PARTIAL DIFFERENTIAL EQUATIONS (PDE's)

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The course provides a foundation to solve PDE's with special emphasis on wave, heat and Laplace equations, formulation and some theory of these equations are also intended.

RECOMMENDED BOOK:

Linear Partial Differential Equations for Scientists and Engineers by Tyn Myint - U Lokenath Debnath

DEDICATION

To

My

Dear Teacher

Syed Hasnaat Raza (IT Instructor)

According's

"EVERYONE CAN DO EVERYTHING BUT THE ONLY DIFFERENCE IS OF SPEED AND QUALITY"

INTRODUCTION

Basic Concepts and Definitions, Superposition Principle, Exercises

FIRST-ORDER QUASI-LINEAR EQUATIONS AND METHOD OF CHARACTERISTICS

Classification of first-order equations, Construction of a First-Order Equation, Method of Characteristics and General Solutions, Canonical Forms of First-Order Linear Equations, Method of Separation of Variables, Exercises

MATHEMATICAL MODELS

Heat Equations and consequences, Wave Equations and consequences.

CLASSIFICATION OF SECOND-ORDER LINEAR EQUATIONS

Second-Order Equations in Two Independent Variables, Canonical Forms, Equations with Constant Coefficients, General Solutions, Summary and Further Simplification, Exercises

FOURIER SERIES, FOURIER TRANSFORMATION AND INTEGRALS WITH APPLICATIONS

Introduction, Fourier Transform, Properties of Fourier Transform , Convolution theorem, Fourier Sine and Fourier Cosine, Exercises, Fourier Series and its complex form

LAPLACE TRANSFORMS

Properties of Laplace Transforms, Convolution Theorem of the Laplace Transform, Laplace Transforms of the Heaviside and Dirac Delta Functions, Hankel Transforms, Properties of Hankel Transforms and Applications, Few results about finite fourier transforms.
Exercises

BASIC CONCEPTS AND DEFINITIONS

DIFFERENCE EQUATION

An equation involving differences (derivatives) is called difference equation.

DIFFERENTIAL EQUATION

An equation that relate a function to its derivative in such a way that the function itself can be determined.

OR an equation containing the derivatives of one dependent variable with respect to one or more independent variables is said to be a differential equation.

It has two types:

i. **Ordinary differential equation (ODE)**
ii. **Partial differential equation (PDE)**

ORDINARY DIFFERENTIAL EQUATION

A differential equation that contains only one independent variable is called ODE.

EXAMPLES:

$y_x + xy = x^2 \quad y''(x) - y'(x) + 6y = 0$

And in general $y = f(x)$

PARTIAL DIFFERENTIAL EQUATION

A differential equation that contains, in addition to the dependent variable and the independent variables, one or more partial derivatives of the dependent variable is called a partial differential equation.

In general, it may be written in the form

$$f\,(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0$$

involving several independent variables x,y, an unknown function 'u' of these variables, and the partial derivatives $u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots$**, of the function. Subscripts on dependent variables denote differentiations, e.g.,**

$u_x = \frac{\partial u}{\partial x}$ *and* $u_y = \frac{\partial u}{\partial y}$.

SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

If a functions $u = u\,(x, y)$ satisfy equation PDE $f\,(x, y, . . , u, u_x, u_y, . . , u_{xx}, u_{xy}, \ldots) = 0$ then it is called solutions of PDE.

EXAMPLES:

$uu_{xy} + u_{xy} = y$

$uu_{xx} + 2yu_{xy} + 3xu_{yy} = 4sinx,$

$(u_x)^2 + (u_y)^2 = 1,$

$u_{xx} - u_{yy} = 0,$ **are partial differential equations. In general $u = u\,(x, y)$**
The functions $u(x, y) = (x + y)^3$ and $u(x, y) = \sin(x - y)$, are solutions of the last equation of above and we can easily be verified.

THE ORDER OF A PARTIAL DIFFERENTIAL EQUATION

The order of a partial differential equation is the power of the highest ordered partial derivative appearing in the equation.
For example $u_{xx} + 2xu_{xy} + u_{yy} = e^y$ is a second-order partial differential equation,
And $u_{xxy} + xu_{yy} + 8u = 7y$ is a third-order partial differential equation.

THE DEGREE OF A PARTIAL DIFFERENTIAL EQUATION

The degree of PDE is the highest power of variable appear in PDE.
For example $u_x + u_y = u + xy$ is of degree one.
And $(u_{xx})^2 = (1+u_y)^{1/2}$ is of degree two.

LINEAR PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is said to be linear if it is linear in the unknown function (dependent variable) and all its derivatives with coefficients depending only on the independent variables.
For example, the equation
$yu_{xx} + 2xyu_{yy} + u = 1$ is a second-order linear partial differential equation

QUASI LINEAR PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is said to be quasi-linear if it is linear in the highest-ordered derivative of the unknown function.
For example, the equation $u_x u_{xx} + xuu_y = \sin y$
is a second-order quasi-linear partial differential equation.

NON LINEAR PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is said to be nonlinear if the unknown function (dependent variable) and all its derivatives with coefficients depending only on the independent variables do not occur linearly.

For example, the equation $u_{xx} + uu_y = 1$ is nonlinear partial differential equation

HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is said to be homogeneous if it always possess the trivial solution i.e. u = 0.

For example, the equation $u_{xx} + u_{yy} = 0$ is a Homogeneous partial differential equation

NON HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is said to be nonhomogeneous if it does not possess the trivial solution.

For example, the equation $u_{xx} + u_{yy} = f(x,y)$ is a nonhomogeneous partial differential equation

EXERCISES

For each of the following, state whether the partial differential equation is linear, quasi-linear or nonlinear. If it is linear, state whether it is homogeneous or nonhomogeneous, and gives its order.

(a) $u_{xx} + xu_y = y$, **(b)** $uu_x - 2xyu_y = 0$, **(c)** $u_x^2 + uu_y = 1$,

(d) $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$, **(e)** $u_{xx} + 2u_{xy} + u_{yy} = \sin x$,

(f) $u_{xxx} + u_{xyy} + \log u = 0$, **(g)** $u_{xx}^2 + u_x^2 + \sin u = e^y$,

(h) $u_t + uu_x + u_{xxx} = 0$.

ANS: Linear (a,d,e) Non – linear (c,f,g) Qausi – linear (b,h)

INITIAL CONDITIONS:

If all conditions are given at the same value of the independent variable, then they are called initial conditions .

For example for a differential equation of order one

$$a(x,y)u_x = f(x,y) \Rightarrow u_x = g(x,y)$$

Then $u_x = g(x,y)$ with $u(a) = u_0$ then x = a is an initial condition.

INITIAL VALUE PROBLEM (IVP):

A DE along with initial conditions defines an IVP. Or Cauchy Problem.

For example, the partial differential equation (PDE)

$u_t - u_{xx} = 0,\ 0 < x < l,\ t > 0,$

with. $u(x, 0) = \sin x,\ 0 \le x \le l,\ t > 0,$ **is IVP**

BOUNDRY CONDITIONS:

If the conditions are given at the end points of the intervals of definition (i.e. for different value of the independent variables) are at the boundary of the domain of definition then they are called boundary conditions.

For example $u'' + 2u' + 3u = 0$ **with** $u(0) = 0, u(2) = 1$ **is a BVP**

BOUNDRY VALUE PROBLEM (BVP):

A DE along with boundary conditions defines an IVP.

For example, the partial differential equation (PDE)

$u_t - u_{xx} = 0, \qquad 0 < x < l,\ t > 0,$

with

B.C. $\qquad u(0, t) = 0,\ t \ge 0,$

B.C. $\qquad u(l, t) = 0,\ t \ge 0,$

PRINCIPLE OF SUPERPOSITION:

According to this principle, if we know 'n' solutions "$u_1, u_2, u_3, \dots \dots \dots u_n$" we can construct other as linear combination.

Statement:

if $u_1, u_2, u_3, \dots \dots \dots u_n$ are solutions of a linear, homogeneous PDE then $W = c_1u_1 + c_2u_2 + \cdots \dots \dots + c_nu_n$ where $c_1, c_2, \dots \dots \dots, c_n$ are constant is also a solution of the equation.

GENERAL SOLUTION:

In the case of partial differential equations, the general solution depends on arbitrary functions rather than on arbitrary constants. To illustrate this, consider the equation

$u_{xy} = 0.$

integrate this equation with respect to y**, we obtain** $u_x(x, y) = f(x)$ **.**
A second integration with respect to x **yields** $u(x, y) = g(x) + h(y)$ **,**
where $g(x)$ **and** $h(y)$ **are arbitrary functions.**

EXAMPLE:

Suppose u **is a function of three variables,** x**,** y**, and** z**. Then, for the Equation** $u_{yy} = 2,$
one finds the general solution $u(x, y, z) = y^2 + yf(x, z) + g(x, z)$ **,**
where f **and** g **are arbitrary functions of two variables** x **and** z**.**

REMARK:

For linear homogeneous ordinary differential equations of order n, a linear combination of n linearly independent solutions is a solution. Unfortunately, this is not true, in general, in the case of partial differential equations. This is due to the fact that the solution space of every homogeneous linear partial differential equation is infinite dimensional.
For example, the partial differential equation
$u_x - u_y = 0$ **can be transformed into the equation** $2u_\eta = 0$
by the transformation of variables $\xi = x + y, \quad \eta = x - y.$
The general solution is $u(x, y) = f(x + y)$ **, where** $f(x + y)$ **is an arbitrary function.**
Thus, we see that each of the functions
$(x + y)^n$ **,** $\sin n(x + y)$ **,** $\cos n(x + y)$ **,** $e^{n(x+y)}$**,** $n = 1, 2, 3, \ldots$ **is a solution of equation** $u_x - u_y = 0$ **.**

WELL POSED PROBLEM:

A mathematical problem is said to be well-posed if it satisfies the following requirements:
1. Existence: There is at least one solution.
2. Uniqueness: There is at most one solution.
3. Continuity: The solution depends continuously on the data.

EXERCISES

1. (a) Verify that the functions

$u(x, y) = x^2 - y^2$,, $u(x, y) = e^x \sin y$,, $u(x, y) = 2xy$

are the solutions of the equation $u_{xx} + u_{yy} = 0$.

(b) Verify that the function $u(x, y) = \text{Log}(\sqrt{x^2 + y^2})$ **satisfies the equation** $u_{xx} + u_{yy} = 0$.

2. Show that $u = f(xy)$, **where** f **is an arbitrary differentiable function Satisfies** $xu_x - yu_y = 0$ **and verify that the functions sin** (xy), **cos** (xy), **log** (xy), e^{xy}, **and** $(xy)^3$ **are solutions.**

3. Show that $u = f(x)\,g(y)$ **where** f **and** g **are arbitrary twice differentiable functions satisfies** $uu_{xy} - u_x u_y = 0$.

4. Determine the general solution of the differential equation $u_{yy} + u = 0$.

5. Find the general solution of $u_{xx} + u_x = 0$, **by setting** $u_x = v$.

6. Find the general solution of $u_{xx} - 4u_{xy} + 3u_{yy} = 0$,

by assuming the solution to be in the form $u(x, y) = f(\lambda x + y)$, **where** λ **is an unknown parameter.**

7. Find the general solution of $u_{xx} - u_{yy} = 0$.

8. (a) Show that the general solution of $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}$

is $u(x, t) = f(x - ct) + g(x + ct)$, **where** f **and** g **are arbitrary twice differentiable functions.**

(b) Show that $u(x, t) = f(x - ct) + g(x + ct)$ **is solution of wave equation.**

(c) whether $u(x, t) = f(x + ct) - g(x - ct)$ **is solution of wave equation or not.**

9. Verify that the function $u = \phi(xy) + x\psi\left(\frac{y}{x}\right)$ **is the general solution of the equation** $x^2u_{xx} - y^2u_{yy} = 0$.

10. If $u_x = v_y$ **and** $v_x = -u_y$, **show that both** u **and** v **satisfy the Laplace Equations** $\nabla^2 u = 0$ **and** $\nabla^2 v = 0$.

11. If $u(x, y)$ **is a homogeneous function of degree** n, **show that** u **satisfies the first-order equation** $xu_x + yu_y = nu$.

12. Verify that

$u(x, y, t) = A\cos(kx)\cos(ly)\cos(nct) + B\sin(kx)\sin(ly)\sin(nct)$,

where $k^2 + l^2 = n^2$, **is a solution of the equation** $u_{tt} = c^2(u_{xx} + u_{yy})$.

13. Show that $u(x, y; k) = e^{-ky}\sin(kx)$, $x \in \mathbf{R}$, $y > 0$, **is a solution of the equation** $\nabla^2 u \equiv u_{xx} + u_{yy} = 0$ **for any real parameter** k**. Verify that** $u(x, y) = \int_0^\infty c(k)e^{-kx}Sin(kx)dk$ **is also a solution of the above equation.**

14. Show, by differentiation that $u(x, y) = \frac{1}{\sqrt{4\pi kt}}exp\left(-\frac{x^2}{4kt}\right)$; $x \in \mathbf{R}$, $t > 0$, **is a solution of the diffusion equation** $u_t = ku_{xx}$ **where** k **is a constant.**

15. (a) Verify that $u(x, y) = \log(x^2 + y^2)$ **satisfies the equation** $u_{xx} + u_{yy} = 0$ **for all** $(x, y) \neq (0, 0)$**.**

(b) Show that $u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ **is a solution of the Laplace equation** $u_{xx} + u_{yy} + u_{zz} = 0$ **except at the origin.**

(c) Show that $u(r) = ar^n$ **satisfies the equation** $r^2u'' + 2ru' - n(n+1)u = 0$**.**

16. Show that $u_n(r, \theta) = r^n\cos(n\theta)$ **and** $u^n(r, \theta) = r^n\sin(n\theta)$, $n = 0, 1, 2, 3, \ldots$

are solutions of the Laplace equation $\nabla^2 u \equiv u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

17. Verify by differentiation that $u(x, y) = \cos x \cosh y$ **satisfies the Laplace equation** $u_{xx} + u_{yy} = 0$**.**

18. Show that $u(x, y) = f(2y + x^2) + g(2y - x^2)$ **is a general solution of the equation** $u_{xx} - \frac{1}{x}u_x - x^2u_{yy} = 0$**.**

20. Show that $u(x, y, t) = f(x + iky - i\omega t) + g(x - iky - i\omega t)$ **is a general solution of the wave equation** $u_{tt} = c^2(u_{xx} + u_{yy})$ **where** f **and** g **are arbitrary twice differentiable functions, and** $\omega^2 = c^2(k^2 - 1)$, k, ω, c **are constants.**

21. Verify that $u(x, y) = x^3 + y^2 + e^x(\cos x \sin y \cosh y - \sin x \cos y \sinh y)$ **is a classical solution of the Poisson equation** $u_{xx} + u_{yy} = (6x+2)$ **.**

22. Show that $u(x, y) = \exp(-\frac{x}{b})\, f(ax - by)$ **satisfies the equation** $bu_x + au_y + u = 0$**.**

23. Show that $u_{tt} - c^2u_{xx} + 2bu_t = 0$ **has solutions of the form** $u(x, t) = (A\cos kx + B\sin kx)V(t)$, **where** c, b, A **and** B **are constants.**

FIRST-ORDER, QUASI-LINEAR EQUATIONS AND METHOD OF CHARACTERISTICS

Many problems in mathematical, physical, and engineering sciences deal with the formulation and the solution of first-order partial differential equations.
From a mathematical point of view, first-order equations have the advantage of providing a conceptual basis that can be utilized for second-,third-, and higher-order equations. This chapter is concerned with first-order, quasi-linear and linear partial differential equations and their solution by using the Lagrange method of characteristics and its generalizations.

CLASSIFICATION OF FIRST-ORDER EQUATIONS

The most general, first-order, partial differential equation in two independent variables x and y is of the form

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset R^2,$$

Where F is a given function of its arguments, and $u = u(x, y)$ is an unknown function of the independent variables x and y which lie in some given domain D in R^2
Equation is often written in terms of standard notation $p = u_x$ and $q = u_y$ so that takes the form $F(x, y, u, p, q) = 0$.
Similarly, the most general, first-order, partial differential equation in three independent variables x, y, z can be written as

$$F(x, y, z, u, u_x, u_y, u_z) = 0.$$

Equation is called a <u>quasi-linear</u> partial differential equation if it is linear in first-partial derivatives of the unknown function $u(x, y)$.
So, the most general quasi-linear equation must be of the form

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u)$$

where its coefficients a, b, and c are functions of x, y, and u.
Examples of quasi linear equations are

$$uu_x + u_t + n u^2 = 0$$

$$x(y^2+u)\, u_x + y(x^2+u)\, u_y = (x^2 - y^2)\, u,$$

A quasi linear Equation is called a semi linear partial differential equation if its coefficients a and b are independent of u, and hence, the semi linear equation can be expressed in the form

$a(x, y)u_x + b(x, y)u_y = c(x, y, u)$

Examples of semi linear equations are

$xu_x + yu_y = u^2 + x^2,$

$(x+1)^2 u_x + (y-1)^2 u_y = (x+y)u^2,$

$u_t + au_x + u^2 = 0$ **where a is a constant.**

Equation $F(x, y, u, u_x, u_y) = 0$ is said to be linear if F is linear in each of the variables

u, u_x, and u_y, and the coefficients of these variables are functions only of the independent variables x and y. The most general, first-order, linear partial differential equation has the form

$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$

Examples of linear equations are

$xu_x + yu_y - nu = 0,$

$nu_x + (x+y)u_y - u = e^x,$

$yu_x + xu_y = xy,$

$(y-z)u_x + (z-x)u_y + (x-y)u_z = 0.$

An equation which is not linear is often called a nonlinear equation. So, first-order equations are often classified as linear and nonlinear.

CONSTRUCTION OF A FIRST-ORDER EQUATION

METHOD I: BY ELIMINATING ARBITRARY CONSTANT.

Let $f(x, y, z, a, b) = 0$ .... (i)

where a and b are arbitrary parameters. And $z = \varphi(x, y)$

We differentiate with respect to x and y to obtain

$f_x + pf_x = 0$ .... (ii) and $f_y + q f_y = 0$ .... (iii)

where $p = z_x$ and $q = z_y$

eliminating a and b from (i), (ii) and (iii) we get

$F(x, y, z, p, q) = 0$

Thus , an equation of the form $f(x, y, z, a, b) = 0$ containing two arbitrary parameters is called a complete solution or a complete integral of equation $F(x, y, z, p, q) = 0$
Its role is somewhat similar to that of a general solution for the case of an ordinary differential equation.

EXAMPLE:
Obtain PDE $z = x + ax^2y^2 + b$ where a,b are arbitrary constants.
Solution:
Given PDE $z = x + ax^2y^2 + b$ .... (i)
We differentiate with respect to x and y to obtain
$z_x = p = 1 + 2axy^2 \Rightarrow 2a = \frac{p-1}{xy^2}$.......... (ii)
and $z_y = q = 2ax^2y$ .... (iii)
(iii) $\Rightarrow z_y = q = (\frac{p-1}{xy^2})x^2y \Rightarrow q = (\frac{p-1}{y})x$
$\Rightarrow px - qy = x$ **which is linear PDE**
eliminating a and b from (i), (ii) and (iii) we get
$F(x, y, z, p, q) = 0$

METHOD II: BY ELIMINATING ARBITRARY FUNCTION

EXAMPLE:
Obtain PDE $z = xy + F(x^2 + y^2)$ where F is arbitrary function.
Solution:
Given PDE $z = xy + F(x^2 + y^2)$ .... (i)
We differentiate with respect to x and y to obtain
$z_x = p = y + F'(x^2 + y^2).2x \Rightarrow F'(x^2 + y^2) = \frac{p-y}{2x}$.......... (ii)
and $z_y = q = x + F'(x^2 + y^2).2y \Rightarrow F'(x^2 + y^2) = \frac{p-x}{2y}$ .... (iii)
equating both $\Rightarrow \frac{p-x}{2y} = \frac{p-y}{2x}$
$\Rightarrow py - qx = y^2 - x^2$

Example: Show that a family of spheres $x^2 + y^2 + (z - c)^2 = r^2$ satisfies the first-order linear partial differential equation $yp - xq = 0$.
Solution:
Given PDE $x^2 + y^2 + (z - c)^2 = r^2$ **……….** (i)
Differentiating the equation with respect to x **and** y **gives**
$x + p(z - c) = 0$ **……….** (ii) **and** $y + q(z - c) = 0$ **……….** (iii)
Eliminating the arbitrary constant c **from these equations, we obtain the first-order, partial differential equation**
$yp - xq = 0$.

Example: Show that the family of spheres $(x - a)^2 + (y - b)^2 + z^2 = r^2$ satisfies the first-order, nonlinear, partial differential equation $z^2(p^2 + q^2 + 1) = r^2$
Solution:
Given PDE $(x - a)^2 + (y - b)^2 + z^2 = r^2$ **……….** (i)
Differentiating the equation with respect to x **and** y **gives**
$(x - a) + zp = 0$**……….** (ii) **and** $(y - b) + zq = 0$**……….** (iii)
Eliminating the two arbitrary constants a **and** b**, we find the nonlinear partial differential equation** $z^2(p^2 + q^2 + 1) = r^2$

Example: Show that All surfaces of revolution with the z-axis as the axis of symmetry satisfy the equation $z = f(x^2 + y^2)$ where f is an arbitrary function.
Solution:
Given PDE $z = f(x^2 + y^2)$ **……….** (i)
Writing $u = x^2 + y^2$ **and differentiating with respect to** x **and** y**, respectively, we obtain** $p = 2xf'(u)$, $q = 2yf'(u)$.
Eliminating the arbitrary function $f(u)$ **from these results, we find the Equation** $yp - xq = 0$.

METHOD III: FOR THE FORM $f(\phi, \psi) = 0$

Theorem:

If $\phi = \phi(x, y, z)$ and $\psi = \psi(x, y, z)$ are two given functions of x, y, and z and if $f(\phi, \psi) = 0$, where f is an arbitrary function of ϕ and ψ, then $z = z(x, y)$ satisfies a first-order, partial differential equation

$$p\frac{\partial(\varphi,\psi)}{\partial(y,z)} + q\frac{\partial(\varphi,\psi)}{\partial(z,x)} = \frac{\partial(\varphi,\psi)}{\partial(x,y)} \quad \text{where} \quad \frac{\partial(\varphi,\psi)}{\partial(x,y)} = \begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix}$$

Proof. We differentiate $f(\phi, \phi) = 0$ with respect to x and y respectively to obtain the following equations:

$$\frac{\partial f}{\partial \varphi}\left(\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z}\right) = 0$$

$$\frac{\partial f}{\partial \varphi}\left(\frac{\partial \varphi}{\partial y} + q\frac{\partial \varphi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z}\right) = 0$$

Nontrivial solutions for $\frac{\partial f}{\partial \varphi}$ and $\frac{\partial f}{\partial \psi}$ can be found if the determinant of the coefficients of these equations vanishes, that is,

$$\begin{vmatrix} \varphi_x + p\varphi_z & \psi_x + p\psi_z \\ \varphi_y + q\varphi_z & \psi_y + p\psi_z \end{vmatrix} = 0$$

Expanding this determinant gives the first-order, quasi-linear equation

$$p\frac{\partial(\varphi,\psi)}{\partial(y,z)} + q\frac{\partial(\varphi,\psi)}{\partial(z,x)} = \frac{\partial(\varphi,\psi)}{\partial(x,y)} \quad \text{where} \quad \frac{\partial(\varphi,\psi)}{\partial(x,y)} = \begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix}$$

METHOD OF CHARACTERISTICS AND GENERAL SOLUTIONS

Theorem:

The general solution of a first-order, quasi-linear partial differential equation

$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u)$

is $f(\phi, \psi) = 0$ where f is an arbitrary function of $\phi(x, y, u)$ and $\psi(x, y, u)$, and ϕ = constant = c_1 and ϕ = constant = c_2 are solution curves of the characteristic equations $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$

The solution curves defined by $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are called the families of characteristic curves of equation

Example: Find the general solution of the first-order linear partial differential equation $xu_x + y\ u_y = u$

Solution: The characteristic curves of this equation are the solutions of the characteristic equations $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$

This system of equations gives the integral surfaces

$\phi = \frac{y}{x} = C_1$ **and** $\psi = \frac{u}{x} = C_2$ **where** C_1 **and** C_2 **are arbitrary constants.**

Thus, the general solution of $f(\frac{y}{x}, \frac{u}{x}) = 0$

where f **is an arbitrary function. This general solution can also be written as** $u\ (x,\ y) = xg(\frac{y}{x})$ **where** g **is an arbitrary function.**

Example: Obtain the general solution of the linear Euler equation

$xu_x + y\ u_y = nu$

Solution: The characteristic curves of this equation are the solutions of the characteristic equations $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{nu}$

This system of equations gives the integral surfaces

$\phi = \frac{y}{x} = C_1$ **and** $\psi = \frac{u}{x^n} = C_2$ **where** C_1 **and** C_2 **are arbitrary constants.**

Thus, the general solution of $f(\frac{y}{x}, \frac{u}{x^n}) = 0$

where f **is an arbitrary function. This general solution can also be written as** $\frac{u}{x^n} = g(\frac{y}{x})$

or $u\ (x,\ y) = x^n g(\frac{y}{x})$ **where** g **is an arbitrary function.**

This shows that the solution $u\ (x,\ y)$ **is a homogeneous function of** x **and** y **of degree** n**.**

Example: Find the general solution of the linear equation

$x^2 u_x + y^2 u_y = (x + y) u$ .... (i)

Solution: The characteristic curves of this equation are the solutions of the characteristic equations $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$ .... (ii)

From the first two of these equations, we find $x^{-1} - y^{-1} = C_1$

where C_1 **is an arbitrary constant.**

It follows from (ii) that $\frac{dx-dy}{x^2-y^2} = \frac{d(x-y)}{(x+y)u}$ **or** $\frac{d(x-y)}{(x-y)} = \frac{du}{u}$

This gives $\frac{(x-y)}{u} = \frac{d(x-y)}{du} = c_2$ .... (iii) **where** C_2 **is a constant.**

Furthermore, (ii) and (iii) also give $\frac{xy}{u} = c_3$ **where** C_3 **is a constant.**

Thus, the general solution of $f\left(\frac{xy}{u}, \frac{(x-y)}{u}\right) = 0$
where f **is an arbitrary function. This general solution can also be written as** $u(x, y) = xy\,g\left(\frac{(x-y)}{u}\right)$ **where** g **is an arbitrary function.**
Or equivalently $u(x, y) = xy\,h\left(\frac{(x-y)}{xy}\right)$ **where** h **is an arbitrary function.**

Example: Show that the general solution of the linear equation
$(y - z)u_x + (z - x)u_y + (x - y)u_z = 0$ .... (i)
is $u(x, y, z) = f(x + y + z, \ x^2 + y^2 + z^2)$ .... (ii)
where f **is an arbitrary function.**
Solution: The characteristic curves of this equation are the solutions of the characteristic equations $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0}$ .... (iii)
Or $du = 0$ **,** $dx + dy + dz = 0$ **,** $xdx + ydy + zdz = 0$
On integrating $u = c_1$ **,** $x + y + z = c_2$ **,** $x^2 + y^2 + z^2 = C_3$
where C_1, C_2, C_3 **are arbitrary constant.**
Thus, the general solution can be written in terms of an arbitrary function f **in the form** $u(x, y, z) = f(x + y + z, \ x^2 + y^2 + z^2)$

NOTE: IVP or BVP also called Cauchy data.

Example: Find the solution of the equation
$u(x + y)u_x + u(x - y)u_y = x^2 + y^2$ .... (i)
with the Cauchy data $u = 0$ **on** $y = 2x$.
Solution: The characteristic curves of this equation are the solutions of the characteristic equations
$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2} = \frac{ydx+xdy-udu}{0} = \frac{xdx-ydy-udu}{0} \quad \text{............. (ii)}$$
These give two integrals $\int d\left[xy - \frac{1}{2}u^2\right] = 0$ **and** $\int d\left[\frac{1}{2}(x^2 - y^2 - u^2)\right] = 0$
$u^2 - x^2 + y^2 = C_1$ **and** $2xy - u^2 = C_2$ .... (iii)
where C_1 **and** C_2 **are constants. Hence, the general solution is**
$f(x^2 - y^2 - u^2, \ 2xy - u^2) = 0$ **where** f **is an arbitrary function.**
Using the Cauchy data in (iii), we obtain $4C_1 = 3C_2$**. Therefore**
$4(u^2 - x^2 + y^2) = 3(2xy - u^2)$
Thus, the solution of equation (i) is given by $7u^2 = 6xy + 4(x^2 - y^2)$

Example: Obtain the solution of the linear equation

$u_x - u_y = 1$ **………….** (i) **with the Cauchy data** $u(x, 0) = x^2$.

Solution: The characteristic curves of this equation are the solutions of the characteristic equations $\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}$ **………….** (ii)

Clearly,

$x + y$ **= constant =** C_1 **and** $u - x$ **= constant =** C_2.

Thus, the general solution is given by

$u - x = f(x + y)$ **………….** (iii) **where** f **is an arbitrary function.**

We now use the Cauchy data to find $f(x) = x^2 - x$, **and hence, the solution is** $u(x, y) = (x + y)^2 - y$

Example: Obtain the solution of the equation

$(y - u)u_x + (u - x)u_y = x - y$ **………….** (i)

with the condition $u = 0$ **on** $xy = 1$.

Solution: The characteristic curves of this equation are the solutions of the characteristic equations $\frac{dx}{y-u} = \frac{dy}{u-x} = \frac{du}{x-y}$ **………….** (ii)

The parametric forms of these equations are

$\frac{dx}{dt} = y - u, \quad \frac{dy}{dt} = u - x, \quad \frac{du}{dt} = x - y.$

These lead to the following equations:

$\dot{x} + \dot{y} + \dot{u} = 0$ **and** $x\dot{x} + y\dot{y} + u\dot{u} = 0$ **………….** (iii)

where the dot denotes the derivative with respect to t.

Integrating (iii), we obtain

$x + y + u$ **= const. =** C_1 **and** $x^2 + y^2 + u^2$ **= const. =** C_2. **………….** (iv)

These equations represent circles.

Using the Cauchy data, we find that

$C_1^2 = (x + y)^2 = x^2 + y^2 + 2xy = C_2 + 2.$

Thus, the integral surface is described by

$(x + y + u)^2 = x^2 + y^2 + u^2 + 2.$

Hence, the solution is given by $u(x, y) = \frac{1-xy}{x+y}$

Example:Solve the linear equation $yu_x + xu_y = u$
with the Cauchy data $u(x, 0) = x^3$ **and** $u(0, y) = y^3$
Solution: The characteristic curves of this equation are the solutions of the characteristic equations $\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}$ or $\frac{du}{u} = \frac{dx-dy}{y-x} = \frac{dx+dy}{y+x}$
Solving these equations, we obtain
$u = \frac{c_1}{x-y} = C_2(x + y)$ **or** $u = C_2(x + y)$, $x^2 - y^2 = \frac{c_1}{c_2}$ **= constant =** C.
So the characteristics are rectangular hyperbolas for $C > 0$ **or** $C < 0$.
Thus, the general solution is given by
$f\left(\frac{u}{x+y}, x^2 - y^2\right) = 0$ **or, equivalently,** $u(x, y) = (x + y)\, g(x^2 - y^2)$
Using the Cauchy data, we find that $g(x^2) = x^2$, **that is,** $g(x) = x$.
Consequently, the solution becomes
$u(x, y) = (x + y)(x^2 - y^2)$ **on** $x^2 - y^2 = C > 0$.
Similarly, $u(x, y) = (x + y)(y^2 - x^2)$ **on** $y^2 - x^2 = C > 0$.

Example: Determine the integral surfaces of the equation
$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$
with the data $x + y = 0$, $u = 1$

Solution: The characteristic equations are
$$\frac{dx}{x(y^2+u)} = \frac{dy}{-y(x^2+u)} = \frac{du}{(x^2-y^2)u} \quad \ldots\ldots\ldots\ldots (i)$$
or $$\frac{dx/x}{(y^2+u)} = \frac{dy/y}{-(x^2+u)} = \frac{du/u}{(x^2-y^2)} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{du}{u}}{0}$$
Consequently, $\log(xyu) = \log C_1$ **or** $xyu = C_1$
From (i), we obtain $\frac{xdx}{x^2(y^2+u)} = \frac{ydy}{-y^2(x^2+u)} = \frac{udu}{(x^2-y^2)u^2}$
whence we find that $x^2 + y^2 - 2u = C_2$
Using the given data, we obtain $C_1 = -x^2$ **and** $C_2 = 2x^2 - 2$
so that $C_2 = -2(C_1 + 1)$.
Thus the integral surface is given by $x^2 + y^2 - 2u = -2 - 2xyu$
Or $2xyu + x^2 + y^2 - 2u + 2 = 0$

Example: Obtain the solution of the equation

$xu_x + y\ u_y = x\ \exp(-u)$ **with the data** $u = 0$ **on** $y = x^2$

Solution: The characteristic equations are

$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{x\exp(-u)}$ .... (i) or $\frac{y}{x} = C_1$

We also obtain from (i) that $dx = e^u du$ **which can be integrated to Find** $e^u = x + C_2$

Thus, the general solution is given by $f(e^u - x, \frac{y}{x}) = 0$

or, equivalently, $e^u = x + g\left(\frac{y}{x}\right)$

Applying the Cauchy data i.e. $u = 0$ **on** $y = x^2$. **we obtain** $g(x) = 1 - x$

Thus, the solution is given by

$e^u = x + 1 - \frac{y}{x}$ **or** $u = \log\left(x + 1 - \frac{y}{x}\right)$

Example: Solve the initial-value problem

$u_t + uu_x = x,\quad u(x, 0) = f(x)$ **where (a)** $f(x) = 1$ **and (b)** $f(x) = x$

Solution: The characteristic equations are

$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{x} = \frac{d(x+u)}{x+u}$

Integration gives $t = \log(x + u) - \log C_1$

Or $(u + x)\,e^{-t} = C_1$. **Similarly, we get** $u^2 - x^2 = C_2$

For case (a) we obtain

$1 + x = C_1$ **and** $1 - x^2 = C_2$, **and hence** $C_2 = 2C_1 - C_1^2$

Thus, $(u^2 - x^2) = 2(u + x)\,e^{-t} - (u + x)^2\,e^{-2t}$

Or $u - x = 2e^{-t} - (u + x)\,e^{-2t}$

A simple manipulation gives the solution

$u(x, t) = x\tanh t + \operatorname{sech} t$

Case (b) is left as an exercise.

EXERCISES

1. (a) Show that the family of right circular cones whose axes coincide with the z-axis; $x^2 + y^2 = (z - c)^2 \tan^2 \alpha$ satisfies the first-order, PDE $yp - xq = 0$

(b) Show that all the surfaces of revolution, $z = f(x^2 + y^2)$ with the z-axis as the axis of symmetry, where f is an arbitrary function, satisfy the partial differential equation $yp - xq = 0$.

(c) Show that the two-parameter family of curves $u - ax - by - ab = 0$ satisfies the nonlinear equation $xp + yq + pq = u$.

2. Find the partial differential equation arising from each of the following surfaces:

(a) $z = x + y + f(xy)$ **(b)** $z = f(x - y)$ **(c)** $z = xy + f(x^2 + y^2)$
(d) $2z = (\alpha x + y)^2 + \beta$

3. Find the general solution of each of the following equations:

(a) $u_x = 0$, **(b)** $au_x + bu_y = 0$; a, b, **are constant,**
(c) $u_x + y u_y = 0$ **(d)** $(1 + x^2) u_x + u_y = 0$
(e) $2xy u_x + (x^2 + y^2) u_y = 0$ **(f)** $yu_y - xu_x = 1$

4. Find the general solution of the equation $u_x + 2xy^2 u_y = 0$.

5. Find the solution of the following Cauchy problems:

(a) $3u_x + 2u_y = 0$, **with** $u(x, 0) = \sin x$
(b) $y u_x + xu_y = 0$, **with** $u(0, y) = \exp(-y^2)$
(c) $xu_x + y u_y = 2xy$, **with** $u = 2$ **on** $y = x^2$,
(d) $u_x + xu_y = 0$, **with** $u(0, y) = \sin y$,
(e) $yu_x + xu_y = xy$, $x \geq 0$, $y \geq 0$, **with** $u(0, y) = \exp(-y^2)$ **for** $y > 0$, **and** $u(x, 0) = \exp(-x^2)$ **for** $x > 0$,

6. Solve the equation $u_x + xu_y = y$ with the Cauchy data
(a) $u(0, y) = y^2$, (b) $u(1, y) = 2y$.

7. Solve the Cauchy problem $(y + u) u_x + y u_y = (x - y)$, with $u = 1 + x$ on $y = 1$.

8. Show that the solution of the equation $yu_x - xu_y = 0$ containing the curve $x^2 + y^2 = a^2$, $u = y$, does not exist.

9. Find the solution of the equation $yu_x - 2xy\ u_y = 2xu$ with the condition $u(0, y) = y^3$.

10. Solve the following Cauchy problems:

(a) $3u_x + 2u_y = 0$, $u(x, 0) = f(x)$,

(b) $au_x + bu_y = cu$, $u(x, 0) = f(x)$, **where** a, b, c **are constants,**

(c) $xu_x + y\ u_y = cu$, $u(x, 0) = f(x)$,

(d) $uu_x + u_y = 1$, $u(s, 0) = \alpha s$, $x(s, 0) = s$, $y(s, 0) = 0$.

BOUNDRY CONDITIONS

- **The boundry conditions $\alpha u(a) + \alpha' u'(a) = 0$ and $\beta u(b) + \beta' u'(b) = 0$ are called Separated Boundry Conditions are Unmixed Boundry Conditions.**
- **If the Separated Boundry conditions are of the form $u(a) = c_1$ *and* $u(b) = c_2$ then they are called Drichlet BC's**
- **If the Separated Boundry conditions are of the form $u'(a) = c'_1$ *and* $u'(b) = c'_2$ then they are called Neumann BC's**
- **If the Separated Boundry conditions are of the form $u(a) = u(b)$ *and* $u'(a) = u'(b)$ then they are called Periodic BC's**

OR: **There must be one initial and two boundry conditions to solve a problem uniquely. Such conditions give initial temperature distribution.**

i. **THE DRICHLET BC's or BC's OF 1st KIND:**
Booundry conditions of the form $u(0, t) = u_0(t)$ and $u(l, t) = u_1(t)$; $t > 0$ are called Drichlet boundry conditions.
Physical Meaning: This condition tells that the temperature at the boundry of a body may be controlled in some way without being held constant.

ii. **THE NEUMANN BC's or BC's OF 2nd KIND:**
Booundry conditions of the form $u_x(0, t) = \gamma(t)$ and $u_x(l, t) = \delta(t)$ are called Neumann boundry conditions. Where γ and δ are functions of time. And in particular, γ and δ may be zero. If $\gamma = 0$ then there is no flow at x = 0
Physical Meaning: This condition tells that the rate of flow of heat is specified at one or more boundry.

iii. **THE ROBBIN BC's or BC's OF 3rd KIND:**
Booundry conditions of the form $\propto_1 u(0, t) + \propto_2 u_x(0, t) = constant$ and $\beta_1 u(l, t) + \beta_2 u_x(l, t) = constant$ are called Robbin boundry conditions.
Physical Meaning: This condition tells about the proportionality between the rate of transfer of heat to the difference of temperature between the two bodies. i.e. both will be Proportional.

REMEMBER:

- **An equation of the form $f\,(x, y, z, a, b) = 0$ containing two arbitrary parameters is called a <u>complete solution or a complete integral</u> of equation $f\,(x, y, z, p, q) = 0$**
  Its role is somewhat similar to that of a general solution for the case of an ODE.
- **Any relationship of the form $f(\varphi, \psi) = 0$ with $\varphi = \varphi(x, y, z)$ and $\psi = \psi(x, y, z)$ and provides a solution of first order PDE is called a <u>general solution or general integral</u> of this equation.**
- **The general solution of first order PDE depends on arbitrary function while The general solution of first order ODE depends on arbitrary constant.**
- **In practice only one solution satisfying prescribed conditions is required for a physical problem, such solution is called <u>Particular Solution.</u>**
- **For a function $F(x, y, z, p, q) = 0$ if the envelope of two parameteric system $f(x, y, z, a, b) = 0$ of surfaces exists, represents a solution of given equation $F(x, y, z, p, q) = 0$ then the envelope is called Singular solution of such equation.**
- **A function is called <u>smooth function</u> if all of its derivatives exist and are continuous.**
- **A solution which is not everywhere differentiable is called a <u>weak solution</u>.**
- **Solutions of Qausi linear and non – linear PDE's may develop discontinuities as they move away from initial state. So their solutions are called weak solutions or shock waves.**

METHOD OF SEPARATION OF VARIABLES

During the last two centuries several methods have been developed for solving partial differential equations. Among these, a technique known as the method of separation of variables is perhaps the oldest systematic method for solving partial differential equations.

- **Its essential feature is to transform the partial differential equations by a set of ordinary differential equations.**
- **The required solution of the partial differential equations is then exposed as a product $u(x, y) = X(x)\,Y(y) \neq 0$**
  or as a sum $u(x, y) = X(x) + Y(y)$

where $X(x)$ and $Y(y)$ are functions of x and y, respectively.

IMPORTANCE: Many significant problems in partial differential equations can be solved by the method of separation of variables. This method has been considerably refined and generalized over the last two centuries and is one of the classical techniques of applied mathematics, mathematical physics and engineering science.
Usually, the first-order partial differential equation can be solved by separation of variables without the need for Fourier series.
This method is used to convert PDE into ODE.

Example: Solve the initial-value problem

$u_x + 2u_y = 0$, $u(0, y) = 4e^{-2y}$. ………. (i)

Solution: (i) $\Rightarrow \frac{\partial}{\partial x}\mathbf{u}(\mathbf{x},\mathbf{y}) + 2\frac{\partial}{\partial y}\mathbf{u}(\mathbf{x},\mathbf{y}) = 0$ ………. (ii)

let $u \equiv u(x, y) = X(x)\,Y(y) \equiv XY$

(ii) $\Rightarrow \frac{\partial}{\partial x}(\mathbf{XY}) + 2\frac{\partial}{\partial y}(\mathbf{XY}) = 0 \Rightarrow X'(x)\,Y(y) + 2\,X(x)\,Y'(y) = 0$

Dividing XY on both sides $\Rightarrow \frac{X'Y}{XY} + 2\frac{XY'}{XY} = 0$

$\Rightarrow \frac{X'}{X} + 2\frac{Y'}{Y} = 0 \Rightarrow \frac{X'}{2X} = -\frac{Y'}{Y} \Rightarrow \frac{X'}{2X} = -\frac{Y'}{Y}$

Since the L.H.S of this equation is a function of x only and the R.H.S is a function of y only

$\Rightarrow \frac{X'}{2X} = -\frac{Y'}{Y} = \lambda \Rightarrow \frac{X'}{2X} = \lambda$ and $-\frac{Y'}{Y} = \lambda$ **where λ is separation constant.**

Consequently, gives two ordinary differential equations

$X'(x) - 2\lambda X(x) = 0$ and $Y'(y) + \lambda Y(y) = 0$

These equations have solutions given, respectively, by

$X(x) = Ae^{2\lambda x}$ **and** $Y(y) = Be^{-\lambda y}$

where A and B are arbitrary integrating constants.

Consequently, the general solution is given by

$u(x, y) = AB\exp(2\lambda x - \lambda y) = C\exp(2\lambda x - \lambda y)$

where $C = AB$ is an arbitrary constant.

Using the condition $u(0, y) = 4e^{-2y}$ we find $4e^{-2y} = u(0, y) = Ce^{-y}$ and hence, we deduce that $C = 4$ and $\lambda = 2$. Therefore, the final solution is $u(x, y) = 4\exp(4x - 2y)$

Example: Solve the equation $y^2 u_x^2 + x^2 u_y^2 = (xyu)^2$. ………. (i)

with $u(x, 0) = 3\exp\left(\frac{x^2}{4}\right)$

Solution: (i) $\Rightarrow y^2\left[\frac{\partial}{\partial x}\mathbf{u}(\mathbf{x},\mathbf{y})\right]^2 + x^2\left[\frac{\partial}{\partial y}\mathbf{u}(\mathbf{x},\mathbf{y})\right]^2 = (xyu)^2$ ……. (ii)

let $u \equiv u(x, y) = X(x)\,Y(y) \equiv XY$

(ii) $\Rightarrow y^2\left[\frac{\partial}{\partial x}(\mathbf{XY})\right]^2 + x^2\left[\frac{\partial}{\partial y}(\mathbf{XY})\right]^2 = x^2y^2(XY)^2$

$\Rightarrow y^2[X'Y]^2 + x^2[XY']^2 = x^2y^2(XY)^2$

Dividing $x^2y^2(XY)^2$ on both sides $\Rightarrow \frac{1}{x^2}\left[\frac{X'}{X}\right]^2 + \frac{1}{y^2}\left[\frac{Y'}{Y}\right]^2 = 1$

$\Rightarrow \quad \frac{1}{x^2}\left[\frac{X'}{X}\right]^2 = 1 - \frac{1}{y^2}\left[\frac{Y'}{Y}\right]^2 \Rightarrow \frac{1}{x^2}\left[\frac{X'}{X}\right]^2 = 1 - \frac{1}{y^2}\left[\frac{Y'}{Y}\right]^2 = \lambda^2$

Thus , we obtain

$\frac{1}{x^2}\left[\frac{X'}{X}\right]^2 = \lambda^2$ **and** $1 - \frac{1}{y^2}\left[\frac{Y'}{Y}\right]^2 = \lambda^2$ **where** λ^2 **is separation constant.**

$\Rightarrow \quad \frac{1}{x}\frac{X'}{X} = \lambda$ **and** $\frac{1}{y}\frac{Y'}{Y} = \sqrt{1-\lambda^2}$

Solving these ODE's we find

$X(x) = Ae^{\frac{\lambda x^2}{2}}$ **and** $Y(y) = Be^{\frac{1}{2}y^2\sqrt{1-\lambda^2}}$

where A **and** B **are arbitrary constant. Thus, the general solution is**

$u \equiv u(x,y) = X(x)Y(y) \equiv \left(Ae^{\frac{\lambda x^2}{2}}\right)\left(Be^{\frac{1}{2}y^2\sqrt{1-\lambda^2}}\right) = Ce^{\frac{\lambda x^2}{2}+\frac{1}{2}y^2\sqrt{1-\lambda^2}}$

where C **=** AB **is an arbitrary constant.**

Using the condition u**(x,** 0**) = 3**$\exp\left(\frac{x^2}{4}\right)$ **we determine that** C **= 3 and**

λ **= (1/2), and the solution becomes** $u(x,y) = 3e^{\frac{1}{4}(x^2+y^2\sqrt{3})}$

Example: Use the separation of variable $u(x,y) = f(x) + g(y)$

Solve the equation $u_x^2 + u_y^2 = 1$ ………. (i)

Solution: (i)$\Rightarrow \left[\frac{\partial}{\partial x}u(x,y)\right]^2 + \left[\frac{\partial}{\partial y}u(x,y)\right]^2 = 1$ ……. (ii)

let $u \equiv u(x, y) = f(x) + g(y) \equiv f + g$

(ii)$\Rightarrow \quad \left[\frac{\partial}{\partial x}\left(f(x) + g(y)\right)\right]^2 + \left[\frac{\partial}{\partial y}\left(f(x) + g(y)\right)\right]^2 = 1$

$\Rightarrow \quad [f'(x)]^2 + [g'(y)]^2 = 1 \Rightarrow \quad [f'(x)]^2 = 1 - [g'(y)]^2$

$\Rightarrow \quad [f'(x)]^2 = 1 - [g'(y)]^2 = \lambda^2$

Thus , we obtain

$[f'(x)]^2 = \lambda^2$ **and** $1 - [g'(y)]^2 = \lambda^2$ **where** λ^2 **is separation constant.**

$\Rightarrow \quad f'(x) = \lambda$ **and** $g'(y) = \sqrt{1-\lambda^2}$

Solving these ODE's we find

$f(x) = \lambda x + A$ **and** $g(y) = y\sqrt{1-\lambda^2} + B$**where** A **and** B **are arbitrary constant. Thus, the general solution is**

$u \equiv u(x,y) = f(x) + g(y) \equiv \lambda x + A + y\sqrt{1-\lambda^2} + B$

$u(x,y) = \lambda x + y\sqrt{1-\lambda^2} + C$

where C **=** A **+** B **is an arbitrary constant.**

Example: Use $u(x, y) = f(x) + g(y)$ **to solve the equation**
$u_x^2 + u_y + x^2 = 0$ ………. (i)

Solution: (i) $\Rightarrow \left[\frac{\partial}{\partial x}u(x,y)\right]^2 + \left[\frac{\partial}{\partial y}u(x,y)\right] + x^2 = 0$ ……. (ii)

let $u \equiv u(x, y) = f(x) + g(y) \equiv f + g$

(ii) $\Rightarrow \left[\frac{\partial}{\partial x}\big(f(x) + g(y)\big)\right]^2 + \left[\frac{\partial}{\partial y}\big(f(x) + g(y)\big)\right] + x^2 = 0$

$\Rightarrow [f'(x)]^2 + [g'(y)] + x^2 = 0 \quad \Rightarrow \quad [f'(x)]^2 + x^2 = -[g'(y)]$

$\Rightarrow [f'(x)]^2 + x^2 = -[g'(y)] = \lambda^2$

Thus , we obtain

$[f'(x)]^2 + x^2 = \lambda^2$ **and** $-[g'(y)] = \lambda^2$ **where** λ^2 **is separation constant.**

$\Rightarrow \quad f'(x) = \sqrt{\lambda^2 - x^2}$ **and** $g'(y) = -\lambda^2$

Integrating above both we obtain

$\Rightarrow \quad f(x) = \frac{1}{2}\lambda^2\left[Sin^{-1}\left(\frac{x}{\lambda}\right) + \frac{x}{\lambda}\sqrt{1 - \frac{x^2}{\lambda^2}}\right] + A$ **and** $g(y) = -\lambda^2 y + B$

where A **and** B **are arbitrary constant. Thus, the general solution is**

$u \equiv u(x,y) = f(x) + g(y) \equiv \frac{1}{2}\lambda^2\left[Sin^{-1}\left(\frac{x}{\lambda}\right) + \frac{x}{\lambda}\sqrt{1 - \frac{x^2}{\lambda^2}}\right] + A - \lambda^2 y + B$

$u(x,y) = \frac{1}{2}\lambda^2\left[Sin^{-1}\left(\frac{x}{\lambda}\right) + \frac{x}{\lambda}\sqrt{1 - \frac{x^2}{\lambda^2}}\right] - \lambda^2 y + C$

where $C = A + B$ **is an arbitrary constant.**

Example: Use $v = \ln u$ **and** $v = f(x) + g(y)$ **to solve the equation**
$x^2 u_x^2 + y^2 u_y^2 = u^2$. ………. (i)

Solution:

In view of $v = \ln u$, $v_x = \frac{1}{u}u_x$ **and** $v_y = \frac{1}{u}u_y$**, and hence, equation (i) becomes** $x^2 v_x^2 + y^2 v_y^2 = 1$. ………. (ii)

(ii) $\Rightarrow x^2\left[\frac{\partial}{\partial x}v(x,y)\right]^2 + y^2\left[\frac{\partial}{\partial y}v(x,y)\right] = 1$ ……. (iii)

let $v \equiv v(x, y) = f(x) + g(y) \equiv f + g$

(iii) $\Rightarrow x^2\left[\frac{\partial}{\partial x}\big(f(x) + g(y)\big)\right]^2 + y^2\left[\frac{\partial}{\partial y}\big(f(x) + g(y)\big)\right] = 1$

$\Rightarrow x^2[f'(x)]^2 + y^2[g'(y)] = 1 \quad \Rightarrow \quad x^2[f'(x)]^2 = 1 - y^2[g'(y)]$

$\Rightarrow x^2[f'(x)]^2 = 1 - y^2[g'(y)] = \lambda^2$

Thus , we obtain

$x^2[f'(x)]^2 = \lambda^2$ **and** $1 - [g'(y)] = \lambda^2$ **where** λ^2 **is separation constant.**

$\Rightarrow \quad f'(x) = \frac{\lambda}{x}$ and $g'(y) = \frac{1}{y}\sqrt{1-\lambda^2}$

Integrating above both $\Rightarrow \quad f(x) = \lambda ln x + A$ **and** $g(y) = \sqrt{1-\lambda^2} ln y + B$

where A **and** B **are arbitrary constant. Thus, the general solution is**

$v \equiv v(x,y) = f(x) + g(y) \equiv \lambda ln x + A + \sqrt{1-\lambda^2} ln y + B$

$v(x,y) = \lambda ln x + \sqrt{1-\lambda^2} ln y + lnC = ln\left(x^{\lambda}.y^{\sqrt{1-\lambda^2}}.C\right)$

Where $lnC = A + B$ **is an arbitrary constant.**

Now $v(x,y) = ln u = ln\left(x^{\lambda}.y^{\sqrt{1-\lambda^2}}.C\right)$

Therefore, the final solution is

$u(x,y) = e^{v} = x^{\lambda}.y^{\sqrt{1-\lambda^2}}.C$ **where C is integrating constant.**

EXERCISES

1. Apply the method of separation of variables $u(x,y) = f(x)\,g(y)$ **to solve the following equations:**

(a) $u_x + u = u_y$, $u(x,0) = 4e^{-3x}$ **(b)** $u_x u_y = u^2$

(c) $u_x + 2u_y = 0$, $u(0,y) = 3e^{-2y}$

(d) $x^2 u_{xy} + 9y^2 u = 0$, $u(x,0) = \exp\left(\frac{1}{x}\right)$ **(e)** $yu_x - xu_y = 0$

(f) $u_t = c^2(u_{xx} + u_{yy})$ **(g)** $u_{xx} + u_{yy} = 0.$

2. Use a separable solution $u(x,y) = f(x) + g(y)$ **to solve the following**
equations:

(a) $u_x^2 + u_y^2 = 1$ **(b)** $u_x^2 + u_y^2 = u$ **(c)** $u_x^2 + u_y + x^2 = 0$

(d) $x^2 u_x^2 + y^2 u_y^2 = 1$ **(e)** $yu_x + xu_y = 0$, $u(0,y) = y^2$.

3. Apply $v = \ln u$ **and then** $v(x,y) = f(x) + g(y)$ **to solve the following equations:** $x^2 u_x^2 + y^2 u_y^2 = u^2$

4. **Apply** $\sqrt{u} = v$ **and** $v(x,y) = f(x) + g(y)$ **to solve the equation** $x^4 u_x^2 + y^2 u_y^2 = 4u$.

5. Using $v = \ln u$ **and** $v = f(x) + g(y)$**, show that the solution of the Cauchy problem**

$y^2 u_x^2 + x^2 u_y^2 = (xyu)^2$ with $u(x,0) = \exp(x^2)$ **is** $u(x,y) = \exp\left(x^2 + i\frac{\sqrt{3}}{2}y^2\right)$

6. Solve v $u_t = ku_{xx}$ **with** $u(x,0) = x$, $u(0,t) = 0 = u(L,t)$

7. Find De alembert's solution of $u_t = c^2 u_{xx}$ **with** $u(0,t) = 0 = u(a,t)$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

Find the solution of each of the following equations by the method of separation of variables:

(a) $u_x - u_y = 0$, $u(0, y) = 2e^{3y}$ **(b)** $u_x - u_y = u$, $u(x, 0) = 4e^{-3x}$

(c) $au_x + bu_y = 0$, $u(x, 0) = \alpha e^{\beta x}$ **where** a, b, α **and** β **are constants.**

CANONICAL/STANDARD/NORMAL FORMS OF FIRST-ORDER LINEAR EQUATIONS

For the general first-order linear partial differential equation

$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$

the canonical (or standard) form is given as follows

$u_\xi + \propto(\xi,\eta)u = \beta(\xi,\eta)$ **where** $\propto(\xi,\eta) = \frac{c}{A}$ **and** $\beta(\xi,\eta) = \frac{d}{A}$

also $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ **are continuously differentiable functions. Also A = a** ξ_x **+b** ξ_y

DERIVATION: consider $u = u(\xi,\eta)$ **where** $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$

Then $u_x = u_\xi \xi_x + u_\eta \eta_x$ and $u_y = u_\xi \xi_y + u_\eta \eta_y$ **put these in 1st order PDE**

We get $A u_\xi + B u_\eta + cu = d$ **..........** (i)

where A = a ξ_x **+b** ξ_y **and B = a** η_x **+b** η_y

now B = 0 if η **is solution of a** η_x **+b** η_y **= 0**

now dividing (i) with A we get

$u_\xi + \propto(\xi,\eta)u = \beta(\xi,\eta)$ **where** $\propto(\xi,\eta) = \frac{c}{A}$ **and** $\beta(\xi,\eta) = \frac{d}{A}$

Example: Reduce the equation $u_x - u_y = u$ **to canonical form, and obtain the general solution.**

Solution: here $a = 1$, $b = -1$, $c = -1$ **and** $d = 0$.

The characteristic equations are $\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{u}$

The characteristic curves are $\xi = x + y = c_1$ **and we choose** $\eta = y = c_2$ **where** c_1 **and** c_2 **are constants. Consequently,** $u_x = u_\xi$ **and** $u_y = u_\xi + u_\eta$ **and hence, given equation becomes** $-u_\eta = u$

Integrating this equation gives $\ln u(\xi, \eta) = -\eta + \ln f(\xi)$

where $f(\xi)$ **is an arbitrary function of** ξ **only.**

Equivalently, $u(\xi, \eta) = f(\xi)e^{-\eta}$

In terms of the original variables x **and** y**, the general solution of equation is** $u(x, y) = f(x + y)e^{-y}$ **where** f **is an arbitrary function.**

Example: Reduce the following equation $yu_x + u_y = x$ to canonical form, and obtain the general solution.

Solution: here $a = y$, $b = 1$, $c = 0$ and $d = x$.

The characteristic equations are $\frac{dx}{y} = \frac{dy}{1} = \frac{du}{x}$

It follows from the first two equations that $\xi(x, y) = x - \frac{y^2}{2} = c_1$

we choose $\eta(x, y) = y = c_2$.

Consequently, $u_x = u_\xi$ **and** $u_y = -yu_\xi + u_\eta$

And hence, given equation reduces to

$u_\eta = \xi + \frac{1}{2}\eta^2$

Integrating this equation gives the general solution

$u(\xi, \eta) = \xi\eta + \frac{1}{6}\eta^3 + f(\xi)$

where f is an arbitrary function.

Thus, the general solution of in terms of x and y is

$u(x, y) = xy - \frac{1}{3}y^3 + f\left(x - \frac{y^2}{2}\right)$

Reduce each of the following equations into canonical form and find the general solution:

(a) $u_x + u_y = u$

(b) $u_x + xu_y = y$

(c) $u_x + 2xyu_y = x$

(d) $u_x - yu_y - u = 1$

MATHEMATICAL MODELS

Usually, in almost all physical phenomena (or physical processes), the dependent variable $u = u(x, y, z, t)$ is a function of three space variables, x, y, z and time variable t.
The three basic types of second-order partial differential equations are:

(a) The wave equation $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0 \Rightarrow u_{tt} - c^2\nabla^2 u = 0$

(b) The heat equation $u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0 \Rightarrow u_t - k\nabla^2 u = 0$

(c) The Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0 \Rightarrow \nabla^2 u = 0$

In this section, we list a few more common linear partial differential equations of importance in applied mathematics, mathematical physics, and engineering science. Such a list naturally cannot ever be complete. Included are only equations of most common interest:

(d) The Poisson equation $\nabla^2 u = f(x, y, z)$

(e) The Helmholtz equation (reduced wave equation) $\nabla^2 u + \lambda u = 0$

(f) The biharmonic equation $\nabla^4 u = \nabla^2(\nabla^2 u) = 0$

(g) The biharmonic wave equation $u_{tt} + c^2\nabla^4 u = 0$

(h) The telegraph equation $u_{tt} + au_t + bu = c^2 u_{xx}$

(i) The Schr*ö*dinger equations in quantum physics

$$i\hbar\psi = \left[\left(-\frac{\hbar^2}{2m}\right)\nabla^2 + V(x,y,z)\right]\psi \quad \text{And} \quad \nabla^2\Psi + \frac{2m}{\hbar^2}[E - V(x,y,z)]\psi = 0$$

(j) The Klein–Gordon equation $\Box u + \lambda^2 u = 0$

Where $\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ **is the d'Alembertian, and in all equations** λ, a, b, c, m, E **are constants and** $h = 2\pi\hbar$ **is the Planck constant.**

(k) For a compressible fluid flow, Euler's equations
$u_t + (u.\nabla)u = -\frac{1}{\rho}\nabla p$ and $\rho_t + \text{div}(\rho u) = 0$ **where u =** (u, v, w) **is the fluid velocity vector,** ρ **is the fluid density, and** $p = p(\rho)$ **is the pressure that relates** p **and** ρ **(the constitutive equation or equation of state).**

ONE DIMENSIONAL WAVE EQUATION USING THE VIBRATING STRING

An equation of the form $u_{tt} = c^2 u_{xx}$ where $c^2 = \frac{T}{\rho}$ is called the one-dimensional wave equation. Where $u\ (x,\ t)$ is a function of displacement at position x in time 't' and 'c' denotes the velocity of wave equation.

PROOF

Let us consider a stretched string of length l fixed at the end points. The problem here is to determine the equation of motion which characterizes the position $u\ (x, t)$ of the string at time t after an initial disturbance is given.

In order to obtain a simple equation, we make the following assumptions:

1. The string is flexible and elastic, that is the string cannot resist bending moment and thus the tension in the string is always in the direction of the tangent to the existing profile of the string.

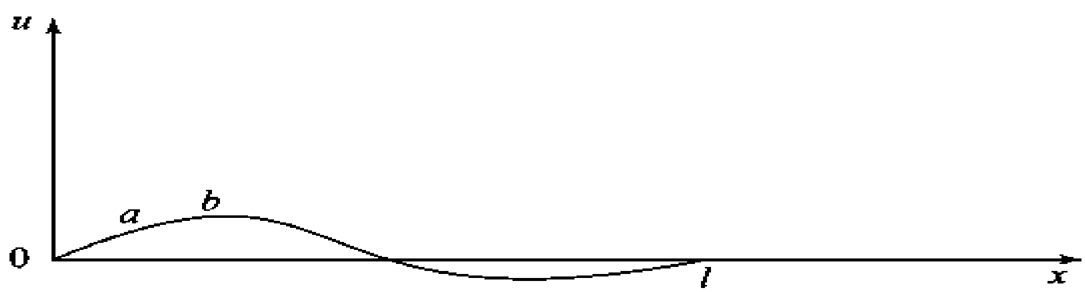
2. There is no elongation of a single segment of the string and hence, by Hooke's law, the tension is constant.

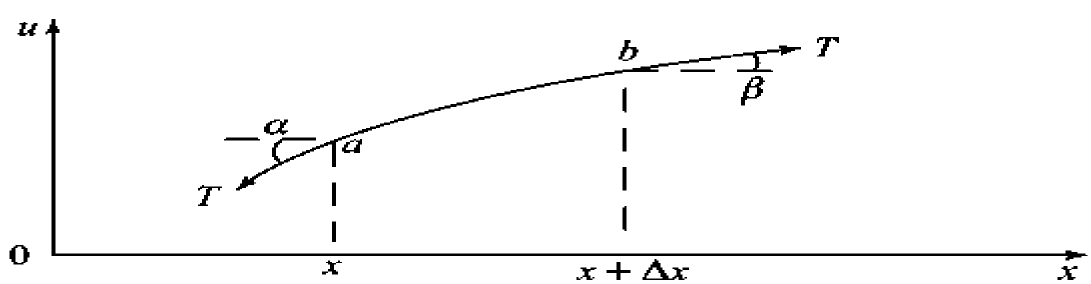
3. The weight of the string is small compared with the tension in the string.

4. The deflection is small compared with the length of the string.

5. The slope of the displaced string at any point is small compared with unity.

6. There is only pure transverse vibration.





We consider a differential element of the string. Let T be the tension at the end points as shown in Figure. The forces acting on the element of the string in the vertical direction are $T\ \sin\ \beta\ - T\ \sin\ \alpha$

By Newton's second law of motion, the resultant force is equal to the mass times the acceleration. Hence,

$T\ \sin\ \beta\ - T\ \sin\ \alpha\ = \rho \delta s u_{tt}$ ……………(i) $\qquad \therefore \rho = m/\delta s$

where ρ is the line density and δs is the smaller arc length of the string.

Since the slope of the displaced string is small, we have $\delta s \simeq \delta x$

Since the angles α and β are small sin $\alpha \simeq$ tan α, sin $\beta \simeq$ tan β

Thus, equation (i) becomes $\tan\beta - \tan\alpha = \frac{\rho}{T}\delta x\, u_{tt}$ ……………(ii)

But, from calculus we know that $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and $x + \delta x$:

$\tan\alpha = u_x(x, t)$ and $\tan\beta = u_x(x + \delta x, t)$ at time t.

Then Equation (ii) may thus be written as

$\frac{1}{\delta x}[(u_x)_{x+\delta x} - (u_x)_x] = \frac{\rho}{T}u_{tt}$

$\frac{1}{\delta x}[u_x(x + \delta x, t) - u_x(x, t)] = \frac{\rho}{T}u_{tt}$

$\lim_{\delta x \to 0}\frac{1}{\delta x}[u_x(x + \delta x, t) - u_x(x, t)] = \frac{\rho}{T}u_{tt}$ **∴limit have no effect on R.H.S**

$u_{tt} = c^2 u_{xx}$ ……………(iii)

where $c^2 = \frac{T}{\rho}$. This is called the one-dimensional wave equation.

If there is an external force f per unit length acting on the string.

Equation (iii) assumes the form

$u_{tt} = c^2 u_{xx} + F,\ F = \frac{f}{\rho}$ **where f may be pressure, gravitation, resistance, and so on.**

TWO DIMENSIONAL WAVE EQUATION USING THE VIBRATING MEMBRANE <u>(JUST STATEMENT)</u>

An equation of the form $u_{tt} = c^2(u_{xx} + u_{yy})$ where $c^2 = \frac{T}{\rho}$ is called the two-dimensional wave equation. Where $u(x, t)$ is a function of displacement at position x in time 't' and 'c' denotes the velocity of wave equation.

PROOF

The equation of the vibrating membrane occurs in a large number of problems in applied mathematics and mathematical physics. Before we derive the equation for the vibrating membrane we make certain simplifying assumptions as in the case of the vibrating string:

1. The membrane is flexible and elastic, that is, the membrane cannot resist bending moment and the tension in the membrane is always in the direction of the tangent to the existing profile of the membrane.

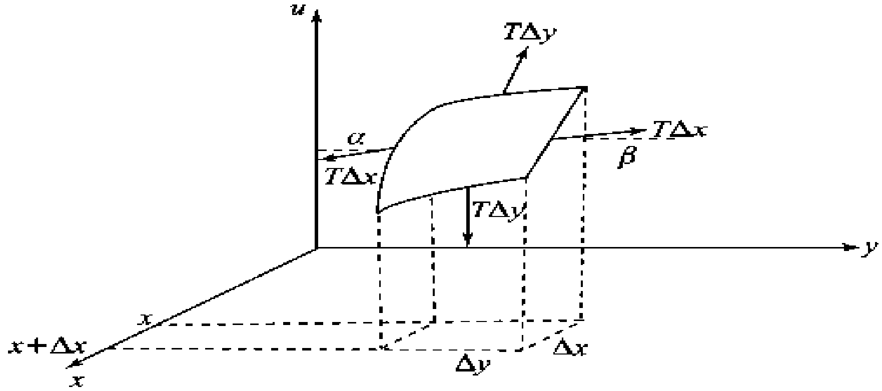
2. There is no elongation of a single segment of the membrane and hence, by Hooke's law, the tension is constant.

3. The weight of the membrane is small compared with the tension in the membrane.

4. The deflection is small compared with the minimal diameter of the membrane.

5. The slope of the displayed membrane at any point is small compared with unity.

6. There is only pure transverse vibration.



We consider a small element of the membrane. Since the deflection and slope are small, the area of the element is approximately equal to $\delta x \delta y$. If T is the tensile force per unit length, then the forces acting on the sides of the element are $T\delta x$ and $T\delta y$, as shown in Figure

The forces acting on the element of the membrane in the vertical direction are

$T\ \delta x \sin\beta - T\ \delta x \sin\alpha + T\ \delta y \sin\delta - T\ \delta y \sin\gamma$.

Since the slopes are small, sines of the angles are approximately equal to their tangents. Thus, the resultant force becomes

$T\ \delta x(\tan\beta - \tan\alpha) + T\ \delta y(\tan\delta - \tan\gamma)$.

By Newton's second law of motion, the resultant force is equal to the mass times the acceleration. Hence,

$T\delta x(\tan\beta - \tan\alpha) + T\delta y(\tan\delta - \tan\gamma) = \rho\ \delta A\ u_{tt}$ ……………(i)

where ρ is the mass per unit area, $\delta A \simeq \delta x \delta y$ is the area of this element, and u_{tt} is computed at some point in the region under consideration. But from calculus, we have

tan $\alpha = u_y(x_1, y)$ **tan** $\beta = u_y(x_2, y + \delta y)$

tan $\gamma = u_x(x, y_1)$ **tan** $\delta = u_x(x + \delta x, y_2)$

where x_1 and x_2 are the values of x between x and $x+\delta x$, and y_1 and y_2 are the values of y between y and $y + \delta y$.

Substituting these values in (i) we obtain

$T\delta x[u_y(x_2, y + \delta y) - u_y(x_1, y)] + T\delta y[u_x(x + \delta x, y_2) - u_x(x, y_1)] = \rho\ \delta x \delta y\ u_{tt}$

Division by $\rho\ \delta x \delta y$ yields

$\frac{T}{\rho}\left[\frac{u_y(x_2, y+\delta y) - u_y(x_1, y)}{\delta y} + \frac{u_x(x+\delta x, y_2) - u_x(x, y_1)}{\delta x}\right] = u_{tt}$ ……………(ii)

In the limit as δx approaches zero and δy approaches zero, we obtain

$u_{tt} = c^2(u_{xx} + u_{yy})$ ……………(iii)

where $c^2 = T/\rho$. This equation is called the two-dimensional wave equation.

If there is an external force f per unit area acting on the membrane. Equation (iii) takes the form

$u_{tt} = c^2(u_{xx} + u_{yy}) + F$ **where** $F = f/\rho$.

THREE DIMENSIONAL WAVE EQUATION USING THE VIBRATING MEMBRANE

Equation of the form $u_{tt} = c_T^2 \nabla^2 u$ where $c_T = \sqrt{\mu/\rho}$ is called transverse wave velocity.

And Equation of the form $u_{tt} = c_L^2 \nabla^2 u$ where $c_T = \sqrt{\lambda + 2\mu/\rho}$ is called longitudinal wave velocity.

Both type of above equations are called Wave equations in three dimension.

GENERAL FORM OF WAVE EQUATION : In general, the wave equation may be written as $u_{tt} = c^2\nabla^2 u$ where the Laplace operator may be one, two, or three dimensional.

The importance of the wave equation stems from the facts that this type of equation arises in many physical problems; for example, sound waves in space, electrical vibration in a conductor, torsional oscillation of a rod, shallow water waves, linearized supersonic flow in a gas, waves in an electric transmission line, waves in magnetohydrodynamics, and longitudinal vibrations of a bar.

**WAVE: A wave is a disturbance that carries energy from one place to another.
For example, wave produced on the string.**

There are two types of waves.

MECHANICAL WAVE: Waves which required any medium for their propogation.
e.g. (i) Sound waves (ii) water waves.
ELECTROMEGNATIC WAVE: Waves which do not required any medium for their propogation. e.g. (i) Radio waves (ii) X - Rays.

Mechanical waves have two types

TRANSVERSE WAVES: In the case of transverse waves, the motion of particles of the medium is perpendicular to the motion of waves.
e.g. Waves produced on water surface
LONGITUDINAL WAVES: In the case of longitudinal waves, the particles of the medium move back and forth along the direction of propogation of wave.
e.g. Waves produced in an elastic spring.

UNIQUENESS THEOREM FOR WAVE EQUATION (UoS Past Papers)
According to this theorem: the solution to the wave equation $u_{tt} = c^2 u_{xx}$ satisfying the IC's $u(x,0) = f(x)\ ; 0 \le x \le L$ and $u_t(x,0) = g(x)\ ; 0 \le x \le L$ and the BC's $u(0,t) = u(L,t) = 0$ where $u(x,t)$ is twise continuously differentiable with respect to 'x' and 't' is unique.

HEAT:
Heat is a form of energy that transferred from hot body to the cold body, by means of thermal contact. It is denoted by 'q'

CONDUCTION OF HEAT:
In this mode heat is transmitted through actual contact between particles (molecules) of the medium.

CONVECTION OF HEAT:
In this mode heat is transmitted through gases or liquids by actual motion of particles (molecules) of the medium.

RADIATION OF HEAT:
In this mode heat is transmitted through electromagnetic waves. Or by means of heat waves or thermal radiations. Medium is not essential for it. i.e. heat can take places in vaccume also.

SPECIFIC HEAT OF SUBSTANCE (MATERIAL) :
The quantity of heat required to raise the temperature of 1g of material by $1C^{\circ}$ and it is denoted by C and mathematically could be written as $\Delta q = Cm\Delta u$

HEAT FLUX (THERMAL FLUX) :
Is the rate of heat energy transfer through a given surface per unit surface area. Its unit is watt or Js^{-1}

THERMAL CONDUCTIVITY:
The quantity of heat flowing per second across a plate (of the material) of unit area and unit thickness, when the temperature difference between opposite sides is $1C^{\circ}$
It determines how good a conductor the material is . It is large for good conductors and small for bad conductors.

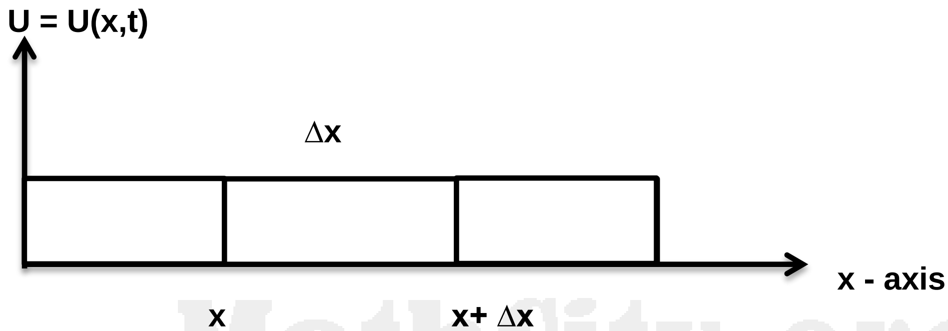
SOME FACTORS ON WHICH RATE OF FLOW OF HEAT DEPENDS

- **Area as** $q(x,t) \propto A$
- **Length as** $q(x,t) \propto \frac{1}{L}$
- **Change in temperature as** $q(x,t) \propto \Delta u$

ONE DIMENSIONAL HEAT EQUATION

An equation of the form $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$ is called heat equation. Where U= U(x,t) is a temperature of a body at 'x' position in time 't' and 'K' is called diffusivity or thermal conductivity of the material.

PROOF:



Let us consider the flow of heat through a uniform rod of length 'l' and cross sectional area 'A' then
Density of rod = $\rho = mass/volume = m/A\Delta x$ $i.e$ $m = \rho A\Delta x$
We choose the x – axis along the length of the rod with origin at one end of the rod. Then temperature at point 'x' from origin at time 't' will be U = U(x,t)
Let flow of heat = q (x,t)
(quantity of heat entering per second through unit area perpendicular to the direction of flow)
Also Heat generation = γ and heat stored per second = cm $\frac{\partial u}{\partial t}$ = c $\rho A\Delta x \frac{\partial u}{\partial t}$
Now using law of conservation of heat energy
(Quantity of heat which entered) + (heat generated inside the rod)
= (Quantity of heat which leave) + (quantity of heat stored)

$q(x,t)A + \gamma A\Delta x = q(x+\Delta x,t)A + c\rho A\Delta x\frac{\partial u}{\partial t}$

dividing both sides by 'A' we get $q(x,t) + \gamma\Delta x = q(x+\Delta x,t) + c\rho\Delta x\frac{\partial u}{\partial t}$

dividing both sides by Δx we get $\frac{1}{\Delta x}[q(x,t) - q(x+\Delta x,t)] + \gamma = c\rho\frac{\partial u}{\partial t}$

Applying $\Delta x \to 0$ $-\frac{\partial q}{\partial x} + \gamma = c\rho\frac{\partial u}{\partial t}$

Now by using Fourier law of heat conductivity which is $q = -K\Delta u$

Then (i) becomes $-\frac{\partial}{\partial x}(-K\Delta u) + \gamma = c\rho\frac{\partial u}{\partial t}$ **then we get** $K\frac{\partial^2 u}{\partial x^2} + \gamma = c\rho\frac{\partial u}{\partial t}$

For standard form we suppose $\gamma = 0$ and $c\rho = 1$ (i.e. no heat generation)

Then $K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

Or $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$ **which is required heat equation in one dimension**

In general $u_t = \kappa\nabla^2 u$ **or** $\nabla^2 u = \frac{1}{\kappa}u_t$

THE HEAT EQUATION (CONDUCTION OF HEAT IN SOLIDS)

We consider a domain D^* bounded by a closed surface B^*. Let $u\ (x,\ y,\ z,\ t)$ be the temperature at a point $(x,\ y,\ z)$ at time t. If the temperature is not constant, heat flows from places of higher temperature to places of lower temperature. Fourier's law states that "the rate of flow is proportional to the gradient of the temperature". Thus the velocity of the heat flow in an isotropic body is

$$\boldsymbol{v = -K grad u}$$

where K is a constant, called the thermal conductivity of the body.

Let D be an arbitrary domain bounded by a closed surface B in D^*. Then the amount of heat leaving D per unit time is $\iint_B v_n ds$ where v_n = v.n is the component of v in the direction of the outer unit normal n of B. Thus, by Gauss' theorem (Divergence theorem)

$$\iint_B v_n ds = \iiint_D div\,(-K grad u)\, dxdydz = -K\iiint_D \nabla^2 u\ dxdydz$$

But the amount of heat in D is given by $\iiint_D \sigma\ \rho\, u\ dxdydz$

where ρ is the density of the material of the body and σ is its specific heat. Assuming that integration and differentiation are interchangeable, the rate of decrease of heat in D is $\iiint_D \sigma\ \rho \frac{\partial u}{\partial t}\ dxdydz$

Since the rate of decrease of heat in D must be equal to the amount of heat leaving D per unit time, we have

$$\iiint_D \sigma\ \rho\, u_t\, dxdydz = -K \iiint_D \nabla^2 u\ dxdydz$$

or

$$\iiint_D [\sigma\ \rho\, u_t - K\nabla^2 u]\ dxdydz = 0 \quad \ldots\ldots\ldots\ldots\text{(i)}$$

for an arbitrary D in D^*. We assume that the integrand is continuous. If we suppose that the integrand is not zero at a point (x_0, y_0, z_0) in D, then, by continuity, the integrand is not zero in a small region surrounding the point $(x_0,\ y_0,\ z_0)$. Continuing in this fashion we extend the region encompassing D. Hence the integral must be nonzero. This contradicts (i). Thus, the integrand is zero everywhere, that is, $u_t = \kappa\, \nabla^2 u$ where $\kappa = K/\sigma\rho$. This is known as <u>the heat equation.</u>

This type of equation appears in a great variety of problems in mathematical physics, for example the concentration of diffusing material, the motion of a tidal wave in a long channel, transmission in electrical cables, and unsteady boundary layers in viscous fluid flows.

CLASSIFICATION OF SECOND-ORDER LINEAR EQUATIONS

SECOND-ORDER EQUATIONS IN ONE INDEPENDENT VARIABLE

The general linear second-order partial differential equation in one dependent variable u may be written as

$\sum_{i,j=1}^{n} A_{ij}u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G$ ………. (i)

in which we assume $A_{ij} = A_{ji}$ and A_{ij}, B_i, F, and G are real-valued functions defined in some region of the space ($x_1, x_2, \ . \ . \ . \ , x_n$)

SECOND-ORDER EQUATIONS IN TWO INDEPENDENT VARIABLES

Second-order equations in the dependent variable u and the independent variables x, y can be put in the form

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ ………. (ii)

where the coefficients are functions of 'x' and 'y' and do not vanish simultaneously. We shall assume that the function 'u' and the coefficients are twice continuously differentiable in some domain in R^2.

CLASSIFICATION OF SECOND-ORDER LINEAR EQUATIONS

The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry. The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, **represents**

hyperbola if $B^2 - 4AC$ is positive i.e. $B^2 - 4AC > 0$

parabola if $B^2 - 4AC$ is zero i.e. $B^2 - 4AC = 0$

or ellipse if $B^2 - 4AC$ is negative i.e. $B^2 - 4AC < 0$

for example:

(i) The heat equation $\frac{1}{K}u_t = u_{xx}$ is parabolic.

(ii) The wave equation $\frac{1}{c^2}u_{tt} = u_{xx}$ is hyperbolic.

(iii) The potential (Laplace) equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ is elliptic.

TRANSFORMATION OF SECOND-ORDER EQUATIONS TO A CANONICAL FORM

To transform equation

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ ………. (ii)

to a canonical form we make a change of independent variables. Let the new variables be $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ ………. (iii)

Assuming that ξ and η are twice continuously differentiable and that the Jacobian $J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$ ………. (iv)

is nonzero in the region under consideration, then x and y can be determined uniquely from the system (iii). Let x and y be twice continuously differentiable functions of ξ and η. Then we have

$u_x = u_\xi \xi_x + u_\eta \eta_x$, $u_y = u_\xi \xi_y + u_\eta \eta_y$

$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$

$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$

$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}$

Substituting these values in equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ **we obtain**

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^* \qquad \text{..........(v)}$$

where

$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$

$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$ **.......... (vi) for all**

$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$

$F^* = F, \quad G^* = G$

The resulting equation (v) is in the same form as the original equation (ii) under the general transformation (iii). The nature of the equation remains invariant under such a transformation if the Jacobian does not vanish. This can be seen from the fact that the sign of the discriminant does not alter under the transformation, that is, $B^{*2} - 4A^*C^* = J^2(B^2 - 4AC)$ **.......... (vii)**

Now The classification of equation (ii) depends on the coefficients $A(x, y)$**,** $B(x, y)$**, and** $C(x, y)$ **at a given point** (x, y)**. We shall, therefore, rewrite equation (ii) as**

$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y)$ **.......... (viii)**

and equation (v) as

$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta)$ **.......... (ix)**

Now suppose that none of A, B, C**, is zero. Let** ξ **and** η **be new variables such that the coefficients** A^* **and** C^* **in equation (ix) vanish. Thus, from (vi), we have**

$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$ **and** $C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$

These two equations are of the same type and hence we may write them in the form $A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0$ **.......... (x)**

in which ζ **stand for either of the functions** ξ **or** η**. Dividing through by** ζ_y^2 **equation (x) becomes**

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0 \qquad \text{..........(xi)}$$

Along the curve ζ **= constant, we have** $d\zeta = \zeta_x dx + \zeta_y dy = 0$.

Thus $\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}$ **.......... (xii) and therefore, equation (ii) may be written in the form** $A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$ **.......... (xiii)**

the roots of which are

$\frac{dy}{dx} = \frac{(B+\sqrt{B^2-4AC})}{2A}$ ………. (xiv) and $\frac{dy}{dx} = \frac{(B-\sqrt{B^2-4AC})}{2A}$ ………. (xv)

These equations, which are known as the characteristic equations, are ordinary differential equations for families of curves in the xy-plane along which ξ = constant and η = constant.

The integrals of equations (xiv) and (xv) are called the <u>characteristic curves</u>. Since the equations are first-order ordinary differential equations, the solutions may be written as

$\phi_1(x, y) = c_1$, c_1 = constant and $\phi_2(x, y) = c_2$, c_2 = constant

Hence the transformations

$\xi = \phi_1(x, y)$, $\eta = \phi_2(x, y)$ will transform equation (viii) to a canonical form.

(A) CANONICAL TRANSFORMATION OF HYPERBOLIC TYPE

If $B^2 - 4AC > 0$, then integration of equations

$\frac{dy}{dx} = \frac{(B+\sqrt{B^2-4AC})}{2A}$ and $\frac{dy}{dx} = \frac{(B-\sqrt{B^2-4AC})}{2A}$

yield two real and distinct families of characteristics.

Equation $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta)$ reduces to $u_{\xi\eta} = H_1$

where $H_1 = H^*/B^*$. It can be easily shown that $B^* \neq 0$. This form is called <u>the first canonical form</u> of the hyperbolic equation.

Now if new independent variables $\alpha = \xi + \eta$, $\beta = \xi - \eta$ are introduced, then equation $u_{\xi\eta} = H_1$ is transformed into $u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta)$

This form is called <u>the second canonical form</u> of the hyperbolic equation.

(B) CANONICAL TRANSFORMATION OF PARABOLIC TYPE

If $B^2 - 4AC = 0$, and equations $\frac{dy}{dx} = \frac{(B+\sqrt{B^2-4AC})}{2A}$ and $\frac{dy}{dx} = \frac{(B-\sqrt{B^2-4AC})}{2A}$ coincide. Thus, there exists one real family of characteristics, and we obtain only a single integral ξ = constant (or η = constant).

Since $B^2 = 4AC$ and $A^* = 0$, we find that $A^* = A\zeta_x^2 + B\xi_x\xi_y + C\zeta_y^2 = (\sqrt{A}\xi_x + \sqrt{C}\xi_y)^2 = 0$

From this it follows that

$A^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 2(\sqrt{A}\xi_x + \sqrt{C}\xi_y)(\sqrt{A}\eta_x + \sqrt{C}\eta_y) = 0$

for arbitrary values of $\eta(x, y)$ which is functionally independent of $\xi(x, y)$; for instance, if $\eta = y$, the Jacobian does not vanish in the domain of parabolicity

Division of equation $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta)$ by C^* yields

$u_{\eta\eta} = H_3(\xi, \eta, u, u_\xi, u_\eta)$, $C^* \neq 0$ This is called the <u>canonical form</u> of the parabolic equation. Equation $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta)$ may also assume the form $u_{\xi\xi} = H_3^*(\xi, \eta, u, u_\xi, u_\eta)$ if we choose η = constant as the integral of equation $\frac{dy}{dx} = \frac{(B+\sqrt{B^2-4AC})}{2A}$

(C) CANONICAL TRANSFORMATION OF ELLIPTIC TYPE

For an equation of elliptic type, we have $B^2 - 4AC < 0$. Consequently, the quadratic equation $A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$ has no real solutions, but it has two complex conjugate solutions which are continuous complex-valued functions of the real variables x and y. Thus, in this case, there are no real characteristic curves. However, if the coefficients A, B, and C are analytic functions of x and y, then one can consider equation $A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$ for complex x and y. A function of two real variables x and y is said to be analytic in a certain domain if in some neighborhood of every point (x_0, y_0) of this domain, the function can be represented as a Taylor series in the variables $(x-x_0)$ and $(y - y_0)$.

Since ξ and η are complex, we introduce new real variables

$\alpha = \frac{1}{2}(\xi + \eta)$, $\beta = \frac{1}{2i}(\xi - \eta)$ **……….(i)**

so that $\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$ **……….(ii)**

First, we transform equations $Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y)$ We then have

$A^{**}(\alpha, \beta)u + B^{**}(\alpha, \beta)u + C^{**}(\alpha, \beta)u = H_4(\alpha, \beta, u, u_\alpha, u_\beta)$ **……….(iii)**

in which the coefficients assume the same form as the coefficients in equation $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta)$ With the use of $\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$ the equations $A^* = C^* = 0$ become

$(A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2) - (A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2) + i[2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y] = 0$

$(A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2) - (A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2) - i[2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y] = 0$

or, $(A^{**} - C^{**}) + iB^{**} = 0$, $(A^{**} - C^{**}) - iB^{**} = 0$

These equations are satisfied if and only if $A^{**} = C^{**}$ and $B^{**} = 0$

Hence, equation (iii) transforms into the form $A^{**}u + A^{**}u = H_4(\alpha, \beta, u, u_\alpha, u_\beta)$

Dividing through by A^{**}, we obtain

$u_{\alpha\alpha} + u_{\beta\beta} = H_5(\alpha, \beta, u, u_\alpha, u_\beta)$ **where $H_5 = (H_4/A^{**})$.**

This is called the <u>canonical form</u> of the elliptic equation.

NOTE:

i. **When given equation is Hyperbolic then roots will be real and distinct and there wil be two characteristic curves i.e. $\xi(x,y) = C_1$, and $\eta(x,y) = C_2$**
ii. **When given equation is Parabolic then roots will be real and equal and there wil be only one characteristic curve i.e. $\xi(x,y) = C_1$**
iii. **When given equation is Elliptic then roots will be real and Complex and there wil be no characteristic curves in reals so will need to make following approximations i.e. $\alpha = \frac{\xi+\eta}{2}$ and $\beta = \frac{\xi-\eta}{2i}$**

REMEBER: for $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$

i. **Canonical form of Hyperbolic equation is** $u_{\xi\eta} = f(u, \xi, \eta, u_\xi, u_\eta)$
ii. **Canonical form of Parabolic equation is** $u_{\xi\xi} = f(u, \xi, \eta, u_\xi, u_\eta)$ **or** $u_{\eta\eta} = f(u, \xi, \eta, u_\xi, u_\eta)$ **but not both in same time.**
iii. **Canonical form of Elliptic equation is** $u_{\alpha\alpha} + u_{\beta\beta} = f(u, \alpha, \beta, u_\alpha, u_\beta)$ **where** $\alpha = \frac{\xi+\eta}{2}$ and $\beta = \frac{\xi-\eta}{2i}$

Example: Transform the equation $y^2u_{xx} - x^2u_{yy} = 0$ into canonical form.

Solution: Here $A = y^2$, $B = 0$, $C = -x^2$. Thus, $B^2 - 4AC = 4x^2y^2 > 0$.

The equation is hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$. From the characteristic equations $\frac{dy}{dx} = \frac{(B+\sqrt{B^2-4AC})}{2A} = \frac{(0+\sqrt{0^2+4y^2x^2})}{2(y^2)} = \frac{(2yx)}{2(y^2)}$

and $\frac{dy}{dx} = \frac{(B-\sqrt{B^2-4AC})}{2A} = \frac{(0-\sqrt{0^2+4y^2x^2})}{2(y^2)} = \frac{-(2yx)}{2(y^2)}$ we have $\frac{dy}{dx} = \frac{x}{y}$ and $\frac{dy}{dx} = -\frac{x}{y}$

After integration of these equations, we obtain

$y^2 - x^2 = C_1$ and $y^2 + x^2 = C_2$

To transform the given equation to canonical form, we consider

$\xi = y^2 - x^2$ and $\eta = y^2 + x^2$

$\xi_x = 2x$, $\eta_x = -2x$, $\xi_{xx} = 2$, $\eta_{xx} = -2$, $\xi_{xy} = 0$, $\eta_{xy} = 0$

$\xi_y = 2y$, $\eta_y = 2y$, $\xi_{yy} = 2$, $\eta_{yy} = 2$

$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$ (after putting values)

$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = -16x^2y^2$ (after putting values)

$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$ (after putting values)

$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = 2(y^2 - x^2)$ (after putting values)

$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = -2(y^2 + x^2)$ (after putting values)

$F^* = F = 0$, $G^* = G = 0$

Now $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*$

$-16x^2y^2u_{\xi\eta} + 2(y^2 - x^2)u_\xi - 2(y^2 + x^2)u_\eta = 0$ ....(i)

Now as $\xi = y^2 - x^2$ and $\eta = y^2 + x^2$

Adding above $y^2 = \frac{\xi+\eta}{2}$ and subtracting above $x^2 = \frac{\xi-\eta}{2}$ then $x^2y^2 = \frac{\xi^2-\eta^2}{4}$

$(i) \Rightarrow -16\left(\frac{\xi^2-\eta^2}{4}\right)u_{\xi\eta} + 2\xi u_\xi - 2\eta u_\eta = 0$

$-4(\xi^2 - \eta^2)u_{\xi\eta} + 2\xi u_\xi - 2\eta u_\eta = 0$

Thus, the given equation assumes the canonical form

$u_{\xi\eta} = \frac{\eta}{2(\xi^2-\eta^2)}u_\xi - \frac{\xi}{2(\xi^2-\eta^2)}u_\eta$

Example: Transform the equation $x^2u_{xx} + 2xy\, u_{xy} + y^2u_{yy} = 0$ into canonical form.

Solution: In this case, the discriminant is $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$.

The equation is therefore parabolic everywhere.

And the characteristic equations is $\frac{dy}{dx} = \frac{y}{x}$

hence, the characteristics are $\frac{y}{x} = c$ which is the equation of a family of straight lines.

Consider the transformation $\xi = \frac{y}{x}, \eta = y$, where η is chosen arbitrarily.

The given equation is then reduced to the canonical form $y^2\, u_{\eta\eta} = 0$.

Thus $u_{\eta\eta} = 0$ for $y \neq 0$

Example: Transform the equation $u_{xx} + x^2u_{yy} = 0$ into canonical form.

Solution: The equation is elliptic everywhere except on the coordinate axis $x = 0$ because $B^2 - 4AC = -4x^2 < 0,\ x \neq 0$

The characteristic equations are

$\frac{dy}{dx} = ix$ **and** $\frac{dy}{dx} = -ix$

Integration yields $2y - ix^2 = c_1$, $2y + ix^2 = c_2$

Thus, if we write $\xi = 2y - ix^2$ $\quad \eta = 2y + ix^2$

hence, $\alpha = \frac{1}{2}(\xi + \eta) = 2y, \beta = \frac{1}{2i}(\xi - \eta) = -x^2$

we obtain the canonical form $u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta}u_\beta$

NOTE: It should be remarked here that a given partial differential equation may be of a different type in a different domain. Thus, for example, <u>Tricomi's equation</u> $u_{xx} + xu_{yy} = 0$ is elliptic for $x > 0$ and hyperbolic for $x < 0$, since $B^2 - 4AC = -4x$

EQUATIONS WITH CONSTANT COEFFICIENTS

In this case of an equation with real constant coefficients, the equation is of a single type at all points in the domain. This is because the discriminant $B^2 - 4AC$ is a constant.

From the characteristic equations $\frac{dy}{dx} = \frac{(B+\sqrt{B^2-4AC})}{2A}$ and $\frac{dy}{dx} = \frac{(B-\sqrt{B^2-4AC})}{2A}$ we can see that the characteristics

$y = \left(\frac{(B+\sqrt{B^2-4AC})}{2A}\right)x + C_1$ **and** $y = \left(\frac{(B-\sqrt{B^2-4AC})}{2A}\right)x + C_2$ **are two families of straight lines. Consequently, the characteristic coordinates take the form**

$\xi = y - \lambda_1 x$ and $\eta = y - \lambda_2 x$ **Where** $\lambda_{1,2} = \frac{(B\pm\sqrt{B^2-4AC})}{2A}$

The linear second-order partial differential equation with constant coefficients may be written in the general form as $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$

In particular, the equation $Au_{xx} + Bu_{yy} + Cu_{yy} = 0$ is called the <u>Euler equation</u>.

(A) HYPERBOLIC TYPE

If $B^2 - 4AC > 0$, the equation is of hyperbolic type, in which case the characteristics form two distinct families.

Using $\xi = y - \frac{\left(B+\sqrt{B^2-4AC}\right)}{2A}x$ and $\eta = y - \frac{\left(B-\sqrt{B^2-4AC}\right)}{2A}x$

equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$ **becomes**

$u_{\xi\eta} = D_1 u_\xi + E_1 u_\eta + F_1 u + G_1(\xi, \eta)$ **where** D_1, E_1, **and** F_1 **are constants.**

Here, since the coefficients are constants, the lower order terms are expressed explicitly.

When $A = 0$, **equation** $\frac{dy}{dx} = \frac{\left(B\pm\sqrt{B^2-4AC}\right)}{2A}$ **does not hold.**

To remove this difficulty consider $-B\left(\frac{dx}{dy}\right) + C\left(\frac{dx}{dy}\right)^2 = 0$

which may again be rewritten as $\frac{dx}{dy} = 0$ **and** $-B + C\left(\frac{dx}{dy}\right) = 0$

Integration gives $x = c_1$, $x = \left(\frac{B}{C}\right)y + c_2$

where c_1 **and** c_2 **are integration constants. Thus, the characteristic coordinates are**

$\xi = x$, $\eta = x - \left(\frac{B}{C}\right)y$ we may use $\eta = By - Cx$

Under this transformation, equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$ **reduces to the canonical form**

$u_{\xi\eta} = D_1^* u_\xi + E_1^* u_\eta + F_1^* u + G_1^*(\xi, \eta)$ **where** D_1^*, E_1^*, **and** F_1^* **are constants.**

The canonical form of the <u>Euler equation</u> is $u_{\xi\eta} = 0$

Integrating this equation gives the general solution

$u = \varphi(\xi) + \psi(\eta) = \phi(y - \lambda_1 x) + \psi(y - \lambda_2 x)$

where ϕ **and** ψ **are arbitrary functions, and** $\lambda_{1,2} = \frac{\left(B\pm\sqrt{B^2-4AC}\right)}{2A}$

(B) PARABOLIC TYPE

When $B^2 - 4AC = 0$, the equation is of parabolic type, in which case only one real family of characteristics exists. From equation $\lambda_{1,2} = \frac{\left(B\pm\sqrt{B^2-4AC}\right)}{2A}$, **we find**

That $\lambda_1 = \lambda_2 = \left(\frac{B}{2A}\right)$, **so that the single family of characteristics is given by**

$y = \frac{B}{2A}x + c_1$ **where** c_1 **is an integration constant. Thus, we have**

$\xi = y - \left(\frac{B}{2A}\right)x, \eta = hy + kx$ (arbitrary)

where η **is chosen arbitrarily such that the Jacobian of the transformation is not zero, and** h **and** k **are constants.**

With the proper choice of the constants h **and** k **in the transformation equation** $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$ **reduces to**

$u_\eta = D_2 u_\xi + E_2 u_\eta + F_2 u + G_2(\xi, \eta)$ **where** D_2, E_2, **and** F_2 **are constants.**

If $B = 0$, we can see at once from the relation $B^2 - 4AC = 0$ that C or A vanishes. The given equation is then already in the canonical form. Similarly, in the other cases when A or C vanishes, B vanishes. The given equation is then also in canonical form.
The canonical form of the <u>Euler equation</u> is $u_{\xi\eta} = 0$
Integrating twice gives the general solution $u = \varphi(\xi) + \eta\psi(\xi)$ where ξ and η are given by $\xi = y - (\frac{B}{2A})x, \eta = hy + kx$. Choosing $h = 1$, $k = 0$ and $\lambda = \frac{B}{2A}$ for simplicity, the general solution of the Euler equation in the parabolic case is $u = \varphi(y - \lambda x) + y\psi(y - \lambda x)$

(C) ELLIPTIC TYPE

When $B^2 - 4AC < 0$, the equation is of elliptic type. In this case, the characteristics are complex conjugates i.e. $\lambda_{1,2} = \frac{(B \pm \sqrt{B^2 - 4AC})}{2A}$. The characteristic equations yield
$y = \lambda_1 x + c_1$, $y = \lambda_2 x + c_2$ **(after integration)**
where λ_1 and λ_2 are complex numbers. Accordingly, c_1 and c_2 are allowed to take on complex values. Thus,
$\xi = y - (a + ib)x, \qquad \eta = y - (a - ib)x$
where $\lambda_{1,2} = a \pm ib$ in which a and b are real constants, and
$a = \frac{B}{2A}$ **and** $b = \frac{1}{2A}\sqrt{4AC - B^2}$
Introduce the new variables
$\alpha = \frac{1}{2}(\xi + \eta) = y - ax, \qquad \beta = \frac{1}{2i}(\xi - \eta) = -bx$
Application of this transformation readily reduces equation
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$ **to the canonical form**
$u_{\alpha\alpha} + u_{\beta\beta} = D_3 u_\alpha + E_3 u_\beta + F_3 u + G_3(\alpha, \beta)$ **where D_3, E_3, F_3 are constants.**
We note that $B^2 - AC < 0$, so neither A nor C is zero. In this elliptic case, the Euler equation gives the complex characteristics
$\xi = y - (a + ib)x, \qquad \eta = y - (a - ib)x$ **which are**
$\xi = (y - ax) - ibx, \eta = (y - ax) + ibx = \bar{\xi}$
Consequently, the Euler equation becomes $u_{\xi\bar{\xi}} = 0$
with the general solution $u = \varphi(\xi) + \psi(\bar{\xi})$
The appearance of complex arguments in the general solution (above) is a general feature of elliptic equations.

Example: Solve the equation $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$. Also find its canonical form.
Solution: Since $A = 4$, $B = 5$, $C = 1$, and $B^2 - 4AC = 9 > 0$, the equation is hyperbolic. Thus, the characteristic equations take the form
$\frac{dy}{dx} = 1, \quad \frac{dy}{dx} = \frac{1}{4} \qquad \therefore \frac{dy}{dx} = \frac{(B \pm \sqrt{B^2 - 4AC})}{2A}$
and hence, the characteristics are $y = x + c_1$, $y = (\frac{x}{4}) + c_2$ **(on integrating)**

The linear transformation $\xi = y - x, \eta = y - (\frac{x}{4})$

Then $\xi_x = -1, \xi_{xx} = 0, \xi_{xy} = 0, \xi_y = 1, \xi_{yy} = 0$

and $\eta_x = -\frac{1}{4}, \eta_{xx} = 0, \eta_{xy} = 0, \eta_y = 1, \eta_{yy} = 0$

$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$ **(after putting values)**

$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = -\frac{9}{4}$ **(after putting values)**

$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$ **(after putting values)**

$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = 0$ **(after putting values)**

$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = \frac{3}{4}$ **(after putting values) and** $F^* = F = 0$, $G^* = G = 2$

Now $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_{\xi} + E^*u_{\eta} + F^*u = G^*$

$u_{\xi\eta} = \frac{1}{3}u_{\eta} - \frac{8}{9}$ **(after putting the values and solving)**

This is the first canonical form.

The second canonical form may be obtained by the transformation

$\alpha = \xi + \eta, \beta = \xi - \eta$

in the form $u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3}u_{\alpha} - \frac{1}{3}u_{\beta} - -\frac{8}{9}$

Example: Solve the equation $u_{xx} - 4u_{xy} + 4u_{yy} = e^y$. **Also find its canonical form.**

Solution: Since $A = 1$, $B = -4$, $C = 4$, **and** $B^2 - 4AC = 0$, **the equation is parabolic.**

Thus, we have from equation $\xi = y - (\frac{B}{2A})x, \eta = hy + kx$ **we have**

$\xi = y + 2x, \eta = y$ **in which** η **is chosen arbitrarily. By means of this mapping the equation transforms into** $u_{\eta\eta} = \frac{1}{4}e^{\eta}$

Example: Solve the equation $u_{xx} + u_{xy} + u_{yy} + u_x = 0$. **Also find its canonical form.**

Solution: Since $A = 1$, $B = 1$, $C = 1$, **and** $B^2 - 4AC = -3 < 0$, **the equation is elliptic.**

We have $\lambda_{1,2} = \frac{\left(B \pm \sqrt{B^2 - 4AC}\right)}{2A} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

and hence, $\xi = y - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x$, $\eta = y - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x$

Introducing the new variables $\alpha = \frac{1}{2}(\xi + \eta) = y - \frac{1}{2}x$, $\beta = \frac{1}{2i}(\xi - \eta) = -\frac{\sqrt{3}}{2}x$

the given equation is then transformed into canonical form $u_{\alpha\alpha} + u_{\beta\beta} = \frac{2}{3}u_{\alpha} + \frac{2}{\sqrt{3}}u_{\beta}$

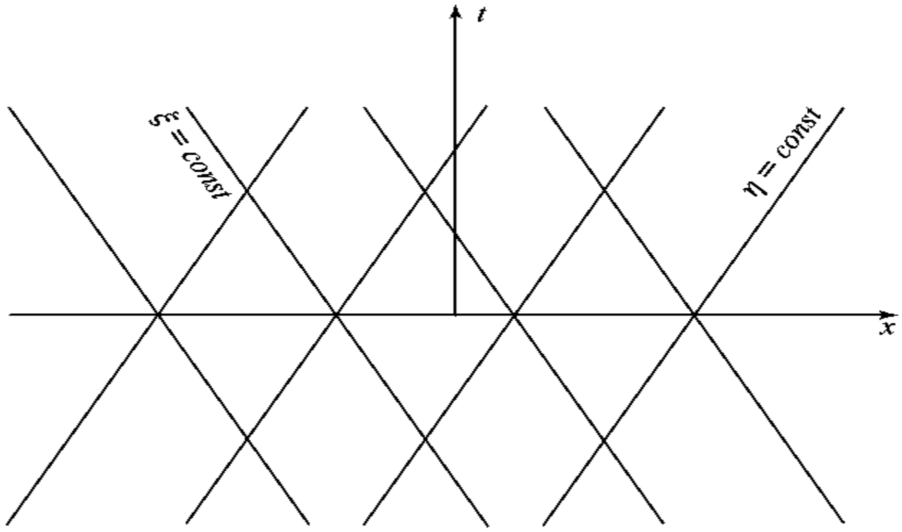
Example: Consider the wave equation $u_{tt} - c^2u_{xx} = 0$, c **is constant. Find its canonical form.**

Solution: Since $A = -c^2$, $B = 0$, $C = 1$, **and** $B^2 - 4AC = 4c^2 > 0$, **the wave equation is hyperbolic everywhere.**

According to $A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$ **the equation of characteristics is**

$-c^2\left(\frac{dt}{dx}\right)^2 + 1 = 0$ **or** $dx^2 - c^2dt^2 = 0$

Therefore, $x + ct = \xi$ **= constant,** $x - ct = \eta$ **= constant.**
Thus, the characteristics are straight lines, which are shown in Figure.



The characteristics form a natural set of coordinates for the hyperbolic equation.
In terms of new coordinates ξ **and** η **defined above, we obtain**
$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ and $u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$
so that the wave equation becomes $-4c^2 u_{\xi\eta} = 0$.
Since $c \neq 0$**, we have** $u_{\xi\eta} = 0$
Integrating with respect to ξ**, we obtain** $u_\eta = \psi_1(\eta)$
where ψ_1 **is the arbitrary function of** η. **Integrating with respect to** η**, we obtain**
$u(\xi,\eta) = \int \psi_1(\eta)\, d\eta + \varphi(\xi)$
If we set $\psi(\eta) = \int \psi_1(\eta)\, d\eta$) **the general solution becomes**
$u(\xi,\eta) = \varphi(\xi) + \psi(\eta)$
which is, in terms of the original variables x **and** t
$u(x,t) = \varphi(x + ct) + \psi(x - ct)$
provided ϕ **and** ψ **are arbitrary but twice differentiable functions.**

Example: Find the characteristic equations and characteristics, and then reduce the equations $u_{xx} - (\text{sech}^4 x)\, u_{yy} = 0$ **to the canonical forms.**

Solution: In equation $A = 1$, $B = 0$ **and** $C = -\text{sech}^4 x$. **Hence,** $B^2 - 4AC = 4\,\text{sech}^4 x > 0$.
Hence, the equation is hyperbolic.

The characteristic equations are $\frac{dy}{dx} = \frac{(B \pm \sqrt{B^2 - 4AC})}{2A} = \pm\, \text{sech}^2 x$
Integration gives $y \mp \tanh x = constant$
Hence, $\xi = y + \tanh x, \eta = y - \tanh x$
Using these characteristic coordinates, the given equation can be transformed into the canonical form $u_{\xi\eta} = \frac{\eta - \xi}{[4 - (\xi - \eta)^2]}(u_\xi - u_\eta)$

Example: Find the characteristic equations and characteristics, and then reduce the equations $u_{xx} + (\text{sech}^4 x) u_{yy} = 0$ to the canonical forms.

Solution: In equation $A = 1$, $B = 0$ and $C = \text{sech}^4 x$. Hence, $B^2 - 4AC = +i\ \text{sech}^2 x$

Integrating gives $y + i \tanh x = constant$

Thus, $\xi = y + i \tanh x \qquad \eta = y - i \tanh x$

The new real variables α and β are $\alpha = \frac{1}{2}(\xi + \eta) = y, \beta = \frac{1}{2i}(\xi - \eta) = \tanh x$

In terms of these new variables, equation can be transformed into the canonical form $u_{\alpha\alpha} + u_{\beta\beta} = \frac{2\beta}{1-\beta^2} u_\beta, \ |\beta| < 1$

Example: Consider the wave equation $u_{xx} + (2 \text{ cosec} y) u_{xy} + (\text{cosec}^2 y) u_{yy} = 0$. Find its canonical form.

Solution: In this case, $A = 1$, $B = 2 \text{ cosec} y$ and $C = \text{cosec}^2 y$. Hence, $B^2 - 4AC = 0$,

And $\frac{dy}{dx} = \frac{B}{2A} = cosec\, y$

The characteristic curves are therefore given by $\xi = x + \cos y$ and $\eta = y$

Using these variables, the canonical form of equation is $u_{\eta\eta} = (\sin^2 \eta \ \cos \eta)\, u_\eta$

GENERAL SOLUTIONS

In general, it is not so simple to determine the general solution of a given equation. Sometimes further simplification of the canonical form of an equation may yield the general solution. If the canonical form of the equation is simple, then the general solution can be immediately ascertained.

Example: Find the general solution of $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0$

Solution: using the transformation $\xi = \frac{y}{x}$, $\eta = y$ this equation reduces to the canonical form $u_{\eta\eta} = 0$, **for** $y \neq 0$

Integrating twice with respect to η, we obtain

$u(\xi,\eta) = \eta f(\xi) + g(\xi)$ **where $f(\xi)$ and $g(\xi)$ are arbitrary functions.**

In terms of the independent variables x and y, we have

$u(x,y) = y f(\frac{y}{x}) + g(\frac{y}{x})$

Example: Determine the general solution of $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

Solution: Since $A = 4$, $B = 5$, $C = 1$, and $B^2 - 4AC = 9 > 0$, the equation is hyperbolic. Thus, the characteristic equations take the form

$\frac{dy}{dx} = 1, \ \frac{dy}{dx} = \frac{1}{4} \qquad \therefore \frac{dy}{dx} = \frac{(B \pm \sqrt{B^2-4AC})}{2A}$

and hence, the characteristics are $y = x + c_1$, $y = (\frac{x}{4}) + c_2$ **(on integrating)**

The linear transformation $\xi = y - x, \eta = y - (\frac{x}{4})$

Then $\xi_x = -1, \xi_{xx} = 0, \xi_{xy} = 0, \xi_y = 1, \xi_{yy} = 0$

and $\eta_x = -\frac{1}{4}, \eta_{xx} = 0, \eta_{xy} = 0, \eta_y = 1, \eta_{yy} = 0$

$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$ **(after putting values)**

$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = -\frac{9}{4}$ **(after putting values)**

$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$ **(after putting values)**

$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = 0$ **(after putting values)**

$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = \frac{3}{4}$ **(after putting values) and** $F^* = F = 0$, $G^* = G = 2$

Now $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_{\xi} + E^*u_{\eta} + F^*u = G^*$

$u_{\xi\eta} = \frac{1}{3}u_{\eta} - \frac{8}{9}$ **(after putting the values and solving)**

By means of the substitution $v = u_{\eta}$ **the preceding equation reduces to**

$v_{\xi} = \frac{1}{3}v - \frac{8}{9}$

Integrating with respect to ξ, we have $v = \frac{8}{3} + \frac{1}{3}e^{\left(\frac{\xi}{3}\right)}F(\eta)$

Integrating with respect to η, we obtain $u(\xi,\eta) = \frac{8}{3}\eta + \frac{1}{3}g(\eta)\,e^{\left(\frac{\xi}{3}\right)} + f(\xi)$

where $f(\xi)$ and $g(\eta)$ are arbitrary functions.

The general solution of the given equation becomes

$u(x,y) = \frac{8}{3}(y - \frac{1}{4}) + \frac{1}{3}g(y - \frac{x}{4})\,e^{\frac{1}{3}(y-x)} + f(y - x)$

Example: Obtain the general solution of $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$.

Solution: Since $B^2 - 4AC = 64 > 0$, the equation is hyperbolic. Thus, from equation

$\frac{dy}{dx} = \frac{(B \pm \sqrt{B^2-4AC})}{2A}$ **the characteristics are** $y = 3x + c_1$, $y = \frac{1}{3}x + c_2$

Using the transformations $\xi = y - 3x, \eta = y - \frac{1}{3}x$

the given equation can be reduced to the form $(\frac{64}{3})u_{\xi\eta} = 0$

Hence, we obtain $u_{\xi\eta} = 0$

Integration yields $u(\xi,\eta) = f(\xi) + g(\eta)$

In terms of the original variables, the general solution is

$u(x,y) = f(y - 3x) + g(y - \frac{x}{3})$

Example: Find the general solution of the following equations

$yu_{xx} + 3yu_{xy} + 3u_x = 0$, $y \neq 0$

Solution: In equation $A = y$, $B = 3y$, $C = 0$, $D = 3$, $E = F = G = 0$.

Hence $B^2 - 4AC = 9y^2 > 0$ and the equation is hyperbolic for all points (x, y) with

$y \neq 0$**. Consequently, the characteristic equations using** $\frac{dy}{dx} = \frac{(B \pm \sqrt{B^2-4AC})}{2A}$ **are**

$\frac{dy}{dx} = \frac{3y \pm 3y}{2y} = 3, 0$

Integrating gives $y = c_1$ **and** $y = 3x + c_2$

The characteristic curves are $\xi = y \text{ and } \eta = y - 3x$

In terms of these variables, the canonical form of equation is $\xi u_{\xi\eta} + u_{\eta} = 0$**.**

Writing $v = u_\eta$ and using the integrating factor gives
$v = u_\eta = \frac{1}{\xi} C(\eta)$ where $C(\eta)$ is an arbitrary function.
Integrating again with respect to η gives
$\boldsymbol{u(\xi,\eta) = \frac{1}{\xi}\int C(\eta)d\eta + g(\xi) = \frac{1}{\xi} f(\eta) + g(\xi)}$
where f and g are arbitrary functions. Finally, in terms of the original variables, the general solution is $\boldsymbol{u(x,y) = \frac{1}{y} f(y - 3x) + g(y)}$

Example: Find the general solution of the following equations
$u_{xx} + 2u_{xy} + u_{yy} = 0$
Solution: Equation has coefficients $A = 1$, $B = 2$, $C = 1$, $D = E = F = G = 0$.
Hence, $B^2 - 4AC = 0$, the equation is parabolic. The characteristic equation is $\frac{dy}{dx} = 1$ and the characteristics are $\xi = y - x = c_1$ and $\eta = y$
Using these variables, equation takes the canonical form $u_{\eta\eta} = 0$
Integrating twice gives the general solution $\boldsymbol{u(\xi,\eta) = \eta f(\xi) + g(\xi)}$
where f and g are arbitrary functions.
In terms of x and y, this solution becomes $\boldsymbol{u(x,y) = y f(y - x) + g(y - x)}$

Example: Find the general solution of the following equations
$u_{xx} + 2u_{xy} + 5u_{yy} + u_x = 0$
Solution: The coefficients of equation are $A = 1$, $B = 2$, $C = 5$, $E = 1$, $F = G = 0$ and hence $B^2 - 4AC = -16 < 0$, equation is elliptic.
The characteristic equations are $\frac{dy}{dx} = (1 + 2i)$
The characteristics are $\boldsymbol{y = (1 - 2i)x} + c_1$, $\boldsymbol{y = (1 + 2i)x} + c_2$
and hence, $\boldsymbol{\xi = y - (1 - 2i)x, \eta = y - (1 + 2i)x}$
and new real variables α and β are
$\boldsymbol{\alpha = \frac{1}{2}(\xi + \eta) = y - x,} \qquad \boldsymbol{\beta = \frac{1}{2i}(\xi - \eta) = 2x}$
The canonical form is given by $u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{4}(u_\alpha - 2u_\beta)$
It is not easy to find a general solution of given equation.

THINGS TO REMEMBER

LAPLACE EQUATION IN CYLINDRICAL COORDINATES

The laplace equation in cylindrical coordinates (r,θ,z) is wrriten as follows

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$$

With most general solution $u(r,\theta,z) = J_n(\lambda r)\left(c_1e^{\lambda z} + c_2e^{-\lambda z}\right)(c_3Cosn\theta + c_4Sinn\theta)$

LAPLACE EQUATION IN SPHERICAL COORDINATES

The laplace equation in spherical coordinates (r,θ,φ) is wrriten as follows

$$\nabla^2 u = \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{Sin\theta}\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{Sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0$$

With most general solution $u(r,\theta,\varphi) = \left(c_1r^{\propto} + c_2r^{-(\propto+1)}\right)c_3\left(AP_n(Cos\theta)\right)$

DIFFUSION (HEAT) EQUATION IN CYLINDRICAL COORDINATES

The Heat equation in cylindrical coordinates (r,θ,z) is wrriten as follows

$$\frac{1}{K}u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

With most general solution

$$u(r,\theta,z,t) = c_1J_v\left(\sqrt{\lambda^2+\mu^2 r}\right)(cCosv\theta + Dsinv\theta)(Ae^{\mu z} + Be^{-\mu z})\left(e^{-\lambda^2Kt}\right)$$

DIFFUSION (HEAT) EQUATION IN SPHERICAL COORDINATES

The Heat equation in spherical coordinates (r,θ,φ) is wrriten as follows

$$\frac{1}{K}u_t = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2Sin\theta}\frac{\partial}{\partial\theta}(Sin\theta . u_\theta) + \frac{1}{r^2Sin^2\theta}u_{\varphi\varphi}$$

With most general solution

$$u(r,\theta,z,t) = \sum_{\lambda,m,n} A_{\lambda mn}(\lambda r)^{-1/2}J_{n+\frac{1}{2}}(\lambda r)\,P_n^m(Cos\theta)e^{\pm im\varphi-\lambda^2Kt}$$

WAVE EQUATION IN CYLINDRICAL COORDINATES

The Wave equation in cylindrical coordinates (r,θ,z) is wrriten as follows

$$\frac{1}{c^2}u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

And for one dimension when $u = u(r)$ only. i.e. u depends only on 'r' then

$$\frac{1}{c^2}u_{tt} = u_{rr} + \frac{1}{r}u_r \quad \text{or} \quad \frac{1}{c^2}u_{tt} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \quad ; r > 0$$

With most general solution $u = \sqrt{\frac{2c}{\pi\omega}}\left[c_1e^{-i\frac{\pi}{4}}\frac{exp[i(\omega/c)(r+ct)]}{\sqrt{r}} + c_2e^{i\frac{\pi}{4}}\frac{exp[i(\omega/c)(r-ct)]}{\sqrt{r}}\right]$

WAVE EQUATION IN SPHERICAL COORDINATES

The Wave equation in spherical coordinates (r,θ,φ) is wrriten as follows

$$\frac{1}{c^2}u_{tt} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2Sin\theta}\frac{\partial}{\partial\theta}(Sin\theta . u_\theta) + \frac{1}{r^2Sin^2\theta}u_{\varphi\varphi}$$

And one dimension when $u = u(r)$ only. i.e. u depends only on 'r' then

$$\frac{1}{c^2}u_{tt} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) \quad ; r > 0$$

With most general solution $u = \left[c_1\frac{exp[i(\omega/c)(r+ct)]}{r} + c_2\frac{exp[-i(\omega/c)(r-ct)]}{r}\right]$

LAPLACE EQUATION IN CYLINDRICAL COORDINATES

The laplace equation in cylindrical coordinates (r,θ,z) is wrriten as follows

$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$

For solution consider $u(r,\theta,z) = F(r,\theta)Z(z)$

$\Rightarrow \nabla^2 u = \frac{\partial^2 F}{\partial r^2}Z + \frac{1}{r}\frac{\partial F}{\partial r}Z + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}Z + F\frac{d^2 Z}{dz^2} = 0$

$\Rightarrow \left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\frac{1}{F} = -\frac{d^2 Z}{dz^2}\frac{1}{Z} = k(say)$ **where k is separation constant.**

$\Rightarrow \left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\frac{1}{F} = k$

$\Rightarrow \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - kF = 0$ **……………(i)**

Or $\Rightarrow -\frac{d^2 Z}{dz^2}\frac{1}{Z} = k$

$\Rightarrow \frac{d^2 Z}{dz^2} + kZ = 0$

$\Rightarrow Z = c_1 Cos\sqrt{k}z + c_2 Sin\sqrt{k}z$ **if 'k' is real and positive.**

$\Rightarrow Z = c_1 e^{\sqrt{k}z} + c_2 e^{-\sqrt{k}z}$ **if 'k' is negative.**

$\Rightarrow Z = c_1 z + c_2$ **if 'k' is equal to zero.**

For physical consideration, one would expect a solution which decay with increasing 'z' and therefore the solution corresponding to negative 'k' is acceptable. Therefore $k = -\lambda^2$ then

$\Rightarrow Z = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$

Also $(i) \Rightarrow \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \lambda^2 F = 0$

$\Rightarrow f''H + \frac{1}{r}f'H + \frac{1}{r^2}fH'' + \lambda^2 fH = 0$ **choosing** $F(r,\theta) = f(r)H(\theta)$

$\Rightarrow (r^2 f'' + rf' + \lambda^2 r^2 f)\frac{1}{f} = -\frac{H''}{H} = k'(say)$

$\Rightarrow -\frac{H''}{H} = k'$ **also** $\Rightarrow (r^2 f'' + rf' + \lambda^2 r^2 f)\frac{1}{f} = k$

For physical consideration, we expect the solution to be periodic in θ which can be obtained when k' will positive. Therefore the acceptable solution will be

$H = c_3 Cosn\theta + c_4 Sinn\theta$

$\Rightarrow r^2 f'' + rf' + (\lambda^2 r^2 - n^2)f = 0$ **when $k' = n^2$ a Bessel's equation.**

$\Rightarrow f = AJ_n(\lambda r) + BY_n(\lambda r)$ **is general solution with $J_n(\lambda r), Y_n(\lambda r)$ Bessel's functions**

Since $Y_n(\lambda r) \to \infty;\ as\ r \to 0$ therefore $Y_n(\lambda r)$ becomes unbounded at $r = 0$.

Continuity of the solution demands B = 0.

Hence the most general and acceptable solution of $\nabla^2 u = 0$ is as follows;

$u(r,\theta,z) = J_n(\lambda r)\left(c_1 e^{\lambda z} + c_2 e^{-\lambda z}\right)(c_3 Cosn\theta + c_4 Sinn\theta)$

LAPLACE EQUATION IN SPHERICAL COORDINATES

The laplace equation in spherical coordinates (r,θ,φ) is wrriten as follows

$$\nabla^2 u=\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{Sin\theta}\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{Sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0$$

For solution consider $u(r,\theta,\varphi)=R(r)F(\theta,\varphi)$

$$\Rightarrow \nabla^2 u=F\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{Sin\theta}\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{R}{Sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}=0$$

$\Rightarrow \frac{1}{R}\left[\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right]=\frac{1}{F}\left[\frac{-1}{Sin\theta}\left\{\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{R}{Sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}\right\}\right]=-\mu$ **with μ a separation constant**

$\Rightarrow \frac{1}{R}\left[\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right]=-\mu$ **…….(i) And** $\Rightarrow \frac{1}{F}\left[\frac{1}{Sin\theta}\left\{\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{Sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}\right\}\right]=\mu$ **……..(ii)**

Now $(i)\Rightarrow r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}+\mu R=0$ **……….(iii)**

this is Euler equation. So using transformation $r=e^z$ the auxiliary solution can be written as $\Rightarrow \Delta(\Delta-1)+2\Delta+\mu=\Delta^2+\Delta+\mu=0\Rightarrow \Delta=\frac{-1\pm\sqrt{1-4\mu}}{2}$

Let $\mu=-\propto(\propto+1)\Rightarrow \Delta=\frac{-1\pm 2\sqrt{\left(\propto+\frac{1}{2}\right)^2}}{2}=\frac{-1}{2}\pm\left(\propto+\frac{1}{2}\right)\Rightarrow \Delta=\propto,-(\propto+1)$

$(iii)\Rightarrow R=c_1r^{\propto}+c_2r^{-(\propto+1)}$

Taking $\mu=-\propto(\propto+1)$ $\quad (ii)\Rightarrow \frac{1}{F}\left[\frac{1}{Sin\theta}\left\{\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{R}{Sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}\right\}\right]=-\propto(\propto+1)$

$(ii)\Rightarrow \frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{Sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}+\propto(\propto+1)FSin\theta=0$

choosing $F(\theta,\varphi)=H(\theta)\Phi(\varphi)$ **and separating variables**

$\Rightarrow \frac{Sin\theta}{H}\left[\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right)+\propto(\propto+1)HSin\theta\right]=-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=v^2$ **with v^2 a separation constant**

$\Rightarrow -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=v^2$ **……(iv) also** $\Rightarrow \frac{Sin\theta}{H}\left[\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right)+\propto(\propto+1)HSin\theta\right]=v^2$ **……(v)**

$(iv)\Rightarrow \frac{d^2\Phi}{d\varphi^2}+v^2\Phi=0\Rightarrow \Phi=c_3Cosv\varphi+c_4Sinv\varphi$ **with $v\neq 0$**

If $v=0$ the solution will be independent of φ which corresponds to the axisymmetric case.

Now for axisymmetric case $(v)\Rightarrow \frac{Sin\theta}{H}\left[\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right)+\propto(\propto+1)HSin\theta\right]=0$

$\Rightarrow \frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right)+\propto(\propto+1)HSin\theta=0$

Transforming the independent variable θ to 'x' and letting $x=Cos\theta$

$\Rightarrow \frac{d}{dx}\left(Sin\theta\frac{dH}{dx}\frac{dx}{d\theta}\right)\frac{dx}{d\theta}+\propto(\propto+1)HSin\theta=0\Rightarrow \frac{d}{dx}\left((1-Cos^2\theta)\frac{dH}{dx}\right)+\propto(\propto+1)H=0$

$\Rightarrow \frac{d}{dx}\left((1-x^2)\frac{dH}{dx}\right)+\propto(\propto+1)H=0$ **this is well known Legendre's equation.**

General solution of above is $H=AP_n(Cos\theta)+BQ_n(Cos\theta)$

Then $F(\theta,\varphi)=c_3\left(AP_n(Cos\theta)\right)$ **put $B=0$ also $\varphi=0$ for axisymmetric case**

Thus $u(r,\theta,\varphi)=\left(c_1r^{\propto}+c_2r^{-(\propto+1)}\right)c_3\left(AP_n(Cos\theta)\right)$

DIFFUSION (HEAT) EQUATION IN CYLINDRICAL COORDINATES

The Heat equation in cylindrical coordinates (r,θ,z) is wrriten as follows

$\frac{1}{K}u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$ **………(i)**

For solution consider $u(r,\theta,z,t) = R(r)H(\theta)Z(z)T(t)$ **………(ii)**

$(i) \Rightarrow \frac{T'}{K}RHZ = R''HZT + \frac{1}{r}R'HZT + \frac{1}{r^2}RH''ZT + RHZ''T$

$\Rightarrow \frac{1}{K}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \frac{Z''}{Z} = -\lambda^2$

where $-\lambda^2$ is a separation constant.

$\Rightarrow \frac{1}{K}\frac{T'}{T} = -\lambda^2$ **……(iii) and** $\Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \frac{Z''}{Z} = -\lambda^2$ **……(iv)**

$(iii) \Rightarrow T' + \lambda^2 KT = 0 \Rightarrow T = e^{-\lambda^2 Kt}$

$(iv) \Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 = -\frac{Z''}{Z} = -\mu^2$ **(say)**

$\Rightarrow -\frac{Z''}{Z} = -\mu^2$ **……(v) and** $\Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 = -\mu^2$ **……(vi)**

$(v) \Rightarrow Z'' - \mu^2 Z = 0 \Rightarrow Z = Ae^{\mu z} + Be^{-\mu z}$

$(vi) \Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 + \mu^2 = 0$

$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} + (\lambda^2+\mu^2)r^2 = -\frac{H''}{H} = v^2$ **(say)**

$\Rightarrow -\frac{H''}{H} = v^2$ **……(vii) and** $\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} + (\lambda^2+\mu^2)r^2 = v^2$ **……(viii)**

$(vii) \Rightarrow H'' + v^2 H = 0 \Rightarrow H = cCosv\theta + Dsinv\theta$

$(viii) \Rightarrow R'' + \frac{1}{r}R' + \left[(\lambda^2+\mu^2) - \frac{v^2}{r^2}\right]R = 0$ **this differential equation is called Bessel's equation of order 'v' and its general solution is as follows;**

$R(r) = c_1 J_v\left(\sqrt{\lambda^2+\mu^2 r}\right) + c_2 Y_v\left(\sqrt{\lambda^2+\mu^2 r}\right)$

Where $J_v(r), Y_v(r)$ are Bessel's functions

$\Rightarrow R(r) = c_1 J_v\left(\sqrt{\lambda^2+\mu^2 r}\right)$ **for singular equation $r = 0$**

Thus general solution of equation (ii) will be

$u(r,\theta,z,t) = c_1 J_v\left(\sqrt{\lambda^2+\mu^2 r}\right)(cCosv\theta + Dsinv\theta)(Ae^{\mu z} + Be^{-\mu z})\left(e^{-\lambda^2 Kt}\right)$

DIFFUSION (HEAT) EQUATION IN SPHERICAL COORDINATES

The Heat equation in spherical coordinates (r,θ,φ) is wrriten as follows

$$\frac{1}{K}u_t = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial\theta}(Sin\theta . u_\theta) + \frac{1}{r^2 Sin^2\theta}u_{\varphi\varphi} \quad \text{.........(i)}$$

For solution consider $u(r,\theta,z,t) = R(r)H(\theta)\Phi(\varphi)T(t)$ **.........(ii)**

$$(i) \Rightarrow \frac{T'}{K}RHZ = R''HZT + \frac{1}{r}R'HZT + \frac{1}{r^2}RH''ZT + RHZ''T$$

$$\Rightarrow \frac{1}{K}\frac{T'}{T} = \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2 Sin\theta}\frac{1}{H}\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right) + \frac{1}{\Phi r^2 Sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -\lambda^2$$

where $-\lambda^2$ is a separation constant.

$$\Rightarrow \frac{1}{K}\frac{T'}{T} = -\lambda^2 \quad \text{........(iii)}$$

and $\Rightarrow \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2 Sin\theta}\frac{1}{H}\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right) + \frac{1}{\Phi r^2 Sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -\lambda^2$ **........(iv)**

$$(iii) \Rightarrow T' + \lambda^2 KT = 0 \Rightarrow T = e^{-\lambda^2 Kt}$$

$$(iv) \Rightarrow r^2 Sin\theta\left[\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{Hr^2 Sin\theta}\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right) + \lambda^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = m^2 \quad \text{(say)}$$

$$\Rightarrow -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = m^2 \quad \text{.............(v)}$$

and $\Rightarrow r^2 Sin\theta\left[\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{Hr^2 Sin\theta}\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right) + \lambda^2\right] = m^2$ **......(vi)**

$$(v) \Rightarrow \Phi'' + \mu^2\Phi = 0 \Rightarrow \Phi = c_1 e^{im\varphi} + c_2 e^{-im\varphi}$$

$$(vi) \Rightarrow \frac{r^2}{R}\left(R'' + \frac{2}{r}R'\right) + \lambda^2 r^2 = \frac{m^2}{Sin\theta} - \frac{1}{HSin\theta}\frac{d}{d\theta}\left(Sin\theta\frac{dH}{d\theta}\right) = n(n+1) \quad \text{(say)}$$

$$\Rightarrow R'' + \frac{2}{r}R' + \left\{\lambda^2 - \frac{n(n+1)}{r^2}\right\}R = 0 \quad \text{......(vii)}$$

And $\Rightarrow -\frac{1}{HSin\theta}(Sin\theta H'' + Cos\theta H') + \frac{m^2}{Sin^2\theta} = n(n+1)$

$$\Rightarrow (H'' + Cot\theta H') + \left\{n(n+1) - \frac{m^2}{Sin^2\theta}\right\}H = 0 \quad \text{......(viii)}$$

$(vii) \Rightarrow (\lambda r)^{-1/2}\left[\Psi'' + \frac{2}{r}\Psi' + \left\{\lambda^2 - \frac{(n+\frac{1}{2})^{1/2}}{r^2}\right\}\Psi\right] = 0$ **for** $R = (\lambda r)^{-1/2}\Psi(r)$

$\Rightarrow (\lambda r)^{-1/2} \neq 0 \Rightarrow \left[\Psi'' + \frac{2}{r}\Psi' + \left\{\lambda^2 - \frac{(n+\frac{1}{2})^{1/2}}{r^2}\right\}\Psi\right] = 0$ **this differential equation is called Bessel's equation of order '$n+\frac{1}{2}$' and its general solution is as follows;** $\Psi(r) = AJ_{n+\frac{1}{2}}(\lambda r) + BY_{n+\frac{1}{2}}(\lambda r)$

$$\Rightarrow R = (\lambda r)^{-1/2}\left(AJ_{n+\frac{1}{2}}(\lambda r) + BY_{n+\frac{1}{2}}(\lambda r)\right)$$

Where J_n, Y_n are Bessel's functions.
Now introducing $\mu = Cos\theta$

$$\Rightarrow Cot\theta = \frac{\mu}{\sqrt{1-\mu^2}} \Rightarrow H'(\theta) = -\sqrt{1-\mu^2}H'(\mu)$$

$$\Rightarrow H''(\theta) = (1-\mu^2)H''(\mu) - \mu H'(\mu)$$

$(viii) \Rightarrow (1-\mu^2)H''(\mu) - 2\mu H'(\mu) + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}H = 0$ **this is an associated Legendre's DE whose solution is as follows;**

$\Rightarrow H(\theta) = A'P_n^m(\mu) + B'Q_n^m(\mu)$ **where P_n^m, Q_n^m are associated Legendre's solutions of degree 'n' and order 'm'**

Then

$$u(r,\theta,z,t) = (\lambda r)^{-1/2}\left(AJ_{n+\frac{1}{2}}(\lambda r) + BY_{n+\frac{1}{2}}(\lambda r)\right)\left(A'P_n^m(\mu) + B'Q_n^m(\mu)\right)\left(c_1e^{im\varphi} + c_2e^{-im\varphi}\right)\left(e^{-\lambda^2Kt}\right)$$

In the general solution the functions $Q_n^m(\mu)$ and $(\lambda r)^{-1/2}BY_{n+\frac{1}{2}}(\lambda r)$ are excluded because these functions have poles at $\mu = \pm 1$ and $r = 0$
Thus general solution of equation (ii) will be

$$u(r,\theta,z,t) = (\lambda r)^{-1/2}\left(AJ_{n+\frac{1}{2}}(\lambda r)\right)\left(A'P_n^m(\mu)\right)\left(c_1e^{im\varphi} + c_2e^{-im\varphi}\right)\left(e^{-\lambda^2Kt}\right)$$

Or in general

$$u(r,\theta,z,t) = \sum_{\lambda,m,n} A_{\lambda mn}(\lambda r)^{-1/2}J_{n+\frac{1}{2}}(\lambda r)\, P_n^m(Cos\theta)e^{\pm im\varphi-\lambda^2Kt}$$

ONE DIMENSIONAL WAVE EQUATION IN CYLINDRICAL COORDINATES

In cylindrical coordinates, the wave equation assumes the following form

$$\frac{1}{c^2}u_{tt} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \quad ; r > 0 \ \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

To find solution consider $u = F(r)e^{i\omega t}$

$$\Rightarrow \frac{\partial u}{\partial r} = F'(r)e^{i\omega t} \quad \text{and} \quad u_{tt} = -\omega^2 F(r)e^{i\omega t}$$

$$(i) \Rightarrow \frac{1}{c^2}\left(-\omega^2 F(r)e^{i\omega t}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)e^{i\omega t}\right)$$

$$\Rightarrow F''(r) + \frac{F'(r)}{r} + \frac{\omega^2}{c^2}F(r) = 0$$ **which has the form of Bessel's equation and hence its solution can be written as follows;**

$$F(r) = AJ_0\left(\frac{\omega r}{c}\right) + BY_0\left(\frac{\omega r}{c}\right)$$

In complex form we can write this equation as

$$F(r) = c_1\left(J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right)\right) + c_2\left(J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right)\right)$$

It can be rewritten as

$$F(r) = c_1 H_0^1\left(\frac{\omega r}{c}\right) + c_2 H_0^2\left(\frac{\omega r}{c}\right)$$ **with** H_0^1, H_0^2 **as hankel transform.**

Defined as follows;

$$H_0^1 = J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right) \quad \text{and} \quad H_0^2 = J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right)$$

$$(i) \Rightarrow u = F(r)e^{i\omega t} = c_1 e^{i\omega t}H_0^1\left(\frac{\omega r}{c}\right) + c_2 e^{i\omega t}H_0^2\left(\frac{\omega r}{c}\right)$$

Using asymptotic expression as follows

$$H_0^1(x) = \sqrt{\frac{2}{\pi x}}\,e^{i\left(x-\frac{\pi}{4}\right)} \quad \text{and} \quad H_0^2(x) = \sqrt{\frac{2}{\pi x}}\,e^{-i\left(x-\frac{\pi}{4}\right)}$$

Then our required solution will be as follows;

$$u = \sqrt{\frac{2c}{\pi\omega}}\left[c_1 e^{-i\frac{\pi}{4}}\frac{exp[i(\omega/c)(r+ct)]}{\sqrt{r}} + c_2 e^{i\frac{\pi}{4}}\frac{exp[i(\omega/c)(r-ct)]}{\sqrt{r}}\right]$$

WAVE EQUATION IN SPHERICAL COORDINATES

The Wave equation inone dimension when $u = u(r)$ only. i.e. u depends only on 'r' will be of the form

$$\frac{1}{c^2}u_{tt} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) \quad ; r > 0 \quad \text{………..(i)}$$

To find solution consider $u = F(r)e^{i\omega t}$

$\Rightarrow \frac{\partial u}{\partial r} = F'(r)e^{i\omega t}$ **and** $u_{tt} = -\omega^2 F(r)e^{i\omega t}$

$$(i) \Rightarrow \frac{1}{c^2}\left(-\omega^2 F(r)e^{i\omega t}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F'(r)e^{i\omega t}\right)$$

$\Rightarrow F''(r) + 2\frac{F'(r)}{r} + \frac{\omega^2}{c^2}F(r) = 0$ **which has the form of Bessel's equation and hence its solution can be written as follows;**

$$F(r) = \frac{A}{\sqrt{r}}J_{\frac{1}{2}}\left(\frac{\omega r}{c}\right) + \frac{B}{\sqrt{r}}J_{-\frac{1}{2}}\left(\frac{\omega r}{c}\right)$$

But we know that

$J_{\frac{1}{2}}\left(\frac{\omega r}{c}\right) = \sqrt{\frac{2}{\pi x}}Sinx$ **and** $J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}Cosx$

Then

$$u = \sqrt{\frac{2c}{\pi\omega}}\left[A\frac{Sin(\omega r/c)}{r} + B\frac{Cos(\omega r/c)}{r}\right]$$

And in complex form $F(r) = c_1\frac{exp[i(\omega r/c)]}{r} + c_2\frac{exp[-i(\omega r/c)]}{r}$

Hence tour required solution is as follows;

$$u = \left[c_1\frac{exp[i(\omega/c)(r+ct)]}{r} + c_2\frac{exp[-i(\omega/c)(r-ct)]}{r}\right]$$

EXERCISES

1. Determine the region in which the given equation is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to canonical form.

(a) $xu_{xx} + u_{yy} = x^2$ **(b)** $u_{xx} + y^2u_{yy} = y$

(c) $u_{xx} + xyu_{yy} = 0$ **(d)** $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$

(e) $u_{xx} + u_{xy} - xu_{yy} = 0$, **(f)** $e^xu_{xx} + e^yu_{yy} = u$

2. Obtain the general solution of the following equations:

(i) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + xyu_x + y^2u_y = 0$, **(ii)** $ru_{tt} - c^2ru_{rr} - 2c^2u_r = 0$, c = constant

(iii) $4u_x + 12u_{xy} + 9u_{yy} - 9u = 9$ **(iv)** $u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 6u_y = 9(2x - y)$

(v) $yu_x + 3yu_{xy} + 3u_x = 0$, $y \neq 0$. **(vi)** $u_{xx} + u_{yy} = 0$ **(vii)** $4u_{xx} + u_{yy} = 0$,

(viii) $u_{xx} - 2u_{xy} + u_{yy} = 0$ **(ix)** $2u_{xx} + u_{yy} = 0$ **(x)** $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

(xi) $3u_{xx} + 4u_{xy} - \frac{3}{4}u_{yy} = 0$

3. Find the characteristics and characteristic coordinates, and reduce the following equations to canonical form:

(a) $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$

(b) $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$

(c) $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$ **(d)** $u_{xx} + u_{yy} + 2u_x + 8u_y + u = 0$

(e) $u_{xy} + 2u_{yy} + 9u_x + u_y = 2$ **(f)** $6u_{xx} - u_{xy} + u = y^2$

(g) $uxy + ux + uy = 3x$, **(h)** $uyy - 9ux + 7uy = \cos y$,

(i) $x^2u_{xx} - y^2u_{yy} - u_x = 1 + 2y^2$ **(j)** $u_{xx} + yu_{yy} + \frac{1}{2}uy + 4yu_x = 0$

(k) $x^2y^2u_{xx} + 2xyu_{xy} + u_{yy} = 0$ **(l)** $u_{xx} + yu_{yy} = 0$.

4. Determine the general solutions of the following equations:

(i) $u_{xx} - 1/c^2\, u_{yy} = 0$, c= constant **(ii)** $u_{xx} + u_{yy} = 0$

(iii) $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$ **(iv)** $u_{xx} - 3u_{xy} + 2u_{yy} = 0$

(v) $u_{xx} + u_{xy} = 0$, **(vi)** $u_{xx} + 10u_{xy} + 9u_{yy} = y$

5. Classify each of the following equations and reduce it to canonical form:

(a) $y\,u_{xx} - xu_{yy} = 0$, $x > 0$, $y > 0$ **(b)** $u_{xx} + (\text{sech}^4 x)u_{yy} = 0$

(c) $y^2u_{xx} + x^2u_{yy} = 0$ **(d)** $u_{xx} - (\text{sech}^4 x)u_{yy} = 0$

(e) $u_{xx} + 6u_{xy} + 9u_{yy} + 3yu_y = 0$ **(f)** $y^2u_{xx} + 2xyu_{xy} + 2x^2u_{yy} + xu_x = 0$

(g) $u_{xx} - (2\cos x)u_{xy} + (1 + \cos^2 x)u_{yy} + u = 0$

(h) $u_{xx} + (2\,\text{cosec}\, y)u_{xy} + (\text{cosec}^2 y)u_{yy} = 0$

(i) $u_{xx} - 2u_{xy} + u_{yy} + 3u_x - u + 1 = 0$

(j) $u_{xx} - y^2u_{yy} + u_x - u + x^2 = 0$

(k) $u_{xx} + yu_{yy} - xu_y + y = 0$

FOURIER TRANSFORMATION AND INTEGRALS WITH APPLICATIONS

FOURIER TRANSFORMATION

If $u\,(x,t)$ is a continuous, piecewise smooth, and absolutely integrable function, then the Fourier transform of $u\,(x,t)$ with respect to $x \in R$ is denoted by $U\,(k,t)$ and is defined by

$$\mathcal{F}\,\{u\,(x,t)\} = U\,(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, u\,(x,t)\, dx$$

where k is called the Fourier transform variable and $exp\,(-ikx)$ is called the kernel of the transform.

Then, for all $x \in R$, the INVERSE FOURIER TRANSFORM of $U\,(k,t)$ is defined by

$$\mathcal{F}^{-1}\,\{U\,(k,t)\} = u\,(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, U(k,t)\, dk$$

CONDITION FOR EXISTENCE OF FOURIER TRANSFORMATION

Fourier Transforamtion and Inverse Fourier Transformation exist if

- **(i)** **The function $f(x)$ or $F(k)$ is continuous or piecewise continuous over $(-\infty,\infty)$ and bounded.**
- **(ii)** **The function $f(x)$ or $F(k)$ are absolutely integrable i.e. $\int_{-\infty}^{\infty}|f(x)|dx$ or $\int_{-\infty}^{\infty}|F(k)|dk$ this condition is sufficient for existence of Fourier Transforamtion and Inverse Fourier Transformation.**

Example: Show that $\mathcal{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{2a}} e^{\left(-\frac{k^2}{4a}\right)}$ $\quad ; a > 0$

Solution. We have, by definition

$$\mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f\,(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,.\,e^{-ax^2}\, dx$$

$$\mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-ax^2}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(x-\frac{ik}{2a}\right)^2+\frac{k^2}{4a^2}\right]}\, dx$$

Consider $\quad ikx - ax^2$

$$= -a\left(x^2 - \frac{ikx}{a}\right)$$

$$= -a\left[\left(x^2 - \frac{2ikx}{2a}\right) + \left(\frac{ik}{2a}\right)^2 - \left(\frac{ik}{2a}\right)^2\right]$$

$$= -a\left[\left(x - \frac{ik}{2a}\right)^2 + \frac{k^2}{4a^2}\right]$$

$$\mathcal{F}\,\{f\,(x)\} == \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2}\, dx$$

Put $a\left(x - \frac{ik}{2a}\right)^2 = P^2 \Rightarrow \sqrt{a}\left(x - \frac{ik}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$$\Rightarrow \mathcal{F}\,\{f\,(x)\} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2}\, dx = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-P^2}\,.\frac{dP}{\sqrt{a}}$$

$$\Rightarrow \mathcal{F}\,\{f\,(x)\} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}}.\sqrt{\pi} \qquad \therefore \int_{-\infty}^{\infty} e^{-P^2}\, dP = \sqrt{\pi}$$

$$\Rightarrow \mathcal{F}\,\{f\,(x)\} = \mathcal{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{2a}} e^{\left(-\frac{k^2}{4a}\right)}$$

NOTE: sometime appears in the form $\mathcal{F}\left\{Ne^{-ax^2}\right\}$ known as Guassian Function.

Example: Show that $\mathcal{F}\left\{e^{-a|x|}\right\} = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}$ $\quad ; a > 0$

Solution. We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} . e^{-a|x|} dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-a|x|} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a+ik)x} dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a-ik)x} dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}$$

Example: Show that $\mathcal{F}\left\{\mathcal{X}_{[-a,a]}(x)\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$

where $\mathcal{X}_{[-a,a]}(x) = H(a-|x|) = \begin{cases} 1, & |x| < a \text{ or } -a < x < a \\ 0, & |x| > a \end{cases}$

Solution. Let us consider $f(x) = \mathcal{X}_{[-a,a]}(x)$ **then We have, by definition**

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} f(x)dx + \int_{-a}^{a} e^{ikx} f(x)dx + \int_{a}^{\infty} e^{ikx} f(x)dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} .0dx + \int_{-a}^{a} e^{ikx} .1dx + \int_{a}^{\infty} e^{ikx} .0dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ikx} dx = \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ika} - e^{-ika}}{2i}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$$

FIND FOURIER TRANSFORMS OF THE FOLLOWINGS;

i. $f_a(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$ **the gate function. Where 'a' is positive constant.**

ii. $f(x) = \frac{1}{|x|}$

iii. $f(x) = \mathcal{X}_{[-a,a]}(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

iv. $f(x) = \begin{cases} 1 - \frac{|x|}{a}, & |x| \leq a \\ 0, & |x| \geq a \end{cases}$

v. $f(x) = \frac{1}{(x^2+a^2)}$

vi. $f(x) = Sin(x^2)$

vii. $f(x) = Cos(x^2)$

KEEP IN MIND

IN THIS BOOK (MYINT) THERE IS A SIGN DIFFERENCE FOR THE DEFINATION OF FOURIER AND INVERSE OF FOURIER. BUT IN SO MANY BOOKS SIGN IS DIFFERENT TO THIS BOOK. SO I PROCEED ACCORDING TO OTHER BOOKS. IF SOME QUESTION APPEAR WITH DIFFERENT SIGN (+VE OR –VE) STUDENTS JUST CHANGE THE SIGN OF EXPONENT IN DEFINATION.

PROPERTIES OF FOURIER TRANSFORMS

LINEARITY PROPERTY: THE FOURIER TRANSFORMATION $\mathcal{F}$ IS LINEAR.
Proof.

Let $u(x) = af(x) + bg(x)$ where a and b are constants.

We have, by definition

$\mathcal{F}\{u(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, u(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,[af(x) + bg(x)]dx$

$\mathcal{F}\{u(x)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f(x)dx + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, g(x)dx = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$

$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$ **hence proved.**

LINEARITY PROPERTY: THE INVERSE FOURIER TRANSFORMATION $\mathcal{F}^{-1}$ IS LINEAR.
Proof.

Let $U(k) = aF(k) + bG(k)$ where a and b are constants.

We have, by definition

$\mathcal{F}^{-1}\{U(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, U(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,[aF(k) + bG(k)]dk$

$\mathcal{F}^{-1}\{U(k)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, F(k)dk + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, G(k)dk$

$\mathcal{F}^{-1}\{aF(k) + bG(k)\} = a\mathcal{F}^{-1}\{F(k)\} + b\mathcal{F}^{-1}\{G(k)\}$ **hence proved.**

SHIFTING PROPERTY: Let $\mathcal{F}\{f(x)\}$ be a Fourier transform of $f(x)$. Then
$\mathcal{F}[f(x - c)] = e^{ikc}F(k)$ **where c is a real constant.**

Proof. From the definition, we have, for $c > 0$,

$\mathcal{F}[f(x - c)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f(x - c)dx$

Put $x - c = x' \Rightarrow dx = dx'$ also as $x \to \pm\infty$ then $x' \to \pm\infty$

$$\mathcal{F}[f(x - c)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(x'+c)}\, f(x')dx' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}.e^{ikc}\, f(x')dx'$$

$\mathcal{F}[f(x - c)] = e^{ikc}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}\, f(x')dx' = e^{ikc}\mathcal{F}\{f(x)\} = e^{ikc}F(k)$

SCALING PROPERTY: If $\mathcal{F}$ is the Fourier transform of f, then
$\mathcal{F}[f(cx)] = \left(\frac{1}{|c|}\right)F\left(\frac{k}{c}\right)$ **where c is a real nonzero constant.**

Proof. For $c \neq 0$

$\mathcal{F}[f(cx)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, f(cx)dx$

$\mathcal{F}[f(cx)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik\left(\frac{x'}{c}\right)}\, f(x')\frac{dx'}{c} = \frac{1}{c}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik\left(\frac{x'}{c}\right)}\, f(x')dx' = \frac{1}{c}F\left(\frac{k}{c}\right)$

Since $c \neq 0$ then either $c < 0$ or $c > 0$

If $c > 0$ then $\mathcal{F}[f(cx)] = \frac{1}{+c}F\left(\frac{k}{c}\right)$

If $c < 0$ then $\mathcal{F}[f(cx)] = \frac{1}{-c}F\left(\frac{k}{c}\right)$

Hence $\mathcal{F}[f(cx)] = \left(\frac{1}{|c|}\right)F\left(\frac{k}{c}\right)$

DIFFERENTIATION PROPERTY: Let f be continuous and piecewise smooth in $(-\infty,\infty)$. Let $f(x)$ approach zero as $|x| \to \infty$. If f and f' are absolutely integrable, then $\mathcal{F}[f'(x)] = (-ik)\mathcal{F}[f(x)] = (-ik)F(k)$

Proof.

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f'(x)dx$$

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}}\left[\left|e^{ikx} f(x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{ikx}(ik) f(x)dx\right] = \frac{1}{\sqrt{2\pi}}\left[0 + (-ik)\int_{-\infty}^{\infty} e^{ikx} f(x)dx\right]$$

$$\mathcal{F}[f'(x)] = (-ik)\mathcal{F}[f(x)] = (-ik)F(k)$$

This result can be easily extended. If f and its first (n − 1) derivatives are continuous, and if its nth derivative is piecewise continuous, then

$$\mathcal{F}[f^n(x)] = (-ik)^n\mathcal{F}[f(x)] = (-ik)^n F(k) \qquad n = 0,1,2,\ldots$$

provided f and its derivatives are absolutely integrable. In addition, we assume that f and its first (n − 1) derivatives tend to zero as |x| tends to infinity.

CONJUGATION PROPERTY: Let f is real then $F(-k) = \overline{F(k)}$

Proof.

Since f is real therefore $f(x) = \overline{f(x)}$ then by defination

$$F(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

And $\overline{F(k)} = \mathcal{F}\left[\overline{f(x)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\overline{f(x)}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(-k)x} f(x)dx = F(-k)$

Hence $F(-k) = \overline{F(k)}$

ATTENUATION PROPERTY: For a function , $\mathcal{F}[e^{ax}f(x)] = F(k - ai)$

Proof.

By definition $F(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$

Then $\mathcal{F}[e^{ax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} e^{ax}f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} e^{-i^2ax}f(x)dx$

$\mathcal{F}[e^{ax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k-ai)x} f(x)dx$ **……….(i)**

Also $F(k - ai) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k-ai)x} f(x)dx$ **……….(ii)**

Thus from (i) and (ii) $\mathcal{F}[e^{ax}f(x)] = F(k - ai)$

MODULATION PROPERTY(i): $\mathcal{F}[Cosaxf(x)] = \frac{1}{2}[F(k + a) + F(k - a)]$

Proof.

By definition $\mathcal{F}[Cosaxf(x)] = \mathcal{F}[\left(\frac{e^{iax} + e^{-iax}}{2}\right)f(x)]$

$$\mathcal{F}[Cosaxf(x)] = \frac{1}{2}[\mathcal{F}\{e^{iax}f(x)\} + \mathcal{F}\{e^{-iax}f(x)\}] = \frac{1}{2}[F(k + a) + F(k - a)]$$

MODULATION PROPERTY (ii): $\mathcal{F}[Sinaxf(x)] = \frac{1}{2i}[F(k + a) - F(k - a)]$

Proof.

By definition $\mathcal{F}[Sinaxf(x)] = \mathcal{F}[\left(\frac{e^{iax} - e^{-iax}}{2i}\right)f(x)]$

$$\mathcal{F}[Cosaxf(x)] = \frac{1}{2i}[\mathcal{F}\{e^{iax}f(x)\} - \mathcal{F}\{e^{-iax}f(x)\}] = \frac{1}{2i}[F(k + a) - F(k - a)]$$

CONVOLUTION FUNCTION / FAULTUNG FUNCTION

The function $(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$

is called the convolution of the functions f **and** g **over the interval** $(-\infty, \infty)$

NOTE: The convolution satisfies the following properties:
1. $f * g = g * f$ **(commutative)**
2. $f * (g * h) = (f * g) * h$ **(associative)**
3. $* (ag + bh) = a(f * g) + b(f * h)$ **, (distributive)**
where a **and** b **are constants.**

PROPERTY: $f * g = g * f$
PROOF: since by definition $(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$
Put $x - \xi = \propto \Rightarrow d\xi = -d\propto$ $also\ \xi = x - \propto$ **and if** $\xi \to \pm\infty$ **then** $\propto \to \mp\infty$ **then**
$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(\propto) g(x-\propto)(-d\propto) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-\propto) f(\propto)\ (d\propto) = g * f$
Hence $f * g = g * f$

PROVE THE FOLLOWING PROPERTIES OF THE FOURIER CONVOLUTION:
$(a)\ f(x) * g(x) = g(x) * f(x), (b)\ f * (g * h) = (f * g) * h$
$(c)\ f * (ag + bh) = a(f * g) + b(f * h),$ *where a and b are constants*
$(d)\ f * 0 = 0 * f = 0, (e)\ f * 1 \neq f$
$(f)\ f * \sqrt{2\pi}\,\delta = f = \sqrt{2\pi}\,\delta * f$

CONVOLUTION / FAULTUNG THEOREM
If $F(k)$ **and** $G(k)$ **are the Fourier transforms of** $f(x)$ **and** $g(x)$ **respectively, then the Fourier transform of the convolution** $(f * g)$ **is the product** $F(k)G(k)$**. That is,**
$\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$
Or, equivalently, $\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$
Or
$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk = (f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$

PROOF: By definition, we have
$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk$
$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} g(x')\,dx'\right\}dk$
By changing the order of integration
$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x')} F(k)\,dk\right] g(x')\,dx'$
$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x') g(x')\,dx' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi = (f * g)(x)$
Where we replace ξ **with** x'
Hence $\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$
Or $\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$

PARSEVAL'S FORMULA

According to this formula $\int_{-\infty}^{\infty}|f(\mathrm{x})|^2\,dx = \int_{-\infty}^{\infty}|F(\mathrm{k})|^2\,dk$

PROOF: The convolution formula gives

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\,F(k)G(k)\mathrm{dk} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi$

$\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi = \int_{-\infty}^{\infty} e^{-\mathrm{ik}\xi}\,F(k)G(k)\mathrm{dk}$

which is, by putting $x = 0$

$\int_{-\infty}^{\infty} f(\xi)g(-\xi)\,d\xi = \int_{-\infty}^{\infty} F(k)G(k)\mathrm{dk}$

$\int_{-\infty}^{\infty} f(x)g(-x)\,dx = \int_{-\infty}^{\infty} F(k)G(k)\mathrm{dk}$

Putting $g(-x) = \overline{f(x)}$ **then** $g(x) = \overline{f(-x)} \Rightarrow \mathcal{F}\{g(x)\} = \mathcal{F}\{\overline{f(-x)}\}$

$\Rightarrow G(k) = \overline{F(k)}$ **then above equation becomes** $\therefore \mathcal{F}\{\overline{f(-x)}\} = \overline{F(k)}$ for complex f.

$\int_{-\infty}^{\infty} f(x)\overline{f(x)}\,dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}\mathrm{dk}$ **where the bar denotes the complex conjugate.**

$\Rightarrow \int_{-\infty}^{\infty}|f(\mathrm{x})|^2\,dx = \int_{-\infty}^{\infty}|F(\mathrm{k})|^2\,dk$

In terms of the notation of the norm, this is $\|f\| = \|F\|$

<u>JUST READ THE FOLLOWING AND KEEP IN MIND THE RESULTS</u>

<u>DIREC DELTA FUNCTION</u>

The dirac delta function is defined as follows;

$$\delta(x) = \lim_{\epsilon\to 0}\delta_\epsilon(x) = \begin{cases}\infty & if\ x = 0\\ 0 & if\ x \neq 0\end{cases}$$

PROPERTIES:

i. $\int_{-\infty}^{\infty}\delta(x)dx = 1$

ii. **For any continuous function $f(x)$;** $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

iii. $\delta(x) = \delta(-x)$

iv. $\delta(ax) = \frac{1}{a}\delta(x) \quad ; a > 0$

v. **SHIFTING PROPERTY: For any continuous function $f(x)$;**
$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

vi. **If $\delta(x)$ is continuous differentiable. Then** $\int_{-\infty}^{\infty} f(x)\delta'(x)dx = -f'(0)$

THE FOURIER TRANSFORMS OF STEP AND IMPULSE FUNCTIONS

The Heaviside unit step function is defined by

$$H(x-a) = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases} \quad \text{where } a \geq 0$$

The Fourier transform of the Heaviside unit step function can be easily determined. We consider first

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, H(x-a)dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{ikx}\, H(x-a)dx + \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, H(x-a)dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{ikx}\,.0dx + \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\,.1dx = \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, dx$$

This integral does not exist. However, we can prove the existence of this integral by defining a new function

$$H(x-a)e^{-\alpha x} = \begin{cases} 0 & x < a \\ e^{-\alpha x} & x \geq a \end{cases}$$

This is evidently the unit step function as $\alpha \to 0$. Thus, we find the Fourier transform of the unit step function as

$$\mathcal{F}[H(x-a)] = \lim_{\alpha\to 0}\mathcal{F}[H(x-a)e^{-\alpha x}] = \lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, H(x-a)e^{-\alpha x}dx$$

$$\mathcal{F}[H(x-a)] = \lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, e^{-\alpha x}dx = \lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{i(k-\alpha)x}\, dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, dx = \frac{e^{ika}}{\sqrt{2\pi}ik} \quad \textbf{For } a = 0 \Rightarrow \mathcal{F}[H(x)] = \frac{1}{\sqrt{2\pi}ik}$$

An impulse function is defined by

$$p(x) = \begin{cases} h & a-\varepsilon < x < a+\varepsilon \\ 0 & x \leq a-\varepsilon \text{ or } x \geq a+\varepsilon \end{cases}$$

where h is large and positive, $a > 0$, and ε is a small positive constant, This type of function appears in practical applications; for instance, a force of large magnitude may act over a very short period of time.

The Fourier transform of the impulse function is

$$\mathcal{F}[p(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, p(x)dx$$

$$\mathcal{F}[p(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a-\varepsilon} e^{ikx}\, p(x)dx + \frac{1}{\sqrt{2\pi}}\int_{a-\varepsilon}^{a+\varepsilon} e^{ikx}\, p(x)dx + \frac{1}{\sqrt{2\pi}}\int_{a+\varepsilon}^{\infty} e^{ikx}\, p(x)dx$$

$$\mathcal{F}[p(x)] = \frac{1}{\sqrt{2\pi}}\int_{a-\varepsilon}^{a+\varepsilon} he^{ikx}\, dx = \frac{h}{\sqrt{2\pi}}\left|\frac{e^{ikx}}{ik}\right|_{a-\varepsilon}^{a-\varepsilon} = \frac{h}{\sqrt{2\pi}}.\frac{1}{ik}\left(e^{ik(a+\varepsilon)} - e^{ik(a-\varepsilon)}\right)$$

$$\mathcal{F}[p(x)] = \frac{h}{\sqrt{2\pi}}.\frac{e^{ika}}{ik}\left(e^{ik\varepsilon} - e^{-ik\varepsilon}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{ika}\left(\frac{e^{ik\varepsilon}-e^{-ik\varepsilon}}{2ik\varepsilon}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{ika}\left(\frac{Sink\varepsilon}{k\varepsilon}\right)$$

Now if we choose the value of $h = \left(\frac{1}{2\varepsilon}\right)$ then the impulse defined by

$$I(\varepsilon) = \int_{-\infty}^{\infty} p(x)dx = \int_{a-\varepsilon}^{a+\varepsilon}\frac{1}{2\varepsilon}dx = 1$$

which is a constant independent of ε. In the limit as $\varepsilon \to 0$, this particular function $p_\varepsilon(x)$ with h = (1/2 ε) satisfies $\lim_{\varepsilon\to 0} p_\varepsilon(x) = 0 \;;\, x \neq 0$
and $\lim_{\varepsilon\to 0} I(\varepsilon) = 1$
Thus, we arrive at the result

$\delta(x-a) = 0, \;\; x \neq a,$ and $\int_{-\infty}^{\infty}\delta(x-a)dx = 1$ **This is the Dirac delta function**

We now define the Fourier transform of $\delta(x)$ as the limit of the transform of $p_\varepsilon(x)$. We then consider

$$\mathcal{F}[\delta(x-a)] = \lim_{\varepsilon\to 0}\mathcal{F}[p_\varepsilon(x)] = \lim_{\varepsilon\to 0}\frac{e^{ika}}{\sqrt{2\pi}}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) = \frac{e^{ika}}{\sqrt{2\pi}}$$

in which we note that, by L'Hospital's rule, $\lim_{\varepsilon\to 0}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) = 1$

When $a = 0$, we obtain $\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}$

REMARK:

- $\mathcal{F}[f^n(x)] = (-ik)^n\mathcal{F}[f(x)] = (-ik)^n F(k)$ $\quad n = 0, 1, 2, \ldots$
- **If** $\mathcal{F}\{u_t\} = \mathcal{F}\{u_x\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}\{u(x,t)\} = (-ik)\mathcal{F}\{u(x,t)\}$ **when 'x' varies not 't'**
- **When range of spatial variable is infinite then Fourier transform is used rather than the sine or cosine.**

EXAMPLE:Obtain the solution of the initial-value <u>problem of heat conduction</u> in an infinite rod

$u_t = \kappa u_{xx}, \quad -\infty < x < \infty, \quad t > 0$

$u(x,0) = f(x), -\infty < x < \infty$ **with** $u(x,t) \to 0, as\ |x| \to \infty$ **where** $u(x, t)$ **represents the temperature distribution and is bounded, and** κ **is a constant of diffusivity.**

Solution:

After taking The Fourier transform of given equation we get

$U_t + \kappa k^2 U = 0$ and $U(k, 0) = F(k)$

The solution of the transformed system is

$$U(k,t) = F(k)\, e^{-k^2\kappa t}$$

The inverse Fourier transformation gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k) e^{-k^2\kappa t} dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{-k^2\kappa t - ikx} dk \quad \ldots\ldots(i)$$

But if a > 0 and b is real or complex and we know that $\int_{-\infty}^{\infty} e^{-k^2 a - 2bk} dk = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{b^2}{a}}$

If we use $a = \kappa t, 2b = ix$

then $\int_{-\infty}^{\infty} e^{-k^2\kappa t - ikx} dk = \int_{-\infty}^{\infty} e^{-k^2 a - 2bk} dk = \frac{\sqrt{\pi}}{\sqrt{\kappa t}} e^{\frac{-x^2}{4\kappa t}}$

then If $g(x)$ is the inverse transform of $G(k) = e^{-k^2\kappa t}$ and has the form

$$g(x) = \mathcal{F}^{-1}\left\{e^{-k^2\kappa t}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2\kappa t - ikx} dk = \frac{1}{\sqrt{2\pi}}.\frac{\sqrt{\pi}}{\sqrt{\kappa t}} e^{\frac{-x^2}{4\kappa t}} = \frac{1}{\sqrt{2\kappa t}} e^{\frac{-x^2}{4\kappa t}}$$

Now using the Convolution theorem

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi) g(x-\xi)\, d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi).\frac{1}{\sqrt{2\kappa t}} e^{\frac{-(x-\xi)^2}{4\kappa t}}\, d\xi$$

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi).G(x-\xi,t)\, d\xi$$

Where $G(x-\xi,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{\frac{-(x-\xi)^2}{4\kappa t}}$ **is called the Green's function (or the fundamental solution) of the diffusion equation.**

ERROR FUNCTION:

The error function is defined by $erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}\,d\eta$

This is a widely used and tabulated function.

Example: Slowing-down of Neutrons Consider the following physical problem

$u_t = u_{xx} + \delta(x)\,\delta(t)$

$u(x,0) = \delta(x)$; $lim_{|x|\to\infty} u(x,t) = 0$

This is the problem of an infinite medium which slows neutrons, in which a source of neutrons is located. Here $u\ (x,\ t)$ **represents the number of neutrons per unit volume per unit time and** $\delta(x)\,\delta(t)$ **represents the source function.**

Solution: The Fourier transformation of equation yields

$U_t + k^2 U = \frac{1}{\sqrt{2\pi}}\delta(t)$ $\therefore \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$ **or** $\mathcal{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}}$

The solution of this, after applying the condition $U(k,0) = \frac{1}{\sqrt{2\pi}}$ **is** $U(k,t) = \frac{1}{\sqrt{2\pi}}e^{-k^2 t}$

Hence, the inverse Fourier transform gives the solution of the problem

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2 t - ikx}dk = \frac{1}{\sqrt{4\pi t}}e^{\frac{-x^2}{4t}}$$

FOURIER COSINE TRANSFORMATION AND INVERSE

Let $f(x)$ **be defined for** $0 \le x < \infty$, **and extended as an even function in** $(-\infty,\infty)$ **satisfying the conditions of Fourier Integral formula. Then, at the points of continuity, the <u>Fourier cosine transform</u> of** $f(x)$ **and its <u>inverse transform</u> are defined by**

$$\mathcal{F}_C\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,Coskxdx$$

$$\mathcal{F}^{-1}{}_C\{F_c(k)\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(k)\,Coskxdk$$

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$$\mathcal{F}^{-1}{}_s\{F_s(k)\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(k)\,Sinkxdk$$

Example: Show that $\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right) \quad ; a > 0$

Solution: We have, by definition

$$\mathcal{F}_C\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\, Coskxdx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[e^{-(a-ik)x} + e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} + \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right) \quad ; a > 0$$

Example: Show that $\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right) \quad ; a > 0$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\, Sinkxdx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)dx = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[e^{-(a-ik)x} - e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} - \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right) \quad ; a > 0$$

Example: Show that $\mathcal{F}_s^{-1}\left\{\frac{1}{k}e^{-sk}\right\} = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right)$

Solution: To prove this we use the standard definite integral

$$\sqrt{\frac{\pi}{2}}\mathcal{F}_s^{-1}\{e^{-sk}\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-sk}\, Sinkxdk = \frac{x}{s^2+x^2}$$

Integrating both sides w.r.to 's' from 's' to '∞'

$$\int_0^{\infty}\frac{e^{-sk}}{k}Sinkxdk = \int_s^{\infty}\frac{xds}{s^2+x^2} = \left|\tan^{-1}\left(\frac{x}{s}\right)\right|_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{s}\right)$$

Consequently

$$\mathcal{F}_s^{-1}\left\{\frac{1}{k}e^{-sk}\right\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-sk}}{k}Sinkxdk = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right)$$

Theorem : Let $f(x)$ **and its first derivative vanish as** $x \to \infty$**. If** $F_c(k)$ **is the Fourier cosine transform, then** $\mathcal{F}_C\{f''(x)\} = -k^2F_c(k) - \sqrt{\frac{2}{\pi}}f'(0)$

PROOF:

Consider $f(x)$ **is real and** $\lim_{x\to\infty}|f(x)| = 0$ **then**

$$\mathcal{F}_C\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f''(x)\, Coskxdx$$

$$\mathcal{F}_C\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[|Coskxf'(x)|_0^{\infty} - \int_0^{\infty} f'(x)(-kSinkx)dx\right]$$

$$\mathcal{F}_C\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[\lim_{x\to\infty}|Coskxf'(x)| - \lim_{x\to 0}|Coskxf'(x)| + k\int_0^\infty f'(x)\,Sinkxdx\right]$$

$$\mathcal{F}_C\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[0 - f'(0) + k\int_0^\infty f'(x)\,Sinkxdx\right]$$

$$\mathcal{F}_C\{f''(x)\} = \left[-\sqrt{\frac{2}{\pi}}f'(0) + k\left\{\sqrt{\frac{2}{\pi}}|Sinkxf(x)|_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,(kCoskx)dx\right\}\right]$$

$$\mathcal{F}_C\{f''(x)\} = \left[-\sqrt{\frac{2}{\pi}}f'(0) + k\left\{\sqrt{\frac{2}{\pi}}|Sinkxf(x)|_0^\infty - k\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,(Coskx)dx\right\}\right]$$

$$\mathcal{F}_C\{f''(x)\} = \left[-\sqrt{\frac{2}{\pi}}f'(0) + k\left\{\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Sinkxf(x)| - \lim_{x\to 0}|Sinkxf(x)|) - kF_c(k)\right\}\right]$$

$$\mathcal{F}_C\{f''(x)\} = -k^2F_c(k) - \sqrt{\frac{2}{\pi}}f'(0)$$

In a similar manner, the Fourier cosine transforms of higher-order derivatives of f (x) can be obtained.

Theorem : Let f (x) and its first derivative vanish as x $\to\infty$. If $F_s(k)$ is the Fourier cosine transform, then $\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}kf(0) - k^2F_s(k)$

PROOF:

Consider $f(x)$ is real and $\lim_{x\to\infty}|f(x)| = 0$ then

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\,Sinkxdx$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[|Sinkxf'(x)|_0^\infty - \int_0^\infty f'(x)\,(kCosx)dx\right]$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[\lim_{x\to\infty}|Sinkxf'(x)| - \lim_{x\to 0}|Sinkxf'(x)| - k\int_0^\infty f'(x)\,Coskxdx\right]$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\left[0 - 0 - k\int_0^\infty f'(x)\,Coskxdx\right]$$

$$\mathcal{F}_s\{f''(x)\} = -k\left[\sqrt{\frac{2}{\pi}}|Coskxf(x)|_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,(-kSinkx)dx\right]$$

$$\mathcal{F}_s\{f''(x)\} = -k\left[\sqrt{\frac{2}{\pi}}\left(\lim_{x\to\infty}|Coskxf(x)| - \lim_{x\to 0}|Coskxf(x)|\right) + k\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,(Sinkx)dx\right]$$

$$\mathcal{F}_s\{f''(x)\} = -k\left[\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Coskxf(x)| - \lim_{x\to 0}|Coskxf(x)|) + kF_s(k)\right]$$

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}kf(0) - k^2F_s(k)$$

In a similar manner, the Fourier sine transforms of higher-order derivatives of $f(x)$ can be obtained.

REMARK:

- $\mathcal{F}\,[f^n(x)] = (-ik)^n \mathcal{F}\,[f\,(x)] = (-ik)^n\, F\,(k)\;\; n = 0, 1, 2, \ldots\ldots\ldots$
- **If** $\mathcal{F}\,\{u_t\} = \mathcal{F}\,\{u_x\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}\,\{u\,(x,t)\} = (-ik)\,\mathcal{F}\,\{u\,(x,t)\}$ **when 'x' varies not 't'**
- **When range of spatial variable is infinite then Fourier transform is used rather than the sine or cosine.**
- **If boundry conditions are of the form** $u(0,t) = value$ **then use Sine transform, while conditions are of the form** $u_x(0,t) = value$ **then use Cosine transform,**

EXAMPLE: Solve the potential equation for the potential $u(x,y)$ **in the semi infinite strip** $0 < x < c;\; y > 0$ **that satisfies the following conditions;**
$u(0,y) = 0\,;\quad u_y(x,0) = 0\;;\; u_x(c,y) = f(y)$

Solution: the potential equation is given as $u_{xx} + u_{yy} = 0\;;\; 0 < x < c;\; y > 0$
Since the BC's are in the form $u_y(x,0) = constant$ **therefor we use fourier cosine transform w.r.to 'y'**

$$\mathcal{F}_C\,\{u_{xx}\} + \mathcal{F}_C\,\{u_{yy}\} = 0 \Rightarrow \frac{d^2}{dx^2}\mathcal{F}_C\,\{u(x,y)\} + \mathcal{F}_C\,\{u_{yy}\} = 0$$

$$\Rightarrow \frac{d^2}{dx^2}U_C\,(x,k) + \left[-k^2 U_c\,(x,k) - \sqrt{\frac{2}{\pi}}\,u_y(x,0)\right] = 0$$

$$\Rightarrow \frac{d^2}{dx^2}U_C\,(x,k) - k^2 U_c\,(x,k) = 0$$

Then general solution will be $U_c(x,k) = c_1 e^{kx} + c_2 e^{-kx}$ **…………(i)**

Now using BC's $u\,(0,y) = 0 \Rightarrow \mathcal{F}_C\{u\,(0,y)\} = 0 \Rightarrow U_c(0,k) = 0$

$(i) \Rightarrow U_c(0,k) = 0 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 = -c_2$

Now $\frac{d}{dx}U_C\,(x,k) = c_1 k e^{kx} - c_2 k e^{-kx}$ **…………(ii)**

using BC's $u_x(c,y) = f(y) \Rightarrow \mathcal{F}_C\{u_x(c,y)\} = f(y) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k)$

$$(ii) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = c_1 k e^{kc} - c_2 k e^{-kc}$$

$$\Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = -c_2 k e^{kc} - c_2 k e^{-kc} \qquad \text{since } c_1 = -c_2$$

$$\Rightarrow F_c(k) = -c_2 k\left(e^{kc} + e^{-kc}\right) \Rightarrow c_2 = -\frac{F_c(k)}{2k\left(\frac{e^{kc}+e^{-kc}}{2}\right)} = -\frac{F_c(k)}{2kCoshkc}$$

$$\Rightarrow c_2 = -\frac{F_c(k)}{2kCoshkc} \Rightarrow c_1 = \frac{F_c(k)}{2kCoshkc} \qquad \text{since } c_1 = -c_2$$

Then $(i) \Rightarrow U_c(x,k) = \frac{F_c(k)}{2kCoshkc}e^{kx} - \frac{F_c(k)}{2kCoshkc}e^{-kx}$

$$U_c(x,k) = \frac{F_c(k)}{kCoshkc}\left(\frac{e^{kx} - e^{-kx}}{2}\right) = \frac{F_c(k)}{kCoshkc}Sinhkx$$

$$\Rightarrow \mathcal{F}_C^{-1}\{U_c(x,k)\} = \mathcal{F}_C^{-1}\left\{\frac{F_c(k)}{kCoshkc}Sinhkx\right\}$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{F_c(k)}{kCoshkc}Sinhkx\,Coskxdk = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc}F_c(k)dk$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f\,(y')\,Cosky'dy'\right]dk$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{SinhkxCoskxCosky'}{kCoshkc} f\,(y')dy'dk$$

EXAMPLE: Solve the problem using Fourier Transformation method
$u_t = u_{xx}$ **with** $u(0,t) = u_0$; $u(x,0) = 0$; $x > 0, t > 0, u_0 > 0$

Solution: BC's suggest that we should use fourier sine transform w.r.to 'x'

$$\mathcal{F}_s\,\{u_t\} = \mathcal{F}_s\,\{u_{xx}\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}_s\,\{u(x,t)\} = \mathcal{F}_s\,\{u_{xx}\}$$

$$\Rightarrow \frac{d}{dt}U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku(0,t) - k^2U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku_0 - k^2U_s\,(k,t)$$

$$\Rightarrow \frac{d}{dt}U_s\,(k,t) + k^2U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku_0 \quad \text{................(i)}$$

This is 1st order, linear, non – homogeneous ODE

Therefore I.F. $= e^{\int k^2dt} = e^{k^2t}$

$$(i) \Rightarrow e^{k^2t}\frac{\partial}{\partial t}U_s\,(k,t) + k^2U_s\,(k,t)e^{k^2t} = \sqrt{\frac{2}{\pi}}ku_0e^{k^2t}$$

$$\Rightarrow \int\frac{d}{dt}e^{k^2t}\,U_sdt = \int\sqrt{\frac{2}{\pi}}ku_0e^{k^2t}dt \quad + \text{Cosntant}$$

$$\Rightarrow e^{k^2t}U_s = \sqrt{\frac{2}{\pi}}ku_0\frac{e^{k^2t}}{k^2} + c \Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} + ce^{-k^2t} \quad \text{................(ii)}$$

Now using IC's $u(x,0) = 0 \Rightarrow \mathcal{F}_s\{u\,(x,0)\} = 0 \Rightarrow U_s(k,0) = 0$

$$(ii) \Rightarrow U_s(k,0) = 0 = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} + ce^0 \Rightarrow c = -\sqrt{\frac{2}{\pi}}\frac{u_0}{k}$$

Thus $(ii) \Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} - \sqrt{\frac{2}{\pi}}\frac{u_0}{k}e^{-k^2t} = \sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)$

$$\Rightarrow \mathcal{F}_s{}^{-1}\{U_s(k,t)\} = \mathcal{F}_s{}^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)\right\}$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)Sinkxdk = \frac{u_0}{k}\frac{2}{\pi}\int_0^\infty\left(1 - e^{-k^2t}\right)Sinkxdk$$

EXAMPLE: Solve the problem using Fourier Transformation method
$u_t = u_{xx}$ **with** $u_x(0,t) = 0$, $u(x,0) = f(x)$; $0 < x < \infty$, $t > 0$

Solution: BC's suggest that we should use fourier cosine transform w.r.to 'x'
$\mathcal{F}_C\{u_t\} = \mathcal{F}_C\{u_{xx}\} \Rightarrow \frac{d}{dt}\mathcal{F}_C\{u(x,y)\} = \mathcal{F}_C\{u_{xx}\}$
$\Rightarrow \frac{d}{dt}U_C(k,t) = \left[-k^2U_c(k,t) - \sqrt{\frac{2}{\pi}}u_x(0,t)\right] = -k^2U_c(k,t) - 0$
$\Rightarrow \frac{d}{dt}U_C(k,t) + k^2U_c(k,t) = 0$ **…………….(i)**
This is 1st order, linear, homogeneous ODE
Then general solution will be $U_c(k,t) = Ae^{-k^2t}$ **………….(ii)**
Now using IC's
$u(x,0) = f(x) \Rightarrow \mathcal{F}_c\{u(x,0)\} = \mathcal{F}_c\{f(x)\} \Rightarrow U_c(k,0) = F_c(k)$
Thus $(i) \Rightarrow U_c(k,0) = F_c(k) = Ae^0 \Rightarrow A = F_c(k)$
$(i) \Rightarrow U_c(k,t) = F_c(k)e^{-k^2t}$
$\Rightarrow \mathcal{F}_C^{-1}\{U_c(k,t)\} = \mathcal{F}_C^{-1}\{F_c(k)e^{-k^2t}\}$
$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)e^{-k^2t}\,Coskxdk$
$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x')\,Coskx'dx'\right]e^{-k^2t}\,Coskxdk$
$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(x')\,Coskx'dx'\right]e^{-k^2t}\,Coskxdk$

Example:: Solve the problem using Fourier Transformation method
$u_{xx} = u_t$; $0 < x < \infty$, $t \geq 0$
with $u(x,0) = e^{-ax^2}$; $u(x), u'(x) \to 0$ as $x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**
$\mathcal{F}\{u_{xx}\} = \mathcal{F}\{u_t\}$
$\Rightarrow (-ik)^2\mathcal{F}\{u(x,t)\} = \frac{d}{dt}\mathcal{F}\{u(x,t)\} \Rightarrow -k^2U(k,t) = \frac{d}{dt}U(k,t)$
$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -k^2 \Rightarrow \int\frac{dU}{U} = -k^2\int dt \Rightarrow lnU = -k^2t + A$
$\Rightarrow U(k,t) = e^{-k^2t+A} \Rightarrow U(k,t) = ce^{-k^2}$ **…………….(i) where** $e^A = c$
Now using IC's
$u(x,0) = e^{-ax^2} \Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{e^{-ax^2}\}$
$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}.e^{-ax^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx-ax^2}dx$

$$\Rightarrow U(k,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(x-\frac{ik}{2a}\right)^2+\frac{k}{4a^2}^2\right]}\,dx$$

$$\Rightarrow U(k,0)=\frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2}\,dx$$

Put $a\left(x-\frac{ik}{2a}\right)^2=P^2 \Rightarrow \sqrt{a}\left(x-\frac{ik}{2a}\right)=P \Rightarrow \sqrt{a}dx=dP \Rightarrow dx=\frac{dP}{\sqrt{a}}$

$$\Rightarrow U(k,0)=\frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2}\,dx=\frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-P^2}\,.\frac{dP}{\sqrt{a}}$$

$$\Rightarrow U(k,0)=\frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}}.\sqrt{\pi} \qquad \therefore \int_{-\infty}^{\infty} e^{-P^2}\,dP=\sqrt{\pi}$$

$$\Rightarrow U(k,0)=\frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)} \quad \ldots\ldots\ldots\ldots\ldots(ii)$$

Consider $ikx-ax^2$

$=-a\left(x^2-\frac{ikx}{a}\right)$

$=-a\left[\left(x^2-\frac{2ikx}{2a}\right)+\left(\frac{ik}{2a}\right)^2-\left(\frac{ik}{2a}\right)^2\right]$

$=-a\left[\left(x-\frac{ik}{2a}\right)^2+\frac{k}{4a^2}^2\right]$

$$(i)\Rightarrow U(k,0)=ce^0 \Rightarrow c=\frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$$

$$\text{Thus } \Rightarrow U(k,t)=\frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}e^{-k^2}=\frac{1}{\sqrt{2a}}e^{-k^2\left(t+\frac{1}{4a}\right)}$$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\}=\mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2a}}e^{-k^2\left(t+\frac{1}{4a}\right)}\right\}$$

$$\Rightarrow u(x,t)=\frac{1}{\sqrt{2a}}.\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-ikx}.e^{-k^2\left(t+\frac{1}{4a}\right)}dk$$

$$\Rightarrow u(x,t)=\frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left\{k^2-\frac{ikx}{\left(t+\frac{1}{4a}\right)}\right\}\right]dk \quad \ldots\ldots\ldots\ldots\ldots(iii)$$

$$\text{Since } k^2-\frac{ikx}{\left(t+\frac{1}{4a}\right)}=k^2-2(k)\left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)+\left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2-\left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2$$

$$k^2-\frac{ikx}{\left(t+\frac{1}{4a}\right)}=\left(k-\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2+\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}$$

$$(iii)\Rightarrow u(x,t)=\frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left(k-\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2\right]e^{-\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}dk$$

$$\Rightarrow u(x,t)=\frac{e^{-\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left(k-\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2\right]dk \quad \ldots\ldots\ldots\ldots(iv)$$

$$\text{Now put } \left(t+\frac{1}{4a}\right)\left(k-\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2=m^2 \Rightarrow \sqrt{\left(t+\frac{1}{4a}\right)}\left(k-\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)=m$$

$$\Rightarrow \sqrt{\left(\mathrm{t}+\frac{1}{4a}\right)}\, dk = dm \Rightarrow dk = \frac{1}{\sqrt{\left(\mathrm{t}+\frac{1}{4a}\right)}}\, dm$$

$$(iv) \Rightarrow u(x,t) = \frac{e^{-\left(\frac{x^2}{4\left(\mathrm{t}+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}.\sqrt{\left(\mathrm{t}+\frac{1}{4a}\right)}}\int_{-\infty}^{\infty} e^{-\mathrm{m}^2}\, dm = \frac{e^{-\left(\frac{x^2}{4\left(\mathrm{t}+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}.\frac{1}{\sqrt{4\mathrm{a}}}\sqrt{4\mathrm{at}+1}}.\sqrt{\pi}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\mathrm{at}+1}}\, e^{-\left(\frac{ax^2}{4\mathrm{at}+1}\right)}$$

Example: Solve the problem using Fourier Transformation method

$u_t(x,t) = \propto^2 u_{xx}(x,t)\ ;\ -\infty < x < \infty\ , t > 0$

with $u_x(x,0) = f(x);\ |u(x,0)| < \infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\,\{u_t\} = \propto^2 \mathcal{F}\,\{u_{xx}\}$

$\Rightarrow \frac{d}{dt}\mathcal{F}\,\{u(x,t)\} = \propto^2 (-ik)^2\mathcal{F}\,\{u(x,t)\} \Rightarrow \frac{d}{dt}U(k,t) = -\propto^2 k^2 U(k,t)$

$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -\propto^2 k^2 \Rightarrow \int \frac{dU}{U} = -\propto^2 k^2 \int dt \Rightarrow lnU = -\propto^2 k^2 t + A$

$\Rightarrow U(k,t) = e^{-\propto^2 k^2 t + A} \Rightarrow U(k,t) = ce^{-\propto^2 k^2}$ ……………(i) **where** $e^A = c$

Now using IC's

$u_x(x,0) = f(x)\ and\ |u(x,0)| < \infty \Rightarrow u(x,0) = f(x)$

$\Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} \quad \Rightarrow U(k,0) = F(k)$

$(i) \Rightarrow U(k,0) = ce^0 \Rightarrow c = F(k)$

Thus $(i) \Rightarrow U(k,t) = F(k)e^{-\propto^2 k^2 t}$

$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\{F(k)e^{-\propto^2 k^2}\}$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,.F(k)e^{-\propto^2 k^2 t}dk$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}\, f(x')dx'\right] e^{-\propto^2 k^2 t}dk$

$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t}dk\right] f(x')dx'$ ……………(iii)

Now consider $I = \int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t}dk$

$I = \int_{-\infty}^{\infty} e^{-iku-\beta k^2}dk$ **put** $x - x' = u$ **and** $\propto^2\ t = \beta$

$I = \int_{-\infty}^{\infty} e^{-\beta\left(k^2+\frac{ik}{\beta}\right)}dk$

$I = \int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{\mathrm{i}u}{2\beta}\right)^2}.\, e^{-\frac{u^2}{4\beta}}dk$

$I = e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{\mathrm{i}u}{2\beta}\right)^2} dk$ ……….(iv)

Consider $k^2 + \frac{\mathrm{ik}}{\beta}u$

$= k^2 + 2\mathrm{k}\left(\frac{\mathrm{i}u}{2\beta}\right) + \left(\frac{\mathrm{i}u}{2\beta}\right)^2 - \left(\frac{\mathrm{i}u}{2\beta}\right)^2$

$= \left(k + \frac{\mathrm{i}u}{2\beta}\right)^2 - \left(\frac{\mathrm{i}u}{2\beta}\right)^2$

$= \left(k + \frac{\mathrm{i}u}{2\beta}\right)^2 + \frac{u^2}{4\beta^2}$

Put $\beta\left(k+\frac{iu}{2\beta}\right)^2 = P^2 \Rightarrow \sqrt{\beta}\left(k+\frac{iu}{2\beta}\right) = P \Rightarrow \sqrt{\beta}dk = dP \Rightarrow dk = \frac{dP}{\sqrt{\beta}}$

$(iv) \Rightarrow I = e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-P^2}\,.\frac{dP}{\sqrt{\beta}} = \frac{dP}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-P^2}\;dp = \frac{1}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}.\sqrt{\pi}$

$(iii) \Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}f(x')dx'$

$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi}.\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\propto^2 t}}e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$

$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi\propto^2 t}.}\int_{-\infty}^{\infty}e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$

Example:

Solve the problem using Fourier Transformation method $u_{xxxx} = \frac{1}{a^2}u_{tt}$
with $u(x,0) = f(x); u_t(x,0) = ag'(x)$ **and** $g, u, u_x, u_{xx}, u_{xxx} \to 0\ as\ x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\{u_{xxxx}\} = \frac{1}{a^2}\mathcal{F}\{u_{tt}\}$

$\Rightarrow (-ik)^4\mathcal{F}\{u(x,t)\} = \frac{1}{a^2}\frac{d^2}{dt^2}\mathcal{F}\{u(x,t)\} \Rightarrow a^2k^4U(k,t) = \frac{d^2}{dt^2}U(k,t)$

$\Rightarrow \frac{d^2}{dt^2}U - a^2k^4U = 0$

$\Rightarrow U(k,t) = Ae^{ak^2t} + Be^{-ak^2t}$ ……………(i)

$\Rightarrow \frac{d}{dt}U(k,t) = Aak^2e^{ak^2t} - Bak^2e^{-ak^2t}$ ……………(ii)

Now using IC's $u(x,0) = f(x) \Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} \Rightarrow U(k,0) = F(k)$

Then $(i) \Rightarrow U(k,0) = Ae^0 + Be^0 \Rightarrow A + B = F(k)$ ……………(iii)

Also $u_t(x,0) = ag'(x) \Rightarrow \mathcal{F}\{u_t(x,0)\} = \mathcal{F}\{ag'(x)\}$

$\Rightarrow \frac{d}{dt}U(k,0) = a(-ik)^1\mathcal{F}\{g'(x)\} \Rightarrow \frac{d}{dt}U(k,0) = -iakG(k)$

Then $(ii) \Rightarrow \frac{d}{dt}U(k,0) = Aak^2e^0 - Bak^2e^0 \Rightarrow -iakG(k) = Aak^2 - Bak^2$

$\Rightarrow -iG(k) = (A-B)k \Rightarrow A - B = -\frac{i}{k}G(k)$ ……………(iv)

Adding (iii) and (iv) $A = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]$

Subtracting (iii) and (iv) $B = \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]$

Then (i) becomes

$\Rightarrow U(k,t) = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]e^{ak^2t} + \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]e^{-ak^2t}$

$\Rightarrow U(k,t) = F(k)\left[\frac{e^{ak^2t}+e^{-ak^2t}}{2}\right] - \frac{i}{k}G(k)\left[\frac{e^{ak^2t}-e^{-ak^2t}}{2}\right]$

$\Rightarrow U(k,t) = F(k)Coshak^2t - \frac{i}{k}G(k)Sinhak^2t$

$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\{F(k)Coshak^2t\} - \mathcal{F}^{-1}\left\{\frac{i}{k}G(k)Sinhak^2t\right\}$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-ikx}\, F(k)Coshak^2tdk - \int_{-\infty}^{\infty} e^{-ikx}\,\frac{i}{k}G(k)Sinhak^2tdk\right]$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, U(k,t)dk$ **is our required solution.**

Example: Solve the problem using Fourier Transformation method
$u_{xx} = \frac{1}{c^2}u_{tt}$ **with** $u(x,0) = p(x); u_t(x,0) = q(x)$ **and** $u, u_x \to 0\ as\ x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\{u_{xx}\} = \frac{1}{c^2}\mathcal{F}\{u_{tt}\}$

$\Rightarrow (-ik)^2\mathcal{F}\{u(x,t)\} = \frac{1}{c^2}\frac{d^2}{dt^2}\mathcal{F}\{u(x,t)\} \Rightarrow -c^2k^2U(k,t) = \frac{d^2}{dt^2}U(k,t)$

$\Rightarrow \frac{d^2}{dt^2}U + c^2k^2U = 0 \Rightarrow U(k,t) = c_1Cosxkt + c_2Sinckt$

$\Rightarrow U(k,t) = c_1\left(\frac{e^{ickt}+e^{-ickt}}{2}\right) + c_2\left(\frac{e^{ickt}-e^{-ickt}}{2}\right)$

$\Rightarrow U(k,t) = \left(\frac{c_1+c_1}{2}\right)e^{ickt} + \left(\frac{c_1-c_1}{2}\right)e^{-ickt}$

$\Rightarrow U(k,t) = Ae^{ickt} + Be^{-ickt}$ **……………(i)**

$\Rightarrow \frac{d}{dt}U(k,t) = Aice^{ickt} - Bice^{-ickt}$ **……………(ii)**

Now using IC's $u(x,0) = p(x) \Rightarrow \mathcal{F}\{u\,(x,0)\} = \mathcal{F}\{p(x)\} \Rightarrow U(k,0) = P(k)$

Then $(i) \Rightarrow U(k,0) = Ae^0 + Be^0 \Rightarrow A + B = P(k)$ **……………(iii)**

Also $u_t(x,0) = q(x) \Rightarrow \mathcal{F}\{u_t\,(x,0)\} = \mathcal{F}\{q(x)\} \Rightarrow \frac{d}{dt}U(k,0) = Q(k)$

Then $(ii) \Rightarrow \frac{d}{dt}U(k,0) = Aicke^0 - Bicke^0$

$\Rightarrow Q(k) = ick(A - B)k \Rightarrow A - B = \frac{1}{ick}Q(k)$ **……………(iv)**

Adding (iii) and (iv) $A = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]$

Subtracting (iii) and (iv) $B = \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]$

Then (i) becomes

$\Rightarrow U(k,t) = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]e^{ickt} + \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]e^{-ickt}$

$\Rightarrow U(k,t) = P(k)\left[\frac{e^{ickt}+e^{-ickt}}{2}\right] + \frac{1}{ick}Q(k)\left[\frac{e^{ickt}-e^{-ickt}}{2}\right]$

$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} =$
$\frac{1}{2}\left[\mathcal{F}^{-1}\{P(k)e^{ickt}\} + \mathcal{F}^{-1}\{P(k)e^{-ickt}\}\right] + \frac{1}{2ick}\mathcal{F}^{-1}\{Q(k)(e^{ickt} - e^{-ickt})\}$ **…..(A)**

$$\mathcal{F}^{-1}\{P(k)e^{ickt}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, P(k)e^{ickt}dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(x-ct)k}\, P(k)dk$$

$$\mathcal{F}^{-1}\{P(k)e^{ickt}\} = P(x-ct)$$

Similarly $\quad \mathcal{F}^{-1}\{P(k)e^{-ickt}\} = P(x+ct)$

And consider $\quad q(x) = \mathcal{F}^{-1}\{Q(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, Q(k)dk$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{x-ct}^{x+ct}\int_{-\infty}^{\infty} e^{-ikx}\, Q(k)dkdx$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{x-ct}^{x+ct} e^{-ikx'}dx'Q(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{e^{-ikx'}}{-ik}\right|_{x-ct}^{x+ct} Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{-ik}\left[e^{-ik(x+ct)} - e^{-ik(x-ct)}\right] Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{ik}\left[e^{-ik(x-ct)} - e^{-ik(x+ct)}\right] Q(k)dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[e^{ickt} - e^{-ickt}\right]\frac{Q(k)}{k}dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\mathcal{F}^{-1}\left\{\left(e^{ickt} - e^{-ickt}\right)\frac{Q(k)}{k}dk\right\}$$

$$(A) \Rightarrow u(x,t) = \frac{1}{2}[P(x+ct) + P(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} q(x')dx'$$

THE DOUBLE FOURIER TRANSFORM AND ITS INVERSE

Let $f(x_1, x_2)$ be a function defined over the whole plane i.e. $-\infty < x_1, x_2 < \infty$ then its fourier transform and inverse are defined as follows;

$$\mathcal{F}\{f(x_1,x_2)\} = F(k_1,k_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)e^{i(k_1x_1+k_2x_2)}\, dx_1dx_2$$

$$\mathcal{F}^{-1}\{F(k_1,k_2)\} = f(x_1,x_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1,k_2)e^{-i(k_1x_1+k_2x_2)}\, dk_1dk_2$$

n - DIMENSIONAL FOURIER TRANSFORM AND ITS INVERSE

$$\mathcal{F}\{f(\textstyle\sum_{i=1}^{n} x_i)\} = F(\textstyle\sum_{i=1}^{n} k_i) = \frac{1}{(\sqrt{2\pi})^n}\int_{all\ space} f(\textstyle\sum_{i=1}^{n} x_i)e^{i(\sum_{i=1}^{n} k_ix_i)}\, d\textstyle\sum_{i=1}^{n} x_i$$

$$\mathcal{F}^{-1}\{F(\textstyle\sum_{i=1}^{n} k_i)\} = f(\textstyle\sum_{i=1}^{n} x_i) =$$

$$\frac{1}{(\sqrt{2\pi})^n}\int_{all\ space} F(\textstyle\sum_{i=1}^{n} k_i)e^{-i(\sum_{i=1}^{n} k_ix_i)}\, d\textstyle\sum_{i=1}^{n} k_i$$

EXAMPLE : Find the temperature distribution in a semi-infinite rod for the following cases with zero initial temperature distribution:
(a) The heat supplied at the end $x = 0$ at the rate $g(t)$; i.e. $u_x(0,t) = g(t)$
(b) The end $x = 0$ is kept at a constant temperature T_0. i.e. $u(0,t) = T_0, t \geq 0$
The problem here is to solve the heat conduction equation
$u_t = \kappa u_{xx}, \quad x > 0, t > 0, \quad \text{with} \quad u(x,0) = 0, x > 0.$
Here we assume that $u(x, t)$ and $u_x(x, t)$ vanish as $x \to \infty$.

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Here we assume that $u(x, t)$ and $u_x(x, t)$ vanish as $x \to \infty$.

SOLUTION:

let $U(k,t)$ be the Fourier cosine transform of $u(x,t)$. Then the transformation of the heat conduction equation yields

$$\mathcal{F}_C\{u_t\} = \kappa \mathcal{F}_C\{u_{xx}\}$$

$$\frac{\partial}{\partial t}\mathcal{F}_C\{u(x,t)\} = \kappa\left[-k^2 U_c(k,t) - \sqrt{\frac{2}{\pi}}u_x(o,t)\right]$$

$$\frac{\partial}{\partial t}U_c(k,t) = \kappa\left[-k^2 U_c(k,t) - \sqrt{\frac{2}{\pi}}g(t)\right]$$

$$\frac{\partial}{\partial t}U_c(k,t) + \kappa k^2 U_c(k,t) = -\sqrt{\frac{2}{\pi}}g(t)\,\kappa \quad \ldots\ldots\ldots\ldots (i)$$

This is linear differentiable and non – homogeneous equation so by using

$$I.F = e^{\int \kappa k^2 dt} = e^{\kappa k^2 t}$$

$$(i) \Rightarrow e^{\kappa k^2 t}\frac{\partial}{\partial t}U_c(k,t) + e^{\kappa k^2 t}\kappa k^2 U_c(k,t) = -e^{\kappa k^2 t}\sqrt{\frac{2}{\pi}}g(t)\,\kappa$$

$$\int \frac{d}{dt}e^{\kappa k^2 t}U_c(k,t)dt = -\sqrt{\frac{2}{\pi}}\int e^{\kappa k^2 t}g(t)dt + c$$

$$U_c(k,t) = -\sqrt{\frac{2}{\pi}}e^{-\kappa k^2 t}\int_0^t e^{\kappa k^2 t'}g(t')dt' + ce^{-\kappa k^2 t} \quad \ldots\ldots\ldots\ldots (ii)$$

Now using $u(x,0) = 0 \Rightarrow U_c(k,0) = 0 \Rightarrow c = 0$

$$(ii) \Rightarrow U_c(k,t) = -\sqrt{\frac{2}{\pi}}e^{-\kappa k^2 t}\int_0^t e^{\kappa k^2 \tau}g(\tau)d\tau$$

$$\mathcal{F}^{-1}{}_C\{U_c(k,t)\} = u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\left[-\sqrt{\frac{2}{\pi}}e^{-\kappa k^2 t}\int_0^t e^{\kappa k^2 \tau}g(\tau)d\tau\right]Coskxdk$$

$$u(x,t) = -\frac{2}{\pi}\int_0^t g(\tau)d\tau\int_0^\infty e^{-\kappa k^2(t-\tau)}\,Coskxdk$$

$$u(x,t) = -\sqrt{\frac{\kappa}{\pi}}\int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}e^{-x^2/4\kappa(t-\tau)}d\tau$$

where $\int_0^\infty e^{-\kappa k^2(t-\tau)}\,Coskxdk = \frac{1}{2}\sqrt{\frac{\pi}{\kappa(t-\tau)}}e^{-x^2/4\kappa(t-\tau)}$

EXAMPLE : Find the temperature distribution in a semi-infinite rod for the following cases with zero initial temperature distribution:
The end $x = 0$ **is kept at a constant temperature** T_0**. i.e.** $u(0,t) = T_0, t \geq 0$
The problem here is to solve the heat conduction equation
$u_t = \kappa u_{xx}$, $x > 0, t > 0$, with $u(x,0) = 0, x > 0$
Here we assume that $u(x, t)$ **and** $u_x(x, t)$ **vanish as** $x \to \infty$
SOLUTION:
We apply the Fourier sine transform $U(k,t)$ **of** $u(x,t)$ **to obtain the transformed equation**

$$\mathcal{F}_s\{u_t\} = \kappa \mathcal{F}_s\{u_{xx}\}$$

$$\frac{\partial}{\partial t}\mathcal{F}_s\{u(x,t)\} = \kappa\left[-k^2 U_s(k,t) + \sqrt{\frac{2}{\pi}}\, k u(o,t)\right]$$

$$\frac{\partial}{\partial t}U_s(k,t) = \kappa\left[-k^2 U_s(k,t) + \sqrt{\frac{2}{\pi}}\, kT_0\right]$$

$$\frac{\partial}{\partial t}U_s(k,t) + \kappa k^2 U_s(k,t) = \sqrt{\frac{2}{\pi}}\, T_0\, k\kappa \quad \ldots\ldots\ldots\ldots (i)$$

This is linear differentiable and non – homogeneous equation so by using

$$I.F = e^{\int \kappa k^2 dt} = e^{\kappa k^2 t}$$

$$(i) \Rightarrow e^{\kappa k^2 t}\frac{\partial}{\partial t}U_s(k,t) + e^{\kappa k^2 t}\kappa k^2 U_s(k,t) = e^{\kappa k^2 t}.\sqrt{\frac{2}{\pi}}\, T_0\, k\kappa$$

$$\int \frac{d}{dt} e^{\kappa k^2 t} U_s(k,t) dt = \sqrt{\frac{2}{\pi}} \int e^{\kappa k^2 t} T_0\, k\kappa dt + c$$

$$e^{\kappa k^2 t} U_s(k,t) = \sqrt{\frac{2}{\pi}} \int e^{\kappa k^2 t} T_0\, k\kappa dt + c = \sqrt{\frac{2}{\pi}}\frac{e^{\kappa k^2 t}}{k} T_0 \kappa + c$$

$$U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{T_0}{k}\kappa + c e^{-\kappa k^2 t} \quad \ldots\ldots\ldots\ldots (ii)$$

Now using $u(x,0) = 0 \Rightarrow U_s(k,0) = 0 \Rightarrow c = -\sqrt{\frac{2}{\pi}}\frac{T_0}{k}\kappa$

$$(ii) \Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{T_0}{k}\kappa - \sqrt{\frac{2}{\pi}}\frac{T_0}{k}\kappa e^{-\kappa k^2 t} = T_0.\sqrt{\frac{2}{\pi}}\frac{1-e^{-\kappa k^2 t}}{k}\kappa$$

$$\mathcal{F}^{-1}{}_s\{U_s(k,t)\} = u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[T_0.\sqrt{\frac{2}{\pi}}\frac{1-e^{-\kappa k^2 t}}{k}\kappa\right] Sinkxdk$$

$$u(x,t) = \frac{2T_0}{\pi}\int_0^\infty \frac{Sinkx}{k}\left(1 - e^{-\kappa k^2 t}\right) dk$$

Now making the use of integral

$$\int_0^\infty e^{-a^2x^2}\left(\frac{Sinkx}{k}\right) dk = \frac{\pi}{2} erf\left(\frac{x}{2a}\right)$$

Then the solution will be

$$u(x,t) = \frac{2T_0}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2} erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] = T_0 erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)$$

Where $erfc(x) = 1 - erf(x)$ **is the complementary error function.Defiend as follows**

$$erf\,c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\eta^2} d\eta$$

Example : Find the solution of the Dirichlet problem in the half-plane $y > 0$

$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, y > 0$ with $u(x,0) = f(x), -\infty < x < \infty,$

u **and** u_x **vanish as** $|x| \to \infty$ **and** u **is bounded as** $y \to \infty$

SOLUTION:

Let $U(k,y)$ be the Fourier transform of $u(x,y)$ with respect to x.

$\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{yy}\} = 0$

$(-ik)^2\mathcal{F}\{u(x,y)\} + \frac{\partial^2}{\partial y^2}\mathcal{F}\{u(x,y)\} = 0$

Application of the Fourier transform with respect to x gives

$U_{yy} - k^2U = 0$

$U(k,0) = F(k)$ **and** $U(k,y) \to 0$ **as** $y \to \infty$

The solution of this transformed system is

$U(k,y) = F(k)\, e^{-|k|y}$

The inverse Fourier transform of $U(k,y)$ gives the solution in the form

$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\, e^{-|k|y}e^{ik\xi}\, d\xi\right]e^{-ikx}dk$

$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{-\infty}^{\infty} f(\xi)\, e^{-k[i(x-\xi)]-|k|y}\, dk$

It follows from the proof of the result $\mathcal{F}\{e^{-a|x|}\} = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}$ $\quad ; a > 0$

$\int_{-\infty}^{\infty} f(\xi)\, e^{-k[i(x-\xi)]-|k|y}\, dk = \frac{2y}{(\xi-x)^2+y^2}$

Hence, the solution of the Dirichlet problem in the half-plane $y > 0$ is

$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(\xi-x)^2+y^2}$

From this solution, we can readily deduce a solution of the Neumann problem in the half-plane $y > 0$.

Example: Find the solution of Neumann's problem in the half-plane $y > 0$

$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, y > 0$ with $u_y(x,0) = g(x), -\infty < x < \infty$

u **is bounded as** $y \to \infty$, u**and** u_x **vanish as** $|x| \to \infty$

SOLUTION:

Let $v(x,y) = u_y(x, y)$. **Then** $u(x,y) = \int^{y} v(x,\eta)d\eta$

and the Neumann problem becomes

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}(u_{xx} + u_{yy}) = 0$ and $v(x,0) = u_y(x,0) = g(x)$

This is the Dirichlet problem for $v(x,y)$, and its solution is given by

$v(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{g(\xi)d\xi}{(\xi-x)^2+y^2}$

Thus, we have

$u(x,y) = \frac{1}{\pi}\int^{y}\eta\int_{0}^{\infty}\frac{g(\xi)d\xi}{(\xi-x)^2+\eta^2}d\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)d\xi\int^{y}\frac{2\eta d\eta}{(\xi-x)^2+\eta^2}$

$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\log[(\xi-x)^2+y^2]d\xi$

where an arbitrary constant can be added to this solution. In other words, the solution of any Neumann's problem is uniquely determined up to an arbitrary constant.

EXERCISES

1. Determine the solution of the initial-value problem

$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, t > 0,$

$u(x,0) = f(x), \; u_t(x,0) = g(x), -\infty < x < \infty$

2. Solve $u_t = u_{xx}, \quad x > 0, t > 0$

$u(x,0) = f(x), u(0,t) = 0$

3. Solve $u_{tt} = c^2 u_{xxxx} = 0, \quad -\infty < x < \infty, t > 0$

$u(x,0) = f(x), \; u_t(x,0) = 0, -\infty < x < \infty$

4. Solve $u_{tt} + c^2 u_{xxxx} = 0, \quad x > 0, t > 0,$

$u(x,0) = 0, \; u_t(x,0) = 0, x > 0$

$u(0, t) = g(t), \; u_{xx}(0, t) = 0, \; t > 0.$

5. Solve $u_t = u_{xx} + tu, \quad -\infty < x < \infty, t > 0$

$u(x,0) = f(x), u(x,t)$ **is bounded,** $-\infty < x < \infty.$

6. Solve $u_{xx} + u_{yy} = 0, \quad 0 < x < \infty, 0 < y < \infty$

$u(x,0) = f(x), 0 \le x < \infty$ and $u_x(0,y) = g(y), 0 \le y < \infty$

$u(x,y) \to 0$ **uniformly in** x **as** $x \to \infty$ **and uniformly in** y **as** $x \to \infty$

7. Solve $u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, 0 < y < a$

$u(x,0) = f(x), u(x,a) = 0, -\infty < x < \infty$ and $u(x, y) \to 0$ **uniformly in** y **as** $|x| \to \infty$

8. Solve $u_t = u_{xx}, \quad x > 0, t > 0$

$u(x,0) = 0, x > 0, u(0,t) = f(t), t > 0,$

$u(x, t)$ **is bounded for all** x **and** t.

9. Solve $u_{xx} + u_{yy} = 0, \quad x > 0, 0 < y < 1$

$u(x,0) = f(x), u(x,1) = 0, x > 0,$

$u(0,y) = 0, u(x,y) \to 0$ **uniformly in** y **as** $x \to \infty$

10. Solve $u_{xx} + u_{yy} = 0, \quad x > 0, 0 < y < 1$

11. Solve using Fourier Transformation $u_{xx} = \frac{1}{\kappa} u_t$ **Semi infinite rod with** $u(0,t) = 0$

and $u(x,0) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \text{elswhere} \end{cases}$

FOURIER SERIES, FOURIER TRANSFORMATION AND INTEGRALS WITH APPLICATIONS

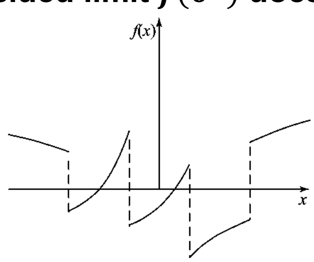
Introduction, Piecewise Continuous Functions and Periodic Functions, Systems of Orthogonal Functions, Fourier Series, Convergence of Fourier Series, Examples and Applications of Fourier Series , Examples and Applications of Cosine and Sine Fourier
Series, Complex Fourier Series , Fourier Series on an Arbitrary Interval, The Riemann–Lebesgue Lemma and Point wise Convergence Theorem, Uniform Convergence, Differentiation, and Integration , Double Fourier Series, Fourier Integrals , Exercises

The Fourier theory of trigonometric series is of great practical importance because certain types of discontinuous functions which cannot be expanded in power series can be expanded in Fourier series. More importantly, a wide class of problems in physics and engineering possesses periodic phenomena and, as a consequence, Fourier's trigonometric series become an indispensable tool in the analysis of these problems.

PIECEWISE CONTINUOUS FUNCTIONS AND PERIODIC FUNCTIONS

A single-valued function f is said to be piecewise continuous in an interval $[a, b]$ if there exist finitely many points $a = x_1 < x_2 < \ . \ . \ . < x_n = b$, such that f is continuous in the intervals $x_j < x < x_{j+1}$ and the one-sided limits $f(x_j+)$ and $f(x_{j+1}-)$ exist for all $j = 1, 2, 3, \ldots, n - 1$.

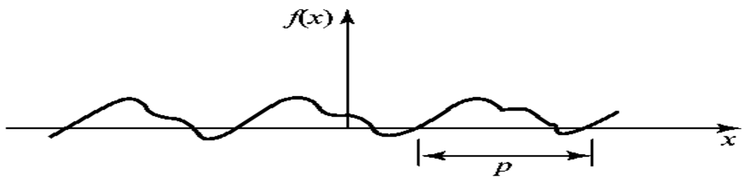
A piecewise continuous function is shown in Figure. Functions such as $\frac{1}{x}$ and $\sin\left(\frac{1}{x}\right)$ fail to be piecewise continuous in the closed interval $[0, 1]$ because the one-sided limit $f(0^+)$ does not exist in either case.



If f is piecewise continuous in an interval $[a, b]$, then it is necessarily bounded and integrable over that interval. Also, it follows immediately that the product of two piecewise continuous functions is piecewise continuous on a common interval.

If f is piecewise continuous in an interval $[a, b]$ and if, in addition, the first derivative f' is continuous in each of the intervals $x_j < x < x_{j+1}$ and the limits $f'(x_j+)$ and $f'(x_j-)$ exist, then f is said to be piecewise smooth; if, in addition, the second derivative f'' is continuous in each of the intervals $x_j < x < x_{j+1}$, and the limits $f''(x_j+)$ and $f''(x_j-)$ exist, then f is said to be piecewise very smooth.

PERIODIC FUNCTION: A piecewise continuous function $f(x)$ in an interval [a, b] is said to be periodic if there exists a real positive number p such that $f(x+p) = f(x)$ for all x, p is called the period of f, and the smallest value of p is termed the fundamental period. A sample graph of a periodic function is given in Figure



If f is periodic with period p, then

$f(x+p) = f(x)$

$f(x+2p) = f(x+p+p) = f(x+p)$

$f(x+3p) = f(x+2p+p) = f(x+2p)$

$f(x+np) = f(x+(n-1)p+p) = f(x+(n-1)p) = f(x)$

for any integer n. Hence, for all integral values of n $f(x+np) = f(x)$

It can be readily shown that if $f_1, f_2, \ldots, f_k$ have the period p and c_k are the constants, then $f = c_1f_1 + c_2f_2 + \ldots + c_kf_k$ has the period p.

SYSTEMS OF ORTHOGONAL FUNCTIONS

A sequence of functions $\{\phi_n(x)\}$ is said to be orthogonal with respect to the weight function $q(x)$ on the interval [a, b] if $\int_a^b \varphi_m(x)\varphi_n(x)q(x)dx = 0 \quad ; m \neq n$

If m = n then we have $\|\varphi_n\| = \sqrt{\int_a^b \varphi_n^2(x)q(x)dx}$

which is called the norm of the orthogonal system $\{\phi_n(x)\}$.

Example: The sequence of functions $\{\sin mx\}, m = 1, 2, \ldots,$ form an orthogonal system on the interval $[-\pi, \pi]$, because $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & ; m \neq n \\ \pi & ; m = n \end{cases}$

In this example we notice that the weight function is equal to unity, and the value of the norm is $\sqrt{\pi}$

ORTHONORMAL SYSTEM OF FUNCTIONS

An orthogonal system $\phi_1, \phi_2, \ldots, \phi_n$, where n may be finite or infinite, which satisfies the relations $\int_a^b \varphi_m(x)\varphi_n(x)q(x)dx = \begin{cases} 0 & ; m \neq n \\ 1 & ; m = n \end{cases}$ is called an orthonormal system of functions on [a, b].

It is evident that an orthonormal system can be obtained from an orthogonal system by dividing each function by its norm on [a, b].

Example: The sequence of functions $1, cos\ x, sinx, \ldots, cos\ nx, sin\ nx$ forms an orthogonal system on $[-\pi, \pi]$ since

$$\int_{-\pi}^{\pi} \text{sinmx sin nx dx} = \begin{cases} 0 & ; m \neq n \\ \pi & ; m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \text{sinmx cos nx d} = 0, \text{for all m, n}$$

$$\int_{-\pi}^{\pi} \text{cosmx cos nx dx} = \begin{cases} 0 & ; m \neq n \\ \pi & ; m = n \end{cases}$$

FOURIER SERIES

A trigonometric series with any piecewise continuous periodic function $f(x)$ of period 2π and of the form

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \text{kx} + b_k \sin \text{kx})$$

is called the Fourier Series of a real valued function $f(x)$ where the symbol $\sim$ indicates an association of a_0, a_k, and b_k to f in some unique manner.

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx$

And are called Fourier Coefficiets.

We may also write

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \text{kx} + b_k \sin \text{kx})$$

COMPLEX FORM OF FOURIER SERIES

Fourier Series expansion for in complex form is given as follows

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad ; \quad -\pi < x < \pi$$

Where $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$

Example:Find the Fourier series expansion for the function
$f(x) = x + x^2, \quad -\pi < x < \pi$

Solultion: Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2\pi^2}{3}$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{4}{k^2}Cosk\pi = \frac{4}{k^2}(-1)^k \ ; \ k = 1, 2, 3, \ldots \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = -\frac{2}{k}Cosk\pi = -\frac{2}{k}(-1)^k \ ; \ k = 1, 2, 3, \ldots \ldots$$

Therefore, the Fourier series expansion for f is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \text{kx} + b_k \sin \text{kx})$$

$$f(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(\frac{4}{k^2}(-1)^k \cos \text{kx} - \frac{2}{k}(-1)^k \sin \text{kx})$$

$$f(x) = \frac{\pi^2}{3} - 4\cos x + 2sinx + \cos 2x - \sin 2x - \ldots.$$

Example: Find the Fourier series expansion for the function

$f\,(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; 0 < x < \pi \end{cases}$

Solultion: Here

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] = -\frac{\pi}{2}$

$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Coskxdx + \int_{0}^{\pi} f(x)Coskxdx\right]$

$a_k = \frac{1}{k^2\pi}(Cosk\pi - 1) = \frac{1}{k^2\pi}\left[(-1)^k - 1\right]\ ;\ \ k = 1, 2, 3, \dots \dots$

$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Sinkxdx + \int_{0}^{\pi} f(x)Sinkxdx\right]$

$b_k = \frac{1}{k}(1 - 2Cosk\pi) = \frac{1}{k}\left[1 - 2(-1)^k\right];\ \ k = 1, 2, 3, \dots \dots$

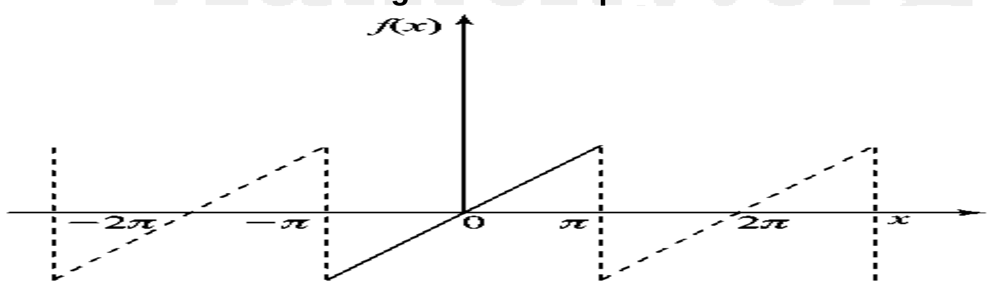
Therefore, the Fourier series expansion for f is

$f\,(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$

$f\,(x) = -\frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{1}{k^2\pi}\left[(-1)^k - 1\right]\cos kx + \frac{1}{k}\left[1 - 2(-1)^k\right]\sin kx)\right]$

Example: Find the Fourier series expansion for the sawtooth wave function $f\,(x) = x$ in the interval $-\pi < x < \pi, f\,(x) = f\,(x \pm 2k\pi)$ for $k = 1, 2, \dots$.

Solultion: This is a periodic function with period 2π and represents a sawtooth wave function as shown in Figure and it is piecewise continuous.



$a_k = 0$ **for** $k = 0, 1, 2, \dots$

$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} xSinkxdx = \frac{2}{k}(-1)^{k+1}$

Therefore, the Fourier series expansion for f is

$f\,(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$

$f\,(x) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{Sinkx}{k} = 2\left(\frac{Sinx}{1} - \frac{Sin2x}{2} + \frac{Sin3x}{3} - \frac{Sin4x}{4} + \cdots \dots \dots\right)$

FOURIER INVERSION FORMULA

The proper inversion formula is given as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{iwx}}\, \hat{f}(w)\, \mathrm{dw}$$

The formula nearly states that f is the fourier transform of $\hat{f}$

where $\hat{f} = \mathcal{F}\{f\ (x)\}$

PROOF:

If $f(x)$ is defined in the interval $(-l, l)$ then it can be represented by its complex fourier series as

$f\ (x) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{i\frac{k\pi x}{l}}$ **Where** $c_k = \frac{1}{2l}\int_{-l}^{l} f(y)\mathrm{e}^{-i\frac{\pi y}{l}} dy$

Let $\hat{f}_l(w) = \frac{1}{2l}\int_{-l}^{l} f(y)\mathrm{e}^{-iwy} dy$

Then for sufficiently regular function we clearly have $\hat{f}_l(w) \to \hat{f}(w)$ as $l \to \infty$

Furthermore $f\ (x) = \frac{1}{2l}\sum_{k=-\infty}^{\infty} \hat{f}_l(w_k)\mathrm{e}^{iw_k x}$ **.............(i) where** $w_k = \frac{k\pi}{l}$

Let $\Delta w_k = \frac{\pi}{l}$ denotes the distance between grid points i.e. $\{w_k = k(\Delta w)\}_{k=-\infty}^{\infty}$ defines a uniform partition of the real line. Therefore it is more convenient to rewrite (i) in the form

$$f\ (x) = \frac{1}{2\pi}\left[\Delta w \sum_{k=-\infty}^{\infty} \hat{f}_l(w_k)\mathrm{e}^{iw_k x}\right]$$

We observe that this formula resembles the inversion formula.

Of course if $l \to \infty$ then $\Delta w \to 0$

Hence if $l \to \infty$ then $\left[\Delta w \sum_{k=-\infty}^{\infty} \hat{f}_l(w_k)\mathrm{e}^{iw_k x}\right] \to \int_{-\infty}^{\infty} e^{\mathrm{iwx}}\, \hat{f}(w)\, \mathrm{dw}$

Thus we obtained our required formula as follows

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{iwx}}\, \hat{f}(w)\, \mathrm{dw}$$

LAPLACE TRANSFORMATION WITH APPLICATIONS

Because of their simplicity, Laplace transforms are frequently used to solve a wide class of partial differential equations. Like other transforms, Laplace transforms are used to determine particular solutions. In solving partial differential equations, the general solutions are difficult, if not impossible, to obtain. The transform technique sometimes offers a useful tool for finding particular solutions. The Laplace transform is closely related to the complex Fourier transform, so the Fourier integral formula can be used to define the Laplace transform and its inverse.

LAPLACE TRANSFORMATION

If $f(t)$ is defined for all values of $t > 0$, then the Laplace transform of $f(t)$ is denoted by $\bar{f}(s)$ or $\mathcal{L}\{f(t)\}$ and is defined by the integral

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{h\to\infty} \int_0^h e^{-st} f(t)dt$$

If $\bar{f}(s)$ is laplace transform of $f(t)$ then $f(t)$ is called the <u>INVERSE LAPLACE TRANSFORM</u> of $\bar{f}(s)$ i.e. $\quad \mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$

QUESTION: Show that $\mathcal{L}\{c\} = \frac{c}{s}$ where 'c' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{c\} = \int_0^\infty e^{-st}\, c dt = c\int_0^\infty e^{-st}\, dt = c\left|-\frac{e^{-st}}{s}\right|_0^\infty = \frac{c}{s}$

QUESTION: Show that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ where 'a' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left|-\frac{e^{-(s-a)t}}{(s-a)}\right|_0^\infty = \frac{1}{s-a}$

QUESTION: Show that $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ where 'a' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\, t^2\, dt = \left|-\frac{t^2 e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} 2t\, dt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \frac{2}{s}\int_0^\infty \frac{e^{-st}}{s} dt = \frac{2}{s^3}$

In above $t^2 e^{-st}, te^{-st} \to 0$ **as** $t \to \infty$

QUESTION: Show that $\mathcal{L}\{Sinwt\} = \frac{w}{s^2+w^2}$ where 'a' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{Sinwt\} = \int_0^\infty e^{-st}\, Sinwtdt = \left|-\frac{Sinwt.e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} wCoswt\, dt$

$\mathcal{L}\{Sinwt\} = \frac{w}{s}\left|-\frac{Coswte^{-st}}{s}\right|_0^\infty - \frac{w}{s}\int_0^\infty \frac{e^{-st}}{s} wSinwtdt = \frac{w}{s^2} - \frac{w^2}{s^2}\mathcal{L}\{Sinwt\} = \frac{w}{s^2+w^2}$

QUESTION: Show that $\mathcal{L}\{c\} = \frac{c}{s}$ where 'c' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{c\} = \int_0^\infty e^{-st}\, cdt = c\int_0^\infty e^{-st}\, dt = c\left|-\frac{e^{-st}}{s}\right|_0^\infty = \frac{c}{s}$

QUESTION: Show that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ where 'a' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at} dt = \int_0^\infty e^{-(s-a)t}\, dt = \left|-\frac{e^{-(s-a)t}}{(s-a)}\right|_0^\infty = \frac{1}{s-a}$

QUESTION: Show that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ where '$n > 0$'

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Then for n = 1;

$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, tdt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} dt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s}$

In above $te^{-st} \to 0$ as $t \to \infty$

for n = 2;

$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\, t^2\, dt = \left|-\frac{t^2 e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} 2t\, dt = \left|-\frac{t^2 e^{-st}}{s}\right|_0^\infty +$
$\frac{2}{s}\int_0^\infty e^{-st}\, dt = \frac{2}{s^3}$ In this part $t^2 e^{-st}, te^{-st} \to 0$ as $t \to \infty$

And in general

$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt = \left|-\frac{t^n e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} nt^{n-1}\, dt$

$\mathcal{L}\{t^n\} = \left|-\frac{t^n e^{-st}}{s}\right|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\} =$
$\frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}\ldots\ldots\ldots..\frac{2}{s}.\frac{1}{s}\mathcal{L}\{t^0\}$

$\mathcal{L}\{t^n\} = \frac{(n-1)(n-1)(n-1)\ldots\ldots\ldots.3.2.1}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^n}.\frac{1}{s}$

Hence $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ where '$n > 0$'

QUESTION: Show that $\mathcal{L}\{Sinat\} = \frac{a}{s^2+a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Sinat\} = \int_0^\infty e^{-st}\, Sinatdt$

$\therefore \int_0^\infty e^{at}\, Sinbtdt = \frac{e^{at}}{a^2+b^2}[aSinbt - bCosbt]$ therefore

$$\mathcal{L}\{Sinat\} = \left|\frac{e^{-st}}{s^2+a^2}[-sSinat - aCosat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-a)\right] = \frac{a}{s^2+a^2}$$

QUESTION: Show that $\mathcal{L}\{Cosat\} = \frac{s}{s^2+a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Cosat\} = \int_0^\infty e^{-st}\, Cosatdt$

$\therefore \int_0^\infty e^{at}\, Cosbtdt = \frac{e^{at}}{a^2+b^2}[aCosbt + bSinbt]$ therefore

$$\mathcal{L}\{Cosat\} = \left|\frac{e^{-st}}{s^2+a^2}[-sCosat + aSinat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-s)\right] = \frac{s}{s^2+a^2}$$

QUESTION: Show that $\mathcal{L}\{Sinhat\} = \frac{a}{s^2-a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Sinhat\} = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\left[\int_0^\infty e^{-st}\, e^{at}dt - \int_0^\infty e^{-st}\, e^{-at}dt\right]$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{a}{s^2-a^2}$$

QUESTION: Show that $\mathcal{L}\{Coshat\} = \frac{s}{s^2-a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then

$$\mathcal{L}\{Coshat\} = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{1}{2}\left[\int_0^\infty e^{-st}\, e^{at}dt + \int_0^\infty e^{-st}\, e^{-at}dt\right]$$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt + \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{s}{s^2-a^2}$$

FUNCTION OF EXPONENTIAL ORDER: A function $f\ (t)$ **is said to be <u>of exponential order</u> as** $t\ \to\ \infty$ **if there exist real constants** M **and** a **such that** $|f\ (t)|\ \le\ Me^{at}$ **for** $0\ \le\ t\ <\ \infty$.

THEOREM: Let f **be piecewise continuous in the interval** $[0,T]$ **for every positive** T**, and let** f **be of exponential order, that is,** $f\ (t)\ =\ O\ (e^{at})$ **as** $t\ \to\ \infty$ **for some** $a\ >\ 0$**. Then, the Laplace transform of** $f\ (t)$ **exists for** $Res\ >\ a$**.**

Proof: Since f **is piecewise continuous and of exponential order, we have**
$|\mathcal{L}\{f(t)\}| = \left|\int_0^{\infty} e^{-st}\ f(t)dt\right| \le \int_0^{\infty} e^{-st}\ |f(t)|dt \le \int_0^{\infty} e^{-st}\ Me^{at}\ dt = M\int_0^{\infty} e^{-(s-a)t}\ dt$
$|\mathcal{L}\{f(t)\}| \le \frac{M}{s-a}$
Thus the Laplace transform of $f\ (t)$ **exists for** $Res\ >\ a$**.**

PROPERTIES OF LAPLACE TRANSFORMS

LINEARITY PROPERTY: THE LAPLACE TRANSFORMATION $\mathcal{L}$ **IS LINEAR.**

Proof. Let $u\ (t) = af(t) + bg(t)$ where a and b are constants.

We have, by definition
$\mathcal{L}\ \{u\ (t)\} = \int_0^{\infty} e^{-st}\ f(t)dt = \int_0^{\infty} e^{-st}\ [af(t) + bg(t)]dt$
$\mathcal{L}\ \{u\ (t)\} = a\int_0^{\infty} e^{-st}\ f(t)dt + b\int_0^{\infty} e^{-st}\ g(t)dt = a\mathcal{L}\ \{f(t)\} + b\mathcal{L}\ \{g(t)\}$
$\mathcal{L}\ \ \{af(t) + bg(t)\} = a\mathcal{L}\ \{f(t)\} + b\mathcal{L}\ \{g(t)\}$ **hence proved.**

SHIFTING PROPERTY: If $\bar{f}(s)$ **is the laplace transformation of** $f(t)$
Then $\qquad \mathcal{L}\{e^{at}\ f(t)\} = \bar{f}(s-a)$
Proof. By definition, we have
$\mathcal{L}\ \{e^{at}\ f(t)\} = \int_0^{\infty} e^{-st} e^{at}\ f(t)dt = \int_0^{\infty} e^{-(s-a)t}\ f(t)dt = \bar{f}(s-a)$
This result also known as 1st shifting theorem or 1st translation theorem.
EXAMPLES:

- i. **If** $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ **then** $\mathcal{L}\{t^2\ e^{t}\} = \frac{2}{(s-1)^3}$
- ii. **If** $\mathcal{L}\{Sinwt\} = \frac{w}{s^2+w^2}$ **then** $\mathcal{L}\{e^{at}\ Sinwt\} = \frac{w}{(s-a)^2+w^2}$
- iii. **If** $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2}$ **then** $\mathcal{L}\{e^{at}\ Coswt\} = \frac{s-a}{(s-a)^2+w^2}$
- iv. **If** $\mathcal{L}\{t^{n}\} = \frac{n!}{s^{n+1}}$ **then** $\mathcal{L}\{e^{at}\ t^{n}\} = \frac{n!}{(s-a)^{n+1}}$

Question: Find $\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$
Answer: in this question we will use the first shifting theorem according to which
$\mathcal{L}\{e^{at}\ f(t)\} = \bar{f}(s-a) \Rightarrow e^{at}\ f(t) = e^{at}\ \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{\bar{f}(s-a)\}$
Thus $\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}\ \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = e^{-4t}.1 = e^{-4t}\ \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$

Question: Find $\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$

Answer: in this question we will use the first shifting theorem according to which
$\mathcal{L}\{e^{at} f(t)\} = \bar{f}(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{\bar{f}(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = -2Cos2t + 3Sin2t$

Question: Find $\mathcal{L}^{-1}\left\{\frac{s-4}{(s-4)^2+4}\right\}$

Answer: in this question we will use the first shifting theorem according to which
$\mathcal{L}\{e^{at} f(t)\} = \bar{f}(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{\bar{f}(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{s-4}{(s-4)^2+4}\right\} = e^{4t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = e^{4t} Cos4t$

Question: Find $\mathcal{L}^{-1}\left\{\frac{9}{(s-3)^2+9}\right\}$

Answer: in this question we will use the first shifting theorem according to which
$\mathcal{L}\{e^{at} f(t)\} = \bar{f}(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{\bar{f}(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{9}{(s-3)^2+9}\right\} = 3e^{3t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = 3e^{3t} Sin3t$

SCALING PROPERTY: If $\bar{f}(s)$ is the laplace transformation of (t) , then
$\mathcal{L}\,[f\,(ct)] = \frac{1}{c}\,\bar{f}\,(\frac{s}{c})$ with $c > 0$

Proof. By definition we have

$\mathcal{L}\,\{f(ct)\} = \int_0^\infty e^{-st}\, f(ct)dt = \frac{1}{c}\int_0^\infty e^{-\left(\frac{s}{c}\right)t'}\, f(t')dt' = \frac{1}{c}\,\bar{f}\,(\frac{s}{c})$ putting $ct = t'$

This result also known as Rule of Scale.

EXAMPLES:

i. If $\mathcal{L}\{Cost\} = \frac{s}{s^2+1}$ then $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2} = \frac{1}{w}\left[\frac{s/w}{(s/w)^2+1}\right]$

ii. If $\mathcal{L}\{e^{t}\} = \frac{1}{s-1}$ then $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} = \frac{1}{a}\left[\frac{1}{\left(\frac{s}{a}-1\right)}\right]$

DIFFERENTIATION PROPERTY: Let f be continuous and f' piecewise continuous, in $0 \le t \le T$ for all $T > 0$. Let f also be of exponential order as $t \to \infty$ Then, the Laplace transform of $f'(t)$ exists and is given by

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) = s\bar{f}(s) - f(0)$$

Proof. If $f(t)$ is continuous and $f'(t)$ is sectionally continuous on the interval $[0,\infty)$ and both are of exponential order then

$\mathcal{L}\,\{f'(t)\} = \int_0^\infty e^{-st}\, f'(t)dt = |e^{-st} f(t)|_0^\infty - (-s)\int_0^\infty e^{-st}\, f(t)dt = [0 - f(0)] + s\mathcal{L}\,\{f(t)\}$

$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) = s\bar{f}(s) - f(0)$

If f' and f'' satisfy the same conditions imposed on f and f' respectively, then, the Laplace transform of $f''(t)$ can be obtained immediately by applying the preceding theorem; that is

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f'(0) = s^2\bar{f}(s) - sf(0) - f'(0)$$

Proof. If $f(t), f'(t)$ are continuous and $f''(t)$ is sectionally continuous on the interval $[0, \infty)$ and all are of exponential order then

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)dt = |e^{-st} f'(t)|_0^\infty - (-s)\int_0^\infty e^{-st} f'(t)dt$$

$$\mathcal{L}\{f''(t)\} = [0 - f'(0)] + s\mathcal{L}\{f'(t)\} = -f'(0) + s[s\bar{f}(s) - f(0)]$$

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f'(0) = s^2\bar{f}(s) - sf(0) - f'(0)$$

Clearly, the Laplace transform of $f^n(t)$ can be obtained in a similar manner by successive application.The result may be written as

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - \cdots \ldots \ldots - sf^{n-2}(0) - f^{n-1}(0)$$

INTEGRATION PROPERTY : If $\bar{f}(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{\bar{f}(s)}{s}$$

PROOF:

Consider $g(\tau) = \int_0^t f(\tau)\, d\tau \Rightarrow g'(\tau) = f(t) \Rightarrow \mathcal{L}[g'(\tau)] = \mathcal{L}[f(t)]$

$\Rightarrow s\bar{g}(s) - g(0) = \mathcal{L}[f(t)]$

$\Rightarrow s\mathcal{L}[g(\tau)] - 0 = \mathcal{L}[f(t)]$

$\Rightarrow \mathcal{L}[g(\tau)] = \frac{\bar{f}(s)}{s}$

$\Rightarrow \mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{\bar{f}(s)}{s}$

In solving problems by the Laplace transform method, the difficulty arises in finding inverse transforms. Although the inversion formula exists, its evaluation requires a knowledge of functions of complex variables. However, for some problems of mathematical physics, we need not use this inversion formula. We can avoid its use by expanding a given transform by the method of partial fractions in terms of simple fractions in the transform variables. With these simple functions, we refer to the table of Laplace transforms given in the end of the book and obtain the inverse transforms. Here, we should note that we use the assumption that there is essentially a one-to-one correspondence between functions and their Laplace transforms.

This may be stated as follows:

LERCH THEOREM: let f and g be piecewise continuous functions of exponential order. If there exists a constant s_0, such that $\mathcal{L}[f] = \mathcal{L}[g]$ for all $s > s_0$, then $f(t) = g(t)$ for all $t > 0$ except possibly at the points of discontinuity.

In order to find a solution of linear partial differential equations, the following formulas and results are useful.

If $\mathcal{L}[u(x,t)] = \bar{u}(x,s)$ then

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\,\bar{u}(x,s) - u(x,0)$$

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\,\bar{u}(x,s) - su(x,0) - u_t(x,0) \quad \text{and so on.}$$

Similarly, it is easy to show that

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial \bar{u}}{\partial x}\ ,\ \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\partial^2 \bar{u}}{\partial x^2}\ ,\ \ldots\ldots\ldots\ldots\ldots,\ \mathcal{L}\left\{\frac{\partial^n u}{\partial x^n}\right\} = \frac{\partial^n \bar{u}}{\partial x^n}$$

The following results are useful for applications:

$$\mathcal{L}\left\{erfc\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}e^{(-a\sqrt{s})} \qquad ;a \geq 0$$

$$\mathcal{L}\{e^{at}erf(\sqrt{at})\} = \frac{\sqrt{a}}{\sqrt{s}(s-a)} \qquad ;a > 0$$

Example: Consider the motion of a semi-infinite string with an external force $f(t)$ acting on it. One end is kept fixed while the other end is allowed to move freely in the vertical direction. If the string is initially at rest, the motion of the string is governed by

$$u_{tt} = c^2u_{xx} + f(t)\ ,0 < x < \infty, t > 0$$

$$u(x,0) = 0,\ \ u_t(x,0) = 0 \ \text{ and } \ u(0,t) = 0,\ \ u_x(x,t) \to 0, \ \text{ as } x \to \infty$$

Solution: Let $u(x,s)$ be the Laplace transform of $u(x,t)$. Transforming the equation of motion and using the initial conditions, we obtain

$$\bar{u}_{xx} - \left(\frac{s^2}{c^2}\right)\bar{u} = -\bar{f}(s)\Big/c^2$$

The solution of this ordinary differential equation is $u(x,s) = Ae^{\frac{sx}{c}} + Be^{\frac{-sx}{c}} + \bar{f}(s)\Big/s^2$

The transformed boundary conditions are given by

$u(0,s) = 0$, **and** $\lim_{x\to\infty} \bar{u}_x(x,s) = 0$

In view of the second condition, we have $A = 0$. Now applying the first condition, we obtain $\bar{u}(0,s) = B + \bar{f}(s)\Big/s^2 = 0$ **Hence** $u(x,s) = \bar{f}(s)\Big/s^2\left[1 - e^{\frac{-sx}{c}}\right]$

SPECIAL CASES: (a) When $f(t) = f_0$, a constant, then $u(x,s) = f_0\left[\frac{1}{s^3} - \frac{1}{s^3}e^{\frac{-sx}{c}}\right]$

The inverse Laplace transform gives the solution

$$u(x,t) = \begin{cases} \frac{f_0}{2}\left[t^2 - \left(t - \frac{x}{c}\right)^2\right] & ;\ t \geq x/c \\ \frac{f_0}{2}[t^2] & ;\ t \leq x/c \end{cases}$$

(b) When $f(t) = \cos\omega t$, where ω is a constant, then

$$\bar{f}(s) = \int_0^\infty e^{-st}Coswtdt = \frac{s}{s^2+w^2}$$

Thus, we have $\bar{u}(x,s) = \frac{1}{s(s^2+w^2)}\left[1 - e^{\frac{-sx}{c}}\right]$ **............(i)**

By the method of partial fractions, we write $\frac{1}{s(s^2+w^2)} = \frac{1}{w^2}\left[\frac{1}{s} - \frac{1}{s^2+w^2}\right]$

Hence $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+w^2)}\right\} = \frac{1}{w^2}[1 - Coswt] = \frac{2}{w^2}\left[Sin^2\left(\frac{wt}{2}\right)\right]$

If we denote $\psi(t) = Sin^2\left(\frac{wt}{2}\right)$ **then the Laplace inverse of equation (i) may be written in the form**

$$u(x,t) = \begin{cases} \frac{2}{w^2}\left[\psi(t) - \psi\left(t - \frac{x}{c}\right)\right] & ; \; t \geq x/c \\ \frac{2}{w^2}\psi(t) & ; \; t \leq x/c \end{cases}$$

UNIT STEP FUNCTION: A real valued function $H: R \to R$ **is defined as**

$$H(t-\xi) = \begin{cases} 1 & ; \; t \geq \xi \\ 0 & ; \; t < \xi \end{cases} \quad \text{When } \xi = 0 \; ; \; H(t) = \begin{cases} 1 & ; \; t \geq 0 \\ 0 & ; \; t < 0 \end{cases}$$

CONVOLUTION FUNCTION / FAULTUNG FUNCTION OF LAPLACE TRANSFORMATION.

The function $(f * g)(t) = \int_0^t f(t-\xi)\, g(\xi)\, d\xi$ **is called the convolution of the functions** f **and** g. regarding laplace transformation.

USEFUL RESULT:

$$(f * g)(t) = \int_0^t e^{-st} f(t-\xi) g(\xi) d\xi = \int_0^\infty H(t-\xi) f(t-\xi)\, g(\xi)\, d\xi$$

CONVOLUTION / FAULTUNG THEOREM OF LAPLACE TRANSFORMATION

If $\bar{f}(s)$ **and** $\bar{g}(s)$**are the Laplace transforms of** $f(t)$ **and** $g(t)$ **respectively, then the Laplace transform of the convolution** $(f * g)(t)$ **is the product** $\bar{f}(s)\bar{g}(s)$

PROOF: By definition, we have

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st} (f * g)\, dt$$

$$\mathcal{L}\{f * g\} = \int_0^\infty \int_0^t e^{-st} f(t-\xi)\, g(\xi)\, d\xi\, dt$$

$$\mathcal{L}\{f * g\} = \int_0^\infty \int_0^t e^{-st} f(\xi) g(t-\xi)\, d\xi\, dt \qquad \text{since } f * g = g * f$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}\left[\int_0^\infty H(t-\xi) f(\xi) g(t-\xi)\, d\xi\right] dt$$

By reversing the order of integration, we have

$$\mathcal{L}\{f * g\} = \int_0^\infty \left[\int_0^\infty e^{-st} H(t-\xi) g(t-\xi)\, dt\right] f(\xi) d\xi$$

If we introduce the new variable $\eta = (t - \xi)$ **in the inner integral, we obtain**

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\, d\xi \left[\int_{-\xi}^\infty e^{-s(\xi+\eta)} H(\eta) g(\eta)\, d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\, d\xi \left[\int_{-\xi}^0 e^{-s(\xi+\eta)} H(\eta) g(\eta)\, d\eta + \int_0^\infty e^{-s(\xi+\eta)} H(\eta) g(\eta)\, d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\, d\xi \left[\int_{-\xi}^0 e^{-s(\xi+\eta)} 0.\, g(\eta)\, d\eta + \int_0^\infty e^{-s(\xi+\eta)}. 1.\, g(\eta)\, d\eta\right] \quad \text{by step function}$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\, d\xi \left[\int_0^\infty e^{-s(\xi+\eta)} g(\eta)\, d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-s\xi} f(\xi) d\xi \int_0^\infty e^{-s\eta} g(\eta)\, d\eta$$

$$\mathcal{L}\{f * g\} = \bar{f}(s)\bar{g}(s)$$

THE CONVOLUTION SATISFIES THE FOLLOWING PROPERTIES:

1. $f * g = g * f$ **(commutative).**
2. $f * (g * h) = (f * g) * h$ **(associative).**
3. $f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h)$ **(distributive),**

where α and β are constants.

Example: Find the temperature distribution in a semi-infinite radiating rod. The temperature is kept constant at $x = 0$, while the other end is kept at zero temperature. If the initial temperature distribution is zero, the problem is governed by

$u_t = ku_{xx} - hu$, $0 < x < \infty, t > 0$, h **= constant,**

$u(x,0) = 0, u(0,t) = u_0$, $t > 0$, u_0 **= constant,** where $u(x,t) \to 0, as\ x \to \infty$

Solution: Let $\bar{u}(x,s)$ be the Laplace transform of $u(x,t)$. Then the transformation with respect to t yields

$$\bar{u}_{xx} - \left(\frac{s+h}{k}\right)\bar{u} = 0 \quad , \quad \bar{u}(0,s) = \frac{u_0}{s} \quad , \quad \lim_{x\to\infty} \bar{u}(x,s) = 0$$

The solution of this equation is $u(x,s) = Ae^{x\sqrt{(s+h)/k}} + Be^{-x\sqrt{(s+h)/k}}$

The boundary condition at infinity requires that $A = 0$. Applying the other boundary condition gives $u(0,s) = B = \frac{u_0}{s}$

Hence, the solution takes the form $\bar{u}(x,s) = \left(\frac{u_0}{s}\right)e^{-x\sqrt{(s+h)/k}}$

We find (by using the Table of Laplace Transforms) that $\mathcal{L}^{-1}\left(\frac{u_0}{s}\right) = u_0$

And $\mathcal{L}^{-1}\left\{e^{-x\sqrt{(s+h)/k}}\right\} = \frac{xe^{\left[-ht-\left(\frac{x^2}{4kt}\right)\right]}}{2\sqrt{\pi kt^3}}$

Thus, the inverse Laplace transform of $u(x,s)$ is $u(x,t) = \mathcal{L}^{-1}\left\{\frac{u_0}{s}e^{-x\sqrt{(s+h)/k}}\right\}$

By using the Integration Theorem $\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{\bar{f}(s)}{s}$ **we have**

$$u(x,t) = \int_0^t \frac{u_0 xe^{\left[-h\tau-\left(\frac{x^2}{4k\tau}\right)\right]}}{2\sqrt{\pi k}\tau^{3/2}}\,d\tau$$

Substituting the new variable $\eta = \frac{x}{2\sqrt{k\tau}}$ **yields**

$$u(x,t) = \frac{2u_0}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^{\infty} e^{\left[-\eta^2+\left(\frac{hx^2}{4k\eta^2}\right)\right]}\,d\eta$$

For the case $h = 0$, the solution $u(x,t)$ becomes

$$u(x,t) = \frac{2u_0}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^{\infty} e^{-\eta^2}\,d\eta = \frac{2u_0}{\sqrt{\pi}}\int_0^{\infty} e^{-\eta^2}\,d\eta - \frac{2u_0}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-\eta^2}\,d\eta$$

$$u(x,t) = u_0\left[1 - erf\left(\frac{x}{2\sqrt{kt}}\right)\right] = u_0 erfc\left(\frac{x}{2\sqrt{kt}}\right)$$

LAPLACE TRANSFORMS OF THE HEAVISIDE AND DIRAC DELTA FUNCTIONS

The Heaviside unit step function is defined by

$$H\,(t - a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \quad \text{where } a \geq 0$$

Now, we will find its Laplace transform.

$$\mathcal{L}\{H\,(t - a)\} = \int_0^\infty e^{-st} H\,(t - a)\,dt = \int_0^a e^{-st} H\,(t - a)\,dt + \int_a^\infty e^{-st} H\,(t - a)\,dt$$

$$\mathcal{L}\{H\,(t - a)\} = \int_a^\infty e^{-st}\,dt = \left|\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s} \quad ; \; s > 0$$

SECOND SHIFTING (TRANSLATION) THEOREM: If $\bar{f}(s)$ and $\bar{g}(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then

$(a)\; \mathcal{L}[H\,(t - a)\, f\,(t - a)] = e^{-as}\,\bar{f}(s) = e^{-as}\,\mathcal{L}\{f\,(t)\}$

$(b)\; \mathcal{L}\{H\,(t - a)\, g\,(t)\} = e^{-as}\mathcal{L}\{g\,(t + a)\}$

Proof: (a) By definition

$$\mathcal{L}\{H\,(t - a)\, f\,(t - a)\} = \int_0^\infty e^{-st} H\,(t - a)\, f\,(t - a)\,dt$$

$$\mathcal{L}\{H\,(t - a)\, f\,(t - a)\} = \int_0^a e^{-st} H\,(t - a) f\,(t - a)\,dt + \int_a^\infty e^{-st} H\,(t - a) f\,(t - a)\,dt$$

$$\mathcal{L}\{H\,(t - a)\, f\,(t - a)\} = \int_a^\infty e^{-st}\, f\,(t - a)\,dt$$

Introducing the new variable $\xi = t - a$, we obtain

$$\mathcal{L}\{H\,(t - a)\, f\,(t - a)\} = \int_0^\infty e^{-(\xi + a)s}\, f\,(\xi)\,d\xi = e^{-as}\int_0^\infty e^{-\xi s}\, f\,(\xi)\,d\xi$$

$$\mathcal{L}\{H\,(t - a)\, f\,(t - a)\} = e^{-as}\mathcal{L}\{f\,(t)\} = e^{-as}\bar{f}(s)$$

To prove (b), we write

$$\mathcal{L}\{H\,(t - a)\, g(t)\} = \int_0^\infty e^{-st} H\,(t - a)\, g(t)\,dt$$

$$\mathcal{L}\{H\,(t - a)\, g(t)\} = \int_0^a e^{-st} H\,(t - a)\, g(t)\,dt + \int_a^\infty e^{-st} H\,(t - a)\, g(t)\,dt$$

$$\mathcal{L}\{H\,(t - a)\, g(t)\} = \int_a^\infty e^{-st}\, g(t)\,dt$$

Now using $(t - a = \tau)$

$$\mathcal{L}\{H\,(t - a)\, g(t)\} = \int_0^\infty e^{-s(a+\tau)}\, g(a + \tau)\,d\tau = e^{-as}\int_0^\infty e^{-s\tau}\, g(a + \tau)\,d\tau$$

$$\mathcal{L}\{H\,(t - a)\, g\,(t)\} = e^{-as}\mathcal{L}\{g\,(t + a)\}$$

Example: Given that $f\,(t) = \begin{cases} 0 & ; t < 2 \\ t - 2 & ; t \geq 2 \end{cases} = (t - 2)H\,(t - 2)$ **then find the Laplace transform of** $f\,(t)$

Solution: We have

$$\mathcal{L}\{f\,(t)\} = \mathcal{L}\{(t - 2)H\,(t - 2)\} = e^{-2s}\mathcal{L}\{t\} = \left(\frac{1}{s^2}\right)e^{-2s}$$

Example: Find the inverse Laplace transform of $\bar{f}(s) = \frac{1+e^{-2s}}{s^2}$

Solution: $\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1+e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = t + (t - 2)H\,(t - 2)$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \begin{cases} t & ; 0 \leq t < 2 \\ 2(t - 1) & ; t \geq 2 \end{cases}$$

THEOREM: **Let $f(t)$ be a piecewise continuous function for $t \geq 0$ and of exponential order. If $f(t)$ is periodic with period T then show that**

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

PROOF: **By definition, we have**

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$

In the 2nd integral on the right put $t = u + T \Rightarrow dt = du$

$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+T)} f(u+T)\,du$

$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^\infty e^{-su} f(u+T)\,du$

Since given functionis periodic with period T therefore $f(u+T) = f(u)$ then

$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^\infty e^{-su} f(u)\,du$

$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\mathcal{L}\{f(u)\}$

$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\mathcal{L}\{f(t)\}$

$(1-e^{-sT})\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt$

$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

As required the result.

THEOREM: **If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$**

PROOF: **By definition, we have**

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\int_s^\infty F(s)\,ds = \int_s^\infty\left[\int_0^\infty e^{-st} f(t)\,dt\right]ds$ integrating.

$\int_s^\infty F(s)\,ds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right]dt$ changing the order of integration.

$\int_s^\infty F(s)\,ds = \int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st}dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$

Hence $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

In order to find a solution of linear partial differential equations, the following formulas and results are useful.

If $\mathcal{L}[u(x,t)] = U(x,s)$ **then**

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\,U(x,s) - u(x,0)$$

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\,U(x,s) - su(x,0) - u_t(x,0)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mathcal{L}\left\{\frac{\partial^n u}{\partial t^n}\right\} = s^n\,U(x,s) - s^{n-1}u(x,0) - \cdots \dots \dots \dots . - su_{t_{n-2}}(x,0) - u_{t_{n-1}}(x,0)$$

Similarly, it is easy to show that

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial}{\partial x}\,U(x,s)\,,\ \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2}\,U(x,s)\,,\ \dots\dots\dots\dots\dots,\ \mathcal{L}\left\{\frac{\partial^n u}{\partial x^n}\right\} = \frac{\partial^n}{\partial x^n}U(x,s)$$

EXAMPLE: Use Laplace Transformation method to solve BVP

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t};\ 0 < x < a;\quad 0 \le t < \infty$$

$$u\,(0,t) = 1,\ u\,(1,t) = 1\ \ ;t > 0\ \ ,u\,(x,0) = 1 + Sin\pi x$$

Solution:

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} \Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s\,U(x,s) - u(x,0)$

$$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s\,U(x,s) - (1 + Sin\pi x)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - s\,U(x,s) = -1 - Sin\pi x \qquad \dots\dots\dots\dots\text{(i)}$$

Which is non – homogeneous 2nd order DE with solution

$$U(x,s) = U_c(x,s) + U_p(x,s) \qquad \dots\dots\dots\dots\text{(ii)}$$

For Chractristic (auxiliary) solution

$$(i) \Rightarrow (D^2 - s)U(x,s) = -1 - Sin\pi x \Rightarrow D^2 - s = 0 \Rightarrow D = \pm\sqrt{s}$$

Then $U_c(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$

For Particular solution

Consider $U_p(x,s) = \frac{-1-Sin\pi x}{D^2-s} = \frac{-e^{0x}}{D^2-s} - img\,\frac{e^{i\pi x}}{D^2-s} = \frac{-1}{0^2-s} - \frac{Sin\pi x}{(i\pi)^2-s} = \frac{1}{s} - \frac{Sin\pi x}{-\pi^2-s}$

Then $U_p(x,s) = \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s}$

$\Rightarrow U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s}$ ………..(iii)

Now using BC's

$u\,(0,t) = 1 \Rightarrow \mathcal{L}\{u\,(0,t)\} = \mathcal{L}\{1 = t^0\} \Rightarrow U(0,s) = \frac{1}{s}$

$u\,(1,t) = 1 \Rightarrow \mathcal{L}\{u\,(1,t)\} = \mathcal{L}\{1 = t^0\} \Rightarrow U(1,s) = \frac{1}{s}$

$(iii) \Rightarrow U(0,s) = \frac{1}{s} = c_1 e^0 + c_2 e^0 + \frac{1}{s} + \frac{Sin(0)}{\pi^2+s} \Rightarrow c_1 + c_2 + \frac{1}{s} = \frac{1}{s} \Rightarrow c_1 = -c_2$

$(iii) \Rightarrow U(1,s) = \frac{1}{s} = c_1 e^{\sqrt{s}(1)} + c_2 e^{-\sqrt{s}(1)} + \frac{1}{s} + \frac{Sin\pi}{\pi^2+s} \Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s} - \frac{1}{s} = 0$

$\Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \Rightarrow -c_2 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0$ $\quad \therefore c_1 = -c_2$

$\Rightarrow c_2\left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right] = 0 \Rightarrow c_2 = 0\ ,\ \left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right] \neq 0$

$\Rightarrow c_2 = 0 \Rightarrow c_1 = 0$ $\quad \therefore c_1 = -c_2$

$(iii) \Rightarrow U(x,s) = \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s}$ $\quad \therefore c_1 = c_2 = 0$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{Sin\pi x}{\pi^2+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + Sin\pi x\mathcal{L}^{-1}\left\{\frac{1}{s-(-\pi^2)}\right\}$

$\Rightarrow u\,(x,t) = 1 + Sin\pi x e^{-\pi^2 t}$ **required solution.**

EXAMPLE:

Use Laplace Transformation method to solve BVP $u_{tt}\,(x,t) = \alpha^2 u_{xx}\,(x,t); t > 0, x > 0$

$u\,(x,0) = u_t\,(x,0) = 0,\ u\,(0,t) = f(t)\ , \lim_{x\to\infty} u\,(x,t) = 0$

Solution:

Given $u_{tt}\,(x,t) = \alpha^2 u_{xx}\,(x,t) \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\}$

$\Rightarrow s^2\,U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow s^2\,U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) \Rightarrow s^2\,U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - \frac{s^2}{\alpha^2}\,U(x,s) = 0$

This is Homogeneous DE of 2nd order therefore

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$ ………..(i)

Now using BC's

$u(0,t) = f(t) \Rightarrow \mathcal{L}\{u(0,t)\} = \mathcal{L}\{f(t)\} \Rightarrow U(0,s) = F(s)$

$\lim_{x\to\infty} u(x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} U(x,s) = 0$

$(i) \Rightarrow U(0,s) = F(s) = c_1 e^{\frac{s}{a}(0)} + c_2 e^{-\frac{s}{a}(0)} \Rightarrow c_1 + c_2 = F(s)$

$(i) \Rightarrow \lim_{x\to\infty} U(x,s) = 0 = \lim_{x\to\infty}\left[c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}\right] = c_1 e^{\infty} + c_2 e^{-\infty}$

$\Rightarrow c_1 = 0 \quad then \quad c_2 = F(s) \qquad\qquad \therefore c_1 + c_2 = F(s)$

Thus $(i) \Rightarrow U(x,s) = F(s)e^{-\frac{s}{a}x}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{F(s)e^{-\frac{s}{a}x}\right\}$

$\Rightarrow u(x,t) = H\left(t-\frac{x}{a}\right)f\left(t-\frac{x}{a}\right)$ **where** $H\left(t-\frac{x}{a}\right)f\left(t-\frac{x}{a}\right) = \begin{cases} 0 & t < \frac{x}{a} \\ f(t) & t \geq \frac{x}{a}\end{cases}$

EXAMPLE: Use Laplace Transformation method to solve BVP

$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) - g$

$u(x,0) = u_t(x,0) = 0,\ u(0,t) = 0\ , \lim_{x\to\infty} u_x(x,t) = 0$

Solution: Given $u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) - g \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2\mathcal{L}\{u_{xx}\} - g\mathcal{L}\{1\}$

$\Rightarrow s^2 U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2}U(x,s) - \frac{g}{s}$

$\Rightarrow s^2 U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2}U(x,s) - \frac{g}{s} \Rightarrow s^2 U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2}U(x,s) - \frac{g}{s}$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - \frac{s^2}{\alpha^2}U(x,s) = \frac{g}{\alpha^2 s}$ **…………(i)**

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ **…………(ii)**

For Chractristic (auxiliary) solution $\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right)U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U_c(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$

For Particular solution Consider $U_p(x,s) = \frac{\frac{g}{\alpha^2 s}}{D^2 - \frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}e^{0x}}{0^2 - \frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}}{-\frac{s^2}{\alpha^2}} = -\frac{g}{s^3}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$ **…………(iii)**

Now using BC's $u(0,t) = 0 \Rightarrow \mathcal{L}\{u(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$\lim_{x\to\infty} u_x\,(x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u_x\,(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} \frac{\partial}{\partial x} U\,(x,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1e^0 + c_2e^{-0} - \frac{g}{s^3} \Rightarrow c_1 + c_2 = \frac{g}{s^3}$

$(iii) \Rightarrow \lim_{x\to\infty} \frac{\partial}{\partial x} U\,(x,s) = 0 = \lim_{x\to\infty}\left[c_1\frac{s}{a}e^{\frac{s}{a}x} - \frac{s}{a}c_2e^{-\frac{s}{a}x}\right] = c_1\frac{s}{a}e^{\infty} + c_2\frac{s}{a}e^{-\infty}$

$\Rightarrow c_1\frac{s}{a}e^{\infty} = 0 \Rightarrow c_1 = 0 \;\; since\;\; \frac{s}{a}e^{\infty} \neq 0, \;\; then \;\; c_2 = \frac{g}{s^3} \qquad \therefore c_1 + c_2 = \frac{g}{s^3}$

Thus $(iii) \Rightarrow U(x,s) = \frac{g}{s^3}e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \frac{g}{2!}\mathcal{L}^{-1}\left\{e^{-\frac{x}{a}s}.\frac{2!}{s^{2+1}}\right\} - \frac{g}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}$

$\Rightarrow u\,(x,t) = \frac{g}{2}H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 - \frac{g}{2}(t^2)$

$\Rightarrow u\,(x,t) = \frac{g}{2}\left[H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 - (t^2)\right]$ where $H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 = \begin{cases} 0 & t < \frac{x}{a} \\ t^2 & t \geq \frac{x}{a} \end{cases}$

EXAMPLE: Use Laplace Transformation method to solve BVP

$u_{xx}\,(x,t) = u_{tt}\,(x,t);\, t > 0;\; 0 < x < 1$

$u\,(0,t) = 0 = u\,(1,t),\;\; u\,(x,0) = Sin\pi x \;\; ,u_t\,(x,0) = -Sin\pi x$

Solution: Given $u_{xx}\,(x,t) = u_{tt}\,(x,t) \Rightarrow \mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s^2\,U(x,s) - su(x,0) - u_t(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s^2\,U(x,s) - sSin\pi x + Sin\pi x$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - s^2\,U(x,s) = -sSin\pi x + Sin\pi x$ …………(i)

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ …………(ii)

For Chractristic (auxiliary) solution

$\Rightarrow (D^2 - s^2)U(x,s) = 0 \Rightarrow D^2 - s^2 = 0 \Rightarrow D = \pm s$ Then $U_c(x,s) = c_1e^{sx} + c_2e^{-sx}$

For Particular solution

$U_p(x,s) = \frac{(1-s)Sin\pi x}{D^2-s^2} = (1-s)img\frac{e^{i\pi x}}{D^2-s^2} = (1-s)\frac{Sin\pi x}{(i\pi)^2-s} = (1-s)\frac{Sin\pi x}{-\pi^2-s}$

$U_p(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1e^{sx} + c_2e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow U(x,s) = c_1e^{sx} + c_2e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$ …………(iii)

Now using BC's $u\,(0,t) = 0 \Rightarrow \mathcal{L}\{u\,(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$u\,(1,t) = 0 \Rightarrow \mathcal{L}\{u\,(1,t)\} = 0 \Rightarrow U(1,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1e^0 + c_2e^{-0} + \frac{(s-1)Sin\pi(0)}{\pi^2+s} \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$(iii) \Rightarrow U(1,s) = 0 = c_1e^s + c_2e^{-s} + \frac{(s-1)Sin\pi}{\pi^2+s} \Rightarrow c_1e^s + c_2e^{-s} = 0 \Rightarrow c_1e^s - c_1e^{-s} = 0$

$\Rightarrow c_1(e^s - e^{-s}) = 0 \Rightarrow c_1 = 0\ \ as\,(e^s - e^{-s}) \neq 0 \Rightarrow c_2 = 0$

Thus $(iii) \Rightarrow U(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = Sin\pi x\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} - \frac{Sin\pi x}{\pi}\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$

$\Rightarrow u\,(x,t) = Sin\pi xCos\pi t - \frac{Sin\pi x}{\pi}Sin\pi t = Sin\pi x\left[Cos\pi t - \frac{Sin\pi x}{\pi}\right]$

EXAMPLE:

Use Laplace Transformation method to solve BVP

$u_{xx}\,(x,t) = u_{tt}\,(x,t); t > 0;\ 0 < x < 1$

$u\,(0,t) = 0 = u\,(1,t),\ u(x,0) = 0\ \ , u_t\,(x,0) = Sin\pi x$

Solution:

Given $u_{xx}\,(x,t) = u_{tt}\,(x,t) \Rightarrow \mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s^2\,U(x,s) - su(x,0) - u_t(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s^2\,U(x,s) - Sin\pi x$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - s^2\,U(x,s) = Sin\pi x$ …………(i)

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ …………(ii)

For Chractristic (auxiliary) solution $\Rightarrow (D^2 - s^2)U(x,s) = 0 \Rightarrow D^2 - s^2 = 0 \Rightarrow D = \pm s$

Then $U_c(x,s) = c_1e^{sx} + c_2e^{-sx}$

For Particular solution

$U_p(x,s) = \frac{Sin\pi x}{D^2-s^2} = img\,\frac{e^{i\pi x}}{D^2-s^2} = \frac{Sin\pi x}{(i\pi)^2-s} = \frac{Sin\pi x}{-\pi^2-s}$

$U_p(x,s) = \frac{Sin\pi x}{\pi^2+s}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1e^{sx} + c_2e^{-sx} + \frac{Sin\pi x}{\pi^2+s}$

$\Rightarrow U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{Sin\pi x}{\pi^2+s}$ ....(iii)

Now using BC's $u\,(0,t) = 0 \Rightarrow \mathcal{L}\{u\,(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$u\,(1,t) = 0 \Rightarrow \mathcal{L}\{u\,(1,t)\} = 0 \Rightarrow U(1,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1 e^0 + c_2 e^{-0} + \frac{Sin\pi(0)}{\pi^2+s} \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$(iii) \Rightarrow U(1,s) = 0 = c_1 e^s + c_2 e^{-s} + \frac{Sin\pi}{\pi^2+s}$

$\Rightarrow c_1 e^s + c_2 e^{-s} = 0 \Rightarrow c_1 e^s - c_1 e^{-s} = 0$

$\Rightarrow c_1(e^s - e^{-s}) = 0 \Rightarrow c_1 = 0\ \ as\ (e^s - e^{-s}) \neq 0 \Rightarrow c_2 = 0$

Thus

$(iii) \Rightarrow U(x,s) = \frac{Sin\pi x}{\pi^2+s}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \frac{Sin\pi x}{\pi}\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$

$\Rightarrow u\,(x,t) = \frac{Sin\pi x}{\pi} Sin\pi t$

EXAMPLE:

A uniform bar of length 'l' is fixed at one end. Let the force

$f(\,t) = \begin{cases} 0 & t < 0 \\ f_0 & t > 0 \end{cases}$ **be suddenly applied at the end $= \ l$, if the bar is initially at rest, find the longitudinal displacement for $t > 0$ using Laplace Transformation the motion of bar is govern by the differential system $u_{tt} = \alpha^2 u_{xx}; t > 0, 0 < x < 1$ and α is constant.**

$u(x,0) = u(0,t) = u_t(x,0) = 0, \quad u_x(l,t) = \frac{f_0}{E}$ **where E is constant.**

Solution:

Given $u_{tt}\,(x,t) = \alpha^2 u_{xx}\,(x,t) \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\}$

$\Rightarrow s^2\,U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow s^2\,U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow s^2\,U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - \frac{s^2}{\alpha^2}\,U(x,s) = 0$

This is Homogeneous DE of 2^{nd} order therefore

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$ ....(i)

Now using BC's

$u(0,t) = 0 \Rightarrow \mathcal{L}\{u(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$u_x(l,t) = \frac{f_0}{E} \Rightarrow \mathcal{L}\{u_x(l,t)\} = \mathcal{L}\left\{\frac{f_0}{E}\right\} \Rightarrow \frac{\partial}{\partial x} U(l,s) = \frac{F_0}{E}$

$(i) \Rightarrow U(0,s) = F(s) = c_1 e^{0} + c_2 e^{-0} \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

Then $U(x,s) = c_1 e^{\frac{s}{a}x} - c_1 e^{-\frac{s}{a}x}$ ....(ii)

$\Rightarrow \frac{\partial}{\partial x} U(x,s) = c_1 \frac{s}{a} e^{\frac{s}{a}x} + c_1 \frac{s}{a} e^{-\frac{s}{a}x}$

Then using $\frac{\partial}{\partial x} U(l,s) = \frac{F_0}{E}$ **we get**

$\Rightarrow \frac{\partial}{\partial x} U(x,s) = \frac{F_0}{E} = c_1 \frac{s}{a} e^{\frac{s}{a}x} + c_1 \frac{s}{a} e^{-\frac{s}{a}x} \Rightarrow c_1 = \frac{F_0}{E\left(\frac{s}{a} e^{\frac{s}{a}x} + \frac{s}{a} e^{-\frac{s}{a}x}\right)}$

Hence $(ii) \Rightarrow U(x,s) = \frac{F_0}{E\left(\frac{s}{a} e^{\frac{s}{a}x} + \frac{s}{a} e^{-\frac{s}{a}x}\right)} \cdot \left(e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}\right) = \frac{F_0}{E} \frac{\left(e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}\right)}{\frac{s}{a}\left(e^{\frac{s}{a}x} + e^{-\frac{s}{a}x}\right)}$

Taking Laplace inverse on both sides

$u(x,t) = \mathcal{L}^{-1}\left\{\frac{F_0}{E} \frac{\left(e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}\right)}{\frac{s}{a}\left(e^{\frac{s}{a}x} + e^{-\frac{s}{a}x}\right)}\right\}$

which is required longitudinal displacement for $t > 0$

An impulse function is defined by

$$p\,(t) = \begin{cases} h & a-\varepsilon < t < a+\varepsilon \\ 0 & t \le a-\varepsilon \text{ or } t \ge a+\varepsilon \end{cases}$$

where h **is large and positive,** $a > 0$**, and** ε **is a small positive constant, This type of function appears in practical applications; for instance, a force of large magnitude may act over a very short period of time.**

The Laplace transform of the impulse function $p(t)$ **is given by**

$$\mathcal{L}[p\,(t)] = \int_0^\infty e^{-st}\, p(t)dt = \int_0^{a-\varepsilon} e^{-st}\, p(t)\,dt + \int_{a-\varepsilon}^{a+\varepsilon} e^{-st}\, p(t)\,dt + \int_{a+\varepsilon}^{\infty} e^{-st}\, p(t)\,dt$$

$$\mathcal{L}[p\,(t)] = \int_0^\infty e^{-st}\, p(t)dt = \int_{a-\varepsilon}^{a+\varepsilon} he^{-st}dt = h\left|\frac{e^{-st}}{-s}\right|_{a-\varepsilon}^{a-\varepsilon} = \frac{he^{-as}}{s}(e^{\varepsilon s} - e^{-\varepsilon s})$$

$$\mathcal{L}[p\,(t)] = 2\frac{he^{-as}}{s}Sinh(\varepsilon s)$$

Example:(The Heat Conduction Equation in a Semi-Infinite Medium and Fractional Derivatives). Solve the one-dimensional diffusion equation

$u_t = \kappa\, u_{xx}, \quad x>0, \quad t>0$

with the initial and boundary conditions

$u\,(x,0) = 0, x>0, \; u\,(0,t) = f\,(t), t>0, \; u\,(x,t) \to 0, as\; x \to \infty, t>0$

Solution: by using Laplace transforamation with respect to 't' we get

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\bar{u} = 0$$

The general solution of this equation is

$$\bar{u}(x,s) = Aexp\left(-x\sqrt{\frac{s}{\kappa}}\right) + Bexp\left(x\sqrt{\frac{s}{\kappa}}\right)$$

where A **and** B **are integrating constants. For bounded solutions,** $B \equiv 0$**, and using** $u\,(0,s) = f\,(s)$**, we obtain the solution**

$$\bar{u}(x,s) = \bar{f}(s)\, exp\left(-x\sqrt{\frac{s}{\kappa}}\right)$$

The Laplace inversion theorem gives the solution

$$u\,(x,t) = \frac{x}{2\sqrt{\pi\kappa}}\int_0^t f(t-\tau)\,\tau^{-3/2}e^{\left(-\frac{x^2}{4\kappa\tau}\right)}d\tau$$

which, by setting $\lambda = \frac{x}{2\sqrt{\kappa\tau}}$ **or** $d\lambda = \frac{-x}{4\sqrt{\kappa}}\tau^{-3/2}d\tau$

$$u\,(x,t) = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\kappa\tau}}}^{\infty} f\left(t - \frac{x^2}{4\kappa\lambda^2}\right)e^{-\lambda^2}d\lambda$$

This is the formal solution of the heat conduction problem.

Example: (Diffusion Equation in a Finite Medium). Solve the diffusion equation
$u_t = \kappa u_{xx}, \quad 0 < x < a, t > 0$
with the initial and boundary conditions
$u(x,0) = 0, 0 < x < a, u(0,t) = U, t > 0, \; u_x(a,t) = 0, t > 0$
where U **is a constant.**
Solution: by using Laplace transforamation with respect to 't' we get

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\bar{u} = 0 \quad ; \quad 0 < x < a$$

$$\bar{u}(0,s) = \frac{U}{s}, \quad \left(\frac{d\bar{u}}{dx}\right)_{x=a} = 0$$

The general solution of this equation is

$$\bar{u}(x,s) = A Cosh\left(x\sqrt{\frac{s}{\kappa}}\right) + B Sinh\left(x\sqrt{\frac{s}{\kappa}}\right)$$

where A **and** B **are integrating constants. and using** $\bar{u}(0,s) = \frac{U}{s}, \quad \left(\frac{d\bar{u}}{dx}\right)_{x=a} = 0$ **we obtain the solution**

$$\bar{u}(x,s) = \frac{U}{s} \cdot \frac{Cosh\left((a-x)\sqrt{\frac{s}{\kappa}}\right)}{Cosh\left(\sqrt{a\frac{s}{\kappa}}\right)}$$

The inverse Laplace transform gives the solution

$$u(x,t) = U\mathcal{L}^{-1}\left\{\frac{Cosh\left((a-x)\sqrt{\frac{s}{\kappa}}\right)}{sCosh\left(\sqrt{a\frac{s}{\kappa}}\right)}\right\}$$

The inversion can be carried out by the Cauchy Residue Theorem to obtain the solution

$$u(x,t) = U\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)} Cos\left(\frac{(2n-1)(a-x)\pi}{2a}\right) \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2 \kappa t\right\}\right]$$

By expanding the cosine term, this becomes

$$u(x,t) = U\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)} Sin\left\{\left(\frac{(2n-1)}{2a}\right)\pi x\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2 \kappa t\right\}\right]$$

This result can be obtained by solving the problem by the method of separation of variables.

Example: (The Wave Equation for the Transverse Vibration of a Semi-Infinite String). Find the displacement of a semi-infinite string, which is initially at rest in its equilibrium position. At time $t = 0$**, the end** $x = 0$ **is constrained to move so that the displacement is** $u(0,t) = Af(t)$ **for** $t \geq 0$ **where** A **is a constant. The problem is to solve the one-dimensional wave equation**
$u_{tt} = c^2 u_{xx}, \quad 0 \leq x < \infty \quad ; t > 0$
with the boundary and initial conditions
$u(x,t) = Af(t) \; at \; x = 0, t \geq 0$ and $u(x,t) \to 0 \; as \; x \to \infty, t \geq 0,$
$u(x,t) = 0 = \frac{\partial u}{\partial t}$ **at** $t = 0$ **for** $0 < x < \infty$

Solution: Application of the Laplace transform of $u\,(x,t)$ with respect to t gives

$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = 0$; **for** $0 \le x < \infty$

$\bar{u}(x,s) = A\bar{f}(s)$, $at \;\; x = 0$ **and** $\bar{u}(x,s) \to 0 \; as \; x \to \infty$,

The solution of this differential equation system is

$$\bar{u}\,(x,s) = A\bar{f}\,(s)\,exp\left(-\frac{xs}{c}\right)$$

Inversion gives the solution

$$u\,(x,t) = Af\left(t - \frac{xs}{c}\right)H\left(t - \frac{xs}{c}\right)$$

In other words, the solution is

$$u\,(x,t) = \begin{cases} Af\left(t - \frac{xs}{c}\right) & t > x \\ 0 & t < \frac{x}{c} \end{cases}$$

This solution represents a wave propagating at a velocity c with the characteristic $x = ct$.

THE GAUSSIAN INTEGRAL

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \quad \text{or} \quad \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Solution: consider $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ **and** $I = \int_{-\infty}^{\infty} e^{-y^2}\,dy$ **then multiplying both**

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$$

Now using polar coordinates $I^2 = \int_0^{2\pi}\int_{-\infty}^{\infty} e^{-r^2}\,rdrd\theta = \pi \Rightarrow I = \sqrt{\pi}$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \Rightarrow 2\int_0^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

FINITE FOURIER SINE TRANSFORMS

Let $f\,(x)$ be a piecewise continuous in a finite interval, say $(0,\pi)$. Then finite Fourier Sine Transform denoted by $F_s(k)$ of the function $f\,(x)$ is defined by

$$F_s(k) = \mathcal{F}_s\,\{f\,(x)\} = \frac{2}{\pi}\int_0^{\pi} f\,(x)\,Sinkxdx \qquad k = 1, 2, 3, \ldots\ldots\ldots$$

$$\mathcal{F}^{-1}{}_s\,\{F_s\,(k)\} = f\,(x) = \sum_{k=1}^{\infty} F_s(k)Sinkx$$

FINITE FOURIER COSINE TRANSFORMS

Let $f\,(x)$ be a piecewise continuous in a finite interval, say $(0,\pi)$. Then finite Fourier Sine Transform denoted by $F_c(k)$ of the function $f\,(x)$ is defined by

$$F_c(k) = \mathcal{F}_c\,\{f\,(x)\} = \frac{2}{\pi}\int_0^{\pi} f\,(x)\,Coskxdx \qquad k = 0, 1, 2, 3, \ldots\ldots\ldots$$

$$\mathcal{F}^{-1}{}_c\,\{F_c\,(k)\} = f\,(x) = \frac{F_c(0)}{2} + \sum_{k=1}^{\infty} F_c(k)Coskx$$

Theorem : Let $f'(x)$ be continuous and $f''(x)$ be peicewise continuous in $[0,\pi]$ if $F_s(k)$ is the finite Fourier Sine Transform of $f\,(x)$ then

$$\mathcal{F}_s\{f''\,(x)\} = \frac{2k}{\pi}\left[f(0) - (-1)^k f(\pi)\right] - k^2 F_s\,(k)$$

PROOF:

By definition

$$\mathcal{F}_C\{f''\,(x)\} = \frac{2}{\pi}\int_0^\pi f''(x)\,Sinkxdx = \frac{2}{\pi}|f'(x)Sinkx|_0^\pi - \frac{2k}{\pi}\int_0^\pi f'(x)\,Coskxdx$$

$$\mathcal{F}_C\{f''\,(x)\} = -\frac{2k}{\pi}|f(x)Coskx|_0^\pi - \frac{2k^2}{\pi}\int_0^\pi f(x)\,Sinkxdx$$

$$\mathcal{F}_s\{f''\,(x)\} = \frac{2k}{\pi}\left[f(0) - (-1)^k f(\pi)\right] - k^2 F_s\,(k)$$

Theorem : Let $f'(x)$ be continuous and $f''(x)$ be peicewise continuous in $[0,\pi]$ if $F_c(k)$ is the finite Fourier Cosine Transform of $f\,(x)$ then

$$\mathcal{F}_c\{f''\,(x)\} = \frac{2}{\pi}\left[(-1)^k f'(\pi) - f'(0)\right] - k^2 F_c(k)$$

PROOF: By Yourself. Same as previous.

HANKEL TRANSFORMS

$\tilde{f}_n\,(\kappa)$ is called the Hankel transform of $f(r)$ and is defined formally by

$$\mathcal{H}_n\{f\,(r)\} = \tilde{f}_n\,(\kappa) = \int_0^\infty rJ_n(\kappa r)\,f\,(r)dr$$

The inverse Hankel transform is defined by

$$\mathcal{H}_n^{-1}\{\tilde{f}_n\,(\kappa)\} = f\,(r) = \int_0^\infty \kappa J_n(\kappa r)\,\tilde{f}_n\,(\kappa)dk$$

Alternatively, the famous Hankel integral formula

$$f\,(r) = \int_0^\infty \kappa J_n(\kappa r)dk \int_0^\infty \rho J_n(\kappa\rho) f(\rho)d\rho$$

can be used to define the Hankel transform and its inverse

In particular, the Hankel transforms of zero order ($n = 0$) and of order one ($n = 1$) are often useful for the solution of problems involving Laplace's equation in an axisymmetric cylindrical geometry.

REMARK: For Bessel Functions

i. $J_0(\kappa r) = \frac{1}{\pi}\int_0^\pi Cos(\kappa r Sin\theta)d\theta$

ii. $J'_0(\kappa r) = -J_1(\kappa r)$ **also** $J_{n+1} = J_{n-1} - 2J'_n$ **for** $J_0(0) = 1, J_n(0) = 0$ **; n>0**

Example: Obtain the zero-order Hankel transforms of

(a) r^{-1} **exp (**$-ar$**) , (b)** $\frac{\delta(r)}{r}$ **(c)** $H(a-r)$

where H (r) **is the Heaviside unit step function.**

Solution:

(a)

$$\mathcal{H}_0\left\{\frac{1}{r}e^{-ar}\right\} = \tilde{f}_0(\kappa) = \int_0^\infty rJ_n(\kappa r)f(r)dr = \int_0^\infty r.\frac{1}{r}e^{-ar}J_0(\kappa r)\,dr = \frac{1}{\kappa^2+a^2}$$

(b)

$$\mathcal{H}_0\left\{\frac{\delta(r)}{r}\right\} = \tilde{f}_0(\kappa) = \int_0^\infty rJ_n(\kappa r)f(r)dr = \int_0^\infty r.\frac{\delta(r)}{r}J_0(\kappa r)\,dr = 1$$

(c)

$$\mathcal{H}_0\{H(a-r)\} = \tilde{f}_0(\kappa) = \int_0^\infty rJ_n(\kappa r)f(r)dr = \int_0^\infty H(a-r)J_0(\kappa r)\,dr$$

$$\mathcal{H}_0\{H(a-r)\} = \tilde{f}_0(\kappa) = \int_0^a rJ_0(\kappa r)\,dr = \frac{1}{\kappa^2}\int_0^{a\kappa}\rho J_0(\rho)\,d\rho = \frac{1}{\kappa^2}|\rho J_1(\rho)|_0^{a\kappa} = \frac{a}{\kappa}J_1(a\kappa)$$

Example: Find the first-order Hankel transform of the following functions:

(a) $f(r) = e^{-ar}$, **(b)** $f(r) = \frac{1}{r}e^{-ar}$

Solution:

(a)

$$\mathcal{H}_1\{e^{-ar}\} = \tilde{f}(\kappa) = \int_0^\infty rJ_1(\kappa r)f(r)dr = \int_0^\infty re^{-ar}J_1(\kappa r)\,dr = \frac{\kappa}{(\kappa^2+a^2)^{3/2}}$$

(b)

$$\mathcal{H}_1\left\{\frac{1}{r}e^{-ar}\right\} = \tilde{f}(\kappa) = \int_0^\infty rJ_1(\kappa r)f(r)dr = \int_0^\infty r\frac{1}{r}e^{-ar}J_1(\kappa r)\,dr$$

$$\mathcal{H}_1\left\{\frac{1}{r}e^{-ar}\right\} = \int_0^\infty e^{-ar}J_1(\kappa r)\,dr = \frac{1}{\kappa}\left[1 - a(\kappa^2+a^2)^{-1/2}\right]$$

Example: Find the n**th-order Hankel transforms of**

(a) f (r) = $r^n H(a-r)$, **(b)** f (r) = $r^n e^{-ar^2}$

Solution:

(a)

$$\mathcal{H}_n\{r^nH(a-r)\} = \tilde{f}(\kappa) = \int_0^\infty rJ_n(\kappa r)f(r)dr = \int_0^a r^{n+1}J_n(\kappa r)\,dr$$

$$\mathcal{H}_n\{r^nH(a-r)\} = \tilde{f}(\kappa) = \frac{a^{n+1}}{\kappa}J_{n+1}(a\kappa)$$

(b)

$$\mathcal{H}_n\left\{r^ne^{-ar^2}\right\} = \tilde{f}(\kappa) = \int_0^\infty rJ_n(\kappa r)f(r)dr = \int_0^\infty r^{n+1}J_n(\kappa r)e^{-ar^2}\,dr$$

$$\mathcal{H}_n\left\{r^ne^{-ar^2}\right\} = \tilde{f}(\kappa) = \frac{\kappa^n}{(2a)^{n+1}}exp\left(-\frac{\kappa^2}{4a}\right)$$

PROPERTIES OF HANKEL TRANSFORMS AND APPLICATIONS

(i) THE HANKEL TRANSFORM OPERATOR, "H_n" IS A LINEAR INTEGRAL OPERATOR for any constants a and b.

i.e. $\mathcal{H}_n\{af(r)+bg(r)\} = a\mathcal{H}_n\{f(r)\} + b\mathcal{H}_n\{g(r)\}$

Proof: by using definition

$\mathcal{H}_n\{af(r)+bg(r)\} = \int_0^\infty rJ_n(\kappa r)\{af(r)+bg(r)\}dr$

$\mathcal{H}_n\{af(r)+bg(r)\} = a\int_0^\infty rJ_n(\kappa r)\,f(r)dr + b\int_0^\infty rJ_n(\kappa r)\,g(r)dr$

$\mathcal{H}_n\{af(r)+bg(r)\} = a\mathcal{H}_n\{f(r)\} + b\mathcal{H}_n\{g(r)\}$

(ii) THE HANKEL TRANSFORM SATISFIES THE PARSEVAL RELATION

$$\int_0^\infty rf(r)\,g(r)dr = \int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk$$

where $\tilde{f}(\kappa)$ and $\tilde{g}\,(\kappa)$ are Hankel transforms of $f(r)$ and $g(r)$ respectively.

Proof:

$\int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk = \int_0^\infty k\tilde{f}(\kappa)dk\int_0^\infty rJ_n(\kappa r)\,g(r)dr$

$\int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk = \int_0^\infty rg(r)dr\int_0^\infty kJ_n(\kappa r)\tilde{f}(\kappa)dk$

$\int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk = \int_0^\infty rf(r)\,g(r)dr$

(iii) (SCALING PROPERTY). If $\mathcal{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$ then $\mathcal{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right)$; a > 0

Proof. We have, by definition,

$\mathcal{H}_n\{f(ar)\} = \int_0^\infty rJ_n(\kappa r)f(ar)dr = \frac{1}{a^2}\int_0^\infty sJ_n\left(\frac{\kappa}{a}s\right)f(s)ds \quad \therefore ar = s$

$\mathcal{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right)$; a > 0

These results are used very widely in solving partial differential equations in the axisymmetric cylindrical configurations.

Exercises

1. Find the Laplace transform of each of the following functions:

(a) t^n, **(b)** $\cos \omega t$, **(c)** $\sinh kt$, **(d)** $\cosh kt$, **(e)** te^{at}, **(f)** $e^{at} \sin \omega t$, **(g)** $e^{at} \cos \omega t$,

(h) $t \sinh kt$, **(i)** $t \cosh kt$ **(j)** $\sqrt{\frac{1}{t}}$ **(k)** $\sqrt{t}$, **(l)** $\frac{\sin at}{t}$

2. Find the inverse transform of each of the following functions:

(a) $\frac{s}{(s^2+a^2)(s^2+b^2)}$ **(b)** $\frac{1}{(s^2+a^2)(s^2+b^2)}$ **(c)** $\frac{1}{(s-a)(s-b)}$ **(d)** $\frac{1}{s(s+a)^2}$ **(e)** $\frac{1}{s(s+a)}$ **(f)** $\frac{s^2-a^2}{(s^2+a^2)^2}$

3. Obtain the solution of the problem

$u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, t > 0,$

$u(x,0) = f(x), \ u_t(x,0) = 0, u(0,t) = 0, u(x,t) \to 0$ **uniformly in** t **as** $x \to \infty$

4. Solve $u_{tt} = c^2 u_{xx}, \quad 0 < x < l, t > 0,$

$u(x,0) = 0, \ u_t(x,0) = 0, u(0,t) = f(t), u(l,t) = 0, t \geq 0.$

5. Solve $u_t = \kappa u_{xx}, \quad 0 < x < \infty, t > 0,$

$u(x,0) = f_0, \ 0 < x < \infty, \ u(0,t) = f_1, \ u(x,t) \to f_0$ **uniformly in** t **as** $x \to \infty, t > 0.$

6. Solve $u_t = \kappa u_{xx}, \quad 0 < x < \infty, t > 0,$

$u(x,0) = x, x > 0, \ u(0,t) = 0, u(x,t) \to x$ **uniformly in** t **as** $x \to \infty, t > 0.$

7. Solve $u_t = \kappa u_{xx}, \quad 0 < x < \infty, t > 0,$

$u(x,0) = 0, 0 < x < \infty, u(0,t) = t^2, \ u(x,t) \to 0$ **uniformly in** t **as** $x \to \infty, t \geq 0.$

8. Solve $u_t = \kappa u_{xx} - hu, \quad 0 < x < \infty, t > 0, h = constant,$

$u(x,0) = f_0, \ x > 0, \ u(0,t) = 0, \ u_x(0,t) \to 0$ **uniformly in** t **as** $x \to \infty, \ t > 0.$

9. Solve $u_t = \kappa u_{xx}, \quad 0 < x < \infty, t > 0,$

$u(x,0) = 0, 0 < x < \infty, u(0,t) = f_0, \ u(x,t) \to 0$ **uniformly in** t **as** $x \to \infty, t > 0.$

10. Solve $u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, t > 0,$

$u(x,0) = 0, u_t(x,0) = f_0, \ 0 < x < \infty, u(0,t) = 0, u_x(x,t) \to 0$ **uniformly in** t **as**
$x \to \infty, t > 0.$

11. Solve $u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, t > 0,$

$u(x,0) = f(x), \ u_t(x,0) = 0, \ 0 < x < \infty,$

$u(0,t) = 0, \ u_x(x,t) \to 0$ **uniformly in** t **as** $x \to \infty, \ t > 0.$

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خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالٰی آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔(اٰمین)

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