MATHEMATICAL METHODS

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The main objective of this course is to provide the students with a range of mathematical methods that are essential to the solution of advanced problems encountered in the fields of applied physics and engineering. In addition this course is intended to prepare the students with mathematical tools and techniques that are required in advanced courses offered in the applied physics and engineering programs.

Recommended Books:

Powers D. L., Boundary Value Problems and Partial Differential Equations,

Boyce W. E., Elementary Differential Equations, John Wiley and Sons

Krasnov M. L. Makarenko G. I. and Kiselev A. I, Problems and Exercises in the Calculus of Variations, Imported Publications, Inc.

J. W. Brown and R. V. Churchil, Fourier Series and Boundary Value Problems, McGraw Hill

A. D. Snider, Partial Differential Equations: Sources and Solutions, Prentice Hall Inc.

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Available at MathCity.org

Course Contents:

- **Fourier Methods:**
  The Fourier transforms. Fourier analysis of the generalized functions. The Laplace transforms. Hankel transforms for the solution of PDEs and their application to boundary value problems.
- **Green's Functions and Transform Methods:**
  Expansion for Green's functions. Transform methods. Closed form Green's functions.
- **Variational Methods:**
  Euler-Lagrange equations. Integrand involving one, two, three and *n* variables. Special cases of Euler-Lagrange's equations. Necessary conditions for existence of an extremum of a functional. Constrained maxima and minima.
- **Perturbation Techniques:**
  Perturbation methods for algebraic equations. Perturbation methods for differential equations.

FOURIER TRANSFORMATION AND INTEGRALS WITH APPLICATIONS

FOURIER TRANSFORMATION: **If $f\ (x)$ is a continuous, piecewise smooth, and absolutely integrable function, then the Fourier transform of $f\ (x)$with respect to $x \in R$ is denoted by $F\ (k)$and is defined by**

$$\mathcal{F}\{f\ (x)\} = F\ (k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\ \mathrm{dx}$$

where k is called the Fourier transform variable and $exp\ (-ikx)$ is called the kernel of the transform.

Then, for all $x\ \in\ R$, the INVERSE FOURIER TRANSFORM of $F\ (k)$ is defined by

$$\mathcal{F}^{-1}\{F\ (k)\} = f\ (x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ F\ (k)\ \mathrm{dk}$$

CONDITION FOR EXISTENCE OF FOURIER TRANSFORMATION

Fourier Transforamtion and Inverse Fourier Transformation exist if

- **(i)** **The function $f(x)$ or $F(k)$ is continuous or piecewise continuous over $(-\infty, \infty)$ and bounded.**
- **(ii)** **The function $f(x)$ or $F(k)$ are absolutely integrable i.e. $\int_{-\infty}^{\infty} |f(x)|dx$ or $\int_{-\infty}^{\infty} |F(k)|dk$ this condition is sufficient for existence of Fourier Transforamtion and Inverse Fourier Transformation.**

Example:

Show that for a Guassian Function $\mathcal{F}\left\{Ne^{-ax^2}\right\} = \frac{N}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$; $a > 0$, N **is constant.**

Solution. We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} . e^{-ax^2} dx$$

$$\mathcal{F}\{f(x)\} = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-ax^2} dx = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(x-\frac{ik}{2a}\right)^2+\frac{k}{4a^2}^2\right]} dx$$

$$\mathcal{F}\{f(x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2} dx$$

Consider $ikx - ax^2$

$$= -a\left(x^2 - \frac{ikx}{a}\right)$$

$$= -a\left[\left(x^2 - \frac{2ikx}{2a}\right) + \left(\frac{ik}{2a}\right)^2 - \left(\frac{ik}{2a}\right)^2\right]$$

$$= -a\left[\left(x - \frac{ik}{2a}\right)^2 + \frac{k}{4a^2}^2\right]$$

Put $a\left(x - \frac{ik}{2a}\right)^2 = P^2 \Rightarrow \sqrt{a}\left(x - \frac{ik}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2} dx = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-P^2} . \frac{dP}{\sqrt{a}}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}}.\sqrt{\pi} \qquad \therefore \int_{-\infty}^{\infty} e^{-P^2} dP = \sqrt{\pi}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \mathcal{F}\left\{Ne^{-ax^2}\right\} = \frac{N}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$$

Example: Find the Fourier transform of a box function

$$f(x) = \begin{cases} 1 , & |x| < a \text{ or } -a < x < a \\ 0 & , |x| > a \end{cases}$$

Solution. Let we have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} f(x)dx + \int_{-a}^{a} e^{ikx} f(x)dx + \int_{a}^{\infty} e^{ikx} f(x)dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} .0dx + \int_{-a}^{a} e^{ikx} .1dx + \int_{a}^{\infty} e^{ikx} .0dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ikx} dx = \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ika} - e^{-ika}}{2i}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$$

Example:

Find the Fourier transform of $g(\mathrm{x}) = \frac{\mathrm{a}}{x^2+a^2}$

Solution. Let we have, by definition

$$\mathcal{F}\{g(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, g(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, \frac{\mathrm{a}}{x^2+a^2}\,\mathrm{dx}$$

$$\mathcal{F}\{g(z)\} = \frac{a}{\sqrt{2\pi}}\oint_c \frac{e^{\mathrm{ikz}}}{z^2+a^2}\,\mathrm{dz}$$ **replacing 'x' with 'z'**

$\therefore e^{\mathrm{ikz}} = e^{\mathrm{ik}(x+iy)} = e^{\mathrm{ikx}}.e^{\mathrm{i^2ky}} = e^{\mathrm{ikx}}.e^{-\mathrm{ky}} \to 0 \quad as \quad y \to \infty \Rightarrow e^{\mathrm{ikz}} \to 0\ ; k > 0$

Similarly $e^{\mathrm{ikz}} \to 0\ ; k < 0$ **when** $y \to -\infty$

Let $g(z) = \frac{e^{\mathrm{ikz}}}{z^2+a^2} \Rightarrow z = \pm ai$ **are the simple poles of** $g(z)$

Now using $R(f,\alpha) = \lim_{s\to\alpha} \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$$R(g,\alpha i) = R_1 = \lim_{z\to\alpha i}(z-\alpha i)\frac{e^{\mathrm{ikz}}}{(z-\alpha i)(z+\alpha i)} = \lim_{z\to\alpha i}\frac{e^{\mathrm{ikz}}}{(z+\alpha i)} = \frac{e^{\mathrm{ik}(\alpha i)}}{2\alpha i} = \frac{e^{-\alpha\mathrm{k}}}{2\alpha i}$$

Similarly

$$R(g,-\alpha i) = R_2 = \lim_{z\to-\alpha i}(z+\alpha i)\frac{e^{\mathrm{ikz}}}{(z-\alpha i)(z+\alpha i)} = \lim_{z\to\alpha i}\frac{e^{\mathrm{ikz}}}{(z-\alpha i)} = \frac{e^{\mathrm{ik}(-\alpha i)}}{-2\alpha i} = \frac{e^{\alpha\mathrm{k}}}{-2\alpha i}$$

Now $\Rightarrow \mathcal{F}\{g(x)\} = \frac{a}{\sqrt{2\pi}}\oint_c \frac{e^{\mathrm{ikz}}}{z^2+a^2}\mathrm{dz} = \frac{a}{\sqrt{2\pi}}.2\pi i\sum_j R_j = \frac{a}{\sqrt{2\pi}}.2\pi i[R_1+R_2]$

Now we use $2\pi i$ **for the contour as a semi circle in upper half plane and** $-2\pi i$ **for the contour as a semi circle in lower half plane**

$$\Rightarrow \mathcal{F}\{g(x)\} = \frac{a}{\sqrt{2\pi}}.[(2\pi i)R_1 + (2\pi i)R_2] = \frac{a}{\sqrt{2\pi}}.2\pi i[R_1 - R_2]$$

$$\Rightarrow \mathcal{F}\{g(x)\} = \frac{a}{\sqrt{2\pi}}.2\pi i\left[\frac{e^{-\alpha\mathrm{k}}}{2\alpha i} + \frac{e^{\alpha\mathrm{k}}}{2\alpha i}\right] = \sqrt{\frac{\pi}{2}}\left[e^{-\alpha\mathrm{k}} + e^{\alpha\mathrm{k}}\right]$$

$$\Rightarrow \mathcal{F}\{g(x)\} = \sqrt{\frac{\pi}{2}}\left[e^{\alpha|\mathrm{k}|} + e^{\alpha|\mathrm{k}|}\right] \qquad \therefore k > 0, k < 0 \Rightarrow |\mathrm{k}| = \pm k$$

$$\Rightarrow \mathcal{F}\{g(x)\} = \sqrt{\frac{\pi}{2}}.2e^{\alpha|\mathrm{k}|} = \sqrt{2\pi}e^{\alpha|\mathrm{k}|}$$

Example: Show that $\mathcal{F}\left\{e^{-\mathrm{ax}^2}\right\} = \frac{1}{\sqrt{2\mathrm{a}}} e^{\left(-\frac{k^2}{4a}\right)} \quad ; a > 0$

Solution. We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}} f(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}} . e^{-\mathrm{ax}^2} \mathrm{dx}$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}-\mathrm{ax}^2} \mathrm{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left[\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2 + \frac{\mathrm{k}^2}{4a^2}\right]} \mathrm{dx}$$

$$\mathcal{F}\{f(x)\} == \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2} \mathrm{dx}$$

Put $a\left(\mathrm{x} - \frac{\mathrm{ik}}{2a}\right)^2 = \mathrm{P}^2 \Rightarrow \sqrt{a}\left(\mathrm{x} - \frac{\mathrm{ik}}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2} \mathrm{dx} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{P}^2} . \frac{dP}{\sqrt{a}}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}} . \sqrt{\pi} \qquad \therefore \int_{-\infty}^{\infty} e^{-\mathrm{P}^2} dP = \sqrt{\pi}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \mathcal{F}\left\{e^{-\mathrm{ax}^2}\right\} = \frac{1}{\sqrt{2\mathrm{a}}} e^{\left(-\frac{k^2}{4a}\right)}$$

Consider $\mathrm{ikx} - \mathrm{ax}^2$

$$= -\mathrm{a}\left(x^2 - \frac{\mathrm{ikx}}{a}\right)$$

$$= -\mathrm{a}\left[\left(x^2 - \frac{2\mathrm{ikx}}{2a}\right) + \left(\frac{\mathrm{ik}}{2a}\right)^2 - \left(\frac{\mathrm{ik}}{2a}\right)^2\right]$$

$$= -a\left[\left(\mathrm{x} - \frac{\mathrm{ik}}{2a}\right)^2 + \frac{\mathrm{k}^2}{4a^2}\right]$$

Example: Show that $\mathcal{F}\left\{e^{-\mathrm{a}|\mathrm{x}|}\right\} = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)} \quad ; a > 0$

Solution. We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}} f(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}} . e^{-\mathrm{a}|\mathrm{x}|} \mathrm{dx}$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}-\mathrm{a}|\mathrm{x}|} \mathrm{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(\mathrm{a}+\mathrm{ik})\mathrm{x}} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(\mathrm{a}-\mathrm{ik})\mathrm{x}} dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\mathrm{a}+\mathrm{ik}} + \frac{1}{\mathrm{a}-\mathrm{ik}}\right] = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}$$

Example: Show that $\mathcal{F}\left\{\mathcal{X}_{[-a,a]}(x)\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$

where $\mathcal{X}_{[-a,a]}(x) = \mathrm{H}(\mathrm{a} - |\mathrm{x}|) = \begin{cases} 1 \ , & |\mathrm{x}| < a \text{ or } -\mathrm{a} < x < a \\ 0 & , \ |\mathrm{x}| > a \end{cases}$

Solution. Let us consider $f(x) = \mathcal{X}_{[-a,a]}(x)$ **then We have, by definition**

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{ikx}} f(x)\mathrm{dx}$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} f(x)dx + \int_{-a}^{a} e^{ikx} f(x)dx + \int_{a}^{\infty} e^{ikx} f(x)dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} .0dx + \int_{-a}^{a} e^{ikx} .1dx + \int_{a}^{\infty} e^{ikx} .0dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ikx} dx = \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ika} - e^{-ika}}{2i}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$$

PROPERTIES OF FOURIER TRANSFORMS

LINEARITY PROPERTY: THE FOURIER TRANSFORMATION $\mathcal{F}$ IS LINEAR.

Proof. Let $u\,(x) = af(x) + bg(x)$ where a and b are constants.

We have, by definition

$$\mathcal{F}\,\{u\,(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ \mathrm{u\,(x)dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ [af(x) + bg(x)]\mathrm{dx}$$

$$\mathcal{F}\,\{u\,(x)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f(x)\mathrm{dx} + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ g(x)\mathrm{dx}$$

$$\mathcal{F}\,\{u\,(x)\} = \mathrm{a}\mathcal{F}\,\{f(x)\} + b\mathcal{F}\,\{g(x)\}$$

$$\mathcal{F}\,\{af(x) + bg(x)\} = \mathrm{a}\mathcal{F}\,\{f(x)\} + b\mathcal{F}\,\{g(x)\}$$ **hence proved.**

LINEARITY PROPERTY: THE INVERSE FOURIER TRANSFORMATION $\mathcal{F}^{-1}$ IS LINEAR.

Proof. Let $U\,(k) = aF(k) + bG(k)$ where a and b are constants.

We have, by definition

$$\mathcal{F}^{-1}\,\{U\,(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ \mathrm{U(k)dk} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ [aF(k) + bG(k)]\mathrm{dk}$$

$$\mathcal{F}^{-1}\,\{U\,(k)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ F(k)\mathrm{dk} + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ G(k)\mathrm{dk}$$

$$\mathcal{F}^{-1}\,\{aF(k) + bG(k)\} = \mathrm{a}\mathcal{F}^{-1}\,\{F(k)\} + \mathrm{b}\mathcal{F}^{-1}\,\{G(k)\}$$ **hence proved.**

SHIFTING PROPERTY: Let $\mathcal{F}\,\{f\,(x)\}$ be a Fourier transform of $f(x)$. Then

(i) $\mathcal{F}\,[f\,(x\,-\,a)] = e^{\mathrm{ika}}\,F(k)$ where 'a' is a real constant.

Proof. From the definition, we have, for a > 0,

$$\mathcal{F}\,[f\,(x\,-\,a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\,(x\,-\,a)\mathrm{dx}$$

Put $x\,-a = x' \Rightarrow dx = dx'$ also as $x \to \pm\infty$ then $x' \to \pm\infty$

$$\mathcal{F}\,[f\,(x\,-\,a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}(x'+a)}\ f\,(x')\mathrm{dx'} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx'}}.\,e^{\mathrm{ika}}\ f\,(x')\mathrm{dx'}$$

$$\mathcal{F}\,[f\,(x\,-a)] = e^{\mathrm{ika}}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx'}}\ f\,(x')\mathrm{dx'} = e^{\mathrm{ika}}\,\mathcal{F}\,\{f\,(x)\} = e^{\mathrm{ika}}\,F(k)$$

(ii) $\mathcal{F}\,[e^{\mathrm{iax}} f\,(x)] = F(k+a)$ **where 'a' is a real constant.**

Proof. From the definition, we have, for a > 0,

$\mathcal{F}\,[e^{\mathrm{iax}} f\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, e^{\mathrm{iax}} f\,(x)\,\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k+a)\mathrm{x}} f\,(x)\,\mathrm{dx} = F(k+a)$

SCALING PROPERTY: If $\mathcal{F}$ is the Fourier transform of f, then

$\mathcal{F}\,[f\,(cx)] = (\frac{1}{|c|})\, F\,(\frac{k}{c})$ **where c is a real nonzero constant.**

Proof. For $c \neq 0$ we have $\mathcal{F}\,[f\,(cx)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, f\,(cx)\mathrm{dx}$

$\mathcal{F}\,[f\,(cx)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}\left(\frac{x'}{c}\right)}\, f\,(x')\,\frac{dx'}{c} = \frac{1}{c}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}\left(\frac{x'}{c}\right)}\, f\,(x')dx' = \frac{1}{c}F\,(\frac{k}{c})$

Since $c \neq 0$ then either $c < 0$ or $c > 0$

If $c > 0$ then $\mathcal{F}\,[f\,(cx)] = \frac{1}{+c}F\,(\frac{k}{c})$ If $c < 0$ then $\mathcal{F}\,[f\,(cx)] = \frac{1}{-c}F\,(\frac{k}{c})$

Hence $\mathcal{F}\,[f\,(cx)] = (\frac{1}{|c|})\, F\,(\frac{k}{c})$

CONJUGATION PROPERTY: Let f is real then $F\,(-k) = \overline{F\,(k)}$

Proof. Since f is real therefore $f\,(x) = \overline{f\,(x)}$ then by defination

$F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, f\,(x)\mathrm{dx}$

$\overline{F\,(k)} = \mathcal{F}\,[\overline{f\,(x)}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, \overline{f\,(x)}\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(-k)\mathrm{x}}\, f\,(x)\mathrm{dx} = F\,(-k)$

Hence $F\,(-k) = \overline{F\,(k)}$

ATTENUATION PROPERTY:

For a function $f\,(x)$ the result will be , $\mathcal{F}\,[e^{ax} f\,(x)] = F\,(k - ai)$

Proof. By definition $F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, f\,(x)\mathrm{dx}$

Then $\mathcal{F}\,[e^{ax} f\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, e^{ax} f\,(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, e^{-i^2 ax} f\,(x)\mathrm{dx}$

$\mathcal{F}\,[e^{ax} f\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k-ai)\mathrm{x}}\, f\,(x)\mathrm{dx}$ **……….(i)**

Also $F\,(k - ai) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k-ai)\mathrm{x}}\, f\,(x)\mathrm{dx}$ ……….(ii)

Thus from (i) and (ii) $\mathcal{F}\,[e^{ax} f\,(x)] = F\,(k - ai)$

MODULATION PROPERTY(i):

$$\mathcal{F}\,[Cosaxf\,(x)] = \frac{1}{2}[F\,(k+a) + F\,(k-a)]$$

Proof. By definition $\mathcal{F}\,[Cosaxf\,(x)] = \mathcal{F}\,[\left(\frac{e^{iax}+e^{-\text{iax}}}{2}\right)f\,(x)]$

$$\mathcal{F}\,[Cosaxf\,(x)] = \frac{1}{2}[\,\mathcal{F}\,\{e^{iax}f\,(x)\} + \mathcal{F}\,\{e^{-iax}f\,(x)\}] = \frac{1}{2}[F\,(k+a) + F\,(k-a)]$$

MODULATION PROPERTY (ii):

$$\mathcal{F}\,[Sinaxf\,(x)] = \frac{1}{2i}[F\,(k+a) - F\,(k-a)]$$

Proof. By definition $\mathcal{F}\,[Sinaxf\,(x)] = \mathcal{F}\,[\left(\frac{e^{iax}-e^{-\text{iax}}}{2i}\right)f\,(x)]$

$$\mathcal{F}\,[Cosaxf\,(x)] = \frac{1}{2i}[\,\mathcal{F}\,\{e^{iax}f\,(x)\} - \mathcal{F}\,\{e^{-iax}f\,(x)\}] = \frac{1}{2i}[F\,(k+a) - F\,(k-a)]$$

ROPERTY: if $f\,(x)$ **is real and even then** $F\,(k)$ **is real.**

Proof. Since f **is real therefore** $f\,(x) = \overline{f\,(x)}$ **....(i) and** $f\,(-x) = f\,(x)$ **……(ii) then by defination**

$$F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\,f\,(x)\text{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\,f\,(-x)\text{dx}$$

$$F\,(k) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\text{ikx}'}\,f\,(x')(-dx') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\text{ikx}'}\,\overline{f\,(x')}dx'$$

Hence $F\,(k) = \overline{F\,(k)}$ **then** $F\,(k)$ **is real.**

ROPERTY: if $f\,(x)$ **is real and odd then** $F\,(k)$ **is pure imaginary.**

Proof. Since f **is real therefore** $f\,(x) = \overline{f\,(x)}$ **………..(i)**

and is odd $f\,(-x) = -f\,(x)$ **………(ii) then by defination**

$$F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\,f\,(x)\text{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\,(-f\,(-x))\text{dx}$$

$$F\,(k) = \mathcal{F}\,[f(x)] = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\,f\,(-x)\text{dx}$$

$$F\,(k) = \frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\text{ikx}'}\,f\,(x')(-dx') = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\text{ikx}'}\,f\,(x')dx'$$

Hence $F\,(k) = -\overline{F\,(k)}$ **or** $\overline{F\,(k)} = -F\,(k)$ **then** $F\,(k)$ **is pure imaginary.**

ROPERTY: if $f\,(x)$ is complex then $\mathcal{F}\left[\overline{f(-x)}\right] = \overline{F\,(k)}$

Proof. by definition

$$\mathcal{F}\left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\overline{f(-x)}dx = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{ik(-x')}\,\overline{f(x')}(-dx')$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-ikx'}\,\overline{f(x')}dx' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'}\,\overline{f(x')}dx'$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\overline{f(x)}dx \qquad \textbf{replacing } x' \textbf{ with x}$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,f(x)dx} = \overline{F\,(k)}$$

$$\mathcal{F}\left[\overline{f(-x)}\right] = \overline{F\,(k)} \qquad \textbf{as required.}$$

DIFFERENTIATION PROPERTY (higher derivative theorem):

Let f be continuous and piecewise smooth in $(-\infty,\infty)$. Let $f(x)$ approach zero as $|x| \to \infty$. If f and f' are absolutely integrable, then

$$\mathcal{F}\,[f'\,(x)] = (-ik)\mathcal{F}\,[f\,(x)] = (-ik)F\,(k)$$

Proof.

$$\mathcal{F}\,[f'\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,f'\,(x)dx$$

$$\mathcal{F}\,[f'(x)] = \frac{1}{\sqrt{2\pi}}\left[\left|e^{ikx}f\,(x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{ikx}\,(ik)\,f\,(x)dx\right]$$

$$\mathcal{F}\,[f'(x)] = \frac{1}{\sqrt{2\pi}}\left[0 + (-ik)\int_{-\infty}^{\infty} e^{ikx}\,f\,(x)dx\right]$$

$$\mathcal{F}\,[f'\,(x)] = (-ik)\mathcal{F}\,[f\,(x)] = (-ik)F\,(k)$$

For n = 2

$$\mathcal{F}\,[f''\,(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,f''\,(x)dx$$

$$\mathcal{F}\,[f''(x)] = \frac{1}{\sqrt{2\pi}}\left[\left|e^{ikx}f'(x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{ikx}\,(ik)\,f'(x)dx\right]$$

$$\mathcal{F}\,[f''(x)] = \frac{1}{\sqrt{2\pi}}\left[0 + (-ik)\int_{-\infty}^{\infty} e^{ikx}\,f'(x)dx\right]$$

$$\mathcal{F}\,[f''(x)] = (-ik)\mathcal{F}\,[f'(x)] = (-ik)(-ik)F\,(k) = (-ik)^2\,F\,(k)$$

This result can be easily extended. If f and its first (n − 1) derivatives are continuous, and if its nth derivative is piecewise continuous, then

$\mathcal{F}\,[f^n(x)] = (-ik)^n \mathcal{F}\,[f\,(x)] = (-ik)^n\, F\,(k) \quad n = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots$

provided f and its derivatives are absolutely integrable. In addition, we assume that f and its first (n − 1) derivatives tend to zero as |x| tends to infinity.

CONVOLUTION FUNCTION / FAULTUNG FUNCTION

The function $(f * g)\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\,(x - \xi)\, g\,(\xi)\, d\xi$

is called the convolution of the functions f and g over the interval (−∞,∞)

NOTE: The convolution satisfies the following properties:

1. $f * g = g * f$ (commutative)
2. $f * (g * h) = (f * g) * h$ (associative)
3. $* \,(ag + bh) = a\,(f * g) + b\,(f * h)$, (distributive)

where a and b are constants.

PROPERTY: $f * g = g * f$

PROOF: since by definition $(f * g)\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\,(x - \xi)\, g\,(\xi)\, d\xi$

Put $x - \xi = \propto \Rightarrow d\xi = -d \propto also\ \xi = x - \propto$ and if $\xi \to \pm\infty$ then $\propto \to \mp\infty$ then

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f\,(\propto) g\,(x - \propto)\,(-d\propto) = g * f$$

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g\,(x - \propto) f\,(\propto)\ \ (d\propto) = g * f$$

Hence $f * g = g * f$

CONVOLUTION / FAULTUNG THEOREM

If $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution $(f * g)$ is the product $F(k)G(k)$. That is,

$\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$

Or, equivalently, $\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$

Or

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk = (f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$$

PROOF: By definition, we have

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk$$

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} g(x')\,dx'\right\} dk$$

By changing the order of integration

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x')} F(k)\,dk\right] g(x')\,dx'$$

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x')g(x')\,dx'$$

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi = (f * g)(x)$$

Where we replace ξ with x'

Hence $\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$

Or $\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$

PARSEVAL'S FORMULA OF 1ST AND 2ND KIND

Theorem given by Marc Anotoine des Chenes Parseval (1755 – 1836)

1ST KIND: According to this formula $\int_{-\infty}^{\infty} |f(\mathrm{x})|^2\, dx = \int_{-\infty}^{\infty} |F(\mathrm{k})|^2\, dk$

PROOF: The convolution formula gives

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, F(k)G(k)\mathrm{dk} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi$

$\int_{-\infty}^{\infty} f(\xi) g(x-\xi)\, d\xi = \int_{-\infty}^{\infty} e^{-\mathrm{ik}x}\, F(k)G(k)\mathrm{dk}$

which is, by putting $x = 0$ $\quad \int_{-\infty}^{\infty} f(\xi) g(-\xi)\, d\xi = \int_{-\infty}^{\infty} F(k)G(k)\mathrm{dk}$

$\int_{-\infty}^{\infty} f(x) g(-x)\, dx = \int_{-\infty}^{\infty} F(k)G(k)\mathrm{dk}$

Putting $g(-x) = \overline{f(x)}$ **then** $g(x) = \overline{f(-x)} \Rightarrow \mathcal{F}\{g(x)\} = \mathcal{F}\left\{\overline{f(-x)}\right\}$

$\Rightarrow G(k) = \overline{F(k)}$ $\quad \therefore \mathcal{F}\left\{\overline{f(-x)}\right\} = \overline{F(k)}$ **for complex** f.

$\int_{-\infty}^{\infty} f(x)\overline{f(x)}\, dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}\mathrm{dk}$

where the bar denotes the complex conjugate.

$\Rightarrow \int_{-\infty}^{\infty} |f(\mathrm{x})|^2\, dx = \int_{-\infty}^{\infty} |F(\mathrm{k})|^2\, dk$

In terms of the notation of the norm, this is $\|f\| = \|F\|$

2ND KIND: According to this formula

$\int_{-\infty}^{\infty} F(k)G(k)\, dk = \int_{-\infty}^{\infty} f(u) g(-u)\, du$

PROOF: The convolution formula gives

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, F(k)G(k)\mathrm{dk} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) g(x-u) du$

by putting $x = 0$ **we get** $\int_{-\infty}^{\infty} F(k)G(k)\, \mathrm{dk} = \int_{-\infty}^{\infty} f(u) g(-u) du$

BOUNDEDNESS AND CONTINUITY OF FOURIER TRANSFORMATION

If $f(x)$ is piecewise smooth and absolutely integrable function on the interval $(-\infty,\infty)$ then its fourier transformation $F(k)$ is bounded and continuous.

PROOF: given that $f(x)$ is piecewise smooth and absolutely integrable function i.e. $J=\int_{-\infty}^{\infty}|f(x)|dx$

now by definition $F(k)=\mathcal{F}[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx$

For boundedness taking mod on both sides

$\Rightarrow |F(k)|=\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx\right|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|e^{ikx}\right||f(x)|dx$

$\Rightarrow |F(k)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(x)|\,dx$ **since** $\left|e^{ikx}\right|=1$

$\Rightarrow |F(k)|\le\frac{1}{\sqrt{2\pi}}.J$ **since** $J=\int_{-\infty}^{\infty}|f(x)|dx$

$\Rightarrow |F(k)|\le\lambda$ **where** $\lambda=\frac{1}{\sqrt{2\pi}}.J\in R$

$\Rightarrow F(k)$ **is bounded.**

Now for continuity of $F(k)$ we have

$F(k+h)-F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(k+h)x}f(x)dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx$

$F(k+h)-F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\left(e^{ihx}-1\right)f(x)dx=I(k,h)$ **say**

$\lim_{h\to0}[F(k+h)-F(k)]=\lim_{h\to0}I(k,h)$ **..........(i)**

Now $\lim_{h\to0}I(k,h)$ exists if $I(k,h)$ is uniformly convergent.

For this consider

$\Rightarrow |I(k,h)|=\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\left(e^{ihx}-1\right)f(x)dx\right|$

$\Rightarrow |I(k,h)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|e^{ikx}\right|\left|e^{ihx}-1\right||f(x)|dx$

$\Rightarrow |I(k,h)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1)\,|Coshx+iSinhx-1||f(x)|dx$

$\Rightarrow |I(k,h)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|(Coshx-1)+iSinhx||f(x)|\,dx$

$$\Rightarrow |\mathrm{I}(k,h)| \leq \frac{1}{\sqrt{2\pi}}.\sqrt{2}\int_{-\infty}^{\infty}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx}$$

$$\Rightarrow |\mathrm{I}(k,h)| \leq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx}$$

$$\Rightarrow \lim_{h\to 0}|\mathrm{I}(k,h)| \leq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{h\to 0}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx} \quad \to 0$$

$$\Rightarrow \lim_{h\to 0}|\mathrm{I}(k,h)| \leq 0 \Rightarrow \lim_{h\to 0}\mathrm{I}(k,h) = 0$$

$$(i) \Rightarrow \lim_{h\to 0}[F\ (k+h) - F\ (k)] = 0$$

$\Rightarrow \lim_{h\to 0} F\ (k+h) = F\ (k) \Rightarrow F\ (k)$ **is continuous.**

Hence If $f(x)$ is piecewise smooth and absolutely integrable function on the interval $(-\infty,\infty)$ then its fourier transformation $F(k)$ is bounded and continuous.

RIEMANN LEBESQUE THEOREM

If $f(x)$ is piecewise smooth and absolutely integrable function then

$$\lim_{|\mathrm{k}|\to\infty} F\ (k) = 0$$

PROOF: given that $f(x)$ is piecewise smooth and absolutely integrable function i.e. $J = \int_{-\infty}^{\infty}|f(x)|dx$

now by definition $F\ (k) = \mathcal{F}\ [f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$

$$F\ (k) = \frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]$$

$$\Rightarrow |F\ (k)| = \left|\frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]\right|$$

$$\Rightarrow |F\ (k)| = \left|\frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]\right|$$

$$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left|\,|f(x)|\frac{|e^{\mathrm{ikx}}|}{|ik|}\right|_{-\infty}^{\infty} + \left|-\int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right|$$

$$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left[\lim_{x\to\infty}\frac{|f(x)|}{|k|} - \lim_{x\to-\infty}\frac{|f(x)|}{|k|}\right] + \int_{-\infty}^{\infty}\frac{|e^{\mathrm{ikx}}|}{|ik|}|f'(x)|\mathrm{dx}$$

$$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left[\lim_{x\to\infty}\frac{|f(x)|}{|k|} - \lim_{x\to-\infty}\frac{|f(x)|}{|k|}\right] + \int_{-\infty}^{\infty}\frac{1}{|k|}|f'(x)|\mathrm{dx} \quad \text{........(i)}$$

Since $f(x)$ is absolutely integrable function then $\lim_{x\to\pm\infty}|f(x)| = 0$

$(i) \Rightarrow |F\,(k)| \le \frac{1}{\sqrt{2\pi}}.\frac{1}{|k|}\int_{-\infty}^{\infty}|f'(x)|\,\mathrm{dx}$ ……..(ii)

Since $f(x)$ is piecewise smooth then $f'(x)$ will be piecewise continuous and therefore $\int_{-\infty}^{\infty}|f'(x)|\,dx = I$

$(ii) \Rightarrow \lim_{|\mathrm{k}|\to\infty}|F\,(k)| \le \lim_{|\mathrm{k}|\to\infty}\frac{1}{\sqrt{2\pi}}.\frac{1}{|k|}.I = 0 \Rightarrow \lim_{|\mathrm{k}|\to\infty}|F\,(k)| = 0$

FOURIER TRANSFORM OF THE FUNCTION OF THE FORM $[x^n f(x)]$

Let f be piecewise continuous on the interval $[-l, l]$ for every positive 'l' and $\int_{-\infty}^{\infty}|x^n f(x)|$ converges then

$\mathcal{F}\,[x^n f(x)] = \frac{1}{i^n}F^n(k) = i^{-n}F^n(k)$; $n = 0, 1, 2, \ldots$……………

Proof. By definition $\mathcal{F}\,[f\,(x)] = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, f(x)\mathrm{dx}$

$\Rightarrow F'(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,(ix)f(x)\mathrm{dx}$ **diff. w.r.to 'k'**

$\Rightarrow i^{-1}F'(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,(x)f(x)\mathrm{dx} = \mathcal{F}\,[xf(x)] = i^{-1}F^1(k)$

$\Rightarrow F''(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,(ix)^2 f(x)\mathrm{dx}$ **again diff. w.r.to 'k'**

$\Rightarrow i^{-2}F''(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,(x^2)f(x)\mathrm{dx} = \mathcal{F}\,[x^2 f(x)] = i^{-2}F^2(k)$

Continuing in this mannar we can get the required result as follows;

$\mathcal{F}\,[x^n f(x)] = i^{-n}F^n(k) = \frac{1}{i^n}F^n(k)$; $n = 0, 1, 2, \ldots$……………

$\mathcal{F}\,[x^n f(x)] = (-i)^n\frac{d^n}{dk^n}F(k)$; $n = 0, 1, 2, \ldots$……………

Where we use the result $i^{-n} = \left(\frac{1}{i}\right)^n = \left(\frac{1}{i}\times\frac{i}{i}\right)^n = \left(\frac{i}{i^2}\right)^n = (-i)^n$

FOURIER TRANSFORM OF AN INTEGRAL

Let f be piecewise continuous on the interval $(-\infty,\infty)$ and that $\int_{-\infty}^{\infty}|f(x)|<\infty$ also $F(0)=0$ with $\mathcal{F}\,[f\,(x)]=F(k)$ then

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,\mathrm{d}x'\right\}=\frac{1}{-ik}F(k)=\frac{i}{k}F(k)$$

Proof. Let $g(x)=\int_{-\infty}^{x} f(x')\,\mathrm{d}x'$ **……………..(i)**

Given that $\mathcal{F}\,[f\,(x)]=F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,f(x)\mathrm{dx}$

$\Rightarrow F(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ **putting k = 0 also** $e^{0}=1$

$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}=0$ **since** $F(0)=0$

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\mathrm{dx}=0 \Rightarrow \lim\limits_{x\to\infty}\int_{-\infty}^{x} f(x')\,\mathrm{dx}'=0 \Rightarrow \lim\limits_{x\to\infty} g(x)=0$

Now from (i) we get by using Leibniz Rule

$g'(x)=f(x') \Rightarrow \mathcal{F}\,\{g'(x)\}=\mathcal{F}\,\{f(x')\} \Rightarrow (-ik)\mathcal{F}\,\{g(x)\}=F(k)$

$\Rightarrow \mathcal{F}\,\{g(x)\}=\frac{1}{-i\mathrm{k}}F(k)$

$\Rightarrow \mathcal{F}\,\{g(x)\}=\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,\mathrm{dx}'\right\}=\frac{1}{-i\mathrm{k}}F(k)=\frac{i}{\mathrm{k}}F(k)$

FOURIER INTEGRAL THEOREM

If $f\,(x)$ is real valued function over $(-\infty,+\infty)$ and the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ is absolutely convergent then $f\,(x)=\frac{1}{\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

PROOF: Since $\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ is absolutely convergent then F.T and I.F.T of function exists.

$f\,(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\,F(k)\mathrm{dk}$ **since** $\mathcal{F}^{-1}\,\{F\,(k)\}=f\,(x)$

$f\,(x)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-\mathrm{ikx}}\,F(k)\mathrm{dk}+\int_{0}^{\infty} e^{-\mathrm{ikx}}\,F(k)\mathrm{dk}\right]$ **…………(i)**

Put in 1st term $-k = k' \Rightarrow dk = -dk'$ **also if** $k \to -\infty, 0$ **then** $k' \to \infty, 0$

$(i) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{ik'x} F(-k')(-dk') + \int_{0}^{\infty} e^{-ikx} F(k)dk\right]$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{ik'x} F(-k')dk' + \int_{0}^{\infty} e^{-ikx} F(k)dk\right]$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{ikx} F(-k)dk + \int_{0}^{\infty} e^{-ikx} F(k)dk\right]$ **replacing** k' **with k**

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\left[e^{ikx}\overline{F(k)} + e^{-ikx}F(k)\right]dk$ **…………(ii)** $\quad \therefore F(-k) = \overline{F(k)}$

Consider $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')dx'$

$\Rightarrow \overline{F(k)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'} \overline{f(x')}dx'$ **taking conjugate**

Then $e^{-ikx}F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x')} f(x')dx'$

Also $e^{ikx}\overline{F(k)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(x-x')} \overline{f(x')}dx'$

Since $f(x)$ **is real therefore** $\overline{f(x')} = f(x')$

Now $e^{ikx}\overline{F(k)} + e^{-ikx}F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{ik(x-x')} + e^{-ik(x-x')}\right]f(x')dx'$

$e^{ikx}\overline{F(k)} + e^{-ikx}F(k) = \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}\left[e^{ik(x-x')} + e^{-ik(x-x')}\right]f(x')dx'$

$e^{ikx}\overline{F(k)} + e^{-ikx}F(k) = \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Cos\,k(x-x')\,f(x')dx'$

$(ii) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Cos\,k(x-x')\,f(x')dx'\,dk$

$f(x) = \frac{2}{2\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} Cos\,k(x-x')f(x')dx'$

$f(x) = \frac{1}{\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} Cos\,k(x-x')f(x')dx'$ **as required.**

THE FOURIER TRANSFORMS OF STEP AND IMPULSE FUNCTIONS

The Heaviside unit step function is defined by

$$H(x-a) = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases} \qquad \text{where } a \geq 0$$

The Fourier transform of the Heaviside unit step function can be easily determined. We consider first

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, H(x-a)dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{ikx}\, H(x-a)dx + \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, H(x-a)dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{ikx}\,.0dx + \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\,.1dx = \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, dx$$

This integral does not exist. However, we can prove the existence of this integral by defining a new function

$$H(x-a)e^{-\alpha x} = \begin{cases} 0 & x < a \\ e^{-\alpha x} & x \geq a \end{cases}$$

This is evidently the unit step function as $\alpha \to 0$. Thus, we find the Fourier transform of the unit step function as

$$\mathcal{F}[H(x-a)] = \lim_{\alpha \to 0} \mathcal{F}[H(x-a)e^{-\alpha x}]$$

$$\mathcal{F}[H(x-a)] = \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, H(x-a)e^{-\alpha x}dx$$

$$\mathcal{F}[H(x-a)] = \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, e^{-\alpha x}dx = \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{i(k-\alpha)x}\, dx$$

$$\mathcal{F}[H(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ikx}\, dx = \frac{e^{ika}}{\sqrt{2\pi}ik} \qquad \text{For } a = 0 \Rightarrow \mathcal{F}[H(x)] = \frac{1}{\sqrt{2\pi}ik}$$

An impulse function is defined by

$$p(x) = \begin{cases} h & a-\varepsilon < x < a+\varepsilon \\ 0 & x \leq a-\varepsilon \text{ or } x \geq a+\varepsilon \end{cases}$$

where h is large and positive, $a > 0$, and ε is a small positive constant, This type of function appears in practical applications; for instance, a force of large magnitude may act over a very short period of time.

The Fourier transform of the impulse function is

$$\mathcal{F}\,[p\,(x\,)\,] \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\; p\,(x\,)\mathrm{dx}$$

$$\mathcal{F}\,[p\,(x\,)\,] \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\mathrm{a}-\varepsilon} e^{\mathrm{ikx}}\; p\,(x\,)\mathrm{dx} + \frac{1}{\sqrt{2\pi}}\int_{\mathrm{a}-\varepsilon}^{\mathrm{a}+\varepsilon} e^{\mathrm{ikx}}\; p\,(x\,)\mathrm{dx} + \frac{1}{\sqrt{2\pi}}\int_{\mathrm{a}+\varepsilon}^{\infty} e^{\mathrm{ikx}}\; p\,(x\,)\mathrm{dx}$$

$$\mathcal{F}\,[p\,(x\,)\,] \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathrm{a}-\varepsilon}^{\mathrm{a}+\varepsilon} he^{\mathrm{ikx}}\;\mathrm{dx} = \frac{h}{\sqrt{2\pi}}\left|\frac{e^{\mathrm{ikx}}}{ik}\right|_{\mathrm{a}-\varepsilon}^{\mathrm{a}-\varepsilon}$$

$$\mathcal{F}\,[p\,(x\,)\,] \;=\; \frac{h}{\sqrt{2\pi}}.\frac{1}{ik}\left(e^{\mathrm{ik}(\mathrm{a}+\varepsilon)} - e^{\mathrm{ik}(\mathrm{a}-\varepsilon)}\right)$$

$$\mathcal{F}\,[p\,(x\,)\,] \;=\; \frac{h}{\sqrt{2\pi}}.\frac{e^{\mathrm{ika}}}{ik}\left(e^{\mathrm{ik}\varepsilon} - e^{-\mathrm{ik}\varepsilon}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{\mathrm{ika}}\left(\frac{e^{\mathrm{ik}\varepsilon} - e^{-\mathrm{ik}\varepsilon}}{2ik\varepsilon}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{\mathrm{ika}}\left(\frac{Sink\varepsilon}{k\varepsilon}\right)$$

Now if we choose the value of $h \;=\; (\frac{1}{2\varepsilon})$ then the impulse defined by

$$I\,(\varepsilon) = \int_{-\infty}^{\infty} p\,(x\,)\mathrm{dx} = \int_{\mathrm{a}-\varepsilon}^{\mathrm{a}+\varepsilon}\frac{1}{2\varepsilon}\mathrm{dx} = 1$$

which is a constant independent of ε. In the limit as $\varepsilon \to 0$, this particular function $p_{\varepsilon}\,(x\,)$ with h = (1/2ε) satisfies $\lim_{\varepsilon\to 0} p_{\varepsilon}\,(x\,) = 0\;\; ; x \neq 0$

and $\lim_{\varepsilon\to 0} I\,(\varepsilon) = 1$

Thus, we arrive at the result $\delta\,(x\,-a) = 0, x \neq\; a,$ and $\int_{-\infty}^{\infty}\delta(x-a)\mathrm{dx} = 1$

This is the Dirac delta function

We now define the Fourier transform of $\delta(x)$ as the limit of the transform of $p_{\varepsilon}\,(x\,)$. We then consider

$$\mathcal{F}\,[\delta\,(x\,-\,a)\,] \;=\; \lim_{\varepsilon\to 0}\mathcal{F}\,[p_{\varepsilon}\,(x\,)\,] \;=\lim_{\varepsilon\to 0}\frac{e^{\mathrm{ika}}}{\sqrt{2\pi}}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) \;=\frac{e^{\mathrm{ika}}}{\sqrt{2\pi}}$$

in which we note that, by L'Hospital's rule, $\lim_{\varepsilon\to 0}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) \;=1$

When a = 0, we obtain $\qquad \mathcal{F}\,[\delta\,(x)\,] \;=\frac{1}{\sqrt{2\pi}}$

FOURIER COSINE TRANSFORMATION AND INVERSE

Let $f\,(x)$ be defined for $0 \le x < \infty$, and extended as an even function in $(-\infty,\infty)$ satisfying the conditions of Fourier Integral formula. Then, at the points of continuity, the <u>Fourier cosine transform</u> of $f(x)$ and its <u>inverse transform</u> are defined by

$$\mathcal{F}_C\,\{f\,(x)\} = F_c\,(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f\,(x)\,Coskxdx$$

$$\mathcal{F}^{-1}{}_C\,\{F_c\,(k)\} = f\,(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c\,(k)\,Coskxdk$$

FOURIER SINE TRANSFORMATION AND INVERSE

Let $f\,(x)$ be defined for $0 \le x < \infty$, and extended as an odd function in $(-\infty,\infty)$ satisfying the conditions of Fourier Integral formula. Then, at the points of continuity, the <u>Fourier sine transform</u> of $f(x)$ and its <u>inverse transform</u> are defined by

$$\mathcal{F}_s\,\{f\,(x)\} = F_s\,(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f\,(x)\,Sinkxdx$$

$$\mathcal{F}^{-1}{}_s\,\{F_s\,(k)\} = f\,(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s\,(k)\,Sinkxdk$$

Example:

Show that $\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right) \quad ; \ a > 0$

Solution: We have, by definition

$$\mathcal{F}_C\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, Coskxdx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-(a-ik)x} + e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} + \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right) \quad ; \ a > 0$$

Example:

Show that $\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right) \quad ; \ a > 0$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, Sinkxdx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)dx = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-(a-ik)x} - e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} - \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right) \quad ; \ a > 0$$

Example:

Show that $\mathcal{F}_s^{-1}\left\{\frac{1}{k}e^{-sk}\right\} = \sqrt{\frac{2}{\pi}}\, tan^{-1}\left(\frac{x}{s}\right)$

Solution: To prove this we use the standard definite integral

$$\sqrt{\frac{\pi}{2}}\mathcal{F}_s^{-1}\{e^{-sk}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-sk}\, Sinkxdk = \frac{x}{s^2+x^2}$$

Integrating both sides w.r.to 's' from 's' to '∞'

$$\int_0^\infty \frac{e^{-sk}}{k} Sinkxdk = \int_s^\infty \frac{xds}{s^2+x^2} = \left| tan^{-1}\left(\frac{x}{s}\right)\right|_s^\infty = \frac{\pi}{2} - tan^{-1}\left(\frac{x}{s}\right)$$

Consequently

$$\mathcal{F}_s^{-1}\left\{\frac{1}{k}e^{-sk}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-sk}}{k} Sinkxdk = \sqrt{\frac{2}{\pi}}\, tan^{-1}\left(\frac{x}{s}\right)$$

Example:

Show that $\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(a^2+k^2)^2} \quad ;\ a > 0$

Solution: We have, by definition

$$\mathcal{F}_C\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, Coskxdx$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\, Coskxdx$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[|x(\int e^{-ax}Coskxdx)|_0^\infty - \int_0^\infty (\int e^{-ax}Coskxdx)\, dx\right] \text{ ....(i)}$$

Now using formula $\int e^{ax}Cosbxdx = \frac{e^{ax}}{a^2+b^2}[aCosbx + bSinbx]$ **one becomes**

$$\mathcal{F}_C\{xe^{-ax}\} =$$
$$\sqrt{\frac{2}{\pi}}\left[\left|x.\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^\infty - \int_0^\infty \left(\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right) dx\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[(0-0) + \frac{a}{a^2+k^2}\int_0^\infty e^{-ax}Coskxdx - \frac{k}{a^2+k^2}\int_0^\infty e^{-ax}Sinkxdx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^\infty - \frac{k}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^\infty\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left\{0 - \left(\frac{-a}{a^2+k^2}\right)\right\} - \frac{k}{a^2+k^2}\left\{0 - \left(\frac{-k}{a^2+k^2}\right)\right\}\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a^2}{(a^2+k^2)^2} + \frac{k^2}{(a^2+k^2)^2}\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-k^2}{(a^2+k^2)^2}\right]$$ **as required.**

Example:

Show that $\mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2ak}{(a^2+k^2)^2} \quad ;\ a > 0$

Solution: We have, by definition

$$\mathcal{F}_s\,\{f\,(x)\} = F_s\,(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f\,(x)\,Sinkxdx$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\,Sinkxdx$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[|x(\int e^{-ax}Sinkxdx)|_0^\infty - \int_0^\infty(\int e^{-ax}Sinkxdx)\,dx\right] \text{ ....(i)}$$

Now using formula $\int e^{ax}Sinbxdx = \frac{e^{ax}}{a^2+b^2}[aSinbx - bCosbx]$ **one becomes**

$$\mathcal{F}_s\,\{xe^{-ax}\} =$$

$$\sqrt{\frac{2}{\pi}}\left[\left|x.\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^\infty - \int_0^\infty\left(\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right)dx\right]$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[(0-0) + \frac{a}{a^2+k^2}\int_0^\infty e^{-ax}Sinkxdx + \frac{k}{a^2+k^2}\int_0^\infty e^{-ax}Coskxdx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^\infty + \frac{k}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^\infty\right]$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left\{0 - \left(\frac{-k}{a^2+k^2}\right)\right\} + \frac{k}{a^2+k^2}\left\{0 - \left(\frac{-a}{a^2+k^2}\right)\right\}\right]$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{ak}{(a^2+k^2)^2} + \frac{ak}{(a^2+k^2)^2}\right]$$

$$\mathcal{F}_s\,\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2ak}{(a^2+k^2)^2} \quad ;\ a > 0$$ **as required.**

Example:

Calculate Fourier Sine Transform of the function $f\ (x) = e^{-x}Cosx$

Solution: We have, by definition

$$\mathcal{F}_s\ \{f\ (x)\} = F_s\ (k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f\ (x)\ Sinkxdx$$

$$\mathcal{F}_s\ \{e^{-x}Cosx\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}Cosx\ Sinkxdx = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\ (2SinkxCosx)dx$$

$$\mathcal{F}_s\ \{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-x}\ [Sin(kx + x) + Sin(kx - x)]dx$$

$$\mathcal{F}_s\ \{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-x}\ Sin(k+1)xdx + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-x}\ Sin(k-1)xdx$$

$$\mathcal{F}_s\ \{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}I_1 + \frac{1}{\sqrt{2\pi}}I_2 \qquad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now using formula $\int e^{ax}Sinbxdx = \frac{e^{ax}}{a^2+b^2}[aSinbx - bCosbx]$

$$I_1 = \int_0^{\infty} e^{-x}\ Sin(k+1)xdx = \left|\frac{e^{-x}}{(-1)^2+(k+1)^2}[(-1)Sin(k+1)x - (k+1)Cos(k+1)x]\right|_0^{\infty}$$

$$I_1 = \left[0 - \frac{e^0}{1+(k+1)^2}\{-0 - (k+1)(1)\}\right] = \frac{1}{1+k^2+2k+1}(k+1) = \frac{(k+1)}{k^2+2k+2}$$

Similarly

$$I_2 = \int_0^{\infty} e^{-x}\ Sin(k-1)xdx = \left|\frac{e^{-x}}{(-1)^2+(k-1)^2}[(-1)Sin(k-1)x - (k-1)Cos(k-1)x]\right|_0^{\infty}$$

$$I_2 = \left[0 - \frac{e^0}{1+(k-1)^2}\{-0 - (k-1)(1)\}\right] = \frac{1}{1+k^2-2k+1}(k-1) = \frac{(k-1)}{k^2-2k+2}$$

$$(i) \Rightarrow \mathcal{F}_s\ \{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\left[\frac{(k+1)}{k^2+2k+2} + \frac{(k-1)}{k^2-2k+2}\right]$$

$$\Rightarrow \mathcal{F}_s\ \{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\left[\frac{2k^3}{k^4+4}\right]$$

$$\Rightarrow \mathcal{F}_s\ \{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{k^3}{k^4+4}\right]$$

Example:

Calculate Fourier Sine Transform of the function $f(x) = \begin{cases} Sinx & 0 \le x < \pi \\ 0 & x > \pi \end{cases}$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\, Sinkxdx$$

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\pi} Sinx\, Sinkxdx + \sqrt{\frac{2}{\pi}}\int_{\pi}^{\infty} 0.Sinkxdx$$

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\pi} Sinx\, Sinkxdx = \sqrt{\frac{2}{\pi}}\left(-\frac{1}{2}\right)\int_0^{\pi}(-2SinxSinkx)\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\int_0^{\pi}[Cos(kx+x) - Cos(kx-x)]\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\int_0^{\pi} Cos(k+1)x\,dx + \frac{1}{\sqrt{2\pi}}\int_0^{\pi} Cos(k-1)x\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\left|\frac{Sin(k+1)x}{k+1}\right|_0^{\pi} + \frac{1}{\sqrt{2\pi}}\left|\frac{Sin(k-1)x}{k-1}\right|_0^{\pi}$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left|\frac{Sin(kx-x)}{k-1} - \frac{Sin(kx+x)}{k+1}\right|_0^{\pi} = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{Sin(k\pi-\pi)}{k-1} - \frac{Sin(k\pi+\pi)}{k+1}\right) - 0\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{Sink\pi Cos\pi - Cosk\pi Sin\pi}{k-1} - \frac{Sink\pi Cos\pi + Cosk\pi Sin\pi}{k+1}\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{-Sink\pi}{k-1} + \frac{Sink\pi}{k+1}\right] \qquad \text{since } Cos\pi = -1$$

$$\mathcal{F}_s\{f(x)\} = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{1}{k+1} - \frac{1}{k-1}\right]$$

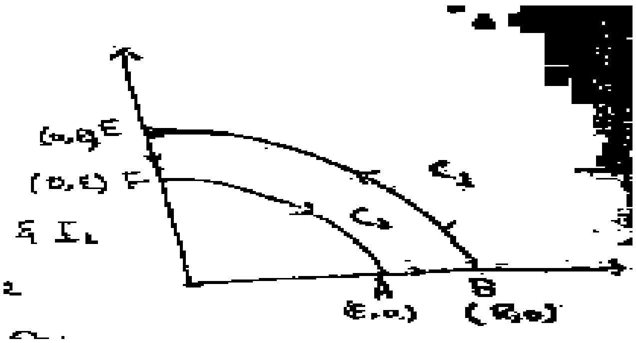
$$\mathcal{F}_s\{f(x)\} = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{k-1-k-1}{(k+1)(k-1)}\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{-2}{k^2-1}\right]$$

$$\mathcal{F}_s\{f(x)\} = -\sqrt{\frac{2}{\pi}}\left[\frac{Sink\pi}{k^2-1}\right]$$

Example: Evaluate $\mathcal{F}_c\{x^{\propto-1}\}$ *and* $\mathcal{F}_s\{x^{\propto-1}\}$

Solution:



We have by definition

$$\mathcal{F}_c\{x^{\propto-1}\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1} Coskxdx \quad \ldots\ldots\ldots\ldots\ldots (I_1)$$

$$\mathcal{F}_s\{x^{\propto-1}\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1} Sinkxdx \quad \ldots\ldots\ldots\ldots\ldots (I_2)$$

Firstly we calculate I_1, I_2 for this we consider the complex valued function

$f(z) = z^{\propto-1}e^{-kz}$; $0 < \propto < 1$

Which is analytic in the closed contour I_1 then by Cauchy Theorem

$$\oint_c f(z)dz = 0$$

$$\int_A^B f(z)dz + \int_{c_1} f(z)dz + \int_E^F f(z)dz + \int_{c_2} f(z)dz = 0$$

If $\epsilon \to 0, R \to 0$then by Jordan theorem $\int_{c_1} f(z)dz = 0$, $\int_{c_2} f(z)dz = 0$

$$\int_A^B f(z)dz + \int_E^F f(z)dz = 0$$

$$\int_\epsilon^R x^{\propto-1}e^{-kx}dx + \int_R^\epsilon (iy)^{\propto-1}e^{-k(iy)}(idy) = 0$$

$$\int_0^\infty x^{\propto-1}e^{-kx}dx = -\int_\infty^0 (i)^{\propto-1}(y)^{\propto-1}e^{-k(iy)}(idy)$$

$$\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (i)^{\propto}(y)^{\propto-1}e^{-k(iy)}dy$$

$$(i)^{-\propto}\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}e^{-k(iy)}dy$$

$$\therefore (i)^{-\propto} = \left(Cos\frac{\pi}{2} + iSin\frac{\pi}{2}\right)^{-\propto} = \left(e^{i\frac{\pi}{2}}\right)^{-\propto} = e^{-i\frac{\pi}{2}\propto}$$

$$e^{-i\frac{\pi}{2}\propto}\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}e^{-k(iy)}dy$$

$$\left(Cos\frac{\pi}{2}\propto - iSin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}(Cosky - iSinky)dy$$

Comparing real and imaginary parts

$\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}(Cosky)dy$ …………(i)

$\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx = \int_0^\infty (y)^{\propto-1}(Sinky)dy$ …………(ii)

Put x = y in both above

$(i) \Rightarrow \int_0^\infty x^{\propto-1}(Coskx)dx = \left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$ …………(iii)

$(ii) \Rightarrow \int_0^\infty x^{\propto-1}(Sinkx)dx = \left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$ …………(iv)

Multiplying $\sqrt{\frac{2}{\pi}}$ on both sides of (iii)

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1}(Coskx)dx = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty \left(\frac{t}{k}\right)^{\propto-1} e^{-t}\frac{dt}{k} \quad \therefore kx = t, x = \frac{t}{k}$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \frac{1}{k^{\propto}}\sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\int_0^\infty (t)^{\propto-1}e^{-t}dt$$

$$\Rightarrow \mathcal{F}_c\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Cos\frac{\pi}{2}\propto\right)\frac{\Gamma(\propto)}{k^{\propto}}$$

Multiplying $\sqrt{\frac{2}{\pi}}$ on both sides of (iv)

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\propto-1}(Sinkx)dx = \sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty x^{\propto-1}e^{-kx}dx$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \frac{1}{k}\sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty \left(\frac{t}{k}\right)^{\propto-1} e^{-t}dt \quad \therefore kx = t, x = \frac{t}{k}$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \frac{1}{k^{\propto}}\sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\int_0^\infty (t)^{\propto-1}e^{-t}dt$$

$$\Rightarrow \mathcal{F}_s\{x^{\propto-1}\} = \sqrt{\frac{2}{\pi}}\left(Sin\frac{\pi}{2}\propto\right)\frac{\Gamma(\propto)}{k^{\propto}}$$

Theorem : Let f (x) and its first derivative vanish as x → ∞. If $F_c\ (k)$ is the Fourier cosine transform, then $\mathcal{F}_C\ \{f''\ (x)\} = -k^2 F_c\ (k) - \sqrt{\frac{2}{\pi}} f'\ (0)$

PROOF: Consider $f\ (x)$ is real and $\lim_{x \to \infty} |f\ (x)| = 0$ then

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''\ (x)\ Coskxdx$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \left[|Coskxf'\ (x)|_0^\infty - \int_0^\infty f'\ (x)\ (-kSinkx) dx \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \left[\lim_{x \to \infty} |Coskxf'\ (x)| - \lim_{x \to 0} |Coskxf'\ (x)| + k \int_0^\infty f'\ (x)\ Sinkxdx \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \left[0 - f'\ (0) + k \int_0^\infty f'\ (x)\ Sinkxdx \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\ (0) + k \left\{ \sqrt{\frac{2}{\pi}} |Sinkxf\ (x)|_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty f\ (x)\ (kCoskx) dx \right\} \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\ (0) + k \left\{ \sqrt{\frac{2}{\pi}} |Sinkxf\ (x)|_0^\infty - k \sqrt{\frac{2}{\pi}} \int_0^\infty f\ (x)\ (Coskx) dx \right\} \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\ (0) + k \left\{ \sqrt{\frac{2}{\pi}} (\lim_{x \to \infty} |Sinkxf\ (x)| - \lim_{x \to 0} |Sinkxf\ (x)|) - kF_c\ (k) \right\} \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = -k^2 F_c\ (k) - \sqrt{\frac{2}{\pi}} f'\ (0)$$

In a similar manner, the Fourier cosine transforms of higher-order derivatives of f (x) can be obtained.

Theorem : Let f (x) and its first derivative vanish as x → ∞. If $F_s\,(k)$ is the Fourier cosine transform, then $\mathcal{F}_s\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}kf\,(0) - k^2F_s\,(k)$

PROOF: Consider $f\,(x)$ is real and $\lim_{x\to\infty}|f\,(x)| = 0$ then

$$\mathcal{F}_s\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''\,(x)\,Sinkxdx$$

$$\mathcal{F}_s\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\left[|Sinkxf'\,(x)|_0^\infty - \int_0^\infty f'\,(x)\,(kCosx)dx\right]$$

$$\mathcal{F}_s\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\left[\lim_{x\to\infty}|Sinkxf'\,(x)| - \lim_{x\to 0}|Sinkxf'(x)| - k\int_0^\infty f'\,(x)\,Coskxdx\right]$$

$$\mathcal{F}_s\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}\left[0 - 0 - k\int_0^\infty f'\,(x)\,Coskxdx\right]$$

$$\mathcal{F}_s\,\{f''\,(x)\} = -k\left[\sqrt{\frac{2}{\pi}}|Coskxf\,(x)|_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty f\,(x)\,(-kSinkx)dx\right]$$

$$\mathcal{F}_s\,\{f''\,(x)\} =$$
$$-k\left[\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Coskxf\,(x)| - \lim_{x\to 0}|Coskxf\,(x)|) + k\sqrt{\frac{2}{\pi}}\int_0^\infty f\,(x)\,(Sinkx)dx\right]$$

$$\mathcal{F}_s\,\{f''\,(x)\} = -k\left[\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Coskxf\,(x)| - \lim_{x\to 0}|Coskxf\,(x)|) + kF_s\,(k)\right]$$

$$\mathcal{F}_s\,\{f''\,(x)\} = \sqrt{\frac{2}{\pi}}kf\,(0) - k^2F_s\,(k)$$

In a similar manner, the Fourier sine transforms of higher-order derivatives of $f\,(x)$ can be obtained.

REMARK:

- $\mathcal{F}\,[f^n(x)] = (-ik)^n\mathcal{F}\,[f\,(x)] = (-ik)^n\,F\,(k)\;\; n = 0, 1, 2, \ldots\ldots\ldots$
- **If** $\mathcal{F}\,\{u_t\} = \mathcal{F}\,\{u_x\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}\,\{u\,(x,t)\} = (-ik)\,\mathcal{F}\,\{u\,(x,t)\}$ **when 'x' varies not 't'**
- **When range of spatial variable is infinite then Fourier transform is used rather than the sine or cosine.**
- **If boundry conditions are of the form $u(0,t) = value$ then use Sine transform, while conditions are of the form $u_x(0,t) = value$ then use Cosine transform,**

EXAMPLE: Solve the potential equation for the potential $u(x,y)$ in the semi infinite strip $0 < x < c;\ y > 0$ that satisfies the following conditions;

$u(0,y) = 0\ ;\quad u_y(x,0) = 0\ ;\ u_x(c,y) = f(y)$

Solution: the potential equation is given as $u_{xx} + u_{yy} = 0\ ;\ 0 < x < c;\ y > 0$

Since the BC's are in the form $u_y(x,0) = constant$ **therefor we use fourier cosine transform w.r.to 'y'**

$\mathcal{F}_C\{u_{xx}\} + \mathcal{F}_C\{u_{yy}\} = 0 \Rightarrow \frac{d^2}{dx^2}\mathcal{F}_C\{u(x,y)\} + \mathcal{F}_C\{u_{yy}\} = 0$

$\Rightarrow \frac{d^2}{dx^2}U_C(x,k) + \left[-k^2U_c(x,k) - \sqrt{\frac{2}{\pi}}u_y(x,0)\right] = 0$

$\Rightarrow \frac{d^2}{dx^2}U_C(x,k) - k^2U_c(x,k) = 0$

Then general solution will be $U_c(x,k) = c_1e^{kx} + c_2e^{-kx}$ **…………(i)**

Now using BC's $u(0,y) = 0 \Rightarrow \mathcal{F}_C\{u(0,y)\} = 0 \Rightarrow U_c(0,k) = 0$

$(i) \Rightarrow U_c(0,k) = 0 = c_1e^0 + c_2e^0 \Rightarrow c_1 = -c_2$

Now $\frac{d}{dx}U_C(x,k) = c_1ke^{kx} - c_2ke^{-kx}$ **…………(ii)**

using BC's $u_x(c,y) = f(y) \Rightarrow \mathcal{F}_C\{u_x(c,y)\} = f(y) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k)$

$(ii) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = c_1ke^{kc} - c_2ke^{-kc}$

$\Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = -c_2ke^{kc} - c_2ke^{-kc}$ **since** $c_1 = -c_2$

$\Rightarrow F_c(k) = -c_2k\left(e^{kc} + e^{-kc}\right) \Rightarrow c_2 = -\frac{F_c(k)}{2k\left(\frac{e^{kc}+e^{-kc}}{2}\right)} = -\frac{F_c(k)}{2kCoshkc}$

$\Rightarrow c_2 = -\frac{F_c(k)}{2kCoshkc} \Rightarrow c_1 = \frac{F_c(k)}{2kCoshkc}$ **since** $c_1 = -c_2$

Then $(i) \Rightarrow U_c(x,k) = \frac{F_c(k)}{2kCoshkc}e^{kx} - \frac{F_c(k)}{2kCoshkc}e^{-kx}$

$U_c(x,k) = \frac{F_c(k)}{kCoshkc}\left(\frac{e^{kx}-e^{-kx}}{2}\right) = \frac{F_c(k)}{kCoshkc}Sinhkx$

$\Rightarrow \mathcal{F}_C^{-1}\{U_c(x,k)\} = \mathcal{F}_C^{-1}\left\{\frac{F_c(k)}{kCoshkc}Sinhkx\right\}$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{F_c(k)}{kCoshkc}Sinhkx\, Coskxdk=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc}F_c(k)dk$$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f\,(y')\,Cosky'dy'\right]dk$$

$$\Rightarrow u(x,y)=\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{SinhkxCoskxCosky\prime}{kCoshkc}f\,(y')dy'dk$$

EXAMPLE: Solve the problem using Fourier Transformation method
$u_t=u_{xx}$ **with** $u(0,t)=u_0\ ;\quad u(x,0)=0\ ; x>0, t>0, u_0>0$
Solution: BC's suggest that we should use fourier sine transform w.r.to 'x'

$$\mathcal{F}_s\,\{u_t\}=\mathcal{F}_s\,\{u_{xx}\}\Rightarrow\frac{\partial}{\partial t}\mathcal{F}_s\,\{u(x,t)\}=\mathcal{F}_s\,\{u_{xx}\}$$

$$\Rightarrow\frac{d}{dt}U_s\,(k,t)=\sqrt{\frac{2}{\pi}}ku(0,t)-k^2U_s\,(k,t)=\sqrt{\frac{2}{\pi}}ku_0-k^2U_s\,(k,t)$$

$$\Rightarrow\frac{d}{dt}U_s\,(k,t)+k^2U_s\,(k,t)=\sqrt{\frac{2}{\pi}}ku_0\quad \text{................(i)}$$

This is 1st order, linear, non – homogeneous ODE
Therefore I.F. = $e^{\int k^2dt}=e^{k^2t}$

$$(i)\Rightarrow e^{k^2t}\frac{\partial}{\partial t}U_s\,(k,t)+k^2U_s\,(k,t)e^{k^2t}=\sqrt{\frac{2}{\pi}}ku_0e^{k^2t}$$

$$\Rightarrow\int\frac{d}{dt}e^{k^2t}\,U_sdt=\int\sqrt{\frac{2}{\pi}}ku_0e^{k^2t}dt\quad + \text{Cosntant}$$

$$\Rightarrow e^{k^2t}U_s=\sqrt{\frac{2}{\pi}}ku_0\frac{e^{k^2t}}{k^2}+c\Rightarrow U_s(k,t)=\sqrt{\frac{2}{\pi}}\frac{u_0}{k}+ce^{-k^2t}\quad \text{................(ii)}$$

Now using IC's $u(x,0)=0\Rightarrow\mathcal{F}_s\{u\,(x,0)\}=0\Rightarrow U_s(k,0)=0$

$$(ii)\Rightarrow U_s(k,0)=0=\sqrt{\frac{2}{\pi}}\frac{u_0}{k}+ce^0\Rightarrow c=-\sqrt{\frac{2}{\pi}}\frac{u_0}{k}$$

Thus $(ii)\Rightarrow U_s(k,t)=\sqrt{\frac{2}{\pi}}\frac{u_0}{k}-\sqrt{\frac{2}{\pi}}\frac{u_0}{k}e^{-k^2t}=\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1-e^{-k^2t}\right)$

$$\Rightarrow\mathcal{F}_s{}^{-1}\{U_s(k,t)\}=\mathcal{F}_s{}^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1-e^{-k^2t}\right)\right\}$$

$$\Rightarrow u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1-e^{-k^2t}\right)Sinkxdk=\frac{u_0}{k}\frac{2}{\pi}\int_0^\infty\left(1-e^{-k^2t}\right)Sinkxdk$$

EXAMPLE: Solve the problem using Fourier Transformation method

$u_t = u_{xx}$ **with** $u_x(0,t) = 0$, $u(x,0) = f(x)$; $0 < x < \infty$, $t > 0$

Solution: BC's suggest that we should use fourier cosine transform w.r.to 'x'

$$\mathcal{F}_C\{u_t\} = \mathcal{F}_C\{u_{xx}\} \Rightarrow \frac{d}{dt}\mathcal{F}_C\{u(x,y)\} = \mathcal{F}_C\{u_{xx}\}$$

$$\Rightarrow \frac{d}{dt}U_C(k,t) = \left[-k^2 U_c(k,t) - \sqrt{\frac{2}{\pi}}u_x(0,t)\right] = -k^2 U_c(k,t) - 0$$

$$\Rightarrow \frac{d}{dt}U_C(k,t) + k^2 U_c(k,t) = 0 \quad \text{................(i)}$$

This is 1st order, linear, homogeneous ODE

Then general solution will be $U_c(k,t) = Ae^{-k^2 t}$ **............(ii)**

Now using IC's

$$u(x,0) = f(x) \Rightarrow \mathcal{F}_c\{u(x,0)\} = \mathcal{F}_c\{f(x)\} \Rightarrow U_c(k,0) = F_c(k)$$

Thus $(i) \Rightarrow U_c(k,0) = F_c(k) = Ae^0 \Rightarrow A = F_c(k)$

$$(i) \Rightarrow U_c(k,t) = F_c(k)e^{-k^2 t}$$

$$\Rightarrow \mathcal{F}_C^{-1}\{U_c(k,t)\} = \mathcal{F}_C^{-1}\{F_c(k)e^{-k^2 t}\}$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)e^{-k^2 t}\, Coskxdk$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x')\, Coskx'dx'\right] e^{-k^2 t}\, Coskxdk$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(x')\, Coskx'dx'\right] e^{-k^2 t}\, Coskxdk$$

Example: **Solve the problem using Fourier Transformation method**

$u_{xx} = u_t\,;\ \ 0 < x < \infty\ ,t \geq 0$

with $u(x,0) = e^{-ax^2};\ u(x), u'(x) \to 0\ \ as\ \ x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\{u_{xx}\} = \mathcal{F}\{u_t\}$

$\Rightarrow (-ik)^2 \mathcal{F}\{u(x,t)\} = \frac{d}{dt}\mathcal{F}\{u(x,t)\} \Rightarrow -k^2 U(k,t) = \frac{d}{dt}U(k,t)$

$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -k^2 \Rightarrow \int \frac{dU}{U} = -k^2 \int dt \Rightarrow lnU = -k^2 t + A$

$\Rightarrow U(k,t) = e^{-k^2 t + A} \Rightarrow U(k,t) = ce^{-k^2}$ **...............(i) where** $e^A = c$

Now using IC's

$u(x,0) = e^{-ax^2} \Rightarrow \mathcal{F}\{u\,(x,0)\} = \mathcal{F}\{e^{-ax^2}\}$

$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}.\,e^{-ax^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-ax^2}dx$

$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(x-\frac{ik}{2a}\right)^2+\frac{k}{4a^2}^2\right]}\ dx$

$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2}\ dx$

Put $a\left(x-\frac{ik}{2a}\right)^2 = P^2 \Rightarrow \sqrt{a}\left(x-\frac{ik}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{ik}{2a}\right)^2}\ dx = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-P^2}\ .\frac{dP}{\sqrt{a}}$

$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}}.\sqrt{\pi} \quad \therefore \int_{-\infty}^{\infty} e^{-P^2}\ dP = \sqrt{\pi}$

$\Rightarrow U(k,0) = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$ **...............(ii)**

$(i) \Rightarrow U(k,0) = ce^0 \Rightarrow c = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$

Thus $\Rightarrow U(k,t) = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}e^{-k^2} = \frac{1}{\sqrt{2a}}e^{-k^2\left(t+\frac{1}{4a}\right)}$

Consider $ikx - ax^2$

$= -a\left(x^2 - \frac{ikx}{a}\right)$

$= -a\left[\left(x^2 - \frac{2ikx}{2a}\right) + \left(\frac{ik}{2a}\right)^2 - \left(\frac{ik}{2a}\right)^2\right]$

$= -a\left[\left(x - \frac{ik}{2a}\right)^2 + \frac{k}{4a^2}^2\right]$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2a}} e^{-k^2\left(t+\frac{1}{4a}\right)}\right\}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2a}}.\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-ikx}.e^{-k^2\left(t+\frac{1}{4a}\right)}\,dk$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left\{k^2+\frac{ikx}{\left(t+\frac{1}{4a}\right)}\right\}\right]dk \;\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Since $k^2 + \frac{ikx}{\left(t+\frac{1}{4a}\right)} = k^2 + 2(k)\left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right) + \left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2 - \left(\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2$

$$k^2 - \frac{ikx}{\left(t+\frac{1}{4a}\right)} = \left(k+\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2 + \frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}$$

$$(iii) \Rightarrow u(x,t) = \frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left(k+\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2\right] e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}\,dk$$

$$\Rightarrow u(x,t) = \frac{e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(t+\frac{1}{4a}\right)\left(k+\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2\right]dk \;\ldots\ldots\ldots\ldots\text{(iv)}$$

Now put $\left(t+\frac{1}{4a}\right)\left(k+\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right)^2 = m^2 \Rightarrow \sqrt{\left(t+\frac{1}{4a}\right)}\left(k+\frac{ix}{2\left(t+\frac{1}{4a}\right)}\right) = m$

$$\Rightarrow \sqrt{\left(t+\frac{1}{4a}\right)}\,dk = dm \Rightarrow dk = \frac{1}{\sqrt{\left(t+\frac{1}{4a}\right)}}\,dm$$

$$(iv) \Rightarrow u(x,t) = \frac{e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}.\sqrt{\left(t+\frac{1}{4a}\right)}}\int_{-\infty}^{\infty} e^{-m^2}\,dm = \frac{e^{\left(\frac{x^2}{4\left(t+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4a\pi}.\frac{1}{\sqrt{4a}}\sqrt{4at+1}}.\sqrt{\pi}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4at+1}}e^{\left(\frac{ax^2}{4at+1}\right)}$$

Example: **Solve the problem using Fourier Transformation method**

$u_t(x,t) = \propto^2 u_{xx}(x,t)\ ;\ -\infty < x < \infty\ ,\ t > 0$

with $u_x(x,0) = f(x);\ |u(x,0)| < \infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\{u_t\} = \propto^2 \mathcal{F}\{u_{xx}\}$

$\Rightarrow \frac{d}{dt}\mathcal{F}\{u(x,t)\} = \propto^2 (-ik)^2 \mathcal{F}\{u(x,t)\} \Rightarrow \frac{d}{dt}U(k,t) = -\propto^2 k^2 U(k,t)$

$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -\propto^2 k^2 \Rightarrow \int \frac{dU}{U} = -\propto^2 k^2 \int dt \Rightarrow lnU = -\propto^2 k^2 t + A$

$\Rightarrow U(k,t) = e^{-\propto^2 k^2 t + A} \Rightarrow U(k,t) = ce^{-\propto^2 k^2}$ **……………(i) where** $e^A = c$

Now using IC's

$u_x(x,0) = f(x)\ and\ |u(x,0)| < \infty \Rightarrow u(x,0) = f(x)$

$\Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} \quad \Rightarrow U(k,0) = F(k)$

$(i) \Rightarrow U(k,0) = ce^0 \Rightarrow c = F(k)$

Thus $(i) \Rightarrow U(k,t) = F(k)e^{-\propto^2 k^2 t}$

$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\left\{F(k)e^{-\propto^2 k^2}\right\}$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} . F(k)e^{-\propto^2 k^2 t} dk$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')dx'\right] e^{-\propto^2 k^2 t} dk$

$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t} dk\right] f(x')dx'$ **……………(iii)**

Now consider $I = \int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t} dk$

$I = \int_{-\infty}^{\infty} e^{-iku-\beta k^2} dk$ **put** $x - x' = u$ **and** $\propto^2 t = \beta$

$I = \int_{-\infty}^{\infty} e^{-\beta\left(k^2+\frac{ik}{\beta}\right)} dk$

$I = \int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{iu}{2\beta}\right)^2} . e^{-\frac{u^2}{4\beta}} dk$

$I = e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{iu}{2\beta}\right)^2} dk$ **……….(iv)**

Consider $k^2 + \frac{ik}{\beta}u$

$= k^2 + 2k\left(\frac{iu}{2\beta}\right) + \left(\frac{iu}{2\beta}\right)^2 - \left(\frac{iu}{2\beta}\right)^2$

$= \left(k + \frac{iu}{2\beta}\right)^2 - \left(\frac{iu}{2\beta}\right)^2$

$= \left(k + \frac{iu}{2\beta}\right)^2 + \frac{u^2}{4\beta^2}$

Put $\beta\left(k+\frac{iu}{2\beta}\right)^2 = P^2 \Rightarrow \sqrt{\beta}\left(k+\frac{iu}{2\beta}\right) = P \Rightarrow \sqrt{\beta}dk = dP \Rightarrow dk = \frac{dP}{\sqrt{\beta}}$

$(iv) \Rightarrow I = e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-P^2}\ .\frac{dP}{\sqrt{\beta}} = \frac{dP}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-P^2}\ dp = \frac{1}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}.\sqrt{\pi}$

$(iii) \Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}f(x')dx'$

$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi}.\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\propto^2 t}}e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$

$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi\propto^2 t}.}\int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$

Example:

Solve the problem using Fourier Transformation method $u_{xxxx} = \frac{1}{a^2}u_{tt}$

with $u(x,0) = f(x); u_t(x,0) = ag'(x)$ **and** $g, u, u_x, u_{xx}, u_{xxx} \to 0\ as\ \ x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\{u_{xxxx}\} = \frac{1}{a^2}\ \mathcal{F}\{u_{tt}\}$

$\Rightarrow (-ik)^4\mathcal{F}\{u(x,t)\} = \frac{1}{a^2}\frac{d^2}{dt^2}\mathcal{F}\{u(x,t)\} \Rightarrow a^2k^4U(k,t) = \frac{d^2}{dt^2}U(k,t)$

$\Rightarrow \frac{d^2}{dt^2}U - a^2k^4U = 0$

$\Rightarrow U(k,t) = Ae^{ak^2t} + Be^{-ak^2t}$ **...............(i)**

$\Rightarrow \frac{d}{dt}U(k,t) = Aak^2e^{ak^2t} - Bak^2e^{-ak^2t}$ **...............(ii)**

Now using IC's $u(x,0) = f(x) \Rightarrow \mathcal{F}\{u\ (x,0)\} = \mathcal{F}\{f(x)\} \Rightarrow U(k,0) = F(k)$

Then $(i) \Rightarrow U(k,0) = Ae^0 + Be^0 \Rightarrow A + B = F(k)$ **...............(iii)**

Also $u_t(x,0) = ag'(x) \Rightarrow \mathcal{F}\{u_t\ (x,0)\} = \mathcal{F}\{ag'(x)\}$

$\Rightarrow \frac{d}{dt}U(k,0) = a(-ik)^1\mathcal{F}\{g'(x)\} \Rightarrow \frac{d}{dt}U(k,0) = -iakG(k)$

Then $(ii) \Rightarrow \frac{d}{dt}U(k,0) = Aak^2e^0 - Bak^2e^0 \Rightarrow -iakG(k) = Aak^2 - Bak^2$

$\Rightarrow -iG(k) = (A-B)k \Rightarrow A - B = -\frac{i}{k}G(k)$ **...............(iv)**

Adding (iii) and (iv) $\quad A = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]$

Subtracting (iii) and (iv) $\quad B = \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]$

Then (i) becomes

$\Rightarrow U(k,t) = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]e^{ak^2t} + \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]e^{-ak^2t}$

$\Rightarrow U(k,t) = F(k)\left[\frac{e^{ak^2t}+e^{-ak^2t}}{2}\right] - \frac{i}{k}G(k)\left[\frac{e^{ak^2t}-e^{-ak^2t}}{2}\right]$

$\Rightarrow U(k,t) = F(k)Coshak^2t - \frac{i}{k}G(k)Sinhak^2t$

$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\{F(k)Coshak^2t\} - \mathcal{F}^{-1}\left\{\frac{i}{k}G(k)Sinhak^2t\right\}$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-ikx}\,F(k)Coshak^2tdk - \int_{-\infty}^{\infty} e^{-ikx}\,\frac{i}{k}G(k)Sinhak^2tdk\right]$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,U(k,t)dk$ **is our required solution.**

Example: **Solve the problem using Fourier Transformation method**

$u_{xx} = \frac{1}{c^2}u_{tt}$

with $u(x,0) = p(x); u_t(x,0) = q(x)$ **and** $u, u_x \to 0 \; as \; x \to \pm\infty$

Solution: since $x \to \pm\infty$ **therefore we should use fourier transform w.r.to 'x'**

$\mathcal{F}\{u_{xx}\} = \frac{1}{c^2}\,\mathcal{F}\{u_{tt}\}$

$\Rightarrow (-ik)^2\mathcal{F}\{u(x,t)\} = \frac{1}{c^2}\frac{d^2}{dt^2}\mathcal{F}\{u(x,t)\} \Rightarrow -c^2k^2U(k,t) = \frac{d^2}{dt^2}U(k,t)$

$\Rightarrow \frac{d^2}{dt^2}U + c^2k^2U = 0 \Rightarrow U(k,t) = c_1Cosxkt + c_2Sinckt$

$\Rightarrow U(k,t) = c_1\left(\frac{e^{ickt}+e^{-ickt}}{2}\right) + c_2\left(\frac{e^{ickt}-e^{-ickt}}{2}\right)$

$\Rightarrow U(k,t) = \left(\frac{c_1+c_1}{2}\right)e^{ickt} + \left(\frac{c_1-c_1}{2}\right)e^{-ickt}$

$\Rightarrow U(k,t) = Ae^{ickt} + Be^{-ickt}$ **...............(i)**

$\Rightarrow \frac{d}{dt}U(k,t) = Aice^{ickt} - Bice^{-ickt}$ **...............(ii)**

Now using IC's $\quad u(x,0) = p(x) \Rightarrow \mathcal{F}\{u\,(x,0)\} = \mathcal{F}\{p(x)\} \Rightarrow U(k,0) = P(k)$

Then $(i) \Rightarrow U(k,0) = Ae^{0} + Be^{0} \Rightarrow A + B = P(k)$ **..............(iii)**

Also $u_t(x,0) = q(x) \Rightarrow \mathcal{F}\{u_t\,(x,0)\} = \mathcal{F}\{q(x)\} \Rightarrow \frac{d}{dt}U(k,0) = Q(k)$

Then $(ii) \Rightarrow \frac{d}{dt}U(k,0) = Aicke^{0} - Bicke^{0}$

$\Rightarrow Q(k) = ick(A - B)k \Rightarrow A - B = \frac{1}{ick}Q(k)$ **..............(iv)**

Adding (iii) and (iv) $\qquad A = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]$

Subtracting (iii) and (iv) $\qquad B = \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]$

Then (i) becomes

$$\Rightarrow U(k,t) = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]e^{ickt} + \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]e^{-ickt}$$

$$\Rightarrow U(k,t) = P(k)\left[\frac{e^{ickt}+e^{-ickt}}{2}\right] + \frac{1}{ick}Q(k)\left[\frac{e^{ickt}-e^{-ickt}}{2}\right]$$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} =$$

$$\frac{1}{2}\left[\mathcal{F}^{-1}\left\{P(k)e^{ickt}\right\} + \mathcal{F}^{-1}\left\{P(k)e^{-ickt}\right\}\right] + \frac{1}{2ick}\mathcal{F}^{-1}\left\{Q(k)\left(e^{ickt} - e^{-ickt}\right)\right\} \;....(\text{A})$$

$$\mathcal{F}^{-1}\left\{P(k)e^{ickt}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,P(k)e^{ickt}dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(x-ct)k}\,P(k)dk$$

$$\mathcal{F}^{-1}\left\{P(k)e^{ickt}\right\} = P(x - ct)$$

Similarly $\quad \mathcal{F}^{-1}\left\{P(k)e^{-ickt}\right\} = P(x + ct)$

And consider $\qquad q(x) = \mathcal{F}^{-1}\{Q(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,Q(k)dk$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{x-ct}^{x+ct}\int_{-\infty}^{\infty} e^{-ikx}\,Q(k)dkdx$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{x-ct}^{x+ct} e^{-ikx'}\,dx'Q(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{e^{-ikx'}}{-ik}\right|_{x-ct}^{x+ct} Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{-ik}\left[e^{-ik(x+ct)} - e^{-ik(x-ct)}\right]Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{ik}\left[e^{-ik(x-ct)} - e^{-ik(x+ct)}\right]Q(k)dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[e^{ickt} - e^{-ickt}\right]\frac{Q(k)}{k}dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\mathcal{F}^{-1}\left\{\left(e^{ickt} - e^{-ickt}\right)\frac{Q(k)}{k}dk\right\}$$

$$(A) \Rightarrow u(x,t) = \frac{1}{2}\left[P(x + ct) + P(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} q(x')dx'$$

THE DOUBLE FOURIER TRANSFORM AND ITS INVERSE

Let $f(x_1, x_2)$ be a function defined over the whole plane i.e. $-\infty < x_1, x_2 < \infty$ then its fourier transform and inverse are defined as follows;

$$\mathcal{F}\{f(x_1,x_2)\} = F(k_1,k_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)e^{i(k_1x_1+k_2x_2)}\,dx_1dx_2$$

$$\mathcal{F}^{-1}\{F(k_1,k_2)\} = f(x_1,x_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1,k_2)e^{-i(k_1x_1+k_2x_2)}\,dk_1dk_2$$

THREE DIMENSIONAL FOURIER TRANSFORM AND ITS INVERSE

Let $f(x_1, x_2, x_3)$ be a function defined over the whole plane i.e. $-\infty < x_1, x_2, x_3 < \infty$ then its fourier transform and inverse are defined as follows;

$$\mathcal{F}\{f(x_1,x_2,x_3)\} = F(k_1,k_2,k_3) = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)e^{i(k_1x_1+k_2x_2+k_3x_3)}\,dx_1dx_2dx_3$$

$$\mathcal{F}^{-1}\{F(k_1,k_2,k_3)\} = f(x_1,x_2,x_3) = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1,k_2,k_3)e^{-i(k_1x_1+k_2x_2+k_3x_3)}\,dk_1dk_2dk_3$$

n - DIMENSIONAL FOURIER TRANSFORM AND ITS INVERSE

$$\mathcal{F}\{f(\textstyle\sum_{i=1}^{n} x_i)\} = F(\sum_{i=1}^{n} k_i) = \frac{1}{(\sqrt{2\pi})^n}\int_{all\ space} f(\sum_{i=1}^{n} x_i)e^{i(\sum_{i=1}^{n} k_ix_i)}\,d\sum_{i=1}^{n} x_i$$

$$\mathcal{F}^{-1}\{F(\textstyle\sum_{i=1}^{n} k_i)\} = f(\sum_{i=1}^{n} x_i) = \frac{1}{(\sqrt{2\pi})^n}\int_{all\ space} F(\sum_{i=1}^{n} k_i)e^{-i(\sum_{i=1}^{n} k_ix_i)}\,d\sum_{i=1}^{n} k_i$$

FOURIER SERIES

A trigonometric series with any piecewise continuous periodic function $f\ (x)$ of period 2π and of the form $f\ (x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx\ + b_k \sin kx)$ is called the Fourier Series of a real valued function $f(x)$ where the symbol $\sim$ indicates an association of a_0, a_k, and b_k to f in some unique manner.

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\ \ ,\ \ a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx\ \ ,\ \ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx$$

And are called Fourier Coefficiets.

We may also write $f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx\ + b_k \sin kx)$

COMPLEX FORM OF FOURIER SERIES

Fourier Series expansion for in complex form is given as follows

$f\ (x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$; $-\pi < x < \pi$ **Where**

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

OR $f\ (x) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi x}{l}}$ **Where** $c_k = \frac{1}{2l}\int_{-l}^{l} f(y)e^{-i\frac{\pi y}{l}}dy$

Example (just read) :Find the Fourier series expansion for the function $f(x) = x + x^2$, $-\pi < x < \pi$

Solultion: Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2\pi^2}{3}$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{4}{k^2}Cosk\pi = \frac{4}{k^2}(-1)^k\ ;\ k = 1, 2, 3, \ldots\ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = -\frac{2}{k}Cosk\pi = -\frac{2}{k}(-1)^k\ ;\ k = 1, 2, 3, \ldots\ldots$$

Therefore, the Fourier series expansion for f is

$$f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx\ + b_k \sin kx)$$

$$f\ (x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(\frac{4}{k^2}(-1)^k \cos kx\ - \frac{2}{k}(-1)^k \sin kx)$$

$$f\ (x) = \frac{\pi^2}{3} - 4\cos x + 2sinx\ +\ cos\, 2x\ -\ sin\, 2x - \ldots.$$

Example (just read): Find the Fourier series expansion for the function

$$f\,(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; 0 < x < \pi \end{cases}$$

Solultion: Here

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] = -\frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Coskxdx + \int_{0}^{\pi} f(x)Coskxdx\right]$$

$$a_k = \frac{1}{k^2\pi}(Cosk\pi - 1) = \frac{1}{k^2\pi}\left[(-1)^k - 1\right] \ ; \ \ k = 1,2,3, \dots \dots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Sinkxdx + \int_{0}^{\pi} f(x)Sinkxdx\right]$$

$$b_k = \frac{1}{k}(1 - 2Cosk\pi) = \frac{1}{k}\left[1 - 2(-1)^k\right] ; \ \ k = 1,2,3, \dots \dots$$

Therefore, the Fourier series expansion for f is

$$f\,(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$f\,(x) = -\frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{1}{k^2\pi}\left[(-1)^k - 1\right]\cos kx + \frac{1}{k}\left[1 - 2(-1)^k\right]\sin kx)\right]$$

FOURIER INVERSION FORMULA:

The proper inversion formula is given as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, F(k)\, dk$$

The formula nearly states that f is the fourier transform of $F(k)$

where $F(k) = \mathcal{F}\{f(x)\}$

PROOF:

by Fourier integral theorem $f(x) = \frac{1}{\pi}\int_0^{\infty} dk \int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty} dk \int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

$\Rightarrow f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')dx' \int_0^{\infty} Cosk(x-x')dk$ **changing the order**

$\Rightarrow f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')dx' . \lim_{m\to\infty} \int_0^{m} Cosk(x-x')dk$ **.........(i)**

Since $\int_{-m}^{m} Cosk(x'-x)dk = 2\int_0^{m} Cosk(x-x')dk$ **.........(ii)**

Also $\int_{-m}^{m} Sink(x'-x)dk = 0 \Rightarrow i\int_{-m}^{m} Sink(x-x')dk = 0$ **.........(iii)**

On subtraction from (ii) and (iii) we have

$\int_{-m}^{m}[Cosk(x-x') - iSink(x-x')]dk = 2\int_0^{m} Cosk(x-x')dk$

$\Rightarrow \int_{-m}^{m} e^{-ik(x-x')}dk = 2\int_0^{m} Cosk(x-x')dk$

$\Rightarrow \int_0^{m} Cosk(x-x')dk = \frac{1}{2}\int_{-m}^{m} e^{-ik(x-x')}dk$ **.........(iv)**

Hence from (i) and (iv)

$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')dx' . \lim_{m\to\infty} \int_{-m}^{m} e^{-ik(x-x')}dk$

$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')dx' \int_{-\infty}^{\infty} e^{-ik(x-x')}dk$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, dk . \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')dx'$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, F(k)\, dk$ **as required.**

LAPLACE TRANSFORMATION WITH APPLICATIONS

Because of their simplicity, Laplace transforms are frequently used to solve a wide class of partial differential equations. Like other transforms, Laplace transforms are used to determine particular solutions. In solving partial differential equations, the general solutions are difficult, if not impossible, to obtain. The transform technique sometimes offers a useful tool for finding particular solutions. The Laplace transform is closely related to the complex Fourier transform, so the Fourier integral formula can be used to define the Laplace transform and its inverse.

INTEGRAL TRANSFORMATION

Consider a set $K(x,y) = \{f(x); f\ is\ function\ of\ x\ over\ [a,b]\}$ then integral transformation is defined as

$T\{f(x)\} = F(y) = \int_a^b f(x)K(x,y)dx$ **where $K(x,y)$ is kernel of T.**

LAPLACE TRANSFORMATION

If $f(t)$ is defined for all values of $t > 0$, then the Laplace transform of $f(t)$ is denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$ and is defined by the integral

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt$$

If $F(s)$ is laplace transform of $f(t)$ then $f(t)$ is called the <u>INVERSE LAPLACE TRANSFORM</u> of $F(s)$ i.e. $\mathcal{L}^{-1}\{F(s)\} = f(t)$

QUESTION: Show that $\mathcal{L}\{c\} = \frac{c}{s}$ where 'c' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\mathrm{st}} f(t)dt$

Then $\mathcal{L}\{c\} = \int_0^\infty e^{-\mathrm{st}}\, cdt = c\int_0^\infty e^{-\mathrm{st}}\, dt = c\left|-\frac{e^{-st}}{s}\right|_0^\infty = \frac{c}{s}$

QUESTION: Show that $\mathcal{L}\{e^{\mathrm{at}}\} = \frac{1}{s-a}$ where 'a' is constant.

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\mathrm{st}} f(t)dt$

Then $\mathcal{L}\{e^{\mathrm{at}}\} = \int_0^\infty e^{-\mathrm{st}}\, e^{\mathrm{at}}\, dt = \int_0^\infty e^{-(s-a)\mathrm{t}}\, dt = \left|-\frac{e^{-(s-a)\mathrm{t}}}{(s-a)}\right|_0^\infty = \frac{1}{s-a}$

QUESTION: Show that $\mathcal{L}\{t^{\mathrm{n}}\} = \frac{n!}{s^{n+1}}$ where '$n > 0$'

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\mathrm{st}} f(t)dt$

Then for n = 1;

$$\mathcal{L}\{t\} = \int_0^\infty e^{-\mathrm{st}}\, tdt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} dt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s}$$

In above $te^{-st} \to 0$ **as** $t \to \infty$

for n = 2;

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-\mathrm{st}}\, t^2\, dt = \left|-\frac{t^2 e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} 2t\, dt = \left|-\frac{t^2 e^{-st}}{s}\right|_0^\infty +$$

$\frac{2}{s}\int_0^\infty e^{-st}\, dt = \frac{2}{s^3}$ **In this part** $t^2 e^{-st}, te^{-st} \to 0$ **as** $t \to \infty$

And in general

$$\mathcal{L}\{t^{\mathrm{n}}\} = \int_0^\infty e^{-\mathrm{st}}\, t^{\mathrm{n}}\, dt = \left|-\frac{t^{\mathrm{n}} e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} nt^{\mathrm{n}-1}\, dt$$

$$\mathcal{L}\{t^{\mathrm{n}}\} = \left|-\frac{t^{\mathrm{n}} e^{-st}}{s}\right|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{\mathrm{n}-1}\, dt = \frac{n}{s}\mathcal{L}\{t^{\mathrm{n}-1}\} =$$

$$\frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}\ldots\ldots\ldots..\frac{2}{s}.\frac{1}{s}\mathcal{L}\{t^0\}$$

$$\mathcal{L}\{t^{\mathrm{n}}\} = \frac{(n-1)(n-1)(n-1)\ldots\ldots\ldots..3.2.1}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^n}.\frac{1}{s}$$

Hence $\mathcal{L}\{t^{\mathrm{n}}\} = \frac{n!}{s^{n+1}}$ where '$n > 0$'

QUESTION: **Show that** $\mathcal{L}\{Sinat\} = \frac{a}{s^2+a^2}$

SOLUTION: **Since** $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Sinat\} = \int_0^\infty e^{-st}\, Sinatdt$

$\therefore \int_0^\infty e^{at}\, Sinbtdt = \frac{e^{at}}{a^2+b^2}[aSinbt - bCosbt]$ **therefore**

$\mathcal{L}\{Sinat\} = \left|\frac{e^{-st}}{s^2+a^2}[-sSinat - aCosat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-a)\right] = \frac{a}{s^2+a^2}$

QUESTION: **Show that** $\mathcal{L}\{Cosat\} = \frac{s}{s^2+a^2}$

SOLUTION: **Since** $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Cosat\} = \int_0^\infty e^{-st}\, Cosatdt$

$\therefore \int_0^\infty e^{at}\, Cosbtdt = \frac{e^{at}}{a^2+b^2}[aCosbt + bSinbt]$ **therefore**

$\mathcal{L}\{Cosat\} = \left|\frac{e^{-st}}{s^2+a^2}[-sCosat + aSinat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-s)\right] = \frac{s}{s^2+a^2}$

QUESTION: **Show that** $\mathcal{L}\{Sinhat\} = \frac{a}{s^2-a^2}$

SOLUTION: **Since** $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then $\mathcal{L}\{Sinhat\} = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\left[\int_0^\infty e^{-st}\, e^{at}dt - \int_0^\infty e^{-st}\, e^{-at}dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{a}{s^2-a^2}$

QUESTION: Show that $\mathcal{L}\{Coshat\} = \frac{s}{s^2-a^2}$

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

Then

$$\mathcal{L}\{Coshat\} = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{1}{2}\left[\int_0^\infty e^{-st}\, e^{at}dt + \int_0^\infty e^{-st}\, e^{-at}dt\right]$$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt + \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{s}{s^2-a^2}$$

FUNCTION OF EXPONENTIAL ORDER: A function f (t) is said to be <u>of exponential order</u> as $t \to \infty$ **if there exist real constants** M **and** c **such that** $|f\,(t)| \le Me^{ct}$ **for** $0 \le t < \infty$.

FUNCTION OF CLASS 'A': A function *f* (t) which is peicewise continuous and is of exponential order is said to be function of class A.

EXISTENCE THEOREM OF LAPLACE TRANSFORMATION:

Let f **be piecewise continuous in the interval** $[0, T]$ **for every positive** T**, and let** f **be of exponential order, that is,** $f\,(t) = O\,(e^{at})$ **as** $t \to \infty$ **for some** $a > 0$**.**
Then, the Laplace transform of $f\,(t)$ **exists for** $Res > a$**.**

OR sufficient condition for the existence of Laplace transformation is that it should be a function of class A.

Proof: Since f **is piecewise continuous and of exponential order, we have**

$$|\mathcal{L}\{f(t)\}| = \left|\int_0^\infty e^{-st}\, f(t)dt\right| \le \int_0^\infty e^{-st}\, |f(t)|dt \le \int_0^\infty e^{-st}\, Me^{at}\, dt = M\int_0^\infty e^{-(s-a)t}\, dt$$

$|\mathcal{L}\{f(t)\}| \le \frac{M}{s-a}$ **Thus the Laplace transform of** $f\,(t)$ **exists for** $Res > a$**.**

Remark: $F(s) = s^2$ **is not L.T. of any piecewise continuous function of exponential order, because** s^2 **does not approaches to zero as** $s \to \infty$ **i.e.** $\mathcal{L}^{-1}\{s^2\}$ **does not exists.**

QUESTION: Show that $\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ **where** α **is any real.**

SOLUTION: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$$\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-st}\, t^\alpha\, dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\alpha \frac{1}{s} du = \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u}\, u^\alpha\, du \text{ ....(i)}$$

Since by definition of Gamma function we have

$$\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1}\, du \Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-u} u^\alpha\, du \quad (i) \Rightarrow \mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

USEFUL RESULTS:

- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ **then** $\mathcal{L}\{t^\alpha\} = \frac{\alpha\Gamma(\alpha)}{s^{\alpha+1}}$
- $\mathcal{L}\{t^\alpha\} = \frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\}$

QUESTION: Find $\mathcal{L}\left\{t^{1/2}\right\}$ **and** $\mathcal{L}\left\{t^{-1/2}\right\}$

SOLUTION: Since $\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

Put $\alpha = \frac{1}{2}$ **Then** $\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ **now using** $\mathcal{L}\{t^\alpha\} = \frac{\alpha\Gamma(\alpha)}{s^{\alpha+1}}$

we have $\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s.s^{\frac{1}{2}}}$

Then $\mathcal{L}\left\{t^{1/2}\right\} = \frac{1}{2s}\frac{\sqrt{\pi}}{\sqrt{s}}$ **as** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **thus** $\mathcal{L}\left\{t^{1/2}\right\} = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$

Put $\alpha = -\frac{1}{2}$ **Then** $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$ **now we have** $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}$

Then $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}}$ **as** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **thus** $\mathcal{L}\left\{t^{-1/2}\right\} = \sqrt{\frac{\pi}{s}}$

QUESTION: Find $\mathcal{L}\{t^{k/2}\}$ where 'k' is an odd positive integer. $\mathcal{L}\{t^{5/2}\}$ =?

SOLUTION: Suppose k = m + 1 where 'm' is any positive integer.

Then using $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\}$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\} = \mathcal{L}\left\{t^{\frac{2m+1}{2}}\right\} = \mathcal{L}\left\{t^{m+\frac{1}{2}}\right\} = \frac{m+\frac{1}{2}}{s}\mathcal{L}\left\{t^{m+\frac{1}{2}-1}\right\} = \frac{m+\frac{1}{2}}{s}.\frac{m-\frac{1}{2}}{s}.\mathcal{L}\left\{t^{m-\frac{1}{2}-1}\right\}$$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\} = \frac{2m+1}{2s}.\frac{2m-1}{2s}.\frac{2m-3}{2s}\ldots\ldots..\frac{3}{2s}.\frac{1}{2s}.\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} = \frac{(2m+1).(2m+1).(2m+1)\ldots\ldots..3.1}{(2s)^{m+1}}\sqrt{\frac{\pi}{s}}$$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\} = \frac{(2m+1).(2m+1-2).(2m+1-4)\ldots\ldots..3.1}{(2s)^{m+1}}\sqrt{\frac{\pi}{s}} = \frac{(k).(k-2).(k-4)\ldots\ldots..3.1}{(2)^{\frac{k+1}{2}}}\sqrt{\frac{\pi}{s^{k+2}}}$$

Where we use $2m + 1 = k \Rightarrow m = (k-1)/2$

If k = 5 then $\mathcal{L}\left\{t^{\frac{5}{2}}\right\} = \frac{5.3.1}{(2)^{\frac{5+1}{2}}}\sqrt{\frac{\pi}{s^{5+2}}} = \frac{15}{(2)^3}\sqrt{\frac{\pi}{s^7}}$

PROPERTIES OF LAPLACE TRANSFORMS

LINEARITY PROPERTY: THE LAPLACE TRANSFORMATION $\mathcal{L}$ IS LINEAR.

Proof. Let $u(t) = af(t) + bg(t)$ where a and b are constants.

We have, by definition

$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st}[af(t) + bg(t)]dt$$

$$\mathcal{L}\{u(t)\} = a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ **hence proved.**

1st SHIFTING PROPERTY (1st TRANSLATION THEOREM):

If $F(s)$ is the laplace transformation of $f(t)$ Then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Proof. By definition, we have

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$$

This result also known as 1st shifting theorem or 1st translation theorem.

EXAMPLES:

i. If $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ then $\mathcal{L}\{t^2 e^{t}\} = \frac{2}{(s-1)^3}$

ii. If $\mathcal{L}\{Sinwt\} = \frac{w}{s^2+w^2}$ then $\mathcal{L}\{e^{at} Sinwt\} = \frac{w}{(s-a)^2+w^2}$

iii. If $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2}$ then $\mathcal{L}\{e^{at} Coswt\} = \frac{s-a}{(s-a)^2+w^2}$

iv. If $\mathcal{L}\{t^{n}\} = \frac{n!}{s^{n+1}}$ then $\mathcal{L}\{e^{at} t^{n}\} = \frac{n!}{(s-a)^{n+1}}$

Question: Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\}$

Answer: in this question we will use the first shifting theorem according to which $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2-1^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2-1^2}\right\} = e^{-t} Cosht$

Question: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\}$

Answer: in this question we will use the first shifting theorem according to which $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-1^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-1^2}\right\} = e^{-t} Sinht$

Question: Find $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\}$

Answer: in this question we will use the first shifting theorem according to which $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \Rightarrow e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$

Thus $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{2}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\}\left\{\frac{2}{s}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} - e^{-2t}\mathcal{L}^{-1}\left\{\frac{2}{s}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = 3e^{-t} - 2e^{-2t}$ **since** $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

SCALING PROPERTY: If $F(s)$ is the laplace transformation of (t) , then

$\mathcal{L}\,[f\,(at)] = \frac{1}{a}\,F\,(\frac{s}{a})$ **with a > 0**

Proof. By definition we have

$\mathcal{L}\,\{f(at)\} = \int_0^\infty e^{-st}\,f(at)dt = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t'}\,f(t')dt' = \frac{1}{a}\,F\,(\frac{s}{a})$

putting $at = t'$ This result also known as Rule of Scale.

EXAMPLES:

i. **If $\mathcal{L}\{Cost\} = \frac{s}{s^2+1}$ then $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2} = \frac{1}{w}\left[\frac{s/w}{(s/w)^2+1}\right]$**

ii. **If $\mathcal{L}\{e^{t}\} = \frac{1}{s-1}$ then $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} = \frac{1}{a}\left[\frac{1}{\left(\frac{s}{a}-1\right)}\right]$**

DIFFERENTIATION PROPERTY:

Let f be continuous and f' piecewise continuous, in $0 \le t \le T$ for all $T > 0$. Let f also be of exponential order as $t \to \infty$ Then, the Laplace transform of $f'\,(t)$ exists and is given by

$$\mathcal{L}[f'\,(t)] = s\mathcal{L}[f(t)] - f\,(0) = sF(s) - f\,(0)$$

Proof. If $f(t)$ is continuous and $f'(t)$ is sectionally continuous on the interval $[0,\infty)$ and both are of exponential order then

$\mathcal{L}\,\{f'(t)\} = \int_0^\infty e^{-st}\,f'(t)dt = |e^{-st}f(t)|_0^\infty - (-s)\int_0^\infty e^{-st}\,f(t)dt$

$\mathcal{L}\,\{f'(t)\} = [0 - f(0)] + s\mathcal{L}\,\{f(t)\}$

$\mathcal{L}[f'\,(t)] = s\mathcal{L}[f(t)] - f\,(0) = sF(s) - f\,(0)$

If f' and f'' satisfy the same conditions imposed on f and f' respectively, then, the Laplace transform of $f''\,(t)$ can be obtained immediately by applying the preceding theorem; that is

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f\,'(0) = s^2F(s) - sf(0) - f'\,(0)$$

Proof. If $f(t), f'(t)$ are continuous and $f''(t)$ is sectionally continuous on the interval $[0,\infty)$ and all are of exponential order then

$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)dt = |e^{-st} f'(t)|_0^\infty - (-s)\int_0^\infty e^{-st} f'(t)dt$

$\mathcal{L}\{f''(t)\} = [0 - f'(0)] + s\mathcal{L}\{f'(t)\} = -f'(0) + s[sF(s) - f(0)]$

$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f'(0) = s^2F(s) - sf(0) - f'(0)$

Clearly, the Laplace transform of $f^n(t)$ can be obtained in a similar manner by successive application.The result may be written as

$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - \cdots \ldots \ldots - sf^{n-2}(0) - f^{n-1}(0)$

INTEGRATION PROPERTY :

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

PROOF:

Consider $g(\tau) = \int_0^t f(\tau)\, d\tau \Rightarrow g'(\tau) = f(t) \Rightarrow \mathcal{L}[g'(\tau)] = \mathcal{L}[f(t)]$

$\Rightarrow sG(s) - g(0) = \mathcal{L}[f(t)] \Rightarrow s\mathcal{L}[g(\tau)] - 0 = \mathcal{L}[f(t)]$

$\Rightarrow \mathcal{L}[g(\tau)] = \frac{F(s)}{s} \qquad \Rightarrow \mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$

Question: Solve the initial value problem $u' - 2u = 0$ with $u(0) = 1$

Answer: Given $u' - 2u = 0$

$\Rightarrow \mathcal{L}\{u'\} - 2\mathcal{L}\{u\} = 0 \Rightarrow sU(s) - u(0) - 2U(s) = 0$

Using $u(0) = 1$ $\quad \Rightarrow sU(s) - 1 - 2U(s) = 0 \Rightarrow U(s) = \frac{1}{s-2}$

$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \qquad \Rightarrow u(t) = e^{2t}$ **required answer.**

Question:

Solve the initial value problem $u'' + 4u' + 3u = 0$ **with** $u(0) = 1, u'(0) = 0$

Answer: Given $u'' + 4u' + 3u = 0$

$\Rightarrow \mathcal{L}\{u''\} + 4\,\mathcal{L}\{u'\} + 3\mathcal{L}\{u\} = 0$

$\Rightarrow s^2U(s) - su(0) - u'(0) + 4sU(s) - 4u(0) + 3U(s) = 0$

$\Rightarrow s^2U(s) - s + 4sU(s) - 4 + 3U(s) = 0$ **since** $u(0) = 1, u'(0) = 0$

$\Rightarrow U(s) = \frac{s+4}{s^2+4s+2} \Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+2}\right\}$

$\Rightarrow u(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s+1} - \frac{1/2}{s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1/2}{s+3}\right\}$

$\Rightarrow u(t) = e^{-\mathrm{t}}\,\mathcal{L}^{-1}\left\{\frac{3/2}{s}\right\} - e^{-3\mathrm{t}}\,\mathcal{L}^{-1}\left\{\frac{1/2}{s}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \frac{3}{2}e^{-\mathrm{t}} - \frac{1}{2}e^{-3\mathrm{t}}$ **since** $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

UNIT STEP FUNCTION: A real valued function $H: R \to R$ **is defined as**

$H\,(t - \xi) = \begin{cases} 1 & ; \quad t \geq \xi \\ 0 & ; \quad t < \xi \end{cases}$ **When** $\xi = 0$; $H\,(t) = \begin{cases} 1 & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases}$

CONVOLUTION FUNCTION / FAULTUNG FUNCTION OF LAPLACE TRANSFORMATION.

The function $(f * g)\,(t) = \int_0^t f\,(t - \xi)\,g\,(\xi)\,d\xi$ **is called the convolution of the functions** f **and** g **regarding laplace transformation.**

THE CONVOLUTION SATISFIES THE FOLLOWING PROPERTIES:

1. $f * g = g * f$ **(commutative).**
2. $f * (g * h) = (f * g) * h$ **(associative).**
3. $f * (\alpha g + \beta h) = \alpha\,(f * g) + \beta\,(f * h)$ **(distributive),**

where α **and** β **are constants.**

USEFUL RESULT:

$$(f * g)(t) = \int_0^t f\,(t - \xi)g\,(\xi)d\xi = \int_0^\infty H\,(t - \xi)f\,(t - \xi)\,g\,(\xi)\,d\xi$$

CONVOLUTION / FAULTUNG THEOREM OF LAPLACE TRANSFORMATION

If $F(s)$ and $G(s)$are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then the Laplace transform of the convolution $(f * g)(t)$ is the product $F(s)G(s)$

OR $\mathcal{L}^{-1}\{F(s)G(s)\} = f * g \Rightarrow \mathcal{L}\{f * g\} = F(s)G(s)$

PROOF: By definition, we have

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}\,(f * g)\,dt$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}\int_0^t f\,(t-\xi)\,g\,(\xi)\,d\xi\ dt$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}\int_0^t f(\xi)g\,(t-\xi)\,d\xi\ dt \qquad \text{since } f * g = g * f$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}\left[\int_0^\infty H\,(t-\xi)f(\xi)g(t-\xi)\,d\xi\right]dt$$

By reversing the order of integration, we have

$$\mathcal{L}\{f * g\} = \int_0^\infty\left[\int_0^\infty e^{-st} H\,(t-\xi)g(t-\xi)\,dt\right]f(\xi)d\xi$$

If we introduce the new variable $\eta = (t-\xi)$ in the inner integral, we obtain

$$\mathcal{L}\{f * g\} = \int_0^\infty f\,(\xi)\,d\xi\left[\int_{-\xi}^\infty e^{-s(\xi+\eta)}H\,(\eta)g(\eta)\,d\eta\right]$$

$$\mathcal{L}\{f * g\} =$$

$$\int_0^\infty f\,(\xi)\,d\xi\left[\int_{-\xi}^0 e^{-s(\xi+\eta)}H\,(\eta)g(\eta)\,d\eta + \int_0^\infty e^{-s(\xi+\eta)}H\,(\eta)g(\eta)\,d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\,d\xi\left[\int_{-\xi}^0 e^{-s(\xi+\eta)}0.\,g(\eta)\,d\eta + \int_0^\infty e^{-s(\xi+\eta)}.\,1.\,g(\eta)\,d\eta\right] \text{ by step function}$$

$$\mathcal{L}\{f * g\} = \int_0^\infty f(\xi)\,d\xi\left[\int_0^\infty e^{-s(\xi+\eta)}g(\eta)\,d\eta\right]$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-s\xi}f(\xi)d\xi\int_0^\infty e^{-s\eta}g\,(\eta)\,d\eta$$

$$\mathcal{L}\{f * g\} = F(s)G(s)$$

PROBLEM: Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s^2+9)}\right\}$

Solution: Here we have $H(s) = F(s)G(s)$

then taking $F(s) = \frac{1}{s^2} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \Rightarrow f(t) = t$

$G(s) = \frac{3}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)}\right\} \Rightarrow g(t) = Sin3t$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(t-\xi)g(\xi)d\xi = \int_0^t (t-\xi)\, Sin3(\xi)\, d\xi$$

$$h(t) = \int_0^t t\, Sin3(\xi)\, d\xi - \int_0^t \xi\, Sin3(\xi) d\xi = \left| -\frac{tCos3\xi}{3} + \frac{\xi Cos3\xi}{3} - \frac{Sin3\xi}{9} \right|_0^t$$

$$h(t) = -\frac{Sin3t}{9} + \frac{t}{3} = \frac{1}{9}(3t - Sin3t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2(s^2+9)}\right\}$$

PROBLEM: Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$

Solution: Here we have $H(s) = F(s)G(s)$

then taking $F(s) = \frac{s}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)}\right\} \Rightarrow f(t) = Cos3t$

$G(s) = \frac{1}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)}\right\} \Rightarrow g(t) = \frac{1}{3}Sin3t$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(t-\xi)g(\xi)d\xi = \frac{1}{3}\int_0^t Cos3(t-\xi)\, Sin3(\xi)\, d\xi$$

$$h(t) = \frac{1}{3}\int_0^t (Cos3tCos3\,\xi + Sin3tSin3\,\xi)\, Sin3(\xi)\, d\xi$$

$$h(t) = \frac{1}{3}\int_0^t Cos3tCos3\xi Sin3\xi + Sin3tSin^2 3\xi\, d\xi$$

$$h(t) = \frac{1}{6}Cos3t\int_0^t 2Cos3\xi Sin3\xi d\xi + \frac{1}{3}Sin3t\int_0^t Sin^2 3\xi\, d\xi$$

$$h(t) = \frac{1}{6}Cos3t\int_0^t Sin6\xi d\xi + \frac{1}{6}Sin3t\int_0^t \left(\frac{1-Cos6\xi}{2}\right)\, d\xi$$

$$h(t) = \frac{1}{6}Cos3t\left| -\frac{Cos6\xi}{6} \right|_0^t + \frac{1}{6}Sin3t\left| \xi - \frac{Sin6\xi}{6} \right|_0^t$$

$$h(t) = \frac{1}{36}Cos3t(1 - Cos6t) + \frac{1}{6}Sin3t\left(t - \frac{Sin6t}{6}\right)$$

$$h(t) = \frac{1}{36}Cos3t - \frac{1}{36}Cos3tCos6t + \frac{1}{6}tSin3t - \frac{1}{36}Sin3tSin6t$$

$$h(t) = -\frac{1}{36}[Cos3tCos6t + Sin3tSin6t] + \frac{1}{6}tSin3t + \frac{1}{36}Cos3t$$

$$h(t) = -\frac{1}{36}[Cos(6t - 3t)] + \frac{1}{6}tSin3t + \frac{1}{36}Cos3t$$

$$h(t) = -\frac{1}{36}Cos3t + \frac{1}{6}tSin3t + \frac{1}{36}Cos3t = -\frac{1}{6}tSin3t = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$$

PROBLEM:

Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$

Solution: Here we have $H(s) = F(s)G(s) = \frac{1}{s^2+6s+13} = \frac{1}{(s+3+2i)(s+3-2i)}$

$$F(s) = \frac{1}{s+3+2i} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3+2i}\right\} \Rightarrow f(t) = e^{-(3+2i)t}$$

$$G(s) = \frac{1}{s+3-2i} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3-2i}\right\} \Rightarrow g(t) = e^{-(3-2i)t}$$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t e^{-(3+2i)\xi}\, e^{-(3-2i)(t-\xi)}d\xi$$

$$h(t) = e^{-(3-2i)t}\int_0^t e^{-(3+2i)\xi}\, e^{(3-2i)\xi}d\xi$$

$$h(t) = e^{-(3-2i)t}\int_0^t e^{-4i\xi}\, d\xi$$

$$h(t) = e^{-(3-2i)t}\left|\frac{e^{-4i\xi}}{-4i}\right|_0^t = \frac{e^{-(3-2i)t}}{-4i}\left|e^{-4it} - e^0\right| = \frac{e^{-(3-2i)t}}{-4i}\left|e^{-4it} - 1\right|$$

$$h(t) = \frac{e^{-3t}}{2}\left[\frac{e^{-2it} - e^{2it}}{-2i}\right] = \frac{e^{-3t}}{2}\left[\frac{e^{2it} - e^{-2it}}{2i}\right]$$

$$h(t) = \frac{e^{-3t}}{2}Sin2t$$

PROBLEM:

Use covolution theorem to calculate laplace transform of

$$f(t) = \int_0^t (t - \beta)^3 e^{\beta} Sin\beta d\beta$$

Solution:

Let $f(t) = g * h = \int_0^t (t - \beta)^3 e^{\beta} Sin\beta d\beta$ **…………..(i)**

Comparing with $g * h = \int_0^t g(t - \beta) h(\beta) d\beta$ **…………..(ii) we get**

$g(t - \beta) = (t - \beta)^3 \Rightarrow g(t) = t^3$ **and** $h(\beta) = e^{\beta} Sin\beta \Rightarrow h(t) = e^t Sint$

Now $\mathcal{L}\{f(t)\} = \mathcal{L}\{g * h\} = F(s)G(s) = \mathcal{L}\{g(t)\}.\mathcal{L}\{h(t)\} = \mathcal{L}\{t^3\}.\mathcal{L}\{e^t Sint\}$

$$\mathcal{L}\{f(t)\} = \frac{3!}{s^{3+1}} \cdot \frac{1}{(s-1)^2 + 1^2} = \frac{6}{s^4(s^2 - 2s + 1)}$$

THE GAUSSIAN INTEGRAL

Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ **or** $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Solution: consider $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ **and** $I = \int_{-\infty}^{\infty} e^{-y^2} dy$

then multiplying both $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} dxdy$

Now using polar coordinates

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} rdrd\theta = \int_0^{2\pi} d\theta \left(-\frac{1}{2}\right) \int_0^{\infty} e^{-r^2} (-2r)dr = \pi \Rightarrow I = \sqrt{\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \Rightarrow 2\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

LAPLACE TRANSFORM OF STEP FUNCTION:

The <u>Heaviside unit step</u> function is defined by

$$H\,(t - a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \quad \text{where } a \geq 0$$

Now, we will find its Laplace transform.

$$\mathcal{L}\{H\,(t - a)\} = \int_0^\infty e^{-st}\,H\,(t - a)\,dt$$

$$\mathcal{L}\{H\,(t - a)\} = \int_0^a e^{-st}\,H\,(t - a)\,dt + \int_a^\infty e^{-st}\,H\,(t - a)\,dt$$

$$\mathcal{L}\{H\,(t - a)\} = \int_0^a e^{-st}\,.\,0\,dt + \int_a^\infty e^{-st}\,.\,1\,dt$$

$$\mathcal{L}\{H\,(t - a)\} = \int_a^\infty e^{-st}\,dt = \left|\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s} \quad ; \; s > 0$$

THEOREM:

If $f(t)$ is a function of exponential order 'c' then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \;\; ; s > a$$

PROOF: Consider $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-\mathrm{st}}\; f(t)dt$

Differentiating w.r.to 's'

$$\Rightarrow \frac{d}{ds} F(s) = (-1) \int_0^\infty e^{-\mathrm{st}}\; tf(t)dt = (-1)\mathcal{L}\{tf(t)\} \Rightarrow (-1)\frac{d^1}{ds^1} F(s) = \mathcal{L}\{t^1 f(t)\}$$

Again differentiating w.r.to 's'

$$\Rightarrow \frac{d^2}{ds^2} F(s) = (-1)(-1) \int_0^\infty e^{-\mathrm{st}}\,(-t)\,tf(t)dt = (-1)^2 \int_0^\infty e^{-\mathrm{st}}\; t^2 f(t)dt = (-1)^2\mathcal{L}\{t^2 f(t)\}$$

$$\Rightarrow (-1)^2 \frac{d^2}{ds^2} F(s) = \mathcal{L}\{t^2 f(t)\}$$

Continuing this process, we get the required

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \;\; ; s > a \qquad \therefore (-1)^n = (-1)^{-n}$$

REMARK: $\mathcal{L}\{t^{-n} f(t)\} = \frac{d^n}{ds^n} F(s)$

LAPLACE TRANSFORMATION OF LOGRITHMIC FUNCTION:

Show that $\mathcal{L}\{lnt\} = \frac{1}{s}(\Gamma'(1) - lns)$

SOLUTION: by using definition

$\mathcal{L}\{lnt\} = \int_0^\infty e^{-st}\ lntdt = \int_0^\infty e^{-u}\ ln\left(\frac{u}{s}\right)\frac{du}{s}$ **by putting** $st = u$

$\mathcal{L}\{lnt\} = \frac{1}{s}\int_0^\infty e^{-u}\ lnudu - \frac{1}{s}\int_0^\infty e^{-u}\ lnsdu = \frac{1}{s}(I) - \frac{1}{s}lns\int_0^\infty e^{-u}\ du$

$\mathcal{L}\{lnt\} = \frac{1}{s}(I) - \frac{1}{s}lns(1) = \frac{1}{s}(I) - \frac{1}{s}lns$ **…………..(i)**

Now consider $I = \int_0^\infty e^{-u}\ lnudu$

Since $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du \Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-u}u^{\alpha}\ du \Rightarrow \Gamma'(1) = \int_0^\infty e^{-u}u^{\alpha}\,lnu\,du$

Put $\alpha = 0 \Rightarrow \Gamma'(1) = \int_0^\infty e^{-u}\,lnu\,du = I$

Thus $\mathcal{L}\{lnt\} = \frac{1}{s}(\Gamma'(1) - lns)$

where $\Gamma'(1) \approx 0.57721$ **is called Euler's constant.**

THE GAMMA FUNCTION:

Gamma function can be defined as follows $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du$

USEFUL RESULTS:

- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

  Proof: since $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du \Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-u}u^{\alpha}\ du$

  $\Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-u}u^{\alpha}\ du = \left|u^{\alpha}\frac{e^{-u}}{-1}\right|_0^\infty - \int_0^\infty \left|\frac{e^{-u}}{-1}\right| \alpha u^{\alpha-1}\ du$

  $\Rightarrow \Gamma(\alpha+1) = 0 + \alpha\int_0^\infty e^{-u}u^{\alpha-1}\ du = \alpha\Gamma(\alpha)$

- $\Gamma(1) = 1$ **we can prove it using** $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\ du$ **with** $\alpha = 1$

- $\Gamma(\alpha+1) = \alpha!$

Proof: since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$put\ \ \alpha = 1 \Rightarrow \Gamma(2) = 1.\Gamma(1) = 1.1 = 1!$

$put\ \ \alpha = 2 \Rightarrow \Gamma(3) = 2.\Gamma(2) = 2.1 = 2!$

$put\ \ \alpha = 3 \Rightarrow \Gamma(4) = 3.\Gamma(3) = 3.2.1 = 3!$

: :

Then $\Gamma(\alpha) = \alpha - 1! \Rightarrow \Gamma(\alpha+1) = \alpha!$

SECOND SHIFTING (TRANSLATION) THEOREM:

If $F(s)$ **and** $H(s)$ **are the Laplace transforms of** $f(t)$ **and** $h(t)$ **respectively, then**

$\mathcal{L}[H(t-a)\,f(t-a)] = e^{-as}F(s) = e^{-as}\mathcal{L}\{f(t)\}$

Or $\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$

Proof: By definition

$\mathcal{L}\{H(t-a)\,f(t-a)\} = \int_0^\infty e^{-st} H(t-a)\,f(t-a)\,dt$

$\mathcal{L}\{H(t-a)\,f(t-a)\} = \int_0^a e^{-st} H(t-a) f(t-a)\,dt + \int_a^\infty e^{-st} H(t-a) f(t-a)\,dt$

$\mathcal{L}\{H(t-a)\,f(t-a)\} = \int_a^\infty e^{-st} f(t-a)\,dt$

Introducing the new variable $\xi = t - a$**, we obtain**

$\mathcal{L}\{H(t-a)\,f(t-a)\} = \int_0^\infty e^{-(\xi+a)s} f(\xi)\,d\xi = e^{-as}\int_0^\infty e^{-\xi s} f(\xi)\,d\xi$

$\mathcal{L}\{H(t-a)\,f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s)$

REMARK:

1st Shifting theorem enables us to calculate Laplace transform of the function of the form $e^{kt}f(t)$ **where the 2nd Shifting theorem in similar way enables us to calculate inverse Laplace transform of the function of the form** $e^{-as}F(s)$

COROLLARY: Prove that $\mathcal{L}\{p(t)f(t)\} = P(-D)F(s)$ where $p(t)$ is a polynomial in 't'.

SOLUTION:

Since $p(t) = a_0 + a_1 t + a_2 t^2 + \dots\dots\dots\dots + a_n t^n = \sum_{i=1}^{n} a_i t^i$ **Then**

$$\mathcal{L}\{p(t)f(t)\} = \mathcal{L}\{\textstyle\sum_{i=1}^{n} a_i t^i f(t)\} = \sum_{i=1}^{n} a_i \mathcal{L}\{t^i f(t)\} = \sum_{i=1}^{n} a_i (-1)^i \frac{d^i}{ds^i} F(s)$$

$$\mathcal{L}\{p(t)f(t)\} = \sum_{i=1}^{n} a_i (-1)^i D^i F(s) = \sum_{i=1}^{n} a_i (-D)^i F(s) = P(-D)F(s)$$

LAPLACE TRANSFORMATION OF BESSEL'S FUNCTION

EXAMPLE:

Find Laplace Tranformation of $J_0(t) = \frac{1}{\pi}\int_0^{\pi} Cos(tSin\theta)d\theta$ **also find** $\mathcal{L}\{J_0(at)\}$

Solution: By definition

$$\mathcal{L}\{J_0(t)\} = \int_0^{\infty} e^{-st} J_0(t)\,dt = \int_0^{\infty} e^{-st}\left[\frac{1}{\pi}\int_0^{\pi} Cos(tSin\theta)d\theta\right] dt$$

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\pi}\int_0^{\pi}\left[\int_0^{\infty} e^{-st}Cos(tSin\theta)\,dt\right] d\theta = \frac{1}{\pi}\int_0^{\pi} I\,d\theta \quad \dots\dots\dots\dots\dots\dots(i)$$

Now $I = \int_0^{\infty} e^{-st}Cos(Sin\theta)t\,dt = \mathcal{L}\{Cos(Sin\theta)t\} = \frac{s}{s^2+(Sin\theta)^2}$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{1}{\pi}\int_0^{\pi}\frac{s}{s^2+Sin^2\theta}d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{s}{s^2+Sin^2\theta}d\theta \qquad \because \int_0^{a} f(x)dx = 2\int_0^{a/2} f(x)dx$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2}\frac{s}{s^2+Sin^2\left(\frac{\pi}{2}-\theta\right)}d\theta \quad \because \int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2}\frac{s}{s^2+Cos^2\theta}d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{sSec^2\theta}{Sec^2\theta(s^2+Cos^2\theta)}d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{sSec^2\theta}{s^2Sec^2\theta+1}d\theta$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2}\frac{sSec^2\theta}{s^2(1+Tan^2\theta)+1}d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{sSec^2\theta}{(s^2+1)+(s^2Tan^2\theta)}d\theta$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi}\int_0^{\pi/2}\frac{sSec^2\theta}{s^2\left[\frac{s^2+1}{s^2}+Tan^2\theta\right]}d\theta = \frac{2}{\pi s}\int_0^{\pi/2}\frac{Sec^2\theta}{\left[\frac{s^2+1}{s^2}+Tan^2\theta\right]}d\theta$$

$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi s}\int_0^{\infty}\frac{dx}{a^2+x^2}$ **by putting** $x = Tan\theta \Rightarrow dx = Sec^2\theta d\theta$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{2}{\pi s}\left|\frac{1}{a}Tan^{-1}\frac{x}{a}\right|_0^{\infty} = \frac{2}{\pi s}\left(\frac{\pi}{2a}\right) = \frac{1}{as} = \frac{1}{s}.\frac{s}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}} \qquad \because a^2 = \frac{s^2+1}{s^2}$$

$$\Rightarrow \mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

- **To find $\mathcal{L}\{J_0(at)\}$ see last portion of next example.**

EXAMPLE: Given the Bessel's functions of the first kind and positive integral order satisfy the recurrence relations $J_1 = -J'_0,\ \ J_{n+1} = J_{n-1} - 2J'_n\ ; n \geq 1$

with $J_0(0) = 1\ , J_n(0) = 0\ ; n > 0$ then show that $\mathcal{L}\{J_n(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}$

also find $\mathcal{L}\{J_n(at)\}\quad ; a > 0$

Solution: We will prove the result by mathematical induction.

Using first recurrence relation:

$$\mathcal{L}\{J_1(t)\} = \mathcal{L}\{-J'_0(t)\} = -\mathcal{L}\{J'_0(t)\} = -[s\mathcal{L}\{J_0(t)\} - J_0(0)] = -\frac{1}{\sqrt{s^2+1}} + 1$$

$\mathcal{L}\{J_1(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^1}{\sqrt{s^2+1}}$ **result is true for 0**

Now $J_2 = J_0 - 2J'_1 \Rightarrow \mathcal{L}\{J_2(t)\} = \mathcal{L}\{J_0(t)\} - 2\mathcal{L}\{J'_1(t)\} = \frac{1}{\sqrt{s^2+1}} - 2[s\mathcal{L}\{J_1(t)\} - J_1(0)]$

$$\Rightarrow \mathcal{L}\{J_2(t)\} = \frac{1}{\sqrt{s^2+1}} - 2\left[s.\frac{\left(\sqrt{s^2+1}-s\right)^1}{\sqrt{s^2+1}} - 0\right] = \frac{1}{\sqrt{s^2+1}} - \frac{2s\left(\sqrt{s^2+1}-s\right)}{\sqrt{s^2+1}} = \frac{\left(\sqrt{s^2+1}-s\right)^2}{\sqrt{s^2+1}}$$

$\Rightarrow \mathcal{L}\{J_2(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^2}{\sqrt{s^2+1}}$ **result is true for 1**

Suppose that result is true for k. $\Rightarrow \mathcal{L}\{J_k(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^k}{\sqrt{s^2+1}}$

Now we will check the result For k+1:

$$J_{k+1} = J_{k-1} - 2J'_k \Rightarrow \mathcal{L}\{J_{k+1}\} = \mathcal{L}\{J_{k-1}\} - 2\mathcal{L}\{J'_k\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}} - 2[s\mathcal{L}\{J_k(t)\} - J_k(0)]$$

$$\Rightarrow \mathcal{L}\{J_{k+1}\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}} - 2\left[s.\frac{\left(\sqrt{s^2+1}-s\right)^k}{\sqrt{s^2+1}} - 0\right] = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}} - \frac{2s\left(\sqrt{s^2+1}-s\right)^k}{\sqrt{s^2+1}}$$

$$\Rightarrow \mathcal{L}\{J_{k+1}\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}}\left[1 - 2s\left(\sqrt{s^2+1}-s\right)\right] = \frac{\left(\sqrt{s^2+1}-s\right)^{k-1}}{\sqrt{s^2+1}}\left(\sqrt{s^2+1}-s\right)^2$$

$\Rightarrow \mathcal{L}\{J_{k+1}\} = \frac{\left(\sqrt{s^2+1}-s\right)^{k+1}}{\sqrt{s^2+1}}$ **result is true for k+1**

So induction complete and result is proved.i.e. $\mathcal{L}\{J_n(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}$

Now to find $\mathcal{L}\{J_0(at)\}\quad ; a > 0$ we will use rule of scale. i.e $\mathcal{L}\,[f\,(at)] = \frac{1}{a}\,F\,(\frac{s}{a})$

$$\textbf{Then } \mathcal{L}\,[J_n(at)] = \frac{1}{a}\,F_n\left(\frac{s}{a}\right) = \frac{1}{a}.\frac{\left(\sqrt{\left(\frac{s}{a}\right)^2+1}-\left(\frac{s}{a}\right)\right)^n}{\sqrt{\left(\frac{s}{a}\right)^2+1}} = \frac{\left(\sqrt{s^2+a^2}-s\right)^n}{a^n\sqrt{s^2+a^2}}$$

$$\textbf{Then for n = 0 } \mathcal{L}\,[J_0(at)] = \frac{\left(\sqrt{s^2+a^2}-s\right)^0}{a^0\sqrt{s^2+a^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

EXAMPLE:

Show that $\mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \frac{1}{2} log_e \left\{1 + \frac{a^2}{s^2}\right\}$

Solution: We will use the result $\mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_s^\infty F(s')ds'$ **…………(i)**

provided $\lim_{x\to 0}\left\{\frac{f(x)}{x}\right\}$ **exists**

now $\lim_{x\to 0}\left\{\frac{f(x)}{x}\right\} = \lim_{x\to 0}\left\{\frac{1-Cosax}{ax}\right\} = \lim_{x\to 0}\left\{\frac{aSinax}{1}\right\} = 0$

$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{1 - Cosax\} = \mathcal{L}\{1\} - \mathcal{L}\{Cosax\} = \frac{1}{s} - \frac{s}{s^2+a^2} \quad ; s > a$

Hence

$(i) \Rightarrow \mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_s^\infty F(s')ds' \; (i) \Rightarrow \mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \int_s^\infty \left(\frac{1}{s'} - \frac{s'}{s'^2+a^2}\right) ds'$

$\Rightarrow \mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \left| lns' - \frac{1}{2} ln(s'^2 + a^2)\right|_s^\infty = \left| ln\frac{s'}{\sqrt{s'^2+a^2}}\right|_s^\infty = 0 - ln\sqrt{\frac{s^2}{s^2+a^2}} = ln\sqrt{\frac{s^2+a^2}{s^2}}$

Thus $\mathcal{L}\left\{\frac{1-Cosax}{ax}\right\} = \frac{1}{2} ln\left\{1 + \frac{a^2}{s^2}\right\} = \frac{1}{2} log_e\left\{1 + \frac{a^2}{s^2}\right\}$

EXAMPLE: Find $\mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\}$ **and deduce** $\mathcal{L}\left\{\frac{Sin^2 t}{t}\right\} = \frac{1}{2} ln\left(\frac{s^2+4}{s^2}\right) \quad ; s > 1$

Solution: We will use the result $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$ **…………(i)**

provided $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ **exists**

now $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\} = \lim_{t\to 0}\left\{\frac{e^{at}-Cosbt}{t}\right\} = \lim_{t\to 0}\left\{\frac{ae^{at}+bSinbt}{1}\right\} = a$

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at} - Cosbt\} = \mathcal{L}\{e^{at}\} - \mathcal{L}\{Cosbt\} = \frac{1}{s-a} - \frac{s}{s^2+b^2}$

$F(u) = \frac{1}{u-a} - \frac{u}{u^2+b^2}$

Hence $(i) \Rightarrow \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du \Rightarrow \mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = \int_s^\infty \left(\frac{1}{u-a} - \frac{u}{u^2+b^2}\right) du$

$\Rightarrow \mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = \left| ln(u-a) - \frac{1}{2} ln(u^2+b^2)\right|_s^\infty = \left| ln\frac{u-a}{\sqrt{u^2+b^2}}\right|_s^\infty = \left| ln\frac{1-\frac{a}{u}}{u\sqrt{1+\left(\frac{b}{u}\right)^2}}\right|_s^\infty$

$\Rightarrow \mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = 0 - ln\frac{s-a}{\sqrt{s^2+b^2}} = ln\frac{\sqrt{s^2+b^2}}{s-a}$

Thus $\mathcal{L}\left\{\frac{e^{at}-Cosbt}{t}\right\} = ln\frac{\sqrt{s^2+b^2}}{s-a}$

Now putting a = 0, b = 2 we get $\mathcal{L}\left\{\frac{e^0-Cos2t}{t}\right\} = ln\frac{\sqrt{s^2+2^2}}{s-0} \Rightarrow \mathcal{L}\left\{\frac{1-Cos2t}{t}\right\} = ln\frac{\sqrt{s^2+4}}{s}$

Hence $\mathcal{L}\left\{\frac{Sin^2t}{t}\right\} = \frac{1}{2}ln\left(\frac{s^2+4}{s^2}\right) \quad ; s > 1$

NULL FUNCTION: A function $N(x)$ **is called Null Function if** $\int_0^\infty N(x)dx = 0$

HEAVISIDE EXPANSION THEOREMS

THEOREM – I :

If $M(s)$ **and** $N(s)$ **are polynomials of degree 'm' and 'n' respectively with m < n and** $N(s)$ **has 'n' distinct zeros** a_i $; i = 1,2,3\ldots\ldots$ **none of which is zero of** $M(s)$ **then**

$$\mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n}\frac{M(a_i)}{N'(a_i)}e^{a_i t}$$

Proof: Given $M(s)$ **and** $N(s)$ **are polynomials of degree 'm' and 'n' respectively**

Let $N(s) = a_0 + a_1s + a_2s^2 + \ldots\ldots\ldots + a_ns^n = (s-a_1)(s-a_2)\ldots\ldots\ldots(s-a_n)$

Then consider $\frac{M(s)}{N(s)} = \frac{c_1}{s-a_1} + \frac{c_2}{s-a_2} + \cdots\ldots\ldots + \frac{c_n}{s-a_n} = \sum_{i=1}^{n}\frac{c_i}{s-a_i}$ **...........(i)**

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \mathcal{L}^{-1}\left\{\sum_{i=1}^{n}\frac{c_i}{s-a_i}\right\} = \sum_{i=1}^{n} c_i\,\mathcal{L}^{-1}\left\{\frac{1}{s-a_i}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n} c_i\,e^{a_i t}$ **...........(ii)**

$(i) \Rightarrow c_i = \lim_{s\to a_i}\left[(s-a_i)\frac{M(s)}{N(s)}\right] =$

$\lim_{s\to a_i}[M(s)]\,.\lim_{s\to a_i}\left[\frac{(s-a_i)}{N(s)}\right] = M(a_i).\lim_{s\to a_i}\left[\frac{1}{N'(s)}\right]$

$(i) \Rightarrow c_i = \frac{M(a_i)}{N'(a_i)}$

$(ii) \Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n}\frac{M(a_i)}{N'(a_i)}e^{a_i t}$

THEOREM – II :

If $M(s)$ and $N(s)$ are polynomials of degree 'm' and 'n' respectively with m < n and if $N(s)$ has a repeated root a_1 of multiplicity 'r' while othere roots $\sum_{i=2}^{n} a_i$ are not repeated then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=2}^{n} \frac{M(a_i)}{N'(a_i)} e^{a_i t} + \sum_{j=1}^{r} \frac{1}{(j\text{-}1!)}\left\{\frac{d^{j-1}}{ds^{j-1}}(s-a_i)^j \frac{M(s)}{N(s)}\right\} e^{a_1 t}\Bigg|_{s=a_1}$$

Proof: Since $N(s)$ has a repeated root a_1 of multiplicity 'r' while othere roots $a_2, a_3, \ldots . a_n$ are not repeated it means

$$N(s) = (s-a_1)^r (s-a_2) \ldots \ldots . . (s-a_n)$$

$$\Rightarrow \frac{M(s)}{N(s)} = \frac{M(s)}{(s-a_1)^r (s-a_2) \ldots\ldots (s-a_n)}$$

Then in terms of Partial fraction we will be as follows

$$\frac{M(s)}{N(s)} = \frac{d_r}{(s-a_1)^r} + \frac{d_{r-1}}{(s-a_1)^{r-1}} + \frac{d_{r-2}}{(s-a_1)^{r-2}} + \cdots \ldots \ldots \ldots + \frac{d_{r-l}}{(s-a_1)^{r-l}} + \cdots \ldots . + \frac{1}{(s-a_1)^1} +$$

$$\frac{d_1}{(s-a_1)^1} + \frac{c_2}{(s-a_2)} + \frac{c_3}{(s-a_3)} + \cdots \ldots \ldots \ldots + \frac{c_n}{(s-a_n)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Multiplying $(s-a_1)^r$ on both sides

$$(s-a_1)^r \frac{M(s)}{N(s)} = d_r + d_{r-1}(s-a_1) + \cdots \ldots \ldots \ldots . + d_1 (s-a_1)^{r-1} + \sum_{i=2}^{\infty} c_i \frac{(s-a_1)^r}{(s-a_i)} \ldots\ldots\ldots\ldots (B)$$

Now taking $\lim\limits_{s \to a_1}$ on both sides we get

$$d_r = \lim_{s \to a_1}\left[(s-a_1)^r \frac{M(s)}{N(s)}\right]$$

$$d_{r-1} = \lim_{s \to a_1} \frac{d}{ds}\left[(s-a_1)^r \frac{M(s)}{N(s)}\right] \qquad \text{diff.w.to 's'}$$

$$d_{r-2} = \frac{1}{2!}\lim_{s \to a_1} \frac{d^2}{ds^2}\left[(s-a_1)^r \frac{M(s)}{N(s)}\right] \qquad \text{again diff.w.to 's'}$$

$$d_{r-l} = \frac{1}{l!}\lim_{s \to a_1} \left(\frac{d}{ds}\right)^l \left[(s-a_1)^r \frac{M(s)}{N(s)}\right] \qquad \text{again diff.w.to 's' } l - time$$

Now by second translation theorem

$\mathcal{L}^{-1}\{e^{-as}\,F(s)\} = H\,(t\,-\,a)\,f\,(t\,-\,a)$

or $\mathcal{L}[H\,(t\,-\,a)\,f\,(t\,-\,a)] \;=\; e^{-as}\,F(s) \;=\; e^{-as}\,\mathcal{L}\{f\,(t)\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-a_1)^r}\right\} = e^{a_1 t}\mathcal{L}^{-1}\left\{\frac{1}{s^r}\right\} = e^{a_1 t}\,\frac{t^{r-1}}{(r-1)!}$

Now by 'A' we have $\frac{M(s)}{N(s)} = \sum_{l=1}^{r}\frac{d_l}{(s-a_1)^l} + \sum_{i=2}^{n}\;\frac{c_i}{(s-a_i)}$

Then taking laplace inverse on both sides

$\mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \mathcal{L}^{-1}\left\{\sum_{l=1}^{r}\frac{d_l}{(s-a_1)^l}\right\} + \mathcal{L}^{-1}\left\{\sum_{i=2}^{n}\;\frac{c_i}{(s-a_i)}\right\} = \sum_{l=1}^{r}\mathcal{L}^{-1}\left\{\frac{d_l}{(s-a_1)^l}\right\} + \sum_{i=2}^{n}\;\mathcal{L}^{-1}\left\{\frac{c_i}{(s-a_i)}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{l=1}^{r}\frac{1}{(l-1)!}\lim_{s\to a_1}\left(\frac{d}{ds}\right)^{l-1}\left[(s-a_1)^l\frac{M(s)}{N(s)}\right]e^{a_1 t} + \sum_{i=2}^{n}\;\lim_{s\to a_i}(s-a_i)\,\frac{M(s)}{N(s)}$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=2}^{n}\frac{M(a_i)}{N'(a_i)}e^{a_i t} + \sum_{j=1}^{r}\frac{1}{(j\text{-}1)!}\left\{\frac{d^{j-1}}{ds^{j-1}}(s-a_i)^j\frac{M(s)}{N(s)}\right\}e^{a_1 t}\Big|_{s=a_1}$

EXAMPLE:

Using Heaviside Expansion theorem evaluate $\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2 s^3}\right\}$

Solution: Given that $F(s) = \frac{s+2}{(s-1)^2 s^3}$ **has a pole at s = 1 of order'2' and at s = 0 of order '3'**

Then in terms of Partial fraction we will be as follows

$F(s) = \frac{d_1}{(s-1)^2} + \frac{d_2}{(s-1)} + \frac{c_1}{s^3} + \frac{c_2}{s^2} + \frac{c_3}{s}$

Now using Heaviside formula

$d_1 = \lim_{s\to 1}[(s-1)^2 F(s)] = \lim_{s\to 1}\left[(s-1)^2\,\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 1}\left[\frac{s+2}{s^3}\right] = 3$

$d_2 = \lim_{s\to 1}\frac{d}{ds}[(s-1)^2 F(s)] = \lim_{s\to 1}\frac{d}{ds}\left[(s-1)^2\,\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 1}\frac{d}{ds}\left[\frac{s+2}{s^3}\right] = -8$

$c_1 = \lim_{s\to 0}[s^3 F(s)] = \lim_{s\to 0}\left[s^3\,\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 0}\left[\frac{s+2}{(s-1)^2}\right] = 2$

$c_2 = \lim_{s\to 0}\frac{d}{ds}[s^3 F(s)] = \lim_{s\to 0}\frac{d}{ds}\left[s^3\,\frac{s+2}{(s-1)^2 s^3}\right] = \lim_{s\to 0}\frac{d}{ds}\left[\frac{s+2}{(s-1)^2}\right] = 5$

$c_3 = \frac{1}{2!}\lim_{s\to 0}\frac{d^2}{ds^2}[s^3 F(s)] = \frac{1}{2}\lim_{s\to 0}\frac{d^2}{ds^2}\left[s^3\,\frac{s+2}{(s-1)^2 s^3}\right] = \frac{1}{2}\lim_{s\to 0}\frac{d^2}{ds^2}\left[\frac{s+2}{(s-1)^2}\right] = 8$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = d_1\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + d_2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + c_1\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + c_3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = 3te^t - 8e^t + 2\frac{t^2}{2} + 5t + 8 = (3t-8)e^t + (t^2 + 5t + 8)$

EXAMPLE: Using Heaviside Expansion theorem evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+2\alpha s+b^2)}\right\}$

Solution: Given that $F(s) = \frac{1}{s^2(s^2+2as+b^2)} = \frac{1}{s^2(s-s_1)(s-s_2)}$ **has simple poles at** $s = s_1, s = s_2$ **and a pole of order '2' at s = 0**

Then in terms of Partial fraction we will be as follows

$$F(s) = \frac{1}{s^2(s^2+2as+b^2)} = \frac{1}{s^2(s-s_1)(s-s_2)} = \frac{d_1}{s^2} + \frac{d_2}{s} + \frac{c_1}{(s-s_1)} + \frac{c_2}{(s-s_2)}$$

Where we take $s_1 = -\alpha - i\beta$, $s_2 = -\alpha + i\beta$ **then** $s_1 s_2 = \alpha^2 + \beta^2 = b^2$

Now using Heaviside formula

$$d_1 = \lim_{s\to 0}[s^2F(s)] = \lim_{s\to 0}\left[s^2\frac{1}{s^2(s-s_1)(s-s_2)}\right] = \lim_{s\to 0}\left[\frac{1}{(s-s_1)(s-s_2)}\right] = \frac{1}{b^2}$$

$$d_2 = \lim_{s\to 0}\frac{d}{ds}[s^2F(s)] = \lim_{s\to 0}\frac{d}{ds}\left[s^2\frac{1}{s^2(s-s_1)(s-s_2)}\right] = \lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{(s-s_1)(s-s_2)}\right]$$

$$d_2 = \lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{s^2+2as+b^2}\right] = \lim_{s\to 0}\frac{-(2s+2\alpha)}{(s^2+2as+b^2)^2} = \frac{-2\alpha}{b^4}$$

$$c_1 = \lim_{s\to s_1}[(s-s_1)F(s)] = \lim_{s\to s_1}\left[(s-s_1)\frac{1}{s^2(s-s_1)(s-s_2)}\right] = \lim_{s\to s_1}\left[\frac{1}{s^2(s-s_2)}\right]$$

$$c_1 = \frac{1}{s_1^2(s_1-s_2)}$$

$$c_2 = \lim_{s\to s_2}[(s-s_2)F(s)] = \lim_{s\to s_2}\left[(s-s_2)\frac{1}{s^2(s-s_1)(s-s_2)}\right] = \lim_{s\to s_2}\left[\frac{1}{s^2(s-s_1)}\right]$$

$$c_2 = \frac{1}{s_2^2(s_2-s_1)}$$

Now as $s_1 = -\alpha - i\beta = \sqrt{\alpha^2+\beta^2}e^{-i\theta} \Rightarrow s_1^2 = (\alpha^2+\beta^2)e^{-2i\theta} = b^2e^{-2i\theta}$

$s_2 = -\alpha + i\beta = \sqrt{\alpha^2+\beta^2}e^{i\theta} \Rightarrow s_2^2 = (\alpha^2+\beta^2)e^{2i\theta} = b^2e^{2i\theta}$ **then**

$s_1 - s_2 = -2i\beta$

Then $c_1 = \frac{1}{s_1^2(s_1-s_2)} = -\frac{e^{2i\theta}}{2ib^2\beta}$ **and** $c_2 = \frac{1}{s_2^2(s_2-s_1)} = \frac{e^{-2i\theta}}{2ib^2\beta}$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = d_1\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + d_2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + c_1\mathcal{L}^{-1}\left\{\frac{1}{(s-s_1)}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{1}{(s-s_2)}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{t}{b^2} - \frac{2\alpha}{b^4} + c_1e^{s_1t} + c_2e^{s_2t} = \frac{t}{b^2} - \frac{2\alpha}{b^4} - \frac{e^{2i\theta}}{2ib^2\beta}.e^{s_1t} + \frac{e^{-2i\theta}}{2ib^2\beta}e^{s_2t}$$

EXAMPLE:

Find the general solution of the differential equation evaluate

$y''(t) + k^2 y(t) = f(t)$

Solution: Given that $y''(t) + k^2 y(t) = f(t)$

$\Rightarrow \mathcal{L}\{y''(t)\} + k^2 \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$

$\Rightarrow s^2Y(s) - sy(0) - y'(0) + k^2Y(s) = F(s) \Rightarrow s^2Y(s) + k^2Y(s) = F(s) + sy(0) + y'(0)$

$\Rightarrow Y(s) = \frac{c_1 + c_2 s + F(s)}{s^2 + k^2}$ **where we use** $y'(0) = c_1, y(0) = c_2$

Now $\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{c_1}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{c_2 s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{c_1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{c_1}{k} Sinkt + c_2 Coskt + \frac{1}{k} Sinkt * f(t)$

$\Rightarrow y(t) = \frac{c_1}{k} Sinkt + c_2 Coskt + \frac{1}{k}\int_0^t e^{-st}\, Sink(t - \xi) f(\xi)\, d\xi$

EXAMPLE:

Slove the IVP $y''(t) + ty'(t) - y(t) = 0$ **with** $y(0) = 0, y'(0) = 1$

Solution: Given that $y''(t) + ty'(t) - y(t) = 0$

$\Rightarrow \mathcal{L}\{y''(t)\} + \mathcal{L}\{ty'(t)\} - \mathcal{L}\{y(t)\} = 0$

$\Rightarrow s^2Y(s) - sy(0) - y'(0) + \left(-\frac{d}{ds}\right)\mathcal{L}\{y'(t)\} - Y(s) = 0$

$\Rightarrow s^2Y(s) - 1 - \left(\frac{d}{ds}\right)\{sY(s) - y(0)\} - Y(s) = 0$

where we use $y(0) = 0, y'(0) = 1$

$\Rightarrow s^2Y(s) - 1 - sY'(s) - Y(s) - Y(s) = 0$ **where we use** $y(0) = 0, y'(0) = 1$

$\Rightarrow Y'(s) + \frac{2-s^2}{s}Y(s) = -\frac{1}{s}$ **this will have in** $I.F = s^2 e^{-\frac{s^2}{2}}$

Thus $\Rightarrow Y(s) = \frac{1}{s^2} + ce^{\frac{s^2}{2}} \Rightarrow Y(s) = \frac{1}{s^2}$ **when** $s \to \infty$ $then\ c = 0$

Now $\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \Rightarrow y(t) = t$

EXAMPLE:

Slove the IVP $u'' - au = f(t)$ **with** $u(0) = u_0, u'(0) = u_1$

Solution: Given that $u'' - au = f(t)$

$\Rightarrow \mathcal{L}\{u''\} - a\mathcal{L}\{u\} = \mathcal{L}\{f(t)\}$

$\Rightarrow s^2U(s) - su(0) - u'(0) - aU(s) = F(s)$

$\Rightarrow s^2U(s) - su_0 - u_1 - aU(s) = F(s)$ **where we use** $u(0) = u_0, u'(0) = u_1$

$\Rightarrow (s^2 - a)U(s) = F(s) + su_0 + u_1$

$\Rightarrow U(s) = \frac{F(s)}{s^2-a} + u_0 . \frac{s}{s^2-a} + u_1 . \frac{1}{s^2-a}$

Now $\Rightarrow \mathcal{L}^{-1}\{U(s)\} = u(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2-a}\right\} + u_0\mathcal{L}^{-1}\left\{\frac{s}{s^2-a}\right\} + u_1\mathcal{L}^{-1}\left\{\frac{1}{s^2-a}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = u(t) = \frac{1}{\sqrt{a}}\mathcal{L}^{-1}\left\{F(s).\frac{\sqrt{a}}{s^2-(\sqrt{a})^2}\right\} + u_0\mathcal{L}^{-1}\left\{\frac{s}{s^2-(\sqrt{a})^2}\right\} + \frac{u_1}{\sqrt{a}}\mathcal{L}^{-1}\left\{\frac{\sqrt{a}}{s^2-(\sqrt{a})^2}\right\}$

$\Rightarrow u(t) = \frac{1}{\sqrt{a}}\mathcal{L}^{-1}\{f(t) * Sinh\sqrt{a}t\} + u_0Cosh\sqrt{a}t + \frac{u_1}{\sqrt{a}}Sinh\sqrt{a}t$

$\Rightarrow u(t) = \frac{1}{\sqrt{a}}\int_0^t e^{-st}\, Sinh\sqrt{a}(t - \xi)f(\xi)\, d\xi + u_0Cosh\sqrt{a}t + \frac{u_1}{\sqrt{a}}Sinh\sqrt{a}t$

MELLIN INTEGRAL TRANSFORMATION:

For a well behaved function 'f' Mellin Integral Transformation is defined as

$M\{f(t): s\} = f^*(s) = \int_0^\infty f(t)t^{s-1}dt$

INVERSE MELLIN INTEGRAL TRANSFORMATION:

For a well behaved function 'f' Inverse Mellin Integral Transformation is defined as

$M^{-1}\{f^*(s): t\} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} f^*(s)t^{-s}ds \;\; ; t > 0; r = R(s)$

THE LAPLACE INVERSION INTEGRAL
or THE FOURIER MELLIN INTEGRAL
or DERIVATION OF INVERSION INTEGRAL

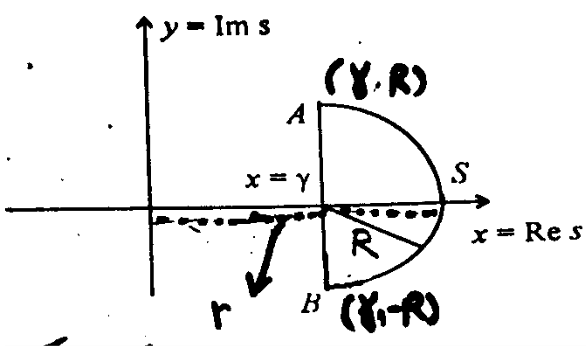
STATEMENT :

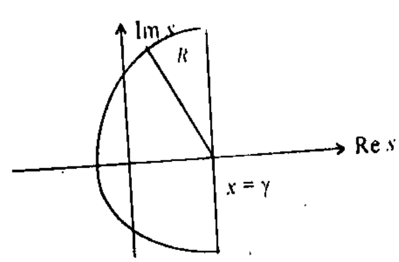
If $f(t)$ is inverse Laplace Transformation of $F(s)$ and all singularties of $F(s)$ in the complex plane 'S' lie to the left of the line $x=\gamma$ then

$$f(t)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR}e^{st}F(s)ds$$

Proof:

Draw the line $x=\gamma$ in the 'S' plane and mark the points $A=(\gamma,R)$ and $B=(\gamma,-R)$ on this line and draw a semicircle S of radius R to the right of the line $x=\gamma$. Let $C=\overline{AB}\cup S$ be the closed contour consisting of the line segment $\overline{AB}$ and S.





Let the function $F(z)=\int_0^\infty e^{-zt}f(t)dt$ is an analytic function on and within the contour C. if 's' is any point inside C then by Cauchy Integral Theorem

$$F(s)=\frac{1}{2\pi i}\oint\frac{F(z)}{z-s}dz\Rightarrow F(s)=\frac{1}{2\pi i}\oint\frac{1}{z-s}\int_0^\infty e^{-zt}f(t)dt\,dz$$

$$\Rightarrow F(s)=\frac{1}{2\pi i}\int_0^\infty f(t)\left[\oint\frac{1}{z-s}e^{-zt}dz\right]dt$$ **interchanging the order of integration.**

$$\Rightarrow F(s)=\frac{1}{2\pi i}\int_0^\infty f(t)\left[\int_{-s}\frac{e^{-zt}}{z-s}dz+\int_A^B\frac{e^{-zt}}{z-s}dz\right]dt=\frac{1}{2\pi i}\int_0^\infty f(t)\int_A^B\frac{e^{-zt}}{z-s}dz\,dt$$ **by Jordan's**

Also $\int_A^B \frac{e^{-zt}}{z-s} dz = \lim_{R\to\infty} \int_{\gamma-iR}^{\gamma+iR} \frac{e^{-zt}}{z-s} dz = -\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-zt}}{z-s} dz$

$\Rightarrow F(s) = \frac{-1}{2\pi i}\int_0^\infty f(t)\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-zt}}{z-s} dz\, dt = \frac{1}{2\pi i}\int_0^\infty f(t)\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-zt}}{s-z} dz\, dt$

$\Rightarrow F(s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[\int_0^\infty e^{-zt} f(t) dt\right]\frac{1}{s-z} dz$ again changing the order of integration.

$\Rightarrow F(s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(z)}{s-z} dz$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z)\mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\} dz = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}F(z)dz$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} e^{st}F(s)ds$

SPECIAL CASE:

Now suppose $F(s)$ has poles only to the left of the line $x = ReS = \gamma$ then we can enclose all those poles in a contour C on the left of $x = \gamma$ then

$\Rightarrow f(t) = \frac{1}{2\pi i}\oint_c e^{st}F(s)ds = \frac{1}{2\pi i}\sum_j\left(2\pi i R_j\right) = \sum_j R_j$

where $R_j = residue\ of e^{st}F(s)$ at the poles $s = s_j$

EXAMPLE:

Use Laplace Inversion Intgral (or Rasidue method) evaluate $\mathcal{L}^{-1}\left\{\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$

Solution:

Given $F(s) = \frac{s^3+2s^2+1}{s^2(s^2+1)} = \frac{s^3+2s^2+1}{s^2(s+i)(s-i)}$ has simple poles at $s = \pm i$ and a pole of order '2' at $s = 0$

Now using $R(f,\alpha) = \lim_{s\to\alpha}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$R(f,0) = R_0 = \lim_{s\to 0}\frac{d}{ds}[s^2e^{st}F(s)] = \lim_{s\to 0}\frac{d}{ds}\left[s^2e^{st}.\frac{s^3+2s^2+1}{s^2(s^2+1)}\right] = t$

$R(f,i) = R_1 = \lim_{s\to i}[(s-i)e^{st}F(s)] = \lim_{s\to i}\left[(s-i)e^{st}\frac{s^3+2s^2+1}{s^2(s+i)(s-i)}\right]$

$R(f,i) = R_1 = \frac{1-i}{2i}e^{it}$

$R(f,-i) = R_2 = \lim_{s\to -i}[(s+i)e^{st}F(s)] = \lim_{s\to -i}\left[(s+i)e^{st}\frac{s^3+2s^2+1}{s^2(s+i)(s-i)}\right]$

$R(f,-i) = R_2 = \frac{1+i}{2i}e^{-it}$

Now $\Rightarrow f(t) = \sum_j R_j = R_0 + R_1 + R_2$

$\Rightarrow f(t) = \sum_j R_j = t + \frac{1-i}{2i}e^{it} + \frac{1+i}{2i}e^{-it}$

$\Rightarrow f(t) = \sum_j R_j = t + \frac{1-i}{2i}(Cost + iSint) + \frac{1+i}{2i}(Cost - iSint)$

$\Rightarrow f(t) = \sum_j R_j = t + Cost + Sint$ **after solving.**

EXAMPLE:

Use Laplace Inversion Intgral (or Rasidue method) evaluate $\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s^2+1)}\right\}$

Solution: Given $F(s) = \frac{2s+1}{s(s^2+1)} = \frac{2s+1}{s(s+i)(s-i)}$ **has simple poles at** $s = 0, \pm i$

Now using $R(f,\alpha) = \lim_{s\to\alpha}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$R(f,0) = R_0 = \lim_{s\to 0}[se^{st}F(s)] = \lim_{s\to 0}\left[se^{st}.\frac{2s+1}{s(s^2+1)}\right] = 1$

$R(f,i) = R_1 = \lim_{s\to i}[(s-i)e^{st}F(s)] = \lim_{s\to i}\left[(s-i)e^{st}\frac{2s+1}{s(s+i)(s-i)}\right]$

$R(f,i) = R_1 = \frac{1+2i}{-2i}e^{it}$

$R(f,-i) = R_2 = \lim_{s\to -i}[(s+i)e^{st}F(s)] = \lim_{s\to -i}\left[(s+i)e^{st}\frac{2s+1}{s(s+i)(s-i)}\right]$

$R(f,-i) = R_2 = \frac{1-2i}{-2i}e^{-it}$

Now $\Rightarrow f(t) = \sum_j R_j = R_0 + R_1 + R_2$

$\Rightarrow f(t) = \sum_j R_j = 1 - \frac{1+2i}{2i}e^{it} - \frac{1-2i}{2i}e^{-it}$

$\Rightarrow f(t) = \sum_j R_j = t - \frac{1+2i}{2i}(Cost + iSint) - \frac{1-2i}{2i}(Cost - iSint)$

$\Rightarrow f(t) = \sum_j R_j = 1 + 2Sint - Cost$ **after solving.**

EXAMPLE:

Use Laplace Inversion Intgral (or Rasidue method) evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$

Solution: Given $F(s)=\frac{1}{s^2(s+1)}$ **has simple pole at** $s=-1$ **and a pole of order '2' at** $s=0$

Now using $R(f,\alpha)=\lim_{s\to\alpha}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$$R(f,0)=R_0=\lim_{s\to 0}\frac{d}{ds}[s^2e^{st}F(s)]=\lim_{s\to 0}\frac{d}{ds}\left[s^2e^{st}.\frac{1}{s^2(s+1)}\right]=t-1$$

$$R(f,-1)=R_1=\lim_{s\to -1}[(s-i)e^{st}F(s)]=\lim_{s\to i}\left[(s+1).e^{st}.\frac{1}{s^2(s+1)}\right]=e^{-t}$$

Now $\Rightarrow f(t)=\sum_j R_j=R_0+R_1$

$\Rightarrow f(t)=\sum_j R_j=t-1+e^{-t}$

In order to find a solution of linear partial differential equations, the following formulas and results are useful.

If $\mathcal{L}[u(x,t)]=\ U(x,s)$ **then**

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\}=s\,U(x,s)-u(x,0)$$

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}=s^2\,U(x,s)-su(x,0)-u_t(x,0)$$

: : :

: : :

$$\mathcal{L}\left\{\frac{\partial^n u}{\partial t^n}\right\}=s^n\,U(x,s)-s^{n-1}u(x,0)-\cdots\ldots\ldots\ldots.-su_{t_{n-2}}(x,0)-u_{t_{n-1}}(x,0)$$

Similarly, it is easy to show that

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\}=\frac{\partial}{\partial x}\,U(x,s)\ ,\ \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\}=\frac{\partial^2}{\partial x^2}\,U(x,s)\ ,\ \ldots\ldots\ldots\ldots\ldots..,\ \mathcal{L}\left\{\frac{\partial^n u}{\partial x^n}\right\}=\frac{\partial^n}{\partial x^n}U(x,s)$$

EXAMPLE:

Use Laplace Transformation method to solve BVP

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; 0 < x < a; \quad 0 \le t < \infty$$

$$u\,(0,t) = 1,\ u\,(1,t) = 1 \quad ; t > 0 \ , u\,(x,0) = 1 + Sin\pi x$$

Solution:

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} \Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = s\,U(x,s) - u(x,0)$

$$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = s\,U(x,s) - (1 + Sin\pi x)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - s\,U(x,s) = -1 - Sin\pi x \qquad \text{...........(i)}$$

Which is non – homogeneous 2nd order DE with solution

$$U(x,s) = U_c(x,s) + U_p(x,s) \qquad \text{...........(ii)}$$

For Chractristic (auxiliary) solution

$$(i) \Rightarrow (D^2 - s)U(x,s) = -1 - Sin\pi x \Rightarrow D^2 - s = 0 \Rightarrow D = \pm\sqrt{s}$$

Then $U_c(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$

For Particular solution

Consider $U_p(x,s) = \frac{-1-Sin\pi x}{D^2-s} = \frac{-e^{0x}}{D^2-s} - img\frac{e^{i\pi x}}{D^2-s} = \frac{-1}{0^2-s} - \frac{Sin\pi x}{(i\pi)^2-s} = \frac{1}{s} - \frac{Sin\pi x}{-\pi^2-s}$

Then $U_p(x,s) = \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s}$

$$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s}$$

$$\Rightarrow U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s} \qquad \text{...........(iii)}$$

Now using BC's

$$u\,(0,t) = 1 \Rightarrow \mathcal{L}\{u\,(0,t)\} = \mathcal{L}\{1 = t^0\} \Rightarrow U(0,s) = \frac{1}{s}$$

$$u\,(1,t) = 1 \Rightarrow \mathcal{L}\{u\,(1,t)\} = \mathcal{L}\{1 = t^0\} \Rightarrow U(1,s) = \frac{1}{s}$$

$$(iii) \Rightarrow U(0,s) = \frac{1}{s} = c_1 e^0 + c_2 e^0 + \frac{1}{s} + \frac{Sin(0)}{\pi^2+s} \Rightarrow c_1 + c_2 + \frac{1}{s} = \frac{1}{s} \Rightarrow c_1 = -c_2$$

$(iii) \Rightarrow U(1,s) = \frac{1}{s} = c_1 e^{\sqrt{s}(1)} + c_2 e^{-\sqrt{s}(1)} + \frac{1}{s} + \frac{Sin\pi}{\pi^2+s} \Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s} - \frac{1}{s} = 0$

$\Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \Rightarrow -c_2 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \qquad \therefore c_1 = -c_2$

$\Rightarrow c_2\left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right] = 0 \Rightarrow c_2 = 0\ ,\ \left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right] \neq 0$

$\Rightarrow c_2 = 0 \Rightarrow c_1 = 0 \qquad \therefore c_1 = -c_2$

$(iii) \Rightarrow U(x,s) = \frac{1}{s} + \frac{Sin\pi x}{\pi^2+s} \qquad \therefore c_1 = c_2 = 0$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{Sin\pi x}{\pi^2+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + Sin\pi x\mathcal{L}^{-1}\left\{\frac{1}{s-(-\pi^2)}\right\}$

$\Rightarrow u(x,t) = 1 + Sin\pi x e^{-\pi^2 t}$ **required solution.**

EXAMPLE:

Use Laplace Transformation method to solve BVP

$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t); t > 0, x > 0$

$u(x,0) = u_t(x,0) = 0,\ u(0,t) = f(t)\ , \lim_{x\to\infty} u(x,t) = 0$

Solution:

Given $u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\}$

$\Rightarrow s^2 U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow s^2 U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) \Rightarrow s^2 U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - \frac{s^2}{\alpha^2} U(x,s) = 0$

This is Homogeneous DE of 2nd order therefore

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$ **…………(i)**

Now using BC's

$u(0,t) = f(t) \Rightarrow \mathcal{L}\{u(0,t)\} = \mathcal{L}\{f(t)\} \Rightarrow U(0,s) = F(s)$

$\lim_{x\to\infty} u(x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} U(x,s) = 0$

$(i) \Rightarrow U(0,s) = F(s) = c_1 e^{\frac{s}{a}(0)} + c_2 e^{-\frac{s}{a}(0)} \Rightarrow c_1 + c_2 = F(s)$

$(i) \Rightarrow \lim_{x\to\infty} U(x,s) = 0 = \lim_{x\to\infty}\left[c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}\right] = c_1 e^{\infty} + c_2 e^{-\infty}$

$\Rightarrow c_1 = 0 \quad then \quad c_2 = F(s) \qquad\qquad \therefore c_1 + c_2 = F(s)$

Thus $(i) \Rightarrow U(x,s) = F(s) e^{-\frac{s}{a}x}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{F(s) e^{-\frac{s}{a}x}\right\}$

$\Rightarrow u(x,t) = H\left(t - \frac{x}{a}\right) f\left(t - \frac{x}{a}\right)$ **where** $H\left(t - \frac{x}{a}\right) f\left(t - \frac{x}{a}\right) = \begin{cases} 0 & t < \frac{x}{a} \\ f(t) & t \geq \frac{x}{a} \end{cases}$

EXAMPLE:

Use Laplace Transformation method to solve BVP

$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) - g$

$u(x,0) = u_t(x,0) = 0,\ u(0,t) = 0\ , \lim_{x\to\infty} u_x(x,t) = 0$

Solution: Given $u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) - g \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\} - g\mathcal{L}\{1\}$

$\Rightarrow s^2 U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow s^2 U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow s^2 U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - \frac{s^2}{\alpha^2} U(x,s) = \frac{g}{\alpha^2 s}$ **…………(i)**

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ **…………(ii)**

For Chractristic (auxiliary) solution

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm \frac{s}{a}$

Then $U_c(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$

For Particular solution

Consider $U_p(x,s) = \frac{\frac{g}{\alpha^2 s}}{D^2 - \frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}e^{0x}}{0^2 - \frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}}{-\frac{s^2}{\alpha^2}} = -\frac{g}{s^3}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$ ....**(iii)**

Now using BC's

$u(0,t) = 0 \Rightarrow \mathcal{L}\{u(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$\lim_{x\to\infty} u_x(x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u_x(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} \frac{\partial}{\partial x} U(x,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1 e^0 + c_2 e^{-0} - \frac{g}{s^3} \Rightarrow c_1 + c_2 = \frac{g}{s^3}$

$(iii) \Rightarrow \lim_{x\to\infty} \frac{\partial}{\partial x} U(x,s) = 0 = \lim_{x\to\infty}\left[c_1 \frac{s}{a} e^{\frac{s}{a}x} - \frac{s}{a} c_2 e^{-\frac{s}{a}x}\right] = c_1 \frac{s}{a} e^{\infty} + c_2 \frac{s}{a} e^{-\infty}$

$\Rightarrow c_1 \frac{s}{a} e^{\infty} = 0 \Rightarrow c_1 = 0$ *since* $\frac{s}{a} e^{\infty} \neq 0$, *then* $c_2 = \frac{g}{s^3} \quad \because c_1 + c_2 = \frac{g}{s^3}$

Thus $(iii) \Rightarrow U(x,s) = \frac{g}{s^3} e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \frac{g}{2!}\mathcal{L}^{-1}\left\{e^{-\frac{x}{a}s} . \frac{2!}{s^{2+1}}\right\} - \frac{g}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}$

$\Rightarrow u(x,t) = \frac{g}{2} H\left(t - \frac{x}{a}\right)\left(t - \frac{x}{a}\right)^2 - \frac{g}{2}(t^2)$

$\Rightarrow u(x,t) = \frac{g}{2}\left[H\left(t - \frac{x}{a}\right)\left(t - \frac{x}{a}\right)^2 - (t^2)\right]$

where $H\left(t - \frac{x}{a}\right)\left(t - \frac{x}{a}\right)^2 = \begin{cases} 0 & t < \frac{x}{a} \\ t^2 & t \geq \frac{x}{a} \end{cases}$

EXAMPLE:

Use Laplace Transformation method to solve BVP

$u_{xx}(x,t) = u_{tt}(x,t); t > 0;\ 0 < x < 1$

$u(0,t) = 0 = u(1,t),\ u(x,0) = Sin\pi x\ \ , u_t(x,0) = -Sin\pi x$

Solution:

Given $u_{xx}(x,t) = u_{tt}(x,t) \Rightarrow \mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = s^2\, U(x,s) - su(x,0) - u_t(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = s^2\, U(x,s) - sSin\pi x + Sin\pi x$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - s^2\, U(x,s) = -sSin\pi x + Sin\pi x$ **…………(i)**

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ **…………(ii)**

For Chractristic (auxiliary) solution

$\Rightarrow (D^2 - s^2)U(x,s) = 0 \Rightarrow D^2 - s^2 = 0 \Rightarrow D = \pm s$

Then $U_c(x,s) = c_1 e^{sx} + c_2 e^{-sx}$

For Particular solution

Consider

$U_p(x,s) = \frac{(1-s)Sin\pi x}{D^2-s^2} = (1-s)img\frac{e^{i\pi x}}{D^2-s^2} = (1-s)\frac{Sin\pi x}{(i\pi)^2-s} = (1-s)\frac{Sin\pi x}{-\pi^2-s}$

$U_p(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$ **…………(iii)**

Now using BC's $u(0,t) = 0 \Rightarrow \mathcal{L}\{u(0,t)\} = 0 \Rightarrow U(0,s) = 0$

$u(1,t) = 0 \Rightarrow \mathcal{L}\{u(1,t)\} = 0 \Rightarrow U(1,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1 e^0 + c_2 e^{-0} + \frac{(s-1)Sin\pi(0)}{\pi^2+s} \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$(iii) \Rightarrow U(1,s) = 0 = c_1e^{s} + c_2e^{-s} + \frac{(s-1)Sin\pi}{\pi^2+s} \Rightarrow c_1e^{s} + c_2e^{-s} = 0 \Rightarrow c_1e^{s} - c_1e^{-s} = 0$

$\Rightarrow c_1(e^{s} - e^{-s}) = 0 \Rightarrow c_1 = 0 \;\; as\; (e^{s} - e^{-s}) \neq 0 \Rightarrow c_2 = 0$

Thus $(iii) \Rightarrow U(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = Sin\pi x\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} - \frac{Sin\pi x}{\pi}\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$

$\Rightarrow u\,(x,t) = Sin\pi x Cos\pi t - \frac{Sin\pi x}{\pi}Sin\pi t = Sin\pi x\left[Cos\pi t - \frac{Sin\pi x}{\pi}\right]$

EXAMPLE:

A uniform bar of length 'l' is fixed at one end. Let the force

$f(\,t) = \begin{cases} 0 & t < 0 \\ f_0 & t > 0 \end{cases}$ **be suddenly applied at the end $= \; l$, if the bar is initially at rest, find the longitudinal displacement for $t > 0$ using Laplace Transformation the motion of bar is govern by the differential system**

$u_{tt} \; = \alpha^2 u_{xx}; t > 0, 0 < x < 1$ **and α is constant.**

$u(x,0) = u(0,t) = u_t(x,0) = 0, \quad u_x(l,t) = \frac{f_0}{E}$ **where E is constant.**

Solution:

Given $u_{tt}\,(x,t) = \alpha^2 u_{xx}\,(x,t) \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2\mathcal{L}\{u_{xx}\}$

$\Rightarrow s^2\,U(x,s) - su(x,0) - u_t(x,0) = \alpha^2\frac{\partial^2}{\partial x^2}U(x,s)$

$\Rightarrow s^2\,U(x,s) - (0) - (0) = \alpha^2\frac{\partial^2}{\partial x^2}U(x,s) \Rightarrow s^2\,U(x,s) = \alpha^2\frac{\partial^2}{\partial x^2}U(x,s)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - \frac{s^2}{\alpha^2}\,U(x,s) = 0$

This is Homogeneous DE of 2nd order therefore

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right)U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U(x,s) = c_1e^{\frac{s}{a}x} + c_2e^{-\frac{s}{a}x}$ **...........(i)**

Now using BC's

$$u\,(0,t) = 0 \Rightarrow \mathcal{L}\{u\,(0,t)\} = 0 \Rightarrow U(0,s) = 0$$

$$u_x(l,t) = \frac{f_0}{E} \Rightarrow \mathcal{L}\{u_x(l,t)\} = \mathcal{L}\left\{\frac{f_0}{E}\right\} \Rightarrow \frac{\partial}{\partial x}U\,(l,s) = \frac{F_0}{E}$$

$$(i) \Rightarrow U(0,s) = F(\,s) = c_1e^{0} + c_2e^{-0} \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

Then $U(x,s) = c_1e^{\frac{s}{a}x} - c_1e^{-\frac{s}{a}x}$ **………..(ii)**

$$\Rightarrow \frac{\partial}{\partial x}U(x,s) = c_1\frac{s}{a}e^{\frac{s}{a}x} + c_1\frac{s}{a}e^{-\frac{s}{a}x}$$

Then using $\frac{\partial}{\partial x}U\,(l,s) = \frac{F_0}{E}$ **we get**

$$\Rightarrow \frac{\partial}{\partial x}U(x,s) = \frac{F_0}{E} = c_1\frac{s}{a}e^{\frac{s}{a}x} + c_1\frac{s}{a}e^{-\frac{s}{a}x} \Rightarrow c_1 = \frac{F_0}{E\left(\frac{s}{a}e^{\frac{s}{a}x} + \frac{s}{a}e^{-\frac{s}{a}x}\right)}$$

Hence $(ii) \Rightarrow U(x,s) = \frac{F_0}{E\left(\frac{s}{a}e^{\frac{s}{a}x} + \frac{s}{a}e^{-\frac{s}{a}x}\right)}.\left(e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}\right) = \frac{F_0}{E}\frac{\left(e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}\right)}{\frac{s}{a}\left(e^{\frac{s}{a}x} + e^{-\frac{s}{a}x}\right)}$

Taking Laplace inverse on both sides

$$u(x,t) = \mathcal{L}^{-1}\left\{\frac{F_0}{E}\frac{\left(e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}\right)}{\frac{s}{a}\left(e^{\frac{s}{a}x} + e^{-\frac{s}{a}x}\right)}\right\}$$

which is required longitudinal displacement for $t > 0$

THEOREM: Let $f\,(t)$ be a piecewise continuous function for $t \geq 0$ and of exponential order. If $f\,(t)$ is periodic with period T then show that

$$\mathcal{L}\{\,f\,(t\,)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\,f\,(t\,)\,dt$$

PROOF: By definition, we have

$$\mathcal{L}\{\,f\,(t\,)\} = \int_0^\infty e^{-st}\;f\,(t\,)\,dt$$

$$\mathcal{L}\{\,f\,(t\,)\} = \int_0^T e^{-st}\;f\,(t\,)\,dt + \int_T^\infty e^{-st}\;f\,(t\,)\,dt$$

In the 2nd integral on the right put $t = u + T \Rightarrow dt = du$

$$\mathcal{L}\{\,f\,(t\,)\} = \int_0^T e^{-st}\;f\,(t\,)\,dt + \int_0^\infty e^{-s(u+T\,)}\;f\,(u+T\,)\,du$$

$$\mathcal{L}\{\,f\,(t\,)\} = \int_0^T e^{-st}\;f\,(t\,)\,dt + e^{-sT}\int_0^\infty e^{-su}\;f\,(u+T\,)\,du$$

Since given functionis periodic with period T therefore $f\,(u+T\,) = f\,(u)$

$$\mathcal{L}\{\,f\,(t\,)\} = \int_0^T e^{-st}\;f\,(t\,)\,dt + e^{-sT}\int_0^\infty e^{-su}\;f\,(u\,)\,du$$

$$\mathcal{L}\{\,f\,(t\,)\} = \int_0^T e^{-st}\;f\,(t\,)\,dt + e^{-sT}\,\mathcal{L}\{\,f\,(u\,)\}$$

$$\mathcal{L}\{\,f\,(t\,)\} = \int_0^T e^{-st}\;f\,(t\,)\,dt + e^{-sT}\,\mathcal{L}\{\,f\,(t\,)\}$$

$$(1-e^{-sT})\mathcal{L}\{\,f\,(t\,)\} = \int_0^T e^{-st}\;f\,(t\,)\,dt$$

$\mathcal{L}\{\,f\,(t\,)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\,f\,(t\,)\,dt$ **As required the result.**

THEOREM: If $\mathcal{L}\{\,f\,(t\,)\} = F(s)$ then $\mathcal{L}\left\{\frac{f\,(t\,)}{t}\right\} = \int_s^\infty F(s)\,ds$

PROOF: By definition, we have

$$\mathcal{L}\{\,f\,(t\,)\} = F(s) = \int_0^\infty e^{-st}\;f\,(t)\,dt$$

$\int_s^\infty F(s)\,ds = \int_s^\infty\left[\int_0^\infty e^{-st}\;f\,(t)\,dt\right]ds$ integrating.

$\int_s^\infty F(s)\,ds = \int_0^\infty f\,(t)\left[\int_s^\infty e^{-st}\,ds\right]dt$ changing the order of integration.

$$\int_s^\infty F(s)\,ds = \int_0^\infty f\,(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty \frac{f\,(t)}{t}e^{-st}dt = \mathcal{L}\left\{\frac{f\,(t\,)}{t}\right\}$$

Hence $\mathcal{L}\left\{\frac{f\,(t\,)}{t}\right\} = \int_s^\infty F(s)\,ds$

HANKEL TRANSFORMS

HANKEL TRANSFORMS

$\tilde{f}_n(\kappa)$ **is called the Hankel transform of** $f(r)$ **and is defined formally by**

$$F_n(\kappa) = \mathcal{H}_n\{f(r)\} = \tilde{f}_n(\kappa) = \int_0^{\infty} rJ_n(\kappa r)f(r)dr$$

The inverse Hankel transform is defined by

$$\mathcal{H}_n^{-1}\{\tilde{f}_n(\kappa)\} = \mathcal{H}_n^{-1}\{F_n(\kappa)\} = f(r) = \int_0^{\infty} \kappa J_n(\kappa r)\tilde{f}_n(\kappa)dk$$

Alternatively, the famous Hankel integral formula

$f(r) = \int_0^{\infty} \kappa J_n(\kappa r)dk \int_0^{\infty} \rho J_n(\kappa\rho)f(\rho)d\rho$

can be used to define the Hankel transform and its inverse

In particular, the Hankel transforms of zero order ($n = 0$) and of order one ($n = 1$) are often useful for the solution of problems involving Laplace's equation in an axisymmetric cylindrical geometry.

REMARK: For Bessel Functions

i. $J_0(\kappa r) = \frac{1}{\pi}\int_0^{\pi} Cos(\kappa r Sin\theta)d\theta$

ii. $J'_0(\kappa r) = -J_1(\kappa r)$ **also** $J_{n+1} = J_{n-1} - 2J'_n$ **for** $J_0(0) = 1, J_n(0) = 0$ **; n>0**

Example:

Obtain the zero-order Hankel transforms of

(a) $r^{-1}\exp(-ar)$ **, (b)** $\frac{\delta(r)}{r}$ **(c)** $H(a - r)$

where $H(r)$ **is the Heaviside unit step function.**

Solution:

(a)

$\mathcal{H}_0\left\{\frac{1}{r}e^{-ar}\right\} = \tilde{f}_0(\kappa) = \int_0^{\infty} rJ_n(\kappa r)f(r)dr = \int_0^{\infty} r.\frac{1}{r}e^{-ar}J_0(\kappa r)\,dr = \frac{1}{\kappa^2 + a^2}$

(b)

$\mathcal{H}_0\left\{\frac{\delta(r)}{r}\right\} = \tilde{f}_0(\kappa) = \int_0^{\infty} rJ_n(\kappa r)f(r)dr = \int_0^{\infty} r.\frac{\delta(r)}{r}J_0(\kappa r)\,dr = 1$

(c)

$\mathcal{H}_0\{H(a - r)\} = \tilde{f}_0(\kappa) = \int_0^{\infty} rJ_n(\kappa r)f(r)dr = \int_0^{\infty} H(a - r)J_0(\kappa r)\,dr$

$\mathcal{H}_0\{H(a - r)\} = \tilde{f}_0(\kappa) = \int_0^{a} rJ_0(\kappa r)\,dr = \frac{1}{\kappa^2}\int_0^{a\kappa} \rho J_0(\rho)\,d\rho = \frac{1}{\kappa^2}|\rho J_1(\rho)|_0^{a\kappa} = \frac{a}{\kappa}J_1(a\kappa)$

Example:
Find the first-order Hankel transform of the following functions:

(a) $f(r) = e^{-ar}$

(b) $f(r) = \frac{1}{r}e^{-ar}$

Solution:

(a)

$$\mathcal{H}_1\{e^{-ar}\} = \tilde{f}(\kappa) = \int_0^\infty rJ_1(\kappa r) f(r)dr = \int_0^\infty re^{-ar}J_1(\kappa r)\, dr = \frac{\kappa}{(\kappa^2+a^2)^{3/2}}$$

(b)

$$\mathcal{H}_1\left\{\frac{1}{r}e^{-ar}\right\} = \tilde{f}(\kappa) = \int_0^\infty rJ_1(\kappa r) f(r)dr = \int_0^\infty r\frac{1}{r}e^{-ar}J_1(\kappa r)\, dr$$

$$\mathcal{H}_1\left\{\frac{1}{r}e^{-ar}\right\} = \int_0^\infty e^{-ar}J_1(\kappa r)\, dr = \frac{1}{\kappa}\left[1 - a(\kappa^2 + a^2)^{-1/2}\right]$$

Example:
Find the nth-order Hankel transforms of

(a) $f(r) = r^n H(a - r)$

(b) $f(r) = r^n e^{-ar^2}$

Solution:

(a)

$$\mathcal{H}_n\{r^n H(a - r)\} = \tilde{f}(\kappa) = \int_0^\infty rJ_n(\kappa r) f(r)dr = \int_0^a r^{n+1}J_n(\kappa r)\, dr$$

$$\mathcal{H}_n\{r^n H(a - r)\} = \tilde{f}(\kappa) = \frac{a^{n+1}}{\kappa}J_{n+1}(a\kappa)$$

(b)

$$\mathcal{H}_n\left\{r^n e^{-ar^2}\right\} = \tilde{f}(\kappa) = \int_0^\infty rJ_n(\kappa r) f(r)dr = \int_0^\infty r^{n+1}J_n(\kappa r)e^{-ar^2}\, dr$$

$$\mathcal{H}_n\left\{r^n e^{-ar^2}\right\} = \tilde{f}(\kappa) = \frac{\kappa^n}{(2a)^{n+1}}exp\left(-\frac{\kappa^2}{4a}\right)$$

PROPERTIES OF HANKEL TRANSFORMS AND APPLICATIONS
(i) THE HANKEL TRANSFORM OPERATOR, "$\mathcal{H}_n$" IS A LINEAR INTEGRAL OPERATOR for any constants a and b.
i.e. $\mathcal{H}_n\{af(r) + bg(r)\} = a\mathcal{H}_n\{f(r)\} + b\mathcal{H}_n\{g(r)\}$
Proof: by using definition
$\mathcal{H}_n\{af(r) + bg(r)\} = \int_0^\infty rJ_n(\kappa r)\{af(r) + bg(r)\}dr$
$\mathcal{H}_n\{af(r) + bg(r)\} = a\int_0^\infty rJ_n(\kappa r)\,f(r)dr + b\int_0^\infty rJ_n(\kappa r)\,g(r)dr$
$\mathcal{H}_n\{af(r) + bg(r)\} = a\mathcal{H}_n\{f(r)\} + b\mathcal{H}_n\{g(r)\}$

(ii) THE HANKEL TRANSFORM SATISFIES THE PARSEVAL RELATION

$$\int_0^\infty rf(r)\,g(r)dr = \int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk$$

where $\tilde{f}(\kappa)$ *and* $\tilde{g}\,(\kappa)$ **are Hankel transforms of** $f(r)$ **and** $g(r)$ **respectively.**
Proof:
$\int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk = \int_0^\infty k\tilde{f}(\kappa)dk\int_0^\infty rJ_n(\kappa r)\,g(r)dr$
$\int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk = \int_0^\infty rg(r)dr\int_0^\infty kJ_n(\kappa r)\tilde{f}(\kappa)dk$
$\int_0^\infty k\tilde{f}(\kappa)\tilde{g}\,(\kappa)\,dk = \int_0^\infty rf(r)\,g(r)dr$

(iii) (SCALING PROPERTY). If $\mathcal{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$ **then** $\mathcal{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right)$; **a** > 0
Proof. We have, by definition,
$\mathcal{H}_n\{f(ar)\} = \int_0^\infty rJ_n(\kappa r)f(ar)dr = \frac{1}{a^2}\int_0^\infty sJ_n\left(\frac{\kappa}{a}s\right)f(s)ds \qquad \therefore ar = s$
$\mathcal{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right)$; **a** > 0

These results are used very widely in solving partial differential equations in the axisymmetric cylindrical configurations.

GREEN'S FUNCTION AND ASSOCIATED BVP's

THE KRONECKER DELTA FUNCTION:

It is denoted by δ_{ij} and can be defined as follows;

$$\delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

DIRAC DELTA FUNCTION

The dirac delta function is defined as follows;

$$\delta(x) = \lim_{\epsilon\to 0} \delta_{\epsilon}(x) = \begin{cases} \infty & if\ x = 0 \\ 0 & if\ x \neq 0 \end{cases} \quad \text{Or} \quad \delta(x-t) = \begin{cases} \infty & if\ x = t \\ 0 & if\ x \neq t \end{cases}$$

PROPERTIES:

i. $\int_{-\infty}^{\infty} \delta(x)dx = 1$

ii. **For any continuous function $f(x)$;** $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

iii. $\delta(x) = \delta(-x)$

iv. $\delta(ax) = \frac{1}{a}\delta(x) \quad ; a > 0$

v. **SHIFTING PROPERTY: For any continuous function $f(x)$;**

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

vi. **If $\delta(x)$ is continuous differentiable. Then** $\int_{-\infty}^{\infty} f(x)\delta'(x)dx = -f'(0)$

REMARK:

i. **Dirac delta function can be regarded as the generalization of Kronecker delta function. It strictly speaking a "generalized function" or "distribution function" or " a unit impulse function"**

ii. **In kronecker delta function δ_{ij} the indecis i,j, are integral variables, whereas in passing to direc delta function they become real continuous variables.**

1st SHIFTING PROPERTY OF DIRAC DELTA FUNCTION:

For any continuous function $f(x)$; $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

Where $f(x)$ is analytic (regualar or continuous function) at x = 0

Proof: Since $\delta(x)$ has singularity at x = 0, the limits $-\infty$ and ∞ of the integration may be changed to (or replace by) $0-\epsilon$ and $0+\epsilon$ where ϵ is a small positive number.

Since $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} f(x)\delta(x)dx$

Moreover, since $f(x)$ is continuous at x = 0. We obtain in $\lim\limits_{\epsilon\to 0}$ follow;

$f(0-\epsilon) = f(0+\epsilon) = f(0)$

Therefore $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)\lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} f(x)\delta(x)dx$

since $\delta(x)$ has singularity at x = 0. Therefore

$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0).1 = f(0)$

2nd SHIFTING PROPERTY OF DIRAC DELTA FUNCTION:

For any continuous function $f(x)$; $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

Where $f(x)$ is analytic (regualar or continuous function) at x = a

Proof: Consider $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx$

Set $x-a=t$ and write $f(t+a) = g(t) \Rightarrow f(a) = g(0)$

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(t+a)\delta(t)dt = \int_{-\infty}^{\infty} g(t)\delta(t)dt$

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = g(0)$ **by 1st shifting property**

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$ **by hypothesis**

GREEN's FUNCTION

Green's Function is the impulse response of non – homogeneous differential equation with specified initial and boundry conditions.

IMPORTANCE: it provides an important tool in the study of BVP's. it also have an intrinsic value for mathematicians. Such function is the response corresponding to the source unit.

PROPERTIES OF GREEN's FUNCTION:

i. **Green's Function is denoted by** $G(x,x')$

ii. $G(x,x')$ **is symmetric i.e.** $G(x,x') = G(x',x)$

iii. $G(x,x')$ **as a function of 'x' satisfies the D Equation** $\frac{d^2}{dx^2}G(x,x') = 0$ **in each of the interval** $0 \le x \le x'$ **and** $x' < x \le l$

iv. $G(0,x') = 0$ **and** $G(l,x') = 0$ **which are the same BC's as those satisfied by** u

v. $G(x,x')$ **is continuous function of 'x' in the interval** $[0\,,l\,]$
(in constructing the Green's function, we will make use of its continuity at $x = x'$ **and this can be seen from the following**
$\lim_{x\to x'_+} G(x,x') = \lim_{x\to x'_-} G(x,x')$
$\lim_{x\to x'_+} \frac{x'}{l}(x-l) = \lim_{x\to x'_-} \frac{x}{l}(x'-l)$
$\frac{x'}{l}(x'-l) = \frac{x'}{l}(x'-l)$

vi. **If we calculate** $G'(x,x') = \frac{d}{dx}G(x,x')$ **we find that**
$G'(x,x') = \begin{cases} \frac{x'-l}{l} & ;0 \le x \le x' \\ \frac{x'}{l} & ;x' < x \le l \end{cases}$ **and** $G'(x,x')$ **will be discontinuous at** $x = x'$

AN IMPORTANT RESULT: $\int_0^x \int_0^{x_2} \varphi(x_1)\, dx_1 dx_2 = \int_0^x \left[\int_{x_1}^x dx_2\right] \varphi(x_1) dx_1$

EXAMPLE: Solve the problem $\frac{d^2u}{dx^2} = f(x)$ **with** $u(0) = 0 = u(l)\ ; 0 \le x \le l$

SOLUTION: This a Singular SL system with $p(x) = 1$

$\frac{d^2u}{dx^2} = f(x) \Rightarrow u''(x) = f(x) \Rightarrow \int_0^x u''(x)dx = \int_0^x f(x)dx$

$\Rightarrow |u'(x)|_0^x = \int_0^x f(x')dx' \Rightarrow u'(x) - u'(0) = \int_0^x f(x')dx'$

$\Rightarrow \int_0^x [u'(x) - u'(0)]\, dx = \int_0^x \left[\int_0^x f(x')dx'\right] dx$

$\Rightarrow \int_0^x u'(x)\, dx - \int_0^x u'(0)\, dx = \int_0^x \int_0^{x''} f(x')dx'\, dx''$

$\Rightarrow |u(x)|_0^x - u'(0)|x|_0^x = \int_0^x \left[\int_{x'}^x dx''\right] f(x')dx'$

$\Rightarrow [u(x) - u(0)] - u'(0)[x - 0] = \int_0^x |x''|_{x'}^x\, f(x')dx'$

$\Rightarrow u(x) - xA = \int_0^x (x - x')\, f(x')dx'$ **where** $u'(0) = A$ **(say) and** $u(0) = 0$

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' + xA$ **............(i)**

Put $x = l \Rightarrow u(l) = \int_0^l (l - x')\, f(x')dx' + lA$

$\Rightarrow \int_0^l (l - x')\, f(x')dx' + lA = 0$ **since** $u(l) = 0$

$\Rightarrow A = -\frac{1}{l}\int_0^l (l - x')\, f(x')dx'$ **............(ii) use it in (i)**

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' - \frac{x}{l}\int_0^l (l - x')\, f(x')dx'$

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' + \frac{x}{l}\int_0^l (x' - l)\, f(x')dx'$ **............(iii)**

This is the solution of given problem.

Now we can costruct a Green's Function by solving (iii)

$\Rightarrow u(x) = \int_0^x (x - x')\, f(x')dx' + \frac{x}{l}\left[\int_0^x (x' - l)\, f(x')dx' + \int_x^l (x' - l)\, f(x')dx'\right]$

$\Rightarrow u(x) = \int_0^x \left[x - x' + \frac{x}{l}(x' - l)\right] f(x')dx' + \frac{x}{l}\int_x^l (x' - l)\, f(x')dx'$

$\Rightarrow u(x) = \int_0^x \left[x - x' + \frac{xx'}{l} - x\right] f(x')dx' + \frac{x}{l}\int_x^l (x' - l)\, f(x')dx'$

$\Rightarrow u(x) = \frac{x'}{l}\int_0^x (x-l)\, f(x')dx' + \frac{x}{l}\int_x^l (x'-l)\, f(x')dx'$

$\Rightarrow u(x) = \int_0^x G(x,x')\, f(x')dx'$

Where $G'(x,x') = \begin{cases} \frac{x'}{l}(x-l) & ; 0 \le x' < x \\ \frac{x}{l}(x'-l) & ; x < x' \le l \end{cases}$

is called Green's function of given problem.

EXAMPLE: Solve and obtained the associated Green's Function

$\frac{d^2y}{dx^2} + k^2 y = f(x)$ **with** $y(0) = 0 = y(l)\ ; 0 \le x \le l$

SOLUTION: This a linear non – homogeneous DE of order 2 with constant coefficients. Its general solution is as follows;

$y = y_c + y_p$

For Charactristic Solution:

$\frac{d^2y}{dx^2} + k^2 y = 0 \Rightarrow D^2 + k^2 = 0 \Rightarrow D = \pm ik \quad \Rightarrow y_c = c_1 Coskx + c_2 Sinkx$

For Charactristic Solution:

For this we will use Wronskian method (Variation of Parameters)

Let $\Rightarrow y_p = u_1 Coskx + u_2 Sinkx$

Where $u_1 = -\int_{x_0}^x \frac{Sinkxf(x)}{W} dx$ **and** $u_2 = \int_{x_0}^x \frac{Coskxf(x)}{W} dx$

$\Rightarrow Wronskian = W = \begin{vmatrix} Coskx & Sinkx \\ -kSinkx & kCoskx \end{vmatrix} = k$

Then $u_1 = -\int_{x_0}^x \frac{Sinkx' f(x')}{k} dx'$ **and** $u_2 = \int_{x_0}^x \frac{Coskx' f(x')}{W} dx'$

$\Rightarrow y_p = -\int_{x_0}^x \frac{Sinkx' f(x')}{k} dx' Coskx + \int_{x_0}^x \frac{Coskx' f(x')}{W} dx' Sinkx$

$\Rightarrow y_p = \frac{1}{k}\int_{x_0}^x [SinkxCoskx' - CoskxSinkx'] f(x')dx'$

$\Rightarrow y_p = \frac{1}{k}\int_{x_0}^x Sin(kx - kx') f(x')dx' \Rightarrow y_p = \frac{1}{k}\int_{x_0}^x Sink(x-x') f(x')dx'$

$\Rightarrow y_p = \frac{1}{k}\int_0^x Sink(x-x') f(x')dx'$ **with** $x_0 = 0$ **a fixed point**

Thus for $y = y_c + y_p$ we have

$$y(x) = c_1 Coskx + c_2 Sinkx + \frac{1}{k}\int_0^x Sink(x - x')f(x')dx' \quad \text{.............(i)}$$

$$\Rightarrow y(0) = c_1 Cos(0) + c_2 Sin(0) + \frac{1}{k}\int_0^0 Sink(0 - x')f(x')dx'$$

$$\Rightarrow c_1 = 0 \quad \text{since } y(0) = 0$$

$$\Rightarrow y(l) = c_1 Coskl + c_2 Sinkl + \frac{1}{k}\int_0^l Sink(l - x')f(x')dx' \quad \text{put } x = l$$

$$\Rightarrow 0 = 0 + c_2 Sinkl + \frac{1}{k}\int_0^l Sink(l - x')f(x')dx'$$

$$\Rightarrow c_2 = -\frac{1}{kSinkl}\int_0^l Sink(l - x')f(x')dx'$$

Using c_1, c_2 in (i)

$$y(x) = -\frac{Sinkx}{kSinkl}\int_0^l Sink(l - x')f(x')dx' + \frac{1}{k}\int_0^x Sink(x - x')f(x')dx'$$

$$y(x) = -\frac{Sinkx}{kSinkl}\left[\int_0^x Sink(l - x')f(x')dx' + \int_x^l Sink(l - x')f(x')dx' + \frac{1}{k}\int_0^x Sink(x - x')f(x')dx'\right]$$

$$y(x) = \frac{1}{k}\int_0^x \left[Sink(x - x') - \frac{Sinkx}{Sinkl}Sink(l - x')\right]f(x')dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$= \frac{1}{k}\int_0^x \left[Sink(x - x') - \frac{Sinkx}{Sinkl}Sink(l - x')\right]\frac{Sinkl}{Sinkl}.f(x')dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y(x) = \frac{1}{k}\int_0^x [(SinkxCoskx' - CoskxSinkx')Sinkl - Sinkx(SinklCoskx' - CosklSinkx')]\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y(x) = \frac{1}{k}\int_0^x [SinklSinkxCoskx' - SinklCoskxSinkx' - SinklSinkxCoskx' + SinkxCosklSinkx']\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y(x) = \frac{1}{k}\int_0^x [-SinklCoskxSinkx' + SinkxCosklSinkx']\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$$y = \frac{1}{k}\int_0^x [-Sinkx'(SinklCoskx + SinkxCoskl)]\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$$

$y = \frac{1}{k}\int_0^x [Sinkx'(SinkxCoskl - SinklCoskx)]\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$

$y = \frac{1}{k}\int_0^x [Sinkx'Sink(x - l)]\frac{f(x')}{Sinkl}dx' - \frac{Sinkx}{kSinkl}\int_x^l Sink(l - x')f(x')dx'$

$y(x) = \int_0^x \frac{Sinkx'Sink(x-l)}{kSinkl}f(x')dx' + \int_x^l \frac{SinkxSink(x'-l)}{kSinkl}f(x')dx'$

$y(x) = \int_0^l G(x,x')f(x')dx'$

Where $G(x,x') = \begin{cases} \frac{Sinkx'Sink(x-l)}{kSinkl} & ; 0 \le x' < x \\ \frac{SinkxSink(x'-l)}{kSinkl} & ; x < x' \le l \end{cases}$

is called Green's function of given problem.

Note: $Sinkl \neq 0$ **i.e. 'k' is not eigenvalue of associated homogeneous problem.**

PROPERTIES OF PREVIOUS GREEN's FUNCTION

i. $G(x,x')$ **is symmetric i.e.** $G(x,x') = G(x',x)$

ii. $G(x,x')$ **as a function of 'x' satisfies the D Equation** $\frac{d^2}{dx^2}G(x,x') = 0$ **in each of the interval** $0 \le x < x'$ **and** $x' < x \le l$

iii. $G(0,x') = 0$ **and** $G(l,x') = 0$ **are the same BC's as those satisfied by the given Green's function.**

iv. $G(x,x')$ **is continuous function of 'x' in the interval** $[0 , l\,]$ **and particularly at** $x = x'$

v. $G'(x,x') = \frac{d}{dx}G(x,x')$ **exists as**

$G'(x,x') = \begin{cases} \frac{Sinkx'Cosk(x-l)}{kSinkl} & ; 0 \le x' < x \\ \frac{CoskxSink(x'-l)}{kSinkl} & ; x < x' \le l \end{cases}$ **and** $G'(x,x')$ **will be discontinuous at** $x = x'$

REMEMBER: The Greenn's Function technique is used to solve DE of the form $(L_x u)(x) = f(x) + BC's$ **where** L_x **is a linear operator with specified BC's.**

EXITENCE OF GREEN's FUNCTION:

If the homogeneous problem associated with SL system $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ with usual BC's has trivial solution then Green's Function exists.

In other words, if $\lambda = 0$ is not an eigenvalue for $L(u) + \lambda\, r(x)u = 0$ with usual BC's then Green's Function exists.

GREEN's FUNCTION ASSOCIATED WITH REGULAR SL SYSTEM:

Let $L(u) + \lambda\, r(x)u = 0$ be the SL equation with the endpoint conditions $\propto_1 u(a) + \propto_2 u'(a) = 0$ and $\beta_1 u(b) + \beta_2 u'(b) = 0$ which may also be written as $B_1(u) = \propto_1 + \propto_2 \frac{\partial}{\partial x} = 0$ and $B_2(u) = \beta_1 + \beta_2 \frac{\partial}{\partial x} = 0$ where B is a BC's operatior define regular SL system and gives a trivial solution. Then the Green's Function associated with regular SL system has the following properties;

i. **$G(x,t)$ considered as the function of 'x' satisfies the DE $L\{G(x,t)\} = 0$ in each of the interval $a \le x < t$ and $t < x \le b$**
ii. **$B_1(G) = 0$ and $B_2(G) = 0$ are the same BC's as those satisfied by the given Green's function.**
iii. **$G(x,t)$ is continuous function of 'x' in the interval $[a\,,b\,]$**
iv. **$G'(x,t) = \frac{d}{dx}G(x,t)$ will be discontinuous as $x \to t$ and moreover**
   $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$
   but $\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

EXAMPLE: Solve the problem associated with non – homogeneous DE

$L(u) + \lambda\, r(x)u = f(x)$ **where** $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$

SOLUTION: The solution of this non – homogeneous DE subject to BC's is closely related to the existence of Green's function associated with homogeneous equation $L(u) + \lambda\, r(x)u = 0$

If the function $G(x,t,\lambda)$ **which does not depends on the source function** $f(x)$ **exists, then solution of given equation can be written as**

$u(x) = \int_a^b G(x,t,\lambda)f(t)dt$ **where** $G(x,t,\lambda)$ **is called Green's function and satisfies the equation** $L(G) + \lambda\, r(x)G = \delta(x-t)$

EXAMPLE:

Construct Green's function associated with the problem $u'' + \lambda u = 0$ **with the boundry conditions** $u(0) = 0$ **and** $u(1) = 0$

Solution: **here** $p(x) = 1 = p(t)$

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$ **……….(i)**

Now using BC's $u(0) = 0$ **and** $u(1) = 0$ **we have** $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ **which is trivial solution. So** $\lambda = 0$ **is not an eigenvalue.**

ii. $G(x,t)$ **as a function of 'x' satisfies the D Equation** $\frac{d^2}{dx^2}G(x,t) = 0$ **in each of the interval** $0 \le x < t$ **and** $t < x \le 1$ **therefore we have**

$$G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + B' & ; t < x \le 1 \end{cases}$$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\,,1]$ **and particularly at** $x = t$ **therefore**

$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$

$\lim_{x\to t^-}(Ax+B) = \lim_{x\to t^+}(A'x+B')$

$At+B = A't+B' \Rightarrow B' = (A-A')t+B$

Hence $G(x,t) = \begin{cases} Ax+B & ; 0 \le x < t \\ A'x+(A-A')t+B & ; t < x \le 1 \end{cases}$

iv. $G(0,t)=0$ **and** $G(1,t)=0$ **are the same BC's as those satisfied by the given Green's function.i.e.**

$G(0,t)=0 \Rightarrow A(0)+B=0 \Rightarrow B=0$

$G(1,t)=0 \Rightarrow A'(1)+(A-A')t+B=0 \Rightarrow A=\frac{A'(t-1)}{t}$ **with** $B=0$

Then $G(x,t) = \begin{cases} \frac{A'(t-1)}{t}x+0 & ; 0 \le x < t \\ A'x+\left(\frac{A'(t-1)}{t}-A'\right)t+0 & ; t < x \le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} \frac{A'(t-1)}{t}x & ; 0 \le x < t \\ A'x-A' & ; t < x \le 1 \end{cases}$

v. $G'(x,t)=\frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x \to t$ **i.e.**

$\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+}\frac{d}{dx}(A'x-A') - \lim_{x\to t^-}\frac{d}{dx}\left(\frac{A'(t-1)}{t}x\right) = \frac{1}{1}$

$\lim_{x\to t^+}(A') - \lim_{x\to t^-}\left(\frac{A'(t-1)}{t}\right) = 1$

$A' - \frac{A'(t-1)}{t} = 1$

$A'\left(\frac{1}{t}\right) = 1 \Rightarrow A' = t$

Then $G(x,t) = \begin{cases} \frac{t(t-1)}{t}x & ; 0 \le x < t \\ tx-t & ; t < x \le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} (t-1)x & ; 0 \le x < t \\ (x-1)t & ; t < x \le 1 \end{cases}$

This is our required Green's Function.

EXAMPLE:

Construct Green's function associated with the problem $xu'' + u' + \lambda ru = 0$ with the boundry conditions $u(0)$ is finite and $u(1) = 0$

Solution: here $p(x) = x$ then $p(t) = t$

i. **Put $\lambda = 0$ in given equation**

$$xu'' + u' + \lambda ru = 0 \Rightarrow xu'' + u' = 0 \Rightarrow \frac{d}{dx}(xu')$$

$\Rightarrow u(x) = Alnx + B$ ……….(i)

Now using BC's $u(0) = finite$ and $u(1) = 0$ we have $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ which is trivial solution. So $\lambda = 0$ is not an eigenvalue.

ii. **$G(x,t)$ as a function of 'x' satisfies the D Equation $xG'' + G' = 0$ in each of the interval $0 \le x < t$ and $t < x \le 1$ therefore we have**

$$G(x,t) = \begin{cases} Alnx + B & ; 0 \le x < t \\ A'lnx + B' & ; t < x \le 1 \end{cases}$$

iii. **$G(x,t)$ is continuous function of 'x' in the interval $[0\,,1]$ and particularly at $x = t$ therefore**

$$\lim_{x \to t^-} G(x,t) = \lim_{x \to t^+} G(x,t)$$

$$\lim_{x \to t^-}(Alnx + B) = \lim_{x \to t^+}(A'lnx + B')$$

$$Alnt + B = A'lnt + B' \Rightarrow B' = (A - A')lnt + B$$

Hence $G(x,t) = \begin{cases} Alnx + B & ; 0 \le x < t \\ A'lnx + (A - A')lnt + B & ; t < x \le 1 \end{cases}$

iv. **$G(0,t) = finite$ and $G(1,t) = 0$ are the same BC's as those satisfied by the given Green's function.i.e.**

$$G(0,t) = finite \Rightarrow Aln(0) + B = finite \Rightarrow A = 0$$

$$G(1,t) = 0 \Rightarrow A'ln(1) + (A - A')lnt + B = 0 \Rightarrow B = A'lnt$$

$\Rightarrow A' = \frac{B}{lnt}$ **with** $A = 0, ln(1) = 0$

Then $G(x,t) = \begin{cases} A'lnt & ; 0 \le x < t \\ A'lnx & ; t < x \le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x \to t$ **i.e.**

$\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+} A'\left(\frac{1}{x}\right) - \lim_{x\to t^-}(0) = \frac{1}{t}$

$A'\left(\frac{1}{t}\right) = \frac{1}{t} \Rightarrow A' = 1$

Then $G(x,t) = \begin{cases} lnt & ; 0 \le x < t \\ lnx & ; t < x \le 1 \end{cases}$ **is our required Green's Function.**

EXAMPLE: **Construct Green's function associated with the problem** $xu'' + u' - \frac{n^2}{x}u + \lambda ru = 0$ **with the boundry conditions** $u(0)$ **is finite and** $u(1) = 0$

Solution:here $p(x) = x$ **then** $p(t) = t$ **this is regular system with** $q(x) = -\frac{n^2}{x}$

i. **Put** $\lambda = 0$ **in given equation**

$xu'' + u' - \frac{n^2}{x}u + (0)ru = 0 \Rightarrow xu'' + u' - \frac{n^2}{x}u = 0$

$\Rightarrow \left(xD^2 + D - \frac{n^2}{x}\right)u = 0$

$\Rightarrow (x^2D^2 + xD - n^2)u = 0$ **……….(i)** **this is Cauchy Euler equation**

Put $x = e^t \Rightarrow lnx = t \Rightarrow xD = \Delta$ **and** $x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$(i) \Rightarrow (\Delta^2 - \Delta + \Delta - n^2)u = 0 \Rightarrow (\Delta^2 - n^2)u = 0 \Rightarrow \Delta = \pm n$

$\Rightarrow u(x) = Ae^{nt} + Be^{-nt} = A(e^t)^n + B(e^t)^{-n}$

$\Rightarrow u(x) = Ax^n + Bx^{-n}$ **……….(ii)**

Now using BC's $u(0) = finite$ **and** $u(1) = 0$ **we have** $A = 0, B = 0$

$(ii) \Rightarrow u(x) = 0$ **which is trivial solution. So** $\lambda = 0$ **is not an eigenvalue.**

ii. $G(x,t)$ **as a function of 'x' satisfies the Differential Equation** $x^2G''+xG'-n^2G=0$ **in each of the interval** $0\le x<t$ **and** $t<x\le 1$ **therefore we have**

$$G(x,t)=\begin{cases} Ax^n+Bx^{-n} & ;0\le x<t \\ A'x^n+B'x^{-n} & ;t<x\le 1 \end{cases}$$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\,,1]$ **and particularly at** $x = t$ **therefore**

$$\lim_{x\to t^-} G(x,t)=\lim_{x\to t^+} G(x,t)$$

$$\lim_{x\to t^-}(Ax^n+Bx^{-n})=\lim_{x\to t^+}(A'x^n+B'x^{-n})$$

$$At^n+Bt^{-n}=A't^n+B't^{-n}\Rightarrow B'=(A-A')t^{2n}+B$$

Hence $G(x,t)=\begin{cases} Ax^n+Bx^{-n} & ;0\le x<t \\ A'x^n+(A-A')t^{2n}x^{-n}+Bx^{-n} & ;t<x\le 1 \end{cases}$

iv. $G(0,t)=0$ **and** $G(1,t)=0$ **are the same BC's as those satisfied by the given Green's function.i.e.**

$$G(0,t)=finite\Rightarrow A(0)^n+B(0)^{-n}=finite\Rightarrow B=0$$

$$G(1,t)=0\Rightarrow A'(1)^n+(A-A')t^{2n}(1)^{-n}+(0)(1)^{-n}$$

$$\Rightarrow A=A'(1-t^{-2n})\ \text{ with } B=0$$

Then

$$G(x,t)=\begin{cases} A'(1-t^{-2n})x^n+(0)x^{-n} & ;0\le x<t \\ A'x^n+\left(\left(A'(1-t^{-2n})\right)-A'\right)t^{2n}x^{-n}+(0)x^{-n} & ;t<x\le 1 \end{cases}$$

$$G(x,t)=\begin{cases} A'(1-t^{-2n})x^n & ;0\le x<t \\ A'x^n+A'x^{-n} & ;t<x\le 1 \end{cases}$$

Hence $G(x,t)=\begin{cases} A'(1-t^{-2n})x^n & ;0\le x<t \\ A'(x^n-x^{-n}) & ;t<x\le 1 \end{cases}$

v. $G'(x,t)=\frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x\to t$ **i.e.**

$$\lim_{x\to t^+} G'(x,t)\ne \lim_{x\to t^-} G'(x,t)$$

But $\lim_{x\to t^+} G'(x,t)-\lim_{x\to t^-} G'(x,t)=\frac{1}{p(t)}$

$$\lim_{x\to t^+} A'(nx^{n-1} - nx^{-n-1}) - \lim_{x\to t^-} nA'(0 - t^{-2n})x^{n-1} = \frac{1}{t}$$

$A'(nt^{n-1} - nt^{-n-1}) - nA'(-t^{-2n})t^{n-1} = \frac{1}{t} \quad \Rightarrow A' = \frac{t^n}{n(2+t^{2n})}$ **after solving**

Then $G(x,t) = \begin{cases} \frac{t^n}{n(2+t^{2n})}(1 - t^{-2n})x^n & ; 0 \le x < t \\ \frac{t^n}{n(2+t^{2n})}(x^n - x^{-n}) & ; t < x \le 1 \end{cases}$

is our required Green's Function.

EXAMPLE: **Construct Green's function associated with the problem** $\frac{d}{dx}\{(1-x^2)u'\} - \frac{h^2}{1-x^2}u + \lambda r u = 0$ **with the boundry conditions** $u(\pm 1)$ **are finite**

Solution:here $p(x) = 1 - x^2$ **then** $p(t) = 1 - t^2$ **this is singular system**

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d}{dx}\{(1-x^2)u'\} - \frac{h^2}{1-x^2}u = 0 \Rightarrow (1-x^2)u'' - 2xu' - \frac{h^2}{1-x^2}u = 0$

$\Rightarrow (1-x^2)^2u'' - 2x(1-x^2)u' - h^2u = 0$ **……….(i)**

Put $t = ln\left(\frac{1+x}{1-x}\right) = ln(1+x) - ln(1-x) \Rightarrow \frac{dt}{dx} = \frac{2}{1-x^2}$

$u' = \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \frac{2}{1-x^2}\frac{du}{dt} \Rightarrow u'' = \frac{4}{(1-x^2)^2}\left[\frac{d^2u}{dt^2} + x\frac{du}{dt}\right]$ **after solving**

$(i) \Rightarrow (1-x^2)^2\frac{4}{(1-x^2)^2}\left[\frac{d^2u}{dt^2} + x\frac{du}{dt}\right] - 2x(1-x^2)\frac{2}{1-x^2}\frac{du}{dt} - h^2u = 0$

$\Rightarrow 4\frac{d^2u}{dt^2} + 4x\frac{du}{dt} - 4x\frac{du}{dt} - h^2u = 0 \Rightarrow \frac{d^2u}{dt^2} - \frac{h^2}{4}u = 0 \Rightarrow D = \pm\frac{h}{2}$

$\Rightarrow u(x) = Ae^{\frac{h}{2}t} + Be^{-\frac{h}{2}t} = A(e^t)^{h/2} + B(e^t)^{-h/2}$

$\Rightarrow u(x) = A\left(\frac{1+x}{1-x}\right)^{h/2} + B\left(\frac{1+x}{1-x}\right)^{-h/2}$ **……….(ii)**

Now using BC's $u(\pm 1) = finite$ **we have** $A = 0, B = 0$

$(ii) \Rightarrow u(x) = 0$ **which is trivial solution. So** $\lambda = 0$ **is not an eigenvalue.**

ii. $G(x,t)$ **as a function of 'x' satisfies the Differential Equation** $\frac{d}{dx}\{(1-x^2)G'\}-\frac{h^2}{1-x^2}G=0$ **in each of the interval** $0\leq x<t$ **and** $t<x\leq 1$ **therefore we have**

$$G(x,t)=\begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2}+B\left(\frac{1+x}{1-x}\right)^{-h/2} & ;-1\leq x<t \\ A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}}+B'\left(\frac{1+x}{1-x}\right)^{-h/2} & ;t<x\leq 1 \end{cases}$$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\,,1]$ **and particularly at** $x\ =\ t$ **therefore**

$$\lim_{x\to t^-}G(x,t)=\lim_{x\to t^+}G(x,t)$$

$$\lim_{x\to t^-}\left(A\left(\frac{1+x}{1-x}\right)^{h/2}+B\left(\frac{1+x}{1-x}\right)^{-h/2}\right)=\lim_{x\to t^+}\left(A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}}+B'\left(\frac{1+x}{1-x}\right)^{-h/2}\right)$$

$$A\left(\frac{1+t}{1-t}\right)^{h/2}+B\left(\frac{1+t}{1-t}\right)^{-h/2}=A'\left(\frac{1+t}{1-t}\right)^{\frac{h}{2}}+B'\left(\frac{1+t}{1-t}\right)^{-h/2}$$

$$\Rightarrow B'=(A-A')\left(\frac{1+t}{1-t}\right)^{h}+B$$

then

$$G(x,t)=\begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2}+B\left(\frac{1+x}{1-x}\right)^{-h/2} & ;-1\leq x<t \\ A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}}+\left[(A-A')\left(\frac{1+t}{1-t}\right)^{h}+B\right]\left(\frac{1+x}{1-x}\right)^{-h/2} & ;t<x\leq 1 \end{cases}$$

$$G(x,t)=\begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2}+B\left(\frac{1+x}{1-x}\right)^{-h/2} & ;-1\leq x<t \\ A'\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}}+\left[(A-A')\left(\frac{1+t}{1-t}\right)^{h}+B\right]\left(\frac{1+x}{1-x}\right)^{-h/2} & ;t<x\leq 1 \end{cases}$$

iv. $G(\pm,t)=finite$ **are the BC's satisfied by the Green's function.**

$$G(-1,t)=finite\Rightarrow A\left(\frac{1+(-1)}{1-(-1)}\right)^{h/2}+B\left(\frac{1+(-1)}{1-(-1)}\right)^{-h/2}=finite\Rightarrow B=0$$

$$G(1,t)=finite\Rightarrow A'\left(\frac{1+(1)}{1-(1)}\right)^{\frac{h}{2}}+\left[(A-A')\left(\frac{1+t}{1-t}\right)^{h}+B\right]\left(\frac{1+(1)}{1-(1)}\right)^{-\frac{h}{2}}=finite$$

$$\Rightarrow A'=0$$

Then $\quad G(x,t) = \begin{cases} A\left(\frac{1+x}{1-x}\right)^{h/2} & ; -1 \le x < t \\ A\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{-h/2} & ; t < x \le 1 \end{cases}$

v. $\quad G'(x,t) = \frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x \to t$ **i.e.**

$\lim_{x\to t^+} G'(x,t) \ne \lim_{x\to t^-} G'(x,t)$

But $\quad \lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+}\left(-A\frac{h}{2}\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{-\frac{h}{2}-1}\left[\frac{2}{(1-x)^2}\right]\right) - \lim_{x\to t^-}\left(A\frac{h}{2}\left(\frac{1+x}{1-x}\right)^{\frac{h}{2}-1}\left[\frac{2}{(1-x)^2}\right]\right) = \frac{1}{1-t^2}$

$-A\frac{h}{2}\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+t}{1-t}\right)^{-\frac{h}{2}-1}\left[\frac{2}{(1-t)^2}\right] - A\frac{h}{2}\left(\frac{1+t}{1-t}\right)^{\frac{h}{2}-1}\left[\frac{2}{(1-t)^2}\right] = \frac{1}{1-t^2}$

$\Rightarrow A = -\frac{1}{2h}\left(\frac{1-t}{1+t}\right)^{h/2}$ $\qquad$ **after solving**

Then $\; G(x,t) = \begin{cases} -\frac{1}{2h}\left(\frac{1-t}{1+t}\right)^{h/2}\left(\frac{1+x}{1-x}\right)^{h/2} & ; -1 \le x < t \\ -\frac{1}{2h}\left(\frac{1-t}{1+t}\right)^{h/2}\left(\frac{1+t}{1-t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{-h/2} & ; t < x \le 1 \end{cases}$

is our required Green's Function.

EXAMPLE:

Construct Green's function associated with the problem $u'' + \lambda u = 0$ **with the boundry conditions** $u(0) + u'(1) = 0$ **and** $u(1) + 2u'(0) = 0$

Solution: $\quad$ **here** $p(x) = 1 = p(t)$

i. $\quad$ **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$ **……….(i)**

Now using BC's

$u(0) + u'(1) = 0$ **and** $u(1) + 2u'(0) = 0$ **we have** $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ **which is trivial solution. So** $\lambda = 0$ **is not an eigenvalue.**

ii. $G(x,t)$ **as a function of 'x' satisfies the D Equation** $\frac{d^2}{dx^2}G(x,t) = 0$ **in each of the interval** $0 \le x < t$ **and** $t < x \le 1$ **therefore we have**

$$G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + B' & ; t < x \le 1 \end{cases}$$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\,,1]$ **and particularly at** $x = t$ **therefore**

$$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$$

$$\lim_{x\to t^-}(Ax+B) = \lim_{x\to t^+}(A'x+B')$$

$$At + B = A't + B' \Rightarrow B' = (A - A')t + B$$

Hence $G(x,t) = \begin{cases} Ax + B & ; 0 \le x < t \\ A'x + (A - A')t + B & ; t < x \le 1 \end{cases}$

iv. $G(x,t)$ **satisfies the BC's**

$$G(0,t) + G'(1,t) = 0 \Rightarrow A(0) + B + A' = 0 \Rightarrow B = -A'$$

$$G(1,t) + 2G'(0,t) = 0 \Rightarrow A'(1) + (A - A')t + B + 2A = 0$$

$\Rightarrow A = A'\left(\frac{t}{t+2}\right)$ **with** $B = -A'$

Then $G(x,t) = \begin{cases} A'\left(\frac{t}{t+2}\right)x - A' & ; 0 \le x < t \\ A'x + \left(A'\left(\frac{t}{t+2}\right) - A'\right)t - A' & ; t < x \le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} A'\left(\frac{t}{t+2}x - 1\right) & ; 0 \le x < t \\ A'\left[x + \left(\frac{t}{t+2} - 1\right)t - 1\right] & ; t < x \le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x \to t$ **i.e.**

$$\lim_{x\to t^+} G'(x,t) \ne \lim_{x\to t^-} G'(x,t)$$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$$\lim_{x\to t^+}(A') - \lim_{x\to t^-} A'\left(\frac{t}{t+2}\right) = \frac{1}{1}$$

$$A' - A'\left(\frac{t}{t+2}\right) = 1 \qquad \Rightarrow A' = \frac{t+2}{2}$$

Then $G(x,t) = \begin{cases} \frac{t+2}{2}\left(\frac{t}{t+2}x - 1\right) & ;0 \le x < t \\ \frac{t+2}{2}\left[x + \left(\frac{t}{t+2} - 1\right)t - 1\right] & ;t < x \le 1 \end{cases}$

$\Rightarrow G(x,t) = \begin{cases} \frac{t}{2}x - \frac{t+2}{2} & ;0 \le x < t \\ \frac{t+2}{2}x + \frac{t^2}{2} - \frac{t(t+2)}{2} - \frac{t+2}{2} & ;t < x \le 1 \end{cases}$

Hence $\Rightarrow G(x,t) = \begin{cases} \frac{tx-t-2}{2} & ;0 \le x < t \\ \frac{(t+2)x-3t-2}{2} & ;t < x \le 1 \end{cases}$ **required Green's Function.**

EXAMPLE:

Construct Green's function associated with the problem $u'' + \lambda u = 0$ **with the boundry conditions** $u'(0) = 0$ **and** $u(1) = 0$

Solution: **here** $p(x) = 1 = p(t)$

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$ **……….(i)**

Now using BC's $u'(0) = 0$ **and** $u(1) = 0$ **we have** $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ **which is trivial solution. So** $\lambda = 0$ **is not an eigenvalue.**

ii. $G(x,t)$ **as a function of 'x' satisfies the D Equation** $\frac{d^2}{dx^2}G(x,t) = 0$ **in each of the interval** $0 \le x < t$ **and** $t < x \le 1$ **therefore we have**

$G(x,t) = \begin{cases} Ax + B & ;0 \le x < t \\ A'x + B' & ;t < x \le 1 \end{cases}$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\,,1]$ **and particularly at** $x = t$ **therefore**

$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$

$\lim_{x\to t^-}(Ax + B) = \lim_{x\to t^+}(A'x + B')$

$At + B = A't + B' \Rightarrow B' = (A - A')t + B$

Hence $G(x,t) = \begin{cases} Ax + B & ;0 \le x < t \\ A'x + (A - A')t + B & ;t < x \le 1 \end{cases}$

iv. $G(0,t) = 0$ **and** $G(1,t) = 0$ **are the same BC's as those satisfied by the given Green's function.i.e.**

$G'(0,t) = 0 \Rightarrow A = 0$

$G(1,t) = 0 \Rightarrow B = A'(t-1)$ **with** $A = 0$

Then $G(x,t) = \begin{cases} A'(t-1) & ; 0 \leq x < t \\ A'x + (0 - A')t + A'(t-1) & ; t < x \leq 1 \end{cases}$

Hence $G(x,t) = \begin{cases} A'(t-1) & ; 0 \leq x < t \\ A'(x-1) & ; t < x \leq 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x \to t$ **i.e.**

$\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+}(A') - \lim_{x\to t^-}(0) = \frac{1}{1} \Rightarrow A' = 1$

Hence $G(x,t) = \begin{cases} (t-1) & ; 0 \leq x < t \\ (x-1) & ; t < x \leq 1 \end{cases}$ **required Green's Function.**

EXAMPLE:

Construct Green's function associated with the problem $u'' + \lambda u = 0$ **with the boundry conditions** $u'(0) = 0$ **and** $u(2) = 0$

Solution: **here** $p(x) = 1 = p(t)$

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$ **……….(i)**

Now using BC's $u'(0) = 0$ **and** $u(2) = 0$ **we have** $A = 0, B = 0$

$(i) \Rightarrow u(x) = 0$ **which is trivial solution. So** $\lambda = 0$ **is not an eigenvalue.**

ii. $G(x,t)$ **as a function of 'x' satisfies the D Equation** $\frac{d^2}{dx^2}G(x,t) = 0$ **in each of the interval** $0 \leq x < t$ **and** $t < x \leq 1$ **therefore we have**

$G(x,t) = \begin{cases} Ax + B & ; 0 \leq x < t \\ A'x + B' & ; t < x \leq 2 \end{cases}$

iii. $G(x,t)$ **is continuous function of 'x' in the interval** $[0\ ,2]$ **and particularly at** $x = t$ **therefore**

$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$

$\lim_{x\to t^-}(Ax+B) = \lim_{x\to t^+}(A'x+B')$

$At+B = A't+B' \Rightarrow B' = (A-A')t+B$

Hence $G(x,t) = \begin{cases} Ax+B & ; 0 \le x < t \\ A'x+(A-A')t+B & ; t < x \le 2 \end{cases}$

iv. $G(0,t) = 0$ **and** $G(2,t) = 0$ **are the same BC's as those satisfied by the given Green's function.i.e.**

$G'(0,t) = 0 \Rightarrow A = 0$

$G(2,t) = 0 \Rightarrow A'(2)+(0-A')t+B = 0 \Rightarrow A'(2-t)+B = 0$

$\Rightarrow B = A'(t-2)$ **with** $A = 0$

Then $G(x,t) = \begin{cases} A'(t-2) & ; 0 \le x < t \\ A'x+(0-A')t+A'(t-2) & ; t < x \le 1 \end{cases}$

Hence $G(x,t) = \begin{cases} A'(t-2) & ; 0 \le x < t \\ A'(x-2) & ; t < x \le 1 \end{cases}$

v. $G'(x,t) = \frac{d}{dx}G(x,t)$ **exists and will be discontinuous as** $x \to t$ **i.e.**

$\lim_{x\to t^+} G'(x,t) \neq \lim_{x\to t^-} G'(x,t)$

But $\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{1}{p(t)}$

$\lim_{x\to t^+}(A') - \lim_{x\to t^-}(0) = \frac{1}{1} \Rightarrow A' = 1$

Hence $G(x,t) = \begin{cases} (t-2) & ; 0 \le x < t \\ (x-2) & ; t < x \le 1 \end{cases}$ **required Green's Function.**

MODIFIED GREEN's FUNCTION

When $\lambda = 0$ is an eigenvalue of the SL system defined by $L(u) + \lambda r u = 0$ with $\beta_1(u) = 0, \beta_2(u) = 0$ then the associated Green's function is called modified green's function. And is denoted by $G_M(x, t)$

PROPERTIES OF MODIFIED GREEN's FUNCTION: (UoS; S.Q)

Let $u_0(x)$ be the normalized eigenfunction corresponding to $\lambda = 0$ this means that $\langle u_0(x), u_0(x) \rangle = \int_a^b u_0(x).u_0(x)dx = 1$ then $G_M(x, t)$ will have the following properties;

i. **$G_M(x, t)$ satisfies the D Equation $L[G_M(x, t)] = u_0(t).u_0(t)$ in each of the interval $a \le x \le t$ and $t < x \le b$**
ii. **$\beta_1[G_M(x, t)] = 0$ and $\beta_2[G_M(x, t)] = 0$ which are the same BC's as those satisfied by $G_M(x, t)$**
iii. **$G_M(x, t)$ is continuous function of 'x' in the interval $[a, b\ ]$ and particularly at x = t**
iv. **$G'_M(x, t) = \frac{d}{dx} G_M(x, t)$ exists and will be discontinuous as $x \to t$ i.e.**
$\lim_{x\to t^+} G'_M(x, t) \neq \lim_{x\to t^-} G'_M(x, t)$
But $\lim_{x\to t^+} G'_M(x, t) - \lim_{x\to t^-} G'_M(x, t) = \frac{1}{p(t)}$
v. **The modified Green's function $G_M(x, t)$ satisfies the orthogonality condition $\int_a^b G_M(x, t).u_0(x)dx = 0$**

EXAMPLE:Construct Green's function associated with the problem $u''+\lambda r u=0$ with the boundry conditions $u'(0)=0$ and $u'(1)=0$

Solution: here $p(x)=1=p(t)$

i. **Put $\lambda=0$ in given equation**

$\frac{d^2u}{dx^2}+\lambda u=0 \Rightarrow \frac{d^2u}{dx^2}+(0)u=0 \Rightarrow \frac{d^2u}{dx^2}=0 \Rightarrow u(x)=Ax+B$

$\Rightarrow u'(x)=A$ ....(i)

Now using BC's $u'(0)=0$ and $u'(1)=0$ we have $A=0, B\neq 0$

$(i)\Rightarrow u(x)=B$ which is non - trivial solution. So $\lambda=0$ is an eigenvalue.

Therefore we take $u_0(x)=1$ as a normalized function.i.e.

$\langle u_0(x), u_0(x)\rangle=\int_a^b u_0(x).u_0(x)dx=\int_0^1 1dx=1$

ii. **$G_M(x,t)$ as a function of 'x' satisfies the D Equation**

$\frac{d^2}{dx^2}G_M(x,t)=u_0(x)u_0(t)=1$ **in each of the interval $0\leq x<t$ and $t<x\leq 1$ therefore we have $G''_M(x,t)=1 \Rightarrow G'_M(x,t)=x+A$**

$\Rightarrow G_M(x,t)=\frac{x^2}{2}+Ax+B$

$$\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+Ax+B & ;0\leq x<t\\ \frac{x^2}{2}+A'x+B' & ;t<x\leq 1\end{cases}$$

iii. **$G_M(x,t)$ satisfies the BC's i.e. $\Rightarrow G'_M(0,t)=0 \Rightarrow A=0$ and**

$\Rightarrow G'_M(1,t)=0 \Rightarrow A'=-1$

$$\text{thus } \Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+B & ;0\leq x<t\\ \frac{x^2}{2}-x+B' & ;t<x\leq 1\end{cases}$$

iv. **$G_M(x,t)$ is continuous function of 'x' in the interval $[0,1]$ and particularly at x = t i.e.**

$\lim_{x\to t^+} G_M(x,t)=\lim_{x\to t^-} G_M(x,t)$

$$\lim_{x\to t^+}\left(\frac{x^2}{2}-x+B'\right)=\lim_{x\to t^-}G'_M\left(\frac{x^2}{2}+B\right)$$

$$\frac{t^2}{2}-t+B'=\frac{t^2}{2}+B\Rightarrow B'=B+t$$

$$\text{thus }\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+B & ;0\le x<t\\ \frac{x^2}{2}-x+B+t & ;t<x\le 1\end{cases}$$

v. $G'_M(x,t)=\frac{d}{dx}G_M(x,t)$ **exists and will be discontinuous as** $x\to t$ **i.e.**

$$\lim_{x\to t^+}G'_M(x,t)\neq\lim_{x\to t^-}G'_M(x,t)$$

But $\lim_{x\to t^+}G'_M(x,t)-\lim_{x\to t^-}G'_M(x,t)=\frac{1}{p(t)}$

$$\lim_{x\to t^+}\left(\frac{2x}{2}-1\right)-\lim_{x\to t^-}\left(\frac{2x}{2}\right)=\frac{1}{1}$$

$$t-1-t=1\Rightarrow -1\neq 1$$

Thus the discontinuity condition does not help to determining the unknown constant B. so we will use orthogonality condition.

vi. **Using orthogonality condition** $\int_0^1 G_M(x,t).u_0(x)dx=0$

$$\int_0^t G_M(x,t).u_0(x)dx+\int_t^1 G_M(x,t).u_0(x)dx=0$$

$$\int_0^t\left(\frac{x^2}{2}+B\right)dx+\int_t^1\left(\frac{x^2}{2}-x+B+t\right)dx=0=0\quad \text{with } u_0(x)=1$$

$B=\frac{t^2}{2}-t+\frac{1}{3}$ **after solving**

Hence our required Green's function is as follows;

$$\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+\frac{t^2}{2}-t+\frac{1}{3} & ;0\le x<t\\ \frac{x^2}{2}-x+\frac{t^2}{2}-t+\frac{1}{3}+t & ;t<x\le 1\end{cases}$$

$$\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+\frac{t^2}{2}-t+\frac{1}{3} & ;0\le x<t\\ \frac{x^2}{2}-x+\frac{t^2}{2}+\frac{1}{3} & ;t<x\le 1\end{cases}$$

EXAMPLE:Construct Green's function associated with the problem

$u'' + \lambda u = 0$ **with the boundry conditions** $u(0) = u(1)$ **and** $u'(0) = u'(1)$

Solution: **here** $p(x) = 1 = p(t)$

i. **Put** $\lambda = 0$ **in given equation**

$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$

$\Rightarrow u'(x) = A$ **……….(i)**

Now using BC's $u(0) = u(1)$ **and** $u'(0) = u'(1)$ **we have** $A = 0, B \neq 0$

$(i) \Rightarrow u(x) = B$ **which is non - trivial solution. So** $\lambda = 0$ **is an eigenvalue.**

Therefore we take $u_0(x) = 1$ **as a normalized function.i.e.**

$\langle u_0(x), u_0(x)\rangle = \int_a^b u_0(x).u_0(x)dx = \int_0^1 1dx = 1$

ii. $G_M(x,t)$ **as a function of 'x' satisfies the D Equation**

$\frac{d^2}{dx^2}G_M(x,t) = u_0(x)u_0(t) = 1$ **in each of the interval** $0 \leq x < t$ **and** $t < x \leq 1$ **therefore we have** $G''_M(x,t) = 1 \Rightarrow G'_M(x,t) = x + A$

$\Rightarrow G_M(x,t) = \frac{x^2}{2} + Ax + B$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & ; 0 \leq x < t \\ \frac{x^2}{2} + A'x + B' & ; t < x \leq 1 \end{cases}$$

iii. $G_M(x,t)$ **satisfies the BC's i.e.** $\Rightarrow G_M(0,t) = G_M(1,t) \Rightarrow A' = A - 1$

and $\Rightarrow G'_M(0,t) = G'_M(1,t) \Rightarrow B' = B - A + \frac{1}{2}$

$$\text{thus } \Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & ; 0 \leq x < t \\ \frac{x^2}{2} + (A-1)x + B - A + \frac{1}{2} & ; t < x \leq 1 \end{cases}$$

iv. $G_M(x,t)$ **is continuous function of 'x' in the interval** $[0, 1]$ **and particularly at x = t i.e.**

$\lim_{x\to t^+} G_M(x,t) = \lim_{x\to t^-} G_M(x,t)$

$$\lim_{x\to t^+}\left(\frac{x^2}{2}+(A-1)x+B-A+\frac{1}{2}\right)=\lim_{x\to t^-}G'_M\left(\frac{x^2}{2}+Ax+B\right)$$

$$\frac{t^2}{2}+(A-1)t+B-A+\frac{1}{2}=\frac{t^2}{2}+At+B\Rightarrow A=\frac{1}{2}-t$$

thus

$$\Rightarrow G_M(x,t)=$$

$$\begin{cases}\frac{x^2}{2}+\left(\frac{1}{2}-t\right)x+B & ;0\le x<t\\ \frac{x^2}{2}+\left(\frac{1}{2}-t-1\right)x+B-\left(\frac{1}{2}-t\right)+\frac{1}{2} & ;t<x\le 1\end{cases}$$

$$\Rightarrow G_M(x,t)=\begin{cases}\frac{x^2}{2}+\left(\frac{1}{2}-t\right)x+B & ;0\le x<t\\ \frac{x^2}{2}-\left(\frac{1}{2}+t\right)x+t+B & ;t<x\le 1\end{cases}$$

v. $G'_M(x,t)=\frac{d}{dx}G_M(x,t)$ **exists and will be discontinuous as** $x\to t$ **i.e.**

$$\lim_{x\to t^+}G'_M(x,t)\neq\lim_{x\to t^-}G'_M(x,t)$$

But $\lim_{x\to t^+}G'_M(x,t)-\lim_{x\to t^-}G'_M(x,t)=\frac{1}{p(t)}$

$$\lim_{x\to t^+}\left(\frac{2x}{2}+\left(\frac{1}{2}-t\right)\right)-\lim_{x\to t^-}\left(\frac{2x}{2}-\left(\frac{1}{2}+t\right)\right)=\frac{1}{1}$$

$$t+\frac{1}{2}-t-t+\frac{1}{2}+t=1\Rightarrow 1=1$$

Thus the discontinuity condition does not help to determining the unknown constant B. so we will use orthogonality condition.

vi. **Using orthogonality condition** $\int_0^1 G_M(x,t).u_0(x)dx=0$

$$\int_0^t G_M(x,t).u_0(x)dx+\int_t^1 G_M(x,t).u_0(x)dx=0=0$$

$$\int_0^t\left(\frac{x^2}{2}+\left(\frac{1}{2}-t\right)x+B\right)dx+\int_t^1\left(\frac{x^2}{2}-\left(\frac{1}{2}+t\right)x+t+B\right)dx=0$$ **with** $u_0(x)=1$

$B=\frac{t^2}{2}-\frac{t}{2}+\frac{1}{12}$ **after solving**

Hence our required Green's function is as follows;

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + \left(\frac{1}{2} - t\right)x + \frac{t^2}{2} - \frac{t}{2} + \frac{1}{12} & ; 0 \le x < t \\ \frac{x^2}{2} - \left(\frac{1}{2} + t\right)x + t + \frac{t^2}{2} - \frac{t}{2} + \frac{1}{12} & ; t < x \le 1 \end{cases}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + \left(\frac{1}{2} - t\right)x + \frac{t^2}{2} - \frac{t}{2} + \frac{1}{12} & ; 0 \le x < t \\ \frac{x^2}{2} - \left(\frac{1}{2} + t\right)x + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12} & ; t < x \le 1 \end{cases}$$

EXAMPLE:

Construct Green's function associated with the problem

$u'' + \lambda u = 0$ with the boundry conditions $u(-1) = u(1)$ and $u'(-1) = u'(1)$

Solution: here $p(x) = 1 = p(t)$

i. **Put $\lambda = 0$ in given equation**

$$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} + (0)u = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = Ax + B$$

$\Rightarrow u'(x) = A$ ……….(i)

Now using BC's $u(-1) = u(1)$ and $u'(-1) = u'(1)$ we have $A = 0, B \neq 0$

$(i) \Rightarrow u(x) = B$ which is non - trivial solution. So $\lambda = 0$ is an eigenvalue.

Therefore we take $u_0(x) = \frac{1}{\sqrt{2}}$ as a normalized function.i.e.

$$\langle u_0(x), u_0(x)\rangle = \int_a^b u_0(x).u_0(x)dx = \int_{-1}^{1} \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} dx = 1$$

ii. **$G_M(x,t)$ as a function of 'x' satisfies the D Equation**

$\frac{d^2}{dx^2} G_M(x,t) = u_0(x)u_0(t) = \frac{1}{2}$ in each of the interval $-1 \le x < t$

and $t < x \le 1$ therefore we have $G''_M(x,t) = \frac{1}{2} \Rightarrow G'_M(x,t) = \frac{1}{2}x + A$

$$\Rightarrow G_M(x,t) = \frac{x^2}{4} + Ax + B$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B & ; -1 \le x < t \\ \frac{x^2}{4} + A'x + B' & ; t < x \le 1 \end{cases}$$

iii. $G_M(x,t)$ satisfies the BC's i.e.

$\Rightarrow G_M(-1,t) = G_M(1,t) \Rightarrow B' = 1 + B - 2A$

and $\Rightarrow G'_M(-1,t) = G'_M(1,t) \Rightarrow A' = A - 1$

$$\text{thus } \Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B & ; -1 \le x < t \\ \frac{x^2}{4} + (A-1)x + 1 + B - 2A & ; t < x \le 1 \end{cases}$$

iv. $G_M(x,t)$ is continuous function of 'x' in the interval $[-1, 1]$ and particularly at x = t i.e.

$\lim_{x\to t^+} G_M(x,t) = \lim_{x\to t^-} G_M(x,t)$

$\lim_{x\to t^+} \left(\frac{x^2}{4} + (A-1)x + 1 + B - 2A\right) = \lim_{x\to t^-} G'_M\left(\frac{x^2}{4} + Ax + B\right)$

$\frac{t^2}{4} + (A-1)t + 1 + B - 2A = \frac{t^2}{4} + At + B \Rightarrow A = \frac{1-t}{2}$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + B & ; -1 \le x < t \\ \frac{x^2}{4} + \left(\frac{1-t}{2} - 1\right)x + 1 + B - 2\left(\frac{1-t}{2}\right) & ; t < x \le 1 \end{cases}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + B & ; -1 \le x < t \\ \frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + t + B & ; t < x \le 1 \end{cases}$$

v. $G'_M(x,t) = \frac{d}{dx}G_M(x,t)$ exists and will be discontinuous as $x \to t$ and gives no information about unknown. So we will use orthogonality condition.

vi. Using orthogonality condition $\int_{-1}^{1} G_M(x,t).u_0(x)dx = 0$

$\int_{-1}^{t} G_M(x,t).u_0(x)dx + \int_{t}^{1} G_M(x,t).u_0(x)dx = 0 = 0$

$\int_{-1}^{t}\left(\frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + B\right)dx + \int_{t}^{1}\left(\frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + t + B\right)dx = 0$ with

$u_0(x) = \frac{1}{\sqrt{2}} \neq 0$

$B = \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6}$ after solving

Hence our required Green's function is as follows;

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} & ; -1 \le x < t \\ \frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + t + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} & ; t < x \le 1 \end{cases}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1-t}{2}\right)x + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{6} & ; -1 \le x < t \\ \frac{x^2}{4} - \left(\frac{1+t}{2}\right)x + \frac{t^2}{4} + \frac{t}{2} + \frac{1}{6} & ; t < x \le 1 \end{cases}$$

EXAMPLE:

Solve the problem $\frac{d^2u}{dx^2} = f(x)$ **with** $u(0) = \propto, u(l) = \beta$

SOLUTION: Let $G(x,x')$ **be a Green's function for the associated homogeneous equation or BVP. Then it satisfies the equation**

$\frac{d^2G}{dx^2} = \delta(x - x')$ **…………(i) with** $G(0,x') = 0 = G(l,x')$ **therefore**

$$\Rightarrow G(x,x') = \begin{cases} -\frac{x}{l}(l - x') & ; 0 \le x < x' \\ -\frac{x'}{l}(l - x) & ; x' < x \le l \end{cases}$$

Since from Lagrange's identity

$\int_a^b [uL(v) - vL(u)]dx = \left|p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\right|_a^b$ **…………(ii)**

By comparing given equation with SL equation w get

$p(x) = 1, q(x) = 0$ **and from BC's** $a = 0, b = l$

And $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x) = \frac{d}{dx}\left\{1.\frac{d}{dx}\right\} + 0 = \frac{d^2}{dx^2}$

Then $(ii) \Rightarrow \int_0^l \left[u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right]dx = \left|1\big(u(x)v'(x) - u'(x)v(x)\big)\right|_0^l$

Take $v(x) = G(x,x')$

$\Rightarrow \int_0^l \left[u\frac{d^2G}{dx^2} - G\frac{d^2u}{dx^2}\right]dx = \left|\big(u(x)G'(x,x') - u'(x)G'(x,x')\big)\right|_0^l$

Since from (i) $\frac{d^2G}{dx^2} = \delta(x - x')$ **and also given** $\frac{d^2u}{dx^2} = f(x)$ **therefore**

$$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx = \left|\left(u(x)G'(x,x') - u'(x)G'(x,x')\right)\right|_0^l$$

$$= \left(u(l)G'(l,x') - u'(l)G'(l,x')\right) - \left(u(0)G'(0,x') - u'(0)G'(0,x')\right) \text{ ....(iii)}$$

Now using $G(x,x') = \begin{cases} -\frac{x}{l}(l-x') & ; 0 \le x < x' \\ -\frac{x'}{l}(l-x) & ; x' < x \le l \end{cases}$ **and** $u(0) = \propto, u(l) = \beta$

$$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx$$

$$= \left(\beta\left(\frac{x'}{l}\right) - \beta'\left(-\frac{x'}{l}\right)(l-l)\right) - \left(\propto\left(-\frac{1}{l}\right)(l-x') - \propto'\left(-\frac{0}{l}\right)(l-x')\right)$$

$$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx = \beta\left(\frac{x'}{l}\right) + \frac{\propto}{l}(l-x')$$

$$\Rightarrow \int_0^l [u\delta(x-x') - Gf(x)]dx = (\beta - \propto)\frac{x'}{l} + \propto \text{ ....(iv)}$$

Now using property of dirac delta

$$\int \delta(x-x')f(x)dx = f(x') \Rightarrow \int \delta(x-x')u(x)dx = u(x')$$

$$(iv) \Rightarrow \int_0^l u(x)\delta(x-x')dx - \int_0^l G(x,x')f(x)dx = (\beta - \propto)\frac{x'}{l} + \propto$$

$$\Rightarrow u(x') - \int_0^l G(x,x')f(x)dx = (\beta - \propto)\frac{x'}{l} + \propto$$

$$\Rightarrow u(x') = \int_0^l G(x,x')f(x)dx + (\beta - \propto)\frac{x'}{l} + \propto$$

$$\Rightarrow u(x) = \int_0^l G(x'',x)f(x'')dx'' + (\beta - \propto)\frac{x}{l} + \propto$$

Where we replace x'**, with** x **and** x **with** x''

EXAMPLE: Determines the Green's function for the exterior dirichlet problem for a unit circle $\nabla^2 u = 0, r > 1; u = f, r = 1$

Solution: Consider Green's function assume the form

$G(\xi, \eta; x, y) = f(\xi, \eta; x, y) + g(\xi, \eta; x, y)$

where $f(\xi, \eta; x, y)$ known as free space Green's function satisfies

$\nabla^2 f = \delta(\xi - x, \eta - y)$ in domain D and $g(\xi, \eta; x, y)$ satisfies $\nabla^2 g = 0$

so that by superposition $G = f + g$ satisfies the equation

$\nabla^2 G = \delta(\xi - x, \eta - y)$ in domain D

Also $G = 0$ on boundries requires that $g = -f$ on boundries.

Now for Laplace operator f must satisfies $\nabla^2 f = \delta(\xi - x, \eta - y)$ in domain D then for $r = 1$ we have

$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) = 0$ and solution will be $f = c_1 + c_2 logr$

Now applying the condition $\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\partial G}{\partial n}ds = 1$ where n is outward normal to the circle and $C_\epsilon = (\xi - x)^2 + (\eta - y)^2 = \epsilon^2$ We get $f = \frac{1}{2\pi} logr$

Now if we introduce the polar coordinates $\rho, \theta, \sigma, \beta$ by means of the equations

$x = \rho Cos\theta, y = \rho Sin\theta, \xi = \sigma Cos\beta, x = \sigma Sin\beta$

We get $g(\sigma, \beta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\sigma^n(a_n Cosn\beta + b_n Sinn\beta)$

Where $g = \frac{1}{4\pi} log\left(1 + \rho^2 - 2\rho Cos(\beta - \theta)\right)$ on boundry

Now by using the relation $log\left(1 + \rho^2 - 2\rho Cos(\beta - \theta)\right) = 2\sum_{n=1}^{\infty}\frac{\rho^n Cosn(\beta-\theta)}{n}$ and equating the coefficients of $Cosn\beta, Sinn\beta$ to determine a_n, b_n we find $a_n = \frac{\rho^n Cosn\theta}{2\pi n}, b_n = \frac{\rho^n Sinn\theta}{2\pi n}$

It therefor follows that $g(\rho, \theta, \sigma, \beta) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(\rho\sigma)^n Cosn(\beta-\theta)}{n}$

$g(\rho, \theta, \sigma, \beta) = \frac{1}{4\pi} log\left(1 + (\rho\sigma)^2 - 2\rho\sigma Cos(\beta - \theta)\right)$

Hence the required Green's function is as follows;

$$G(\rho, \theta; \sigma, \beta) = \frac{1}{4\pi} log\left(\rho^2 + \sigma^2 - 2\rho\sigma Cos(\beta - \theta)\right) - \frac{1}{4\pi} log\left(1 + (\rho\sigma)^2 - 2\rho\sigma Cos(\beta - \theta)\right)$$

VARIATIONAL METHODS

The subject of calculus of variation or variational method is similar to but more general than the subject of maxima and minima in Calculus.

FUNCTIONAL:

Let M be the set of functions defined over the interval [a,b]

i.e. $\boldsymbol{M = \{f \mid f: [a,b] \to \mathbb{R}\}}$ **such that each function is integrable then a rule of function** $\boldsymbol{I: M \to \mathbb{R}}$ **defined by** $\boldsymbol{I[f(x)] = J \epsilon \mathbb{R}}$ **is called functional.**

STATIONARY VALUE:

The maximum or minimum value of the function or functional is called stationay value OR the point at which the 1st derivative of a function or functional become zero is called Stationary value.

EXTERMAL:

The curve $\boldsymbol{y = f(x)}$ **along which the functional 'I' takes the stationary values is called extermal. i.e. if** $\boldsymbol{\delta I[f(x)] = 0}$ **then** $\boldsymbol{y = f(x)}$ **is extermal curve.**

SOME EXAMPLES OF VARIATIONAL PROBLEMS:

Here we discuss some important problems whose attempted solutions have led to the development of the subject of Calculus of Variation.

Historically there are three such problems;

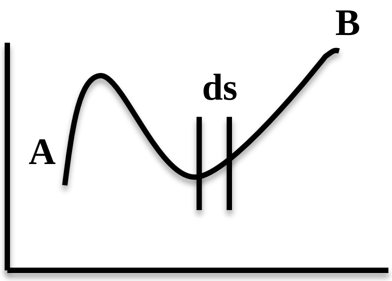
i. **The problems of geodesics: i.e. to find the cuve of minimum length joining two points on given surface.**
ii. **The brachistochrone problems: i.e. to find the path of quickest descent, joining two points in spacew, for a particle moving under gravity.**
iii. **Dido's problems: i.e. the problem of findind curve of given length which encloses maximum area by itself or with a given straight line.**

GEOSDESICS PROBLEM:

Find the curve whose distance between two points is minimum.

EXPLANATION: **Let $y = y(x)$ be a curve C on the surface S which is represented by $z = z(x, y)$. Then suppose that A and B be the two points on the curve C. then distance (length) between two points A and B is given by**

$l = \int_A^B ds$ ……………(i)



In the case of any surface $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

Since curve lies in xy – plane therefore z = 0 then we get

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (y')^2}dx$$

$(i) \Rightarrow l = \int_A^B ds = \int_A^B \sqrt{1 + (y')^2}dx$ **this is our required length.**

BRACHISTOCHRONE PROBLEM:

A particle falls under gravity from A to B. determine the curve along which the time taken by the particle will be minimum.

EXPLANATION: **Consider a particle falls under gravity from A to B . then instantaneous velocity is given by** $V = \frac{ds}{dt}$ **and time taken between A and B is given by** $total\ time = \int_A^B dt = \int_A^B \frac{dt}{ds} ds = \int_A^B \frac{1}{\frac{ds}{dt}} ds = \int_A^B \frac{1}{V} ds$ ……….(i)

Now using 3rd equation of motion under gravity we get $V = \sqrt{2gy}$

$$(i) \Rightarrow total\ time = \frac{1}{\sqrt{2g}} \int_A^B \frac{1}{\sqrt{y}} \sqrt{1 + (y')^2}dx = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1 + (y')^2}{y}}\, dx$$ **required.**

DIDO's PROBLEM:

Find the closed curve of given length which enclosed maximum area.

EXPLANATION:

Suppose that $y = y(x)$ is the curve which meet the x – axis at points x_1 and x_2 and enclosed maximum area $A = \int_{x_1}^{x_2} y dx$ and and the length of the same curve given as $l = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$ then the problem reduces to that of maximizing the area in equation $A = \int_{x_1}^{x_2} y dx$ subject to the condition given in $l = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$

$$(\boldsymbol{Remember})$$

Discuss 3 well known problmes, viz., geodesic, brachistochrone and dido' and formulate them as variational problems.

FUNDAMENTAL THEOREM ON VARIATIONAL CALCULUS:

If $f(x)$ is continuous function in the interval (x_1, x_2) and the integral $\int_{x_1}^{x_2} f(x)g(x)dx$ is identically zero. i.e. $\int_{x_1}^{x_2} f(x)g(x)dx \equiv 0$ where $g(x)$ satisfies the following conditions;

i. **It is an arbitrary function with continuous derivatives in the interval (x_1, x_2)**
ii. **$g(x_1) = g(x_2) = 0$**

Then $f(x) \equiv 0$ for all $x \epsilon [x_1, x_2]$

PROOF: We prove by contradiction. If possible let $f(x) \neq 0$ in (x_1, x_2). Then there is at least one point x_0 in (x_1, x_2) such that $f(x_0) \neq 0$. Then because of continuity of $f(x)$ in (x_1, x_2) there must exists an interval $(x_0 - \delta, x_0 + \delta)$ where $\delta > 0$ surrounding x_0 such that $f(x) > 0$ for all $x \epsilon [x_0 - \delta, x_0 + \delta]$

Since $g(x)$ is arbitrary, it can be taken as

$$g(x) = \begin{cases} (x - x_0 + \delta)^2 (x - x_0 - \delta)^2 & if\ x \epsilon [x_0 - \delta, x_0 + \delta] \\ 0 & otherwise \end{cases}$$

It is clear that $g(x) = 0$ at the endpoints of the interval $(x_0 - \delta, x_0 + \delta)$ and has continuous derivative inside the interval. Then integral $\int_{x_1}^{x_2} f(x) g(x) dx$ becomes $\int_{x_0-\delta}^{x_0-\delta} f(x)(x - x_0 + \delta)^2 (x - x_0 - \delta)^2 dx > 0$

This is contradiction, as $\int_{x_1}^{x_2} f(x) g(x) dx = 0$

Hence $f(x) \equiv 0$ for all $x \epsilon [x_1, x_2]$

EULER LAGRANGE's EQUATION:

Let $I = \int_{x_1}^{x_2} F(x, y, y') dx$ where $y = y(x)$ is a continuous function having continuous 1st and 2nd order derivatives satisfying the following endpoint condtions $y_1 = y(x_1)$ and $y_2 = y(x_2)$, also if F is supposed to be have continuous 1st and 2nd order derivatives w.r.to its arguments, then the function $y = y(x)$ will extremise the given integral if it satisfies the following DE

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

PROOF: given that $I = \int_{x_1}^{x_2} F(x, y, y') dx$

$$\delta I = \int_{x_1}^{x_2} \delta F(x, y, y') dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y'\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \delta y' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \delta \left(\frac{dy}{dx}\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + \left[\left| \frac{\partial F}{\partial y'} (\delta y) \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx \right]$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx - \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx \qquad \text{since } \delta y(x_1) = 0 = \delta y(x_2)$$

For extermal curve $\delta I = 0$ then $\int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx - \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx = 0$

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y dx = 0$$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ $\quad\quad$ $\delta y \neq 0, dx \neq 0$ **being orbitrary values.**

SPECIAL CASES:

i. **When F is independent of 'y''**

Then $\frac{\partial F}{\partial y'} = 0$ **then EL equation becomes as follows;**

$\frac{\partial F}{\partial y} = 0$ **this is an algebraic equation in 'x' and 'y'. the solution may not satisfy the given boundry conditions.**

ii. **When F is independent of 'y'**

Then $\frac{\partial F}{\partial y} = 0$ **then EL equation becomes as follows;**

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = Constant$$

iii. **When F is independent of 'x'**

Then $\frac{\partial F}{\partial x} = 0$ **then EL equation becomes as follows;**

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} \Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right) y'$$

$$\Rightarrow \left(\frac{\partial F}{\partial y}\right) dy = d\left(\frac{\partial F}{\partial y'}\right) y' \quad \ldots\ldots\ldots\ldots(i)$$

Since $F = F(y, y') \Rightarrow dF = \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy'$

$\Rightarrow dF = d\left(\frac{\partial F}{\partial y'}\right) y' + \frac{\partial F}{\partial y'} dy'$ $\quad\quad$ **by (i)**

$$\Rightarrow dF = d\left(y' \frac{\partial F}{\partial y'}\right) \Rightarrow d\left(F - y' \frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow F - y' \frac{\partial F}{\partial y'} = constant$$

iv. Suppose 'F' is linear function in y'

i.e. $F(x,y,y') = M(x,y) + N(x,y)y'$ **………(i)**

$(i) \Rightarrow \frac{\partial F}{\partial y} = \left(\frac{\partial M}{\partial x}\frac{dx}{dy} + \frac{\partial M}{\partial y}\frac{dy}{dy}\right) + \left(\frac{\partial N}{\partial x}\frac{dx}{dy} + \frac{\partial N}{\partial y}\frac{dy}{dy}\right)y'$

$\Rightarrow \frac{\partial F}{\partial y} = \left(\frac{\partial M}{\partial y}\right) + \left(\frac{\partial N}{\partial y}\right)y'$ **………(ii)**

Again $(i) \Rightarrow \frac{\partial F}{\partial y'} = N(x,y)$

$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\big(N(x,y)\big) = \frac{\partial N}{\partial x}\frac{dx}{dx} + \frac{\partial N}{\partial y}\frac{dy}{dx} = \frac{\partial N}{\partial x} + \frac{\partial N}{\partial y}y'$ **………(iii)**

now as $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \left(\frac{\partial M}{\partial y}\right) + \left(\frac{\partial N}{\partial y}\right)y' - \frac{\partial N}{\partial x} - \frac{\partial N}{\partial y}y' = 0$

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow M_y(x,y) = N_x(x,y)$ **this is not a DE which may not satisfy the given boundry conditions.**

EULER's LAGRANGE EQUATION IS SECOND ORDER DE

As we know that $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ **……….(i)**

Since $F = F(x,y,y')$ **then** $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial y'}$ **are also functions of** x, y and y'

Then by using chain rule

$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\frac{dx}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx}$

$\Rightarrow \frac{\partial F}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)y' + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)y''$

$\Rightarrow F_y = F_{xy'} + F_{yy'}y' + F_{y'y'}y''$ **which is 2nd oder Differential equation.**

EXTENSION OF EULER LAGRANGE's EQUATION WITH ONE INDEPENDENT VARIABLE AND MANY DEPENDENT VARIABLES:

Let $I = \int_{x_1}^{x_2} F(x, y_k, y_k{}')dx$; $k = 1, 2, 3, \dots \dots . n$ **with the stationary condtions** $y_k(x_1) = constant$ **and** $y_k(x_2) = constant,$ **then Euler's Lagrange's equation can be written as** $\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k{}'}\right) = 0$

PROOF: given that $I = \int_{x_1}^{x_2} F(x, y_k, y_k{}')dx$

$$\delta I = \int_{x_1}^{x_2} \delta F(x, y_k, y_k{}')dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k}\delta y_k + \frac{\partial F}{\partial y_{k'}}\delta y_k{}'\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y_{k'}}\delta y_k{}' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y_{k'}}\delta\left(\frac{dy_k}{dx}\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y_{k'}}\frac{d}{dx}(\delta y_k) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx + \left[\left|\frac{\partial F}{\partial y_{k'}}(\delta y_k)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}(\delta y_k)\frac{d}{dx}\left(\frac{\partial F}{\partial y_{k'}}\right) dx\right]$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y dx - \int_{x_1}^{x_2}(\delta y_k)\frac{d}{dx}\left(\frac{\partial F}{\partial y_{k'}}\right) dx \qquad \text{since } \delta y_k(x_1) = 0 = \delta y_k(x_2)$$

For extermal curve $\delta I = 0$ **then** $\int_{x_1}^{x_2} \frac{\partial F}{\partial y_k}\delta y_k dx - \int_{x_1}^{x_2}(\delta y_k)\frac{d}{dx}\left(\frac{\partial F}{\partial y_{k'}}\right) dx = 0$

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k{}'}\right)\right]\delta y_k dx = 0$$

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k{}'}\right) = 0 \ ; \ k = 1, 2, 3, \dots \dots . n$$

$\delta y_k \neq 0, dx \neq 0$ **being orbitrary values.**

EXAMPLE: Let $I = \int_{x_1}^{x_2} F(x, \varphi, \psi, \varphi', \psi')dx$ **with the stationary condtions** $\delta\varphi(x_1) = \delta\varphi(x_2) = 0$ **and** $\delta\psi(x_1) = \delta\psi(x_2) = 0$ **then**

$$\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right) = 0$$

PROOF: given that $I = \int_{x_1}^{x_2} F(x, \varphi, \psi, \varphi', \psi')dx$

$$\delta I = \int_{x_1}^{x_2} \delta F(x, \varphi, \psi, \varphi', \psi')dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \varphi}\delta\varphi + \frac{\partial F}{\partial \psi}\delta\psi + \frac{\partial F}{\partial \varphi'}\delta\varphi' + \frac{\partial F}{\partial \psi'}\delta\psi'\right)dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi}\delta\varphi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi'}\delta\varphi' dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi}\delta\psi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi'}\delta\psi' dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi}\delta\varphi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi'}\delta\left(\frac{d\varphi}{dx}\right)dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi}\delta\psi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi'}\delta\left(\frac{d\psi}{dx}\right)dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi}\delta\varphi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi}\delta\psi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi'}\frac{d}{dx}(\delta\varphi)dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi'}\frac{d}{dx}(\delta\psi)dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi}\delta\varphi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi}\delta\psi dx + \left[\left|\frac{\partial F}{\partial \varphi'}(\delta\varphi)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}(\delta\varphi)\frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)dx\right] + \left[\left|\frac{\partial F}{\partial \psi'}(\delta\psi)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}(\delta\psi)\frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)dx\right]$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varphi}\delta\varphi dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \psi}\delta\psi dx - \int_{x_1}^{x_2}(\delta\varphi)\frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)dx - \int_{x_1}^{x_2}(\delta\psi)\frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)dx$$

since $\delta\varphi(x_1) = \delta\varphi(x_2) = 0$ **and** $\delta\psi(x_1) = \delta\psi(x_2) = 0$

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)\right)\delta\varphi dx + \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\right)\delta\psi dx$$

For extermal curve $\delta I = 0$ **then**

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)\right)\delta\varphi dx + \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\right)\delta\psi dx = 0$$

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)\right)\delta\varphi dx = 0 \; and \; \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\right)\delta\psi dx = 0$$

$$\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right) = 0 \;\; \text{and} \; \frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right) = 0$$

$\delta\varphi \neq 0, dx \neq 0, \delta\psi \neq 0$ **being orbitrary values.**

EXTENSION OF EULER LAGRANGE's EQUATION WITH ONE INDEPENDENT VARIABLE AND ONE DEPENDENT VARIABLE WITH ITS HIGHER ORDER DERIVATIVES:

Let $I = \int_{x_1}^{x_2} F\left(x, y, y', y'', y''', \dots \dots y^{(n)}\right) dx$ **with the stationary condtions**

$y(x_1) = y'(x_1) = y''(x_1) = \cdots \dots \dots \dots \dots = y^{(n)}(x_1) = constant$ **and**

$y(x_2) = y'(x_2) = y''(x_2) = \cdots \dots \dots \dots = y^{(n)}(x_2) = constant$**, then Euler's Lagrange's equation can be written as**

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \cdots \dots \dots \dots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

PROOF: given that $I = \int_{x_1}^{x_2} F\left(x, y, y', y'', y''', \dots \dots y^{(n)}\right) dx$

$$\delta I = \int_{x_1}^{x_2} \delta F\left(x, y, y', y'', y''', \dots \dots y^{(n)}\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' + \cdots \dots \dots \dots + \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)}\right) dx$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx + \cdots \dots \dots + \int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)} dx \qquad \dots\dots\dots\dots\dots(i)$$

Consider $\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta\left(\frac{dy}{dx}\right) dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y) dx$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = \left|\frac{\partial F}{\partial y'}(\delta y)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = -\int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx \qquad \textbf{since } \delta y(x_1) = 0 = \delta y(x_2)$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = (-1)^1 \int_{x_1}^{x_2} (\delta y)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

Also $\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta\left(\frac{dy'}{dx}\right) dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\frac{d}{dx}(\delta y') dx$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx = \left|\frac{\partial F}{\partial y''}(\delta y')\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} (\delta y')\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' dx = -\int_{x_1}^{x_2} (\delta y')\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) dx \qquad \textbf{since } \delta y(x_1) = 0 = \delta y(x_2)$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' dx = \left| -\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)(\delta y) \right|_{x_1}^{x_2} + (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' dx = (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx$$

Similarly $\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(n)}} \delta y^{(n)} dx = (-1)^n \int_{x_1}^{x_2} (\delta y) \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) dx$

Then equation (i) becomes

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + (-1)^1 \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx + (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx + \cdots \ldots \ldots + (-1)^n \int_{x_1}^{x_2} (\delta y) \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) dx$$

For extermal curve $\delta I = 0$ then

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y dx + (-1)^1 \int_{x_1}^{x_2} (\delta y) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx + (-1)^2 \int_{x_1}^{x_2} (\delta y) \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx + \cdots \ldots \ldots + (-1)^n \int_{x_1}^{x_2} (\delta y) \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) dx = 0$$

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} + (-1)^1 \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)\right] \delta y dx = 0$$

$$\frac{\partial F}{\partial y} + (-1) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \cdots \ldots \ldots \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

$\delta y \neq 0, dx \neq 0$ being orbitrary values.

EULER LAGRANGE's EQUATION WITH TWO INDEPENDENT VARIABLES:

Let $I = \iint_R F(x, y, u, u_x, u_y)dxdy$ **then Euler's Lagrange's equation can be written as** $\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$

PROOF: given that $I = \iint_R F(x, y, u, u_x, u_y)dxdy$

$$\delta I = \iint_R \delta F(x, y, u, u_x, u_y)dxdy$$

$$\delta I = \iint_R \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y\right)dxdy \qquad \text{..........(i)}$$

Consider $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u)$

$$\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

$$\frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

Similarly $\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$

$$(i) \Rightarrow \delta I = \iint_R \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u\right)dxdy$$

$$\Rightarrow \delta I = \iint_R \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)dudxdy + \iint_R \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right)dxdy$$

$$\Rightarrow \delta I = I_1 + I_2 \qquad \text{..........(ii)}$$

Consider $I_2 = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right)dxdy$

$$I_2 = \oint_c -\left(\frac{\partial F}{\partial u_y}\delta u dx\right) + \left(\frac{\partial F}{\partial u_x}\delta u dy\right) = \oint_c \left(\frac{\partial F}{\partial u_x}dy - \frac{\partial F}{\partial u_y}dx\right)\delta u \quad \text{by Green's theorem}$$

Since u is prescribed on the boundry therefore due to the closed curve δu must be zero. i.e. $I_2 = 0$

$$(ii) \Rightarrow \delta I = \iint_R \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)dudxdy$$

$$\Rightarrow \iint_R \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)dudxdy = 0 \quad \text{for extermal curve } \delta I = 0$$

$$\Rightarrow \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \qquad \text{since } du \neq 0, dx \neq 0, dy \neq 0$$

Hence required.

EULER LAGRANGE's EQUATION WITH THREE INDEPENDENT VARIABLES:

Let $I = \iiint_V F(x, y, z, u, u_x, u_y, u_z)dxdydz$ then Euler's Lagrange's equation can be written as $\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$

PROOF:

given that $I = \iiint_V F(x, y, z, u, u_x, u_y, u_z)dxdydz$

$\delta I = \iiint_V \delta F(x, y, z, u, u_x, u_y, u_z)dxdydz$

$\delta I = \iiint_V \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z\right)dxdydz$ ....(i)

Consider $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u)$

$\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$

$\frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$

Similarly $\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$

And $\frac{\partial F}{\partial u_z}\delta u_z = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u$

$(i) \Rightarrow \delta I = \iiint_V \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u\right)dxdydz$

$\Rightarrow \delta I = \iiint_V \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right)dudxdy + \iiint_V \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right)dxdydz$

$\Rightarrow \delta I = I_1 + I_2$ ....(ii)

Consider $I_2 = \iiint_V \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right)dxdydz$

$I_2 = \iiint_V \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial F}{\partial u_x}\delta u\hat{\imath} + \frac{\partial F}{\partial u_y}\delta u\hat{\jmath} + \frac{\partial F}{\partial u_z}\delta u\hat{k}\right)dv$

$I_2 = \iiint_V \nabla.\vec{G}dv$

$\Rightarrow I_2 = \oiint_s \vec{G}.\vec{n}ds$ **by divergence theorem.**

Since u is prescribed on the boundry therefore due to the closed curve δu must be zero. i.e. $I_2 = 0$

$(ii) \Rightarrow \delta I = \iiint_V \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right)dudxdy$

$\Rightarrow \iiint_V \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right)dudxdy = 0$ **for extermal curve** $\delta I = 0$

$\Rightarrow \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$ **since** $du \neq 0, dx \neq 0, dy \neq 0, dz \neq 0$

Hence required.

PLATEAU'S PROBLEM: (Problem of minimal surface)

In this problem we will find the surface of minimal area which is bounded by a given closed curve.

EXPLANATION:

Consider a surface $z = z(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$ then 1st fundamental form of given surface is

$(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2$

Where $= \vec{r}_u.\vec{r}_u = |\vec{r}_u|^2$, $F = \vec{r}_u.\vec{r}_v$, $G = \vec{r}_v.\vec{r}_v = |\vec{r}_v|^2$ **are fundamental quantities of the surface. If we take parameters (x,y) and put** $u = x, v = y$ **then**

$E = |\vec{r}_x|^2 = \left|\frac{\partial x}{\partial x}\hat{\imath} + \frac{\partial y}{\partial x}\hat{\jmath} + \frac{\partial z}{\partial x}\hat{k}\right|^2 = \left|1\hat{\imath} + 0\hat{\jmath} + z_x\hat{k}\right|^2 = \left(\sqrt{|1 + {z_x}^2|}\right)^2$

$E = 1 + {z_x}^2$

$G = \left|\vec{r}_y\right|^2 = \left|\frac{\partial x}{\partial y}\hat{\imath} + \frac{\partial y}{\partial y}\hat{\jmath} + \frac{\partial z}{\partial y}\hat{k}\right|^2 = \left|0\hat{\imath} + 1\hat{\jmath} + z_y\hat{k}\right|^2 = \left(\sqrt{\left|1 + {z_y}^2\right|}\right)^2$

$G = 1 + {z_y}^2$

$F = \vec{r}_x.\vec{r}_y = \left(\frac{\partial x}{\partial x}\hat{\imath} + \frac{\partial y}{\partial x}\hat{\jmath} + \frac{\partial z}{\partial x}\hat{k}\right)\left(\frac{\partial x}{\partial y}\hat{\imath} + \frac{\partial y}{\partial y}\hat{\jmath} + \frac{\partial z}{\partial y}\hat{k}\right) = \left(1\hat{\imath} + z_x\hat{k}\right)\left(1\hat{\jmath} + z_y\hat{k}\right)$

$F = z_x z_y$

Put v = constant then $(ds_1)^2 = E(du)^2 \Rightarrow ds = \sqrt{E}du$

Put u = constant then $(ds_2)^2 = G(dv)^2 \Rightarrow ds = \sqrt{G}dv$

Then $ds = |ds_1 \times ds_2| = |ds_1||ds_2|Sin\theta$

$ds = \sqrt{E}du\sqrt{G}dvSin\theta \Rightarrow ds = \sqrt{EG}dudvSin\theta$ **.........(i)**

if $Cos\theta = \frac{F}{\sqrt{EG}}$ **and** $Sin\theta = \sqrt{1 - Cos^2\theta} = \frac{\sqrt{EG-F^2}}{\sqrt{EG}}$

$(i) \Rightarrow ds = \sqrt{EG - F^2}dxdy$

$\Rightarrow s = \iiint \sqrt{EG - F^2}dxdy$

$$\Rightarrow s = \iiint \sqrt{(1+z_x{}^2)(1+z_y{}^2)-(z_x z_y)^2}\,dxdy$$

$$\Rightarrow s = \iiint \sqrt{1+z_x{}^2+z_y{}^2}\,dxdy$$

Now let $F = F(x,y,z,z_x,z_y) = \sqrt{1+z_x{}^2+z_y{}^2}$

Then by using EL equation for two independent variables

$$\frac{\partial F}{\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right)=0$$

$$\Rightarrow \frac{\partial}{\partial z}\sqrt{1+z_x{}^2+z_y{}^2}-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z_x}\sqrt{1+z_x{}^2+z_y{}^2}\right)-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_y}\sqrt{1+z_x{}^2+z_y{}^2}\right)=0$$

$$\Rightarrow 0-\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x{}^2+z_y{}^2}}\right)-\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x{}^2+z_y{}^2}}\right)=0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x{}^2+z_y{}^2}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x{}^2+z_y{}^2}}\right)=0$$

$$\Rightarrow \left(\frac{\left(\sqrt{1+z_x{}^2+z_y{}^2}\right)z_{xx}-\frac{z_x z_{xx}}{\sqrt{1+z_x{}^2+z_y{}^2}}z_x}{\left(\sqrt{1+z_x{}^2+z_y{}^2}\right)^2}\right)+\left(\frac{\left(\sqrt{1+z_x{}^2+z_y{}^2}\right)z_{yy}-\frac{z_y z_{yy}}{\sqrt{1+z_x{}^2+z_y{}^2}}z_y}{\left(\sqrt{1+z_x{}^2+z_y{}^2}\right)^2}\right)=0$$

$$\Rightarrow \left(\frac{(1+z_x{}^2+z_y{}^2)z_{xx}-z_x{}^2z_{xx}}{(1+z_x{}^2+z_y{}^2)^{3/2}}\right)+\left(\frac{(1+z_x{}^2+z_y{}^2)z_{yy}-z_y{}^2z_{yy}}{(1+z_x{}^2+z_y{}^2)^{3/2}}\right)=0$$

$$\Rightarrow (1+z_x{}^2+z_y{}^2)z_{xx}-z_x{}^2z_{xx}+(1+z_x{}^2+z_y{}^2)z_{yy}-z_y{}^2z_{yy}=0$$

$\Rightarrow (1+z_y{}^2)z_{xx}+(1+z_x{}^2)z_{yy}=0$ **this is our required.**

CONSTRAIN EXTREMA OR PROBLEMS WITH CONSTRAINTS OR VARIATIONAL PROBLEMS WITH SIDE CONDITIONS OR ISOPERIMETRIC PROBLEMS:

To find the stationary value of a functional $I = \int_{x_1}^{x_2} F(x, y_k, y'_k)dx$ where the argument of F are subjected to constraints or additional conditions such as

i. $G(x, y_k) = constant$
ii. $G(x, y_k, y'_k) = constant$
iii. $\int_{x_1}^{x_2} G(x, y_k, y'_k)dx = constant$

Then we construct a new function involving parameter λ i.e. $H = F + \lambda G$

EULER LAGRANGE EQUATION FOR CONSTRAIN EXTREMA

The extermal curves $y_k = y_k(x)$; $k = 1, 2, 3, \dots \dots \dots .., n$ of the functional

$I = \int_{x_1}^{x_2} F(x, y_k, y'_k)dx$ **with constraints**

$G_j(x, y_k) = constant$; $j = 1, 2, .., n$ **………….(i)**

Then $J = \int_{x_1}^{x_2} F(x, y_k, y'_k)dx + \sum_{i=1}^{m} \lambda_i \int_{x_1}^{x_2} G_i(x, y_k)dx$

$J = \int_{x_1}^{x_2} \left(F(x, y_k, y'_k) + \sum_{i=1}^{m} \lambda_i\, G_i(x, y_k)\right) dx = \int_{x_1}^{x_2} H\, dx$

With $F(x, y_k, y'_k) + \sum_{i=1}^{m} \lambda_i\, G_i(x, y_k) = H$ where $\lambda_i = \lambda_i(x)$ are suitably choosen multiplier. It is clear that the Euler Lagrange's equation in this case will be $\frac{\partial H}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'_k}\right) = 0$; $k = 1, 2, 3, \dots \dots \dots .., n$ **………….(ii)**

Then the curves $y_k = y_k(x)$; $k = 1, 2, 3, \dots \dots \dots .., n$ can be obtained from both equations.i.e. (i) and (ii)

GEODESIC:

A geodesic is the curve of shortest length joining two points in space.

EXAMPLE:

Prove that a straight line is the shortest distance between two points in the plane.

PROOF: Since this is the geodesic problem therefore we use the functional

$I = \int_a^b \sqrt{1+(y')^2}\,dx$ **with** $F = F(x, y, y') = \sqrt{1+(y')^2}$

Since F is not depend on 'y' therefore we use following EL equation;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = 0 \; and \; \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y'} = Constant = C$$

$$\Rightarrow \frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2}\right) = C$$

$$\Rightarrow \frac{y'}{\sqrt{1+(y')^2}} = C \Rightarrow y' = C\sqrt{1+(y')^2}$$

$$\Rightarrow (y')^2 = C^2(1+(y')^2) = C^2 + C^2(y')^2$$

$$\Rightarrow (y')^2 - C^2(y')^2 = C^2 \Rightarrow (1-C^2)(y')^2 = C^2$$

$$\Rightarrow (y')^2 = \frac{C^2}{1-C^2} \Rightarrow y' = \sqrt{\frac{C^2}{1-C^2}}$$

$\Rightarrow y' = \frac{dy}{dx} = a \;\; (say)$ **where** $a = \sqrt{\frac{C^2}{1-C^2}}$

$$\Rightarrow \int \frac{dy}{dx}dx = \int a\,dx$$

$\Rightarrow y = ax + c$ **which is straight line.**

The applications of the Calculus of Variations in Mechanics are based on employing Principle of Least Action and Hamilton's Principle;
stated as below;

PRINCIPLE OF LEAST ACTION

According to this principle:

Let a particle move in an external field of force which is conservative. If the motion takes place in the interval of the time from t_1 to t_2 where $t_2 > t_1$ then the actual path traced by the particle is the one along which $I = \int_{t_1}^{t_2} Ldt$ is minimum. Where L is the Lagrangian and for a conservative system

$$L = T - V = kinetic\ energy - potential\ energy$$

HAMILTON's PRINCIPLE:

According to this principle:

The path of motion of a rigid body in the time interval $t_2 - t_1$ is such that the integral $A = \int_{t_1}^{t_2} Ldt$ has a stationary value, where L is the Lagrangian.

EXAMPLE:

Find the equation of the path in space down which a particle will fall from one point to another in shortest possible time.

Solution:

This is the Brachistochrone problem, therefore we use the following functional

$I = \int_a^b dt \Rightarrow I = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{1+(y')^2}{y}}\, dt$ **with** $F = F(x, y, y') = \frac{1}{\sqrt{2g}} \sqrt{\frac{1+(y')^2}{y}}$

Since F is not depend on 'x' therefore we use following EL equation;

$$F - y'\left(\frac{\partial F}{\partial y'}\right) = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}} \sqrt{\frac{1+(y')^2}{y}} - y' \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{y}} \frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2}\right) = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\left[\sqrt{\frac{1+(y')^2}{y}} - y'\frac{1}{\sqrt{y}}\frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2}\right)\right] = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\left[\sqrt{\frac{1+(y')^2}{y}} - y'\frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+(y')^2}}\right] = constant$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\left[\sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{y}\sqrt{1+(y')^2}}\right] = constant$$

$$\Rightarrow \sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{y(1+(y')^2)}} = \sqrt{2g}(constant) = a(say)$$

$$\Rightarrow \left(\sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{y(1+(y')^2)}}\right)^2 = a^2$$

$$\Rightarrow \frac{1+(y')^2}{y} + \frac{(y')^4}{y(1+(y')^2)} - 2\left(\sqrt{\frac{1+(y')^2}{y}}\cdot\frac{(y')^2}{\sqrt{y(1+(y')^2)}}\right) = a^2$$

$$\Rightarrow \frac{1+(y')^2}{y} + \frac{(y')^4}{y(1+(y')^2)} - \frac{2(y')^2}{y} = a^2$$

$$\Rightarrow \frac{\left(1+(y')^2\right)^2+(y')^4-2(y')^2\left(1+(y')^2\right)}{y(1+(y')^2)} = a^2 \Rightarrow \frac{1}{y(1+(y')^2)} = a^2 \quad \text{after solving}$$

$$\Rightarrow 1 = a^2y(1+(y')^2) \Rightarrow \frac{1}{a^2y} = (1+(y')^2) \Rightarrow (y')^2 = \frac{1}{a^2y} - 1 = \frac{1-a^2y}{a^2y}$$

$$\Rightarrow y' = \frac{dy}{dx} = \sqrt{\frac{1-a^2y}{a^2y}} \Rightarrow \int \frac{\sqrt{a^2y}}{\sqrt{1-a^2y}}dy = \int dx = x + c$$

Put $a^2y = Sin^2\theta \Rightarrow a^2dy = 2Sin\theta Cos\theta d\theta \Rightarrow dy = \frac{2}{a^2}Sin\theta Cos\theta d\theta$

$$\Rightarrow \int \frac{\sqrt{Sin^2\theta}}{\sqrt{1-Sin^2\theta}}\cdot\frac{2}{a^2}Sin\theta Cos\theta d\theta = x + c \Rightarrow \frac{2}{a^2}\int \frac{Sin\theta}{Cos\theta}\cdot Sin\theta Cos\theta d\theta = x + c$$

$$\Rightarrow \frac{2}{a^2}\int Sin^2\theta\, d\theta = x + c \Rightarrow \frac{2}{a^2}\int \frac{1-Cos2\theta}{2}d\theta = \frac{1}{a^2}\int 1 - Cos2\theta\, d\theta = x + c$$

$$\Rightarrow x = \frac{1}{2a^2}(2\theta - Sin2\theta) + b \quad \text{.........(i)}$$

and $y = \frac{1}{2a^2}(2Sin^2\theta) \Rightarrow y = \frac{1}{2a^2}(1 - Cos2\theta)$ **.........(ii)**

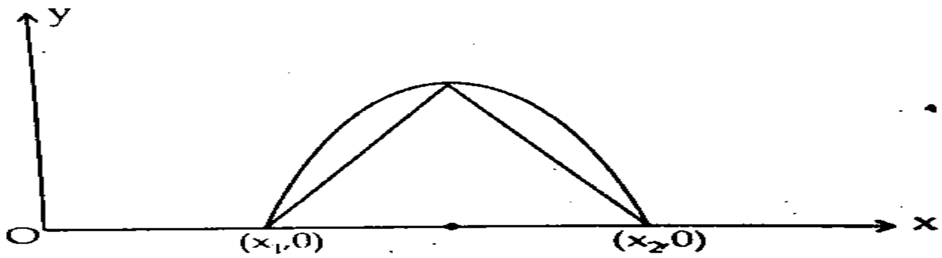
(i) and (ii) are parametric equations of cycloid, where ‘a’ ,’b’ are constants.

Thus the curve downwhich the particle takes the minimum time is cycloid.

DIDO's PROBLEM:

Find the closed curve of given length which enclosed maximum area.

EXPLANATION:



Suppose that $y = y(x)$ is the curve which meet the x – axis at points $A(x_1, 0)$ and $B(x_2, 0)$ and encloses maximum area. Since area enclosed $A = \int_{x_1}^{x_2} y dx$ therefore we have to extremized the functional $l = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$

Here $F = y, G = \sqrt{1 + (y')^2}$ and therefore we construct a new function

$H = F + \lambda G = y + \lambda\sqrt{1 + (y')^2}$

Since there is no explicit dependence on 'x' so we use the special case of EL equation. i.e. $H - y' \frac{\partial H}{\partial y'} = cosntant$

$$\Rightarrow y + \lambda\sqrt{1 + (y')^2} - y' \frac{\partial}{\partial y'}\left(y + \lambda\sqrt{1 + (y')^2}\right) = c_1$$

$$\Rightarrow y + \lambda\sqrt{1 + (y')^2} - \frac{\lambda(y')^2}{\sqrt{1+(y')^2}} = c_1$$

$$\Rightarrow \lambda\left(\frac{1+(y')^2-(y')^2}{\sqrt{1+(y')^2}}\right) = c_1 - y \Rightarrow \lambda\left(\frac{1}{\sqrt{1+(y')^2}}\right) = c_1 - y$$

$$\Rightarrow \frac{c_1-y}{\lambda} = \frac{1}{\sqrt{1+(y')^2}} \Rightarrow \frac{(c_1-y)^2}{\lambda^2} = \frac{1}{1+(y')^2} \Rightarrow 1 + (y')^2 = \frac{\lambda^2}{(c_1-y)^2}$$

$$\Rightarrow (y')^2 = \frac{\lambda^2}{(c_1-y)^2} - 1 \Rightarrow (y')^2 = \frac{\lambda^2-(c_1-y)^2}{(c_1-y)^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\sqrt{\lambda^2-(c_1-y)^2}}{c_1-y}$$

$$\Rightarrow \int \frac{c_1-y}{\sqrt{\lambda^2-(c_1-y)^2}} dy = \int dx \Rightarrow -\int \frac{z}{\sqrt{\lambda^2-z^2}} dz = \int dx \qquad \text{put } c_1 - y = z$$

$$\Rightarrow \frac{1}{2}\int (\lambda^2 - z^2)^{-1/2} (-2z) dz = x + c_2$$

$$\Rightarrow \frac{1}{2}\frac{(\lambda^2-z^2)^{1/2}}{\frac{1}{2}} = x + c_2 \Rightarrow (\lambda^2 - z^2)^{1/2} = x + c_2 \Rightarrow \lambda^2 - z^2 = (x + c_2)^2$$

$$\Rightarrow \lambda^2 = (x + c_2)^2 + z^2 \Rightarrow \lambda^2 = (x + c_2)^2 + (y - c_1)^2$$

This is an equation of circular arc where the constants c_1, c_2 can be determined by using the given conditions $y(x_1) = 0 = y(x_2)$

INVERSE OF DIDO's PROBLEM:

It can be stated as;

The extermal curves of the functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$ with the endpoint conditions $y(x_1) = y_1, y(x_2) = y_2$ and subject to the constraint $J[y] = \int_{x_1}^{x_2} G(x, y, y')dx = constant$ are the same as the extermals of funtioan J with the same endpoint conditions and subject to the constraint $J[y] = constant$

PROOF:

Consider $F \equiv F(t, x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$ and $G \equiv G(t, x, y, \dot{x}, \dot{y}) = \frac{1}{2}(x\dot{y} - \dot{x}y)$

Therefore $H = F + \lambda G = \sqrt{\dot{x}^2 + \dot{y}^2} + \frac{\lambda}{2}(x\dot{y} - \dot{x}y)$

As the EL equations are $\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0$ and $\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0$

In this problem these equations reduce to

$$\lambda\dot{y} - \frac{d}{dt}\left(\frac{x}{\sqrt{\dot{x}^2+\dot{y}^2}} - \lambda y\right) = 0 \quad \text{and} \quad \lambda\dot{x} - \frac{d}{dt}\left(\frac{x}{\sqrt{\dot{x}^2+\dot{y}^2}} - \lambda x\right) = 0$$

Which on simplification and integration yield

$$2\lambda y - \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c_2 \qquad \text{and} \qquad 2\lambda x - \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c_1$$

On eliminating $\dot{x}, \dot{y}$ we obtain $\quad (x - c_1')^2 + (y - c_2')^2 = \left(\frac{1}{2\lambda}\right)^2$

Where $c_1' = \frac{c_1}{2\lambda}$ and $c_2' = \frac{c_2}{2\lambda}$

EXAMPLE:

Find the curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ which give the minimum area of the surface of revolution around y – axis.

Solution:

This is a Dido Problem in xy – plane. We want to find a curve which gives the minimum area of surface of revolution generated around y – axis.

Since curve revolve around y – axis therefore

$Area = \int_A^B 2\pi x ds = 2\pi \int_A^B x\sqrt{1+(y')^2}dx$ **with** $F(x,y,y') = x\sqrt{1+(y')^2}$

Since F is not depend on 'y' therefore we use following EL equation;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = 0 \; and \; \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = Constant$$

$\Rightarrow \frac{\partial}{\partial y'}\left(x\sqrt{1+(y')^2}\right) = a\ (say) \Rightarrow \frac{xy'}{\sqrt{1+(y')^2}} = a$ **after solving**

$$\Rightarrow xy' = a\sqrt{1+(y')^2} \Rightarrow x^2(y')^2 = a^2(1+(y')^2) \Rightarrow (x^2-a^2)(y')^2 = a^2$$

$$\Rightarrow (y')^2 = \frac{a^2}{x^2-a^2} \Rightarrow y' = \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}} \Rightarrow \int dy = \int \frac{a}{\sqrt{x^2-a^2}}dx$$

$\Rightarrow y = aCosh^{-1}\left(\frac{x}{a}\right) + c$ **required.**

EXAMPLE:

Find the curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ which give the minimum area of the surface of revolution around x – axis.

Solution:

This is a Dido Problem in xy – plane. We want to find a curve which gives the minimum area of surface of revolution generated around x – axis.

Since curve revolve around x – axis therefore

$Area = \int_A^B 2\pi y ds = 2\pi \int_A^B y\sqrt{1+(y')^2}dx$ **with** $F(x,y,y') = y\sqrt{1+(y')^2}$

Since F is not depend on 'x' therefore we use following EL equation;

$$F - y'\left(\frac{\partial F}{\partial y'}\right) = Constant$$

$$y\sqrt{1+(y')^2} - y'\left(\frac{\partial}{\partial y'}\left(y\sqrt{1+(y')^2}\right)\right) = Constant$$

$$\Rightarrow y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} = a\ (say)$$

$$\Rightarrow \frac{y\left(1+(y')^2\right) - y(y')^2}{\sqrt{1+(y')^2}} = a \Rightarrow y(1+(y')^2-(y')^2) = a\sqrt{1+(y')^2}$$

$$\Rightarrow y = a\sqrt{1+(y')^2} \Rightarrow y^2 = a^2(1+(y')^2) = a^2 + a^2(y')^2$$

$$\Rightarrow y^2 - a^2 = a^2(y')^2$$

$$\Rightarrow (y')^2 = \frac{y^2-a^2}{a^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\sqrt{y^2-a^2}}{a} \Rightarrow \int \frac{a}{\sqrt{y^2-a^2}}dy = \int dx$$

$\Rightarrow x = aCosh^{-1}\left(\frac{y}{a}\right) + c$ **required.**

EXAMPLE:

On what curves can the functional $I = \int_0^{\frac{\pi}{2}}((y')^2 - y^2)dx$ with condition $y(0) = 0, y\left(\frac{\pi}{2}\right) = 1$ be extremized.

Solution:

$I = \int_0^{\frac{\pi}{2}}\sqrt{(y')^2 - y^2}dx$ **with** $F(x, y, y') = (y')^2 - y^2$

Since F is not depend on 'y' therefore we use following EL equation;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial y}((y')^2 - y^2) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}((y')^2 - y^2)\right) = 0$$

$$\Rightarrow -2y - \frac{d}{dx}(2y') = 0 \Rightarrow -2(y + y'') = 0 \Rightarrow y'' + y = 0$$

Then general solution will be $y = Acosx + Bsinx$

$\Rightarrow y(0) = 0 \Rightarrow A = 0$ **and** $\Rightarrow y\left(\frac{\pi}{2}\right) = 1 \Rightarrow B = 1$

Hence The general solution will be $y = sinx$

EXAMPLE:

Find the extermal for $I = \int_0^{\frac{\pi}{2}}((y')^2 + (z')^2 + 2yz)dx$ **with condition** $y(0) = 0, y\left(\frac{\pi}{2}\right) = 1; z(0) = 0, z\left(\frac{\pi}{2}\right) = -1$ **be extremized.**

Solution:

We have $I = \int_0^{\frac{\pi}{2}}((y')^2 + (z')^2 + 2yz)dx$ **with** $F = (y')^2 + (z')^2 + 2yz$

since there are two unknown functions 'y', 'z' (extermal curves) there will be a pair of EL equations;

Since F is not depend on 'y' therefore we use following EL equation;

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y\prime}\right) = 0$ **……….(i)** **and** $\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z\prime}\right) = 0$ **……….(ii)**

$(i) \Rightarrow \frac{\partial}{\partial y}((y')^2 + (z')^2 + 2yz) - \frac{d}{dx}\left(\frac{\partial}{\partial y\prime}((y')^2 + (z')^2 + 2yz)\right) = 0$

$\Rightarrow 2z - \frac{d}{dx}(2y') = 0 \Rightarrow 2(z - y'') = 0 \Rightarrow y'' = z$ **……….(iii)**

$(ii) \Rightarrow \frac{\partial}{\partial z}((y')^2 + (z')^2 + 2yz) - \frac{d}{dx}\left(\frac{\partial}{\partial z\prime}((y')^2 + (z')^2 + 2yz)\right) = 0$

$\Rightarrow 2y - \frac{d}{dx}(2z') = 0 \Rightarrow 2(y - z'') = 0 \Rightarrow z'' = y$ **……….(iv)**

Using (iii) in (iv) we get $\Rightarrow y^{iv} - y = 0$ **……….(v)**

Then general solution of (v) will be $y = Ae^x + Be^{-x} + Ccosx + Esinx$

And $y'' = z = Ae^x + Be^{-x} - Ccosx - Esinx$

$\Rightarrow y(0) = 0 \Rightarrow A + B + E = 0$ **……….(vi)**

And $\Rightarrow y\left(\frac{\pi}{2}\right) = 1 \Rightarrow Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} + E = 1$ **……….(vii)**

Similarly $\Rightarrow z(0) = 0 \Rightarrow A + B - C = 0$ **……….(viii)**

And $\Rightarrow z\left(\frac{\pi}{2}\right) = -1 \Rightarrow Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} - E = -1$ **……….(ix)**

Adding (v) and (vii) $B = -A$ **also subtraction from (v) and (vii)** $\underline{C = 0}$

Adding (vi) and (viii) $Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} = 0$ **also subtraction from (vi) and (viii)** $\underline{E = 1}$ **then using the relation** $B = -A$ **we get** $\underline{A = 0, B = 0}$

Putting all values we get $y = sinx, z = -sinx$

EXAMPLE:

Find the extermal for $I=\int_0^{\frac{\pi}{2}}((y'')^2-y^2+x^2)dx$ **with condition** $y(0)=1, y\left(\frac{\pi}{2}\right)=0; y'(0)=0, y'\left(\frac{\pi}{2}\right)=1$ **be extremized.**

Solution:

We have $I=\int_0^{\frac{\pi}{2}}((y'')^2-y^2+x^2)dx$

with $F=F(x,y,y',y'')=(y'')^2-y^2+x^2$

therefore the extermal curve $y=y(x)$ is obtained by the solving EL equation

$$\frac{\partial F}{\partial y}+(-1)^1\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+(-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0 \text{ ……….(i)}$$

$$(i)\Rightarrow -2y+0+\frac{d^2}{dx^2}(2y'')=0\Rightarrow -2y+2y^{iv}=0$$

$$\Rightarrow y^{iv}-y=0 \text{ ……….(ii)}$$

Then general solution of (v) will be $y=Ae^x+Be^{-x}+Ccosx+Esinx$

And $y'=Ae^x-Be^{-x}+Ccosx-Esinx$

$$\Rightarrow y(0)=1\Rightarrow A+B+C=1 \text{ ……….(iii)}$$

And $\Rightarrow y\left(\frac{\pi}{2}\right)=0\Rightarrow Ae^{\frac{\pi}{2}}+Be^{-\frac{\pi}{2}}+E=0$ ……….(iv)

Similarly $\Rightarrow y'(0)=0\Rightarrow A-B+E=0$ ……….(v)

And $\Rightarrow y'\left(\frac{\pi}{2}\right)=1\Rightarrow Ae^{\frac{\pi}{2}}-Be^{-\frac{\pi}{2}}-C=-1$ ……….(vi)

Subtracting and simlifying (iv) and (v) $\left(e^{\frac{\pi}{2}}-1\right)A+\left(e^{-\frac{\pi}{2}}+1\right)B=0$

Adding and simlifying (iii) and (vi) $\left(e^{\frac{\pi}{2}}+1\right)A-\left(e^{-\frac{\pi}{2}}-1\right)B-2=0$

$$\Rightarrow \frac{A}{2\left(e^{-\frac{\pi}{2}}+1\right)}=\frac{B}{-2\left(e^{\frac{\pi}{2}}-1\right)}=\frac{1}{-\left(e^{\frac{\pi}{2}}-1\right)\left(e^{-\frac{\pi}{2}}-1\right)-\left(e^{\frac{\pi}{2}}+1\right)\left(e^{-\frac{\pi}{2}}+1\right)}$$

$$\Rightarrow \frac{A}{2\left(e^{-\frac{\pi}{2}}+1\right)}=\frac{B}{-2\left(e^{\frac{\pi}{2}}-1\right)}=-\frac{1}{4}$$

$\Rightarrow A=\frac{1}{2}\left(e^{-\frac{\pi}{2}}+1\right)$ and $\Rightarrow B=-\frac{1}{2}\left(e^{\frac{\pi}{2}}-1\right)$

$$(iii)\Rightarrow \frac{1}{2}\left(e^{-\frac{\pi}{2}}+1\right)-\frac{1}{2}\left(e^{\frac{\pi}{2}}-1\right)+C=1\Rightarrow C=\frac{1}{2}\left(e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}}\right)$$

$$(v)\Rightarrow \frac{1}{2}\left(e^{-\frac{\pi}{2}}+1\right)+\frac{1}{2}\left(e^{\frac{\pi}{2}}-1\right)+E=1\Rightarrow E=-\frac{1}{2}\left(e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}\right)$$

Hence

$$y=\frac{1}{2}\left(e^{-\frac{\pi}{2}}+1\right)e^x-\frac{1}{2}\left(e^{\frac{\pi}{2}}-1\right)e^{-x}+\frac{1}{2}\left(e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}}\right)cosx-\frac{1}{2}\left(e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}\right)sinx$$

EXAMPLE:

Show that the EL equation for the functional $I = \int_a^b F(x, y, z, y', z')dx = 0$ **admit the following 1st integrals;**

i. $\frac{\partial F}{\partial y'} = C$ **if F does not contains 'y'**

ii. $F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = constant$ **if F does not contains 'x'**

Solution: The corresponding EL equations are

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ **……..(i)** **and** $\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$ **……..(ii)**

i. **When F is independent of 'y'**

Then $\frac{\partial F}{\partial y} = 0$ **then EL equation becomes as follows;**

$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = Constant$

ii. **When F is independent of 'x'**

Since $F = F(x, y, z, y', z')$

$\Rightarrow dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y}dz + \frac{\partial F}{\partial y'}dy' + \frac{\partial F}{\partial z'}dz'$ **……..(iii)**

From (i) and (ii) $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ **and** $\frac{\partial F}{\partial z} = \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)$

$\frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$ **and** $\frac{\partial F}{\partial z} = \frac{d}{dz}\left(\frac{\partial F}{\partial z'}\right)\frac{dz}{dx}$

$\frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)y'$ **and** $\frac{\partial F}{\partial z} = \frac{d}{dz}\left(\frac{\partial F}{\partial z'}\right)z'$

$\frac{\partial F}{\partial y}dy = d\left(\frac{\partial F}{\partial y'}\right)y'$ **and** $\frac{\partial F}{\partial z}dz = d\left(\frac{\partial F}{\partial z'}\right)z'$

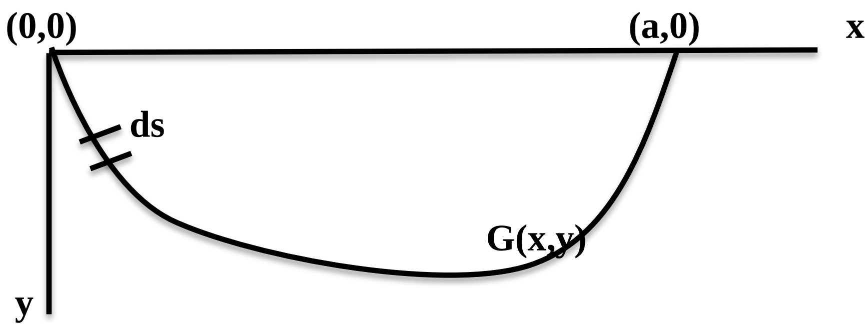
$(iii) \Rightarrow dF = d\left(\frac{\partial F}{\partial y'}\right)y' + d\left(\frac{\partial F}{\partial z'}\right)z' + \frac{\partial F}{\partial y'}dy' + \frac{\partial F}{\partial z'}dz'$

$\Rightarrow dF = d\left(\frac{\partial F}{\partial y'}y'\right) + d\left(\frac{\partial F}{\partial z'}z'\right) \Rightarrow d\left(F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'}\right) = 0$

$\Rightarrow F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = constant$

EXAMPLE: ($BRACHISTOCHRONE\ PROBLEM$):

A uniform cable is fixed at its ends at the same level in space and is allowed to hang under gravity. Find the final shape of the cable.
SOLUTION:



The final shape of the cable wil correspond to the state of a stable equilibrium or minimum P.E. we choose the coordinate axis as shown in the figure. Let (0,0) and (a,0) be the position of the end points of the cable. The P.E. of the cable is given by $V = mg\bar{y}$ where $\bar{y}$ is the y – coordinate of centroid of the cable. The minimum value of V corresponds to the minimum value of $\bar{y}$

Now y – coordinate of centroid of the curve $y = y(x)$ is given by

$$\bar{y} = \frac{V}{mg} = \frac{mgy}{mg} = \frac{my}{m} = \frac{\int_0^a \rho y ds}{\int_0^a \rho ds} = \frac{\int_0^a y ds}{\int_0^a ds} = \frac{1}{l}\int_0^a y\sqrt{1 + (y')^2}dx$$

Where 'l ' is the length of the curve.i.e $l = \int_0^a ds = \int_0^a \sqrt{1 + (y')^2}dx$

And we use $\rho = \frac{m}{l} \Rightarrow m = \rho l = \rho \int_0^a ds$

Here $F = y\sqrt{1 + (y')^2}, G = \sqrt{1 + (y')^2}$ and therefore we construct a new function $H = F + \lambda G = y\sqrt{1 + (y')^2} + \lambda\sqrt{1 + (y')^2}$

$$\Rightarrow H = (y + \lambda)\sqrt{1 + (y')^2}$$

Since there is no explicit dependence on 'x' so we use the special case of EL equation. i.e. $H - y'\frac{\partial H}{\partial y'} = cosntant$

$$\Rightarrow (y + \lambda)\sqrt{1 + (y')^2} - y'\frac{\partial}{\partial y'}\left((y + \lambda)\sqrt{1 + (y')^2}\right) = c_1$$

$$\Rightarrow (y+\lambda)\sqrt{1+(y')^2} - \frac{\lambda(y\prime)^2}{\sqrt{1+(y\prime)^2}} = c_1$$

$$\Rightarrow (y+\lambda)\left(\frac{1+(y')^2-(y')^2}{\sqrt{1+(y')^2}}\right) = c_1 \Rightarrow (y+\lambda)\left(\frac{1}{\sqrt{1+(y')^2}}\right) = c_1$$

$$\Rightarrow \frac{1}{\sqrt{1+(y')^2}} = \frac{c_1}{y+\lambda} \Rightarrow \frac{(c_1)^2}{(y+\lambda)^2} = \frac{1}{1+(y')^2} \Rightarrow 1+(y')^2 = \frac{(y+\lambda)^2}{(c_1)^2}$$

$$\Rightarrow (y')^2 = \frac{(y+\lambda)^2}{(c_1)^2} - 1 \Rightarrow (y')^2 = \frac{(y+\lambda)^2-(c_1)^2}{(c_1)^2} \Rightarrow y' = \frac{dy}{dx} = \frac{\sqrt{(y+\lambda)^2-(c_1)^2}}{c_1}$$

$$\Rightarrow \int \frac{c_1}{\sqrt{(y+\lambda)^2-(c_1)^2}}dy = \int dx \Rightarrow -\int \frac{c_1}{\sqrt{z^2-(c_1)^2}}dz = \int dx \qquad \text{put } y+\lambda = z$$

$$\Rightarrow c_1 Cosh^{-1}\left(\frac{z}{c_1}\right) = x + c_2 \Rightarrow Cosh^{-1}\left(\frac{z}{c_1}\right) = \frac{x+c_2}{c_1} \Rightarrow \frac{z}{c_1} = Cosh\left(\frac{x+c_2}{c_1}\right)$$

$$\Rightarrow \frac{y+\lambda}{c_1} = Cosh\left(\frac{x+c_2}{c_1}\right) \Rightarrow y = c_1 Cosh\left(\frac{x+c_2}{c_1}\right) - \lambda \qquad \text{............(i)}$$

$$\Rightarrow y(0) = 0 \;\; and \;\; y(a) = 0 \Rightarrow \frac{\lambda}{c_1} = c_1 Cosh\left(\frac{c_2}{c_1}\right) \;\; and \;\; \frac{\lambda}{c_1} = c_1 Cosh\left(\frac{a+c_2}{c_1}\right)$$

$$\Rightarrow c_1 Cosh\left(\frac{c_2}{c_1}\right) = c_1 Cosh\left(-\frac{a+c_2}{c_1}\right) \Rightarrow \frac{c_2}{c_1} = -\frac{a+c_2}{c_1} \Rightarrow c_2 = -\frac{a}{2}$$

Using $c_2 = -\frac{a}{2}$ **and** $\lambda = c_1 Cosh\left(\frac{-a}{2c_1}\right)$ **then using in** (i) **we get**

$$\Rightarrow y = c_1 Cosh\left(\frac{x-\frac{a}{2}}{c_1}\right) - c_1 Cosh\left(\frac{-a}{2c_1}\right)$$

$$\Rightarrow y = c_1 C\left[osh\left(\frac{x-\frac{a}{2}}{c_1}\right) - Cosh\left(\frac{-a}{2c_1}\right)\right]$$ **This curve is called Catenary.**

EXAMPLE:

Show that a solid of revolution which for a given surface area has maximum volume is a sphere.

OR find the curve which generates a surface of revolution of a given area which enclosed the maximum volume.

SOLUTION:

Let a curve $y = y(x)$ **with** $y(0) = 0 = y(a)$ **be rotated about x – axis so as to generate a surface of revolution. An element of the surface is . therefore total area will be** $A = 2\pi \int_0^a y ds = 2\pi \int_0^a y\sqrt{1+(y')^2}dx$ **and the volume**

element or solid of revolution is $\pi y^2 dx$ **therefore total volume will be**
$V = \pi \int_0^a y^2 dx$
Here $F = y^2, G = y\sqrt{1+(y')^2}$ **and therefore we construct a new function**
$H = F + \lambda G = y^2 + \lambda y\sqrt{1+(y')^2}$
Since there is no explicit dependence on 'x' so we use the special case of EL equation. i.e. $H - y'\frac{\partial H}{\partial y'} = cosntant$

$$\Rightarrow y^2 + \lambda y\sqrt{1+(y')^2} - y'\frac{\partial}{\partial y'}\left(y^2 + \lambda y\sqrt{1+(y')^2}\right) = c$$

$$\Rightarrow y^2 + \lambda y\sqrt{1+(y')^2} - \frac{\lambda y(y')^2}{\sqrt{1+(y')^2}} = c \Rightarrow y^2 + \lambda y\left[\sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}}\right] = c$$

$$\Rightarrow \lambda y\left[\frac{1+(y')^2-(y')^2}{\sqrt{1+(y')^2}}\right] = c - y^2 \Rightarrow \lambda y = (c-y^2)\sqrt{1+(y')^2} \quad \text{..........(i)}$$

Using $y(0) = 0 \Rightarrow c = 0, \sqrt{1+(y')^2} \neq 0$

$$(i) \Rightarrow \lambda y = -y^2\sqrt{1+(y')^2} \Rightarrow \lambda = -y\sqrt{1+(y')^2} \Rightarrow \lambda^2 = y^2[1+(y')^2]$$

$$\Rightarrow \lambda^2 - y^2 = y^2(y')^2 \Rightarrow (y')^2 = \frac{\lambda^2-y^2}{y^2} \Rightarrow y' = \frac{\sqrt{\lambda^2-y^2}}{y} = \frac{dy}{dx}$$

$$\Rightarrow \int \frac{y}{\sqrt{\lambda^2-y^2}}dy = \int dx \Rightarrow -\sqrt{\lambda^2-y^2} = x + a \Rightarrow \lambda^2 - y^2 = (x+a)^2$$

$\Rightarrow (x+a)^2 + (y-0)^2 = \lambda^2$ **this is an equation of circle centered at** $(a, 0)$ **having radius** λ **and hence the surface of revolution is sphere.**

EXAMPLE: **Find eigenvalue and eigen function of the functional**

$I = \int_0^3 [(2x+3)^2(y')^2 - y^2]dx$ **subjected to the endpoin conditions**

$y(0) = 0 = y(3)$ **and side condition** $\int_0^3 y^2 dx$

SOLUTION:
Here $F = (2x+3)^2(y')^2 - y^2, G = y^2$ **and therefore we construct a new function** $H = F + \lambda G = (2x+3)^2(y')^2 - y^2 + \lambda y^2$

$H = (2x+3)^2(y')^2 + (\lambda - 1)y^2$

Using EL equation $\frac{\partial H}{\partial y}-\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right)=0$

$$\frac{\partial}{\partial y}((2x+3)^2(y')^2+(\lambda-1)y^2)-\frac{d}{dx}\left(\frac{\partial}{\partial y'}((2x+3)^2(y')^2+(\lambda-1)y^2)\right)=0$$

$$\Rightarrow(\lambda-1)2y-\frac{d}{dx}((2x+3)^2 2y')=0$$

$$\Rightarrow -2\left[\frac{d}{dx}((2x+3)^2y')-(\lambda-1)y\right]=0\Rightarrow\frac{d}{dx}((2x+3)^2y')-(\lambda-1)y=0$$

$$\Rightarrow(2x+3)^2y''+2(2x+3)2y'-(\lambda-1)y=0$$

$$\Rightarrow 4\left(x+\frac{3}{2}\right)^2y''+8\left(x+\frac{3}{2}\right)y'+(1-\lambda)y=0$$

$$\Rightarrow\left[4\left(x+\frac{3}{2}\right)^2D^2+8\left(x+\frac{3}{2}\right)D+(1-\lambda)\right]y=0\ \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Put $2x+3=e^t\Rightarrow ln(2x+3)=t$

And $\left(x+\frac{3}{2}\right)D=\Delta\Rightarrow\left(x+\frac{3}{2}\right)^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$

$$(i)\Rightarrow[4\Delta^2-4\Delta+8\Delta+(1-\lambda)]y=0$$

$\Rightarrow 4\Delta^2-4\Delta+8\Delta+(1-\lambda)=0$ **since** $y\neq 0$

$$\Rightarrow 4\Delta^2+4\Delta+(1-\lambda)=0\Rightarrow\Delta=-\frac{1}{2}\pm\frac{1}{2}\sqrt{\lambda}$$

if $\lambda=0$ and $\lambda>0$ **We obtain trivial solution for the given problem**

if $\lambda<0$ **We obtain non – trivial solution for the given problem**

if $\lambda=-\mu^2$ **then** $\Delta=-\frac{1}{2}+\frac{1}{2}\mu i$

and general solution will be $y(x)=e^{-\frac{1}{2}t}\left[c_1Cos\frac{1}{2}\mu t+c_2Sin\frac{1}{2}\mu t\right]$

$$y(x)=(e^t)^{-\frac{1}{2}}\left[c_1Cos\frac{\mu}{2}ln(2x+3)+c_2Sin\frac{\mu}{2}ln(2x+3)\right]$$

$$y(x)=(2x+3)^{-\frac{1}{2}}\left[c_1Cos\frac{\mu}{2}ln(2x+3)+c_2Sin\frac{\mu}{2}ln(2x+3)\right]\ \ldots\ldots\ldots\text{(ii)}$$

Using $y(0)=0$

$$c_1Cos\frac{\mu}{2}ln(3)+c_2Sin\frac{\mu}{2}ln(3)=0\ \ldots\ldots\ldots\text{(iii)}$$

Also Using $y(3) = 0$

$$c_1 Cos\frac{\mu}{2}ln(9) + c_2 Sin\frac{\mu}{2}ln(9) = 0$$

$$\Rightarrow c_1 Cos\mu ln(3) + c_2 Sin\mu ln(3) = 0 \qquad \text{..........(iv)}$$

For non – trivial solution

$$\begin{vmatrix} Cos\frac{\mu}{2}ln(3) & Sin\frac{\mu}{2}ln(3) \\ Cos\mu ln(3) & Sin\mu ln(3) \end{vmatrix} = 0$$

$$\Rightarrow \left(Cos\frac{\mu}{2}ln(3)\right)\left(Sin\mu ln(3)\right) - \left(Cos\mu ln(3)\right)\left(Sin\frac{\mu}{2}ln(3)\right) = 0$$

$$\Rightarrow Sin\left(\mu ln(3) - \frac{\mu}{2}ln(3)\right) = 0 \Rightarrow \mu ln(3) - \frac{\mu}{2}ln(3) = Sin^{-1}(0)$$

$$\Rightarrow \frac{\mu}{2}ln(3) = n\pi \qquad \text{n = 1,2,3,...............}$$

$$\Rightarrow \mu = \frac{2n\pi}{ln(3)} \Rightarrow \mu_n = \frac{2n\pi}{ln(3)} \qquad \text{n = 1,2,3,...............}$$

$$(iv) \Rightarrow c_1 Cos\frac{2n\pi}{ln(3)}ln(3) + c_2 Sin\frac{2n\pi}{ln(3)}ln(3) = 0$$

$$\Rightarrow c_1 Cos2n\pi + c_2 Sin2n\pi = 0 \Rightarrow c_1(1) + c_2(0) = 0 \Rightarrow c_1 = 0$$

But $c_2 \neq 0$ we take $c_2 = c_n$ then eigen solution will be as follows;

$$y_n(x) = \frac{c_n}{\sqrt{2x+3}} Sin\frac{n\pi}{ln(3)}ln(2x+3)$$

GEODESIC:

A geodesic is the curve of shortest length joining two points in space.

EXAMPLE:

Find the curve of shortest length between the given points in a plane using polar coordinates.

Solution:

Since we know that $l = \int_A^B ds$ ....(i)

Also $ds = \sqrt{(dx)^2 + (dy)^2}$ **..............(ii)**

Now usig $x = rCos\theta, y = rSin\theta$

$(dx)^2 = (dr)^2Cos^2\theta + r^2Sin^2\theta(d\theta)^2 - 2rdrCos\theta Sin\theta$

$(dy)^2 = (dr)^2Sin^2\theta + r^2Cos^2\theta(d\theta)^2 + 2rdrCos\theta Sin\theta$

$(ii) \Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dr)^2 + r^2(d\theta)^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$\Rightarrow ds = \sqrt{r^2 + (r')^2}d\theta$

$(i) \Rightarrow l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2}d\theta$ **subjected to** $r(\theta_1) = c_1$ **and** $r(\theta_2) = c_2$

Here $F = \sqrt{r^2 + (r')^2}$ **Since there is no explicit dependence on 'θ' so we use the special case of EL equation. i.e.** $F - r'\frac{\partial F}{\partial r'} = cosntant$

$\Rightarrow \sqrt{r^2 + (r')^2} - r'\frac{\partial}{\partial r'}\left[\sqrt{r^2 + (r')^2}\right] = c_1$

$\Rightarrow \sqrt{r^2 + (r')^2} - r'\frac{r'}{\sqrt{r^2+(r')^2}} = c_1 \Rightarrow \frac{r^2+(r')^2-(r')^2}{\sqrt{r^2+(r')^2}} = c \Rightarrow \frac{r^2}{\sqrt{r^2+(r')^2}} = c_1$

$\Rightarrow \frac{\sqrt{r^2+(r')^2}}{r^2} = \frac{1}{c_1} \Rightarrow \sqrt{r^2 + (r')^2} = \frac{r^2}{c_1} \Rightarrow r^2 + (r')^2 = \frac{r^4}{{c_1}^2} \Rightarrow (r')^2 = \frac{r^4}{{c_1}^2} - r^2$

$\Rightarrow (r')^2 = \frac{r^4 - {c_1}^2r^2}{{c_1}^2} \Rightarrow (r')^2 = \frac{r^2(r^2 - {c_1}^2)}{{c_1}^2} \Rightarrow r' = \frac{dr}{d\theta} = \frac{r\sqrt{r^2-{c_1}^2}}{c_1}$

$\Rightarrow c_1\int \frac{1}{r\sqrt{r^2-{c_1}^2}}dr = \int d\theta \Rightarrow c_1\frac{1}{c_1}Sec^{-1}\left(\frac{r}{c_1}\right) = \theta + c_2 \Rightarrow Sec^{-1}\left(\frac{r}{c_1}\right) = \theta + c_2$

$\Rightarrow \frac{r}{c_1} = Sec(\theta + c_2) \Rightarrow \frac{r}{Sec(\theta+c_2)} = c_1 \Rightarrow c_1 = rCos(\theta + c_2)$

$\Rightarrow c_1 = r(Cos\theta Cosc_2 - Sin\theta Sinc_2)$

$\Rightarrow c_1 = (rCos\theta.Cosc_2 - rSin\theta.Sinc_2)$

$\Rightarrow c_1 = (xCosc_2 - ySinc_2)$

$\Rightarrow -xCosc_2 + ySinc_2 + c_1 = 0$

$\Rightarrow -\propto x + \beta y + \gamma = 0$

Which represent the straight line.

EXAMPLE:

Find the curve of shortest length on the surface of sphere.

Solution:

Let A and Bbe the two points on the sphere S. here the problem is to minimize

Since we know that $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ **.........(i)**

Now usig $x = rSin\theta Cos\varphi, y = rSin\theta Sin\varphi, z = rCos\theta$

$$dx = r[Cos\theta d\theta Cos\varphi - Sin\theta Sin\varphi d\varphi]$$

$$dy = r[Cos\theta d\theta Sin\varphi + Sin\theta Cos\varphi d\varphi]$$

$$dz = -rSin\theta d\theta$$

$$\Rightarrow ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{r^2\left[1 + Sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right]}d\theta$$

$$\Rightarrow ds = r\sqrt{1 + Sin^2\theta(\varphi')^2}d\theta$$

$(i) \Rightarrow l = r\int_{\theta_1}^{\theta_2}\sqrt{1 + Sin^2\theta(\varphi')^2}d\theta$ **subjected to** $r(\theta_1) = c_1$ **and** $r(\theta_2) = c_2$

Here $F = \sqrt{1 + Sin^2\theta(\varphi')^2}$ **then corresponding EL equation will be**

$$\frac{\partial F}{\partial \varphi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \varphi'}\right) = 0$$

$$\Rightarrow 0 - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \varphi'}\right) = 0 \Rightarrow \frac{\partial F}{\partial \varphi'} = Constant \Rightarrow \frac{\partial}{\partial \varphi'}\left(\sqrt{1 + Sin^2\theta(\varphi')^2}\right) = C$$

$$\Rightarrow \frac{Sin^2\theta\varphi'}{\sqrt{1+Sin^2\theta(\varphi')^2}} = C \Rightarrow Sin^2\theta\varphi' = C\sqrt{1 + Sin^2\theta(\varphi')^2}$$

$$\Rightarrow Sin^4\theta(\varphi')^2 = C^2(1 + Sin^2\theta(\varphi')^2) \Rightarrow Sin^4\theta(\varphi')^2 = C^2 + C^2Sin^2\theta(\varphi')^2$$

$$\Rightarrow Sin^4\theta(\varphi')^2 - C^2Sin^2\theta(\varphi')^2 = C^2 \Rightarrow Sin^2\theta(Sin^2\theta - C^2)(\varphi')^2 = C^2$$

$$\Rightarrow (\varphi')^2 = \frac{C^2}{Sin^2\theta(Sin^2\theta - C^2)} \Rightarrow \varphi' = \frac{d\varphi}{d\theta} = \frac{C}{Sin\theta\sqrt{Sin^2\theta - C^2}}$$

$$\Rightarrow \varphi' = \frac{d\varphi}{d\theta} = \frac{C}{Sin^2\theta\sqrt{1 - \frac{C^2}{Sin^2\theta}}} = \frac{C.Cosec^2\theta}{\sqrt{1 - C^2Cosec^2\theta}} = \frac{C.Cosec^2\theta}{\sqrt{1 - C^2(1 + Cot^2\theta)}} = \frac{C.Cosec^2\theta}{\sqrt{1 - C^2 - C^2Cot^2\theta}}$$

$$\Rightarrow \int d\varphi = \int \frac{C.Cosec^2\theta}{\sqrt{1-C^2-C^2Cot^2\theta}} d\theta \Rightarrow \varphi = \int \frac{C.Cosec^2\theta}{C\sqrt{\left(\frac{\sqrt{1-C^2}}{C}\right)^2 - Cot^2\theta}} d\theta$$

$$\Rightarrow \varphi = \int \frac{Cosec^2\theta}{\sqrt{\left(\frac{\sqrt{1-C^2}}{C}\right)^2 - Cot^2\theta}} d\theta$$

$$\Rightarrow \varphi = \int \frac{-1}{\sqrt{a^2-t^2}} dt \qquad \text{with } \frac{\sqrt{1-C^2}}{C} = a \ ; \ Cot\theta = t \ ; \ -Cosec^2\theta d\theta = dt$$

$$\Rightarrow \varphi = Cos^{-1}\left(\frac{t}{a}\right) + \propto \Rightarrow \varphi = Cos^{-1}\left(\frac{Cot\theta}{a}\right) + \propto \Rightarrow \varphi - \propto = Cos^{-1}\left(\frac{Cot\theta}{a}\right)$$

$$\Rightarrow Cos(\varphi - \propto) = \frac{Cot\theta}{a} \Rightarrow Cos\varphi Cos \propto + Sin\varphi Sin \propto = \frac{1}{a} . \frac{Cos\theta}{Sin\theta}$$

$$\Rightarrow raSin\theta Cos\varphi Cos \propto + raSin\theta Sin\varphi Sin \propto = rCos\theta$$

$$\Rightarrow a(rSin\theta Cos\varphi)Cos \propto + a(rSin\theta Sin\varphi)Sin \propto = rCos\theta$$

$$\Rightarrow aCos \propto x + aSin \propto y = z \Rightarrow Ax + By = z$$

This is an equation of the plane through center of sphere. Hence the curve of shortest length joining A and B is the arc of great circle through A and B.

EXAMPLE:

Find the geodesic curve for the cylinder $x^2 + y^2 = a^2$

Solution:

We have to minimize $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ **………(i)**

Now usig $x = rCos\theta, y = rSin\theta, z = z$ **for cylindrical coordinates**

$dx = -rSin\theta d\theta, dy = rCos\theta d\theta, dz = dz$

$$\Rightarrow ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{r^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$(i) \Rightarrow l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (z')^2}\, d\theta$ **subjected to** $r(\theta_1) = c_1$ **and** $r(\theta_2) = c_2$

Here $F = \sqrt{r^2 + (z')^2}$ **then corresponding EL equation will be**

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0$$

$$\Rightarrow 0 - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0 \Rightarrow \frac{\partial F}{\partial z'} = Constant \Rightarrow \frac{\partial}{\partial z'}\left(\sqrt{r^2 + (z')^2}\right) = C$$

$$\Rightarrow \frac{z'}{\sqrt{r^2+(z')^2}} = C \Rightarrow z' = C\sqrt{r^2 + (z')^2} \Rightarrow (z')^2 = C^2(r^2 + (z')^2)$$

$$\Rightarrow (z')^2 - C^2(z')^2 = C^2r^2 \Rightarrow (1 - C^2)(z')^2 = C^2r^2 \Rightarrow (z')^2 = \frac{C^2r^2}{(1-C^2)}$$

$$\Rightarrow z' = \frac{Cr}{\sqrt{1-C^2}} \Rightarrow \frac{dz}{d\theta} = \propto (say) \Rightarrow z = \propto \theta + C' \Rightarrow z - C' = \propto Tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow Tan\left(\frac{z-C'}{\propto}\right) = \frac{y}{x}$$

The intersection of this surface with given cylinder gives required extreme curve.

EXAMPLE:

Find the shortest distance between the points $A(1,-1,0)$ and $B(2,1,-1)$ in the plane $15x - 7y + z - 22 = 0$

Solution:

We have to minimize $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$l = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx = \int_A^B \sqrt{1 + (y')^2 + (z')^2}\, dx$$

$$\Rightarrow l = \int_{x_1}^{x_2} \sqrt{1 + (y')^2 + (z')^2}\, dx$$

subjected to constraint $15x - 7y + z - 22$

Here $F = \sqrt{1 + (y')^2 + (z')^2}, G = 15x - 7y + z - 22$

and therefore we construct a new function

$$H = F + \lambda G = \sqrt{1 + (y')^2 + (z')^2} + \lambda(15x - 7y + z - 22)$$

Using EL equation $\quad \frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$$\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)-$$

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)\right)=0$$

$$\Rightarrow -7\lambda-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \qquad \text{…………..(i)}$$

Also Using EL equation $\frac{\partial H}{\partial z}-\frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right)=0$

$$\frac{\partial}{\partial z}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)-$$

$$\frac{d}{dx}\left(\frac{\partial}{\partial z'}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(15x-7y+z-22)\right)\right)=0$$

$$\Rightarrow \lambda-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \qquad \text{…………..(ii)}$$

Multiplying (ii) with 7 then adding in (i)

$$-\frac{d}{dx}\left(\frac{y'+7z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0\Rightarrow\frac{y'+7z'}{\sqrt{1+(y')^2+(z')^2}}=C \qquad \text{…………..(iii)}$$

Since $15x-7y+z-22=0$

The endpoint conditions satisfied by the functions $y=y(x)$ **and** $z=z(x)$ **are**

$y(1)=-1, y(2)=1, z(1)=0, z(2)=-1$

$\Rightarrow 15-7y'+z'=0\Rightarrow z'=7y'-15$ **diff. w.r.to 'x'**

$$(iii)\Rightarrow\frac{y'+7(7y'-15)}{\sqrt{1+(y')^2+(7y'-15)^2}}=C$$

$$\Rightarrow y'+49y'-105=C\sqrt{1+(y')^2+49(y')^2+225-210y'}$$

$$\Rightarrow 50y'-105=C\sqrt{50(y')^2-210y'+226}$$

$$\Rightarrow [5(10y'-21)]^2=\left[C\sqrt{50(y')^2-210y'+226}\right]^2$$

$$\Rightarrow 25(10y'-21)^2=C^2(50(y')^2-210y'+226)$$

$$\Rightarrow 25(100(y')^2-420y'+441)=C^2(50(y')^2-210y'+226)$$

$$\Rightarrow (2500-50C^2)(y')^2+(210C^2-11000)y'+(11025-226C^2)=0$$

This is the quadratic equation in y'

Since C was arbitray, we can always choose it , so that the equation has real roots. Let $\propto$ be one such root then $y' = \propto = dy/dx$

$\Rightarrow y = \propto x + \beta$

Now using $y(1) = -1, y(2) = 1, z(1) = 0, z(2) = -1$

$\propto + \beta = -1, 2\propto + \beta = 1$ **then** $\propto = 2,, \beta = -3$

Then we get $\Rightarrow y = 2x - 3 \Rightarrow y' = 2$

Also for z' **we have** $\Rightarrow z' = 7y' - 15 = -1$

Then required least distance is $\Rightarrow l = \int_1^2 \sqrt{1 + (y')^2 + (z')^2}\, dx$

$\Rightarrow l = \int_1^2 \sqrt{1 + 4 + 1}\, dx = \sqrt{6}|x|_1^2 = \sqrt{6}$

$\Rightarrow l = \sqrt{6}$ **is required least distance.**

EXAMPLE:

Find the shortest distance between the points $A(1, 0, -1)$ and $B(0, -1, 1)$ in the plane $x + y + z = 0$

Solution:

We have to minimize $l = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$l = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx = \int_A^B \sqrt{1 + (y')^2 + (z')^2}\, dx$$

$$\Rightarrow l = \int_{x_1}^{x_2} \sqrt{1 + (y')^2 + (z')^2}\, dx$$

subjected to constraint $x + y + z$

Here $F = \sqrt{1 + (y')^2 + (z')^2}, G = x + y + z$

and therefore we construct a new function

$$H = F + \lambda G = \sqrt{1 + (y')^2 + (z')^2} + \lambda(x + y + z)$$

Using EL equation $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$$\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(x+y+z=0)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2+(z')^2}+\right.\right.$$

$$\left.\left.\lambda(x+y+z=0)\right)\right)=0$$

$$\Rightarrow \lambda-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \quad \ldots\ldots\ldots\ldots..(i)$$

Also Using EL equation $\frac{\partial H}{\partial z}-\frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right)=0$

$$\frac{\partial}{\partial z}\left(\sqrt{1+(y')^2+(z')^2}+\lambda(x+y+z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial z'}\left(\sqrt{1+(y')^2+(z')^2}+\right.\right.$$

$$\left.\left.\lambda(x+y+z)\right)\right)=0$$

$$\Rightarrow \lambda-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0 \quad \ldots\ldots\ldots\ldots..(ii)$$

Subtracting (i) and (ii) we get

$$\frac{d}{dx}\left(\frac{y'-z'}{\sqrt{1+(y')^2+(z')^2}}\right)=0\Rightarrow\frac{y'-z'}{\sqrt{1+(y')^2+(z')^2}}=C \qquad \ldots\ldots\ldots\ldots..(iii)$$

Since $x+y+z=0$

The endpoint condition satisfied by the functions $y=y(x)$ **is**

$y(1)=0, y(0)=-1$

$\Rightarrow 1+y'+z'=0\Rightarrow z'=-1-y'$ **diff. w.r.to 'x'**

$$(iii)\Rightarrow\frac{y'+1+y'}{\sqrt{1+(y')^2+(-1-y')^2}}=C$$

$$\Rightarrow 2y'+1=C\sqrt{1+(y')^2+(y')^2+1+2y'}$$

$$\Rightarrow [2y'+1]^2=\left[C\sqrt{2+2(y')^2+2y'}\right]^2$$

$$\Rightarrow 1+4(y')^2+4y'=C^2(2+2(y')^2+2y')$$

$$\Rightarrow (4-2C^2)(y')^2+(4-2C^2)y'+(1-2C^2)=0$$

This is the quadratic equation in y'

Since C was arbitray, we can always choose it , so that the equation has real roots. Let $\propto$ be one such root then $y' = \propto = dy/dx$

$\Rightarrow y = \propto x + \beta$

Now using $y(1) = 0, y(0) = -1$

$\propto + \beta = -1, \propto (0) + \beta = -1$ **then** $\propto = 1,, \beta = -1$

Then we get $\Rightarrow y = x - 1 \Rightarrow y' = 1$

Also for z' we have $\Rightarrow z' = -1 - y' = -2$

Then required least distance is $\Rightarrow l = \int_0^1 \sqrt{1 + (y')^2 + (z')^2}\,dx$

$\Rightarrow l = \int_0^1 \sqrt{1 + 1 + 4}\,dx = \sqrt{6}|x|_0^1 = \sqrt{6}$

$\Rightarrow l = \sqrt{6}$ **is required least distance.**

PERTURBATION TECHNIQUES

WHAT ARE PERTURBATION METHODS?

Many physical processes are described by equations which cannot be solved analytically.Working in mathematical modelling, you would have to be exceptionally lucky never tohave this happen to you!

There are two main approaches to dealing with these equations:

- **numerical methods and**
- **analytic approximations.**

The methods all rely on there being a parameter in the problem that is relatively small: $\varepsilon \ll 1$. The most common example you may have seen before1 is that of high-Reynolds number fluid mechanics, in which a viscous boundary layer is found close to a solid surface. Note that in this case the standard physical parameter *Re* is large: our small parameter is $\varepsilon = Re^{-1}$.

WHY USE PERTURBATION METHODS?

There are two major types of use for these methods. The first is in modeling physical applications which, like high-Reynolds number flow, naturally supply such a small parameter. This kind of application is fairly common, and this is one of the reasons that perturbation methods are a cornerstone of applied mathematics.

The second use of perturbation methods is coupled with numerical methods. Although computed solutions to a problem can be very accurate, and available for very complex systems, there are two major drawbacks to numerical computation: and perturbation methods can help with both of these.

There is always a concern with numerical calculations about whether the code is correct.A helpful check can be to push one or more of the physical parameters of the problem toextreme values and compare the numerical results with a perturbation solution worked out when that parameter is small (or large).
There are other ways of checking code, however; more importantly, a numerical calculation does not often provide insight into the underlying physics. Sometimes (surprisinglyoften in practice) the simplified problems presented by taking a limiting case have a simplified physics which nonetheless encapsulates some of the key mechanisms from the full problem – and these mechanisms can then be better understood through perturbation methods

2 Perturbation Theory

In this chapter, we wish to revise perturbation theory. We also focus on *Singular perturbation theory* and *regular perturbation theory*. Perturbation theory leads to an expression for the desired solution in terms of a formal power series in small parameter (ϵ), known as perturbation series that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem and further terms describe the deviation in the solution. Consider,

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

Here, x_0 be the known solution to the exactly solvable initial problem and $x_1, x_2 \ldots$ are the higher order terms. For small ϵ these higher order terms are successively smaller. An approximate *"perturbation solution"* is obtained by truncating the series, usually by keeping only the first two terms.

2.1 Regular Perturbation Theory

Very often, a mathematical problem can not be solved exactly or, if the exact solution is available it exhibits such an intricate dependency in the parameters that it is hard to use as such. It may be the case however, that a parameter can be identified, *say*, ϵ ,such that the solution is available and reasonably simple for $\epsilon = 0$. Then one may wonder how this solution is altered for non zero but small ϵ . Perturbation theory gives a systematic answer to this question.

2.2 Singular Perturbation Theory

It concern the study of problems featuring a parameter for which the solution of the problem at a limiting value of the parameter are different in character from the limit of the solution of the general problem. For regular perturbation problems, the solution of the general problem converge to the solution of the limit problem as the parameter approaches the limit value.

2.1 Perturbation methods for algebraic equations

Let us suppose our algebraic equations depend on a parameter ϵ. Suppose the root can be found for $\epsilon = 0$. We look for roots for small ϵ. The procedure of regular perturbation are the follows:

- Express $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$;
- Plug this expression into equation, make Taylor expansion of coefficients of the equation.
- Equating the coefficients with like power of ϵ;
- Solve x_0, x_1 successively.

Example 1 Consider $x(x-1) = \epsilon$.

- Let $x = x_0 + \epsilon x_1 + \cdots$. Plug this into equation, we get

$$(x_0 + \epsilon x_1 + \cdots)(x_0 + \epsilon x_1 + \cdots - 1) = \epsilon.$$

- Equating the coefficients of like powers:

$$\begin{aligned} \epsilon^0 &: \; x_0(x_0 - 1) = 0, \\ \epsilon^1 &: \; 2x_0 x_1 - x_1 = 1, \\ \vdots &: \; \vdots \end{aligned}$$

This leads to two sets of solutions:

$$x_a^{(1)} = 0 - \epsilon; \; x_a^{(2)} = 1 + \epsilon$$

The true solution is

$$x = \frac{1 \pm \sqrt{1 + 4\epsilon}}{2} \approx \frac{1}{2}(1 \pm (1 + 2\epsilon)).$$

which are consistent to the regular perturbation solutions.

Example 2 Let us consider the equation $x^2 = \epsilon$. You will see that

- The expansion $x = x_0 + \epsilon x_1 + \cdots$ does not work. You should try $x = \sqrt{\epsilon}$.
- For $\epsilon < 0$, the solution becomes imaginary.

So, we should try $x = x_0 + \delta(\epsilon)x_1 + \delta(\epsilon)^2 x_2 + \cdots$.

- Plug this ansatz, we get

$$x_0^2 + 2x_0\delta x_1 + \delta^2 x_1^2 + 2\delta^2 x_0 x_2 + \cdots = \epsilon.$$

By comparing both sides, we see that we should have $x_0 = 0$. Then this gives

$$\delta^2 x_1^2 + \cdots = \epsilon$$

To equate both sides, we need to choose $\delta^2 = \epsilon$. This give $\delta = \sqrt{\epsilon}$ and $x_1 = 1$.

Example 3. Let us consider

$$\epsilon x^2 + ax + b = 0.$$

As $\epsilon \to 0$, there is only one root. Thus, the perturbation method can not recover the other root, which goes to ∞ as $\epsilon \to 0$. If $a = 0$, there is no root as $\epsilon \to 0$, the reduced equation is even inconsistent at all. Thus, the above perturbation method does not work for such case.

Nevertheless, we can try the following thing. We know the other solution goes to infinity as $\epsilon \to 0$, we try $x = \frac{x^*}{\epsilon}$. Plug this into equation, we obtain

$$\frac{x^{*2}}{\epsilon} + a\frac{x^*}{\epsilon} + b = 0.$$

We see that x^* satisfies an equation where the regular perturbation method can handle. We write

$$x^* = x_0^* + \epsilon x_1^* + \cdots,$$

Plug this into the above rescaled equation, we get

$$\begin{aligned} x_0^{*2} + ax^* &= 0 \\ 2x_0^* x_1^* + ax_1^* + b &= 0 \end{aligned}$$

These equations give the $x_0^* = -a$ and $x_1^* = b/a$, lead to the second solution of the orginal equation

$$x^{(2)} = -\frac{a}{\epsilon} + \frac{b}{a} + \cdots.$$

You may think how to handle the case when $a = 0$.

Homework

1. Find the asymptotic behaviors of the equations

   (a) $\epsilon x^3 + x - 2 = 0$;

   (b) $\epsilon^2 x^4 + \epsilon x^3 + x - 1 = 0$

   (c) $\epsilon x^5 + x^3 - 1 = 0$;

2. There are infinite roots of $\tan x = x$. Find their asymptotic formula.

2.1.1 Justification of regular perturbation method for algebraic equations

Implicit Function Theorem

Theorem 2.3. *Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be smooth. Suppose x_0 is a solution of $F(x_0, 0) = 0$ and suppose $\partial F/\partial x(x_0, 0)$ is non-singular. Then there is a smooth solution set $x(\epsilon)$ satisfying $F(x(\epsilon), \epsilon) = 0$ for small ϵ.*

The proof of this theorem is based on method of contraction map. We rewrite the above equation as a perturbtion equation: let us write $x(\epsilon) = x_0 + y(\epsilon)$, then $y(\epsilon)$ satisfies

$$\frac{\partial F}{\partial x}(x_0, 0)y + \frac{\partial F}{\partial \epsilon}(x_0, 0)\epsilon + r(y) = 0.$$

Here,

$$r(y) = F(x_0 + y, \epsilon) - F(x_0, 0) - \frac{\partial F}{\partial x}(x_0, 0)y - \frac{\partial F}{\partial \epsilon}(x_0, 0)\epsilon = O(|y|^2 + \epsilon^2).$$

Since $J := \partial F/\partial x(x_0, 0)$ is non-singular, we take its inversion and get

$$y = Ty := J^{-1}\left(-\frac{\partial F}{\partial \epsilon}(x_0, 0)\epsilon - r(y)\right).$$

We want to find a small number ϵ_0 and another number η such that for any $|\epsilon| \leq \epsilon_0$, the mapping T is a strict contraction map from $|y| \leq \eta$ to itself. Then by the fixed point theorem, we can obtain a fixed point $y(\epsilon)$.

This process also tell us the construction of the perturbed solution. The method breaks down when the Jacobian $J := \frac{\partial F}{\partial x}(x_0, 0)$ is singular, or when it has very small eigenvalue.

2.2 Regular perturbation method for differential equations

We start from some examples.

A falling object with resistivity The model reads

$$m\frac{dv}{dt} = -av + bv^2, v(0) = V_0.$$

We introduce the dimensionless variables $y = v/V_0$, $\tau = at/m$, then the equation becomes

$$\frac{dy}{d\tau} = -y + \epsilon y^2,$$

where

$$\epsilon = \frac{bV_0}{a} << 1.$$

It means the resistivity (damping) is very large, as compared with V_0 and b. This equation has exact solution

$$y(\tau) = \frac{e^{-\tau}}{1 + \epsilon(e^{-\tau} - 1)}.$$

which has the following Taylor expansion in ϵ:

$$y = e^{-\tau} + \epsilon(e^{-\tau} - e^{-2\tau}) + \epsilon^2(e^{-\tau}2e^{-2\tau} + e^{-3\tau}) + \cdots .$$

The regular perturbation method introduces a Taylor expansion of y in terms of ϵ:

$$y(\tau, \epsilon) = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \cdots .$$

We plug this ansatz into the equation, equating the coefficients of like powers of ϵ. We get

$$\begin{aligned} y_0' &= -y_0, \\ y_1' &= -y_1 + y_0^2 \\ y_2' &= -y_2 + 2y_0y_1, \\ &\vdots \end{aligned}$$

Equating the initial conditions, we get

$$y_0(0) = 1, y_1(0) = y_2(0) = \cdots = 0.$$

Solving these equations, we get

$$y_a = e^{-\tau} + \epsilon(e^{-\tau} - e^{-2\tau}) + \epsilon^2(e^{-\tau}2e^{-2\tau} + e^{-3\tau}) + \cdots ,$$

We find this approach does work for this example.

Nonlinear oscillator Consider a nonlinear oscillator

$$m\frac{d^2y}{d\tau^2} = -ky - ay^3, y(0) = A, \frac{dy}{d\tau}(0) = 0.$$

This is so-called hard spring. We rescale it by

$$t = \frac{\tau}{m/k}, u = \frac{y}{A}.$$

Then we get the Duffing equation:

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0$$
$$u(0) = 1, u'(0) = 0.$$

We perform regular perturbation method

$$u(t, \epsilon) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots,$$

Plugging this into equation and the initial conditions, we get

$$\ddot{u}_0 + u_0 = 0, u_0(0) = 1, \dot{u}_0(0) = 0,$$

$$\ddot{u}_1 + u_1 = -u_0^3, u_1(0) = 0, \dot{u}_1(0) = 0,$$

From the fisrt equationm, we obtain

$$u_0(t) = \cos t.$$

The second equation becomes

$$\ddot{u}_1 + u_1 = -\cos^3 t = -\frac{1}{4}(3\cos t + \cos 3t)$$

Solving this equation with initial condition, we get

$$u_1(t) = \frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8}t\sin t.$$

The term $t\sin t$ is a resonant term from $\cos t$. Such a term is called a secular term. It will grow linearly and eventually to infinite as $t \to \infty$. However, by energy method, one can show that the solution is bounded. What wrong is that the expansion is only good for finite time. The estimate $|y_a(t,\epsilon) - y_\epsilon(t,\epsilon)| = O(\epsilon^2)$ is only valid for $t \in [0, T]$ for a finite T.

Asymptotic Expansion First, we give some definitions of some notations.

- The notation $f(\epsilon) = o(g(\epsilon))$ means that $f/g \to 0$ as $\epsilon \to 0$.
- If f and g are also function of t in an interval I, the notation

$$f(t,\epsilon) = o(g(t,\epsilon)) \text{ as } \epsilon \to 0, t \in I$$

means that for every $t \in I$, we have $f(t,\epsilon)/g(t,\epsilon) \to 0$ as $\epsilon \to 0$.

- If the above limit is uniform for $t \in I$, we say that $f(t,\epsilon) = o(g(t,\epsilon))$ as $\epsilon \to 0$ uniformly on $t \in I$.

Definition 2.1. • *A sequence of gauge functions $\{g_n(t,\epsilon)\}$ is an asymptotic sequence on $t \in I$ as $\epsilon \to 0$ if*

$$g_{n+1}(t,\epsilon) = o(g_n(t,\epsilon)) \textit{ for every } t \in I.$$

- *Given a function $y(t,\epsilon)$ and an asymptotic sequence $\{g_n(t,\epsilon)$ on $t \in I$, the formal expansion $\sum_{n=0}^{\infty} a_n g_n(t,\epsilon)$ is said to be an asymptotic expansion of $y(t,\epsilon)$ as $\epsilon \to 0$ if*

$$y(t,\epsilon) - \sum_{n=0}^{N} a_n g_n(t,\epsilon) = o(g_N(t,\epsilon)), \ \textit{as } \epsilon \to 0,$$

for any N. If the limits are uniform for $t \in I$, we say it is a uniform asymptotic expansion on I.

The expansion sequences are usually seperable such as

- $\epsilon^n u_n(t)$
- $\epsilon^{\alpha_n} u_n(x)$, where α_n is a strictly increasing sequence.
- $\epsilon^n \ln \epsilon u_n(x)$.

A rigorous proof for regular perturbation method has been done. Basically, the highest order term is elliptic operator on finite domain, the perturbation should be in the low orders and can be controlled by the elliptic operator. For instance, consider

$$\Delta u = f(u, Du, \epsilon) \text{in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

We assume the equation is solvable for $\epsilon = 0$. We also assume $f(u, Du, \epsilon)$ can be controlled by Δu. This means that $\Delta^{-1} f(u, Du, \epsilon)$ is a compact smooth map from, say H^1 to H^1. Then we can apply implicit function theorem to get the solution for small ϵ. In general, the term Du is harder to control, it relies on Sobolev embedding theorem.

If the underlying operator is wave opertor, or Schrödinger, then it is even harder. There is no such compactness property. Nash-Moser technique is introduced.

2.3 The Poincaré-Lindstedt Method

In the Duffin's equation:

$$\ddot{u} + u + \epsilon u^3 = 0, u(0) = 1, \dot{u}(0) = 0,$$

the regular perturbation method leds to a secular term, which is incorrect for large time. One way to solve this problem is to introduce a change of time scale. Let

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \cdots ,$$

$$\tau = \omega(\epsilon)t, \omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots ,$$

Then

$$\omega^2 u'' + u + \epsilon u^3 = 0, u(0) = 1, u'(0) = 0,$$

where $' = d/d\tau$. We plug the expansion above for u and ω into this equation, equating the coefficients of likely powers of ϵ, we get

$$(1 + 2\epsilon\omega_0\omega_1 + \cdots)(u_0'' + \epsilon u_1'' + \cdots) + (u_0 + \epsilon u_1 + \cdots) + \epsilon(u_0^3 + 3\epsilon u_0^2 u_1 + \cdots) = 0,$$

$$u_0(0) + \epsilon u_1(0) + \cdots = 1, \; u_0'(0) + \epsilon u_1'(0) + \cdots = 0,$$

$$u_0'' + u_0 = 0, u_0(0) = 1, u_0'(0) = 0, \cdots ,$$

$$u_1'' + u_1 = -2\omega_1 u_0'' - u_0^3, u_1(0) = u_1'(0) = 0,$$

This gives

$$u_0(\tau) = \cos\tau.$$

$$u_1'' + u_1 = 2\omega_1 \cos\tau - \cos^3\tau = (2\omega_1 - \frac{3}{4})\cos\tau - \frac{1}{4}\cos 3\tau.$$

If we choose $\omega_1 = 3/8$, then the secular term can be avoided. This leads

$$u_1(\tau) = \frac{1}{32}(\cos 3\tau - \cos\tau).$$

Thus, we get the expansion

$$u(\tau) = \cos\tau + \frac{\epsilon}{32}(\cos 3\tau - \cos\tau) + \cdots,$$

$$\tau = t + \frac{3}{8}\epsilon t + \cdots.$$

Homework.

- pp. 101: 8(a),
- pp. 102: 11, 13. 15.

2.4 Singular perturbation methods

2.4.1 Outer solutions, inner solutions and matched asymptotics

If the small parameter ϵ appears in the highest order term, then this term is unimportant in most of the region except in a small region where the high order derivatives are important. Let us see the following example:

$$au_x = \epsilon u_{xx}, x \in (0,1),$$

$$u(0) = 0, u(1) = 1.$$

Here, we assume $a < 0$. Physically, ϵ is the viscosity, a the advection velocity. In our present situation, the advection direction is toward left. This equation can be solved easily. We integrate it once to get

$$-au + \epsilon u_x = C_1,$$

where C_1 is a constant to be determined. Using method of separation of variable,

$$\frac{du}{C_1 + au} = \frac{dx}{\epsilon}.$$

Integrate this again, we get

$$\frac{1}{a} \ln(au + C_1) = \frac{x}{\epsilon} + C_2.$$

This gives

$$u = C_3 \exp\left(\frac{ax}{\epsilon}\right) + C_4.$$

Putting the boundary conditions, we get

$$C_3 + C_4 = 0, C_3 e^{a/\epsilon} + C_4 = 1.$$

These gives

$$C_3 = \frac{1}{e^{a/\epsilon} - 1}, \quad C_4 = -\frac{1}{e^{a/\epsilon} - 1}.$$

Hence, the exact solution is

$$u_\epsilon(x) = \frac{1 - e^{ax/\epsilon}}{1 - e^{a/\epsilon}}.$$

Next, we shall use perturbation method to find approximate solution. It consists of three steps: (1) finding outer solution, (2) finding inner solution, (3) matching the outer and inner solution.

Finding Outer solution Let us first solve this equation with $\epsilon = 0$:

$$au_x = 0.$$

This leads to $u = constant$. From boundary conditions, they are two possible solutions, $u = 0$ or $u = 1$. As we shall see later that the condition $a < 0$ leads to the flows move toward left and hence we should use the boundary condition $u(1) = 1$. This implies $u(x) \equiv 1$ for the unperturbed equation. This also means that $u(x) \equiv 1$ in the region where u is smooth (hence ϵu_{xx} is small. The solution

$$u_o(x) = 1$$

is called *outer solution*.

Finding inner solution However, u cannot be smooth through out the whole region $(0, 1)$ because $u(0) = 0$. Therefore, we expect there is an abrupt change of u near $x = 0$. This region is called *boundary layer*. Let us suppose its thickness is $\delta(\epsilon)$. We rescale $\xi = x/\delta$. The rescaled equation becomes

$$\frac{\epsilon}{\delta^2}\frac{d^2u}{d\xi^2} - \frac{a}{\delta}\frac{du}{d\xi} = 0.$$

If we want to have these two terms to be equally important, then we should take

$$\frac{\epsilon}{\delta^2} = \frac{1}{\delta}.$$

In this case, $\delta = \epsilon$. The resulting equation is

$$u_{\xi\xi} = au_\xi$$

with boundary condition $u(0) = 0$. Now, we get the same equation without ϵ. This gives

$$u_\xi = au + C_1.$$

The solution is

$$u(\xi) = C\left(e^{a\xi} - 1\right).$$

where C is to be determined. The solution

$$u_i(x) = C\left(e^{ax/\epsilon} - 1\right)$$

is called inner solution. This inner solution is an approximate in the region $(0, \epsilon)$. To determine C, we need to match $u_i(x)$ and $u_o(x)$ for x in some overlapping zone. A natural overlapping zone is when $\eta = x/\sqrt{\epsilon}$ with $\eta = O(1)$. In this case, let us fix η and we expect $u_o(\sqrt{\epsilon}\eta) - u_i(\sqrt{\epsilon}\eta) \to 0$ as $\epsilon \to 0$. This implies

$$\lim_{\epsilon \to 0} C\left(e^{a\eta/\sqrt{\epsilon}} - 1\right) = 1.$$

Since $a < 0$, we get from the above limit that $C = -1$. Thus,

$$u_i(x) = 1 - e^{ax/\epsilon}.$$

Match inner and outer solutions We can define an approximate solution to be the sum of the inner solution and outer solution minus the overlapping value. The overlapping value is 1. Thus, we define

$$u_a(x) = u_o(x) + u_i(x) - 1 = 1 - e^{ax/\epsilon}.$$

We see that $u_a(0) = 0$, $u_a(1) = 1 - e^{a/\epsilon}$. In fact,

$$\begin{aligned} u_\epsilon(x) - u_a(x) &= \left(1 - e^{ax/\epsilon}\right)\left(\frac{1}{1 - e^{a/\epsilon}} - 1\right) \\ &= \left(1 - e^{ax/\epsilon}\right)\frac{e^{a/\epsilon}}{1 - e^{a/\epsilon}} \end{aligned}$$

Remark. Notice that the above approximate solution $u_a(x)$ does not satisfy the boundary condition at $x = 1$, but has a small error. It does satisfy the equation and the boundary condition at $x = 0$. Alternatively, we can choose different approximate solution. For instance, let $\omega(x)$ be a weighted function which is 1 for $0 \le x \le 1/3$ and 0 for $2/3 \le x \le 1$ and smoothly and monotonely connect 1 and 0 for $1/3 \le x \le 2/3$. Using this weighted function, we define

$$u_a(x) = \omega\left(\frac{x}{\epsilon^\beta}\right) u_i(x) + \left(1 - \omega\left(\frac{x}{\epsilon^\beta}\right)\right) u_o(x).$$

Here, β is any number between 0 and 1. This approximate solution satisfies the boundary condition, but does not satisfy the equation, with a small residual.

Example This example is taken from J. Cole's book, Singular Perturbation, pp. 21. Consider

$$\epsilon u_{xx} + \sqrt{x} u_x - u = 0, x \in (0, 1),$$

$$u(0) = 0, u(1) = e^2.$$

1. **Finding outer solution:** just like regular perturbation method, we assume

$$u(x) = u_0(x) + \epsilon u_1(x) + \cdots.$$

Plugging this into equation and the boundary conditions, equating the coefficients of the like power terms of ϵ, we get

$$\sqrt{x} u_{0,x} = u_0, \ u_0(1) = e^2,$$

$$\sqrt{x} u_{1,x} = u_0 - u_{0,xx}, \ u_1(1) = 0.$$

This leads to the outer expansion:

$$u_o(x) = e^{2\sqrt{x}}\left(1 - \epsilon\left(\frac{1}{2x} - \frac{2}{\sqrt{x}} + \frac{3}{2}\right) + O\left(\frac{\epsilon^2}{x^{5/2}}\right)\right)$$

The reason why we only use the boundary condition at $x = 1$ is because the advection velocity is negative ($-\sqrt{x}$), which means that the upwind direction is right. By the method of characteristics, the solution is determined by its upwind data, thus, from the right. Hence the boundary condition for the outer solution is from $x = 1$.

2. **Finding inner solution:** The boundary layer occurs near $x = 0$. We rescale x by introducing the layer variable

$$\xi = \frac{x}{\delta(\epsilon)}.$$

The boundary layer expansion is

$$u = w_0(\xi)\nu_0(\epsilon) + w_1(\xi)\nu_1(\epsilon) + \cdots,$$

Plug this into the equation, we get

$$\frac{\epsilon}{\delta^2}\left(\nu_0 w_{0,\xi\xi} + \nu_1 w_{1,\xi\xi} + \cdots\right) + \frac{\sqrt{\delta}}{\delta}\sqrt{\xi}\left(\nu_0 w_{0,\xi} + \nu_1 w_{1,\xi} + \cdots\right) \quad -\nu_0 w_0 - \nu_1 w_1 - \cdots = 0.$$

We choose $\delta(\epsilon)$ so that

$$\frac{\epsilon}{\delta^2} = \frac{\sqrt{\delta}}{\delta}.$$

This gives

$$\delta(\epsilon) = \epsilon^{2/3}.$$

The dominant boundary equation becomes

$$w_{0,\xi\xi} + \sqrt{\xi} w_{0,\xi} = 0, \; w_0(0) = 0.$$

Its solution is

$$w_0(\xi) = C_0 \int_0^{\xi} \exp\left(-\frac{2}{3}\zeta^{3/2}\right) d\zeta.$$

Thus, the inner solution has the form:

$$u_i(x) = w_0\left(\frac{x}{\epsilon^{2/3}}\right) + \cdots.$$

3. **Matching:** Let the matching scale is

$$x_\eta = \frac{x}{\eta}$$

$\eta(\epsilon)$ is chosen so that, with fixed x_η, as $\epsilon \to 0$,

- the outer variable $x = \eta x_\eta \to 0$;
- the inner variable: $\xi = x/\delta = x_\eta \eta/\delta \to \infty$.

For instance, we can choose $\eta = \epsilon^\beta$ with $0 < \beta < 2/3$. The outer and inner solution should match in the overlapping zone where x_η is fixed. As $\epsilon \to 0$,

- The outer solution $u_0(x) \to 1$, as $x \to 0$.
- The inner solution

$$w_0(\xi) \to C_0 \int_0^{\infty} \exp\left(-\frac{2}{3}\zeta^{3/2}\right) d\zeta, \text{ as } \xi \to \infty.$$

To match these two limits, we should require

$$C_0 = \left(\int_0^\infty \exp\left(-\frac{2}{3}\zeta^{3/2}\right) d\zeta \right)^{-1}.$$

This gives the complete description of inner solution. The approximate solution is then defined to be

outer solution + inner solution − overlapped value

That is,

$$u_a(x) = u_o(x) + w_0\left(\frac{x}{\epsilon^{2/3}}\right) - 1.$$

Remark. The next term is $\nu_1(\epsilon) = \epsilon^{1/3}$. Check by yourself.

2.4.2 The boundary layers, initial layers and interior layers

Initial layer Let us consider a damped spring-mass system:

$$m\ddot{y} + a\dot{y} + ky = 0,$$

with $y(0) = 0$ and $m\dot{y}(0) = I$. This means that we apply an impulse at the mass at $t = 0$. The dimensions of these variables are

$$[m] = M, [a] = MT^{-1}, [k] = MT^{-2}, [I] = MLT^{-1}.$$

Three possible time scales are

$$\frac{m}{a}, \sqrt{\frac{m}{k}}, \frac{a}{k},$$

corresponding to balancing inertia and damping, inertia and spring stiffness, damping and spring stiffness. Possible length scales are

$$\frac{I}{a}, \frac{I}{\sqrt{mk}}, \frac{aI}{mk}$$

We expect that the impulse will cause an abrupt change of the mass position in short time, then relax to its equilibrium. The first time period is called an initial phase, the second is called an relaxation phase. In the initial phase, the dominated terms should be the inertia term and the damping terms. In the relaxation phase, it should be a balancing between damping and spring stiffness. Thus, for the realxation phase, we introduce the time scale

$$\bar{t} = \frac{t}{a/k}.$$

The equation becomes

$$\frac{ma^2}{k^2}\frac{d^2y}{d\bar{t}^2} + \frac{a^2}{k}\frac{dy}{d\bar{t}} + ky = 0,$$

In the initial phase, the amptitude of the mass is related to I and the damping, and the mass too. However, from dimensional analysis, the mass M appears in both I and a. Thus, the amptitude should be related only to I/a. Thus, in the relaxation phase, we rescale the length by

$$\bar{y} = \frac{y}{I/a}.$$

The rescaled equation becomes

$$\epsilon\bar{y}'' + \bar{y}' + \bar{y} = 0,$$

$$\bar{y}(0) = 0, \epsilon\bar{y}'(0) = 1.$$

Here, the dimensionless parameter

$$\epsilon = \frac{mk}{a^2} << 1.$$

The outer solution is a solution of $\bar{y}' + \bar{y} = 0$, this gives the outer solution

$$\bar{y}_o(\bar{t}) = Ce^{-\bar{t}}.$$

During the initial phase, we rescale $\tau = \bar{t}/\epsilon$ and $Y = \bar{y}$. Then

$$\frac{d^2Y}{d\tau^2} + \frac{dY}{d\tau} + \epsilon Y = 0.$$

The conditions $\bar{y}(0) = 0, \epsilon\bar{y}'(0) = 1$ gives the inner solution

$$\bar{y}_i(\bar{t}) = Y(\frac{\bar{t}}{\epsilon}) = 1 - e^{-\bar{t}/\epsilon}.$$

Matching the outer solution and inner slution in an overlapping zone, (i.e. $\bar{t}_\eta = \bar{t}\sqrt{\epsilon}$ is fixed), we should require

$$\lim_{\epsilon\to 0}\bar{y}_0(\bar{t}) = \lim_{\epsilon\to 0}\bar{y}_i(\bar{t}).$$

This leads to $C = 1$ in the outer solution. Thus, the final approximate solution is

$$\begin{aligned}\bar{y}_a(\bar{t}) &= \bar{y}_o(\bar{t}) + \bar{y}_i(\bar{t}) - \lim_{\epsilon\to 0}\bar{y}_i(\bar{t}) \\ &= e^{-\bar{t}} - e^{-\bar{t}/\epsilon}.\end{aligned}$$

In terms of the original variables, it reads

$$y_a(t) = \frac{I}{a}\left(e^{-kt/a} - e^{-at/m}\right).$$

Enzyme Kinetics Consider the following chemical reaction:

$$A + B \rightleftharpoons C \rightarrow P + B$$

Here, A (a substrate) and B (an enzyme) combine to form a molecule C. C breaks into a product P and an orginal enzyme B. Let a, b, etc represent their concentration. The equations for the kinetics are

$$\begin{aligned}
\dot{a} &= -k_1 ab + k_2 c \\
\dot{b} &= -k_1 ab + k_2 c + k_3 c \\
\dot{c} &= k_1 ab - k_2 c - k_3 c \\
\dot{p} &= k_3 c.
\end{aligned}$$

The initial concentrations are

$$a(0) = \hat{a}, b(0) = \hat{b}, c(0) = 0, p(0) = 0.$$

It is easy to see that $b + c = \hat{b}$. The first three equations do not involve p. Thus, we have the reduced equations

$$\begin{aligned}
\dot{a} &= -k_1 a(\hat{b} - c) + k_2 c, \\
\dot{c} &= k_1 a(\hat{b} - c) - (k_2 + k_3)c.
\end{aligned}$$

The dimensions of each quantities are

$$[a] = C, [c] = C, [k_1] = T^{-1}C^{-1}, [k_2] = [k_3] = T^{-1}.$$

Here, C is the dimension of concentration. We can rescale a and c by $\bar{a} = a/\hat{a}$ and $\bar{c} = c/\hat{b}$. We also rescale time by $\bar{t} = t/T$. The dimensionless equation becomes

$$\frac{\hat{a}}{T}\frac{d\bar{a}}{d\bar{t}} = -k_1 \hat{a}\hat{b}\bar{a}(1 - \bar{c}) + k_2 \hat{b}\bar{c},$$

$$\frac{\hat{b}}{T}\frac{d\bar{c}}{d\bar{t}} = k_1 \hat{a}\hat{b}\bar{a}(1 - \bar{c}) - (k_2 + k_3)\hat{b}\bar{c},$$

To determine the time scale T, we notice that our interest is to study how $\mathbf{A}$ is converted to $\mathbf{P}$ through the enzyme $\mathbf{B}$. Thus, the decreasing time scale should be the time scale we should concern. Thus, we should balance $\frac{\hat{a}}{T}$ and $k_1\hat{a}\hat{b}$ in the equation for $\bar{a}$. Thus, T is taken to be

$$T = \frac{1}{k_1\hat{b}}.$$

With this time scale, we obtain the dimensionless equations:

$$\begin{aligned} \frac{d\bar{a}}{d\bar{t}} &= -\bar{a} + (\bar{a} + \lambda)\bar{c}, \\ \epsilon\frac{d\bar{c}}{d\bar{t}} &= \bar{a} - (\bar{a} + \mu)\bar{c}. \end{aligned}$$

Here,

$$\epsilon = \frac{\hat{b}}{\hat{a}} << 1, \lambda = \frac{k_2}{\hat{a}k_1}, \mu = \frac{k_2 + k_3}{\hat{a}k_1}.$$

The initial conditions are

$$\bar{a}(0) = 1, \bar{c}(0) = 0.$$

The outer solution is obtained by setting $\epsilon = 0$. From this, we obtain $\bar{a} - (\bar{a} + \mu)\bar{c} = 0$, which gives

$$\bar{c} = \frac{\bar{a}}{\bar{a} + \mu},$$

and the first equation becomes

$$\frac{d\bar{a}}{d\bar{t}} = -\frac{\mu - \lambda}{1 + \frac{\mu}{\bar{a}}}$$

By separation of variable and integrating it, we get

$$\bar{a} + \mu \ln \bar{a} = -(\mu + \lambda)\bar{t} + K.$$

Here, the constant K will be determined from matiching with the inner solution. These two solutions a and c are our outer solutions. We denote them by a_o and c_o respectively.

In the initial layer, we use the non-dimensional variables

$$\tau = \frac{t}{\epsilon}, A = a, C = c.$$

The resulting equations

$$\begin{aligned} \frac{dA}{d\tau} &= \epsilon(-A + (A + \lambda)C) \\ \frac{dC}{d\tau} &= A - (A + \mu)C \end{aligned}$$

Setting $\epsilon = 0$, we obtain $dA/d\tau = 0$. From the initial condition $\bar{a} = 1$, we should take $A = 1$. Plug this into the second equation, we get

$$\frac{dC}{d\tau} = 1 - (1 + \mu)C.$$

With the initial condition $c(0) = 0$, we obtain

$$C = \frac{1 - e^{-(\mu+1)\tau}}{\mu + 1}.$$

For matching, we should have the outer solution $a_o(0) = A(\infty)$ and $c_o(0) = C(\infty)$. After some calculation, we get $K = 1$ from matching condition. Thus, the final approximate solution is

$$\begin{aligned} a_a(t) &= a_o(\bar{t}) \\ c_a(t) &= c_o(\bar{t}) + C(\tau) - \frac{1}{\mu + 1}. \end{aligned}$$

Boundary layers and internal layers We have seen that the Sturm-Liouville system

$$\epsilon u_{xx} - a(x)u_x - b(x)u = 0 \; x \in (0, 1),$$

$$u(0) = u_0, \; u(1) = u_1,$$

may have boundary layer. We will show that

- if $a(x) > 0$ for $x \in (0, 1)$, then the boundary layer appears at $x = 1$;
- if $a(x) < 0$ for $x \in (0, 1)$, then the boundary layer appears at $x = 0$;
- if $a(x)$ changes sign once in $(0, 1)$ with $a(0) > 0$ and $a(1) < 0$, then an internal layer is formed.

Shock wave Consider the Burgers' equation

$$uu_x = \epsilon u_{xx}, x \in (-1, 1)$$

with $u(-1) = 1$ and $u(1) = -1$. We can see that the outer solution is

$$u_o(x) = \begin{cases} 1 & -1 \le x < x_0 \\ -1 & x_0 < x \le 1 \end{cases}$$

The constant x_0 will be determined later.

For inner solution, we rescale it by $\bar{x} = (x - x_0)/\epsilon$. The equation becomes

$$\left(\frac{u^2}{2}\right)_{\bar{x}} = u_{\bar{x}\bar{x}}.$$

We integrate it once to get

$$\frac{u^2}{2} - u_{\bar{x}} = C.$$

Applying matching condition ($u(\pm\infty) = \mp 1$), we obtain $C = 1/2$. Using separation of variable, we get

$$\frac{2du}{u^2 - 1} = d\bar{x}.$$

Integrating

$$\ln\left(\frac{u + 1}{u - 1}\right) = \bar{x} - \bar{x}_0$$

Here, $\bar{x}_0$ is a constant. We can absorb $\bar{x}_0$ into x_0. Thus, we take $\bar{x}_0 = 0$. Then

$$\frac{u+1}{u-1} = e^{\bar{x}}.$$

This yields

$$u(\bar{x}) = \frac{1+e^{\bar{x}}}{1-e^{\bar{x}}} = -\tanh(\bar{x}).$$

The parameter x_0 is not unique. Any such interior layer solution with the outer solution forms an approximate solution unless we impose an extra condition. Usually, we impose an excess mass based on conservation of mass. This means that

$$\int_{-1}^{1} u(x) - u_o(x)\, dx = m$$

where m is called the excess mass. With this, $\bar{x}_0$ is determined by

$$\int_{-1}^{x_0} -\tanh\left(\frac{x-x_0}{\epsilon}\right) - 1, dx + \int_{x_0}^{1} -\tanh\left(\frac{x-x_0}{\epsilon}\right) + 1, dx = m$$

By taking $\epsilon \to 0$, we can obtain an approximation for x_0.

$$x_0 = \frac{m}{2} + O(\epsilon).$$

2.5 WKB method

The WKB method is a perturbation method for solving problems of the followng form

$$-\epsilon^2 u'' + q(x)u = 0$$

When $q < 0$, we expect exponential decay solution. When $q > 0$, we expect oscillatory solution. In both cases, we look solution of the form $e^{w(x)}$.

Nonoscillatory Case Consider

$$\epsilon^2 u'' - k(x)^2 u = 0, x \in (0, \infty)$$

Let us try regular perturbation for large x. We write

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3),$$

Plug into the equation, we get

$$-k^2 u_0 - \epsilon k^2 u_1 + \epsilon^2 (u_0'' - k^2 u_2) + \cdots = 0,$$

This leads to $u_0 = u_1 = u_2 = \cdots = 0$. We get no information from regular perturbation for large x. If we observe the equation more carefully, suppose $k > 0$, then we expect the solution decays for large x. Thus, let us figure how it decays by trying the ansatz $u = e^{w(x)}$. Then

$$u' = w' e^w, u'' = (w'' + w'^2)e^w,$$

and the equation becomes

$$\left(\epsilon^2 (w'' + w'^2) - k^2\right) e^w = 0.$$

Thus, as long as $w \neq -\infty$, we get

$$\epsilon^2(w'' + w'^2) - k(x)^2 = 0.$$

Let us introduce $v = \epsilon w'$ We have

$$\epsilon v' + v^2 - k(x)^2 = 0.$$

Apply the regular perturbation method

$$v = v_0 + \epsilon v_1 + \cdots$$

We get

$$v_0(x) = \pm k(x), v_1(x) = -\frac{k'}{2k}.$$

This gives

$$\epsilon w' = \pm k(x) - \epsilon \frac{k'}{2k} + O(\epsilon^2)$$

Or

$$w(x) = \frac{1}{\epsilon}\left(\pm \int_a^x k(\xi)\, d\xi - \frac{\epsilon}{2} \ln k(x) + O(\epsilon^2)\right).$$

Thus, we get an expansion

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{k(x)}} \exp\left(\pm \frac{1}{\epsilon} \int_a^x k(\xi)\, d\xi + O(\epsilon)\right) \\ &= \frac{1}{\sqrt{k(x)}} \exp\left(\pm \frac{1}{\epsilon} \int_a^x k(\xi)\, d\xi\right) (1 + O(\epsilon)). \end{aligned}$$

If we require $u(\infty)$ to be bounded, then we can only accept the exponential term. Suppose $k > 0$, for instance, then

$$u(x) = C\frac{1}{\sqrt{k(x)}} \exp\left(-\frac{1}{\epsilon} \int_a^x k(\xi)\, d\xi\right) (1 + O(\epsilon))$$

is admissible.

For $x \sim 0$, we can rescale x by $x' = x/\epsilon$. Expand $u(x')$ in Taylor series near $x' = 0$. We get

$$-2u_2 - 6u_3 x' + (k_0 + k_1 x' + \ldots)(u_0 + u_1 x + \ldots) = 0.$$

This leads to

$$\begin{aligned} k_0 u_0 - 2u_2 &= 0 \\ -6u_3 + k_0 u_1 + k_1 u_0 &= 0, \end{aligned}$$

We can determine u_0 from the boundary condition. We can determine u_1 from the matching with the outer solution. In doing so, we find that the outer solution $u = e^w$ is also suitable for $x \sim 0$.

Oscillatory cases For the Schrödinger equation

$$\epsilon^2 u'' + k(x)^2 u = 0,$$

where $k > 0$, we can approximate u by $e^{iw(x)}$. Then

$$u' = iw'e^{iw}, u'' = (iw'' - w'^2)e^{iw},$$

and equation becomes

$$\epsilon^2(iw'' - w'^2) + k^2 = 0.$$

If we introduce $v = \epsilon w'$, then

$$i\epsilon v' - v^2 + k^2 = 0.$$

The ansatz for the regular perturbation $v = v_0 + \epsilon v_1 + \cdots$ gives

$$v_0 = \pm k, v_1 = \frac{iv_0'}{2v_0} = i(\ln\sqrt{v_0})'.$$

$$w(x) = \pm\frac{1}{\epsilon}\int_a^x k(\xi)\,d\xi + i\ln\sqrt{\pm k(x)} + O(\epsilon).$$

$$u(x) = \exp\left(\pm i\int_a^x k(\xi)\,d\xi\right)\exp\left(-\ln\sqrt{\pm k(x)}\right)(1 + O(\epsilon)).$$

Thus,

$$u(x) = \frac{c_1}{\sqrt{k(x)}}\exp\left(\frac{i}{\epsilon}\int_a^x k(\xi)\,d\xi\right) + \frac{c_2}{\sqrt{k(x)}}\exp\left(-\frac{i}{\epsilon}\int_a^x k(\xi)\,d\xi\right).$$

2.5.1 Method of geometric optics

In optics, the governing equation is

$$u_{tt} = c(x)^2 \triangle u.$$

For waves with a fixed frequency ω, the solution has the form: $u(x,t) = e^{-i\omega t}u(x)$ and $u(x)$ satisfies the Helmholtz equation

$$\triangle u + \frac{\omega^2}{c(x)^2}u = 0.$$

We are interested in the high frequency approximation of its solution.

Let us rewrite $\omega = 1/\epsilon$. The ansatz is

$$u(x) = A(x,\epsilon)e^{i\phi(x)/\epsilon} = (A_0(x) + A_1(x)\epsilon + \cdots)e^{i\phi(x)/\epsilon}$$

where ϕ is a phase function, A the amptitude. We plug u into the Helmholtz equation to get

$$u_{x_i} = A_{x_i}e^{i\phi/\epsilon} + \frac{iA}{\epsilon}\phi_{x_i}e^{i\phi/\epsilon},$$

$$\triangle u = \left(\triangle A + \frac{2i}{\epsilon} \nabla A \cdot \nabla \phi + \frac{iA}{\epsilon} \triangle \phi - \frac{A}{\epsilon^2} |\nabla \phi|^2 \right) e^{i\phi/\epsilon}$$

$$\left(-\frac{A}{\epsilon^2} |\nabla \phi|^2 + \frac{i}{\epsilon} (A \triangle \phi + 2\nabla A \cdot \phi) + \triangle A \right) + \frac{A}{\epsilon^2 c(x)^2} = 0.$$

Expanding A in $A_0 + \epsilon A_1 + \cdots$ and equating the coefficients of ϵ, we get

$$|\nabla \phi|^2 = \frac{1}{c(x)^2},$$

$$A_0 \triangle \phi + 2\nabla A_0 \cdot \nabla \phi = 0.$$

The first equation is called the eikonal equation. The second equation is called the transport equation. In the second equation, if we rename $A_0 = \sqrt{\rho_0}$, then the transport equation becomes

$$\sqrt{\rho_0} \triangle \phi + 2\nabla \sqrt{\rho_0} \cdot \nabla \phi = \sqrt{\rho_0} \triangle \phi + \frac{1}{\sqrt{\rho_0}} \nabla \rho_0 \cdot \nabla \phi = 0.$$

Thus, this is equivalent to

$$\nabla \cdot (\rho_0 \nabla \phi) = 0.$$

WKB method for Schrödinger equation Consider the Schrödinger equation

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi.$$

We look solution of the form: $\psi = e^{w/\hbar}$. We have

$$\begin{aligned}
\partial_t \psi &= \frac{1}{\hbar} \partial_t w e^{w/\hbar}, \\
\nabla \psi &= \frac{1}{\hbar} \nabla w e^{w/\hbar}, \\
\nabla^2 \psi &= \left(\frac{1}{\hbar} \nabla^2 w + \frac{1}{\hbar^2} \sum_k (\partial_{x_k} w)^2 \right) e^{w/\hbar}.
\end{aligned}$$

Thus, we get

$$i\partial_t w = -\frac{1}{2m} \left(\hbar \nabla^2 w + \sum_k (\partial_{x_k} w)^2 \right) + V.$$

We write $w = R + iS$, real part and imaginary part. Then

$$i\partial_t (R + iS) = -\frac{1}{2m} \left(\hbar \nabla^2 (R + iS) + |\nabla R|^2 - |\nabla S|^2 + 2i \nabla R \cdot \nabla S \right) + V.$$

Equating the real part and imaginary part, we get

$$\partial_t R = -\frac{1}{2m} \left(\hbar \nabla^2 S + 2\nabla R \cdot \nabla S \right),$$

$$-\partial_t S = -\frac{1}{2m}\left(\hbar\nabla^2 R + |\nabla R|^2 - |\nabla S|^2\right) + V.$$

[1] Since $[S] = ET = ML^2T^{-1}$, $[\nabla S] = MLT^{-1}$, we can define $p = \nabla S$, $v = (\nabla S)/m$. Multiplying the first equation by $e^{2R/\hbar}$, we get

$$\begin{aligned} e^{2R/\hbar}\partial_t R &= -\frac{1}{2m}\left(\hbar e^{2R/\hbar}\nabla^2 S + 2e^{2R/\hbar}\nabla R\cdot\nabla S\right) \\ &= -\frac{\hbar}{2m}\nabla\cdot\left(e^{2R/\hbar}\nabla S\right) \end{aligned}$$

Next, we define $\rho = e^{2R/\hbar}$, use $v = \nabla S/m$, then the first equation can be written as

$$\partial_t\rho + \nabla\cdot(\rho v) = 0.$$

This is the continuity equation.

For the second equation

$$S_t + \frac{1}{2m}|\nabla S|^2 + V = \frac{1}{2m}(|\nabla R|^2 + \hbar\nabla^2 R)$$

We express the RHS in terms of ρ. We get

$$S_t + \frac{1}{2m}|\nabla S|^2 + V = \frac{\hbar^2}{4m}\frac{\nabla^2\rho}{\rho}.$$

We take gradient of this equation to get

$$mv_t + m\nabla\left(\frac{|v|^2}{2}\right) + \nabla V = \frac{\hbar^2}{4m}\nabla\left(\frac{\nabla^2\rho}{\rho}\right)$$

$$v_t + v\cdot\nabla v + \frac{1}{m}\nabla V = \frac{\hbar^2}{4m^2}\nabla\left(\frac{\nabla^2\rho}{\rho}\right)$$

Multiplying ρ to this equation, we get

$$\rho(v_t + v\cdot\nabla v) + \frac{\rho}{m}\nabla V = \frac{\rho\hbar^2}{4m^2}\nabla\left(\frac{\nabla^2\rho}{\rho}\right)$$

The left-hand side is the pressureless momentum equation for the gas dynamics. The right-hand side is a dispersion term. It regularizes the solution.

Thus, solving the outer solution based on the WKB approach for the Schrödinger equation is equivalent to solve a pressureless Euler equation. Although it is also not an easy job, the WKB approach provides us an insight of the macroscopic structure of the Schrödinger equationm. It also make a link between qnautum mechanics and classical mechanics.

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خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (اٰمین)

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