CHAPTER # 5 {Hilbert spaces}

Def: (5.1): ⓐ An inner product space is also called Pre-Hilbert space.

ⓑ An inner product space (ie a Pre-Hilbert space) X is called a Hilbert space if it is complete in the sense of a metric space.

Notation (5.2): we shall use H to represent a Hilber space.

Examples (5.3):- ① $\mathbb{R}^n$ is a Hilbert space with inner product defined by: $(x,y) = \sum_{i=1}^{n} x_i y_i$; $x, y \in \mathbb{R}^n$.

~~Because~~ we have proved that $\mathbb{R}^n$ is an inner product space and we know from analysis that $\mathbb{R}^n$ is also complete.

② $\mathbb{C}^n$ is ~~an inner~~ a Hilbert space with inner product ~~space~~ defined by: $(x,y) = \sum_{i=1}^{n} x_i \overline{y_i}$; $x, y \in \mathbb{C}^n$.

③ the space l_2 of all complex sequences $x = \{x_i\}$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$ is an inner product space under the inner product: $(x,y) = \sum_{i=1}^{\infty} x_i \overline{y_i}$; $y = \{y_i\} \in l_2$.

we also know that l_2 is complete, hence l_2 is a Hilbert space.

④ Every finite dimensional inner product space is a Hilbert space.

Because every finite dimensional inner product space is a finite dimensional n.l.s and we have proved in CH#1 that every finite dimensional n.l.s is complete.

Theorem (5.4) [Schwarz's Inequality]

If x and y are two vectors in a Hilbert space, then

$$|(x,y)| \leq \sqrt{(x,x)}\cdot\sqrt{(y,y)}$$
$$= \|x\|\,\|y\| \longrightarrow ①$$

and equality holds in ① iff x and y are linear dependent.

Pf: See corresponding Proof in Inner product spaces, only replacing Inner product spaces by Hilbert spaces.

Theorem (5.5): The Inner product in a Hilbert space H is jointly continuous ie is a continuous function.

Proof: See corresponding pf in Inner product spaces.

Theorem (5.6) (Parallelogram law)

If x and y are any vectors in a Hilbert space H, Then

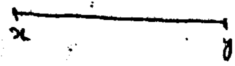
$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Pf:- Same as for Inner product spaces.

Exercise (5.7):- ① let $X = l_p$, $p>1$, $p \neq 2$ is a norm linear space but not a Hilbert space, because we have proved in CH#4 That it is not an Inner product space.

② The space $X = C[a,b]$ is not a Hilbert space, because we have Proved That it is not an Inner product space.

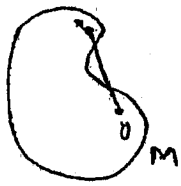
Recall: The line segment joining two given elements x and y of a space X is defined to be the set of all $z \in X$, of the form: $z = tx + (1-t)y$ for every real no: t such that $0 \le t \le 1$.



A subset M of X is said to be Convex if for every $x, y \in M$ the line segment joining x and y is contained in M

ie $z = tx + (1-t)y \in M$ for every t, where $0 \le t \le 1$.

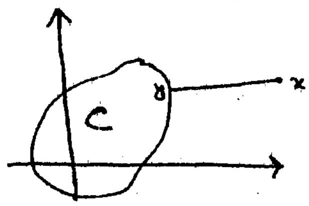
(Convex set)



(Not Convex).

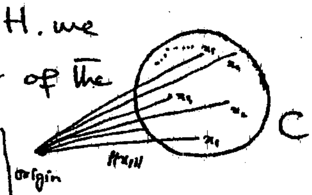
Definition (5.8): If C is any non-empty subset of a Hilbert space H, we define $d(x, C)$ (the distance from x to C) by

$$d(x, C) = \inf_{y \in C} \|x - y\|$$



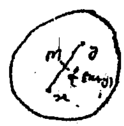
Theorem (5.9): A closed Convex subset C of a Hilbert space H Contains a unique vector of smallest norm.

Proof: let C be a closed Convex subset in H. we show that it Contains a unique vector of the smallest norm.



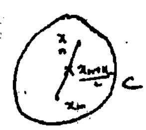
Since C is convex, so by above definition, it is non-empty and Contains $\frac{1}{2}(x+y)$, whenever it Contains x and y.

$d = \inf\{\|x_1\|, \|x_2\|, \cdots\}$
$= \|x_i\|$ (say)



let $d = \inf\{\|x\| : x \in C\}$, Then by def: of an infimum, There exists a sequence $\{x_n\}$ in C, such that $x_n \to d$. (because of a result).

and by the convexity of C; $\frac{1}{2}(x_n + x_m)$ is in C.



and $\left\|\frac{x_n+x_m}{2}\right\| \geq d$ (by def: of d)

$\Rightarrow \|x_n + x_m\| \geq 2d$.

Now using Parallelogram law, we have

$$\|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right).$$

$$\Rightarrow \|x_n - x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right) - \|x_n + x_m\|^2.$$

$$= 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2$$

$$\leq 2\|x_n\|^2 + 2\|x_m\|^2 - (2d)^2 \quad \left[\because \|x_n + x_m\| \geq 2d \Rightarrow -\|x_n+x_m\| \leq -2d\right]$$

$$= 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2.$$

$$\to 2d^2 + 2d^2 - 4d^2 = 0 \text{ as } n, m \to \infty \quad (\text{by above, } \|x_n\| \to d,\ \|x_m\| \to d)$$

$\Rightarrow \|x_n - x_m\|^2 \to 0$ as $n, m \to \infty$

This shows that $\{x_n\}$ is a Cauchy sequence in C.

Now since H is complete and C is a closed subspace of H, so C is complete, so the Cauchy sequence $\{x_n\}$ converges in C

i.e. $x_n \to x \in C$ (say); Then $x = \lim\limits_{n\to\infty} x_n$.

$$\Rightarrow \|x\| = \left\|\lim_{n\to\infty} x_n\right\| = \lim_{n\to\infty}\|x_n\| \quad (\because \text{norm is a contin: function})$$

$$= d \quad (\because \|x_n\| \to d)$$

i.e. x is a vector in C with smallest norm.

Now we show that x is unique. For this let us suppose that x' is another vector in C with $x' \neq x$, which also has norm d

i.e. $\|x'\| = d$.

Now x, x' are in C and C is convex, so that $\frac{1}{2}(x + x')$ is also in C and by applying parallelogram law, we have,

$$\left\|\frac{x+x'}{2}\right\|^2 + \left\|\frac{x-x'}{2}\right\|^2 = 2\left(\left\|\frac{x}{2}\right\|^2 + \left\|\frac{x'}{2}\right\|^2\right).$$

$$\Rightarrow \left\|\frac{x+x'}{2}\right\|^2 = 2\left(\left\|\frac{x}{2}\right\|^2+\left\|\frac{x'}{2}\right\|^2\right) - \left\|\frac{x-x'}{2}\right\|^2.$$
$$= 2\left(\frac{\|x\|^2}{4}+\frac{\|x'\|^2}{4}\right) - \left\|\frac{x-x'}{2}\right\|^2$$
$$= \frac{\|x\|^2}{2}+\frac{\|x'\|^2}{2} - \left\|\frac{x-x'}{2}\right\|^2.$$
$$\leq \frac{\|x\|^2}{2}+\frac{\|x'\|^2}{2}$$
$$= \frac{d^2}{2}+\frac{d^2}{2}$$
$$= d^2$$

ie $\left\|\frac{x+x'}{2}\right\| \leq d$, which is a Contradiction to the definition of d ($\because d = \inf\{\|x\| : x \in C\}$). This contradiction arises due to our wrong supposition that $x \neq x'$. Hence $x = x'$ ie x is unique.

This Completes the proof.

[Polarization identity]

Theorem (5.10): In any Hilbert space, The inner product is related to the norm by the following identity, called Polarization identity, which is,

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$
$$= \sum_{r=0}^{3} i^r \|x+i^r y\|^2.$$

Proof: Same as the proof in Inner product spaces.

Theorem (Assignment): If B is a complex Banach space whose norm obeys the parallelogram law and if an Inner product is defined on B by the polarization identity, then B is a Hilbert space.

Proof:-

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Definitions (5.11):

(1) Two vectors x and y in a Hilbert space H are said to be orthogonal if $(x,y)=0$. we express symbolically the orthogonal vectors x and y by $x \perp y$.

(2) A vector x is said to be orthogonal to a non-empty set A if $(x,y)=0$ for every y in A. we write it as $x \perp A$.

(3) Two non-empty sets A and B in a Hilbert space H are said to be orthogonal if $(x,y)=0$ for every x in A and every y in B. we write it as $A \perp B$.

(4) A set A is said to be orthogonal if for every pair of elements x,y in A with $x \neq y$, we have $(x,y)=0$.

Remark: (I) $x \perp 0$ for every x in a Hilbert space H.

(II) if $x \perp y$, then $y \perp x$.

(III) 0 is only vector orthogonal to itself.

(IV) if $x \perp y$, $x \perp z$, then $x \perp y+z$ and $x \perp \alpha y$ for any scaler α.

(V) if $x \perp y_n$, where $y_n \to y$; then $x \perp y$.

Proof: See proof in Inner product spaces.

Theorem (5.12) [Pythagorean Theorem]

(1) If x and y are orthogonal vectors in a Hilbert space H, then $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$.

(2) Generalized Pathagorean Thm:

If $\{x_1, x_2, \ldots, x_n\}$ is an orthogonal set in a Hilbert space H, then $\|x_1+x_2+\cdots+x_n\|^2 = \|x_1\|^2+\|x_2\|^2+\cdots+\|x_n\|^2$.

OR $\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$

Proof: See proof in Inner product spaces.

Definition (5.13):

If M is any subset of a Hilbert space H, Then the orthogonal Complement of M, denoted by $M^{\perp}$, is defined as :

$$M^{\perp} = \{x \in H : (x, y) = 0 \text{ for every } y \text{ in } M\}$$
$$= \{x \in H : x \perp M\}$$

and also
$$M^{\perp\perp} = (M^{\perp})^{\perp} = \{x \in H : (x, y) = 0 \text{ for every } y \text{ in } M^{\perp}\}$$
$$= \{x \in H : x \perp M^{\perp}\}.$$

Remark: From the above definition, it is clear That

① $\{0\}^{\perp} = H$ ② $H^{\perp} = \{0\}$.

Theorem (5.14):- let M_1, M_2 be subsets of a Hilbert space H, Then prove the following:

(I) $M_1 \subseteq M_1^{\perp\perp}$ ie any subset of H is contained in its double orthogonal Complement.

(II) if $M_1 \subseteq M_2$, Then $M_2^{\perp} \subseteq M_1^{\perp}$.

(III) $(M_1 \cup M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp}$ and $(M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.

(IV) $M_1^{\perp} = M_1^{\perp\perp\perp}$

(V) $M_1 \cap M_1^{\perp} \subseteq \{0\}$

(VI) $M_1^{\perp}$ is a closed linear space.

Proof:- (I)

(II) let $x \in M_2^{\perp} \Rightarrow (x, y) = 0$ for every y in M_2 (by def:)

$\Rightarrow (x, y) = 0$ for every y in M_1 ($\because M_1 \subseteq M_2$).

$\Rightarrow x \perp M_1$

$\Rightarrow x \in M_1^{\perp}$ (by def:)

So that $M_2^{\perp} \subseteq M_1^{\perp}$.

(III) Since $M_1 \subseteq M_1 \cup M_2$ and $M_2 \subseteq M_1 \cup M_2$ (always true)

$\Rightarrow (M_1 \cup M_2)^\perp \subseteq M_1^\perp$ and $(M_1 \cup M_2)^\perp \subseteq M_2^\perp$ (by (II)).

$\Rightarrow (M_1 \cup M_2)^\perp \subseteq M_1^\perp \cap M_2^\perp \longrightarrow$ ⓐ

Now let $x \in M_1^\perp \cap M_2^\perp \Rightarrow x \in M_1^\perp$ and $x \in M_2^\perp$

$\Rightarrow x \perp M_1$ and $x \perp M_2$

So by def:, $(x, u) = 0$ for every u in M_1 and

$(x, v) = 0$ for every v in M_2.

and so $(x, u) = 0$ for every u in $M_1 \cup M_2$.

$\Rightarrow x \in (M_1 \cup M_2)^\perp$

so that $M_1^\perp \cap M_2^\perp \subseteq (M_1 \cup M_2)^\perp \longrightarrow$ ⓑ

From ⓐ & ⓑ; we get: $(M_1 \cup M_2)^\perp = M_1^\perp \cap M_2^\perp$ #.

Next we show that $(M_1 \cap M_2)^\perp \supseteq M_1^\perp \cup M_2^\perp$.

For this since $M_1 \cap M_2 \subseteq M_1$ and $M_1 \cap M_2 \subseteq M_2$

$\Rightarrow M_1^\perp \subseteq (M_1 \cap M_2)^\perp$ and $M_2^\perp \subseteq (M_1 \cap M_2)^\perp$ (by (II)).

$\Rightarrow M_1^\perp \cup M_2^\perp \subseteq (M_1 \cap M_2)^\perp$ as required.

(IV) By part (I), we have: $M_1 \subseteq M_1^{\perp\perp}$

and so part (II), we have: $(M_1^{\perp\perp})^\perp \subseteq M_1^\perp$

ie $M_1^{\perp\perp\perp} \subseteq M_1^\perp \longrightarrow$ ⓐ ($M_1^\perp$ is [illegible] if it is [illegible] o. Complement)

Also by part (I), $M_1^\perp \subseteq (M_1^\perp)^{\perp\perp}$ ie $M_1^\perp \subseteq M_1^{\perp\perp\perp} \longrightarrow$ ⓑ

From ⓐ and ⓑ; we have $M_1^\perp = M_1^{\perp\perp\perp}$.

(V) If $M_1 \cap M_1^\perp = \phi$, then clearly $M_1 \cap M_1^\perp = \phi \subseteq \{0\}$

ie $M_1 \cap M_1^\perp \subseteq \{0\}$

If $M_1 \cap M_1^\perp \neq \phi$, then let $x \in M_1 \cap M_1^\perp \Rightarrow x \in M_1$ and $x \in M_1^\perp$.

Now since $x \in M_1^\perp \overset{\&\, x \in M_1}{\Rightarrow} (x, x) = 0$ ie $\|x\|^2 = 0$ ie $\|x\| = 0$ ie $x = 0 \in \{0\}$

ie $x \in \{0\}$. Therefore $M_1 \cap M_1^\perp \subseteq \{0\}$.

(VI) We show that $M_1^{\perp}$ is a closed linear subspace.

For this we first recall that "A subset M of a linear space X is a subspace of X if for any $x, y \in M$ and any scalars α, β, we have $\alpha x + \beta y \in M$".

Now let x, y be any two elements in $M_1^{\perp}$ and α, β be any scalars, then for any u in M_1, we have:

$(x, u) = 0$ and $(y, u) = 0$ and therefore:

$$\begin{aligned}(\alpha x + \beta y, u) &= (\alpha x, u) + (\beta y, u) \\ &= \alpha (x, u) + \beta (y, u) \\ &= \alpha \cdot 0 + \beta \cdot 0 \\ &= 0\end{aligned}$$

ie $(\alpha x + \beta y, u) = 0$ for any u in M_1.

$\Rightarrow \alpha x + \beta y \in M_1^{\perp}$, which shows that $M_1^{\perp}$ is a subspace of H.

To complete the proof, it remains to show that $M_1^{\perp}$ is closed and in order to prove this, it is enough to show that "if $\{x_n\}$ is any convergent sequence in $M_1^{\perp}$ converging to a point x (say) ie $x_n \to x$, then $x \in M_1^{\perp}$".

Now for any $u \in M_1$, we can write:

$(x, u) = \left(\lim_{n\to\infty} x_n, u\right)$ $\quad [\because x_n \to x$ ie $\lim_{n\to\infty} x_n = x]$

$= \lim_{n\to\infty} (x_n, u)$ $\quad [\because$ Inner product is contin: function$]$

$= 0$, because $x_n \in M_1^{\perp}$ $\forall n$ as $\{x_n\}$ is a seq: in $M_1^{\perp}$.

ie $(x, u) = 0$ for any $u \in M_1$. $\Rightarrow x \perp M_1$

$\Rightarrow x \in M_1^{\perp}$. Thus $M_1^{\perp}$ is closed linear subspace of H. Thus completing the proof.

Theorem (5.11): If M is a (closed) linear subspace of a Hilbert space H, then $M \cap M^{\perp} = \{0\}$.

Proof:- Let $x \in M \cap M^{\perp}$. Then $x \in M$ and $x \in M^{\perp} \Rightarrow x \perp M$

$\Rightarrow (x, y) = 0$ for every y in M.

$\Rightarrow (x, x) = 0$, because $x \in M$.

$\Rightarrow \|x\|^2 = 0 \Rightarrow \|x\| = 0 \Rightarrow x = 0$.

This shows that $0 \in M \cap M^{\perp} \Rightarrow \{0\} \subseteq M \cap M^{\perp}$

But we know that part (V) of previous Thm, $M \cap M^{\perp} \subseteq \{0\}$

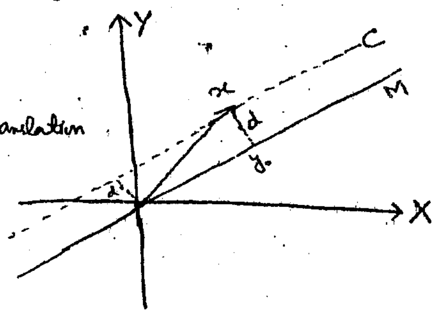
Hence $M \cap M^{\perp} = \{0\}$

Remark:- For sets M and $M^{\perp}$, $M \cap M^{\perp} \subseteq \{0\}$ and for subspaces M and $M^{\perp}$, $M \cap M^{\perp} = \{0\}$.
The reason is that it is not necessary for 0 to present in any subset but ~~for~~ every subspace contains 0.

Recall: ① Any subspace of a linear space X is convex.

② For any subspace M of a linear space X and $x \in X$, the set $x + M = \{x + m : m \in M\}$ is convex.

Theorem (5.12): Let M be a closed linear subspace of a Hilbert space H. Let x be a vector not in M and let $d = d(x, M)$, Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.



Proof:- Let us set $C = x + M$ (The translation of M by x); Then the set $C = x + M$ is a closed convex set and d is the distance from the origin to C (see Figure).

So by Thm (5.8), there exists a unique vector say z_0 in C such that $\|z_0\| = d = \inf\{\|z\| : z \in C\}$.

Since $z_0 \in C$, so by def. of C, $z_0 = x + y$ for some $y \in M$.
Let us put $x - z_0 = y_0$, then the vector $y_0 = x - z_0$ is easily seen to be in M ($\because z_0 = x + y$, $y_0 = x - z_0 = x - x - y = -y \in M$)
and $\|z_0\| = \|x - y_0\|$ ($\because y_0 = x - z_0$)

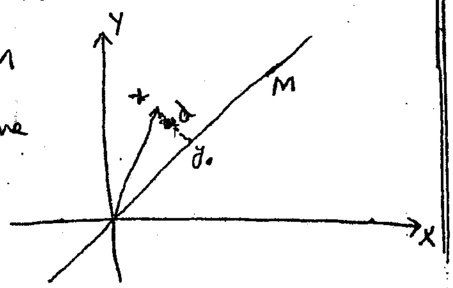
i.e. $\|x - y_0\| = \|z_0\| = d$ (from above)

Thus there exists a vector y_0 in M such that $\|x - y_0\| = d$.
It remains to prove the uniqueness of the vector y_0.
For this let y_1 be another vector in M such that $y_0 \neq y_1$ and $\|x - y_1\| = d$.

Then $z_1 = x - y_1$ is a vector in C ($\because C = x + M$) such that $z_1 \neq z_0$ and $\|z_1\| = \|x - y_1\| = d$ i.e. $\|z_1\| = d$, which is a contradiction to the fact that "there is a unique vector z_0 in C such that $\|z_0\| = d$". This is because of our wrong supposition. So $y_0 = y_1$ and Hence there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

Theorem (5.17):- If M is a proper closed linear subspace of a Hilbert space H, then there exists a non-zero vector z_0 in H such that $z_0 \perp M$.

Proof: Let x be a vector not in M and let $d = d(x, M)$, then by above Thm, there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.



let us take $z_0 = x - y_0$ and observe that since $d > 0$, so z_0 is a non-zero vector in H $\left(\because \|z_0\| = \underset{=d>0}{\|x-y_0\|}\right)$

In order to show that $z_0 \perp M$, it is enough to show that $z_0 \perp y$ for every y in M.

For this let $\lambda \in \mathbb{C}$, then we have:

$$\|z_0 - \lambda y\| = \|x - y_0 - \lambda y\| \quad (\because z_0 = x - y_0)$$
$$= \|x - (y_0 + \lambda y)\|$$
$$\geq d \quad [\because \text{by defi. of } d]$$
$$= \|z_0\| \quad (\because \|z_0\| = \|x - y_0\| = d)$$

ie $\|z_0 - \lambda y\| \geq \|z_0\|$

$$\Rightarrow \|z_0 - \lambda y\|^2 \geq \|z_0\|^2$$

and from this, we have:

$$(z_0 - \lambda y, z_0 - \lambda y) \geq (z_0, z_0)$$

$$\Rightarrow (z_0, z_0) - (z_0, \lambda y) - (\lambda y, z_0) + (\lambda y, \lambda y) \geq (z_0, z_0).$$

$$\Rightarrow -\bar{\lambda}(z_0, y) - \lambda(y, z_0) + \lambda\bar{\lambda}(y, y) \geq 0$$

$$\Rightarrow -\bar{\lambda}(z_0, y) - \lambda\overline{(z_0, y)} + \lambda\bar{\lambda}(y, y) \geq 0.$$

$$\Rightarrow -\bar{\lambda}(z_0, y) - \lambda\overline{(z_0, y)} + |\lambda|^2 (y, y) \geq 0 \longrightarrow (1)$$

Put $\lambda = \mu(z_0, y)$ for an arbitrary real number μ,

Then (1) becomes:

$$-\overline{\mu(z_0, y)}(z_0, y) - \mu(z_0, y)\overline{(z_0, y)} + |\mu(z_0, y)|^2 (y, y) \geq 0.$$

$$\Rightarrow -\mu|(z_0, y)|^2 - \mu|(z_0, y)|^2 + \mu^2 |(z_0, y)|^2 (y, y) \geq 0. \quad [\because \mu \in \mathbb{R}]$$

$$\Rightarrow -2\mu|(z_0, y)|^2 + \mu^2 |(z_0, y)|^2 \|y\|^2 \geq 0 \longrightarrow (2)$$

Now put $a = |(z_0, y)|^2$ and $b = \|y\|^2$, then from (2), we obtain:

$-2\mu a + \mu^2 ab \geq 0, \ \forall$ real nos: μ

$\Rightarrow \mu a(\mu b - 2) \geq 0$; $\forall$ real nos: $\mu \longrightarrow (3)$

However if $a > 0$, then (3) is impossible for all sufficiently small positive μ eg: for $a=1, b=1, \mu=1$ we get $-1 \geq 0$, which is not possible.

we see from this that $a=0$ is only possibility. which means $|(z_0,y)|^2 = 0$ $(\because a = |(z_0,y)|^2)$

$\Rightarrow |(z_0,y)| = 0 \quad \Rightarrow (z_0,y) = 0 \quad \Rightarrow z_0 \perp y$ for all y in M

$\Rightarrow z_0 \perp M$, which completes the proof.

Definition (5.18):- let M and N be two subspaces of a linear space L. we define $M+N = \{x+y : x \in M, y \in N\}$

Since M and N are subspaces, it is easy to see that $M+N$ is also a subspace spaned (generated) by all vectors in M and N together ie $M+N = [M \cup N]$.

Definition (5.19): If $M+N = L$, then we say that L is the sum of the subspaces M and N.

This means that any evector in L is expressible as the sum of a vector in M and a vector in N ie if $z \in L$, then $z = x+y$ where $x \in M$ & $y \in N$.

If each vector z in L is expressible uniquely in the form of $z = x+y$ with $x \in M$ & $y \in M$; then we say that L is the direct sum of the subspace M and N. Symbolically we write it as $L = M \oplus N$.

Theorem (Recall):- Let a linear space L be the sum of two subspaces M and N ie $L = M+N$; Then $L = M \oplus N$ iff $M \cap N = \{0\}$.

Remark:- The condition in this above Thm that the subspaces M and N have only the origin in common is often expressed by saying that M and N are disjoint. (Which is the main diff: b/w the disjoint sets and disjoint spaces).

Remark: Two non-empty sets S_1, S_2 of a Hilbert space H are said to be orthogonal (written as $S_1 \perp S_2$) if $x \perp y$ for all $x \in S_1$ and for all $y \in S_2$.

Theorem (5-20):- If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, Then the linear subspace $M+N$ is also closed.

Proof:- To show that $M+N$ is closed, we need to show that all the limit points of $M+N$ are in $M+N$.
Let $\{z_n\}$ be a sequence in $M+N$ converging to a limit point say z. It is enough to show that z is in $M+N$.
Since $M \perp N$, we see that M and N are disjoint (ie $M \cap N = \{0\}$)
So by above Thm, the sum $M+N$ can be strengthened to the direct sum $M \oplus N$ and

Thus each z_n can be expressed uniquely in the form $z_n = x_n + y_n$; where x_n is in M and y_n is in N.
Since x_n and y_n are orthogonal ($\because M \perp N$), so by Pethagorean Theorem, we have:

$$\begin{aligned} \|z_n - z_m\|^2 &= \|(x_n+y_n) - (x_m+y_m)\|^2 \quad [z_n = x_n + y_n] \\ &= \|(x_n - x_m) + (y_n - y_m)\|^2 \\ &= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 \quad \text{(by Peth: Thm:)} \end{aligned}$$

ie $\|z_n - z_m\|^2 = \|x_n - x_m\|^2 + \|y_n - y_m\|^2$

Since $\{z_n\}$ is a Cauchy sequence (being cgt.), so by def:

$\|z_n - z_m\| < \varepsilon$ for $m, n \geq N$.

So from above, we have:

$\|x_n - x_m\|^2 + \|y_n - y_m\|^2 = \|z_n - z_m\|^2 < \varepsilon^2$ for $m, n \geq N$.

$\Rightarrow \|x_n - x_m\|^2 + \|y_n - y_m\|^2 < \varepsilon^2$ for $m, n \geq N$.

$\Rightarrow \|x_n - x_m\|^2 < \varepsilon^2, \quad \|y_n - y_m\|^2 < \varepsilon^2$ for $m, n \geq N$.

$\Rightarrow \|x_n - x_m\| < \varepsilon, \quad \|y_n - y_m\| < \varepsilon$ for $m, n \geq N$

which shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in M and N respectively.

Also M and N are closed subspace of the complete space (Hilbert space) H, so M and N are complete.

So by the completeness, there exists vectors x and y in M and N respectively such that $x_n \to x$ and $y_n \to y$.

Since $x+y$ is a vector in $M+N$, so we have:

$z = \lim z_n = \lim (x_n + y_n) = \lim x_n + \lim y_n = x + y \in M+N$

$\Rightarrow z \in M+N$. Thus $M+N$ is closed. Thus completing the proof.

Theorem (5.21): [Projection Theorem]

Statement: If M is a closed linear subspace of a Hilbert space H, Then $H = M \oplus M^{\perp}$.

Proof: let M be a closed linear subspace of H, Then $M^{\perp}$ is also a closed linear subspace of H (Proved already).
Also M and $M^{\perp}$ are orthogonal, because if $x \in M^{\perp}$, Then by def: $(x,y)=0$ for all y in M. Since x was choosen arbitrary in $M^{\perp}$, so $(x,y)=0$ for every y in M and every x in $M^{\perp}$
So That $M^{\perp} \perp M$.

Thus M and $M^{\perp}$ are orthogonal closed linear subspaces of H, Therefore $M + M^{\perp}$ is also a closed linear subspace of H (by previous result).

we need to show that $H = M \oplus M^{\perp}$.

First we show That $H = M + M^{\perp}$.

On the Contrary, assume that $H \neq M + M^{\perp}$, Then $M + M^{\perp}$ is a proper closed linear subspace of H. Then by Theorem (5.17), There exists a non-zero vector z_0 such that $z_0 \perp (M + M^{\perp})$. So $z_0 \in (M + M^{\perp})^{\perp}$ [by def. of orthogonal complement of a set].

Now $M \subseteq M + M^{\perp} \Rightarrow (M + M^{\perp})^{\perp} \subseteq M^{\perp}$

and $M^{\perp} \subseteq M + M^{\perp} \Rightarrow (M + M^{\perp})^{\perp} \subseteq M^{\perp\perp}$

So that $(M + M^{\perp})^{\perp} \subseteq M^{\perp} \cap M^{\perp\perp} = \{0\}$ ($\because$ $M^{\perp}$ & $M^{\perp\perp}$ are orthog: because $M \perp M^{\perp}$)

Hence $z_0 \in (M + M^{\perp})^{\perp} \subseteq \{0\} \Rightarrow z_0 \in \{0\} \Rightarrow z_0$ is a zero vector, which is a contradiction to the fact that $z_0 \neq 0$.
So our supposition was wrong and hence $H = M + M^{\perp}$.

To complete the proof, it is enough to observe that since M and $M^\perp$ are orthogonal, so $M \cap M^\perp = \{0\}$. Thus by Theorem (page 15), the statement $H = M + M^\perp$ can be strengthened to the $H = M \oplus M^\perp$.

This completes the required result.

Orthonormal sets in Hilbert spaces:

Def. (5.22): An orthonormal set in a Hilbert space H is a non-empty subset of H which consist of mutually orthogonal unit vector; that is, it is a non-empty subset $\{e_i\}$ of H with the following properties.

(i) $(e_i, e_j) = 0$ if $i \neq j$

(II) $(e_i, e_j) = 1$ if $i = j$.

Examples (5.23): see examples following the definition of orthonormal sets in Inner product spaces.

Remark (5.24): If $H = \{0\}$ ie H contains only the zero element, then it has no orthonormal set.

If H contains a non-zero vector x, then we can construct e by normalizing x, that is $e = \frac{x}{\|x\|}$. Then the single element set $\{e\}$ is clearly an ~~orthogon~~ orthonormal set because $(e,e) = \|e\|^2 = \left\|\frac{x}{\|x\|}\right\|^2 = \frac{\|x\|^2}{\|x\|^2} = 1$.

Generally speaking if $\{x_i\}$ is a non-empty set of mutually orthogonal non-zero vectors in H, and if the

x_i's are normalized by replacing each of them by $e_i = \dfrac{x_i}{\|x_i\|}$, then the resulting set $\{e_i\}$ is orthonormal set.

Theorem (5.25): let $S = \{e_1, e_2, \dots, e_n\}$ be an orthonormal set in a Hilbert space H. If x is any vector in H, then

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2. \quad \text{(Bessel's inequality)} \longrightarrow (1)$$

and $\quad x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j \quad$ for each j

i.e. $\quad x - \sum_{i=1}^{n} (x, e_i) e_i \perp S.$

Proof: We have:

$$0 \le \left\| x - \sum_{i=1}^{n} (x, e_i) e_i \right\|^2 \quad \text{(obvious)}$$

$$= \left(x - \sum_{i=1}^{n} (x, e_i) e_i \,,\; x - \sum_{j=1}^{n} (x, e_j) e_j \right)$$

$$= (x, x) - \left(x, \sum_{j=1}^{n} (x, e_j) e_j \right) - \left(\sum_{i=1}^{n} (x, e_i) e_i, x \right) + \left(\sum_{i=1}^{n} (x, e_i) e_i ; \sum_{j=1}^{n} (x, e_j) e_j \right).$$

$$= (x, x) - \sum_{j=1}^{n} \overline{(x, e_j)} (x, e_j) - \sum_{i=1}^{n} (x, e_i)(e_i, x) + \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i) \overline{(x, e_j)} (e_i, e_j).$$

$$= (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i) \overline{(x, e_j)} (e_i, e_j)$$

$$= (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 + \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)} (e_i, e_i)$$

[∵ S is an o.n.s so by putting values of i, we get inn. prod. equal to zero]

$$= (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 - \sum_{i=1}^{n} |\cancel{(x, e_i)}|^2 + \sum_{i=1}^{n} |\cancel{(x, e_i)}|^2 \cdot 1$$

$$\Rightarrow 0 \le (x, x) - \sum_{j=1}^{n} |(x, e_j)|^2 \Rightarrow \sum_{j=1}^{n} |(x, e_j)|^2 \le (x, x)$$

$$\Rightarrow \sum_{j=1}^{n} |(x, e_j)|^2 \le \|x\|^2,$$ which is equivalent to (1).

In order to show that $x - \sum_{i=1}^{n} (x, e_i) e_i \perp S$, Consider any e_j in S where $j = 1, 2, \ldots, n$.

Then
$$\left(x - \sum_{i=1}^{n} (x, e_i) e_i, e_j\right) = (x, e_j) - \left(\sum_{i=1}^{n} (x, e_i) e_i, e_j\right)$$
$$= (x, e_j) - \sum_{i=1}^{n} (x, e_i)(e_i, e_j)$$
$$= (x, e_j) - (x, e_j)(e_j, e_j)$$
($\because$ for all i's the second term on R.H.S is zero because S is an o.n. set)
$$= (x, e_j) - (x, e_j) \cdot 1$$
$$= 0$$

This shows that $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$ for each j

$$\Rightarrow x - \sum_{i=1}^{n} (x, e_i) e_i \perp S.$$

Thus completing the proof.

<u>Theorem (5.26)</u>: If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is any vector in H, the the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable.

<u>Proof:-</u>

Theorem (5.27) [Generalization of Bessel's Inequality]

If $\{e_i\}$ is an orthonormal set in a Hilbert Space H, then

$$\sum |(x, e_i)|^2 \le \|x\|^2 \quad \text{for every vector } x \text{ in } H. \quad \to (1)$$

Proof: let us define a set S as:

$$S = \{e_i : (x, e_i) \neq 0\}$$

Then by Thm (5.26), S is either empty or Countable.

If S is empty, then $(x, e_i) = 0$, so $\sum |(x, e_i)|^2$ is zero and so in This Case (1) reduces to $0 \le \|x\|^2$ which is obviously true.

If S is Countable, then S is finite or Countably infinite.

When S is finite let it can be written in the form

$S = \{e_1, e_2, \ldots\ldots, e_n\}$ for some positive integer n.

In This Case, we denote $\sum |(x, e_i)|^2$ to be $\sum_{i=1}^{n} |(x, e_i)|^2$ which is clearly independent of the order in which the vectors of S are arranged. So Inequality (1) reduces to

$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2$, which is the Bessel Inequality when $\{e_i\}$ is finite ~~orthogonal~~ orthonormal set and it has been proved already in Theorem (5.25).

When S is Countably infinite: let the vectors in S be arranged in some definite order ie $S = \{e_1, e_2, \ldots, e_n, \ldots\ldots\}$

Now by the Theory of "absolutely Convergent Series" we know that "if $\sum_{i=1}^{\infty} |(x, e_i)|^2$ Converges, then every Series obtained from this series by rearranging its terms also converges and all such series have the same sum".

So we therefore can define: $\Sigma |(x, e_i)|^2$ to be $\sum_{i=1}^{\infty} |(x,e_i)|^2$.
and it follows from the above remark that $\Sigma |(x,e_i)|^2$ is a non-negative extended real number, which depends only on S and not on the arrangement of vectors in S. So in this case ① reduces to:

$$\sum_{i=1}^{\infty} |(x,e_i)|^2 \leq \|x\|^2 \longrightarrow ②$$

Now from Bessel's inequality for finite case, we have:

$$\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$$

It follows that no partial sum of the series on the left of ② can exceedes $\|x\|^2$ and so it is clear that ② is true ($\because \sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$
$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2 \Rightarrow \sum_{i=1}^{\infty} |(x,e_i)|^2 \leq \|x\|^2$)

This completes the proof.

Recall: ① Let P be a set of elements. Suppose there is a binary relation defined between certain pairs a,b of P, expressed symbolically by $a < b$, with the properties
(I) If $a < b$ and $b < c$, then $a < c$ (Transitivity)
(II) If $a \in P$, then $a < a$ (Reflexivity)
(III) If $a < b$ and $b < a$, then $a = b$ (Antisymmetry)
Then P is said to be partially ordered set.
For example if P is the set of all subsets of a given set X, the set inclusion $(A \subseteq B)$ gives a partial ordering of P.
② If P is a partially ordered set. Moreover if for any pair a,b in P either $a < b$ or $b < a$, then P is said to be completely (totally, linearly, simply) ordered set.

A completely ordered set is called a chain.

eg: The real numbers are completely ordered by the relation "a is less than or equal to b ie $a \le b$".

Zorn's Lemma: (only Recall) (5.28)

Let P be a non-empty partially ordered set with the property that every completely ordered subset of P has an upper bound. Then P contains at least one maximal element.

Theorem (5.29) Every non-zero Hilbert space H contains a complete orthonormal set.

Proof:- Let $H \neq \{0\}$ and M be the set of all subsets of H which are orthonormal. We define partially ordering in M by the usual set inclusion, so that M is a partially ordered set.

Since $H \neq \{0\}$, therefore if $x \neq 0$ is a vector in H, then $\{y\} \in M$ where $y = \frac{x}{\|x\|}$. ($\because$ by Previous remark $\{y\}$ is an orthonormal set).

So the set M of all orthonormal sets is non-empty.

Now let $C = \{E_\lambda : \lambda \in \Delta\}$ be an increasing chain of orthonormal subsets in M (ie $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$)

Then $\bigcup_i E_i$ is the upper bound of C.

Now M is a partially ordered set and every chain in M has its upper bound, so by "Zorn's Lemma", there exists a maximal element in M. Let A be that element that is the set which is maximal in M so that H contains a complete orthonormal set.

Theorem (5.30) (Assignment): Let $\{e_i\}$ be an orthonormal set in a Hilbert space H and let x be a vector in H, Then

$$x - \sum (x, e_i) e_i \perp \{e_i\}.$$

Theorem (5.31): Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H, then the following are equivalent.

(a) $\{e_i\}$ is complete.

(b) $x \perp \{e_i\} \Rightarrow x = 0$

(c) if x is an arbitrary vector in H, then $x = \sum (x, e_i) e_i$.

(d) if x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$.

Proof: (a) $\Rightarrow$ (b)

Suppose (a) is true ie $\{e_i\}$ is complete $\Rightarrow$ $\{e_i\}$ is maximal o.n.s.
On contrary suppose that (b) is not true, then there exists a vector $x \neq 0$ such that $x \perp \{e_i\}$.
Define $e = \frac{x}{\|x\|}$ (Normalization of x), then the set $\{e_i, e\}$ is an orthonormal set, which properly contains $\{e_i\}$, but this contradicts the completeness of $\{e_i\}$. Hence (b) is true.

(b) $\Rightarrow$ (c)

Suppose that (b) is true ie $x \perp \{e_i\} \Rightarrow x = 0$.
Now by (5.30), we have $x - \sum (x, e_i) e_i$ is orthogonal to $\{e_i\}$
ie $x - \sum (x, e_i) e_i \perp \{e_i\}$
so by (b), we get: $x - \sum (x, e_i) e_i = 0$

or $x = \sum (x, e_i) e_i$ for any vector x in H. Hence (c) is true.

(c) $\Rightarrow$ (d) Suppose that (c) is true ie $x = \sum (x, e_i) e_i$ for any vector x in H.

Now $x = \sum (x, e_i) e_i = \sum_{i=1}^{\infty} (x, e_i) e_i$

Then $\|x\|^2 = (x, x) = \left(x, \sum_{i=1}^{\infty} (x, e_i) e_i\right)$ $\left[\because x = \sum (x, e_i) e_i = \sum_{i=1}^{\infty} (x, e_i) e_i\right]$

$= \left(x, \lim_{n \to \infty} \sum_{i=1}^{n} (x, e_i) e_i\right)$

$= \lim_{n \to \infty} \sum_{i=1}^{n} \overline{(x, e_i)} (x, e_i)$ [$\because$ Inner product is continuous].

$$\Rightarrow \|x\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |(x,e_i)|^2$$
$$= \sum_{i=1}^{\infty} |(x,e_i)|^2$$

using $\Sigma (x,e_i)e_i$ in place of $\sum_{i=1}^{\infty} (x,e_i)e_i$, we get

$\|x\|^2 = \Sigma |(x,e_i)|^2$. Hence (d) is true.

Finally (d) $\Longrightarrow$ (a)

suppose that (d) is true ie $\|x\|^2 = \Sigma |(x,e_i)|^2$.
we show that (a) is true. on the contrary assume that (a) is not true ie $\{e_i\}$ is not complete, then it is properly contained in an orthonormal set $\{e_i, e\}$.

~~Since~~ so by definition of orthonormal set, we can say that e is orthogonal to e_i's.

$$\text{Now } \|e\|^2 = \Sigma |(e,e_i)|^2 \quad (\because \text{by (d)})$$
$$= \Sigma \|0\|^2 \quad (\text{o is a vector, therefore we take norm})$$
$$= \|0\|$$
$$= 0$$

ie $\|e\| = 0$

and this contradicts the fact that $\|e\| = 1$
so our supposition was wrong and hence $\{e_i\}$ is complete
Hence (a) is true.
This completes the required proof.

<u>Remark (5.32)</u>: let $\{e_i\}$ be a complete orthonormal set and let x be an arbitrary vector in a Hilbert space H. Then the numbers (x,e_i) are called the Fourier coefficients of x, the expression $(x,e_i)e_i$ is called the Fourier expansion of x and the equation $\|x\|^2 = \Sigma |(x,e_i)|^2$ is called Parseval's equation or formula — all w.r.t the particular complete orthonormal set $\{e_i\}$ under consideration.

—*—*—*—*—*—*—*—

The Gram-Schmidt orthogonalization process:

It is a Constructive procedure for converting a linearly independent set $\{x_1, x_2, \cdots, x_n, \cdots\}$ into corresponding orthonormal set $\{e_1, e_2, \cdots, e_n, \cdots\}$ with the property that for each n, the linear subspace spanned by $\{e_1, e_2, \cdots, e_n\}$ is the same as that spaned by $\{x_1, x_2, \cdots, x_n\}$.

We state this process in the form of the following theorem.

Theorem (5.33): Suppose that $\{x_1, x_2, \cdots, x_n, \cdots\}$ is a linearly independent set in a Hilbert space H, Then there exists an orthonormal set $\{e_1, e_2, \cdots, e_n, \cdots\}$ with the property that for each n, the linear subspace spanned by $\{e_1, e_2, \cdots, e_n\}$ is the same as that spanned by $\{x_1, x_2, \cdots, x_n\}$

Proof: Certainly $x_1 \neq 0$, because the set $\{x_1, x_2, \cdots, x_n, \cdots\}$ is linearly independent.

we define $y_1, y_2, \cdots$ and $e_1, e_2, \cdots$ recursively as follows:

$$y_1 = x_1 \qquad e_1 = \frac{y_1}{\|y_1\|}$$

Clearly the subspace spanned by x_1 and e_1 are the same.

$$y_2 = x_2 - (x_2, e_1)e_1 \qquad e_2 = \frac{y_2}{\|y_2\|}$$

$$y_3 = x_3 - (x_3, e_1)e_1 - (x_3, e_2)e_2 \qquad e_3 = \frac{y_3}{\|y_3\|}$$

$$\vdots$$

$$y_n = x_n - (x_n, e_1)e_1 - (x_n, e_2)e_2 - \cdots - (x_n, e_{n-1})e_{n-1} \qquad e_n = \frac{y_n}{\|y_n\|}$$

$$= x_n - \sum_{i=1}^{n-1} (x_n, e_i)e_i$$

$$y_{n+1} = x_{n+1} - \sum_{i=1}^{n} (x_{n+1}, e_i)\, e_i \quad , \quad e_{n+1} = \frac{y_{n+1}}{\|y_{n+1}\|}$$

$$\vdots \qquad\qquad \vdots$$

The process terminates if $\{x_n\}$ is a finite set, otherwise it continues indefinitely.

Also note that $y_n \neq 0$ because $x_1, x_2, \dots, x_n$ are l-I.

Thus e_n is well-definition ie the definition of e_n is valid.

From the construction, it is clear that e_2 is a linear combination of x_1 and x_2 & x_2 is a linear combination of e_1, e_2.

Similarly x_3 is a linear combination of e_1, e_2, e_3 and e_3 is a linear combination of x_1, x_2, x_3.

So by induction each x_n is a linear combination of $e_1, e_2, \dots, e_n$ and each e_n is a linear combination of $x_1, x_2, \dots, x_n$.

Thus the linear subspace spanned by the x's is the same as that spanned by e's.

Now it remains to show that the set of e's is an orthonormal set ie $\{e_1, e_2, \dots, e_n, \dots\}$ is orthonormal.

Now since $e_i = \dfrac{y_i}{\|y_i\|}$

$$\Rightarrow \|e_i\| = \frac{\|y_i\|}{\|y_i\|} = 1$$

Now we show by induction that $(e_i, e_j) = 0$; $i, j = 1, 2, \dots$, $i \neq j$.

Consider $(e_1, e_2) = \left(e_1, \dfrac{y_2}{\|y_2\|}\right) = \dfrac{1}{\|y_2\|}(e_1, y_2)$

$$= \frac{1}{\|y_2\|}\left(e_1, x_2 - (x_2, e_1)e_1\right)$$

$$= \frac{1}{\|y_2\|}\left[(e_1, x_2) - (e_1, (x_2, e_1)e_1)\right].$$

$$= \frac{1}{\|y_2\|}\left[(e_1, x_2) - \overline{(x_2, e_1)}(e_1, e_1)\right]$$

$$\Rightarrow (e_1, e_2) = \frac{1}{\|y_2\|}\left[(e_1, x_2) - (e_1, x_2)(1)\right] \quad (\because \|e_1\| = 1)$$
$$= 0$$
$$\Rightarrow (e_1, e_2) = 0.$$

Suppose that $(e_i, e_j) = 0$ for $i, j = 1, 2, ----, n-1$; $i \neq j$

Now
$$(e_n, e_j) = \left(\frac{y_n}{\|y_n\|}, e_j\right)$$
$$= \frac{1}{\|y_n\|}(y_n, e_j)$$
$$= \frac{1}{\|y_n\|}\left(x_n - \sum_{i=1}^{n-1}(x_n, e_i)e_i \,,\ e_j\right)$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - \left(\sum_{i=1}^{n-1}(x_n, e_i)e_i \,,\ e_j\right)\right].$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - \sum_{i=1}^{n-1}(x_n, e_i)(e_i, e_j)\right]$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - (x_n, e_j)(e_j, e_j)\right] \quad \left(\begin{array}{l}\text{After expansion}\\ \text{all other terms}\\ \text{vanishes by assump.}\end{array}\right)$$
$$= \frac{1}{\|y_n\|}\left[(x_n, e_j) - (x_n, e_j)\cdot 1\right] \quad (\because \|e_j\| = 1).$$
$$= 0$$

Hence by induction $\{e_1, e_2, ----, e_n, ----\}$ form an orthonormal set. Hence the result follows.

The Conjugate space of a Hilbert space H :

Let H be a Hilbert space. By H^*, we denote the conjugate space of H (ie the set of all continuous linear transformations of H into $\mathbb{C}$). The elements of H^* are called continuous linear functionals or briefly functional.

One of the fundamental properties of a Hilbert space H is the fact that there is a natural correspondence between the vectors in H and the functionals in H^* as we shall see below.

If y is a vector in Hilbert space H, then the complex function f_y defined by $f_y(x) = (x, y)$ for x in H is linear, because for any x_1, x_2 in H & scalar α; we have,

$$f_y(x_1 + x_2) = (x_1 + x_2, y) \quad [\text{by def: of } f_y(x)].$$
$$= (x_1, y) + (x_2, y)$$
$$= f_y(x_1) + f_y(x_2)$$

and
$$f_y(\alpha x) = (\alpha x, y)$$
$$= \alpha (x, y)$$
$$= \alpha f_y(x).$$

Moreover,
$$|f_y(x)| = |(x, y)|$$
$$\leq \|x\| \|y\| \quad [\text{by schwarz's inequality}].$$

for all x in H. This inequality shows that f_y is bounded (considering $M = \|y\|$) and hence continuous and is therefore a functional on H ie $f_y \in H^*$.

Since $|f_y(x)| \leq \|x\| \|y\|$ (by above).

Thus we have: $\|f_y\| \leq \|y\|$ (Taking sup over x with $\|x\| = 1$).

Evenmore equality is attained here ie $\|f_y\| = \|y\|$, because this is clear when $y = 0$ (if $y = 0$ then $\|f_y\| \leq 0 \Rightarrow \|f_y\| = 0$ because norm is non-negative)

and if $y \neq 0$, then
$$\|y\|^2 = (y, y) = f_y(y) \quad (\because f_y(x) = (x, y)).$$
$$\leq |f_y(y)|$$
$$\leq \|f_y\| \|y\|$$
$$\Rightarrow \|y\|^2 \leq \|f_y\| \|y\| \Rightarrow \|y\| \leq \|f_y\|$$
so that $\|f_y\| = \|y\|$

We see that for every y in H, there exists a functional f_y in H^* such that $\|f_y\| = \|y\|$

In such case, we say that $y \longrightarrow f_y : H \longrightarrow H^*$ is a norm preserving mapping of H into H^*.

[If $T : X \to Y$ is a linear mapping from a n.l.s X into a n.l.s Y, then T is called norm preserving mapping if $\|Tx\| = \|x\|$ $\forall x \in X$]

Theorem (5.34) **[Riesz Representation Theorem]**

Let H be a Hilbert space and let f be an arbitrary functional in H^*, then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H and $\|f\| = \|y\|$

Proof: Let M be the null space (Kernel) of f, that is
$$M = \{x \in H : f(x) = 0\}.$$

Since f is continuous ($\because f$ is functional), so by the continuity of f, the null space M of f is a closed subspace of H, by a result saying that "the null space of a non-zero continuous linear operator is a closed subspace".

If $M = H$, then $f(x) = 0$ (by def: of M)
$= (x, 0)$ for all x in H and the Thm is proved.

If $M \neq H$, then M is a proper closed subspace of H and so there exists a non-zero vector y_0 in H which is orthogonal to M ie $y_0 \perp M$ (by 5.17).

Since y_0 is not in M, thus $f(y_0) \neq 0$. [by def. of M].

For any vector x in H, the vector $z = x - \frac{f(x)}{f(y_0)} \cdot y_0$ is in M, because $f(z) = f\left(x - \frac{f(x)}{f(y_0)} y_0\right) = f(x) - \frac{f(x)}{\cancel{f(y_0)}} \cancel{f(y_0)} = 0$.

Also since $y_0 \perp M$, so that $y_0 \perp z$ ($\because z \in M$)

$$\Rightarrow (z, y_0) = 0 \Rightarrow \left(x - \frac{f(x)}{f(y_0)} \cdot y_0, y_0\right) = 0$$

$$\Rightarrow (x, y_0) - \left(\frac{f(x)}{f(y_0)} \cdot y_0, y_0\right) = 0 \Rightarrow (x, y_0) - \frac{f(x)}{f(y_0)} (y_0, y_0) = 0$$

$$\Rightarrow \frac{f(x)}{f(y_0)} (y_0, y_0) = (x, y_0) \Rightarrow f(x) = \frac{f(y_0)}{(y_0, y_0)} \cdot (x, y_0)$$

$$\Rightarrow f(x) = \left(x, \frac{\overline{f(y_0)}}{(y_0, y_0)} \cdot y_0\right) = \left(x, \frac{\overline{f(y_0)}}{(y_0, y_0)} y_0\right)$$

Let $y = \dfrac{\overline{f(y_0)}}{(y_0, y_0)}\, y_0$; Then from we have:

$$f(x) = (x, y) \quad \text{For all } x \text{ in } H.$$

To complete the proof, it remains to show that y is unique.

For this if we also have $f(x) = (x, y')$ for all x, then

$$(x, y) = (x, y')$$

$$\Rightarrow (x, y) - (x, y') = 0$$

$$\Rightarrow (x, y - y') = 0 \quad \text{for all } x \text{ in } H.$$

For particular $x = y - y'$, we get:

$$(y - y', y - y') = 0 \Rightarrow \|y - y'\|^2 = 0 \Rightarrow y - y' = 0$$

$\Rightarrow y = y'$. Hence y is unique.

Next we show that $\|f\| = \|y\|$

We have: $f(x) = (x, y)$

$$\text{so } |f(x)| = |(x, y)|$$

$$\leq \|x\| \, \|y\| \quad \text{(Schwarz's inequality)}$$

and thus it follows that

$$\|f\| \leq \|y\| \quad \Big(\text{Taking } \sup_{\|x\|=1} \text{ over both sides}\Big)$$

Also
$$\|y\|^2 = (y, y) = f(y)$$
$$\leq |f(y)|$$
$$\leq \|f\| \, \|y\|$$

$$\Rightarrow \|y\| \leq \|f\|$$

So that $\|f\| = \|y\|$. This completes the proof.