow A \subseteq B \).

Let \( U = B \) and \( V = B' \).

Then, \( U \) is open as \( U \) is in local base.

\( V \) is open because \( V = B' \) and \( B \) is closed.

Further, \( x \in U \cap V = \emptyset \) \((B \cap B' = \emptyset)\).

Hence, it shows that \( \text{0}, \text{3} \) and \( \text{3} \) are equivalent.

**COMPLETELY REGULAR SPACE: (DEF)**

A topological space \((X, T)\) is said to be completely regular space if for any closed set \( A \) in \( X \) and \( x \in X \) such that \( x \notin A \), there exist a continuous function \( f: X \to [0, 1] \) such that \( f(x) = 0 \) and \( f(A) = 1 \).

**EXAMPLE:**

Every metric space is completely regular.

**THEOREM:**

Every completely regular space is regular.

**PROOF:**

Let \( X \) be a completely regular space.

To prove: \( X \) is regular.
Let \( x \in X \) and \( A \) be a closed subset of \( X \) such that \( x \notin A \). Then, as \( X \) is completely regular, there exist a continuous function \( f : X \to [0, 1] \) such that \( f(x) = 0 \) and \( f(A) = 1 \).

Let \( U = [0, \frac{1}{2}] \) and \( V = [\frac{1}{2}, 1] \). Then \( U \) and \( V \) are open in \( [0, 1] \).

As \( f \) is continuous, so \( f^{-1}(U) \) and \( f^{-1}(V) \) are open in \( X \).

And \( x \in f^{-1}(U) \), \( A \subseteq f^{-1}(V) \)

and \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \).

So, \( X \) is regular.

Hence proved.

Available at
www.mathcity.org
Theorem: Every subspace of a completely regular space is completely regular.

Proof: Let \( X \) be a completely regular space and \( Y \) be a subspace of \( X \).

To prove: \( Y \) is completely regular.

Let \( x \in Y \) and \( A \) be a closed subset of \( Y \) such that \( x \notin A \). As \( x \in Y \) and \( Y \subseteq X \), it follows that \( A \) is closed in \( X \) and \( Y \) is a subspace of \( X \). So, there exists a closed subset \( B \) in \( X \) such that \( A = B \cap Y \).

As \( x \notin A \) and \( x \in Y \), \( x \notin B \).

As \( X \) is completely regular, there exists a continuous function \( f : X \to [0, 1] \) such that:

\[
f(x) = 0 \quad \text{and} \quad f(B) = 1.
\]

Now, define \( g : Y \to [0, 1] \) by \( g(y) = f(y) \) for \( y \in Y \).

Then, \( x \notin Y \Rightarrow g(x) = f(x) = 0 \).

Definition: \( T_0 \)-Space

A regular \( T_0 \)-space is called a \( T_0 \)-space.

Definition: \( T_{1 \frac{1}{2}} \)-Space

A completely regular \( T_0 \)-space is called a \( T_{1 \frac{1}{2}} \)-space or a Tychonoff space.

Theorem: A completely regular \( T_1 \)-space is a \( T_{1 \frac{1}{2}} \)-space.
**Proof.** Let $X$ be a completely regular $T_1$-space.

To prove, $X$ is $T_{3\frac{1}{2}}$-space.

Let $x \neq y \in X$ such that $x \neq y$.

As $X$ is $T_1$-space, so each singleton subset in $X$ is closed.

So $A = \{x, y\}$ is closed set in $X$ and $x \in A$.

As $X$ is completely regular so then there exists a continuous function $f : X \to [0, 1]$ such that:

$f(x) = 0$, $f(y) = 1 \Rightarrow f(y) = 1$.

Now let $U = (0, \frac{1}{2}]$ and $V = [\frac{1}{2}, 1)$ be two open sets in $[0, 1]$, then as $f$ is continuous so $f^{-1}(U)$ and $f^{-1}(V)$ are open in $X$. Further as:

$f(x) = 0 \in U \Rightarrow x \in f^{-1}(U)$

$f(y) = 1 \in V \Rightarrow y \in f^{-1}(V)$.

$\Rightarrow f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$.

$\Rightarrow X$ is $T_{3\frac{1}{2}}$-space.

---

**Theorem:** A subspace of $T_{3\frac{1}{2}}$-space is $T_{3\frac{1}{2}}$-space.

**Proof.** Let $X$ be a $T_{3\frac{1}{2}}$-space and $Y$ be a subspace of $X$.

To prove, $Y$ is $T_{3\frac{1}{2}}$-space.

As $X$ is $T_{3\frac{1}{2}}$-space, so $X$ is completely regular and $X$ is $T_1$-space.

A subspace of completely regular space is completely regular. As subspace of $T_1$-space is $T_1$, so $Y$ is completely regular and $T_1$-space. $\Rightarrow Y$ is $T_{3\frac{1}{2}}$-space.

Hence Proved.

Available at

www.mathcity.org
Set of all continuous functions from \( X \to \mathbb{R} \)

Theorem: For any topological space \( X \), if \( C(X, \mathbb{R}) \) separates the points of \( X \), then \( X \) is \( T_2 \)-space.

Proof: Let any \( x, y \in X \) such that \( x \neq y \)

and let \( f \in C(X, \mathbb{R}) \).

i.e. \( f \) is a continuous function from \( X \to \mathbb{R} \).

Now by the given condition, \( f(x) \neq f(y) \).

Say \( f(x) < f(y) \).

As \( f(x), f(y) \in \mathbb{R} \) and \( f(x) < f(y) \).

So there exists \( \epsilon > 0 \) such that

\[ f(x) < f(y) - \epsilon < f(y). \]

Put \( U = \{ u \in X : f(u) < f(y) \} \) and \( V = \{ v \in X : f(v) > f(x) + \epsilon \} \).

Then \( x \in U \) and \( y \in V \).

Further, \( U = f^{-1}((f(x) - \epsilon, f(y))) \) and \( V = f^{-1}((f(y), f(x) + \epsilon)) \).

Now as \( (f(x) - \epsilon, f(y)) \) and \( (f(y), f(x) + \epsilon) \) are open sets in \( \mathbb{R} \) and \( f \) is continuous, so inverse images of open sets in \( \mathbb{R} \) are also open in \( X \) and \( U \) and \( V \) are open in \( X \).

Also by definition of \( U \) and \( V \):

\[ U \cap V = \emptyset. \]

Hence, we have found two open sets \( U \) and \( V \) in \( X \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

\[ \Rightarrow X \text{ is } T_2. \]

Definition: Normal Space.

A topological space \( (X, \mathcal{T}) \) is said to be normal space if for any two closed disjoint subsets \( A \) and \( B \) of \( X \), there are open sets \( U \) and \( V \) in \( X \) such that \( A \subseteq U \), \( B \subseteq V \) and \( U \cap V = \emptyset \).

Theorem: Every discrete space with at least two points is normal.
Proof: Let $X$ be a discrete space with at least two points. To prove, $X$ is normal.

As $X$ is discrete, so, each subset of $X$ is open as well as closed.

Let $A$ and $B$ be two disjoint closed sets in $X$. Let $U = A$ and $V = B$.

Then $U$ and $V$ are open and $U \cap V = \emptyset$.

$A \subseteq U$, $B \subseteq V$.

$\Rightarrow X$ is normal.

Theorem: Every subspace of a regular space is regular.

Proof: Let $X$ be a regular space and $Y$ be a subspace of $X$.

To prove, $Y$ is regular.

Let $x \in Y$ and $A$ be a closed set in $Y$ such that $x \notin A$.

Now as $A$ is closed in $Y$ and $Y$ is subspace of $X$, so then there exists a closed set $B$ in $X$ such that $A = B \cap Y$.

Further $x \notin A \Rightarrow x \notin B$. 

$\Rightarrow x \notin B \cap Y = \emptyset$.

Further $x \in Y \Rightarrow x \in X$.

So $x \in X$ and $B$ is a closed set in $X$ such that $x \notin B$ and $X$ is regular, so then there exists two open sets $U$ and $V$ in $X$ such that $x \in U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Put $U_1 = U \cap Y$ and $V_1 = V \cap Y$.

As $U$ and $V$ are open in $X$, so, $U_1$ and $V_1$ are open in $Y$.

Now as $x \in U_1$, $x \in Y \Rightarrow x \in U \cap Y = U_1$.

$\Rightarrow x \in U_1$.

Also $B \subseteq V \Rightarrow B \cap Y \subseteq V \cap Y = V_1$.

$\Rightarrow A \subseteq V_1$. 


And \[ u \cap v = (u \cap v) \cap (v \cap v) = (u \cap v) \cap (v \cap v) = \emptyset \cap v = \emptyset. \]

\[ \Rightarrow \text{Y is regular.} \]

**Theorem:** Every metric space is normal space.

**Proof:** Let \((X, d)\) be a metric space.

To prove: \( X \) is normal space.

Let \( A \) and \( B \) be the two disjoint non-empty closed sets in \( X \). Let \( a \in A \), then

\[ d(a, B) = \inf d(a, b) \]

definition of distance between a point

\[ A, B \neq \emptyset \text{ and } a \in A \text{ to } a \notin B. \]

Further as \( B \) is closed, \( \inf d(a, B) > 0 \).

Let \( d(a, B) = \gamma \).

Similarly, let \( b \in B \) and \( d(b, a) = \delta \).

Now consider the open balls:

\[ B(a, \frac{\gamma}{2}) \text{ and } B(b, \frac{\delta}{2}). \]

Put \( U = \bigcup_{a \in A} B(a, \frac{\gamma}{2}) \), \( V = \bigcup_{b \in B} B(b, \frac{\delta}{2}). \)

Then \( U \) and \( V \) are open, \((\cup \text{union of open balls})\) is open.

With \( A = U \) and \( B = V \).

Now we prove that \( u \cap v = \emptyset \).

Suppose on the contrary that \( u \cap v \neq \emptyset \).

\[ x \in u \cap v \Rightarrow x \in U \text{ and } x \in V. \]

\[ x \in U \Rightarrow x \in \bigcup_{a \in A} B(a, \frac{\gamma}{2}) \]

\[ x \in V \Rightarrow x \in \bigcup_{b \in B} B(b, \frac{\delta}{2}) \]

\[ \Rightarrow x \in B(a, \frac{\gamma}{2}) \text{ and } x \in B(b, \frac{\delta}{2}). \]

\[ \Rightarrow d(x, a) < \frac{\gamma}{2} \text{ and } d(x, b) < \frac{\delta}{2}. \]

As \( a \in A, b \in B \) and \( A \cap B = \emptyset \Rightarrow a \neq b \).

\[ \Rightarrow d(a, b) > 0. \]

Available at

www.mathcity.org
Let \( d(a_1, b_1) \geq x \).

Clearly, \( x \leq x \) and \( x \leq x \), \( x \) are min in \( d \) from each other.

Now \( x = d(a_1, b_1) \leq d(a_1, x_1) + d(x_1, b_1) \leq \frac{x}{x} + \frac{x}{x} \leq 2 \frac{x}{x} \).

\( \Rightarrow x \leq 2 \frac{x}{x} \).

\( \Rightarrow 3x \leq 2x \).

Which is a contradiction.

So our supposition is wrong.

Hence \( U \cap V = \emptyset \).

\( \Rightarrow X \) is normal.

**Definition:** A normal \( T_4 \) space is called \( T_4 \) space.

**Theorem:** A normal \( T_4 \) space is regular space.

**Proof:** Let \( X \) be a \( T_4 \) space.

To prove \( X \) is regular.

As \( X \) is \( T_4 \), \( X \) is normal and \( X \) is \( T_4 \) as well.

Let \( x \in X \) and \( A \) be a closed set in \( X \). Let \( x \notin A \).

As \( X \) is \( T_4 \), \( x \cup A \) is closed.

Hence \( x \) and \( A \) are two disjoint closed sets in \( X \).

As \( X \) is normal, so there exists two open sets \( U \) and \( V \) in \( X \) such that:

\( x \cup A \subseteq U \), \( A \subseteq V \), and \( U \cap V = \emptyset \).

\( \Rightarrow x \in U \), \( A \subseteq V \) and \( U \cap V = \emptyset \).

\( \Rightarrow X \) is regular.

**Theorem:** A topological space \((X, x)\) is normal if for any closed set \( A \) and open set \( U \) containing \( A \), there...
is at least one open set \( V \) containing \( A \) such that
\[ A \subseteq V \subseteq V \subseteq U. \]

**Proof.** Let \( X \) be a normal space. And let \( U \) be a closed set in \( X \); \( U \) be an open set in \( X \) with
\[ A \subseteq U. \] To prove: there is at least one open set \( V \) in \( X \) with
\[ A \subseteq V \subseteq V \subseteq U. \]

Now, \( A \subseteq U \Rightarrow \exists U' \neq \emptyset \). 

As \( U \) is open so \( U' \) is closed in \( X \). So, we have \( A \) and \( U' \) are two closed, disjoint sets in \( X \). As \( X \) is normal so there exists two open sets \( V \) and \( V' \) in \( X \) such that \( A \subseteq V \), \( U' \subseteq V \), and \( V \cap V' = \emptyset \).

Now, \( V \subseteq V' \Rightarrow V' \subseteq U \). Also as \( V \cap V' = \emptyset \Rightarrow V \subseteq V' \).

\[ \Rightarrow A \subseteq V \subseteq V' \subseteq U. \]

Now, as \( V' \) is open, so \( V' \) is closed in \( X \). 

This means \( V' \) is the smallest closed superset of \( V \).

But as \( V \) is the smallest closed superset of \( V \), so \( V \subseteq V' \).

\[ \Rightarrow A \subseteq V \subseteq V' \subseteq U. \]

Conversely, let it is given that in a topological space \( (X, T) \) for any closed set \( A \) in \( X \), \( U \) is an open set in \( X \) with \( A \subseteq U \). There is at least one open set \( V \) in \( X \) such that \( A \subseteq V \subseteq V \subseteq U \).

To prove: \( X \) is normal. Let \( A \) and \( B \) be the two closed, disjoint sets in \( X \). As \( A \cap B = \emptyset \Rightarrow A \subseteq B \).

As \( B \) is closed, so \( B' \) is open.

Now, by given condition, there is an open set \( V \) in \( X \), such that \( A \subseteq V \subseteq V \subseteq B \).

Now, \( A \subseteq V \) and \( V \subseteq B \Rightarrow A \subseteq V \) and \( B' \subseteq V \). 

Now, as \( V \) is closed so \( V' \) is open.

Further as \( V \subseteq V' \Rightarrow V \cap (V')' = \emptyset \).

Hence, we have found two open sets \( V \) and \( V' \)
in X such that \( A \subseteq V, B \subseteq V \), and \( V \cap V' = \emptyset \).
\[ \Rightarrow X \text{ is normal.} \]

**Theorem:** Every closed subspace of a normal space is normal.

**Proof:** Let \( X \) be a normal space and \( Y \) be a closed subspace of \( X \).
To prove: \( Y \) is normal.
Let \( A_1 \) and \( A_2 \) be the two disjoint closed sets in \( Y \). Thus \( Y \) is a subspace, so then there are two disjoint closed sets \( B_1 \) and \( B_2 \) in \( X \) such that:
\[ A_1 = B_1 \cap Y, \quad A_2 = B_2 \cap Y. \]
Now as \( B_1 \) and \( B_2 \) are two disjoint closed sets in \( X \) and \( X \) is normal, so then there are two open sets \( U_1 \) and \( U_2 \) such that \( B_1 \subseteq U_1 \), \( B_2 \subseteq U_2 \), and \( U_1 \cap U_2 = \emptyset \).
Put \( V_1 = U_1 \cap Y \) and \( V_2 = U_2 \cap Y \).
\[ V_1 \text{ and } V_2 \text{ are open in } Y. \]
As \( B_1 \subseteq U_1 \Rightarrow B_1 \cap Y \subseteq U_1 \cap Y \Rightarrow A_1 \subseteq V_1 \).
Also \( B_2 \subseteq U_2 \Rightarrow B_2 \cap Y \subseteq U_2 \cap Y \Rightarrow A_2 \subseteq V_2 \).
\[ V_1 \cap V_2 = (U_1 \cap Y) \cap (U_2 \cap Y) \]
\[ = (U_1 \cap U_2) \cap Y \]
\[ = \emptyset \cap Y = \emptyset . \]
\[ \Rightarrow Y \text{ is normal.} \]

**Theorem:** Every metric space is completely regular.

**Proof:** Let \((X, d)\) be a metric space.
To prove: \( X \) is completely regular.
Let \( A \) be a closed subset of \( X \) and \( x \in X \)
such that \( x \notin A \). And we have to find out a continuous function \( f: X \to [0, 1] \) such that \( f(x) = 0 \)
and \( f(A) = 1 \).
Define \( g : X \rightarrow [0, 1] \) by \( g(y) = d(y, B) \) where \( B \) is any other closed set in \( X \) with \( A \cap B = \emptyset \) and \( x \in B \).

Then, i) \( g(x) = d(x, B) = 0 \).

ii) \( g(A) = d(A, B) > 0 \). (\( A \) and \( B \) are closed).

Let \( d(A, B) = k \).

iii) Now for any \( \varepsilon > 0 \), we choose \( s = \varepsilon \) such that whenever \( d(y, y') < s \),

\[ |g(y) - g(y')| \leq d(y, y') \leq \varepsilon = \varepsilon \]

\[ |g(y) - g(y')| \leq \varepsilon, (d(y, y') < d(y, x)) \leq \varepsilon \]

Now \( f : [0, 1] \rightarrow X \) by \( f(y) = \frac{1}{k}g(y) \).

As \( g \) is continuous, so \( f \) is continuous with \( f(0) = \frac{1}{k}g(0) = \frac{1}{k}(0) = 0 \).

\[ f(1) = \frac{1}{k}g(1) = \frac{d(A, B)}{k} \]

\[ \Rightarrow X \text{ is completely regular.} \]

Open function:

A function \( f \) is said to be open function if image of each open set is open.

Example:

Let \( X = \{a, b, c, d, e\}, f(x) = x^2 \) for \( x = a, b, c, d, e \), \( f(y) = 1 \) for \( y = a, b, c, d, e \).

Define \( f : X \rightarrow Y \) as \( f(1) = 2, f(2) = 3, f(3) = a \).

\( f(1) = d \). Then \( f \) is open.
A function $f$ is said to be closed if its image of each closed set is closed.

Theorem: A closed and continuous image of a normal space is normal.

Proof: Let $X$ be a normal space and $Y = f(X)$ is its closed continuous image.

To prove: $Y = f(X)$ is normal.

Let $A$ and $B$ be the two disjoint closed in $Y = f(X)$.

As $f$ is continuous, so inverse image of each closed set is closed.

So $f^{-1}(A)$ and $f^{-1}(B)$ are closed in $X$.

Let $A = f^{-1}(A)$ and $B = f^{-1}(B)$.

Then, $A$ and $B$ are closed in $X$ and

$$A \cap B = f^{-1}(A) \cap f^{-1}(B)$$

$$= f^{-1}(A \cap B)$$

$$= f^{-1}(\emptyset) = \emptyset$$

Therefore, $A$ and $B$ are two disjoint closed sets in normal space $X$. So, there exists two open sets $U$ and $V$ in $X$ such that $A \subseteq U$, $B \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$.

As $U$ and $V$ are open in $X$, so, $U$ and $V$ are open in $X$.

Further, as $f$ is closed, so $f(U)$ and $f(V)$ are closed.

As $Y = f(X)$

Put $U \subseteq f(U)$

$V \subseteq f(V)$

$\Rightarrow U$ and $V$ are open in $Y = f(X)$.

Now we show $A \subseteq U_1$. 


Let $x \in A_1$.

$\Rightarrow x \in f(A)$.

$\Rightarrow f^{-1}(x) \subseteq A \subseteq U.$

$\Rightarrow f^{-1}(x) \subseteq U.$

$\Rightarrow f^{-1}(x) \subseteq U'.$

$\Rightarrow x \notin f(U').$

$\Rightarrow x \in f(U)$.

$\Rightarrow x \in U.$

$\Rightarrow x \in U_1.$

$\Rightarrow A_1 \subseteq U_1.$

Similarly, $B_1 \subseteq V_1.$

Now, $U \cap V_1 = (f(U'))' \cap (f(V_1))' = [f(U') \cup f(V_1)]' = [f(U' \cup V_1)]' = [f(X)]' = Y' = \varnothing.$

$\Rightarrow f(X)$ is normal.

These Notes are the lectures delivered by Tahm Mahmood.
"Compactness in Topological Spaces"

**Definition:** Compactness.
A topological space \((X, T)\) is said to be compact if every open cover for \(X\) has a finite subcover.

**Examples:**
1. If \(X\) is any set with discrete topology, then \(X\) is compact.
2. If \(X\) is finite set, then for any topology \(T\) on \(X\), \((X, T)\) is compact.
3. If \(X\) is any set with \(A \subseteq X\), then \(T = \{U \subseteq X : A \subseteq U\}\) then, \((X, T)\) is compact.

**Remark:**
If \((X, T)\) is a compact space, then it is Lindelöf. But converse is not true because, e.g., if \(X = \mathbb{N}\) and \(T = \mathcal{T}_0\)
Then \((X, T)\) is a Lindelöf space but \((X, T)\) is not compact because \(\{x_n: n \in \mathbb{N}\}\) is an open cover for \(X\), which has no finite subcover. All are open.

**Theorem:** Let \(X\) be an infinite set with infinite topology, then \(X\) is compact.

**Proof:** Let \(X = \bigcup_{i \in I} X_i\) be an open cover for \(X\). We have to find a finite subcover of \(X\) for \(X\).

Since \(X\) is an open cover for \(X\),

\[ X = \bigcup_{i \in I} X_i. \]

Now, for any \(U \in X\), \(U\) is an open set.

\[ \Rightarrow U_i \text{ is finite.} \]

Now as \(X\) is an open cover for \(X\), i.e.,

\[ X = \bigcup_{i \in I} U_i, \]

all are open.
So, for any $x \in U$, $i \in I$:

$\Rightarrow x_i \in U_i$

$\Rightarrow x_i \in U_i$ for some $i \in I$

$\Rightarrow \exists x_i \in U_i$

$\Rightarrow \bigcup_{i=1}^{n} x_i \subseteq U_i$

$\Rightarrow \bigcup_{i=1}^{n} U_i \subseteq U_i$

$\Rightarrow \bigcup_{i=1}^{n} U_i \subseteq \bigcup_{i=1}^{n} U_i$

$\Rightarrow \bigcup_{i=1}^{n} U_i \subseteq \bigcup_{i=1}^{n} U_i$

$\Rightarrow X \subseteq \bigcup_{i=1}^{n} U_i)$

$\Rightarrow X = \bigcup_{i=1}^{n} U_i$.

Theorem: The real line $\mathbb{R}$ is not compact with usual topology.

Proof: Let $x = \bigcup_{i=1}^{n} U_i$, $x \subseteq \bigcup_{i=1}^{n} U_i$.

Suppose on the contrary that $\mathbb{R}$ is compact.

Then by the definition of compact space, $\mathbb{R}$ has a finite subcover for $\mathbb{R}$.

Let $\bigcup_{i=1}^{n} U_i \subseteq \bigcup_{i=1}^{n} U_i$ be the finite subcover for $\mathbb{R}$.

Let $m = \max \{n_1, n_2, n_3, \ldots, n_n\}$. Then $\bigcup_{i=1}^{n} U_i \subseteq \bigcup_{i=1}^{n} U_i$.

Then $m \in \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} U_i$. Therefore, $\mathbb{R}$ is not compact.
\[ R = \{ \text{im} \} \text{, which is a contradiction.} \]

So, our supposition is wrong.

Hence, \( R \) is not compact.

**Finite Intersection Property**

Let \((X, T)\) be a topological space and \( \mathcal{F} = \{ \mathcal{F}_i : i \in I \} \) be a collection of some subsets of \( X \), then \( \mathcal{F} \) is said to have finite intersection property if each finite subcollection of \( \mathcal{F} \) has non-empty intersection e.g., Let \( X = \mathbb{N}, T = \text{Fin} \) and \( \mathcal{F} = \{ \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \ldots \} \) then \( \mathcal{F} \) satisfies finite intersection property.

**Theorem:** A topological space \((X, T)\) is compact if and only if every collection of closed sets in \( X \) which satisfy finite intersection property has non-empty intersection.

**Proof:** Let \( X \) be compact and \( \mathcal{F} = \{ \mathcal{F}_i : i \in I \} \) be the collection of closed sets which satisfy finite intersection property.

To prove: \( \bigcap_{i \in I} \mathcal{F}_i \neq \emptyset \)

Suppose on the contrary that \( \bigcap_{i \in I} \mathcal{F}_i = \emptyset \).

\[ \Rightarrow (\bigcap_{i \in I} \mathcal{F}_i) = \emptyset \neq \bigcup_{i \in I} \mathcal{F}_i = X. \]

Now as \( \{ \mathcal{F}_i : i \in I \} \) is the collection of closed sets so \( \{ \mathcal{F}_i : i \in I \} \) is the collection of open sets with \( \bigcup_{i \in I} \mathcal{F}_i = X. \)

\[ \Rightarrow \{ \mathcal{F}_i : i \in I \} \text{ is an open cover of } X, \text{ where } X. \]
is compact space.

⇒ \( \bigcup_{i=1}^{n} U_i \) is a finite subcover for \( X \).

⇒ \( \bigcap_{i=1}^{n} U_i = X \).

⇒ \( \bigcap_{i=1}^{n} U_i' = X' \) ⇒ \( \bigcap_{i=1}^{n} U_i = \emptyset \).

⇒ \( \bigcup_{i=1}^{n} U_i \), \( \forall i, 2, \ldots, n \) is a finite subcollection of \( \bigcup_{i=1}^{n} U_i \) with empty intersection.

⇒ \( \bigcup_{i=1}^{n} U_i \) does not satisfy finite intersection property, which is a contradiction.

⇒ \( \bigcup_{i=1}^{n} U_i \) satisfies finite intersection property.

⇒ our supposition is wrong.

And hence: \( \bigcap_{i=1}^{n} U_i = \emptyset \).

Conversely, let \( (X, T) \) be a topological space. Each collection of closed sets in \( X \) which satisfies finite intersection property has non-empty intersection.

To prove: \( X \) is compact.

Let: \( \mathcal{C} = \{ C_i \} \) be an open cover for \( X \).

\( \forall i \in I \), \( U_i \) \( \supseteq X \) ⇒ \( \bigcup_{i=1}^{n} U_i \supseteq X \) ⇒ \( \bigcap_{i=1}^{n} C_i = \emptyset \).

\( \exists i : C_i \) is a collection of closed sets with empty intersection.

Then, by given hypothesis \( \bigcap_{i=1}^{n} C_i \) does not satisfy finite intersection property. Then, there exists a finite subcollection \( \mathcal{C}_i, C_2, \ldots, C_n \) with empty intersection.

\( \forall i \in I \), \( C_i \) \( \supseteq X \) ⇒ \( \bigcap_{i=1}^{n} C_i = \emptyset \) ⇒ \( \bigcap_{i=1}^{n} C_i = X \).

⇒ \( \bigcup_{i=1}^{n} C_i, C_2, \ldots, C_n \) is a finite subcover for \( X \).

⇒ \( X \) is compact.
**Theorem:** Every closed subspace of a compact space is compact.

**Proof:** Let \( X \) be compact and \( Y \) be a closed subspace of \( X \).

To prove: \( Y \) is compact.

Let \( \mathcal{U} := \{ U_i \}_{i \in I} \) be an open cover for \( Y \).

As \( U_i, \forall i \in I \), is an open set in \( Y \) and \( Y \) is subspace of \( X \), so then there is an open set \( \mathcal{V} \) in \( X \) such that:

\[
U_i \subseteq V_a \cap Y.
\]

\[
\Rightarrow U_i \subseteq V_a.
\]

\[
\Rightarrow U \cup \mathcal{V} = U \cup \mathcal{V}_a.
\]

\[
\Rightarrow U \cup \mathcal{V} = U \cup \mathcal{V}_a.
\]

\[
\Rightarrow U \cup \mathcal{V} = U \cup \mathcal{V}_a.
\]

Now, \( X = U \cup \mathcal{V} = (U \cup \mathcal{V}_a) \cup Y \subseteq X \).

\[
\Rightarrow X = (U \cup \mathcal{V}_a) \cup Y.
\]

As \( Y \) is closed, so \( Y \) is open in \( X \).

\[
\Rightarrow Y \subseteq \bigcup_{i \in I} U_i.
\]

As \( X \) is compact, so this open cover has a finite subcover \( \mathcal{F} \) \( = \{ V_1, V_2, \ldots, V_n \} \) \( \Rightarrow A \subseteq B \) or \( A \subseteq B \).

Now, \( Y \subseteq \bigcup_{i \in I} U_i \).

\[
\Rightarrow Y \subseteq \bigcup_{i \in I} U_i.
\]

\[
\Rightarrow Y \subseteq \bigcup_{i \in I} U_i \cap Y = \emptyset.
\]

\[
\Rightarrow Y = \bigcup_{i \in I} (U_i \cap Y) = \bigcup_{i \in I} U_i.
\]
$\Rightarrow \{ U_\alpha, V_\alpha, \ldots \} \text{ is a finite subcover for } Y. \Rightarrow Y \text{ is compact.}$

**Theorem:** Continuous image of a compact space is compact.

**Proof:** Let $f : X \to Y$ be a continuous function from a compact space $X$ to a topological space $Y$.

To prove, $f(X)$ is compact.

Let $\{ U_\alpha, \alpha \in I \}$ be an open cover for $f(X)$, where $f(X)$ is the subspace of $Y$.

As $\{ U_\alpha, \alpha \in I \}$ is an open set in $f(X)$ and $f(X)$ is a subspace of $Y$.

So, there exists an open set $V_\alpha$ in $Y$ such that $U_\alpha = f^{-1}(V_\alpha) = \cup_{\alpha \in I} V_\alpha$.

$\Rightarrow f(X) \subseteq \cup_{\alpha \in I} V_\alpha$.

$\Rightarrow X \subseteq f^{-1}(\cup_{\alpha \in I} V_\alpha)$.

$\Rightarrow X \subseteq \cup_{\alpha \in I} f^{-1}(V_\alpha)$.

As $V_\alpha, \alpha \in I$ is open in $Y$ and $f : X \to Y$ is continuous.

$\Rightarrow f^{-1}(V_\alpha), \alpha \in I$, is open in $X$ (Inverse image of open set is open).

$\Rightarrow \{ f^{-1}(V_\alpha), \alpha \in I \}$ is an open cover for $X$.

Since, $X$ is compact.

So, this open cover has a finite subcover,

$\{ f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \ldots , f^{-1}(V_{\alpha_n}) \}$ for $X$.

$\Rightarrow \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}) = X$. 

MathCity.org Merging Man and maths
\[ \Rightarrow f^{-1}(\bigcup_{i=1}^{n} V_{i}) = X \]
\[ \Rightarrow X = f^{-1}(\bigcup_{i=1}^{n} V_{i}) \]
\[ \Rightarrow f(X) \subseteq \bigcap_{i=1}^{n} V_{i} \]
\[ \Rightarrow f(X) = \left( \bigcap_{i=1}^{n} V_{i} \right) \cap f(X) \]
\[ \Rightarrow f(X) = \bigcup_{i=1}^{n} (V_{i} \cap f(X)) \quad \text{(Distributive property)} \]
\[ = \bigcup_{i=1}^{n} V_{i} \]
\[ \Rightarrow \bigcup_{i=1}^{n} U_{i} \quad \text{is a finite subcover for } f(X) \]

Hence, \( f(X) \) is compact.

**THEOREM:** Prove that in a T1-space, any countable disjoint compact subspace of \( X \) can be separated by open sets in the sense they have disjoint neighborhoods.

**Proof:** Let \( x \in X \) and \( C \) be a compact subspace of \( X \) such that \( x \notin C \).

To prove: \( X \) and \( C \) can be separated by open sets.

Let \( y \in C \Rightarrow x \neq y \), then as \( x, y \in X \), \( x \in T \), \( X \) is a T1 space, so then there exists two open sets \( U_{y} \) and \( V_{y} \) such that \( x \in U_{y}, y \in V_{y} \) and \( U_{y} \cap V_{y} = \emptyset \).

Now as, \( y \in V_{y} \Rightarrow x \notin V_{y} \]
\[ \Rightarrow y \in V_{y} \cap C \subseteq U_{y} \cap C \]
\[ \Rightarrow C \subseteq \bigcup_{y \in C} U_{y} \Rightarrow C = (\bigcup_{y \in C} U_{y}) \cap C \]
\[ C = \bigcup_{y \in C} (V \cap C) \]

As \( y \in C \), \( V \) is open in \( X \), so \( V \cap C \) is open in \( C \).

\[ \forall y \in C : y \subset C \text{ is an open cover for } C \]

As \( C \) is compact, so this open cover has a finite subcover \( \{ V \cap C \} \), \( V_1 \cap C, V_2 \cap C, \ldots, V_n \cap C \).

\[ C = \bigcap_{i=1}^{n} V_i \]

Put \( U = \bigcap_{i=1}^{n} U_i \) and \( V = \bigcup_{i=1}^{n} V_i \).

\[ x \in U, C \in V \]

Now to prove \( U \cap V \neq \emptyset \):

Suppose, on the contrary, \( U \cap V = \emptyset \).

\[ x \in U \]

\[ x \in U \text{ and } x \in V \]

\[ x \in \bigcap_{i=1}^{n} U_i \text{ and } x \in \bigcup_{i=1}^{n} V_i \]

Then, there is an \( i \), \( 1 \leq i \leq n \), such that

\[ x \in U_i \text{ and } x \in V_i \]

\[ U \cap V = \emptyset \]

Which is a contradiction.

\( \therefore \) our supposition is wrong.

Hence \( U \cap V \neq \emptyset \).

**Theorem:** Compact subspace of a \( T_2 \)-space is closed.

**Proof:** Let \( X \) be a \( T_2 \)-space, and \( C \) be a compact subspace of \( X \).
To prove: \( C \) is closed.

We have \( C' \) is open.

1. If \( C' = \emptyset \), then \( C \) is open.
2. If \( C' \neq \emptyset \), then \( \forall x \in C \implies \exists y \in C \). 

Then, by a well-known theorem in a \( T_1 \)-space any point and a disjoint subspace of \( X \) can be separated by open sets in the sense. They have disjoint neighbourhoods; there exists two open sets \( U_x \) and \( V_x \) such that \( \forall x \in C \), \( C \subseteq V_x \) and \( U_x \cap V_x = \emptyset \).

Now, \( \forall x \in U_x \).

\( \exists x \in U_x \subseteq V_x \) \( \implies U_x \cap V_x = \emptyset \).

Also, \( C \subseteq V_x \)

\( \implies C \subseteq U_x \subseteq V_x \subseteq C' \).

\( \exists x \in U_x \subseteq V_x \subseteq C' \).

\( \exists x \in \neg C \).

\( U_x \cap V_x = \emptyset \).

\( \forall x \in \neg C \).

\( C' = \bigcup_{x \in \neg C} U_x \subseteq C' \).

\( C' = \bigcup_{x \in \neg C} U_x \).

\( C \subseteq \bigcup_{x \in \neg C} U_x \).

\( \forall x \), \( U_x \) is open so \( \bigcup_{x \in \neg C} U_x \) is open.

\( \implies C' \) is open.

\( \implies C \) is closed.

**Definition:** Homeomorphism

A function \( f: X \to Y \) is said to be homeomorphic if:

1) \( f \) is continuous.
2) \( f \) is open.
3) \( f \) is bijective.

**Theorem:** A 1-1 continuous mapping from a compact space \( X \) onto a \( T_1 \)-space \( Y \) is Homeomorphism.
Proof. As given, $f$ is continuous and bijective, so, we have just to prove that $f$ is open.

Let $G$ be an open set in $X$.

$\Rightarrow$ $G'$ is closed set in $X$.

As closed subset of a compact space is compact, $G'$ is compact.

Further, as, continuous image of a compact space is compact, $f(G')$ is compact.

$\Rightarrow$ $f(G')$ is a compact subspace of $Y$.

As compact subspace of $T_5$ space is closed.

And $f(G')$ is a compact subspace of $T_5$-space.

$\Rightarrow f(G)$ is closed in $Y$.

Now, $f(G') = f(X - G) = f(X) - f(G) = \overline{f(G)} - f(G)$.

$\Rightarrow \overline{f(G)}$ is closed in $Y$.

$\Rightarrow f(G)$ is open in $Y$.

$\Rightarrow f$ is open function.

Hence, $f$ is homeomorphism.

Theorem. A topological space $X$ is compact if and only if every class of closed sets with empty intersection has a finite sub class with empty intersection.

Proof. Given $X$ is compact and $\{C_{\alpha} \mid \alpha \in I\}$ be a class of closed sets in $X$ with $\bigcap_{\alpha \in I} C_{\alpha} = \emptyset$. To prove, there is a finite subclass of $\{C_{\alpha} \mid \alpha \in I\}$ with empty intersection.

Now as, $\bigcap_{\alpha \in I} C_{\alpha} = \emptyset$

$\Rightarrow (\bigcap_{\alpha \in I} C_{\alpha})' = \emptyset' \Rightarrow \bigcup_{\alpha \in I} C_{\alpha}' = X.$
As $C_\alpha$ is closed, so $C_\alpha$ is open.

$\Rightarrow \{C_\alpha : \alpha \in I\}$ is an open cover for $X$.

As $X$ is compact, so this open cover has a finite subcover $\{C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}\}$.

$\Rightarrow \bigcap_{i=1}^{n} C_{\alpha_i} = X$.

$\Rightarrow \left(\bigcup_{i=1}^{n} C_{\alpha_i}\right) = X$.

$\Rightarrow \left(\bigcap_{i=1}^{n} C_{\alpha_i}\right) = \phi$.

$\Rightarrow \{C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}\}$ is a finite subcover of $\{C_\alpha : \alpha \in I\}$ with empty intersection.

Conversely, suppose in a topological space $X$ each class $\{C_\alpha : \alpha \in I\}$ of closed sets with empty intersection has a finite subcover with empty intersection.

To prove $X$ is compact.

Now, let $\{U_\alpha : \alpha \in I\}$ be an open cover for $X$.

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha = X \Rightarrow \left(\bigcup_{\alpha \in I} U_\alpha\right) = X \Rightarrow \bigcap_{\alpha \in I} U_\alpha = \phi$.

$\Rightarrow \{\cap_{\alpha \in I} U_\alpha\} = \phi$.

$\Rightarrow \left(\bigcap_{\alpha \in I} U_\alpha\right) = \phi \Rightarrow \bigcup_{\alpha \in I} U_\alpha = X$.

$\Rightarrow \{U_\alpha = \emptyset, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ is a finite open subcover for $X$.

$\Rightarrow X$ is compact.

Hence proved.
THEOREM: Every compact $T_2$-space is normal.

Proof: Let $X$ be a compact $T_2$-space.

To prove: $X$ is normal.

Let $A$ and $B$ be the two closed disjoint subsets of $X$. We have to prove that there exists two open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Let $A = \overline{A}$ and $B$ are disjoint i.e. $A \cap B = \emptyset$.

So $A \not\subseteq B$. Then, as $X$ is compact and $B$ is closed in $X$, so $B$ is also compact.

Further as $X$ is also $T_2$ and in a $T_2$-space, a point and a disjoint compact subspace can be separated by open sets, so there exists two open sets $U_x$ and $V_x$ such that:

$A \subseteq U_x$, $B \subseteq V_x$, and $U_x \cap V_x = \emptyset$.

Now, $A \subseteq \bigcup_{x \in A} U_x$, $\overline{A} \subseteq \bigcup_{x \in A} \overline{U_x}$, $\bigcup_{x \in A} \overline{U_x} = \bigcup_{x \in A} U_x$.

$\Rightarrow A = (\bigcup_{x \in A} U_x) \setminus A \

\Rightarrow A = (\bigcup_{x \in A} U_x) \setminus (\bigcup_{x \in A} \overline{U_x}) \

\Rightarrow A = (\bigcup_{x \in A} U_x) \setminus (\bigcup_{x \in A} \overline{U_x})$.

$\Rightarrow \overline{A} = (\bigcup_{x \in A} \overline{U_x}) \

\Rightarrow \overline{A} = (\bigcup_{x \in A} \overline{U_x}) \setminus A \

\Rightarrow \overline{A} = (\bigcup_{x \in A} \overline{U_x}) \setminus (\bigcup_{x \in A} U_x) \

\Rightarrow \overline{A} = (\bigcup_{x \in A} \overline{U_x}) \setminus (\bigcup_{x \in A} U_x)$.

Put $U = \bigcup_{x \in A} U_x$ and $V = \bigcup_{x \in A} V_x$. 

Then, $A \subseteq U$, $B \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Hence, $X$ is normal.
Then \( U \) and \( V \) are open.
\[
A = U, \ B = V.
\]

Now to prove only \( \cap U = \emptyset \).

Suppose on the contrary \( \cap U \neq \emptyset \).
\[
\exists \ z \in \cap U.
\]
\[
\Rightarrow z \in U \text{ and } z \in V.
\]
\[
\Rightarrow z \in U_{\alpha_i} \text{ and } z \in V_{\alpha_i}
\]
\[
\Rightarrow \exists \ a_i \text{ for some } i.
\]
\[
\forall a_i \text{ for all } i
\]
\[
\Rightarrow U_{\alpha_i} \cap V_{\alpha_i} = \emptyset. \text{ A contradiction to our assumption is wrong and hence } \cap U = \emptyset.
\]
\[
\Rightarrow X \text{ is normal.}
\]

Heine-Borel Theorem

Statement: Every closed and bounded subset of \( \mathbb{R} \) is compact.

Proof: Closed and bounded subset of \( \mathbb{R} \) is some closed interval \([a,b]\).

Case I: If \( a = b \), then \([a,b] = [a,a] \), which is compact.

Case II: If \( a < b \), then the class of all intervals \([a,c] \cap [c,b] \) is an open subbase for \([a,b] \).

Similarly, the class of all intervals \([c,d] \cap [a,b] \) is a closed subbase for \([a,b] \).

Let \( S = \{ [a,c], [c,d] \} \) be the class of those subbasic closed sets which satisfy the finite intersection property.

Now, here arises the following cases:

\( S \) contains only the interval of the form \([a,c] \), then always \( a \leq c \).

\[\Rightarrow [a,b] \text{ is compact.}\]
ii) If $I$ contains only the intervals of the form $[a, b]$, then $I$ is compact.

iii) If $I$ contains the interval of both forms. Then, put $a = \sup \{a_i \mid i \in I\}$

To prove: $a \leq c_i$, $\forall i \in I$.

Suppose on the contrary that it is not true.

Then, for some $i_0$, $d > c_{i_0}$

$\implies c_{i_0} < d = \sup \{a_i \mid i \in I\}$

Then, there exists some $c_{i_0}$ such that $c_{i_0} < a_0$.

Then, $[a, c_{i_0}] \cap [d_0, b] = \emptyset$

$\implies I$ does not satisfy the infinite intersection property.

Which is a contradiction.

So, our supposition is wrong and hence, $a \leq c_i$, $\forall i \in I$.

$\implies \{a, b\}$ is compact. $d = \sup \{a_i \mid i \in I\}$

**Theorem:** Every compact subspace of real line $\mathbb{R}$ is closed and bounded.

**Proof:** Let $C$ be a compact subspace of $\mathbb{R}$.

To prove: $C$ is closed and bounded.

As $\mathbb{R}$ is a space and compact subspace of it.

$\mathbb{R}$ is space is closed. so, $C$ is closed.

Now, it remains only to prove that $C$ is bounded.

Let $U_k = B(a_k, k) \in \mathcal{N}$. Then, $U_k$, $k \in \mathbb{N}$ is an open cover for $\mathbb{R}$.

$B(a_k, k) = [k, k + 1]$.

Now, as $U_k$ is open in $\mathbb{R}$ and $C$ is a subspace of $\mathbb{R}$.

of $\mathbb{R}$, so, $U_k \cap C$ is a open set in $\mathbb{R}$.

Let $S_k \in \mathcal{N}$, $k \in \mathbb{N}$ be an open cover for $C$.

As $C$ is compact, so, this open cover has a finite.
Theorem. Prove that compact subspace of $\mathbb{R}^n$ is closed and bounded.

Proof. Let $C$ be a compact subspace of $\mathbb{R}^n$.

To prove $C$ is closed and bounded:

As $\mathbb{R}^n$ is $T_2$ and $C$ is a compact subspace of $\mathbb{R}^n$, so $C$ is closed.

Now, let $U_k = B(a, k)$ (where $a = (a_1, \ldots, a_n)$ and $k \in \mathbb{N}$),
then $\{U_k : k \in \mathbb{N}\}$ is an open cover for $\mathbb{R}^n$.

As $U_k$ is an open set in $\mathbb{R}^n$,
so, $U_k \cap C$ is open set in $C$.

Let $\{U_k \cap C : k \in \mathbb{N}\}$ be an open cover for $C$.

As $C$ is compact, so this open cover has a finite subcover $\{U_k \cap C : \ldots \}$.

$\Rightarrow C = \bigcup_{k=1}^{n} (U_k \cap C)$,

$\Rightarrow C = \bigcup_{k=1}^{n} U_k$ where $n = \text{max} (k_1, k_2, \ldots, k_m)$. 
\[ C = U^n \]

As \( U^n \) is bounded, so \( C \) is bounded.

\[ \text{Theorem: A continuous real valued function defined on a compact space is bounded and attains its bounds.} \]

\[ \text{Proof: Let } f: X \to \mathbb{R} \text{ be a continuous function from a compact space } X. \]

To prove \( f \) is bounded.

For this, we prove \( f(X) \) is bounded.

As \( X \) is compact and \( f \) is continuous and continuous image of a compact space is compact, so \( f(X) \) is compact. Hence \( f(X) \) is a compact subspace of \( \mathbb{R} \).

As a compact subspace of \( \mathbb{R} \) is closed and bounded, so \( f(X) \) is closed and bounded.

Let \( M = \sup f(X) \) and \( m = \inf f(X) \).

As \( f(X) \) is bounded, so \( M \) and \( m \) exist.

\( M \) and \( m \) are the limit points of \( f(X) \).

As \( f(X) \) is closed.

So \( M, m \in f(X) \).

\[ \text{Definition: Countably Compact Space.} \]

A topological space \( X \) is said to be countably compact if every countable open cover for \( X \) has a finite subcover for \( X \).

Remark: Every compact space is also countably compact space.
Theorem: A topological space $X$ is countably compact if and only if every countable collection of closed sets in $X$, which satisfy finite intersection property, has non-empty intersection.

Proof: Suppose that $X$ is countably compact. Let $\{A_n\}_{n \in \mathbb{N}}$ be the countable collection of closed sets which satisfy finite intersection property.

To prove: $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

Suppose on the contrary, $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

Then $(\bigcap_{n \in \mathbb{N}} A_n)^\complement = X$.

As $A_n$ is closed, $(\bigcap_{n \in \mathbb{N}} A_n)^\complement$ is open for all $n$.

$\Rightarrow \{\bigcap_{n \in \mathbb{N}} A_n\}^\complement = X$.

As $X$ is countably compact, this countable open cover has a finite subcover $U_1, U_2, \ldots, U_n$.

$\Rightarrow \bigcup_{i=1}^{n} U_i = X$.

$\Rightarrow (\bigcap_{i=1}^{n} U_i)^\complement = X$.

$\Rightarrow \bigcap_{i=1}^{n} U_i = \emptyset$.

$\Rightarrow \text{The class } \{\bigcap_{n \in \mathbb{N}} A_n\} \text{ does not satisfy finite intersection property. Which is a contradiction.}$

$\Rightarrow \{\bigcap_{n \in \mathbb{N}} A_n\} \text{ satisfy finite intersection property.}$

So, our supposition is wrong.

Hence, $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

Conversely, suppose that for every countable collection of closed sets $\{A_n\}_{n \in \mathbb{N}}$ which satisfy finite intersection property, has non-empty intersection.

To prove: $X$ is countably compact.
Suppose $X$ is not countably compact.

Let $\{U_n : n \in \mathbb{N}\}$ be a countably open cover for $X$.

As $X$ is not countably compact, there exists a finite subcollection $\{U_1, U_2, \ldots, U_m\}$ of $\{U_n : n \in \mathbb{N}\}$.

\[ \bigcup_{i=1}^{m} U_i \neq X. \]

\[ \Rightarrow \bigcup_{i=1}^{m} U_i \neq X. \]

As $\{U_n : n \in \mathbb{N}\}$ be collection of open sets with

\[ \bigcup_{n \in \mathbb{N}} U_n = X. \]

\[ \Rightarrow \bigcup_{n \in \mathbb{N}} U_n = X. \]

$\{U_n : n \in \mathbb{N}\}$ is the collection of closed sets

\[ \bigcap_{n \in \mathbb{N}} U_n = \emptyset. \]

$\Rightarrow \bigcap_{n \in \mathbb{N}} U_n = \emptyset.$

$\Rightarrow \bigcap_{n \in \mathbb{N}} U_n$ is the class of closed sets.

**Theorem:** Let $X$ be a topological space, then any infinite subset of $X$ has a limit point if and only if every countably infinite subset of $X$ has a limit point.

**Proof:** Let $u$ be a subset in topological space $X$.

If every infinite subset of $X$ has a limit point, then trivially, every countably infinite subset of $X$ also has a limit point.

Conversely, suppose every countably infinite subset of $X$ has the limit point.

To prove, every infinite subset of $X$ has a limit point.
Let \( A \) be an infinite subset of \( X \). Then, by a well-known result of set theory, \( A \) has a countable infinite subset \( B \).

Then, by the hypothesis, \( B \) has the limit point, say \( x \). Then, for every open set \( U \) in \( X \),

\[
\forall U \ni x, \quad U \cap B \neq \emptyset
\]

\[
\Rightarrow \quad U \cap A \neq \emptyset, \quad B = A
\]

\[
\Rightarrow \quad x \text{ is also the limit point of } A.
\]

**Theorem:** Let \( X \) be a countably compact space then every infinite subset of \( X \) has a limit point.

**Proof:** Let \( A \) be an infinite subset of a countably compact space \( X \). To prove \( A \) has a limit point.

Suppose \( A \) has no limit point.

Let \( B = \{a_1, a_2, \ldots\} \) be a countably infinite subset of \( A \). Then \( B \) has no limit point. Now, consider the subset \( C_n = \{|x_n, x_{n+1}, x_{n+2}, \ldots| \}, n \in \mathbb{N} \).

Then, as \( \forall n \), \( D_n = B \subset C_n \), \( D_n \subseteq (E \cap A) = A \).

Hence, \( C_n \) is closed for all \( n \), then \( A \) is closed.

\[
\Rightarrow \quad C_n, n \in \mathbb{N}, \text{ is a class of closed sets which satisfy finite intersection property:}
\]

Because, for every finite subcollection:

\[
\forall C_n, C_{n+1}, \ldots, C_m \quad \bigcap_{i=1}^{m} C_i = C_{i_k} \quad \text{where} \quad i_k = \max (n, n + 1, \ldots, m).
\]

Hence, \( C_n, n \in \mathbb{N}, \text{ is a class of closed sets which satisfy finite intersection property and:} \)

\[
\exists C_n = \emptyset \Rightarrow X \text{ is not countably compact.}
\]

Which is a contradiction.

So, our supposition is wrong.
Hence, A has a limit point.

BOLZANO WEIERSTRASS PROPERTY.
A space $X$ is said to satisfy B-W property if and only if every infinite subset of $X$ has a limit point in $X$.

Corollary. Every countably compact space satisfies B-W property.

Proof. Let $X$ be a countably compact space and $A$ be an infinite subset of $X$. Then, by a well-known theorem, $A$ has a limit point. So, $X$ satisfies B-W property.

SEQUENTIALLY COMPACT SPACE.
A space $X$ is said to be sequentially compact if and only if every sequence in $X$ has a convergent subsequence.

Theorem. A metric space is sequentially compact if and only if it satisfies B-W property.

Proof. A metric space is sequentially compact.
To prove. $X$ satisfies B-W property.

Let $A$ be an infinite subset of $X$.

To prove. $A$ has a limit point in $X$.

Let $\{a_n\}$ be a sequence in $A$. As $A \subseteq X$, so $\{a_n\}$ is also a sequence in $X$.

As $X$ is sequentially compact, so this sequence $\{a_n\}$ has a convergent subsequence.

Let $\{a_{n_k}\}$ be a convergent subsequence of $\{a_n\}$.
such that $x_{nk} \to x \in X$.

Let $B$ be the set of the points of $\{x_{nk}\}$.

Then, $x$ is the limit point of $B$.

As $B = A \cdot \{x\}$, $x$ is the limit point of $A$.

$\Rightarrow X$ satisfy B.W. Property.

Conversely, suppose $X$ satisfies B.W. Property.

To prove: $X$ is sequentially compact.

Let $\{x_n\}$ be a sequence in $X$.

If a point $x$ in $\{x_n\}$ repeated infinitely many times, then $(x_n, x, x, \ldots) \to x$ is a convergent subsequence of $\{x_n\}$.

If no point repeated infinitely many times then let $A$ be the set of the points of sequence $\{x_n\}$.

As $X$ satisfies B.W. Property, $A$ has limit point.

Point $x$ then we can choose a sequence $\{x_n\}$ of $\{x_n\}$ such that $x_{nk} \to x$.

$\Rightarrow X$ is sequentially compact.

**Theorem.** Every compact metric space is sequentially compact.

**Proof.** Let $X$ be a compact metric space.

To prove: $X$ is sequentially compact.

For this, we prove $X$ satisfies B.W. Property.

"Because a metric space is sequentially compact if and only if it satisfies B.W. Property."

Let $A$ be an infinite subset of $X$.

To prove: $A$ has limit point.

Suppose, $A$ has no limit point.

Then, there exists an open set ball $B(x, r)$

which does not contain any point of $A$ different
From \( \mathcal{A} \):

- \( \forall \epsilon > 0, \exists \mathcal{G}_i \) is an open cover for \( X \).
- \( \mathcal{A} \) is compact, so this open cover has a finite subcover.
- \( \forall \epsilon > 0, \exists \mathcal{G}_i \), \( \forall \epsilon > 0, \exists \mathcal{G}_i \), \( \therefore \mathcal{B}(x_i, \epsilon) \).
- As \( \mathcal{A} \) only contains centers of these open balls.
- \( \mathcal{A} \) is finite.
- Which is a contradiction.
- \( \mathcal{A} \) is an infinite set.
- So our supposition is wrong.
- Hence, \( \mathcal{A} \) has a limit point.
- \( \Rightarrow \mathcal{X} \) satisfies \( B_i \) property.
- \( \Rightarrow \mathcal{X} \) is sequentially compact.

**Lebesgue Number:**

Let \((X, d)\) be a metric space and \( \mathcal{G}_i \) be an open cover for \( X \). A real number \( a \) is called Lebesgue number for \( \mathcal{G}_i \) if \( \mathcal{G}_i \) covers \( X \) such that \( \forall x \in X \), \( \exists G \in \mathcal{G}_i \) with diameter less than \( a \) is contained in at least one \( G_i \).

**Big Set:** A subset of \( X \) is called big set if it is not contained in any \( G_i \).

**V-Map**

**Lebesgue Covering Lemma:** In a sequentially compact metric space, every open cover has a Lebesgue number.

**Proof:** Let \( X \) be a sequentially compact space and \( \mathcal{G}_i \) be an open cover for \( X \).

To prove: \( \mathcal{G}_i \) has a Lebesgue number.

Here, arises two cases.
Case I: If big set does not exist, then for all $a \geq a_0$, big set does not exist. So, for every $A \subseteq X$, such that $d(A) < a$, there is an open set $G_i$ such that $A \subseteq G_i$.

$\Rightarrow a$ is the Lebesgue number for open cover $\{G_i\}$.

Case II: If big sets exist then let $a'$ be the greatest lower bound of diameters of these big sets, then $0 < a' \leq a_0$. Then,

i) If $a' = \infty$, then any positive real number $a'$ is the Lebesgue number for the open cover $\{G_i\}$.

$\Rightarrow a'$ is not a big set because $d(A) < a'$ and $a'$ itself is infimum and infimum is less than all the other diameters.

So, for at least one $G_i$, $A \subseteq G_i \Rightarrow a$ is Lebesgue number.

ii) If $0 < a' < a_0$. Then for any $a$, $a_0 < a'$, then $a$ is Lebesgue number for open cover $\{G_i\}$ (By *)

iii) If $a' = 0$. Then, there exists a big set $B_n$ such that $d(B_n) \leq 1/n$. Now, choose $\forall i \in B_n$

$\Rightarrow \exists_i x_i$ is a sequence in $X$.

As $X$ is sequentially compact, so this sequence $\{x_i\}$ has a convergent subsequence $\{x_{i_n}\}$. Let $x \in X$. As $\{G_i\}$ is an open cover for $X$, so $X = \cup G_i \Rightarrow x \in X \Rightarrow x \in G_i$.

$\Rightarrow x \in G_i$ for some $i = i_0$.

As, $x \in G_i$ and $G_i$ is an open set.

So, there exist an open ball $B(x, r)$ such that $x \in B(x, r) = G_i$
Let \( B(x, \frac{1}{2}) \) be the concentric open ball to \( B(x, 1) \).

Then, as \( x_n \to x \in B(x, \frac{1}{2}) \), so \( B(x, \frac{1}{2}) \) contains all the points of \( x_n \) and hence of \( x \), except a finite number of points.

Let \( n_0 \) be the positive integer such that \( n_0 > \frac{1}{2} \).

We claim that \( B_{n_0} = B(x, \frac{1}{2}) \) is \( G_0 \).

Let \( y \in B_{n_0} \), now by triangular inequality:

\[
d(y, x) \leq d(y, x_{n_0}) + d(x_{n_0}, x)\]

\[
\Rightarrow d(y, x_{n_0}) \leq d(y, x) - d(x_{n_0}, x).
\]

As \( y, x_{n_0} \in B_{n_0} \),

\[
d(y, x_{n_0}) \leq d(y, x) - d(x_{n_0}, x) = \frac{1}{2} < \frac{1}{2}.
\]

Also, as \( x_{n_0} \in B(x, \frac{1}{2}) \),

\[
d(x_{n_0}, x) \leq \frac{1}{2}.
\]

So,

\[
d(y, x) \leq d(y, x_{n_0}) + d(x_{n_0}, x) \leq \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}.
\]

\[
\Rightarrow y \in B(x, \frac{1}{2}) \subset G_0.
\]

\[
\Rightarrow B_{n_0} \subset G_0.
\]

\( B_{n_0} \) is not a big set:

Which is a contradiction.

by \( d \to 0 \). Hence, \( d(x, y) \to \infty \).

Thus, \( \exists \) \( n_0 \) and \( n \) (ii) Lebesgue number exists.

Hence Proved.

**Definition:** (E-net)

Let \((X, d)\) be a metric space and \( e > 0 \), a subset \( A \) of \( X \) is called an E-net if:

1) \( A \) is finite,
2) \( X = \bigcup_{a \in A} B(a, e) \).

**Definition:** Totally Bounded.

A metric space \((X, d)\) is said to be
Theorem: Every sequentially compact metric space is totally bounded.

Proof: Let \((X, d)\) be a sequentially compact metric space.

To prove: \(X\) is totally bounded.

Let \(\varepsilon > 0\) and \(a \in X\):

If \(B(a, \varepsilon) = X\), then \(a\) is an \(\varepsilon\)-net.

If \(B(a, \varepsilon) \neq X\),

Let \(a_2 \in X\) such that \(a_2 \notin B(a, \varepsilon)\).

Now, if \(B(a_1, \varepsilon) \cup B(a_2, \varepsilon) = X\), then \(a_1, a_2\) is an \(\varepsilon\)-net for \(X\).

If \(B(a_1, \varepsilon) \neq X\),

Then, \(a_3 \in X\) such that \(a_3 \notin B(a_1, \varepsilon) \cup B(a_2, \varepsilon)\).

Now, if \(B(a_1, \varepsilon) \cup B(a_2, \varepsilon) \cup B(a_3, \varepsilon) = X\), then \(a_1, a_2, a_3\) is an \(\varepsilon\)-net for \(X\).

If \(B(a_1, \varepsilon) \cup B(a_2, \varepsilon) \cup B(a_3, \varepsilon) \neq X\),

Then, continuing this way, we have \(B(a_1, \varepsilon) \cup B(a_2, \varepsilon) \cup \ldots \cup B(a_n, \varepsilon) = X\).

If \(X\) has no convergent subsequence,

\(X\) is not sequentially compact.

Which is a contradiction.

So our supposition is wrong.

And hence \(a_1, a_2, \ldots \) and is an \(\varepsilon\)-net.

\(\Rightarrow X\) is totally bounded.
Theorem: Every sequentially compact metric space is compact.

Proof: Let \( X \) be a sequentially compact metric space. To prove \( X \) is compact.

Let \( \{ G_i \} \) be an open cover for \( X \). As \( X \) is sequentially compact, by the Heine-Borel Lemma, \( \{ G_i \} \) has a Lebesgue number \( a \) such that \( \varepsilon = \frac{a}{3} > 0 \). As every sequentially compact metric space is totally bounded, then for \( \varepsilon > 0 \), it has an \( \varepsilon \)-net.

Let \( A = \{ a_1, a_2, a_3, \ldots, a_n \} \).

\[ X = \bigcup_{k=1}^{n} B(a_k, \varepsilon), \text{ for } a_k \in A. \]

Now, \( d(B(a_k, \varepsilon)) < 2\varepsilon \leq a(\frac{a}{3}) \leq a. \)

\[ \Rightarrow d(B(a_k, \varepsilon)) \leq a. \]

Then, by the definition of a Lebesgue number,
\[ B(a_k, \varepsilon) \subseteq G_k \text{ for some } G_k \in \{ G_i \}. \]

\[ \Rightarrow \bigcup_{k=1}^{n} B(a_k, \varepsilon) \subseteq \bigcup_{k=1}^{n} G_k. \]

\[ \Rightarrow X = \bigcup_{k=1}^{n} B(a_k, \varepsilon) \subseteq \bigcup_{k=1}^{n} G_k \subseteq X. \]

\[ \Rightarrow X = \bigcup_{k=1}^{n} G_k. \]

\[ \Rightarrow \{ G_1, G_2, \ldots, G_n \} \text{ is a finite subcover for } X. \]

\[ \Rightarrow X \text{ is compact.} \]
Theorem: A continuous function from a compact metric space to a metric space is uniformly continuous.

Proof: Let \( f: X \to Y \) be a continuous mapping from a compact metric space \( X \) to a metric space \( Y \).

To prove: \( f \) is uniformly continuous.

Let \( d \) and \( d' \) be the metrics on \( X \) and \( Y \) respectively.

Let \( x \in X \) and \( \varepsilon > 0 \). Now, as \( x \in X \), \( f(x) \in Y \).

Now consider the open ball \( B(f(x), \varepsilon/3) \), which is an open set in \( Y \).

As \( f: X \to Y \) is continuous, so \( f^{-1}(B(f(x), \varepsilon/3)) \) is an open set in \( X \). (Inverse image of each open set is open.)

Then, \( \bigcup_{x \in X} f^{-1}(B(f(x), \varepsilon/3)) = X \).

\[ y \in f^{-1}(B(f(x), \varepsilon/3)) \text{ for some } x \in X \]

As \( X \) is compact and every compact metric space is sequentially compact and as every sequentially compact metric space has a Lebesgue number \( \delta \), \( X \) has a Lebesgue number, say \( \varepsilon/3 \) for open cover \( \mathcal{U} \).

Let \( x_1, x_2 \in X \) such that \( d(x_1, x_2) \leq \delta \).

Now as \( d(x_1, x_2) \leq \delta \)

\[ d(f(x_1), f(x_2)) \leq \varepsilon/3. \]

Then, by the definition of Lebesgue number \( \{U_1, U_2\} \subseteq f^{-1}(B(f(x^*), \varepsilon/3)) \).
\[ f(x, y) = B(f(x), \frac{\varepsilon}{2}) \]

\[ f(x^1, x^2) \subseteq B(f(x^1), \frac{\varepsilon}{2}) \]

\[ f(x), f(x^1) \subseteq B(f(x^1), \frac{\varepsilon}{2}) \]

\[ d_a(f(x), f(x^1)) \leq \frac{\varepsilon}{2} \]

\[ d_a(f(x^1), f(x^2)) \leq \frac{\varepsilon}{2} \]

Now, \[ d_a(f(x), f(x^1)) \leq d_a(f(x^1), f(x^2)) + d_a(f(x^2), f(x)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \]

\[ d_a(f(x), f(x^1)) \leq \varepsilon \]

\[ \Rightarrow f \text{ is uniformly continuous} \]

Theorem. Every sequentially compact metric space is countably compact.

Proof: Let \( X \) be a sequentially compact metric space. To prove: \( X \) is countably compact.

Let \( C = \{ U_1, U_2, \ldots, U_n, \ldots \} \) be a countable open cover for \( X \).

Suppose \( X \) is not countably compact. Then, this open cover has no finite subcover \( U_{i_1}, U_{i_2}, \ldots, U_{i_k} \) of \( C \)

\[ \bigcup_{j=1}^{k} U_{i_j} \neq X \]

Then, there is \( x_n \in X \) such that

\[ x_n \notin \bigcup_{j=1}^{k} U_{i_j} \]
so then, we have an infinite sequence $x_0 x_1 x_2 \ldots$ in $X$.

But as $X$ is sequentially compact, so this sequence has a convergent subsequence $x_{n_k}$ and $x_{n_k} \rightarrow x \in X$.

Now as $X = \bigcup_{n=1}^{\infty} U_n$,

$$\exists x \in \bigcup_{n=1}^{\infty} U_n,$$

$$\Rightarrow x \in U_n \text{ for some } n \in \mathbb{N}.$$  

But, then for sufficiently large $k$, $x_{n_k} \in U_k$.

Which is a contradiction.

So, our supposition is wrong.

Hence, $X$ is countably compact.

**Definition:** $\varepsilon$-net:

Let $(X,d)$ be a metric space, and $\varepsilon > 0$ be any real number and $M \subseteq X$.

Then a finite subset $A$ of $X$ is said to be $\varepsilon$-net for $M$ if for every $x \in M$ there is at least one $a \in A$ such that $d(x,a) \leq \varepsilon$, i.e., $x \in B(a,\varepsilon)$. If $M = X$, then $A$ is called $\varepsilon$-net for $X$.

**Theorem:** A subset $A = \{a_1, a_2, \ldots, a_n\}$ of $X$ is an $\varepsilon$-net for $X$ iff $X = \bigcup B(a_i,\varepsilon)$.

**Proof:** Suppose $A$ is $\varepsilon$-net for $X$.  


To prove: \( X = \bigcup_{i=1}^{n} B(ai, \varepsilon) \).

\[ \bigcup_{i=1}^{n} B(ai, \varepsilon) \subseteq X. \]

Let \( x \in X \). Then, as \( A_i \) is an \( \varepsilon \)-net for \( X \),
so there exists \( a_i, 1 \leq i \leq n \), in \( A_i \) such that
\[ d(x, a_i) \leq \varepsilon, \]
\[ \Rightarrow x \in B(ai, \varepsilon). \]
\[ \Rightarrow x \in \bigcup_{i=1}^{n} B(ai, \varepsilon). \]

As \( x \in X \) is an arbitrary \( x \),
\[ X \subseteq \bigcup_{i=1}^{n} B(ai, \varepsilon). \]
\[ \Rightarrow X = \bigcup_{i=1}^{n} B(ai, \varepsilon). \]

Conversely, suppose \( X = \bigcup_{i=1}^{n} B(ai, \varepsilon) \).

To prove, \( A \) is an \( \varepsilon \)-net for \( X \):

Let \( x \in X \),
\[ \Rightarrow x \in \bigcup_{i=1}^{n} B(ai, \varepsilon). \]
\[ \Rightarrow x \in B(ai, \varepsilon), \text{ for some } a_i \in A. \]
\[ \Rightarrow d(x, a_i) \leq \varepsilon, \text{ for some } a_i \in A. \]
\[ \Rightarrow A \text{ is } \varepsilon \text{-net for } X. \]

**Theorem:** Every totally bounded metric space is bounded.

**Proof:** Let \((X, d)\) be a totally bounded metric space. To prove: \( X \) is bounded.
As $X$ is totally bounded, then, for $\varepsilon > 0$,

$X$ has $\varepsilon$-net $F = \{a_1, a_2, \ldots, a_n\}$.

i.e., $x = \bigcap_{i=1}^n B(a_i, \varepsilon)$. So then, for every

$d(x, x')$, there is $a_i \in F$ such that $d(a_i, x') \leq \varepsilon$.

Similarly, for $y \in X$, there is $a_j \in F$ such that $d(y, a_j) \leq \varepsilon$.

Now, $d(x, y) \leq d(x, a_j) + d(a_j, y)$

$\leq d(x, a_j) + d(a_i, a_j) + d(a_j, y)$

$\leq \varepsilon + d(a_i, a_j) + \varepsilon$

$= d(a_i, y) \leq d(a_i, a_j) + \varepsilon$

$s(X) \leq s(F) + 2\varepsilon$

As $F$ is finite, so, diameter of $F$ is finite.

So, $s(F)$ is finite.

$\Rightarrow s(X)$ is finite.

$\Rightarrow s(X)$ is bounded.

These Notes are the lectures delivered by Tahir Mahmood.
**Definition: Disconnected**

A topological space \((X,F)\) is said to be disconnected if there exists two non-empty open (or closed) sets \(A\) and \(B\) in \(X\) such that \(A\cap B = \emptyset\).

\[\text{e.g. If } X = \{1, 2, 3, 4, 5\}, A = \{1, 2, 3\}, B = \{4, 5\}\]

Then, \((X,F)\) is disconnected because we have \(A \cap B = \emptyset\).

**Definition: Connected Spaces**

A topological space \((X,F)\) is said to be connected if it is not disconnected.

\[\text{e.g. If } X = \{1, 2, 3, 4\}, A = \{1, 2\}, B = \{3, 4\}\]

Then, \((X,F)\) is connected.

**Remark:** For any set \(X\), if \(F = \{\emptyset, X\}\), then \((X,F)\) is connected.

**Theorem:** For any \(X\) with more than two points, \((X,F_0)\) is disconnected.

**Proof:** Since \(F_0\) is the power set of \(X\), so every subset of \(X\) is open (as well as closed).

Proposition: Let \(A \subseteq X\). Then \(A^c = X \setminus A \neq \emptyset\).

Also, \(A^c\) is open (\(A^c\) is also closed).

So we have two open sets \(A\) and \(B = A^c\) such that \(A \cup B = X\) and \(A \cap B = \emptyset\).

So \((X,F_0)\) is disconnected.

Hence proved.
PAGE NOT FOUND

>> ERROR

MATHCITY.ORG
THEOREM: If \( X \) is infinite, then \((X, \mathcal{F}_0)\) is connected.

Proof: On the contrary, suppose \((X, \mathcal{F}_0)\) is disconnected. Then, there exist two open sets (or closed sets) \( A \) and \( B \) such that \( A \cup B = X \) and \( A \cap B = \emptyset \)

Now, as \( A \) and \( B \) are open and \( \mathcal{F}_0 \) is finite, \( \Rightarrow A' \) and \( B' \) are finite. Now \( A \cap B = \emptyset \)

\( \Rightarrow (A \cap B)' = \emptyset \)

\( \Rightarrow A' \cup B' = X \)

\( \Rightarrow X \) is finite (Union of two finite sets is finite).

Which is a contradiction.

\( X \) is infinite.

So, our supposition is wrong.

Hence, \((X, \mathcal{F}_0)\) is connected.

THEOREM: Continuous image of a connected space is connected.

Proof: Let \( X \) be a connected space and \( f: X \to Y \) be a continuous function.

To prove: \( f(X) \) is connected.

Suppose on the contrary that \( f(X) \) is disconnected. Then, there exist two non-empty open sets \( A \) and \( B \) in \( f(X) \) such that \( A \cap B = f(X) \) and \( A \cap B = \emptyset \).

Now, \( A \cup B = f(X) \)

Now, as \( A \) and \( B \) are open in \( f(X) \) and \( f \) is continuous function.
So, $f^{-1}(A)$ and $f^{-1}(B)$ are open in $X$.

Further, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(f(X)) = X$.

And, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$.

$\Rightarrow$ $X$ is disconnected.

Which is a contradiction.

$\Rightarrow$ $X$ is connected.

So, our supposition is wrong.

Hence, $f(X)$ is connected.

**Theorem:** The space $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is disconnected.

**Proof:** Let $r$ be any irrational number.

Then, $J = (J_r \cap \mathbb{Q})$ are open in $\mathbb{R}$.

$\Rightarrow J = (J_r \cap \mathbb{Q}) \cup (J_r \cap \mathbb{Q})$ is a union of open sets in $\mathbb{Q}$ with

$$J = (J_r \cap \mathbb{Q}) \cup (J_r \cap \mathbb{Q}) = (J_r \cap \mathbb{Q}) \cap (J_r \cap \mathbb{Q})$$

$A = B, A \cap B = A$

And $J = (J_r \cap \mathbb{Q}) \cap (J_r \cap \mathbb{Q}) = (J_r \cap \mathbb{Q}) \cap (J_r \cap \mathbb{Q})$

$\emptyset \cap \mathbb{Q} = \emptyset$

$\Rightarrow \mathbb{Q}$ is disconnected.
Theorem. A topological space \( X \) is disconnected if and only if \( X \) contains a non-empty subset \( A \) which is both open and closed.

Proof. Suppose \( X \) is disconnected and \( A \neq \emptyset \) be a subset of \( X \).

To prove: \( A \) is both open and closed.

As \( X \) is disconnected, there exists two open sets \( A \) and \( B \) such that \( A \cup B = X \) and \( A \cap B = \emptyset \).

Now \( A \) is open.

Now, as \( A \cup B = X \) and \( A \cap B = \emptyset \), by law of complements \( B = A' \).

Now \( B \) is open \( \Rightarrow A' \) is open \( \Rightarrow A \) is closed.

\( \Rightarrow \) \( A \) is both open and closed.

Conversely, suppose that for topological space \( X \), there is a non-empty subset \( A \) of \( X \) which is both open and closed.

To prove: \( X \) is disconnected.

Let \( B = A' \), so \( A \) is closed.

\( \Rightarrow A' \) is open \( \Rightarrow B \) is open.

\( \Rightarrow A \) and \( B \) are open in \( X \) with \( A \cup B = A \cup A' = X \)

And \( A \cap B = A \cap A' = \emptyset \).

\( \Rightarrow \) \( X \) is disconnected.
Theorem: A space $X$ is connected if and only if there does not exist a continuous surjective function from $X$ to discrete two point space.

Proof. Let $X$ be connected.

To prove: There does not exist a continuous surjective function from $X$ to discrete two point space $Y = \{a, b\}$.

Suppose on the contrary that there exists a function $f : X \to Y = \{a, b\}$, $Y$ is discrete, which is continuous and onto. As $Y$ is discrete so $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$ are open sets.

As $f$ is continuous so $f^{-1}(\emptyset), f^{-1}(\{a\}), f^{-1}(\{b\})$ and $f^{-1}(\{a, b\})$ all are open in $X$.

Now as $f$ is onto, so $f(X) = Y \Rightarrow X = f^{-1}(Y)$.

$\Rightarrow X = f^{-1}(\{a, b\})$

$= f^{-1}(\{a\} \cup \{b\})$

$= f^{-1}(\{a\}) \cup f^{-1}(\{b\})$.

Further $f^{-1}(\{a\}) \cap f^{-1}(\{b\}) = f^{-1}(\emptyset) = \emptyset$.

$\Rightarrow X$ is disconnected.

Which is a contradiction.

$\therefore X$ is connected.

So, our supposition is wrong.

Hence, there does not exist a continuous function from $X$ onto discrete two point space $Y$. 

Conversely, suppose there does not exist a continuous function from \( X \) onto a discrete two-point space \( Y \).

To prove: \( X \) is connected.

Suppose on the contrary that \( X \) is not connected. Then \( X \) is disconnected, so there exists two non-empty, open (or closed) sets \( A \) and \( B \) in \( X \) such that \( A \cup B = X \) and \( A \cap B = \emptyset \).

Now define a function \( f: X \to Y = \{a, b\} \) by

\[
 f(a) = a \quad \text{and} \quad f(b) = b
\]

\[
 A = f^{-1}(\{a\}) \quad \text{and} \quad B = f^{-1}(\{b\})
\]

Now as \( Y = \{a, b\} \) is discrete, so \( \varnothing, \{a\}, \{b\}, \{a, b\} \) are open in \( Y \).

Now \( f^{-1}(\emptyset) = f^{-1}(\{a\} \cup \{b\}) \)

\[
 = f^{-1}(\{a\}) \cup f^{-1}(\{b\}) = A \cup B = X
\]

So \( f \) is continuous. Inverse image of each open set is open and here all four open sets of \( Y \) have which is a contradiction (open inverse images).

So our supposition is wrong.

Hence \( X \) is connected.

Hence Proved.
**Theorem:** A topological space $X$ is disconnected iff there exist a continuous function from $X$ onto discrete two points space.

**Theorem:** A topological space $X$ is said to be connected iff every continuous function from $X$ to discrete space $Y$ reduces to a constant function.

**Proof:** Suppose $X$ is connected. Then, $X$ has no proper subset which is both open and closed. Let $a$ be an element of $X$ which is discrete. So $a$ is both open and closed.

$\Rightarrow f^{-1}(\{a\})$ is open and closed in $X$.

$\Rightarrow f^{-1}(\{a\})$ is not a proper subset of $X$.

$\Rightarrow f^{-1}(\{a\}) = \emptyset$ or $f^{-1}(\{a\}) = X$.

But $f^{-1}(\{a\}) \neq \emptyset$.

So, $f^{-1}(\{a\}) = X \Rightarrow f(X) = \{a\}$.

$\Rightarrow f$ is constant function.

Conversely, suppose every continuous function $f$ from $X$ to discrete space $Y$ reduces to a constant function.

To prove: $X$ is connected.

Suppose $X$ is disconnected. Then, there exists a continuous function $f: X \to \{0,1\}$. Which is continuous and is onto and $Y$ is discrete. $f(X)=Y \Rightarrow f$ is not constant.
Which is a contradiction.
So our supposition is wrong.
And hence, \( X \) is connected!

**Theorem:**
\[ \text{Let } X \text{ be disconnected space with disconnection } \{ A, B \} \text{ and } C \text{ is a connected subspace of } X. \text{ Then, either } C \subseteq A \text{ or } C \subseteq B. \]

**Proof:**
Suppose on the contrary that \( C \not\subseteq A \) and \( C \not\subseteq B \). Then, \( C \cap A \) and \( C \cap B \) are non-empty open sets in \( C \) (=C is subspace)
so \( C \cap A \) is open in \( C \)
with \( (C \cap A) \cup (C \cap B) = C \cap (A \cup B) \)
\[ = C \cap X = C \]
\[ = C \cap \emptyset = \emptyset \]
\[ \Rightarrow C \text{ is disconnected} \]
A contradiction.
So our supposition is wrong.
Hence, \( C \subseteq A \) or \( C \subseteq B \).

**Theorem:**
\[ \text{Let } X = \bigcup_{a \in I} X_a \text{ where each } X_a \text{ is connected and } \bigcap_{a \in I} X_a \neq \emptyset. \text{ Then } X \text{ is connected.} \]

**Proof:** Suppose \( X \) is disconnected. Then there exists two non-empty sets \( A \) and \( B \) in \( X \) such that \( A \cup B = X \) and \( A \cap B = \emptyset \).
Now as \( X = \bigcup_{a \in I} X_a \Rightarrow \) for each \( a \in I \), \( X_a \subseteq X \).
As for each $x \in I$, $X_x$ is connected, so, by well known theorem, for each $x \in I$,
either $X_x = A$ or $X_x = B$.

But as $\bigcup_{x \in I} U_x \neq \emptyset$,
so $\bigcup_{x \in I} U_x = A$ or $\bigcup_{x \in I} U_x = B$.

$\Rightarrow \quad X = A$ or $X = B$.

If $X = A \Rightarrow A = X \Rightarrow B = \emptyset$.
If $X = B \Rightarrow B = X \Rightarrow A = \emptyset$.

Which is a contradiction.

Both $A$ and $B$ are non-empty.

So, our supposition is wrong.
And, hence $X$ is connected.

THEOREM: A topological space $X$ is connected if, and only if, for every pair of points in $X$ there is some connected subspace of $X$ which contains both.

PROOF: Suppose $X$ is connected and $x, y \in X$ such that $x \neq y$. To prove: there is some connected subspace of $X$ which contains both $x$ and $y$. Then $X$ itself is the connected subspace of $X$ which contains both $x$ and $y$.

Conversely, suppose in a topological space $X$, for every pair of points $x, y \in X$ such that $x \neq y$, there is some connected...
A subspace of $X$, which contains both $x$ and $y$.

To prove: $X$ is connected.

Now let, $a \in X$ be some fixed point such that for $x \in X$, $a \neq x$. Then, by the hypothesis, there is a connected subspace $C_{ax} \subset X$ such that $a \in C_{ax}$.

Then we have a collection $\{C_{ax} : x \in X\}$ of connected subspaces of $X$, such that,

$\bigcap_{x \in x} C_{ax} \neq \emptyset$ and $\bigcup_{x \in x} C_{ax} = X$.

Then, by a well known theorem, $X$ is connected.

**Theorem:** Let $C$ be a connected subspace of $X$ and $A$ a non-empty subset of $X$.

If $C \subseteq A \subseteq C$. Then, $A$ is connected. In particular, $C$ is connected.

**Proof:** Suppose on the contrary that $A$ is disconnected. Then, there exists two non-empty open sets $U$ and $V$ of $A$ such that,

$U \cup V = A$ and $U \cap V = \emptyset$.

As $U$ and $V$ are open in $A$ and $A$ is a subspace of $X$. So, then there exists two disjoint open sets $U'$ and $V'$ in $X$ such that,

$U' \cap V' = \emptyset$ and $U = U' \cap A$ and $V = V' \cap A$.

Now, $C \subseteq A = U \cup V$, $U \cap V = \emptyset$.

$\Rightarrow C \subseteq UV$. and $C$ is connected and $UV = \emptyset$. Then, by a well known theorem, either $C \subseteq U$ or $C \subseteq V$.

Without any loss of generality, suppose.
$C \subseteq U.$

As $U \cap V = \emptyset \Rightarrow U \subseteq V'$

$\Rightarrow C \subseteq U \subseteq V' \Rightarrow C \subseteq V'$

As $V'$ is open $\Rightarrow V'$ is closed.

So $V'$ is the closed superset of $C$.

But $C$ is the smallest closed superset of $C$.

So $C \subseteq V'$

$\Rightarrow C \subseteq U \subseteq C$ (given)

$\Rightarrow C = A \subseteq C \subseteq V'$

$\Rightarrow A \subseteq V' \Rightarrow \text{any} = \emptyset \Rightarrow V_1 = \emptyset$

Which is a contradiction.

$\Rightarrow V_1 \neq \emptyset$

So, our supposition is wrong.

Hence, $A$ is connected.

Now to prove $C$ is connected.

As $C \subseteq C \subseteq C$ so, by the above argument

$C$ is connected.

**Theorem:** A subspace $X$ of a real line $\mathbb{R}$ is connected if and only if $X$ is an interval.

**Proof:** Suppose $X$ is connected. To prove:

$X$ is an interval. Suppose $X$ is not an interval, then there exists $y, x \in X$ such that:

$x < y < x$ and $x, x \in X$ but $y \notin X$.

Now $1 - y$ and $y \in \mathbb{R}$ are open in $\mathbb{R}$.

$\Rightarrow 1 - y \cap [n, x]$ and $y \cap [n, x]$ are open

in $X$: with

$(1 - y \cap [n, x]) \cup (y \cap [n, x]) = (1 - y \cup [n, x] \cap [n, x] = (R \cup [n, x]) \cap [n, x] = X$.
\[
\begin{align*}
\text{and } & \quad \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} I \cap \bigcap_{n=1}^{\infty} I = \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} I \\
& = \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} I \cap I_n = I_n = \emptyset \\
\Rightarrow & \quad X \text{ is disconnected.} \\
& \quad \text{Which is a contradiction.} \\
\text{So, our supposition is wrong.} \\
\text{Hence, } X \text{ is connected.}
\end{align*}
\]

Conversely, suppose \( X \) is an interval.

To prove: \( X \) is connected. On the contrary,
suppose \( X \) is disconnected. Then there exists
two non empty open disjoint subsets \( A \) and \( B \) of \( X \) such that \( A \cup B = X \) and \( A \cap B = \emptyset \).

Let \( a \in A \) and \( b \in B \).

As \( A \cap B = \emptyset \Rightarrow a \neq b \).

Let \( a < b \). Put \( y = \sup \{a, b \mid a, b \in A \} \).

Then, by the definition of supremum for
\( \forall \epsilon > 0 \), there is some point \( a' \) in \( A \) such that
\( y - \epsilon < a' \).

\[ y - a' < \epsilon \]
\[ a' + \epsilon < y \]
\[ a' \in B(y, \epsilon) \]

So, every open ball with centre \( y \)
contains a point of \( A \) different from \( y \).

\[ y \text{ is the limit point of } A. \]

As \( A \) is also closed,
so \( y \in A \). Similarly \( y \in B \Rightarrow A \cap B = \emptyset \).
Which is a contradiction.

\[ A \cap B = \emptyset \]

So, our supposition is wrong.

Hence, \( X \) is connected.

**Component (Def)**

The maximal connected subspace of a topological space \( X \) is called component of \( X \).

i.e., a connected subspace of topological space \( X \) is called component of \( X \) if it is not contained in any other connected subspace of \( X \).

**Theorem.** Let \( X \) be a topological space, then:

i) Each \( x \in X \) is contained in exactly one component of \( X \).

ii) Each connected subspace \( S \) of \( X \) is contained in exactly one component of \( X \).

iii) Each connected subspace \( S \) of \( X \) which is both open and closed is component of \( X \).

iv) Every component of \( X \) is closed in \( X \).

**Proof:** 1) Let \( C = \{ C_x : x \in X \} \) be a collection of all connected subspace \( C_x \) of \( X \) which contains \( x \).

Then, \( \bigcup_{C_x} = X \).

Then, by a well-known theorem,

\[ C = U_{C_x} \text{ is connected.} \]
Subspace of $X$ and $x \in C$ and for every $x \in I$, $C_x = C$. This shows that $C$ is component of $X$.

Now we show that $C$ is the only component of $X$ containing $x$. To this end, let $C^*$ be another component of $X$ containing $x$.

Now as $C^*$ is the component of $X$ containing $x$ and $C$ is connected subspace of $X$, so,

$C = C^*$. Also as $C^*$ is connected subspace of $X$ containing $x$, so $C^* = X$.

$\Rightarrow C^* \subseteq U \cap x = C \\
\Rightarrow C^* \subseteq C \Rightarrow C = C^*$

This shows that $C$ is the only component of $X$ containing $x$.

ii) Let $A$ be a connected subspace of $X$ and to prove $A$ is contained only in one component of $X$. Let $\mathcal{C} = \{C \subseteq X | C$ is connected and $C \cap A \neq \emptyset \}$ and $\mathcal{C} = C \cap A$, which is connected subspace of $X$. Also, $C = C$.

$\Rightarrow C$ is connected subspace of $X$. Also, $C = C$.

$\Rightarrow C$ is connected subspace of $X$ containing $A$. Also $C = U \cap x$ so $C$ is such maximal connected subspace of $X$.

$\Rightarrow C$ is component of $X$ containing $A$.

Now we show $C$ is the only component of $X$ containing $A$.

For this, let $C^*$ be another component of $X$ containing $A$. Now as $C^*$ is maximal.
connected subspace of $X$ containing $A$ and $C$ is connected.

Further, let $C^*$ be the connected subspace of $X$ containing $A$ and $C = C^*$.

Thus, $C^* = \bigcup_{x \in C} x = C$.

Hence, $C = C^*$.

Therefore, $A$ is only component of $X$ containing $C$.

iii) Let $A$ be a connected subspace of $X$.

To prove: $A$ is component of $X$.

Suppose, $A$ is not component of $X$.

Then, $A$ is contained in exactly one component of $X$, say $C$. As $C$ is component of $X$, $A = C$ and $A$ is not component of $X$.

This is a contradiction.

Hence, $A$ is component of $X$.
iv). Let $C$ be a component of $X$.
To prove: $C$ is closed.
For this, we prove $C = C'$.
Suppose $C = C$.
Now as $C = C$ and $C + C = C = C$, then by a well-known theorem, $C$ is connected $\Rightarrow C$ is connected.
Subspace of $X$ containing $C \subseteq C$ is not component of $X$.
A contradiction.
- $C$ is component of $X$.
So our supposition is wrong and hence $C = C'$.
$\Rightarrow C$ is closed.

totally disconnected (def).
A topological space $X$ is called totally disconnected if for each pair of points of $X$, we can find a disconnection $A \cup B$ of $X$ such that $A \cap B$.

Theorem: Every discrete space is totally disconnected.
Proof: Let $X$ be a discrete space.
To prove: $X$ is totally disconnected.
Let $x, y \in X$ such that $x \neq y$.
Let $U = \{x\}$ and $V = X - \{x\}$.
As $X$ is discrete, so $U$ and $V$ are open in $X$.
Also clearly,
$x \in U$, $y \in V$, $U \cap V = \emptyset$, $U \cup V = X$.
$\Rightarrow X$ is totally disconnected.
**Theorem:** Every totally disconnected is $T_0$ space.

**Proof:** Let $X$ be a totally disconnected.
To prove: $X$ is a $T_0$ space.

Let $x, y \in X$ such that $x \neq y$.
As $X$ is totally disconnected, so there exist two open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$, $UV = \emptyset$ and $UV = X$.

i.e., we have two open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$, and $UV = \emptyset$.

$\Rightarrow X$ is a $T_0$ space.

**Theorem:** A subspace $\mathbb{Q}$ of reals in the line $\mathbb{R}$ is totally disconnected.

**Proof:** To prove: $\mathbb{Q}$ is totally disconnected in $\mathbb{R}$.
Let $x, y \in \mathbb{Q}$ such that $x \neq y$.
Without any loss of generality, suppose $x < y$.

Now as by a well-known theorem of calculus, there is an irrational number between every two rational numbers. There is an irrational number $t$ such that $x < t < y$.

Now, $]-\infty, t]$ and $]t, +\infty[$ are two open sets in $\mathbb{R}$. Now as $\mathbb{Q}$ is subspace of $\mathbb{R}$, so

$U = \cap \mathbb{Q}\{]-\infty, t]\}$ and $V = \cap \mathbb{Q}\{]t, +\infty[\}$

are open in $\mathbb{Q}$.

Also, $x \in U$, $y \in V$, $UV = \emptyset$, $UV = \mathbb{Q}$.

$\Rightarrow \mathbb{Q}$ is totally disconnected.

Hence Proved.
THEOREM: The components of totally disconnected space are its singleton subsets.

Proof: Let $X$ be a totally disconnected space. To prove: Components of $X$ are its singleton subsets. For this, we show that no two points subspace of $X$ is connected.

Let $x, y \in X$ such that $x \neq y$ and $C = \{x, y\}$ be a subspace of $X$. As $X$ is totally disconnected and $x, y \notin C$, we then there exist tilto $f$ of $x$ and $y$ in $X$ such that $\nexists U, \ n \in X$ such that $x \in U, \ y \in U, \ UV = X, \ UN \neq \phi$.

Now as $U$ and $V$ are open in $X$ and $C$ is subspace of $X$, so,

$\text{Unc}(C) \cap \text{Unc}(C) = \text{Unc}(C) \cap \text{Unc}(C) = X \cap C = C, \quad (\text{Unc}) \cap \text{Unc}(C) = (\text{Unc}) \cap \text{Unc}(C) = X \cap C = C$

Hence Proved.

THEOREM: If a $T_0$-space has an open base whose sets are also closed, then $X$ is totally disconnected.

Proof: Let $x, y \in X$ such that $x \neq y$. As $X$ is a space, so then, there exists two open sets $U$ and $V$ in $X$ such that $x \in U, \ y \in V$ and $UV = \phi$. 

\[ \text{(Unc)} \cap \text{Unc}(C) = X \cap C = C, \quad (\text{Unc}) \cap \text{Unc}(C) = X \cap C = C \]
Let \( B \) be an open base for \( X \) whose elements are also closed. As \( x \in U \), \( U \) is an open set, \( B \) is base, so there is \( B \in B \) such that \( x \in B \subseteq U \).

Now, as \( B \subseteq U \) and \( \cup V = \emptyset \),

so \( \emptyset = \cup V \Rightarrow \forall \in V \Rightarrow x \in B \subseteq W \).

As \( B \in B \), so \( B \) is closed.

\( \Rightarrow B' \) is open. Now \( B \) and \( W \) are two open sets in \( X \) with \( x \in B \), \( y \in B' \).

\( \Rightarrow y \in B' \Rightarrow y \in W \).

Also \( B \cup W = B \cup B' = X \).

\( \Rightarrow X \) is totally disconnected.

**Theorem:** Let \( X \) be a compact, non-disjoint space. Then \( X \) is totally disconnected if it has an open base whose sets are also closed.

**Proof:** Let the compact \( T \) space has an open base whose sets are closed. To prove:\n\( X \) is totally disconnected.

Let \( x \neq x' \) such that \( x \neq x' \). As \( X \) is \( T \),

so there is an open set \( U \) such that

\( x \in U \) and \( x' \notin U \). Now as \( x \in U \) and \( U \) is open in \( X \) with \( X \) has base \( B \). Then, there is \( G \in B \) such that \( x \in G \subseteq U \).

As \( y \notin U \) and \( G = U \Rightarrow y \notin G \Rightarrow y \notin G' \).

As \( G \in B \), so \( G \) is also closed.

\( \Rightarrow G' \) is open. Put \( G' = H \).

Hence, we have two open sets \( G \) and \( H \)

in \( X \) such that \( x \in G \), \( y \in H \).

\( G \cup H = G \cup H = X \) and \( G \cap H = G \cap H = \emptyset \).
$\Rightarrow \ X \ is \ totally \ disconnected.$

Conversely, suppose $X$ is totally disconnected (where $X$ is also compact and $\exists \ B_x \subset X$ open basis for $X$).

To prove: $X$ has an open base where sets are also closed.

Let $B$ be an open base for $X$.

To prove: elements of $B$ are also closed.

Let $x \in X$ and $G$ be an open set in $X$ such that $x \in G$.

Case I. If $G = X$, then $x \in B$ such that $x \in G$. Clearly, $B_x$ is both open and closed.

Case II. If $G \neq X \Rightarrow G \subset X$. Now as $G$ is an open set, so $G'$ is closed. As $x \in G \Rightarrow x \in G'$.

Now, $G'$ is closed subspace of $X$ and $X$ is compact. So $G'$ is also compact (Closed subspace of a compact space is compact).

As $X$ is totally disconnected, so $\forall y \in G'$ such that $x \neq y$.

Now, there is subset $H_y$ of $X$ which is both open and closed, such that $y \in H_y$ and $x \notin H_y$. Then, the set $\{H_y : y \in X\}$ is an open cover for $G'$.

As $G'$ is compact, so this open cover has a finite subcover $\{H_1, H_2, \ldots, H_n\}$.

$\Rightarrow G' = \bigcup_{i=1}^{n} H_i = H \Rightarrow G' \subset H$.

Clearly, $H$ is both open and closed.

Further as $x \notin H \Rightarrow x \notin \bigcup_{i=1}^{n} H_i = H \Rightarrow x \notin H$.

$\Rightarrow x \in H \subset B_x$. 

Here \( B_x \) is both open and closed.

Now let \( x \in B_x \).

\[ x \in H \Rightarrow x \in G \Rightarrow x \notin G' \Rightarrow x \notin G \]

\[ \Rightarrow B_x \subseteq G' \Rightarrow x \notin B_x \subseteq G \]

The collection of all such \( B_x \) form an open base whose elements are also closed.

Hence proved.

**Theorem:** Let \( X \) and \( Y \) be two topological spaces, then a function \( f : X \to Y \) is continuous if and only if for every subset \( A \subseteq X \), \( f(A) \) is closed.

**Proof:** Given \( f \) is continuous and \( A \subseteq X \).

To prove: \( f(A) \subseteq f(A) \).

As \( f(A) \subseteq f(A) \Rightarrow A \subseteq f^{-1}(f(A)) \).

As \( f \) is continuous and \( f(A) \) is closed.

Then, \( f^{-1}(f(A)) \) is closed in \( X \). (Note: \( f \) is continuous, iff inverse image of each closed set is closed).

\[ A \subseteq f^{-1}(f(A)) \text{ and } f^{-1}(f(A)) \text{ is closed subset of } X \]

\[ \Rightarrow f^{-1}(f(A)) \text{ is the closed superset of } A \]

But, \( A \) is the smallest closed superset of \( A \).

\[ \Rightarrow A = f^{-1}(f(A)) \]

\[ \Rightarrow f(A) \subseteq f(A) \]
Conversely, suppose for any subset $A$ of $X$,

$$f(A) \subseteq f(A).$$

To prove $f$ is continuous,

Let $C$ be a closed set in $Y$.

To prove $f$ is continuous, we have to prove

$f^{-1}(C)$ is closed in $X$.

Let $A = f^{-1}(C)$,

$$A = f^{-1}(C) \Rightarrow f(A) = f(f^{-1}(C)) \subseteq f(f^{-1}(C)) \quad \text{By given condition}.$$

$$f(A) \subseteq f(f^{-1}(C)) = C$$

$$\Rightarrow f(A) \subseteq C \Rightarrow f^{-1}(f(A)) \subseteq f^{-1}(C).$$

$$\Rightarrow \overline{A} \subseteq A \quad (\because f^{-1}(C) = A).$$

But $A \subseteq \overline{A}$,

$$A = \overline{A} \Rightarrow A \text{ is closed in } X.$$

$$\Rightarrow f^{-1}(C) \text{ is closed in } X.$$

$$\Rightarrow f \text{ is continuous.}$$

**DEFINITION:**

Let $X$ be a topological space and $A$ and $B$ are subsets of $X$. Then $A$ and $B$ are said to be separated if and only if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

**THEOREM:** Let $X$ be a topological space and $A$, $B$ are the subsets of $X$ if $A$ and $B$ are separated in $X$ then $A \cup B$ is disconnected.
**Proof:** Let \( Y = A \cup B \).

Now as \( A \) and \( B \) are separated in \( X \),

so, \( A \cap B = \emptyset \) and \( A \cap B = \emptyset \).

Now let \( G = B' \) and \( H = A' \).

Then as \( A \) and \( B \) are closed,

\( \Rightarrow A' \) and \( B' \) are open.

\( \Rightarrow H \) and \( G \) are open in \( X \).

\( \Rightarrow Y_G \) and \( Y_H \) are open in \( Y \) (\( Y \) is subspace).

Now, \( A \cap B = \emptyset \Rightarrow A \subseteq G \).

Further as, \( B \subseteq B \Rightarrow B \cap B' = \emptyset \).

Now, \( Y_G = (A \cup B) \cap G = (A \cap G) \cup (B \cap G) \).

\( \Rightarrow Y_G = A \cup (B \cap B') \).

\( = A \cup \emptyset = A \).

Similarly, \( Y_H = B \).

Now, \( Y_G \) and \( Y_H \) are open in \( Y \) with,

\( (Y_G) \cup (Y_H) = A \cup B = Y \).

and \( (Y_G) \cap (Y_H) = A \cap B = \emptyset \).

\( \Rightarrow Y \) is disconnected.

**Theorem:** Let \( G \) and \( H \) be the disconnection of a subset \( A \) of a topological space \( X \), then, show that \( A \cap G \) and \( A \cap H \) are separated.
Proof: To prove, $\text{Ang}$ and $\text{AnH}$ are separated, i.e., $(\text{Ang}) \cap (\text{AnH}) = \emptyset$ and $(\overline{\text{Ang}}) \cap (\text{AnH}) = \emptyset$.

First we prove, if $x \in D(\text{Ang})$ then $x \notin \text{AnH}$.

Suppose on the contrary, $x \in D(\text{Ang})$.

$\Rightarrow x \in \text{AnH}$.

$\Rightarrow x \in A$ and $x \in H$.

Now, $\text{Ang} \subseteq G$ and $\text{Ang} \subseteq A$.

$\Rightarrow D(\text{Ang}) \subseteq D(G)$ and $D(\text{Ang}) \subseteq D(A)$.

So, $x \in D(\text{Ang}) \Rightarrow x \in D(G) \Rightarrow x \in G$.

($G$ is closed).

Now, $x \in H$ and $x \in G \Rightarrow x \in G \cap H$.

$\Rightarrow G \cap H \neq \emptyset$.

Which is a contradiction.

So, our supposition is wrong.

Hence for $x \in D(\text{Ang}) \Rightarrow x \notin \text{AnH}$.

$\Rightarrow D(\text{Ang}) \cap (\text{AnH}) = \emptyset$.

Also, $(\text{Ang}) \cap (\text{AnH}) = \text{Ang}(G \cap H)$.

$\Rightarrow D(\text{Ang}) \cap (\text{AnH}) = \emptyset$.

$\Rightarrow (\text{Ang}) \cap (\text{AnH}) = \emptyset$.

Similarly, $(\text{Ang}) \cap (\text{AnH}) = \emptyset$.

$\Rightarrow (\text{Ang})$ and $(\text{AnH})$ are separated.
THEOREM: Show that a topological space $X$ is connected if and only if every non-empty proper subspace has a non-empty boundary.

Proof: We know that:

1) A topological space $X$ is disconnected if it has a subset $A$ which is both open and closed.

2) If $(X,A)$ is topological space and $A = X$ then boundary of $A$ is empty if $A$ is both open and closed.

Now given $X$ is connected and $A$ is non-empty proper subspace of $X$.

To prove: boundary of $A$ is non-empty.

Suppose boundary of $A$ is empty: $b(A) = \emptyset$ then, by (i), $A$ is both open and closed, but by (i) $X$ is disconnected which is a contradiction.

So our supposition is wrong: And hence, $b(A) \neq \emptyset$.

Conversely, suppose in a topological space $X$ every non-empty proper subset of $X$ has non-empty boundary.

To prove, $X$ is connected.

Suppose $X$ is disconnected then by (i) there is subset $A$ of $X$ which is both open and closed.
then by (iii), $b(A) = \emptyset$.

A contradiction.

$b(A) \neq \emptyset$

So, our supposition is wrong.

And hence, $X$ is connected.

**Theorem:** If $X$ and $Y$ are connected topological spaces then, $X \times Y$ is also connected.

**Proof:** Let $x \in X$ and $y \in Y$.

Then, $\exists x \in X$ and $x \times y \in Y$ are two topological spaces with $x \times y \subseteq Y$ and $X \times Y \subseteq X$.

$\Rightarrow$ As $X$ and $Y$ are connected.

So $x \times y \subseteq X \times Y$ are connected for all $x \in X$ and $y \in Y$.

Also $(\exists x, y) \subseteq (x \times y) \cap (X \times Y)$.

$\Rightarrow (x \times y) \cap (X \times Y) \neq \emptyset$.

$\Rightarrow (x \times y) \cup (X \times Y)$ is connected.

(-The union of $T\&\&$ is connected provided their intersection $\neq \emptyset$)

Furthermore,

$\forall x \in X$, $\exists T_x \neq \emptyset$ and $\forall x \in X$, $T_x = X \times y$.

where $T_x = \{y \times (x \times y) \cup (x \times y)\}$

$\Rightarrow X \times Y$ is connected.
Theorem. Every normed space is metric space.

Proof. Let \((X, \|\cdot\|)\) be the normed space.

To prove, it is enough to prove that for any \(x, y \in X\) if \(\|x - y\| < 1\), then there exists \(\delta > 0\) such that \(\|x - z\| < \|x - y\| + \delta\) for all \(z \in X\).

The normed space \((X, \|\cdot\|)\) is called complete.

Definition. A normed space \((X, \|\cdot\|)\) is called complete if every Cauchy sequence \((x_n)_{n=1}^{\infty}\) in \(X\) converges to a limit in \(X\).

Examples:
1. \(X = \mathbb{R}\), \(\|\cdot\| = \text{ordinary distance}\)
2. \(X = \mathbb{R^n}\), \(\|\cdot\| = \text{Euclidean norm}\)
3. \(X = \mathbb{R^n}\), \(\|\cdot\| = \text{sup norm}\)

Exercises:
1. If \(X = \mathbb{R}\), \(\|\cdot\| = \text{ordinary distance}\), then \(\|x - y\| < 1\) if \(x, y \in X\).
2. If \(X = \mathbb{R^n}\), \(\|\cdot\| = \text{Euclidean norm}\), then \(\|x - y\| < 1\) if \(x, y \in X\).
3. If \(X = \mathbb{R^n}\), \(\|\cdot\| = \text{sup norm}\), then \(\|x - y\| < 1\) if \(x, y \in X\).
\begin{align*}
\quad d(x, y) &= \|x - y\|, \quad \forall x, y \in X \quad \text{then}.
\quad \text{i) } & \quad \|x - y\| \geq 0 \quad \iff \\
\quad & \quad \text{ii) } \quad d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y \quad \iff \\
\quad & \quad \text{so } \quad d(x, y) = 0 \quad \text{if } x = y. \\
\quad \text{iii) } \quad d(x, y) &= \|x - y\| \\
\quad &= \|x - z + (z - y)\| \\
\quad &= \|x - z\| + \|z - y\| \\
\quad \Rightarrow d(x, y) &= d(x, z) + d(z, y). \\
\quad \text{iv) } \quad \text{Let } x, y, z \in X. \\
\quad \quad d(x, y) &= \|x - y\| = \|x - z + z - y\| \\
\quad \quad &\leq \|x - z\| + \|z - y\| \\
\quad \quad \Rightarrow d(x, y) &\leq d(x, z) + d(z, y). \\
\quad \text{Since, all the conditions are satisfied} \\
\quad \text{so, } (X, d) \text{ is metric space.}
\end{align*}

Remark: Converse of the above theorem is not true in general i.e., a metric space is not necessarily a normed space.

Example: Let \( X \neq \emptyset \) and define \( d: X \times X \to \mathbb{R} \) by
\begin{align*}
\quad d(x, y) &= 0 \quad \text{if } x = y, \\
\quad &= 1 \quad \text{if } x \neq y.
\end{align*}

Then \((X, d)\) is metric space (Discrete metric) but it is not a normed space.
Theorem. Show that a metric $d'$ induced by norm on $X$ satisfies the following:

1) $d'(x+a, y+a) = d'(x, y)$
2) $d'(ax, ay) = |a| d'(x, y)$ for all $a \in \mathbb{R}$ and $x, y \in X$.

Proof. i) $d'(x+a, y+a) = \| (x+a) - (y+a) \| = \| x+y \| = \| x - y \| = d'(x, y)$

ii) $d'(ax, ay) = \| ax - ay \| = \| a(x-y) \| = |a| \| x - y \| = |a| d'(x, y)$

Theorem. Let $X$ be a normed space. Then for all $x, y \in X$

$\| x \| - \| y \| \leq \| x - y \|$

Proof. Let $\| x \| = \| x - y + y \| \\
\leq \| x - y \| + \| y \| \\
\Rightarrow \| x \| - \| y \| \leq \| x - y \| \Rightarrow 0$

Interchanging $x$ and $y$, we get:

$\| y \| - \| x \| \leq \| y - x \| \\
\Rightarrow -(\| x \| - \| y \|) \leq \| x - y \|$

$\Rightarrow -(\| x \| - \| y \|) \leq \| x - y \| \Rightarrow 0$

$\Rightarrow \| x \| - \| y \| \geq \| x - y \| \Rightarrow 0$

0 and 2$' \Rightarrow \| x \| - \| y \| \leq \| x - y \| \leq \| x - y \|$
Uniformly Continuous.

Let \((X, \|\cdot\|)\) be a normed space. Then, \(\|\cdot\|\) defined on \(X\) is said to be uniformly continuous if and only if for every \(\varepsilon > 0\), there exists an \(\delta > 0\) such that \(\|x - y\| < \delta\) whenever \(\|f(x) - f(y)\| < \varepsilon\).

\[
\|x - y\| < \delta \quad \Rightarrow \quad \|f(x) - f(y)\| < \varepsilon
\]

Theorem. Let \(X\) be a normed space and \(f : X \rightarrow \mathbb{R}\) be defined by \(f(x) = \|x\|\). Then, \(f\) is uniformly continuous.

Proof. Let \(\varepsilon > 0\) and choose \(\delta = \varepsilon\). Then, whenever \(\|x - y\| < \delta\), \(\|f(x) - f(y)\| = \|x\| - \|y\|\) \(\leq \|x - y\| < \delta\). Now, \(\|f(x) - f(y)\| = \|x\| - \|y\|\) < \(\delta = \varepsilon\). \(\Rightarrow\ f\) is uniformly continuous.

Theorem. Let \(X\) be a normed space. We defined.

i) \(f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) by \(f(x, y) = x + y\).

ii) \(g : \mathbb{R} \rightarrow \mathbb{R}\) by \(g(x) = ax + b\).

Then, show that \(f\) and \(g\) are continuous.

Proof. i) To prove \(f\) is continuous.

Let \(x_n \rightarrow x\) and \(y_n \rightarrow y\). The two sequences in \(X\) such that \(x_n \rightarrow x\) and \(y_n \rightarrow y\). Now:
\[ \| f(x_n, y_n) - f(x, y) \| = \| (x_n + y_n) - (x + y) \| \\
\leq \| (x_n - x) + (y_n - y) \| \\
\rightarrow 0 \text{ as } n \rightarrow \infty \]

Therefore, \[ f(x_n, y_n) \rightarrow f(x, y) \text{ as } n \rightarrow \infty. \]

\[ \Rightarrow f \text{ is continuous}. \]

ii) Let \( a_n \) and \( b_n \) be two sequences such that \( a_n \rightarrow a \) and \( b_n \rightarrow b \).

Now, \[ \| g(x_n, y_n) - g(x, y) \| = \| (a_n - a) + (b_n - b) \| \]
\[ = \| a_n - a \| + \| b_n - b \| \]
\[ \leq \| a_n - a \| + \| b_n - b \| \]
\[ \rightarrow 0 \text{ as } n \rightarrow \infty. \]

\[ \Rightarrow g(x_n, y_n) \rightarrow g(x, y). \]

\[ \Rightarrow g \text{ is continuous}. \]

**Definition:** Closed Unit Ball.

Let \( X \) be a normed space. Then, the set denoted and defined by:

\[ B(0) = \{ x \in X : \| x \| \leq 1 \} \]

is called closed unit ball.

**Convex Set:** Let \( X \) be a normed space and \( M \subseteq X \). Then, \( M \) is said to be convex if for all \( x, y \in M \) and \( a \in [0, 1] \), \( ax + (1-a)y \in M \).
**Theorem:** Prove that a closed unit ball in a normed space $X$ is convex.

**Proof:** Let $B(0) = \{ x \in X : \|x\| \leq 1 \}$ be the closed unit ball in a normed space $X$.

To prove $B(0)$ is convex, let $x, y \in B(0)$ and $\alpha \in [0, 1]$.

Assume $x, y \in B(0)$, so $\|x\| \leq 1$ and $\|y\| \leq 1$.

Now, $\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|1-\alpha\|\|y\|$

$= \alpha \|x\| + (1-\alpha)\|y\|$

$\leq \alpha (1) + (1-\alpha) (1)$

$= \alpha + 1 - \alpha = 1$

$\Rightarrow \|\alpha x + (1-\alpha)y\| \leq 1$

$\Rightarrow \alpha x + (1-\alpha)y \in B(0)$

$\Rightarrow B(0)$ is a convex set.

---

**Explanation:**

**Definition:** Vector Space.

Let $V$, $\alpha$, and $F$ be the fields, then:

1. $V \ni x, y \in V \Rightarrow \alpha x \in V$.
2. $(V, +)$ is abelian group.
3. $\alpha, \beta \in F$ and $x \in V$.
4. $(\alpha + \beta)x = \alpha x + \beta x$.
5. $\alpha (x + y) = \alpha x + \alpha y$.
6. $(\alpha \beta)x = \alpha (\beta x)$.

---

**Explanation:** Vector Space.
Example: \( V = \mathbb{R}^2, F = \mathbb{R} \).

\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)
\]

\[
d(x, y) = (a, b)
\]

\[
l(x, y) = (1, x, 0) = (a_0) + (a_2 y)
\]

**LINEAR COMBINATION:**

\[
a_1, a_2, a_3, \ldots, a_n \in V, \quad b_1, b_2, b_3, \ldots, b_n \in F,
\]

\[
d_1 a_1 + d_2 a_2 + d_3 a_3 + \ldots + d_n a_n \in V \text{ is called a linear combination of } a_1, a_2, a_3, \ldots, a_n.
\]

**Definition: Spanning Set.**

If \( S \subseteq V \) over \( F \), then the set of all linear combinations of a finite number of elements of \( S \) is called the linear span of \( S \) or the spanning set of \( S \). It is denoted by \( \langle S \rangle \) or \( L(S) \).

**Linearly Dependent and Linearly Independent:**

If \( x_1, x_2, x_3, \ldots, x_n \in V(F) \), then for any choice of scalars \( a_1, a_2, \ldots, a_n \in F \):

i) If \( a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0 \Rightarrow a_i = 0 \), for all \( i, 1 \leq i \leq n \), then \( x_1, x_2, \ldots, x_n \) are called linearly independent.

ii) If \( x_1, x_2, x_3, \ldots + a_n x_n = 0 \Rightarrow a_i = 0 \), for some \( i, 1 \leq i \leq n \), then \( x_1, x_2, \ldots, x_n \) are called linearly dependent.
linearly dependent.

Examples:

Let \( V = \mathbb{R}^n \) and \( F = \mathbb{R} \).

We define \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\)
and \( a(x_1, y_1) = (ax_1, ay_1) \). Then,

1) \( \alpha = (1, -1), \beta = (2, 3) \)

and \( \alpha + \beta y = 0 \)

\[\begin{align*}
\Rightarrow \alpha + \beta y &= 0 \\
\Rightarrow (1, -1) + \beta (2, 3) &= (0, 0) \\
\Rightarrow (a, -a) + (2\beta, 3\beta) &= (0, 0) \\
\Rightarrow (a + 2\beta, -a + 3\beta) &= (0, 0) \\
\Rightarrow a + 2\beta &= 0 \Rightarrow 1 \quad \text{and} \quad -a + 3\beta &= 0 \Rightarrow 2
\end{align*}\]

Adding 1 and 2, we get:

\[5\beta = 0 \Rightarrow \beta = 0 \]

Putting value of \( \beta \) in 1:

\[a + 2(0) = 0 \Rightarrow a = 0 \]

\[\Rightarrow x, y \text{ are linearly independent.}\]

2) \( \alpha = (1, 2), \beta = (3, 6) \)

and \( \alpha + \beta y = 0 \)

\[\begin{align*}
\Rightarrow \alpha + \beta y &= 0 \\
\Rightarrow (1, 2) + \beta (3, 6) &= (0, 0) \\
\Rightarrow (a, 2a) + (3\beta, 6\beta) &= (0, 0) \\
\Rightarrow (a + 3\beta, 2a + 6\beta) &= (0, 0) \\
\Rightarrow a + 3\beta &= 0 \Rightarrow 1 \quad \text{and} \quad 2a + 6\beta &= 0 \Rightarrow 2
\end{align*}\]

\[a + 3\beta = 0 \]

Choose \( \beta = 1 \Rightarrow a = -3 \)

\[\Rightarrow x, y \text{ are linearly dependent.}\]
LINEARLY INDEPENDENT LEMMA

STATEMENT:
Let \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n \) be linearly independent set of vectors in vector space \( X \), then for any real number \( c > 0 \) and for every choice of scalar \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \):
\[
|c\mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 + \ldots + \lambda_n \mathbf{x}_n| \geq c |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n|
\]
This relation is called linearly independent lemma.

PROOF: To prove: \( |c\mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 + \ldots + \lambda_n \mathbf{x}_n| \geq c |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n| \)

Let \( S = |\lambda_1| + |\lambda_2| + |\lambda_3| + \ldots + |\lambda_n| \).
If \( S = 0 \) \( \Rightarrow \lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0 \).
Then, \( S \) is trivially proved.
If \( S \neq 0 \) \( \Rightarrow \lambda_1 \neq 0, \lambda_2 \neq 0, \ldots, \lambda_n \neq 0 \).

\[
|c\mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 + \ldots + \lambda_n \mathbf{x}_n| \geq S \\
= \frac{1}{S} |c\mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 + \ldots + \lambda_n \mathbf{x}_n| \geq C \\
= \frac{1}{S} \left| \frac{\lambda_1}{S} \mathbf{x}_1 + \frac{\lambda_2}{S} \mathbf{x}_2 + \frac{\lambda_3}{S} \mathbf{x}_3 + \ldots + \frac{\lambda_n}{S} \mathbf{x}_n \right| \geq C \\
= \frac{1}{S} \left| \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \beta_3 \mathbf{x}_3 + \ldots + \beta_n \mathbf{x}_n \right| \geq C \text{, where } \beta_j = \frac{\lambda_j}{S} \text{, where } j = 1, 2, 3, \ldots, n.
\]
Further, \( \sum_{j=1}^{n} |\beta_j| = |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n| \) \( \leq S \).
\[ \sum_{j=1}^{n} |B_j| = \frac{1}{S} \left[ |a_1| + |a_2| + \cdots + |a_n| \right] \]

\[ = \frac{S}{S} = 1 \]

\[ \Rightarrow \sum_{j=1}^{n} |B_j| = 1 \]

Hence, we have:

\[ \| \beta_1 \delta_1 + \beta_2 \delta_2 + \cdots + \beta_n \delta_n \| > C, \quad \text{with} \quad \sum_{j=1}^{n} |B_j| = 1 \]

\[ \tag{2} \]

Since (1) and (2) are equivalent.

So, to prove (1), we prove (2).

On the contrary, suppose (2) is wrong.

Then, there exists a sequence \( x_n \) such that

\[ \| x_n \| \rightarrow 0 \quad \text{and} \quad x_n = \beta_1 \delta_1 + \beta_2 \delta_2 + \cdots + \beta_n \delta_n \]

with \( \sum_{j=1}^{n} |B_j| = 1 \).

Now, as, \( \sum_{j=1}^{n} |B_j| = 1 \Rightarrow |B_j| \leq 1 \) for some fixed \( j \),

say for \( j = 1 \), \( \Rightarrow |B_1| \leq 1 \).

\[ \Rightarrow \exists \beta_1 \in \mathbb{R} \text{ is a bounded sequence}. \]

Then, by B.W. property, this sequence has a convergent subsequence \( \{ x_{m_j} \} \)

such that \( x_{m_j} \rightarrow x \).

Then, \( \{ x_{m_j} \} \) has a subsequence \( \{ x_{(m_j)} \} \)

such that

\[ x_{(m_j)} = x_{m_j} \delta_1 + \beta_{m_j} \delta_2 + \cdots + \beta_{m_j} \delta_n \]

\[ \Rightarrow \frac{1}{\sqrt{m_j}} x_{(m_j)} \]

\[ \Rightarrow \sqrt{m_j} x_{(m_j)} \delta_1 + \beta_{m_j} x_2 + \cdots + \beta_{m_j} x_n \]
Continuing in this way after \( n \) steps, we have a subsequence \( x_{m,n} \) of \( x_k \) and

\[
y_{m,n} \to \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = y.
\]

and

\[
\sum_{j=1}^{n} |\beta_j| = 1.
\]

Now as \( \text{norm is a continuous function} \),

\[
y(m,n) \to y \Rightarrow \|y(m,n)\| \to \|y\| \quad (f(1) = \|x\|)
\]

Now as, \( \sum_{j=1}^{n} |\beta_j| = 1 \), so not all \( \beta_j \)'s are zero, so \( y \neq 0 \) \( \Rightarrow \ast \)

Further as, \( \|y\| \to 0 \Rightarrow \|y(m,n)\| \to 0 \)

\[
\Rightarrow y = 0 \Rightarrow \ast \ast
\]

\( \ast \) and \( \ast \ast \) gives the contradiction.

So, our supposition is wrong.

And hence \( \ast \) is true so, ultimately \( \ast \ast \) is true.

**Basis of a Vector Space:**

If \( V(\mathbb{F}) \) is a vector space and \( S \neq \emptyset \) be a subset of \( V \), then \( S \) is said to be the basis for \( V \) if

1) \( S \) is linearly independent.
2) \( \langle S \rangle = V \) (ie, every \( v \in V \) is a linear combination of finite number of elements of \( S \)).
DEFINITION: If $S$ is the basis for $V$, then order of $S$ is called the dimension of $V$. If $\alpha(S)$ is finite, then $V$ is called finite-dimensional.

EXAMPLE: Let $V = \mathbb{R}^2$ and $S = \{ (1,0), (0,1) \}$. Then:

\[
\begin{align*}
\alpha (1,0) + \beta (0,1) &= (0,0) \\
\Rightarrow (1,0) + (0,1) &= (1,1) \\
\Rightarrow \alpha &= 1, \beta = 1 \\
\Rightarrow S \text{ is linearly independent.}
\end{align*}
\]

Now, $V$, $V = \{(x,y) \in V, \text{then:}\}$

\[
\begin{align*}
V &= \{(x,y) \}
= \{(x+0, 0+y)\}
= \{(x,0) + (0,y)\}
= \alpha (1,0) + \beta (0,1)
\end{align*}
\]

$\Rightarrow \forall \in S$.

$\Rightarrow V \subseteq S$

$\Rightarrow S = V \Rightarrow S \text{ is basis for } V$

$\Rightarrow \text{dim}(V) = 2$.

DEFINITION: SUBSPACE. Let $V$ be a vector space over field $F$ and $W$ be a non-empty subset of $V$, then $W$ is said to be subspace of $V$ if $W$ itself is the vector space over the same field $F$. 
Remark:
Let $V$ be a vector space over field $F$ and $W \subseteq V$, then $W$ is a subspace of $V$ if for all $w_1, w_2 \in W$ and $\alpha, \beta \in F$,

$$\alpha w_1 + \beta w_2 \in W.$$ 

Theorem: Let $Y$ be a finite dimensional subspace of a normed space $X$. Then, $Y$ is complete.

Proof. Since $Y$ is finite dimensional, $Y$ has a finite basis. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for $Y$. To prove $Y$ is complete, let $\{x^{(m)}\}$ be the Cauchy sequence in $Y$.

Then, for every $\epsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that:

$$\|x^{(m)} - x^{(p)}\| < \epsilon \quad \text{whenever} \quad m, p \geq n_0.$$ 

Let $\epsilon > \|x^{(m)} - x^{(p)}\|$. For $m, p \geq n_0$,

$$\|x^{(m)} - x^{(p)}\| < \epsilon.$$ 

Now, as $\{x^{(m)}\}$ is a sequence in $Y$ and $\{e_1, e_2, \ldots, e_n\}$ is a basis for $Y$, then there exists scalar $d_1^{(m)}, d_2^{(m)}, \ldots, d_n^{(m)} \in F$ such that:

$$x^{(m)} = d_1^{(m)} e_1 + d_2^{(m)} e_2 + \ldots + d_n^{(m)} e_n.$$ 

Similarly, $x^{(p)} = d_1^{(p)} e_1 + d_2^{(p)} e_2 + \ldots + d_n^{(p)} e_n$.

Now, $\|x^{(m)} - x^{(p)}\| = \|d_1^{(m)} e_1 + d_2^{(m)} e_2 + \ldots + d_n^{(m)} e_n - d_1^{(p)} e_1 - d_2^{(p)} e_2 - \ldots - d_n^{(p)} e_n\|$

$$= \|d_1^{(m)} - d_1^{(p)}\| e_1 + \|d_2^{(m)} - d_2^{(p)}\| e_2 + \ldots + \|d_n^{(m)} - d_n^{(p)}\| e_n\|$$

$$\geq c \left[ |d_1^{(m)} - d_1^{(p)}| + |d_2^{(m)} - d_2^{(p)}| + \ldots + |d_n^{(m)} - d_n^{(p)}| \right],$$

$c > 0$ (by linearly independent Lemma).
\[ \| x^{(m)} - x^{(p)} \| \geq C \sum_{i=1}^{n} |a_i^{(m)} - a_i^{(p)}| \]

\[ E > \| x^{(m)} - x^{(p)} \| \geq C \sum_{i=1}^{n} |a_i^{(m)} - a_i^{(p)}| \quad \forall \; m, p \geq n_0. \]

\[ \forall \; m, p \geq n_0 \]

\[ \frac{\sum_{i=1}^{n} |a_i^{(m)} - a_i^{(p)}|}{E} < \frac{\varepsilon}{C} \quad \forall \; m, p \geq n_0. \]

\[ |a_i^{(m)} - a_i^{(p)}| < \varepsilon/C \quad \forall \; m, p \geq n_0. \]

\[ \{ x^{(p)} \} \text{ is a cauchy sequence in } F = \mathbb{R}. \]

Since, \( R \) is complete, so there exists \( x \in R \) such that \( a_i^{(p)} \to x \).

Put \( x = a_1 e_1 + a_2 e_2 + a_3 e_3 + \ldots + a_n e_n \in Y \). (\( e_1, e_2, \ldots, e_n \) are basis of \( Y \) and all \( e_i \) are orthonormal.)

Now, \( \| x^{(m)} - x_k \| = \| (a_1^{(m)} - a_1) e_1 + (a_2^{(m)} - a_2) e_2 + \ldots + (a_n^{(m)} - a_n) e_n \| \)

\[ \leq |a_1^{(m)} - a_1| + \| a_2^{(m)} - a_2 \| \leq + \| a_n^{(m)} - a_n \| \leq \varepsilon. \]

\[ = k \sum_{i=1}^{n} |a_i^{(m)} - a_i| < k \left( \varepsilon/C \right) = \varepsilon'. \]

\[ \Rightarrow \| x^{(m)} - x_k \| < \varepsilon'. \]

\[ \Rightarrow x^{(m)} \to x \in Y. \]

\[ \Rightarrow Y \text{ is complete}. \]

Hence, Proved.
DEFINITION.
Two norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) on a linear space \( X \) are said to be equivalent norms if there exists two positive real numbers \( a \) and \( b \) such that:
\[
a \| x \|_1 \leq \| x \|_2 \leq b \| x \|_1 \quad \forall x \in X.
\]

THEOREM.
Any two norms defined on a finite dimensional normed space \( X \) are equivalent.

Proof. Let \( X \) be a finite dimensional normed space with basis \( \{ e_1, e_2, \ldots, e_n \} \).

Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be the two norms defined on \( X \).

To prove, \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent.

Let \( x \in X \), then there exists \( a_1, a_2, \ldots, a_n \in F \) such that:
\[
x = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n.
\]

Then:
\[
\| x \|_1 = \| a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \|_1 = c \cdot (|a_1| + |a_2| + \cdots + |a_n|), \quad c > 0, \text{ (By Linearly Independent Lemma.)}
\]

\[
\Rightarrow \| x \|_1 \geq c S \quad \text{where} \quad S = |a_1| + |a_2| + \cdots + |a_n|.
\]

\[
\Rightarrow S \leq \frac{1}{c} \| x \|_1 \Rightarrow 0 \quad (c > 0).
\]

Also,
\[
\| x \|_2 = \| a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \|_2 \leq \sum |a_i| \| e_i \|_2 + \sum |a_i| \| e_i \|_2 + \cdots + \| a_n \| \| e_n \|_2 \quad (\text{By } \| u \| \leq \| u \| \text{ for all } u) \]
\[
\leq \sum |a_i| k + \sum |a_i| k + \cdots + |a_n| k, \quad k = \max_{i=1}^{n} \| e_i \|.
\]

\[
= k \sum_{i=1}^{n} |a_i| = KS, \quad (S = \sum_{i=1}^{n} |a_i|).
\]
\[
\|x\|_2 \leq KS \\
\Rightarrow \frac{1}{K} \|x\|_2 \leq S \leq \frac{1}{C} \|x\|_1 \quad \text{(by using \(4\))}
\]

\[
\Rightarrow \frac{1}{K} \|x\|_2 \leq \frac{1}{C} \|x\|_1
\]

\[
\Rightarrow \|x\|_2 \leq \frac{K}{C} \|x\|_1 \rightarrow 2
\]

Now let \(x \in X\) then there exists \(d_1, d_2, \ldots, d_n \in F\) such that \(x = \sum d_i e_i + d_1 e_2 + \ldots + d_n e_n\).

\[
\Rightarrow \|x\|_2 = \|d_1 e_2 + d_2 e_2 + \ldots + d_n e_n\|_2 \\
\geq C \left[ |d_1| + |d_2| + \ldots + |d_n| \right] \quad (by \text{linearly independent bounds})
\]

\[
\Rightarrow \|x\|_2 \geq C'S \quad \text{where} \quad S = |d_1| + |d_2| + \ldots + |d_n|
\]

\[
\Rightarrow S \leq \frac{1}{C'} \|x\|_2 \rightarrow 3
\]

Also, \(\|x\|_1 = \|d_1 e_1 + d_2 e_2 + \ldots + d_n e_n\|_1\)

\[
\leq |d_1| \|e_1\| + |d_2| \|e_2\| + \ldots + |d_n| \|e_n\|
\]

Let \(k' = \max_{i=1}^{n} \|e_i\|\).

\[
\Rightarrow \|x\|_1 \leq k'(|d_1| + |d_2| + \ldots + |d_n|) \\
\Rightarrow \|x\|_1 \leq k'S \\
\Rightarrow \frac{1}{k'} \|x\|_1 \leq \frac{S}{k'} \leq \frac{1}{C'} \|x\|_2 \quad \text{(using \(3\))}
\]

\[
\Rightarrow \frac{C'}{k'} \|x\|_1 \leq \|x\|_2 \rightarrow 4
\]

\(4\) and \(2\) \[\Rightarrow \frac{C'}{k'} \|x\|_1 \leq \|x\|_2 \leq \frac{K}{C} \|x\|_1 \]

\[\Rightarrow a \|x\|_1 \leq \|x\|_2 \leq b \|x\|_1 \quad \text{with} \quad a = C/k', \quad b = K/C.
\]

Hence, \(\|x\|_1\) and \(\|x\|_2\) are equivalent.
Theorem: Prove that equivalent norms on a named space $X$ define a same topology on $X$.

Proof. Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be the two equivalent norms defined on linear space $X$.

Then, there exists two positive real nos. $a$ and $b$ such that:

$$a ||x||_1 \leq ||x||_2 \leq b ||x||_1 \quad \forall x \in X.$$ 

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies defined on $X$ w.r.t. $|| \cdot ||_1$ and $|| \cdot ||_2$, respectively.

Let $U \in \mathcal{T}_1$, i.e. $U$ is an open set w.r.t. $|| \cdot ||_1$. Then, for each $x \in U$, there exists an open ball $B_1(r,x)$ such that $x \in B_1(r,x) \subseteq U$.

Here, $B_1(r,x) = \{ y \in X : ||y-x||_1 < r \}$. (By Def. of open set).

But $r' = b \cdot r$.

Consider $B_2(r',x') = \{ y \in X : ||y-x'||_2 < r' \}$.

Now, we prove $B_2(r',x') \subseteq B_1(r,x)$.

Let $y \in B_2(r',x') \Rightarrow ||y-x'||_2 < r'$.

\[ \Rightarrow ||x-x'||_2 < a \cdot ||y-x'||_2 < ar. \]

Now, $a ||x-x'||_1 \leq ||y-x'||_2$. (By $\Theta$).

\[ \Rightarrow ||y-x'||_1 \leq \frac{1}{a} \cdot ||x-x'||_1 \leq r. \]

\[ \Rightarrow ||y-x'||_1 \leq r \Rightarrow \] $y \in B_1(x,y)$.

\[ \Rightarrow B_2(r',x') \subseteq B_1(x,y). \]

\[ \Rightarrow x \in B_2(r',x') \subseteq B_1(x,y) = U. \]
\[ x \in B_2(x, r) \subseteq U_1. \]
\[ \Rightarrow \quad \text{For every } x \in U_1, \text{ we have an open ball } B_2(x, r). \]
\[ \text{w.r.t } \| \cdot \|_2 \text{ such that } x \in B_2(x, r) \subseteq U_1. \]
\[ \Rightarrow \quad U_1 \text{ is an open set w.r.t. } \| \cdot \|_2. \]
\[ \Rightarrow \quad U_1 \in \mathcal{F}_2 \Rightarrow \mathcal{F}_1 \subseteq \mathcal{F}_2 \rightarrow \Theta. \]

Now let \( u_2 \in \mathcal{F}_2. \)
\[ \Rightarrow \quad U_2 \text{ is an open set w.r.t. } \| \cdot \|_2, \text{ then for each } x \in U_2, \text{ there exist an open ball } B_2(x, r) \text{ w.r.t. } \| \cdot \|_2 \text{ such that } x \in B_2(x, r) \subseteq U_2. \]
\[ \text{Here } B_2(x, r) = \{ y \in \mathbb{X} : \| x - y \|_2 < r \}. \]
\[ \text{Let } r' = \frac{1}{b} \text{ and consider } B_1(x, r') = \{ y \in \mathbb{X} : \| x - y \|_1 < r' \}. \]

Now we prove that \( B_1(x, r') \subseteq B_2(x, r). \)
\[ \text{Let } y \in B_1(x, r') \Rightarrow \| x - y \|_1 < r'. \]
\[ \Rightarrow \quad \| x - y \|_2 < \frac{1}{b} r. \]

\[ \Rightarrow \quad b \| x - y \|_1 < r. \]

Now \( \frac{1}{b} \| x - y \|_2 < \| x - y \|_1 \) (By *Ω*).

\[ \Rightarrow \quad \| x - y \|_2 < \| x - y \|_1 < r. \]
\[ \Rightarrow \quad \| x - y \|_2 < r. \]
\[ \Rightarrow \quad y \in B_2(x, r). \]
\[ \Rightarrow \quad B_1(x, r') \subseteq B_2(x, r) \subseteq U_2. \]
\[ \Rightarrow \quad B_1(x, r') \subseteq U_2. \]
\[ \Rightarrow \quad \text{for every } U_2, \text{ is an open set w.r.t. } \| \cdot \|_1. \]
\[ \Rightarrow \quad U_2 \in \mathcal{F}_1 = \mathcal{F}_2 \Leftrightarrow \mathcal{F}_1 = \mathcal{F}_2 \rightarrow \Theta. \]

*Ω* and *Ω* \( \Rightarrow \mathcal{F}_1 = \mathcal{F}_2. \)
Theorem: Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be the two equivalent norms defined on linear space $X$. Then, a Cauchy sequence w.r.t. $\| \cdot \|_1$ is also Cauchy sequence w.r.t. $\| \cdot \|_2$.

Proof: Since $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent, so then there exist two positive real numbers $a$ and $b$ such that $a \| x \|_1 \leq \| x \|_2 \leq b \| x \|_1 \quad \forall x \in X$.

Let $(x_n)$ be a Cauchy sequence w.r.t. $\| \cdot \|_1$. Then, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\| x_m - x_n \|_1 < \epsilon \quad \forall m, n \geq n_0.$$ 

Now, $\| x_m - x_n \|_2 \leq b \| x_m - x_n \|_1 \leq b \epsilon \quad \forall m, n \geq n_0$.

$$\Rightarrow \| x_m - x_n \|_2 < \epsilon', \quad \text{where} \quad \epsilon' = b^{-1} \epsilon.$$ 

Then $(x_n)$ is also Cauchy w.r.t. $\| \cdot \|_2$.

Conversely, let $(x_n)$ be Cauchy sequence w.r.t. $\| \cdot \|_2$. Then, for every $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $\| x_m - x_n \|_2 < \epsilon \quad \forall m, n \geq n_0$.

Now, $\| x_m - x_n \|_1 \leq a \| x_m - x_n \|_2 \leq a \epsilon \quad \forall m, n \geq n_0$.

$$\Rightarrow \| x_m - x_n \|_1 < \epsilon' \quad \forall m, n \geq n_0.$$ 

$$\Rightarrow \| x_m - x_n \|_1 < a^{-1} \epsilon' \quad \forall m, n \geq n_0.$$ 

$$\Rightarrow \| x_m - x_n \|_1 < \epsilon, \quad \epsilon = a^{-1} \epsilon', \quad \forall m, n \geq n_0.$$ 

Thus, $(x_n)$ is Cauchy w.r.t. $\| \cdot \|_1$.

Hence Proved.
DEFINITION:
A complete normed space is called a Banach space.

Theorem: Prove that \( \mathbb{R} \) is a Banach space.

Proof: Let \( \| \cdot \| : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( \| x \| = |x| \).

First we show that \( (\mathbb{R}, \| \cdot \|) \) is a normed space.

Let \( x, y \in \mathbb{R} \).

i). \( \| x \| \geq 0 \). (to \( \| x \| = 0 \).

ii). \( x = 0 \). (to \( \| x \| = 0 \). \( \iff \) \( x = 0 \).

iii). \( \| x + y \| = |x + y| = |x| + |y| = \| x \| + \| y \| \).

iv). \( \| x + y \| \leq \| x \| + \| y \| \).

\( \therefore (\mathbb{R}, \| \cdot \|) \) is a normed space.

Next, we prove \( \mathbb{R} \) is complete.

Let \( \{ x_n \} \) be a Cauchy sequence in \( \mathbb{R} \).

Then, for every \( \varepsilon > 0 \), there exist \( N \in \mathbb{N} \) such that \( \| x_m - x_n \| < \varepsilon \) whenever \( m, n > N \).

Let \( \varepsilon = \frac{1}{2} \) and \( n_0 \) be the smallest natural number satisfying \( \star \).

\( \varepsilon = \frac{1}{2^{n_1}} \) and \( n_1 \) be the smallest natural number satisfying \( \star \).

\( \varepsilon = \frac{1}{2^{n_2}} \) and \( n_2 \) be the smallest natural number satisfying \( \star \).

Do on up to \( \varepsilon = \frac{1}{2^{n_k}} \) and \( n_k \) be the smallest natural number satisfying \( \star \).
So then $\{x_{nk}\}$ is a subsequence of $\{x_n\}$.

Now we show that $x_n \to x \in \mathbb{R}$. Consider the closed interval $I_k = [x_{nk} - \frac{1}{2^k}, x_{nk} + \frac{1}{2^k}]$.

Then $I_{kh} = [x_{nk_{k+1}} - \frac{1}{2^k}, x_{nk_{k+1}} + \frac{1}{2^k}]$.

Now we show that $I_{kh} = I_k$.

As $\{x_{nk}\}$ is Cauchy, and $\{x_{nk}\}$ is a subsequence of $\{x_n\}$, so $\{x_{nk}\}$ is also Cauchy. So for some $\varepsilon = \frac{1}{2^k}$, \(n = n_k\), \(m = m_k\), there exists $\varepsilon > 0$ such that $d(x_{nk}, x_{mk}) < \varepsilon$ for $n, m > m_k$.

Thus $x_{nk} = x_{nk_{k+1}} - \frac{1}{2^k} < x_{nk_{k+1}} < x_{nk_{k+1}} + \frac{1}{2^k} \to x$.

Now from first part of (1):

$x_{nk} = \frac{1}{2^k} \leq x_{nk_{k+1}}$.

Now, from second part of (1),

\[
\frac{1}{2^k} + \frac{1}{2^k} < x_{nk_{k+1}} \leq x_{nk_{k+1}} + \frac{1}{2^k} \to x.
\]
\[ a_{nk+1} \leq a_{nk} + \frac{1}{2^k} \leq a_{nk} + \frac{1}{2^k} \]

\[ \Rightarrow a_{nk+1} \leq a_{nk} + \frac{1}{2^k} \leq a_{nk} + \frac{1}{2^k} \]

\[ \Rightarrow a_{nk+1} \leq a_{nk} + \frac{1}{2^k} \leq a_{nk} + \frac{1}{2^k} \]

\[ \Rightarrow a_{nk+1} \leq a_{nk} + \frac{1}{2^k} \leq a_{nk} + \frac{1}{2^k} \to 3 \]

Now from 2 and 3, we get:

\[ a_{nk} - \frac{1}{2^k} \leq a_{nk} - \frac{1}{2^k} \leq a_{nk} + \frac{1}{2^k} \leq a_{nk} + \frac{1}{2^k} \]

\[ \Rightarrow [a_{nk} - \frac{1}{2^k}, a_{nk} + \frac{1}{2^k}] \subseteq [a_{nk} - \frac{1}{2^k}, a_{nk} + \frac{1}{2^k}] \]

\[ \Rightarrow I_{k+1} \subseteq I_k \]

\[ \Rightarrow \{ I_k \} \text{ is a non-empty decreasing sequence of closed sets and } \{ I_k \} \to 0 \text{ when } k \to \infty. \text{ Then, by Cantor's intersection theorem:} \]

\[ \bigcap_{k=1}^{\infty} I_k \text{ contains exactly one point, say } x. \]

\[ \Rightarrow x \in [a_{nk} - \frac{1}{2^k}, a_{nk} + \frac{1}{2^k}] \]
\[\Rightarrow \|x_n - x\| \leq \frac{1}{2^n}.\]

When \(k \to \infty\),

\[\|x_n - x\| \to 0.\]

\[\Rightarrow x_n \to x.\]

\[\Rightarrow x_n \to x \in \mathbb{R} \Rightarrow \mathbb{R} \text{ is complete.}\]

\[\Rightarrow (\mathbb{R}, \|\cdot\|) \text{ is Banach space.}\]

**Theorem:** Prove that \(\mathbb{C}\) is Banach space.

**Proof:** Define \(\|\cdot\| : \mathbb{C} \to \mathbb{R}\) by \(\|x\| = |x|\).

First, we show that \((\mathbb{C}, \|\cdot\|)\) is a normed space.

Let \(x, y \in \mathbb{C}\).

\(i)\) \(|x| \geq 0, \|x\| = 0 \iff x = 0.\)

\(ii)\) \(|x| = 0 \iff \|x\| = 0.\)

\(iii)\) \(|x + y| = |x| + |y| = \|x\| + \|y\| = \|x + y\|.\)

\(iv)\) \(|x| + |y| = \|x + y\| = \|x\| + \|y\| = \|x + y\|.\)

\[\Rightarrow (\mathbb{C}, \|\cdot\|) \text{ is normed space.}\]

Now, we show that \(\mathbb{C}\) is complete.

Let \(\{x_n\}_{n=1}^{\infty}\) be a Cauchy sequence in \(\mathbb{C}\).

Then, for every \(\varepsilon > 0\), there exists some positive integer \(N\) such that:

\[\|x_m - x_n\| \leq \varepsilon, \quad \forall \, m, n > N.\]

\[\Rightarrow |x_m - x_n| \leq \varepsilon, \quad \forall \, m, n > N.\]

Let \(x_n = x_m + y_n\). Then, we have.
\[
\left| (a_{m+n} + i b_{m+n}) - (a_{m+n} + i b_{m+n}) \right| < \varepsilon, \forall m, n \geq n_0
\]

\[
\Rightarrow \left| (a_{m+n} - a_m) + i (b_{m+n} - b_m) \right| < \varepsilon, \forall m, n \geq n_0
\]

\[
\Rightarrow \sqrt{(a_{m+n} - a_m)^2 + (b_{m+n} - b_m)^2} < \varepsilon, \forall m, n \geq n_0
\]

\[
\Rightarrow (a_{m+n} - a_m)^2 + (b_{m+n} - b_m)^2 < \varepsilon^2, \forall m, n \geq n_0
\]

\[
\Rightarrow (a_{m+n} - a_m)^2 < \varepsilon^2, \forall m, n \geq n_0
\]

\[
\Rightarrow |a_{m+n} - a_m| < \varepsilon, \forall m, n \geq n_0
\]

\[
\Rightarrow \{a_m\} \text{ and } \{b_m\} \text{ are the two Cauchy sequences in } \mathbb{R}. \text{ As } \mathbb{R} \text{ is complete, so } a_m \to a \in \mathbb{R} \text{ and } b_m \to b \in \mathbb{R}. \text{ Put } x = a + i b \in \mathbb{C}.
\]

Now as \(a_m \to a\) and \(b_m \to b\), so for every \(\varepsilon > 0\), there exists some positive integers \(m_0\) and \(n_0\) such that:

\[
|a_m - a| < \varepsilon \sqrt{a^2}, \forall m \geq m_0
\]

\[
|b_m - b| < \varepsilon \sqrt{b^2}, \forall n \geq n_0
\]

\[
(x_{m+n} - x)^2 + (y_{m+n} - y)^2 < \varepsilon^2, \forall m, n \geq n_0
\]

\[
\Rightarrow \sqrt{(x_{m+n} - x)^2 + (y_{m+n} - y)^2} < \varepsilon, \forall m, n \geq n_0
\]

\[
\Rightarrow |x_{m+n} - x| < \varepsilon, \forall m \geq m_0
\]
\[ \|x_m - z\| \leq \varepsilon, \forall m \geq m' \]
\[ \Rightarrow z_m \rightarrow z \in C. \]
\[ \Rightarrow z \in C. \]
\[ \Rightarrow C \text{ is Complete.} \]
\[ \Rightarrow (C, \| \cdot \|) \text{ is Banach Space.} \]

**Theorem:** Prove that \( \mathbb{R}^n \) is Banach Space.

**Proof:** Let \( X = \mathbb{R}^n \) and \( \| \cdot \| : X \rightarrow \mathbb{F} \) be defined by:
\[ \|x\| = \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 \right]^{1/2} \text{ where } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n. \]

1. As for all \( i \), \( |x_i| \geq 0 \)
\[ \Rightarrow \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 \right]^{1/2} \geq 0 \]
\[ \Rightarrow \|x\| \geq 0. \]

2. \( \|x\| = 0 \iff \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 \right]^{1/2} = 0 \iff |x_i|^2 = 0 \iff |x_i| = 0 \iff x_i = 0 \forall i, 1 \leq i \leq n. \]
\[ \Rightarrow x = 0. \]

3. Let \( x \in \mathbb{F}, \|x\| = \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 \right]^{1/2} \]
\[ = \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 |x_i|^2 \right]^{1/2} = \left| \sum_{i=1}^{n} \frac{x_i}{|x_i|} |x_i| \right| \]
\[ = \|x\| \|x\|. \]

4. \( \|x + y\| = \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i + y_i|^2 \right]^{1/2} \]
\[ \leq \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 + |y_i|^2 \right]^{1/2}. \]
\[ = \|x\| + \|y\| \]

\[ \Rightarrow \|x+y\| \leq \|x\| + \|y\| \]

Hence, \((\mathbb{R}^n, \| \cdot \|)\) is a normed space.

Now to prove \(\mathbb{R}^n\) is complete.
Let \(\{x^{(m)}\}_m^n\) be a Cauchy sequence in \(\mathbb{R}^n\).

Then, by the definition of a Cauchy sequence, for every \(\varepsilon > 0\) there exists some positive integer \(n_0\) such that:

\[ \| x^{(m)} - x^{(p)} \| < \varepsilon \quad \forall m, p \geq n_0 \]

\[ \Rightarrow \left( \frac{\varepsilon}{\sqrt{n}} \right) \| x^{(m)}_i - x^{(p)}_i \| < \varepsilon \quad \forall m, p \geq n_0 \]

\[ \Rightarrow \| x^{(m)}_i - x^{(p)}_i \|^2 < \varepsilon^2 \quad \forall m, p \geq n_0 \]

\[ \Rightarrow \| x^{(m)}_i - x^{(p)}_i \| < \varepsilon \quad \forall m, p \geq n_0 \quad \text{and} \quad i = 1, 2, \ldots, n \]

\[ \Rightarrow \{x^{(m)}\}_m^n \text{ is a Cauchy sequence in } \mathbb{R}^n. \]

Since \(\mathbb{R}^n\) is complete, \(x^{(m)} \to x\) in \(\mathbb{R}^n\).

\[ \Rightarrow \text{ for every } \varepsilon > 0, \text{ there exists some positive integer } n_1 \text{ such that:} \]

\[ \| x^{(m)}_i - x_i \| < \varepsilon \quad \forall m \geq n_1, \forall i = 1, 2, \ldots, n \]

\[ \Rightarrow \| x^{(m)}_i \| < \varepsilon + \| x_i \| \quad \forall m \geq n_1, \forall i = 1, 2, \ldots, n. \]
\[ \sum_{i=1}^{n} |x_i^{(m)} - x_i| < \varepsilon \Rightarrow \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} |x_i^{(m)} - x_i| + \sum_{i=1}^{n} |x_i| \leq n \varepsilon / n \forall m \equiv \text{mod} (n, n, \ldots, n) \]

\[ \Rightarrow \left\| x^{(m)} - x \right\| \leq \varepsilon \Rightarrow \forall m \equiv \text{mod} (n, n, \ldots, n) \]

\[ \Rightarrow x^{(m)} \to x \in \mathbb{R}^n \]

\[ \Rightarrow \mathbb{R}^n \text{ is complete} \]

Hence, \( \mathbb{R}^n \) is Banach space.

Theorem: Show that \( L^p \) is Banach space.

Proof. Let \( X = L^p \) and \( \| \cdot \| : X \to \mathbb{F} \) be defined by \( \| x \| = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \).

i) As \( \forall i, |x_i| \geq 0 \), then \( \sum_{i=1}^{n} |x_i|^p \geq 0 \Rightarrow \sum_{i=1}^{n} |x_i|^p \geq 0 \).

\[ \Rightarrow \left\| x \right\| = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \geq 0 \]

\[ \Rightarrow \| x \| \geq 0 \]

ii) \( \| x \| = 0 \iff \| x_i \| = 0 \iff x_i = 0 \forall i \iff x = 0 \).

iii) \( \| ax \| = \left( \sum_{i=1}^{n} |ax_i|^p \right)^{1/p} = |a| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} = |a| \| x \| \)

\[ \Rightarrow \| x \| = \sum_{i=1}^{n} |x_i| \leq \| x \| \]
iv). \[ \|x + y\| = \left[ \frac{1}{p} \|x + y\|^p \right]^\frac{1}{p} \]
\[ \leq \left[ \frac{1}{p} \|x\|^p \right]^\frac{1}{p} + \left[ \frac{1}{p} \|y\|^p \right]^\frac{1}{p} \]

\[ = \|x\| + \|y\| \]
\[ \Rightarrow \|x + y\| \leq \|x\| + \|y\| \]

\[ \Rightarrow (X, \|\cdot\|) \text{ is a normed space.} \]

Now to prove $l^p$ is complete.

Let $x^{(n)}$ be a Cauchy sequence in $l^p$.

Then for every $\varepsilon > 0$, there exists some positive integer $N$ such that:

\[ \|x^{(m)} - x^{(n)}\| < \varepsilon, \forall m, n > N \]
\[ \Rightarrow \left[ \frac{1}{p} \|x^{(m)} - x^{(n)}\|^p \right]^\frac{1}{p} < \varepsilon \quad \forall m, n > N \]
\[ \Rightarrow \|x^{(m)} - x^{(n)}\| < \varepsilon \quad \forall m, n > N \]

\[ \Rightarrow x^{(n)} \text{ is a Cauchy sequence in } R^p. \]

Since $R^p$ is complete, $\forall x^{(n)} \in l^p$.

So when $n \to \infty$, then from (i):

\[ \left[ \frac{1}{p} \|x^{(m)} - x\|^p \right]^\frac{1}{p} < \varepsilon \quad \forall m > N \]
\[ \Rightarrow \|x^{(m)} - x\| < \varepsilon \quad \forall m > N, \quad x = (x^{(m)} - x) \in l^p \]

\[ \Rightarrow x^{(m)} \to x \in l^p \Rightarrow l^p \text{ is complete.} \]

Hence, $l^p$ is a Banach space.
Theorem: Prove that $l^\infty$ is a Banach space.

Proof: Let us define $\|\cdot\|: l^\infty \to \mathbb{R}$ by $\|x\| = \sup_{n=1}^{\infty} |x_n|$

$x = \{x_n\} \in l^\infty$

i) $\forall n |x_n| > 0 \implies \sup_{n=1}^{\infty} |x_n| > 0$

$\implies \|x\| > 0$

ii) $\|x\| = 0 \iff \sup_{n=1}^{\infty} |x_n| = 0$

$\iff |x_n| = 0 \quad \forall n$

$\iff x_n = 0 \quad \forall n \iff x = 0$

iii) Let $x \in F = \mathbb{R}$ and $x \in l^\infty$.

New $\|x\| = \sup_{n=1}^{\infty} |x_n|$

$= |x| \sup_{n=1}^{\infty} |x_n|$

$= |x| \|x\|$

iv) $\|x+y\| = \sup_{n=1}^{\infty} |x_n+y_n|$

$\leq \sup_{n=1}^{\infty} |x_n| + \sup_{n=1}^{\infty} |y_n|$

$= \|x\| + \|y\|$

$\implies (l^\infty, \|\cdot\|)$ is a normed space.

Now to prove $x = l^\infty$ is complete.
Let \( (\xi_1^{(m)}, \xi_2^{(m)}, \ldots) \) be a Cauchy sequence in \( l^\infty \).

Then, by the definition of a Cauchy sequence for every \( \varepsilon > 0 \), there exist some positive integer \( m_0 \) such that \( \| \xi^{(m)} - \xi^{(n)} \| < \varepsilon \quad \forall m, n > m_0 \).

\[
\Rightarrow \sup_{m=n}^{\infty} |\xi^{(m)} - \xi^{(n)}| < \varepsilon, \quad \forall m, n > m_0 \tag{1}
\]

\[
\Rightarrow |\xi^{(m)} - \xi^{(n)}| < \varepsilon, \quad \forall m, n > m_0 \tag{2}
\]

\[
\Rightarrow \xi^{(p)} \text{ is a Cauchy sequence in } \mathbb{R}.
\]

As \( \mathbb{R} \) is complete, there exists some \( x \in \mathbb{R} \) such that \( \xi^{(n)} \rightarrow x \) in \( \mathbb{R} \).

Then, when \( p \rightarrow \infty \), then:

\[
\| \xi^{(m)} - x \| < \varepsilon, \quad \forall m > m_0 \text{ for some } m_0 \in \mathbb{N}.
\]

where \( x = \lim_{n \to \infty} \xi^{(n)} \) changes to \( \lim_{m \to \infty} \sup \xi^{(m)} = x_0 \).

\[
\Rightarrow \xi^{(m)} \rightarrow x
\]

Now, \( |x_0| = |x_0 - y^{(m)} + y^{(m)}| \leq |x_0 - y^{(m)}| + |y^{(m)}| \leq \varepsilon + \lambda \text{ for some } \lambda \) by \( (1) \).

\[
\Rightarrow |x_0| < \varepsilon + \lambda
\]

\[
\Rightarrow x = x_0 \in l^\infty \text{. Hence } x \in l^\infty.
\]

\[
\Rightarrow l^\infty \text{ is complete.}
\]

\[
\Rightarrow l^\infty \text{ is a Banach space.}
\]
Theorem: Prove that \( C[a,b] \) is Banach space.

Proof: Let \( X = C[a,b] \) and define \( \| \cdot \| : X \to \mathbb{R} \) by

\[
\| f \| = \sup_{x \in [a,b]} |f(x)|.
\]

Then,

1) For all \( x \in [a,b] \), \( |f(x)| \geq 0 \).

\[
\Rightarrow \sup_{x \in [a,b]} |f(x)| \geq 0 \Rightarrow \sup_{x \in [a,b]} |f(x)| > 0 \Rightarrow \| f \| > 0.
\]

ii) \( \| f \| = 0 \iff \sup_{x \in [a,b]} |f(x)| = 0 \iff f(x) = 0 \iff f = 0 \).

iii) \( \| af \| = \sup_{x \in [a,b]} |af(x)| = |a| \sup_{x \in [a,b]} |f(x)| = |a| \| f \| \).

iv) \( \| f + g \| = \sup_{x \in [a,b]} |(f + g)(x)| \leq \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)| = \| f \| + \| g \| \implies \| f + g \| \leq \| f \| + \| g \| \).
Since all the conditions are satisfied
so, \((X, \| \cdot \|)\) is normed space.

Now we prove that \(X\) is complete.

Let \(x_n = (x_n)\) be a Cauchy sequence in \(X\). Then, by the definition of Cauchy sequence for every \(\epsilon > 0\), there is some integer \(n_0\) such that:

\[
\|x_m - x_n\| < \epsilon \quad \forall \ m, n \geq n_0.
\]

\[
\Rightarrow \sup_{x \in [a,b]} |(x_n - x_m)(x)| \leq \epsilon, \quad \forall \ m, n \geq n_0.
\]

\[
\Rightarrow \sup_{x \in [a,b]} |f_n(x) - f_n(x)| \leq \epsilon, \quad \forall \ m, n \geq n_0.
\]

\[
\Rightarrow |f_n(x) - f_n(x)| \leq \epsilon, \quad \forall \ m, n \geq n_0.
\]

\[
\Rightarrow f_n(x) \text{ is a Cauchy sequence in } \mathbb{R}.
\]

Since \(\mathbb{R}\) is complete, \(\mathbb{R}\) then there exists some function \(f\) such that \(f_n(x) \rightarrow f(x) \rightarrow \infty\).

Further as \(\forall n, f_n\) is continuous.

So, \(f\) being uniform limit of \(f_n\), is also continuous. Then \(f \in C[a, b]\).

From \(1\) and \(2\),

\[
\sup_{x \in [a,b]} |f_n(x) - f(x)| \leq \epsilon, \quad \forall \ m, n \geq n_0, \text{ for some } n_0 \in \mathbb{N}.
\]

\[
\Rightarrow \|f_n - f\| \leq \epsilon, \quad \forall \ m \geq n!
\]

\[
\Rightarrow f_n \rightarrow f \in C[a, b].
\]
\[ \Rightarrow C[a,b] \text{ is complete.} \]

\[ \Rightarrow C[a,b] \text{ is Banach space.} \]

**Theorem:** A subspace \( Y \) of a Banach space \( X \) is complete if \( Y \) is closed in \( X \).

**Proof:** Suppose \( Y \) is complete.

To prove: \( Y \) is closed in \( X \).

Let \( \{ x_n \} \) be a sequence in \( Y \) such that \( x_n \to x \). Then, \( \{ x_n \} \) is also a Cauchy sequence (since every convergent sequence is Cauchy).

As \( Y \) is complete, \( x_n \to x \in Y \).

\[ \Rightarrow Y \text{ is closed in } X. \]

Conversely, let us suppose \( Y \) is closed in \( X \).

To prove: \( Y \) is complete.

Let \( \{ x_n \} \) be a Cauchy sequence in \( Y \).

As \( Y \subseteq X \), \( \{ x_n \} \) is also a Cauchy sequence in \( X \). As \( X \) is Banach space so \( X \) is also complete. Then, \( x_n \to x \in X \).

As \( Y \) is closed \( x \in Y \).

Thus, \( Y \) is complete.

**Theorem:** Prove that every finite dimensional subspace of a Banach space is Banach space.

**Proof:** As every finite dimensional subspace of a normed space is closed so if \( X \) is
The Banach space $X$ is finite dimensional. Let $Y$ be a subspace of $X$. Then $Y$ is closed in $X$.

As a subspace $Y$ of a Banach space $X$ is complete iff $Y$ is closed in $X$.

$\Rightarrow$ $Y$ is also complete.

Hence, $Y$ is a Banach space.

**Theorem:** Let $X$ be a finite dimensional space, then any $M \subset X$ is compact iff $M$ is closed and bounded.

**Proof:** Suppose $M$ is compact. If every sequence in $M$ has a convergent subsequence.

To prove: $M$ is closed and bounded. A contradiction.

Now let $x \in M$. Then there exists a sequence $x_1, x_2, \ldots, x_n$ in $M$ such that $x_n \to x$.

As $x_1, x_2, \ldots, x_n$ is a sequence in $M$ and $M$ is compact, so then this sequence has a convergent subsequence. Then there exists a subsequence $x_{n1}, x_{n2}, \ldots, x_{nk}$ such that $x_{nk} \to x \in M$.

Then we have $x_n \to x \in M \Rightarrow x \in M$.

$\Rightarrow M = M \Rightarrow (2)$

and $M = M \Rightarrow M = M$

$\Rightarrow M$ is closed.
Now, we prove that \( M \) is bounded.

Suppose \( M \) is not bounded and let \( b \in M \) be a fixed point. As \( M \) is not bounded then for some \( n \in \mathbb{N} \), there is some \( x_n \in M \) such that
\[
d(x_n, b) > n.
\]

If \( M \) is bounded then it is closed.

Then, the sequence \( x_n \) in \( M \) has no convergent subsequence. Then, \( M \) is not compact.

Which is a contradiction.

So, our assumption is wrong and hence, \( M \) is bounded.

Conversely, assume \( M \) is closed and bounded.

To prove: \( M \) is compact.

Let \( \{x_m\} \) be a sequence in \( M \).

Now as \( X \) is finite dimensional, \( X \) has the finite basis \( e_1, e_2, \ldots, e_n \). Now as \( x_m \in X \),
\[
x_m = \alpha_{1m} e_1 + \alpha_{2m} e_2 + \ldots + \alpha_{nm} e_n.
\]

Further as \( M \) is bounded, so there exists some positive real number \( k \), such that:
\[
\| x_m \| < k \Rightarrow k \| x_m \| < c [ \| \alpha_{1m} e_1 \| + \| \alpha_{2m} e_2 \| + \ldots + \| \alpha_{nm} e_n \| ] \Rightarrow k \| x_m \| < c [ \| \alpha_{1m} \| + \| \alpha_{2m} \| + \ldots + \| \alpha_{nm} \| ] \Rightarrow \sum_{i=1}^{n} | \alpha_{im} | < \frac{k}{c}.
\]
$|a_i^m| < \frac{1}{c}$, for some fixed $i$.

$\Rightarrow \text{Eq. } m^2 \text{ is a bounded sequence in } \mathbb{R}.$

Then, by B-W property, $\text{Eq. } m^2$ has a convergent subsequence $\text{Eq. } m^2$ such that $\beta_i^m \rightarrow \beta_i$.

Put $x_m = \beta_i x_1 + \beta_2 x_2 + \ldots + \beta_n x_n$.

Then $x_m \rightarrow \beta_i x_1 + \beta_2 x_2 + \ldots + \beta_n x_n = x$. (When $m \rightarrow \infty$)

So $\text{Eq. } m^2$ is a subsequence of $\text{Eq. } x^2$.

and $x_m \rightarrow x$.

As $x_m \in M$ and $x_m \rightarrow x$, and $M$ is closed $\Rightarrow x \in M \Rightarrow M$ is compact.

"REIZ LEMMA"

**Statement**: Let $Y$ and $Z$ be the two subspaces of a normed space $X$ and $Y$ is closed proper subspace of $Z$, then for any real number $a$, $0 < a < 1$, there exist a point $x \in Z$ such that $\exists_{x \in X} \text{ } \|x - z\| > 0 \forall z \in Y$.

Proof: As $Y$ is closed proper subspace of $Z$, so then there exist some $z \in Z$ such that $\exists_{y \in Y} \text{ } \|Y - z\| = 0 \forall y \in Y$.

Put $a = \inf \{ \|Y - z\| : z \in Z \}$. $\forall y \in Y$.

Then $a > 0$ and $a \leq \|Y - z\| \Rightarrow \exists_{z \in Z} \|Y - z\| > 0$.

Also, by the definition of infimum, for $\epsilon > 0$,

There exists some point $y \in Y$ such that

$\|Y - z\| < a + \epsilon$. (By def. of $\inf$)

Let $a + \epsilon = \frac{a}{\delta}$, $\delta \in J_0 [\epsilon].$

then, $\|Y - y\| < \frac{a}{\delta}$.
Put \( x = \frac{u - y_0}{\|u - y_0\|} \in X. \)

Further, \( \|x\| = \frac{\|u - y_0\|}{\|u - y_0\|} = 1. \)

\[ \Rightarrow \|x\| = 1. \]

Now it remains to prove:

\[ \|x - y\| > 0 \quad \forall y \in Y. \]

Now let \( y \in Y \) and \( \alpha = \|u - y_0\|, \) then

\[ \|x - y\| = \frac{\|u - y_0 - \alpha y\|}{\alpha} = \frac{\|u - (y_0 + \alpha y)\|}{\alpha} \]

Now as \( y_0, y \in Y \) and \( Y \) is subspace so,

\[ y_0 + \alpha y \in Y. \]

Then, \( \|x - y\| = \frac{1}{\alpha} \|u - (y_0 + \alpha y)\| = \frac{1}{\alpha} \cdot \alpha \) (by (1))

\[ \Rightarrow \|x - y\| > 0/\alpha. \]

Now \( \|u - y_0\| < \alpha/\beta \Rightarrow \alpha < \alpha/\beta \) (\( \|u - y_0\| = \alpha \))

\[ \Rightarrow 0 < 0/\alpha < \|x - y\| \quad \forall y \in Y. \]

\[ \Rightarrow \|x - y\| > 0 \quad \forall y \in Y. \]

Hence Proved.
Inner Product Space:
Let \( V \) be a vector space over the field \( F \) (\( \mathbb{R} \) or \( \mathbb{C} \)). Then a mapping \( \langle \cdot , \cdot \rangle : V \times V \to F \) is called inner product space if:

i) \( \langle x, x \rangle \geq 0 \) \( \forall x \in V \)

ii) \( \langle x, x \rangle = 0 \) \( \iff \) \( x = 0 \)

iii) \( \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + 2 \langle x, y \rangle \)

iv) \( \langle ax, y \rangle = a \langle x, y \rangle \)

v) \( \langle x, y \rangle = \bar{\langle y, x \rangle} \)

Then, the pair \((V, \langle \cdot , \cdot \rangle)\) is called inner product space.

Example 1:
Let \( V = \mathbb{R}^m \) over the field \( F = \mathbb{R} \).
We define \( \langle \cdot , \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) by:

\[ \langle x, y \rangle = \sum_{i=1}^{m} x_i y_i \]

where \( x = (x_1, x_2, \ldots, x_m) \), \( y = (y_1, y_2, \ldots, y_m) \).

Solution:

i) \( \langle x, x \rangle = \sum_{i=1}^{m} x_i^2 \geq 0 \)

ii) \( \langle x, x \rangle = 0 \implies \sum_{i=1}^{m} x_i^2 = 0 \implies x_i = 0 \implies x = 0 \)

iii) Let \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \) and \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \)
Some Properties of Inner Product

LHS = \[\frac{x_1 + y_1}{x_1 + y_1} = \frac{x_1}{x_1} + \frac{y_1}{y_1} \]

RHS = \[\frac{x_1}{x_1} + \frac{y_1}{y_1} \]

Thus, all the conditions are fulfilled.

Let \( \angle x + y \), then \( \angle xy = \angle (\angle x \angle y) \).

If \( x \perp y \), then \( x \perp y \).
ii) \[ \langle x, ay \rangle = a \langle x, y \rangle \]

\[ \text{L.H.S.} = \langle x, ay \rangle = \frac{\langle x, y \rangle}{y} \]
\[ = \frac{\langle x, y \rangle}{a} \]
\[ = a \langle x, y \rangle = \text{R.H.S.} \]

iii) \[ \langle ax + by, z \rangle \]
\[ = \langle ax, z \rangle + \langle by, z \rangle \]
\[ = a \langle x, z \rangle + b \langle y, z \rangle \]

**Theorem:**
Prove that every inner product space is a normed space.

**Proof:**
Let \( V \) be an inner product space.
Define \( \| \cdot \| : V \to \mathbb{F} \) by \( \| x \| = \sqrt{\langle x, x \rangle} \) \( \forall x \in V \).

Then, as:

i) \[ \text{As } \langle x, x \rangle \geq 0 \quad \forall x \in V, \]
\[ \Rightarrow \exists \lambda x, x > 0 \quad \forall x \in V. \]
\[ \Rightarrow \| x \| > 0 \quad \forall x \in V. \]

ii) \[ \| x \| = 0 \Leftrightarrow \exists \lambda x, x > 0 \Leftrightarrow \lambda x = 0 \]
\[ \Rightarrow \| x \| = 0 \quad \forall x = 0. \]

iii) Let \( a \in \mathbb{F} \) and \( x \in V \), then:
\[ \| ax \| = \sqrt{\langle ax, ax \rangle} \]
\[ = \sqrt{a^2 \langle x, x \rangle} \]
\[ = a \| x \| \]
\[ = a \sqrt{\langle x, x \rangle} \]
\[ = \sqrt{a^2 \langle x, x \rangle} \]
\[ = \sqrt{\langle ax, ax \rangle} \]
\[ \|ax\| = |a| \|x\| \geq |a| \|x\|^{3/2} \]

\[ = |a| \|x\| \cdot \|x\| \]

\[ iv) \quad \|a+y\| = \sqrt{a+y, a+y} \geq \sqrt{a+y, a+y} \]

\[ = \sqrt{a+y, a+y} \]

\[ \Rightarrow \quad \|a+y\|^2 = \langle a+y, a+y \rangle \]

\[ = \langle a, a \rangle + \langle a, y \rangle + \langle y, a \rangle + \langle y, y \rangle \]

\[ = \|a\|^2 + \|y\|^2 + \|a\|^2 + \|y\|^2 \]

\[ = \|a\|^2 + 2 \Re \langle a, y \rangle + \|y\|^2 \]

\[ \leq \|a\|^2 + 2 \|a\| \|y\| + \|y\|^2 \]

\[ \leq \|a\|^2 + 2 \|a\| \|y\| + \|y\|^2 \quad \text{(by C.S.F.)} \]

\[ = (\|a\| + \|y\|)^2 \]

\[ \Rightarrow \quad \|a+y\|^2 \leq (\|a\| + \|y\|)^2 \]

\[ \Rightarrow \quad \|a+y\| \leq \|a\| + \|y\| \]

Since all the conditions are satisfied, so V is also a normed space.

**Parallelogram Law:**
Let V be an inner product space. Then for any \( a, y \in V \),
\[ \|a+y\|^2 + \|a-y\|^2 = 2 \|a\|^2 + 2 \|y\|^2 \]

**Proof:**
\[ \text{LHS} = \|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \]

\[ = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle x, x \rangle - \langle y, y \rangle - \langle x, y \rangle - \langle y, x \rangle \]

\[ = 2\|x\|^2 + 2\|y\|^2 \]

\[ = 2(\|x\|^2 + \|y\|^2) = \text{RHS} \]

Hence Proved.

**Polarization Identity:**

i) \( \text{Re} \langle x, y \rangle = \frac{1}{2} \|x+y\|^2 - \|x-y\|^2 \)

ii) \( \text{Im} \langle x, y \rangle = \frac{1}{2} \|x+y\|^2 - \|x-y\|^2 \) is called the polarization identity.

**Proof:**

\[ \text{RHS} = \frac{1}{2} \|x+y\|^2 - \|x-y\|^2 \]

\[ = \frac{1}{4} \langle x+y, x+y \rangle - \langle x-y, x-y \rangle \]

\[ = \frac{1}{4} \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle - \langle x, x \rangle - \langle y, y \rangle - \langle x, y \rangle - \langle y, x \rangle \]

\[ = \frac{1}{4} \cdot \frac{1}{2} \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle - \langle x, x \rangle - \langle y, y \rangle - \langle x, y \rangle - \langle y, x \rangle \]

\[ = \frac{1}{4} \cdot \frac{1}{2} \cdot 2 \langle x, y \rangle + 2 \langle y, x \rangle \]

\[ = \frac{1}{2} \cdot 2 \cdot \text{Re} \langle x, y \rangle = \text{Re} \langle x, y \rangle = \text{LHS} \]
ii) \[ R.H.S = \frac{1}{4} \frac{\langle \lambda x + y, x + y \rangle - \langle x - iy, x - iy \rangle}{\langle x - iy, x - iy \rangle} \]
\[ = \frac{1}{4} \frac{\langle x + iy, x + iy \rangle - \langle x - iy, x - iy \rangle}{\langle x - iy, x - iy \rangle} \]
\[ = \frac{1}{4} \left( \frac{\langle x, x \rangle + \langle iy, iy \rangle - \langle x, x \rangle + \langle iy, iy \rangle}{\langle x - iy, x - iy \rangle} \right) \]
\[ = \frac{1}{4} \left( \frac{\langle iy, iy \rangle + \langle iy, iy \rangle}{\langle x - iy, x - iy \rangle} \right) \]
\[ = \frac{1}{4} \langle 2 < x, iy > + 2 < iy, x > \rangle \]
\[ = \frac{1}{4} \langle 2 \text{Re} < x, iy > \rangle \]
\[ = \frac{1}{2} \langle \lambda \text{Re} < x, iy > \rangle \]
\[ = \text{Re} \langle \lambda x, y \rangle = \text{L.H.S.} \]

Hence Proved.

**Cauchy-Schwarz Inequality.**

In an inner product space \( V \) for all \( x, y \in V \),
\[ |\langle x, y \rangle| \leq \|x\| \|y\| \]

**Proof:**

For any \( \lambda \in \mathbb{F} \)
\[ \|x - \lambda y\|^2 = 0 \]
\[ \Rightarrow 0 \leq \|x - \lambda y\|^2 \]
\[ = \langle x - \lambda y, x - \lambda y \rangle \]
\[ = \langle x, x \rangle - 2 \text{Re} \langle x, \lambda y \rangle + \|\lambda y\|^2 \]
\[ = \|x\|^2 - 2 \text{Re} \langle x, \lambda y \rangle + \|\lambda y\|^2 \]
\[ = \|x\|^2 - 2 \lambda \text{Re} \langle x, y \rangle + \lambda^2 \|y\|^2 \]
\[ = \|x\|^2 - 2 \lambda \text{Re} \langle x, y \rangle + \lambda^2 \|y\|^2 \]
\[
\begin{align*}
\text{Put } \lambda &= \left(\frac{\langle x, y \rangle}{\|y\|^2}\right) \\
\Rightarrow \langle y, y \rangle - \lambda \|y\|^2 &= 0 \\
\text{Then } 0 &\leq \|x\|^2 - \left(\frac{\langle x, y \rangle}{\|y\|^2}\right)^2 \|y\|^2 \\
0 &\leq \|x\|^2 \|y\|^2 - \langle x, y \rangle^2 \\
0 &\leq \|x\|^2 \|y\|^2 - 1\langle x, y \rangle^2 \\
\Rightarrow \|x, y\|^2 &\leq \|x\|^2 \|y\|^2 \\
\Rightarrow \|x, y\| &\leq \|x\| \|y\|.
\end{align*}
\]

Hence proved.

**Remark:** We know that every inner product space is a normed space, but the converse is not true in general, i.e., a normed space need not to be an inner product space.

*E.g.* if we choose \( X = l^p, p > 2 \),

And we define \( \| \cdot \| : X \to \mathbb{R} \) by

\[
\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, \quad x = (x_1, x_2, \ldots)
\]

Then, \((X, \| \cdot \|)\) is a normed space.

But \( X \) is not an inner product space, because

*E.g.* if we choose,

\[
x = (1, 0, 0, 0, \ldots), \quad y = (1, -1, 0, 0, 0, \ldots) \in l^p
\]
Then, \( \|x\| = 2^{1/4} \), \( \|y\| = 2^{1/4} \).

\[
\begin{align*}
  2[\|x\|^2 + \|y\|^2] &= 2\left[2^{1/4} + 2^{1/4}\right] \\
  &= 2\left[2 \cdot 2^{1/4}\right] = 4\left[2^{1/4}\right].
\end{align*}
\]

Now, \( x+y = (2,0,0,0,\ldots) \).

\[
\|x+y\|^2 = 2 \Rightarrow \|x+y\| = 2.
\]

And \( x-y = (0,2,0,0,\ldots) \).

\[
\|x-y\|^2 = 2 \Rightarrow \|x-y\| = 2.
\]

\[
\|x+y\|^2 + \|x-y\|^2 = 8 + 4(2^{1/4})
\]

\[
\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2].
\]

\[
\Rightarrow \text{an inner product fails to hold.}
\]

Hence, \( X \) is not an inner product space.

**APPOLONIUS IDENTITY.**

Let \( V \) be an inner product space, then \( V \).

\[
\|x-x\|^2 + \|x-y\|^2 = \frac{1}{2} \|x-y\|^2 + 2\|\frac{x-y}{2}\|^2.
\]

This is called **Appolonius identity**.

**Proof:** In the parallelogram law.

\[
\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + \|y\|^2.
\]

Put \( x' = x-x \) and \( y' = x-y \).
\[ \| z - \alpha \|^2 + \| z - y \|^2 = \| z - (\alpha + y) \|^2 + \| \alpha + y \|^2 = \| z - \alpha \|^2 + 2 \| z - y \| \]

\[ \Rightarrow \| z - (\alpha + y) \|^2 = \| z - \alpha \|^2 + 2 \| z - y \|^2 - \| \alpha + y \|^2 \]

\[ \Rightarrow \| z - (\alpha + y) \|^2 = \| z - \alpha \|^2 + 2 \| z - y \|^2 - \frac{1}{2} \| z - \alpha \|^2 + \frac{1}{2} \| z - y \|^2 \]

\[ \Rightarrow \| z - \alpha \|^2 + \| z - y \|^2 = \frac{1}{2} \| z - \alpha \|^2 + \frac{1}{2} \| z - y \|^2 \]

\[ \Rightarrow \| z - \alpha \|^2 + \| z - y \|^2 = \frac{1}{2} \| z - \alpha \|^2 + \frac{1}{2} \| z - y \|^2 \]

Hence Proved.

**Theorem:**
Prove that inner product map is a continuous function.

**Proof:**
Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space. To prove, \(\langle \cdot, \cdot \rangle\) is a continuous function.

For this, let \(x_n \to x\) and \(y_n \to y\) in \(V\).
Then, when \(n \to \infty\), \(\| x_n - x \| \to 0\) and \(\| y_n - y \| \to 0\).

Now, \[ \| x_n - y \| = \| x_n - x + x - y \| \]

\[ \leq \| x_n - x \| + \| x - y \| \]

\[ \leq \| x_n - x \| + \| x_n - y \| + \| y - y_n \| \] (By Cauchy-Schwarz inequality)
Hilbert Space: (Def).
A complete inner product space is called Hilbert space.

Eg: $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces.

Definition:
Let $V$ be an inner product space.
Then, $\forall x, y \in V$ are said to be orthogonal, denoted by $x \perp y$, if $\langle x, y \rangle = 0$.

Definition:
Let $V$ be an inner product space and $A \subseteq V$.
Then, an element $x \in V$ is orthogonal to $A$, $x \perp A$, if $\langle x, a \rangle = 0$, $\forall a \in A$.

Definition:
Let $V$ be an inner product space and $A \subseteq V$.
Then, the element orthogonal complement of $A$ is denoted and defined by

$A^\perp = \{ x \in V : x \perp A \}$.

Pythagorean Theorem:
Let $V$ be an inner product space and $x, y \in V$ such that $x \perp y$, then,

$\| x + y \|^2 = \| x \|^2 + \| y \|^2$. 

\[ \|x+y\|^2 = \|x\|^2 + \|y\|^2 \]

**Proof:**

\[
\begin{align*}
L.H.S &= \|x+y\|^2 \\
&= \langle x+y, x+y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&= \|x\|^2 + 0 + 0 + \|y\|^2 \\
&= R.H.S
\end{align*}
\]

Hence Proved.

**Example:**

Let \( V \) be an inner product space and \( U, V \in V \) such that:

\[ \langle x, U \rangle = \langle x, V \rangle \quad \text{V} \forall x \in V \]

Then, \( U = V \)

**Solution:**

It is given that \( \forall x \in V, \langle x, U \rangle = \langle x, V \rangle \)

To show, \( U = V \)

Now, \[ \langle x, U \rangle = \langle x, V \rangle \]

\[ \Rightarrow \langle x, U \rangle - \langle x, V \rangle = 0 \]

\[ \Rightarrow \langle x, U-V \rangle = 0 \]

Since, this result holds for all \( x \in V \),

so it holds for \( x = U-V \in V \)

\[ \Rightarrow \langle U-V, U-V \rangle = 0 \Leftrightarrow \|U-V\|^2 = 0 \]

\[ \Rightarrow U = V \]
**Definition:**

Let $V$ be an inner product space and $x \in V$ and $M \subseteq V$. Then the distance between $x$ and $M$ is denoted and defined as,

$$S = \inf_{y \in M} \|x - y\|.$$ 

**Theorem: Minimizing Vector.**

Let $X$ be an inner product space and $M$ be a non empty convex subset of $X$ which is complete. Then for every $x \in X$, there exists a unique $y \in M$ such that $S = \inf_{y \in M} \|x - y\|$. 

**Proof:**

By definition $S = \inf_{y \in M} \|x - y\|$. 

Then, by definition of infimum, there exists a sequence $(y_n)$ in $M$ such that,

$$S = \lim_{n \to \infty} \|x - y_n\|.$$ 

Now, we have that $(y_n)$ is in $M$ is Cauchy sequence. Let $V_n = y_n - x$. Then,

$$\|V_n\| = \|y_n - x\| = \|x - y_n\| = S_n \quad \text{(Say)}$$ 

Then,

$$\|V_n + V_m\| = \|y_n - x + y_m - x\|$$

$$= \|y_n + y_m - 2x\|$$

$$= 2\|\frac{1}{2}(y_n + y_m) - x\|.$$ 

Now as $y_n, y_m \in M$ and $M$ is convex, so

$$\frac{1}{2}y_n + \frac{1}{2}y_m \in M.$$
\( \Rightarrow \| (y_n + y_m) - x \| \geq \delta \)
\[ \Rightarrow 2 \| (y_n + y_m) - x \| \geq 2 \delta \]
\[ \Rightarrow \| V_n + V_m \| \geq 2 \delta \]

Now, \( \| V_n - V_m \| = \| y_n - x + y_m - x \| \]
\[ = \| y_n - y_m \| \]
\[ \Rightarrow \| y_n - y_m \| = \| V_n - V_m \| \]
\[ = 2 \| V_n \|^2 + 2 \| V_m \|^2 - \| V_n + V_m \|^2 \quad \text{(By \( \| \|_2 \) law)}
\leq 2 \left( \delta_n^2 + \delta_m^2 \right) - (2 \delta)^2
\rightarrow 2 \left( \delta^2 + \delta^2 \right) = 4 \delta^2 \quad \text{when} \ m, n \to \infty
\Rightarrow \| y_n - y_m \| \to 0 \quad \text{when} \ m, n \to \infty
\Rightarrow \{ y_n \} \text{ is Cauchy sequence}

Now as \( M \) is complete, so \( y_n \to y' \in M \).
So \( \delta = \lim_{n \to \infty} \| x - y_n \| = \| x - y' \| \) \quad \text{(\( \lim_{n \to \infty} y_n = y' \))}

Now we prove the uniqueness of \( y' \). Let us assume \( \delta = \| x - y' \| \) and also \( \delta = \| x - y'' \| \), \( y', y'' \in M \)

Now \( \| y' - y'' \|^2 = \| y' - x + x - y'' \|^2 \)
\[ = \| (y' - x) + (x - y'') \|^2 \]
\[ = 2 \| y' - x \|^2 + 2 \| x - y'' \|^2 - \| (y' - x) - (x - y'') \|^2 \quad \text{(By \( \| \|_2 \) law)}
\leq 2 \| y' - x \|^2 + 2 \| x - y'' \|^2 - 4 \delta^2 \left( \| y' + y'' \| - \| x \| \right)^2
\leq 2 \delta^2 + 2 \delta^2 - 4 \delta^2
\]
\[ \Rightarrow \| y - y' \| ^2 \leq 0 \]
\[ \Rightarrow \| y - y' \| = 0 \]
\[ \Rightarrow \| y - y' \| = 0 \Rightarrow y = y' \]

Hence, the existence of \( y' \) is unique.

**THEOREM:**

Let \( Y \) be a complete subspace of an inner product space \( X \), then for \( x \in X \), there exist a unique \( y \in Y \) such that \( x = y + z \) is orthogonal to \( Y \).

**Proof:**

Let \( x \in X \). Then as \( Y \) is a complete subspace of an inner product space \( X \), then there exist a unique \( y \in Y \) such that \( z = x - y \).

To prove: \( z \perp Y \).

Suppose \( z \) is not orthogonal to \( Y \). Then, there exist some \( y' \in Y \) such that:
\[ \langle z, y' \rangle > 0 \]

Let \( \langle x, y' \rangle = \alpha \in F \)

Now, let \( \beta \in F \). Then,
\[ \| x - y \|^2 = \langle x - y, x - y \rangle \]
\[ = \langle x, x \rangle - \beta \langle x, y \rangle - \beta \langle y, x \rangle + \beta^2 \langle y, y \rangle \]
\[ = \| x \|^2 - \beta \langle x, y \rangle - \beta \langle y, x \rangle + \beta^2 \| y \|^2 \]

Choose \( \beta = \frac{\langle x, y' \rangle}{\| y' \|^2} \).

Then,
\[ \| x - y \|^2 = \| x \|^2 - \frac{\langle x, y' \rangle \langle y', x \rangle}{\| y' \|^2} - \frac{\langle x, y' \rangle \langle y', x \rangle}{\| y' \|^2} = \beta (0) \]
$\forall \|x-\beta y\|^2 = \|x\|^2 - \langle x, x \rangle \leq \|y\|^2$, \\
$\|y\|^2 = \|x\|^2 - 1 < \langle x, y \rangle > 1$ \\
$\|y\|^2 = \|x\|^2 - \|x\|^2 \leq \|z\|^2 = \delta^2, \\
\Rightarrow \|x-\beta y\|^2 \leq \delta^2, \\
\Rightarrow \|x-\beta y\| \leq \delta \Rightarrow \square, \\
Also: \|x-\beta y\| = \|x-\gamma + \beta y\| \\
= \|x-\gamma (1+\beta)\| \geq \delta, \\
\Rightarrow \|x-\beta y\| \geq \delta \Rightarrow \square, \\
\square... \square gives the contradiction, \\
So, our supposition is wrong. \\
And hence, $\perp Y$. \\

**DEFINITION:** 
A vector space $X$ is said to be the direct sum of its subspace $Y$ and $Z$ if for every $x \in X$, there exists $y \in Y$ and $z \in Z$ such that $x = y + z$ and this representation is unique. In this case, we write $X = Y \oplus Z$. (In this case, $V_{XZ} = 0$.) \\

**THEOREM:** 
Let $V$ be a closed subspace of a Hilbert space $H$. Then $H = V \oplus V^\perp$. \\

**PROOF:**
Let $x \in H$. As $Y$ is a closed subspace of a complete space, $H$ is also complete. Hence, there exist a unique $y \in Y$ such that $x - y \in \text{Z}$. Then $x - y \in Y^\perp$ and $x - y = z \Rightarrow x = y + z$ and $y \in Y$ and $z \in Y^\perp$.

Now, we prove the expression $x = y + z$ is unique. Let us assume $x = y_1 + z_1$ and $x = y_2 + z_2$ where $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^\perp$.

Now $x = y_1 + z_1$ and $x = y_2 + z_2$.

$\Rightarrow y_1 + z_1 = y_2 + z_2$.

$\Rightarrow y_1 - y_2 = z_2 - z_1 \in Y \cap Y^\perp = \{0\}$.

$\Rightarrow y_1 - y_2 = 0$, $z_2 - z_1 = 0$.

$\Rightarrow y_1 = y_2$, and $z_1 = z_2$.

$\Rightarrow$ The representation $x = y + z$ is unique.

Hence, $H = Y \oplus Y^\perp$.

$z \in Y \cap Y^\perp$.

$\Rightarrow z \in Y$ and $z \in Y^\perp \Rightarrow z \in Y$ and $z \perp Y$.

As $z \perp Y$, so $\langle z, y \rangle = 0$, $\forall y \in Y$.

In particular as $z \in Y$, so $\langle z, z \rangle = 0$.

$\Rightarrow z = 0$.

$\Rightarrow Y \cap Y^\perp = \{0\}$.

Hence, Proved.
THEOREM:
Let $Y$ be a closed subspace of a Hilbert space $H$. Then, $Y = Y^\perp$.

Proof:
Let $x \in Y$. To prove: $x \in Y^\perp$

Now, for $y \in Y^\perp \Rightarrow y \perp Y$.

Then, as $x \in Y$ and $y \perp Y \Rightarrow \langle x, y \rangle = 0$.

Hence, for any $y \in Y^\perp \Rightarrow \langle x, y \rangle = 0$.

$\Rightarrow x \in (Y^\perp)^\perp \Rightarrow x \in Y^\perp \Rightarrow x \in Y^{\perp\perp} \Rightarrow 0$.

Now let $x \in Y^{\perp\perp} \subset H$.

Then $x \in H = Y \oplus Y^\perp$.

So, then there exists unique $y \in Y$ and $x \in Y^\perp$ such that $x = y + z$. Now as $y \in Y \subset Y^{\perp\perp}$,

$\Rightarrow y \in Y^{\perp\perp} \Rightarrow y \in Y^\perp \Rightarrow x - y \in Y^\perp$.

$\Rightarrow x - y \perp Y^\perp \Rightarrow x - y \perp x \perp Y^\perp$.

$\Rightarrow$ for all element in $Y^\perp$ and so for $z \in Y^\perp$,

$x \perp z \Rightarrow x = 0$.

$\Rightarrow x = y$.

$\Rightarrow x \in Y$ (as $y \in Y$).

$\Rightarrow Y^{\perp\perp} \subset Y \Rightarrow \exists$

$\Rightarrow \exists$ and $\exists \Rightarrow Y = Y^{\perp\perp}$.
Theorem:
For a nonempty subspace $M$ of a Hilbert space $H$, the span of $M$ is dense in $H$ if and only if $M^\perp = \{0\}$.

Proof:
Let $V = \langle M \rangle$. Then to show $V = H$ iff $M^\perp = \{0\}$.

Subcase $V$ is dense in $H$ is $V = H$.

Then, to prove $M^\perp = \{0\}$.

Let $x \in M^\perp \Rightarrow x \in H \Rightarrow x \in V = H$.

Then, there exist a sequence $\{x_n\} \subset V$ such that $x_n \to x$. As $x \in M^\perp \Rightarrow x \perp M$,

then for any $y \in V$ as $V = \langle M \rangle$,

$0 = V = \langle m_1, m_2, \ldots, m_k \rangle + \langle o \rangle$,

where $m_1, m_2, \ldots, m_k \in M$ and $o \in \langle o \rangle$.

Then, $0 = \langle V, x \rangle = \langle \langle m_1, m_2, \ldots, m_k \rangle, x \rangle$.

$= \langle x, (0) \rangle + \langle x, (0) \rangle + \ldots + \langle x, (0) \rangle$.

$= \langle V, x \rangle = 0 \Rightarrow x = 0$.

Thus, $\langle V, x \rangle = 0 \Rightarrow x \in \langle V \rangle$.

Further, as $\langle V, x \rangle = 0$ and $x \in \langle V \rangle$.

$\Rightarrow \langle x, x \rangle = 0$.

But $\langle x, x \rangle = 0$.

$\Rightarrow 0 = \langle x, x \rangle = 0 \Rightarrow M^\perp = \{0\}$.

Conversely, suppose $M^\perp = \{0\}$.

To prove $\langle M \rangle = V = H$.
Obviously, \( \langle M \rangle \subseteq H \Rightarrow \circ \).

Now let \( 0 + x \in H \Rightarrow xH = V \oplus V^\perp \)

\( \Rightarrow x \in V \lor x \in V^\perp \)

If \( x \in V \Rightarrow x \in V^\perp \Rightarrow x \in V \)

If \( x \in V^\perp \Rightarrow x \perp V \Rightarrow x \perp M \) (intersection)

\( \Rightarrow x \perp M \Rightarrow x = 0 \)

Which is impossible.

So, \( x \notin V^\perp \)

\( \Rightarrow x \in V \Rightarrow x \in V^\perp \)

\( \Rightarrow H \subseteq V \Rightarrow \circ \)

\( \circ \) and \( \circ \Rightarrow V = H \)

\( \therefore M = V \) is dense in \( H \).

Hence proved.

Orthogonal Set: (Def).

Let \( X \) be an inner product space and \( M \subseteq X \). Then, \( M \) is said to be orthogonal if for all \( x \neq y \in M \).

\[ \langle x, y \rangle = 0 \text{ if } x \neq y \]

Orthonormal Set: (Def).

Let \( X \) be an inner product space and \( M \subseteq X \). Then, \( M \) is said to be orthonormal if for all \( x \neq y \in M \).

\[ \langle x, y \rangle = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases} \]
THEOREM: An orthonormal set \( \{e_1, e_2, \ldots, e_n\} \) in an inner product space \( X \) is linearly independent.

PROOF: Let \( a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = 0 \), where \( a_1, a_2, \ldots, a_n \) are constants.

Then \( \langle a_1 e_1 + a_2 e_2 + \cdots + a_n e_n, e_i \rangle = 0 \).

\( \Rightarrow a_1 \langle e_1, e_i \rangle + a_2 \langle e_2, e_i \rangle + \cdots + a_n \langle e_n, e_i \rangle = 0 \).

\( \Rightarrow a_i \langle e_i, e_i \rangle = 0 \) for all \( i = 1, 2, \ldots, n \).

\( \Rightarrow a_i = 0 \) for all \( i = 1, 2, \ldots, n \).

\( \Rightarrow \{e_1, e_2, \ldots, e_n\} \) is linearly independent.

BESSEL'S INEQUALITY: Let \( \{e_1, e_2, \ldots, e_n\} \) be an orthonormal set in an inner product space \( X \) and \( x \) be an arbitrary element in \( X \). Then, for the scalar \( a_k = \langle e_k, x \rangle \), the expression

\[ \sum_{k=1}^{n} |\langle e_k, x \rangle|^2 \]

assumes its minimum value for \( a_k = \langle e_k, x \rangle \) when \( k = 1, 2, \ldots, n \). Then, the minimum value equals

\[ \|x\|^2 = \sum_{k=1}^{n} |\langle e_k, x \rangle|^2 \]

and also \( \sum_{k=1}^{n} |\langle e_k, x \rangle|^2 \leq \|x\|^2 \).

PROOF: Obviously, \( 0 \leq \|x\|^2 - \sum_{k=1}^{n} |\langle e_k, x \rangle|^2 \)

\[ \leq \|x - \sum_{k=1}^{n} \frac{\langle e_k, x \rangle}{\|e_k\|^2} e_k\|^2 \]

\[ \leq \|x - \frac{\sum_{k=1}^{n} \langle e_k, x \rangle e_k}{\sum_{k=1}^{n} |\langle e_k, x \rangle|^2} \|^2 \leq \|x - \frac{\sum_{k=1}^{n} \langle e_k, x \rangle e_k}{\|x\|^2} \|^2 \]

\[ \leq \frac{\|x\|^2}{\sum_{k=1}^{n} |\langle e_k, x \rangle|^2} \sum_{k=1}^{n} |\langle e_k, x \rangle|^2 \]
\[ \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle a_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

\[ = \|x\|_2^2 - \sum_{k=1}^n \alpha_k \langle e_k, e_k \rangle - \sum_{k=1}^n a_i \langle e_i, e_k \rangle + \sum_{k=1}^n a_i \langle e_i, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_i, e_k \rangle \]

Obviously, \( \|x - \frac{\alpha_k}{\|e_k\|} a_k e_k \|_2 \) has minimum value when:

\[ \frac{\alpha_k}{\|e_k\|} a_k e_k = 0 \Rightarrow \alpha_k = 0, \|e_k\|^2 = 0 \]

\[ \Rightarrow \alpha_k = 0 \Rightarrow a_k = 0, \langle e_k, e_k \rangle = 0 \]

Hence, \( \|x - \frac{\alpha_k}{\|e_k\|} a_k e_k \|_2 \) has minimum value when \( a_k = 0 = \langle e_k, e_k \rangle \)
\begin{align*}
\forall n, \quad \sum_{k=1}^{n} |x_k|^2 \leq \|x\|^2.
\end{align*}

Since R.H.S of inequality is independent of \( n \), so it holds even when \( n \to \infty \).

\( \therefore \quad \sum_{k=1}^{\infty} |x_k|^2 \leq \|x\|^2. \)

**Total Orthogonal Sets:** (Def).

For orthogonal sets of vectors \( \{e_1, e_2, \ldots, e_n\} \) in an inner product space, it is said to be total orthonormal or closed if for each \( x \in X \) and \( \forall k \in \mathbb{N} \),

\[ D_k = \langle x, e_k \rangle, \quad \|x\|^2 = \sum_{k=1}^{\infty} |D_k|^2. \]

**Parseval's Equality:**

An orthonormal system \( \{e_1, e_2, \ldots, e_n\} \) in an inner product space \( X \) is total orthonormal if for all \( x \in X \),

\[ x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k. \]

**Proof:**

If \( \{e_1, e_2, \ldots, e_n\} \) is total orthonormal, then \( \|x\|^2 = \sum_{k=1}^{\infty} |D_k|^2 \) and it holds if

\[ \lim_{n \to \infty} \left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\| = 0. \]

\[ \iff \quad x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k. \]

**Corollary:**

Let \( A \) be a closed subspace of a Hilbert space \( H \). If \( x \in H \setminus A \), then there is a unique \( y \in A \) such that \( \min_{y \in A} \|x - y\| = \|x - y\| \).
Proof:

As $H$ is Hilbert space, so then as a metric space, $H$ is complete. Now as $A$ is a closed subspace of a complete space, $A$ is compact, so $A$ is convex and hence then, by the minimizing vector, there is a unique $y_A$ such that:

$$ s = \inf_{x \in A} \| x - y \| $$

$$ y_A = \frac{1}{s} \sum_{x \in A} x $$

Proved.

These Notes are the lectures delivered by Taher Mahmood

MathCity.org
Merging Man and maths
**LINEAR OPERATOR**: (DEF)

Let $X$ and $Y$ be the two normed spaces. Then, an operator $T : X \to Y$ is said to be linear if, for all $x, y \in X$ and $\alpha, \beta \in F$, we have

$$ T(\alpha x + \beta y) = \alpha T(x) + \beta T(y). $$

**EXAMPLES**: Let us define $I : N \to N$ by $I(x) = x$. Then $I(\alpha x + \beta y) = \alpha I(x) + \beta I(y)$, so $I$ is linear.

Let $K$ be the space of all analytic functions over $C$ and $D : K \to K$ be defined by $D(f) = f'$, then

$$ D(\alpha f + \beta g) = (\alpha f + \beta g)' = \alpha f' + \beta g' = \alpha D(f) + \beta D(g). $$

**THE KERNEL OR NULL SPACE OF A LINEAR OPERATOR**: (DEF)

Let $X$ and $Y$ be the two normed spaces over field $F$. and $T : X \to Y$ be a linear operator. Then, kernel $\ker(T)$ is denoted and defined by

$$ \ker(T) = \{ x \in X : T(x) = 0_Y \} $$

**CONTINUOUS LINEAR OPERATOR**: (DEF)

A linear operator $T : X \to Y$ is said to be continuous at a point $x_0 \in X$ if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$ \| T(x) - T(x_0) \| < \varepsilon \text{ whenever } \| x - x_0 \| < \delta. $$
**Bounded Linear Operator:** (Def).

A linear operator \( T: N \to M \) is said to be bounded if, for all \( x \in N \),

there is some positive real number \( k \) such that:

\[
\|T(x)\| \leq k\|x\|.
\]

**Theorem:** Let \( T: N \to M \) be a linear operator from a normed space \( N \) to a normed space \( M \). Then:

1. \( T \) is continuous on \( N \) iff \( T \) is bounded.
2. \( T \) is continuous iff \( T \) is continuous at \( 0 \in N \).
3. If \( T \) is continuous on \( N \), then \( \ker T \) is closed in \( N \).

**Proof:**

1. Suppose \( T \) is continuous on \( N \), then \( T \) is continuous at each point in \( N \), say also at \( x_0 \in N \). Then, by the definition of continuity for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that:

\[
\|T(x) - T(x_0)\| < \varepsilon \quad \text{whenever} \quad \|x - x_0\| \leq \delta.
\]

Now let \( y \in N \) and put \( x = x_0 + \left( \frac{y}{\|y\|} \right) \frac{\delta}{2\|y\|} y \).

Then, \( x - x_0 = \left( \frac{\delta}{2\|y\|} \right) y \).

\[
\Rightarrow \quad \|x - x_0\| = \left\| \left( \frac{\delta}{2\|y\|} \right) y \right\| = \frac{\delta}{2\|y\|} \|y\| = \frac{\delta}{2} \leq \delta.
\]

So then by continuity of \( T \), \( \|T(x) - T(x_0)\| < \varepsilon \).
Let \( T \) be a continuous function on \( N \) and \( T \) is continuous on \( N \) as well. To prove: \( T \) is continuous at \( a \in N \).

Case 1: \( T \) is continuous at \( a \) as well.

Then for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \|T(x) - T(a)\| < \epsilon \) whenever \( \|x - a\| < \delta \).

Conversely, suppose \( T \) is bounded on \( N \). Then:

\[
\|T(x) - T(a)\| = \|T(x) - T(y) + T(y) - T(a)\| \\
\leq \|T(x) - T(y)\| + \|T(y) - T(a)\| \\
\leq k \|x - y\| + k \|y - a\| \\
\leq k \|x - a\| \\
\leq k \mathcal{L}(\{x \in N \mid \|x - a\| < \delta\}) \\
\leq k \epsilon
\]

Therefore, \( T \) is bounded on \( N \).

Hence, \( \mathcal{L}(\{x \in N \mid \|x - a\| < \delta\}) \leq \epsilon / k \).

Finally, \( \mathcal{L}(\{x \in N \mid \|x - a\| < \delta\}) \leq \epsilon \).

Therefore, \( T \) is continuous at \( a \).
Let $x \in \mathbb{N}$. Since $T$ is continuous at $0$, so for every $\varepsilon > 0$, there exist a $\delta > 0$ such that,

$$
\|T(x) - T(0)\| \leq \varepsilon \quad \text{whenever} \quad \|x - 0\| < \delta,
$$

$$
\Rightarrow \|T(x) - T(0)\| \leq \varepsilon \quad \text{whenever} \quad \|x\| < \delta,
$$

$$
\Rightarrow \|T(x)\| \leq \varepsilon \quad \text{whenever} \quad \|x\| < \delta.
$$

So, in particular, when for all $x \in \mathbb{N}$,

$$
\|x - 0\| < \delta, \text{ then } \|T(x - 0)\| < \varepsilon.
$$

$$
\Rightarrow \|T(x) - T(0)\| \leq \varepsilon,
$$

$$
\Rightarrow T \text{ is continuous at } 0 \in \mathbb{N}. \text{ Since, } 0 \text{ is arbitrary, so } T \text{ is continuous on } \mathbb{N}.
$$

5). Given $T$ is continuous.

To prove: Ker $T$ is closed.

Let $x \in$ Ker $T$. Then there exists a sequence $x_n \in$ Ker $T$ such that $x_n \to x$.

Since $T$ is continuous and $x_n \to x$, so $T(x_n) \to T(x)$.

As for all $n$, $x_n \in$ Ker $T$,

$$
\Rightarrow T(x_n) = 0,
$$

$$
\Rightarrow \forall n, T(x_n) = 0 \text{ and } T(x_n) \to T(x) \Rightarrow T(x) = 0 \Rightarrow x \in$ Ker $T$.

$$
\Rightarrow \text{ Ker } T \subseteq \text{ Ker } T \to 4.
$$

But Ker $T \subseteq$ Ker $T \to 2$.

1 and 3 $\Rightarrow$ Ker $T =$ Ker $T$.

$\Rightarrow$ Ker $T$ is closed.
Norm of a Bounded Linear Operator:

Let \( T : N \to M \) be a bounded linear operator, then, \( \| T \| \) is denoted and defined by:

\[
\| T \| = \sup_{x \neq 0} \frac{\| T(x) \|}{\| x \|}
\]

Remark:

\[
\| T \| = \sup_{x \neq 0} \frac{\| T(x) \|}{\| x \|} = \sup_{\| x \| = 1} \| T(x) \|
\]

Theorem: Prove that every linear operator on a finite dimensional normed space is bounded.

Proof: Let \( N \) be a finite dimensional normed space with basis \( B = \{e_1, e_2, \ldots, e_n\} \) and let \( T : N \to M \) be a linear operator on \( N \).

To prove: \( T \) is bounded.

Let \( x \in N \) and \( B \) is a basis for \( N \).

\[ x = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n \quad \text{then:} \quad a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n \in E \]

\[
T(x) = T(a_1 e_1 + a_2 e_2 + \ldots + a_n e_n) = a_1 T(e_1) + a_2 T(e_2) + \ldots + a_n T(e_n)
\]

\[
\| T(x) \| = \| a_1 T(e_1) + a_2 T(e_2) + \ldots + a_n T(e_n) \|
\]
$$\|T(x)\| \leq \|\lambda_1 T(e_1)\| + \|\lambda_2 T(e_2)\| + \cdots + \|\lambda_n T(e_n)\| \leq \lambda _1 \|T(e_1)\| + \lambda _2 \|T(e_2)\| + \cdots + \lambda _n \|T(e_n)\| = \lambda S, \quad S = \|x_1\| + \|x_2\| + \cdots + \|x_n\| \quad \Rightarrow \quad \|T(x)\| \leq \lambda S \rightarrow 0$$

Now also,
$$\|x\| = \|a_1 T(e_1) + a_2 T(e_2) + \cdots + a_n T(e_n)\| \leq C \left( \|a_1\| + \|a_2\| + \cdots + \|a_n\| \right) \quad (\text{by linearly independent lemma})$$

$$= C S, \quad S = \|x_1\| + \|x_2\| + \cdots + \|x_n\| \quad \Rightarrow \quad C \leq \|x\| \Rightarrow S \leq \frac{1}{C} \|x\|.$$  

From (4),
$$\|T(x)\| \leq \lambda S \leq \lambda \frac{1}{C} \|x\| \quad \Rightarrow \quad \|T(x)\| \leq \frac{C}{\lambda} \|x\| \quad (k \text{ is some real no.}) \quad \Rightarrow \quad T \text{ is bounded.}$$

**Remark:**
A linear operator defined on a finite dimensional normed space is continuous.

**Theorem:**
Let $T_1: N \rightarrow M$ and $T_2: M \rightarrow K$ be bounded linear operators, then $T_2 T_1$ is also bounded and $\|T_2 T_1\| \leq \|T_2\| \|T_1\|$.  

In particular if $T: N \rightarrow N$ is bounded linear operator, then $\|T^m\| \leq \|T\|^m$.  

**Proof.** As $T_i$ and $T_x$ are bounded so,

$\| T_i(x) \| \leq \| T_i \| \| x \|$ \hspace{1cm} \forall x$

and $\| T_x(x) \| \leq \| T_x \| \| x \|

New, $\| (T_x T_i)(x) \| = \| T_x (T_i(x)) \|

\leq \| T_x \| \| T_i(x) \|$ \hspace{1cm} \hspace{0.5cm} \text{($T_x$ is bounded)}

\leq \| T_x \| \| T_i \| \| x \|$ \hspace{1cm} \text{($T_i$ is bounded)}

$\Rightarrow \| (T_x T_i)(x) \| \leq \| T_x \| \| T_i \| \| x \|

\Rightarrow T_x T_i \text{ is bounded.}$

Further, $\| (T_x T_i)(x) \| \leq \| T_x \| \| T_i \| \| x \|

\Rightarrow \| (T_x T_i)(x) \| \leq \| T_x \| \| T_i \|

\Rightarrow \sup_{x \rightarrow 0} \| (T_x T_i)(x) \| \leq \| T_x \| \| T_i \|

Ab, $\| T_x \| \| T_i \| = \| T_x T_i \|$

$\Rightarrow$ Also $\| T_x T_i \| \leq \| T_i \| \| T_x \|

Next, $\| T^n \| = \| T^{n-1} T \|

\leq \| T^{n-1} \| \| T \|

= \| T^{n-2} T \| \| T \|

\leq \| T^{n-3} T \| \| T \| \| T \|

\hspace{2cm} \cdots$

\leq \| T \| \| T \| \| T \| \cdots n \text{ factors}

= \| T \| ^n$

$\Rightarrow \| T^n \| = \| T \| ^n.$

**Hence Proved.**
**Theorem:** The space $B(N,M)$ of all bounded (and hence continuous) linear functionals from named space $N$ to named space $M$ is a named space under the defined norm,

$$
\|T\| = \sup_{\|x\|=1} \|T(x)\|
$$

**Proof:**

First we show that $B(N,M)$ is a linear space.

Let $T_i, T_j \in B(N,M)$ and define

1. $(T_i + T_j)(x) = T_i(x) + T_j(x)$ and
2. $(\alpha T_i)(x) = \alpha T_i(x)$.

Then

$$
(T_i + T_j)(ax + by) = T_i(ax + by) + T_j(ax + by)
$$

(By definition of $T_i + T_j$)

$$
= \alpha T_i(ax + by) + \beta T_j(ax + by)
$$

(Linear)

$$
= \alpha T_i(x) + \beta T_j(x)
$$

($T_i$ and $T_j$ are linear)

$$
\Rightarrow T_i + T_j \text{ is linear.}
$$

Further, as $\|T_i + T_j\| = \sup_{\|x\|=1} \|(T_i + T_j)(x)\| = \sup_{\|x\|=1} \|T_i(x) + T_j(x)\| 
\leq \sup_{\|x\|=1} \|T_i(x)\| + \sup_{\|x\|=1} \|T_j(x)\| 
\leq \|T_i\| + \|T_j\|

$$
\Rightarrow \|T_i + T_j\| \leq \|T_i\| + \|T_j\|
$$

$\Rightarrow T_i + T_j$ is bounded.

Thus

$$
\Rightarrow T_i + T_j \in B(N,M)
$$

$B(N,M)$ is closed.
As, sum of the mappings is always associative and commutative and so is associative and commutative in $B(N; M)$.

Now define $O : N \to M$ by $O(x) = 0$,
then $O(ax + by) = 0 = a \cdot 0 + b \cdot 0 = a \cdot O(x) + b \cdot O(y)$.
$\Rightarrow$ $O$ is linear.
Also, $\|O(x)\| = 0 \Rightarrow O$ is bounded.
$\Rightarrow O \in B(N; M)$.
And for all $T \in B(N; M), (O + T)(x) = O(x) + T(x) = 0 + T(x) = T(x)$.
$(T + O)(x) = T(x) + O(x) = T(x) + 0 = T(x)$.
$\Rightarrow 0 + T = T + O = T$.
$\Rightarrow 0$ is additive identity in $B(N; M)$.

Further, for any $T \in B(N; M)$, define $(-T) : N \to M$ by $(-T)(x) = -T(x)$.

Then, $(-T)(ax + by) = -T(ax + by) = -[aT(x) + bT(y)] = -aT(x) - bT(y) = a(-T)(x) + b(-T)(y)$.
$\Rightarrow -T$ is linear.

And $\|T(x)\| \leq \|T\| \|x\|$, $\Rightarrow \|T\|$ is bounded.
And $(T + (-T))(x) = T(x) + (-T)(x) = T(x) - T(x) = 0 = O(x)$.
and $(-T + T)(x) = (-T)(x) + T(x) = -T(x) + T(x) = 0 = O(x)$. 
$\Rightarrow (T+\bar{T})=T+(-T)=0$

$\Rightarrow -T$ is additive inverse of $T$ in $B(N,M)$.

$\Rightarrow (B(N,M), +)$ is an abelian group.

2. Let $\alpha, \beta \in F$ and $T \in B(N,M)$

Then, $(T(\alpha + \beta))T(\alpha) = (\alpha + \beta)T(\alpha)$.

$= \alpha T(\alpha) + \beta T(\alpha) = (\alpha T)(\alpha) + (\beta T)(\alpha)$.

$= (\alpha + \beta)T(\alpha) = (\alpha + \beta)T(\alpha)$.

$\Rightarrow (\alpha + \beta)T = \alpha T + \beta T$.

5. Let $\alpha, \beta \in F$ and $T_1, T_2 \in B(N,M)$

$(\alpha T_1 + \beta T_2)(\alpha) = (\alpha T_1)(\alpha) + (\beta T_2)(\alpha)$.

$= \alpha T_1(\alpha) + \beta T_2(\alpha) = (\alpha T_1)(\alpha) + (\beta T_2)(\alpha)$.

$= (\alpha T_1 + \beta T_2)(\alpha)$.

$\Rightarrow (\alpha T_1 + \beta T_2) = \alpha T_1 + \beta T_2$.

4. Let $\alpha, \beta \in F$ and $T \in B(N,M)$

$(\alpha \beta)T(\alpha) = (\alpha \beta)T(\alpha)$.

$= \alpha (\beta T(\alpha)) = (\alpha \beta T)(\alpha)$.

$\Rightarrow (\alpha \beta)T = \alpha (\beta T)$.

5. Let $\alpha \in F$ and $T \in B(N,M)$.

$(1 \cdot T)(\alpha) = 1 \cdot T(\alpha) = T(\alpha)$.

$\Rightarrow 1 \cdot T = T$.

Since, all the conditions are satisfied.
So $B(N; M)$ is linear space.

Now, we show $B(N; M)$ is normed space.

i) As $\forall x \in N, \|T(x)\| \geq 0$

$\Rightarrow \sup_{\|x\| = 1} \|T(x)\| = 0 \Rightarrow \|T\| = 0$

ii) $\|T\| = 0 \iff \sup_{\|x\| = 1} \|T(x)\| = 0$

$\iff \|T(x)\| = 0 \quad \forall x$

$\iff T(x) = 0 \quad \forall x$

$\iff T = 0$

iii) $\|\alpha T\| = \sup_{\|x\| = 1} \|\alpha (T(x))\|$

$= \sup_{\|x\| = 1} \|\alpha T(x)\|$

$= \sup_{\|x\| = 1} |\alpha| \|T(x)\| = |\alpha| \sup_{\|x\| = 1} \|T(x)\|$

$= |\alpha| \|T\|$

iv) $\|T + T(a)\| = \sup_{\|x\| = 1} \|T(x) + T(a)(x)\|$

$= \sup_{\|x\| = 1} \|T(x) + T(a)(x)\|$

$\leq \sup_{\|x\| = 1} \|T(x)\| + \sup_{\|x\| = 1} \|T(a)(x)\| = \|T\| + \|T(a)\|$

Since all the conditions are satisfied,
so $B(N; M)$ is normed space.
**Theorem:** If $M$ is a Banach space, then so is $B(N,M)$ under the norm defined by
\[ \|T\| = \sup_{n \in N} \|T(n)\|, \quad x \in N. \]

**Proof:** $B(N,M)$ is a normed space (already proved)

Now let $\{T_m\}$ be a Cauchy sequence in $B(N,M)$. Then for every $\varepsilon > 0$, there is a natural number $n_0$ such that:

\[ \|T_m - T_n\| < \varepsilon \quad \forall m, n \geq n_0, \]

\[ \sup_{n \geq n_0} \|T_m(n) - T_n(n)\| < \varepsilon, \quad \forall m, n \geq n_0. \]

\[ \sup_{n \geq n_0} \|T_m(n) - T_n(n)\| < \varepsilon, \quad \forall m, n \geq n_0. \]

\[ \Rightarrow \sup_{n \geq n_0} \|T_m(n)\| < \varepsilon, \quad \forall m, n \geq n_0. \]

\[ \Rightarrow \{T_m(n_0)\} \text{ is a Cauchy sequence in } M. \]

Since $M$ is complete,
\[ \Rightarrow T_m(n_0) \rightarrow T(n_0) \in M. \]

\[ \Rightarrow \lim_{m \rightarrow \infty} T_m(n) = T(n). \]

Now, $T(ax + by) = \lim_{m \rightarrow \infty} T_m(ax + by)$

\[ = \lim_{m \rightarrow \infty} [aT_m(x) + bT_m(y)]. \]

\[ = aT(x) + bT(y). \]

\[ \Rightarrow T \text{ is linear}. \]

Now, $\|T\| = \lim_{m \rightarrow \infty} \|T_m\|$

\[ \leq \lim_{m \rightarrow \infty} \sup_{n \in N} \|T_m(n)\| = k \|x\|, \quad k = \sup_{m \in N} \|T_m\|. \]
$\Rightarrow T_\infty$ is bounded $\Rightarrow T \in B(N; M)$.

Hence, from $\sup_{\|T\| = 1} \|T(n) - T(m)\| \leq \varepsilon \forall m, n \in \mathbb{N}$.

When $m \to \infty$,

$\sup_{\|T\| = 1} \|T(n) - T(n)\| \leq \varepsilon \forall n \geq n_0$.

$\Rightarrow \|T_n - T\| \leq \varepsilon \forall n \geq n_0$.

$\Rightarrow T_n \to T \in B(N; M)$,

$\Rightarrow B(N; M)$ is complete.

$\Rightarrow B(N; M)$ is Banach space.

**Linear Functionals:**

Let $N$ be a normed space over the
field $F$. Then, a function $f : N \to F$ is
said to be linear functional if,

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$\forall \alpha, \beta \in F$ and $x, y \in N$.

These notes are the lectures delivered by
Tahir Mahmood.