Topology And Functional Analysis

Paper A/2011

SECTION I

Question # 1(a):- Define a sub-Base and a Base for a topology on a set X. let S be a non-empty collection of subsets of X. Suppose that $X = \cup S$. Then show that S is sub-Base for some topology on X.

Sol:-

Sub-Base:- let (X,τ) be a topological space & $S \subseteq P(X)$, then S is said to be sub-base for this (X,τ), if the collection consisting of all possible intersections of all possible finite sub-families of S forms a base for (X,τ).

Base:- let (X,τ) be a topological space. Then a collection β of subsets of X is said to be base for X if:

(i) $\beta \subseteq \tau$

(ii) For every $U \in \tau$, there is a

sub family γ of β s.t.

$$U = \cup \gamma$$

let β be the collection of all possible finite intersections of members (sub-families) of S.

i.e. $\beta = \{B : B = \bigcap_{i=1}^{n} S_i \,,\, S_i \in S \,,\, 1 \leq i \leq n\}$

then

$$X = \cup S \subseteq \cup B \subseteq X$$

$$\Rightarrow X = \cup B$$

Further let $B_1, B_2 \in \beta$ & $x \in B_1 \cap B_2$

since B_1 & B_2 are finite intersections of sub-families of S.

So $B_1 \cap B_2$ is finite intersection of some sub-family of S

$$\Rightarrow B_1 \cap B_2 = B_3 \in S$$

then $x \in B_3 \subseteq B_1 \cap B_2$.

Then by a well known theorem of base, i.e.

" let X be a non-empty set. A family β of subsets of X is base for some topology T on X if and only if:

(i) $X = \bigcup_{\alpha \in I} B_\alpha$, $B_\alpha \in \beta$

(ii) For $B_1, B_2 \in \beta$ and $x \in B_1 \cap B_2$ then there exists some $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$."

β is base for some topology on X.

Hence S is sub-base for some topology on X.

Question # 1 (b) :-

(i) Give an example of topological space which is not metrizable:

Sol :- let $X = \{a, b\}$ and $T = \{\phi, X, \{a\}\}$

Then T is a topology on X. The space (X, T) is known as the Sierpinski space. In this space, every subset of X is either open or closed.

This space is not a metrizable topological space. That is no metric d on X can be defined such that the topology of (X, d) is equal to T.

For suppose that d is a metric on X such that the induced topology of (X, d) is T. Since $\{a\}$ is open in (X, T) and $a \in \{a\}$, there is an open ball $B(a; r) \subseteq \{a\}$, by definition of an open set. So for every $x \in B(a; r)$, $x \in \{a\}$ so $d(x, a) < r$

$$\Rightarrow x = a$$

Hence $d(a, b) \geq r$

Consider now $B(b; r)$. Then for all y in $B(b; r)$, $d(y, b) < r$

Hence

$$a \notin B(b; r)$$

Thus $B(b; r) = \{b\}$

So that $\{b\}$ is open in (X,d).
But $\{b\}$ is not open in (X,τ).
So (X,τ) is not metrizable.

Question # 1(b):-

(ii) Show that the intervals $(0,1)$ and (a,b) are homeomorphic.

Sol:-

let $X=(a,b)$ & $Y=(0,1)$
Here $f: X \to Y$ defined by

$$f(x) = \frac{x-a}{b-a} \text{ is homeomorphism.}$$

because
(i) f is continuous obviously.
(ii) f is bijective.
To prove f is 1-1
let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1-a}{b-a} = \frac{x_2-a}{b-a}$$

$$\Rightarrow x_1 - a = x_2 - a$$

$$\Rightarrow x_1 = x_2$$

$\Rightarrow f$ is 1-1

Now to prove f is onto.
let $x \in Y$ then we can find an element $x(b-a)+a \in X$
s.t.

$$f[x(b-a)+a] = \frac{x(b-a)+\cancel{a}-\cancel{a}}{b-a}$$

$\Rightarrow f[x(b-a)+a] = x$

$\Rightarrow f$ is onto.

Hence f is bijective.

(iii) Obviously $f^{-1}: Y \to X$ defined by

$$f^{-1}(y) = a(1-y) + by$$

exists and is continuous.

Hence $(a,b) \simeq (0,1)$.

Question # 2(a):- Let X be an arbitrary topological space and Y be a Housdroff space. Let $f: X \to Y$ be a continuous function. Then show that the graph $G = \{(x,y) : y = f(x)\} \subseteq X \times Y$ is closed in $X \times Y$.

Sol:- For this we prove G' is open in $X \times Y$.

: Let $(x,y) \in G'$

then $(x,y) \notin G$

$\Rightarrow y \neq f(x)$

As $y, f(x) \in Y$ & Y is T_2-space. So $\exists$ two open sets U and V in Y s.t. $y \in U$, $f(x) \in V$ and $U \cap V = \phi$.

Now as $f(x) \in V$ so $x \in f^{-1}(V)$

$\Rightarrow (x, y) \in f^{-1}(V) \times U$

As V is open in Y & $f: X \to Y$ is continuous so $f^{-1}(V)$ is open in X.

$\Rightarrow f^{-1}(V) \times U$ is open in $X \times Y$.

Now we prove $f^{-1}(V) \times U \subseteq G'$

let $(\alpha, \beta) \in f^{-1}(V) \times U$

$\Rightarrow \alpha \in f^{-1}(V)$ & $\beta \in U$

$\Rightarrow f(\alpha) \in V$ & $\beta \in U$

$\Rightarrow \beta \neq f(\alpha)$. $\because U \cap V = \phi$

$\Rightarrow (\alpha, \beta) \in G'$,

$\Rightarrow f^{-1}(V) \times U \subseteq G'$

$\Rightarrow (x, y) \in f^{-1}(V) \times U \subseteq G'$

$\Rightarrow G'$ is open in $X \times Y$.

$\Rightarrow G$ is closed in $X \times Y$.

Question # 2(b) :-

Show that every metric space is Regular Housdroff space.

Sol :- First we prove every metric space is Regular.

For this, we first prove that every

metric space is completely regular.

let (X,d) be a metric space. To prove X is completely regular.
let A be a closed set in X and $x \in X$ s.t. $x \notin A$.
Now define a function
$g: X \to \mathbb{R}$ by $g(y) = d(y, B)$.

where B is another closed set in X with $A \cap B = \phi$ & $x \in B$.
then (i) $g(x) = d(x, B) = 0$
(ii) $g(A) = d(A, B) > 0$
let $g(A) = d(A, B) = k$
(iii) Now for $\epsilon > 0$, we can choose $\delta = \epsilon$ s.t.
whenever $d(y, y') < \delta$ then

$$|g(y) - g(y')| = |d(y,B) - d(y',B)|$$

$$\leq d(y, y')$$

$$< \delta = \epsilon$$

$\Rightarrow |g(y) - g(y')| < \epsilon$

$\Rightarrow g$ is continuous

Now define $f: X \to [0,1]$
by $f(z) = \frac{1}{k} g(z)$

then f is continuous.

with $f(x) = \frac{1}{k} g(x) = \frac{1}{k}(0) = 0$

& $f(A) = \frac{1}{k} g(A) = \frac{1}{k}(k)$

$\Rightarrow f(A) = 1$

$\Rightarrow X$ is completely regular.

Now every completely regular space is regular.

let X be a completely regular.
To prove X is regular.
let A be a closed set in X and $x \in X$ s.t. $x \notin A$.
Then as X is completely regular so $\exists$ a continuous function $f: X \to [0,1]$ s.t.
$f(x) = 0$ & $f(A) = 1$

let $U = [0, \frac{1}{2}[$ & $V =]\frac{1}{2}, 1]$
then U & V are open in $[0,1]$.
As f is continuous, so $f^{-1}(U)$ & $f^{-1}(V)$ are open in X.
And $x \in f^{-1}(U)$, $A \subseteq f^{-1}(V)$
and $f^{-1}(U) \cap f^{-1}(V) = \phi$
So X is regular.

Now we prove every metric space is Housdroff.
let (X, d) be a metric space.
To prove X is Housdroff.
let for every $x, y \in X$ s.t. $x \neq y$

we can choose

$U = S_{\frac{r}{2}}(x)$ & $V = S_{\frac{r}{2}}(y)$

where $r = d(x,y) > 0$

then $x \in U$, $y \in V$ & $U \cap V = \phi$.

Because on the contrary if

$U \cap V \neq \phi$, then let $z \in U \cap V$

$\Rightarrow z \in U$ & $z \in V$

$\Rightarrow z \in S_{\frac{r}{2}}(x)$ & $z \in S_{\frac{r}{2}}(y)$

$\Rightarrow d(z,x) < \frac{r}{2}$ & $d(z,y) < \frac{r}{2}$

Now $r = d(x,y) \leq d(z,x) + d(z,y)$

$< \frac{r}{2} + \frac{r}{2}$

$\Rightarrow r < r$

which is contradiction.

so our supposition is wrong.

Hence $U \cap V = \phi$.

$\Rightarrow$ Every metric space is Housdroff.

Thus, we conclude that every metric space is regular housdroff.

Question # 3(a):- Define open cover, sub-cover and compact spaces. Prove that compactness is a topological property.

Sol :-

Open Cover:- let (X,τ) be a topological space and $A \subseteq X$, then a collection $\gamma = \{O_\alpha : \alpha \in I\}$ of open sets is said to be open cover for A if

$$A \subseteq \bigcup_{\alpha \in I} O_\alpha .$$

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Sub-Cover:- If γ is an open cover for A & β is a subfamily of γ s.t.

$A \subseteq \bigcup \beta$ then β is called open subcover for A.

Compact Space:- A topological space (X,τ) is said to be compact if every open cover for X has a finite subcover.

Now

To prove compactness is a topological property.

we prove homeomorphic image of compact space is compact.

let $f: X \to Y$ be a homeomorphism from a compact space X to a topological space Y.

To prove $Y = f(X)$ is compact.

let $\{U_\alpha : \alpha \in I\}$ be an open cover for Y, for all $\alpha \in I$, U_α are open subsets of Y.

$$\Rightarrow \quad Y = \bigcup_{\alpha \in I} U_\alpha$$

Since f, being a homeomorphic is a continuous function and U_α are open subsets of Y, so $V_\alpha = f^{-1}(U_\alpha)$ are open subsets of X.

Now $V_\alpha = f^{-1}(U_\alpha)$

$$\Rightarrow \bigcup_{\alpha\in I} V_\alpha = \bigcup_{\alpha\in I} f^{-1}(U_\alpha)$$

$$\Rightarrow \bigcup_{\alpha\in I} V_\alpha = f^{-1}\left(\bigcup_{\alpha\in I} U_\alpha\right) \quad \because Y = \bigcup_{\alpha\in I} \alpha$$

$$\Rightarrow = f^{-1}(Y)$$

$$\Rightarrow \bigcup_{\alpha\in I} V_\alpha = X$$

This shows that $\{V_\alpha : \alpha \in I\}$ is an open cover of X.

Since X is compact, so this open cover must have a finite subcover.

let the finite subcover be $\{V_{\alpha_i} : i = 1, 2, 3, \dots n\}$

i.e.

$$X = \bigcup_{i=1}^{n} V_{\alpha_i} \quad \because V_{\alpha_i} = f^{-1}(U_{\alpha_i})$$

$$\Rightarrow X = \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$$

$$\Rightarrow X = f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$

$$\Rightarrow f(X) = \bigcup_{i=1}^{n} U_{\alpha_i}$$

$$\Rightarrow Y = \bigcup_{i=1}^{n} U_{\alpha_i} \quad \because f(X) = Y$$

This shows that $\{U_{\alpha_i} : i = 1, 2, 3, \dots n\}$ is a finite subcover of Y.

Thus, an open cover of Y has a finite subcover of Y.

Hence, Y is compact space.

Thus, the homeomorphic image of a compact space is compact.

Hence, compactness is a topological property.

Hence Proved.

Question # 3 (b):- Define connected spaced and prove that a space X is connected if and only if there does not exist a continuous function f from X onto two point discrete space.

Sol:-

Connected Spaces:-

A topological space $(X, \mathcal{T})$ is said to be connected if there exist no non-empty disjoint open sets A and B s.t.

$$A \cup B = X$$

let X is connected.
To prove there does not exist a continuous surjective function f from X to discrete two

point space $Y = \{a, b\}$
Suppose on the contrary that there exists a function $f: X \to Y$ which is continuous and surjective
Now as f is surjective
So $f(X) = Y$
As $Y = \{a, b\}$ is discrete
So $\{a\}, \{b\}$ are open in Y.
As f is continuous so $f^{-1}(\{a\})$ and $f^{-1}(\{b\})$ are open in X.
Also

$$f^{-1}(\{a\}) \cup f^{-1}(\{b\})$$
$$= f^{-1}(\{a\} \cup \{b\})$$
$$= f^{-1}(Y)$$
$$= X$$

&

$$f^{-1}(\{a\}) \cap f^{-1}(\{b\})$$
$$= f^{-1}(\{a\} \cap \{b\})$$
$$= f^{-1}(\phi)$$
$$= \phi$$

$\Rightarrow X$ is disconnected
which is a contradiction
$\because X$ is connected
So, our supposition is wrong.
Hence, there does not exist a continuous surjective function from X to a discrete two point space Y.
Conversely assume there does not exist a continuous surjective function f from X to discrete

two point space $Y = \{a, b\}$.
To prove X is connected.
Suppose on the contrary X is disconnected.
Then, there exist two non-empty open sets A & B in X s.t.

$$A \cup B = X \quad \& \quad A \cap B = \phi$$

Now define $f : X \to Y$ by

$$f(A) = \{a\} \quad \& \quad f(B) = \{b\}.$$

Then
$$\begin{aligned} f(X) &= f(A \cup B) \\ &= f(A) \cup f(B) \\ &= \{a\} \cup \{b\} \end{aligned}$$

$\Rightarrow f(X) = Y$

$\Rightarrow f$ is surjective

Also as open sets in Y are $\phi, \{a\}, \{b\}, Y$.

with
$$\begin{aligned} f^{-1}(\phi) &= \phi \\ f^{-1}(\{a\}) &= A \\ f^{-1}(\{b\}) &= B \\ f^{-1}(Y) &= X \end{aligned}$$

$\Rightarrow$ Inverse image of each open set is open.
$\Rightarrow f$ is continuous.
which is a contradiction.
$\because$ there does not exist a continuous surjective function
So our supposition is wrong.
Hence X is connected.

Question # 4(a) :- let X be a countably compact space. Show that every infinite subset of X has a limit point in X.

Sol :- let A be an infinite subset of X.

To prove A has a limit point.
Suppose on the contrary A has no limit point.
Then every subset of A also has no limit point.
let $B = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite subset of A. Then B has no limit point.
Now consider

$C_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$, $n \in \mathbb{N}$

Then, $\forall\ n \in \mathbb{N}$, $D(C_n) = \phi$

$\Rightarrow D(C_n) \subseteq C_n$

$\Rightarrow \forall\ n \in \mathbb{N}$, C_n is closed.

$\Rightarrow \{C_n : n \in \mathbb{N}\}$ is a class of closed sets which satisfy finite intersection property.
Because for every finite sub collection $\{C_{n_1}, C_{n_2}, \ldots, C_{n_r}\}$

$$\bigcap_{i=1}^{n} C_{n_i} = C_{n'} \neq \phi$$

where $n' = \max(n_1, n_2, \ldots, n_r)$

Hence $\{C_n : n \in \mathbb{N}\}$ is a class of closed sets which satisfy finite intersection property and

$$\bigcap_{i=1}^{\infty} C_{n_i} = \phi \Rightarrow \bigcup_{i=1}^{\infty} C_{n_i}' = X$$

$\Rightarrow X$ is not countably compact which is a contradiction.
So our supposition is wrong.
Hence, A has a limit point.

Question # 4(b) :- Define components of a Topological space. let X be a topological space. Then show that
(i) For each $x \in X$ there is exactly one component of X containing x.
(ii) Each connected subset of X is contained in a component of X.
(iii) Every component is closed.

Sol :-

Component :- let (X, τ) be a topological space & C be a maximal connected subspace of X. Then C is called component of X.

(i) let $\gamma = \{C_\alpha : \alpha \in I \text{ and } x \in C_\alpha\}$ be a collection of all connected subspaces of X which contain x.

Then $\bigcap_{\alpha \in I} C_\alpha \neq \phi$ $\because x \in \bigcap_{\alpha \in I} C_\alpha$

Then by a well known theorem

"let $X = \bigcup_{\alpha \in I} X_\alpha$, where each X_α is connected and

$\bigcap_{\alpha \in I} X_\alpha \neq \phi$. Then X is connected."

$C = \bigcup \gamma = \bigcup_{\alpha \in I} C_\alpha$ is connected

subspace of X and $x \in C$.

Now as $C = \bigcup_{\alpha \in I} C_\alpha$ so for each

$\alpha \in I$, $C_\alpha \subseteq C$,

then C is a component of X.

Now we prove there is no other component of X containing x.

let us assume C^* be any other component of X containing x.

Then C^* is connected subspace of X containing x.

Then $C^* \in \gamma$

$\Rightarrow C^* \subseteq \bigcup \gamma$

$\Rightarrow C^* \subseteq \bigcup_{\alpha \in I} C_\alpha = C$

$\Rightarrow C^* \subseteq C$

$\Rightarrow C^*$ is not component of X

Hence $x \in X$ is contained in only one component of X.

(ii) let A be the connected subspace of X.
To prove A is contained in only one component of X.

let $\gamma = \{C_\alpha : \alpha \in I\}$ be a collection of all connected subspaces of X containing A.
Then

$$\bigcap_{\alpha\in I} C_\alpha \neq \phi \ \& \ \cup\gamma = \bigcup_{\alpha\in I} C_\alpha = C$$

which is connected subspace of X.
Also $A \subseteq C$
$\Rightarrow C$ is connected subspace of X containing A.
Also $C = \cup\gamma = \bigcup_{\alpha\in I} C_\alpha$

So C is such maximal connected subspace of X.
$\Rightarrow C$ is component of X containing A.
Now we show C is the only component of X containing A.
For this, let C^* be another component of X containing A.
Now as C^* is maximal connected subspace of X containing A and C is connected subspace of X containing A so $C \subseteq C^*$.
Further also as C^* is connected subspace of X containing A
so $C^* \in \gamma$

$$\Rightarrow C^* \subseteq \cup \gamma$$

$$\Rightarrow C^* \subseteq \bigcup_{\alpha \in I} C_\alpha = C$$

$$\Rightarrow C^* \subseteq C$$

$$\Rightarrow C = C^*$$

Hence, C is the only component of X containing A.

(iii) let C be a component of X.
To prove C is closed.
For this, we prove $C = \bar{C}$
Suppose on contrary $C \neq \bar{C}$
Now as $C \subseteq \bar{C}$ and $C \neq \bar{C}$
$\Rightarrow C \subset \bar{C}$
Now as C is connected, so by a well known theorem $\bar{C}$ is connected subspace of X containing C.
$\Rightarrow C$ is not component of X.
which is a contradiction.
$\because C$ is component of X.
So our supposition is wrong.
& Hence $C = \bar{C}$
$\Rightarrow C$ is closed.

SECTION II

Question # 5(a) :-

(i) Show that $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

Sol:- let $(\mathbb{R}, d)$ be the real line with usual metric on $\mathbb{R}$ and $\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$ is a set of integers as a subset of the real line $\mathbb{R}$. Since

$$\mathbb{Z}^c = \mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1)$$

is open being union of open intervals.

So $\mathbb{Z}$ is closed in $\mathbb{R}$.

and hence $\bar{\mathbb{Z}} = \mathbb{Z}$

Also $\bar{\mathbb{Z}}$ contains no open ball

i.e. $B(x, r) \not\subseteq \bar{\mathbb{Z}}, \ \forall \ x \in \mathbb{R}$

Hence $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

(ii) Show that Q is of the First category.

Sol:- As Q is countable.

& $Q = \bigcup_{q \in Q} \{q\}$ countable union of singleton $\{q\}$ subsets of Q.

Now we show that each singleton subset $\{q\}$ of Q is nowhere dense.

As

$$\{q\}^c = R \setminus \{q\} = (-\infty, q) \cup (q, \infty)$$

which is open being union of open intervals. Therefore each singleton $\{q\}$ is closed. So

$$\overline{\{q\}} = \{q\}$$

Also $\overline{\{q\}}$ contains no open ball.
Therefore each singleton is nowhere dense.
Hence $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is the countable union of nowhere dense subsets.
Hence $\mathbb{Q}$ is of first category.

Question # 5(b):- Show that l^p is complete space.

Sol:- The space l^p consists of all sequences $X = \{x_i\}$ of real or complex numbers such that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty$$

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then $$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

Thus $X = \{x_i\}$ is in $l^p \Leftrightarrow \sum_{i=1}^{\infty} |x_i|^p$ converges.

To prove l^p is complete.
let $\{x^{(m)}\}$ be a cauchy sequence in l^p, then for every $\epsilon > 0$, $\exists$ a +ve integer n_0 such that

$$\|x^{(m)} - x^{(n)}\| < \epsilon \quad , \forall\, m, n \geq n_0$$

$$\Rightarrow \left[\sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p\right]^{1/p} < \epsilon \quad (1), \forall\, m, n \geq n_0$$

$$\Rightarrow |x_i^{(m)} - x_i^{(n)}| < \epsilon \quad , \forall\, m, n \geq n_0$$

$\Rightarrow \{x_i^{(n)}\}$ is a cauchy sequence in $\mathbb{R}$.
Since $\mathbb{R}$ is complete so
$x_i^{(n)} \to x_i \in \mathbb{R}$, so when $n \to \infty$
then from ①

$$\left[\sum_{i=1}^{\infty} |x_i^{(m)} - x_i|^p\right]^{1/p} < \epsilon \quad \forall\ m \geq n_0$$

$$\Rightarrow \|x^{(m)} - x\| < \epsilon \quad , \forall\ m \geq n_0$$

$x = (x_1, x_2, \ldots, x_n)$

$$\Rightarrow x^{(m)} \to x$$

Now $x = x^{(m)} - (x^{(m)} - x) \in l^p$

$\Rightarrow x^{(m)} \to x \in l^p$

$\Rightarrow l^p$ is complete.

Question # 6(a):- State & prove Baire's Category theorem.

Sol:- Statement:- A complete metric space is of second category.

let X be a complete metric space.
To prove X is of second category.
Suppose on contrary X is of first category.
i.e.

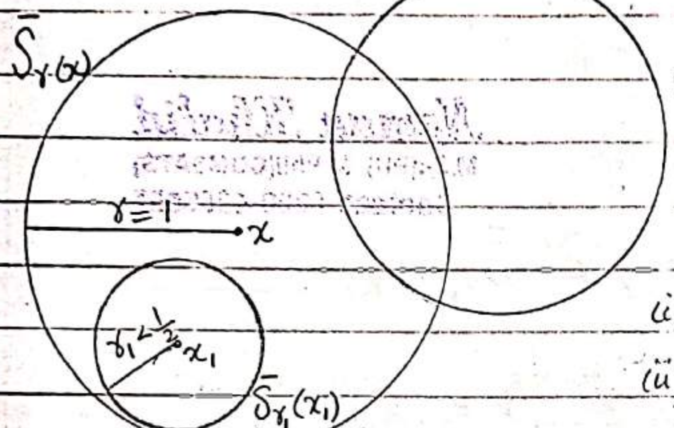
$$X = \bigcup_{n=1}^{\infty} M_n, \text{ where } M_n \text{ are}$$

nowhere dense sets.

let $x \in X$, consider a closed sphere $\bar{S}_r(x)$ of radius $r = 1$ centred at x.
Since M_1 is nowhere dense, $\exists$ a closed sphere $\bar{S}_{r_1}(x_1) \subseteq \bar{S}_r(x)$ disjoint from M_1.

i.e. $\bar{S}_{r_1}(x_1) \cap M_1 = \phi$

which contains no points of the set M_1.
let $r_1 < \frac{1}{2}$, without loss of generality.

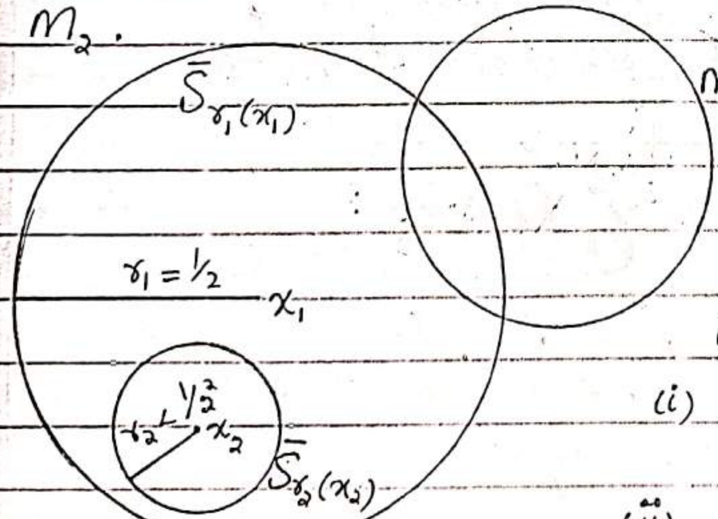


M_1 is nowhere dense

(i) $\bar{S}_{r_1}(x_1) \subseteq \bar{S}_r(x)$

(ii) $\bar{S}_{r_1}(x_1) \cap M_1 = \phi$.

Now M_2 is nowhere dense, therefore $\exists$ a closed sphere $\bar{S}_{r_2}(x_2) \subseteq \bar{S}_{r_1}(x_1)$ with
$r_2 < \frac{1}{2^2}$ and $\bar{S}_{r_2}(x_2) \cap M_2 = \phi$
which contains no points of the set M_2.



M_2 is nowhere dense

(i) $\bar{S}_{r_2}(x_2) \subseteq \bar{S}_{r_1}(x_1)$

(ii) $\bar{S}_{r_2}(x_2) \cap M_2 = \phi$

and so on. Thus

$\bar{S}_{r_1}(x_1), \bar{S}_{r_2}(x_2), \ldots, \bar{S}_{r_n}(x_n)$ is a nested sequence of closed spheres such that

(i) $\bar{S}_{r_{k+1}}(x_{k+1}) \subseteq \bar{S}_{r_k}(x_k)$, $k = 1, 2, 3, \ldots$

(ii) $\bar{S}_{r_k}(x_k) \cap M_k = \phi$, $k = 1, 2, 3, \ldots$

By Cantor's intersection theorem, since $dia\left(\bar{S}_{r_k}(x_k)\right) \to 0$ as $k \to \infty$

$$\bigcap_{k=1}^{\infty} \bar{S}_{r_k}(x_k) = \{x\} \neq \phi$$

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where $x \in X \Rightarrow x \in \bar{S}_{r_k}(x_k)$

Since $\bar{S}_{r_k}(x_k) \cap M_k = \phi$, $k = 1, 2, 3, \ldots$

$\Rightarrow x \notin M_k$ for $k = 1, 2, 3, \ldots$

$\Rightarrow x \notin X$, which is a contradiction.

So our supposition is wrong.

Hence X is of second category.

Question # 6(b):- Show that a normed space N is a Banach space if every absolutely convergent series converges.

Sol:- Let N be a normed linear space in which every absolutely convergent series converges.

Here we have to show that N is is Banach space. For this we prove that N is complete.
let $\{x_n\}$ be a cauchy sequence in N. Then for each natural number k, $\exists$ an integer n_k s.t.

$$\forall\ m, n : m, n \geq n_k$$

$$\Rightarrow \|x_m - x_n\| < 2^{-k}$$

without any loss of generality, we can assume that for $k = 1, 2, 3, \ldots$

$$n_1 < n_2 < n_3 < \ldots$$

Now we form as subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows.
let $S_k = x_{n_k}$

$$y_1 = S_1 \text{ and } y_k = S_k - S_{k-1} \text{ for } k \geq 2.$$

$$y_2 = S_2 - S_1 = S_2 - y_1 \Rightarrow S_2 = y_1 + y_2$$

$$y_3 = S_3 - S_2 = S_3 - (y_1 + y_2) \Rightarrow S_3 = y_1 + y_2 + y_3$$

and so on

$$S_k = y_1 + y_2 + \ldots + y_k$$

Thus $\|y_k\| = \|S_k - S_{k-1}\|$

$$= \|x_{n_k} - x_{n_{k-1}}\| < 2^{-(k-1)}$$

By the choice of number x_{n_k}.
Hence

$$\sum_{k=1}^{\infty} \|y_k\| < \sum_{k=1}^{\infty} 2^{-(k-1)} < \infty$$

By our assumption $\sum_{k=1}^{\infty} y_k$ is convergent.

Hence $\{S_k\}$ is convergent in N.

let $x = \lim\limits_{k\to\infty} S_k$

then $x = \lim\limits_{k\to\infty} x_{n_k}$

So that $\{x_{n_k}\}$ converges to x.
But then $\{x_n\}$ converges to x.
Hence every cauchy sequence in N converges to a point of N.
Therefor N is complete.
Hence N is Banach space.

Question # 7(a):- let $T: N \to M$ be a surjective linear operator.
Then prove that
(i) T^{-1} exists if and only if $Tx = 0$ implies $x = 0$
(ii) If T is bijective and $\dim N = n$ then show that M also has dimension n.

Sol:- (i) Suppose $T: N \to M$ be surjective (onto) linear operator.
let T^{-1} exists, then T^{-1} is linear
Take $Tx = 0$
But $T0 = 0$
we have $Tx = T0$
As T^{-1} exists, so applying T^{-1} on b.S.

$$T^{-1}Tx = T^{-1}T0 \Rightarrow Ix = I0$$
$$\Rightarrow x = 0$$

Conversely suppose that $Tx = 0 \Rightarrow x = 0$

To show that T^{-1} exists, it is sufficient to prove that T is bijective

(i) T is given to be surjective (onto).

(ii) T is injective (one-one)

Let $Tx_1 = Tx_2$

$\Rightarrow Tx_1 - Tx_2 = 0$

$\Rightarrow T(x_1 - x_2) = 0$ $\because T$ is linear

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

$\Rightarrow T$ is injective.

Hence T is bijective.

Therefore T^{-1} exists.

(ii) :- Suppose that $T: N \to M$ be a bijective linear operator. Then $\dim(N) = n$

Let $B = \{e_1, e_2, \dots, e_n\}$ be a basis of N.

We show that $B^* = \{Te_1, Te_2, \dots, Te_n\}$ forms a basis of M.

(a) B^* is linearly independent.

To prove $\{Te_1, Te_2, Te_3, \dots, Te_n\}$ is linearly independent, there exists scalars $a_1, a_2, \dots, a_n$

put

$$\sum_{i=1}^{n} a_i Te_i = 0$$

we prove $a_i = 0, \ \forall \ i = 1, 2, 3, \dots n$

Since T is linear, so

$$T\left(\sum_{i=1}^{m} a_i e_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} a_i e_i = 0$$

Since B is a basis of N, so the vectors $e_1, e_2, \ldots, e_n$ must be linearly independent.

i.e. $a_i = 0 \quad , \forall\, i = 1, 2, 3, \ldots, n$

(b) B^* spans M.
or B^* is spanning set of M.

Since B is basis of N, so for each $x \in N$, there is a unique linear combination of the form

$$x = \sum_{i=1}^{n} a_i e_i$$

Also for each $x \in N$, $\exists\, y \in M$ such that

$$y = Tx$$
$$= T\left(\sum_{i=1}^{n} a_i e_i\right)$$

$$\Rightarrow y = \sum_{i=1}^{n} a_i T(e_i)$$

Thus y can also be written as a unique linear combination of vectors in B^*. Hence B^* is basis of M.

$\therefore \dim(N) = n = \dim(M)$.
Hence proved.

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Question # 7(b):- Prove that every finite dimensional normed space is complete.

Sol:- Let M be a finite dimensional subspace of a normed space $(N, \|\cdot\|)$.
Let $\{x_1, x_2, \dots, x_n\}$ be a basis of M.
Let $\{x^{(m)}\}$ be a cauchy sequence in M. Then each $x^{(m)}$ is of the form

$$x^{(m)} = a_1^{(m)} x_1 + a_2^{(m)} x_2 + \dots + a_n^{(m)} x_n$$
$$a_i^{(m)} \in F, \ 1 \le i \le n.$$

$$\Rightarrow x^{(m)} = \sum_{i=1}^{n} a_i^{(m)} x_i$$

and given for any $\epsilon > 0$, $\exists$ a natural number n_0 s.t. $\forall\ m, p$: $m, p \ge n_0 \Rightarrow \|x^{(m)} - x^{(p)}\| < \epsilon$

By linearly independent lemma, there is a real number $c > 0$ s.t.

$$\epsilon > \|x^{(m)} - x^{(p)}\| = \|\sum_{i=1}^{n} a_i^{(m)} x_i - \sum_{i=1}^{n} a_i^{(p)} x_i\|$$
$$= \|\sum_{i=1}^{n} (a_i^{(m)} - a_i^{(p)}) x_i\|$$
$$\ge c \sum_{i=1}^{n} |a_i^{(m)} - a_i^{(p)}|$$

So that

$$m, p \geq n_0 \Rightarrow \sum_{i=1}^{n} |a_i^{(m)} - a_i^{(p)}| < \frac{\epsilon}{c}$$

that is

$$m, p \geq n_0 \Rightarrow |a_i^{(m)} - a_i^{(p)}| \leq \sum_{i=1}^{n} |a_i^{(m)} - a_i^{(p)}| < \frac{\epsilon}{c}$$

Hence $\{a_i^{(m)}\}$ is a cauchy sequence in F. Since F is $\mathbb{R}$ or $\mathbb{C}$.
So F is complete, $a_i^{(m)} \to a_i$, say as $m \to \infty$ for each $i = 1, 2, 3, \ldots, n$

Take

$$x = \sum_{i=1}^{n} a_i x_i \quad \text{i.e. } x \in M$$

then

$$\|x^{(m)} - x\| = \left\|\sum_{i=1}^{n} (a_i^{(m)} - a_i) x_i\right\|$$

$$\leq \sum_{i=1}^{n} |a_i^{(m)} - a_i| \cdot \|x_i\|$$

$$\Rightarrow \|x^{(m)} - x\| \leq k \sum_{i=1}^{n} |a_i^{(m)} - a_i|$$

where $k = \sup_{i=1}^{n} \|x_i\|$ is fixed.

$$\Rightarrow \|x^{(m)} - x\| \to 0 \quad \text{as} \quad m \to \infty$$

i.e. $\lim_{m \to \infty} x^{(m)} = x$

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Hence M is complete.

Question # 8(a):- Show that a unit ball in a banach space is compact.

Sol:-

Question # 8(b):- (i) State & prove Parallelogram Identity.
(ii) State & prove Polarization Identity.

Sol:- Parallelogram law or Identity:-
(i) let V be a complete inner product space. Then for any $x, y \in V$

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right]$$

Proof:- $L.H.S. = \|x+y\|^2 + \|x-y\|^2$

$$= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + \|y\|^2 + \|x\|^2 + \|y\|^2$$

$$= 2\left[\|x\|^2 + \|y\|^2\right] = R.H.S.$$

Hence proved.

Polarization Identity:-
(ii) let V be a complex inner product space. Then for any $x, y \in V$

(a) $Re\langle x, y\rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$

(b) $Im\langle x, y\rangle = \frac{1}{4}\{\|x+iy\|^2 - \|x-iy\|^2\}$

Proof:- (a)

$$R.H.S. = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$$

$$= \frac{1}{4}\{\langle x+y, x+y\rangle - \langle x-y, x-y\rangle\}$$

$$= \frac{1}{4}\{\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$- \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle\}$$

$$= \frac{1}{4}\{2\langle x, y\rangle + 2\langle y, x\rangle\}$$

$$= \frac{1}{2}\{\langle x, y\rangle + \overline{\langle x, y\rangle}\}$$

$$= \frac{1}{2} \cdot 2\, Re\langle x, y\rangle$$

$$= Re\langle x, y\rangle = L.H.S.$$

Hence Proved.

Proof:- (b)

$$R.H.S = \frac{1}{4}\{\|x+iy\|^2 - \|x-iy\|^2\}$$

$$= \frac{1}{4}\{\langle x+iy, x+iy\rangle - \langle x-iy, x-iy\rangle\}$$

$$= \frac{1}{4}\{\langle x, x\rangle + \langle x, iy\rangle + \langle iy, x\rangle + \langle iy, iy\rangle$$

$$- \langle x, x\rangle + \langle x, iy\rangle + \langle iy, x\rangle - \langle iy, iy\rangle\}$$

$$= \frac{1}{4}\{2\langle x, iy\rangle + 2\langle iy, x\rangle\}$$

$$R.H.S. = \frac{1}{2}\{\bar{i}<x,y> + \bar{i}<x,y>\}$$

$$= \frac{1}{2}\cdot 2\, Im<x,y>$$

$$= Im<x,y> = L.H.S.$$

Hence Proved.

Question # 9(a):- let A, B be subsets of a Hilbert space H. Then show that

(i) $A \subseteq A^{\perp\perp}$ (ii) $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$

(iii) $(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$

(iv) $A^{\perp} = A^{\perp\perp\perp}$

(v) $A^{\perp}$ is closed subspace of H.

Sol:- (i) To prove $A \subseteq A^{\perp\perp}$

let $x \in A$, then $<x, y> = 0 \quad \forall\, y \in A^{\perp}$

Hence $x \in A^{\perp\perp}$

$\Rightarrow A \subseteq A^{\perp\perp}$

(ii) To prove $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$

let $x \in B^{\perp}$

then $<x, y> = 0, \quad \forall\, y \in B$

since $A \subseteq B$, we have

$<x, y> = 0, \quad \forall\, y \in A$

$\Rightarrow x \in A^{\perp}$

$\Rightarrow B^{\perp} \subseteq A^{\perp}$

(iii) To prove $(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$

Since $A \subseteq A \cup B$ & $B \subseteq A \cup B$

$\Rightarrow (A \cup B)^{\perp} \subseteq A^{\perp}$ & $(A \cup B)^{\perp} \subseteq B^{\perp}$ by (ii)

$\Rightarrow (A \cup B)^{\perp} \subseteq A^{\perp} \cap B^{\perp}$ ①

Now let $x \in A^{\perp} \cap B^{\perp}$

$\Rightarrow x \in A^{\perp}$ & $x \in B^{\perp}$

$\Rightarrow \langle x, u \rangle = 0, \ \forall u \in A$

& $\langle x, y \rangle = 0, \ \forall y \in B$

$\Rightarrow \langle x, y \rangle = 0, \ \forall y \in A \cup B$

$\Rightarrow x \in (A \cup B)^{\perp}$

$\Rightarrow A^{\perp} \cap B^{\perp} \subseteq (A \cup B)^{\perp}$ ②

combining ① & ②

$$(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$$

Hence Proved.

(iv) To prove $A^{\perp} = A^{\perp\perp\perp}$

By (i) $A \subseteq A^{\perp\perp}$

$\Rightarrow A^{\perp\perp\perp} \subseteq A^{\perp}$ — Ⓐ by (ii)

Also by (i) replacing A by $A^{\perp}$

$$A^{\perp} \subseteq (A^{\perp})^{\perp\perp} = A^{\perp\perp\perp}$$

$\Rightarrow A^{\perp} \subseteq A^{\perp\perp\perp}$ — Ⓐ'

Combining (A) & (*)

$$A^{\perp} = A^{\perp\perp\perp}$$ Hence Proved.

(v) To prove $A^{\perp}$ is closed subspace of H.

let $y, z \in A^{\perp}$ & $a, b \in F$
Then for any $x \in A$
$$\langle y, x \rangle = 0 \text{ \& } \langle z, x \rangle = 0$$

So $$\langle ay + bz, x \rangle = a\langle y, x \rangle + b\langle z, x \rangle$$
$$= 0$$
Hence $ay + bz \in A^{\perp}$
i.e. $A^{\perp}$ is a subspace of H.
Next let y be a limit pt. of $A^{\perp}$, then $\exists$ a seq. $\{y_n\}$ in $A^{\perp}$ such that
$$\lim_{n\to\infty} y_n = y$$

Now $\langle y_n, x \rangle = 0 \quad \forall\, x \in A$

$$\Rightarrow \lim_{n\to\infty} \langle y_n, x \rangle = 0$$

$$\Rightarrow \langle \lim_{n\to\infty} y_n, x \rangle = 0$$

$$\Rightarrow \langle y, x \rangle = 0$$

$$\Rightarrow y \in A^{\perp}$$

Therefore $A^{\perp}$ is a closed subspace of H.

Question # 9(b):-

let A be a complete subspace of an inner product space V. Then
$$V = A \oplus A^{\perp}$$

Sol:- Let A be a complete subspace of an inner product space V. Then there is a unique $y \in A$ such that
$$\|x - y\| = \inf_{y' \in A} \|x - y'\|.$$

Put $z = x - y$ then $z \perp A$ so that $z \in A^{\perp}$. we know that $A^{\perp}$ is closed subspace of V. Hence
$$x = y + z \quad \text{— ① where } y \in A \;\&\; z \in A^{\perp}$$

Also $A \cap A^{\perp} \subseteq \{0\} \Rightarrow A \cap A^{\perp} = \{0\}$.

To prove the uniqueness of the expression given by ① suppose that
$$x = y_1 + z_1$$

So that $y + z = y_1 + z_1$
$$\Rightarrow y - y_1 = z_1 - z \in A \cap A^{\perp} = \{0\}$$
$$\Rightarrow y = y_1 \quad \& \quad z = z_1$$

Hence $V = A \oplus A^{\perp}$