CHAPTER # 4 [Inner Product Spaces.]

Definition (4.1):- By an Inner product on a complex linear space X, we mean a mapping $(x,y) \longrightarrow \langle x,y\rangle : X \times X \longrightarrow \mathbb{C}$ satisfying the conditions.

(i) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$.

(ii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$, where the bar denotes complex conjugate.

(iii) $\langle \lambda x, y\rangle = \lambda \langle x,y\rangle$

(iv) $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ iff $x=0$

for all x,y,z in X and $\lambda \in \mathbb{C}$.

when $\langle\, , \rangle$ is an Inner product on a complex linear space X, then the pair $(X, \langle\, ,\rangle)$ is called Inner product space, and we reffer to the complex number $\langle x,y\rangle$ as the Inner product of the vectors x and y.

Consequences of the Def: of Inner product:

For all x,y,z in X and $\lambda \in \mathbb{C}$, we have,

(v) $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

(vi) $\langle x, \lambda y\rangle = \bar{\lambda}\langle x,y\rangle$.

(vii) $\langle x,0\rangle = \langle 0,x\rangle = 0$.

Proof: (v)
$$\begin{aligned}\langle x, y+z\rangle &= \overline{\langle y+z, x\rangle} \quad (\text{by (ii)}) \\ &= \overline{\langle y,x\rangle + \langle z,x\rangle} \quad (\text{by (i)}) \\ &= \overline{\langle y,x\rangle} + \overline{\langle z,x\rangle} \quad [\text{Property of Comp: nos:}] \\ &= \overline{\overline{\langle x,y\rangle}} + \overline{\overline{\langle x,z\rangle}} \quad [\text{by (ii)}] \\ &= \langle x,y\rangle + \langle x,z\rangle \quad [\text{Property of Complex nos:}].\end{aligned}$$

(vi)
$$\begin{aligned}\langle x, \lambda y\rangle &= \overline{\langle \lambda y, x\rangle} \quad [\text{by (ii)}] = \overline{\lambda\langle y,x\rangle} \quad [\text{by (iii)}] \\ &= \bar{\lambda}\,\overline{\langle y,x\rangle} \quad (\text{Property of Complex nos:}) \\ &= \bar{\lambda}\langle x,y\rangle.\end{aligned}$$

(VII) $\langle x,0\rangle = \langle x, 0\cdot y\rangle$

$= \bar{0}\langle x,y\rangle$ [by (VI)].

$= 0\cdot\langle x,y\rangle$

$= 0$

Similarly we can show that $\langle 0,x\rangle = 0$.

Remark: For real linear spaces, the definition of Inner product is the same as the one given above, except that scalers and values $\langle x,y\rangle$ are required to be real so that the "bars" denoting the complex conjugation no longer appear in conditions (ii) and (vi).

In order to avoid the complexity of terms, we shall denote an Inner product simply by (x,y) instead of $\langle x,y\rangle$.

Examples (4.2):- ① The space l_2^n with the Inner product of two vectors $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$ defined by: $(x,y)=\sum_{i=1}^{n} x_i\bar{y_i}$ is an Inner product space.

② The space l_2 with the Inner product of two vectors $x=(x_1,x_2,\dots\dots)$ and $y=(y_1,y_2,\dots\dots)$ defined by

$(x,y) = \sum_{i=1}^{\infty} x_i\bar{y_i}$ is an Inner product space.

③ $C[a,b]$, the linear space of continuous functions f on an interval $[a,b]$, with Inner product defined by:

$$(f,g) = \int_a^b f(t)\,\overline{g(t)}\,dt\ ,\quad t\in[a,b]$$

is an Inner product space.

Remark: An Inner product space is also called a Pre-Hilbert space.

Proposition: If X is an Inner product space, then $x \in X$ is the zero element iff $(x,y)=0$ for all y in X.

Proof: Let $x=0$, then for all y in X, we have

$$(x,y) = (0,y) = 0 \quad [\because (x,0)=(0,x)=0].$$

ie $(x,y)=0$; $\forall\, y$ in X.

Conversely if $(x,y)=0$ $\forall\, y$ in X, then let $y=x$ (Particular), so

$$(x,x)=0 \Rightarrow x=0 \quad [\because (x,x)=0 \text{ if } x=0].$$

This Completes the proof.

Theorem (4.3) [Schwarz's Inequality OR Cauchy-Bunyakoski's Inequality]

Statement: If X is an Inner product space, then

$$|(x,y)| \le \sqrt{(x,x)}\cdot\sqrt{(y,y)} \quad ; \text{ for all } x,y \text{ in } X \longrightarrow (1)$$

the equality holds in (1) iff x and y are linearly dependent.

Proof: If $y=0$, then (1) holds because $(x,0)=0$

Also if $x=0$, then (1) holds because $(0,y)=0$

Now let $y \neq 0$. For every scaler α, we have:

$$(x-\alpha y, x-\alpha y) \geq 0 \qquad (\because (x,x)\geq 0)$$

$$\Rightarrow (x,x) + (x,-\alpha y) + (-\alpha y, x) + (-\alpha y, -\alpha y) \geq 0$$

$$\Rightarrow (x,x) - \bar{\alpha}(x,y) - \alpha(y,x) + \alpha\bar{\alpha}(y,y) \geq 0 \longrightarrow (2)$$

Setting $\alpha = \dfrac{(x,y)}{(y,y)}$, we have: $\bar{\alpha} = \dfrac{\overline{(x,y)}}{(y,y)}$ $\left[\because \overline{(y,y)} = \overline{(y,y)} = (y,y)\right]$

and so $\alpha\bar{\alpha} = \dfrac{(x,y)}{(y,y)}\cdot\dfrac{\overline{(x,y)}}{(y,y)} = \dfrac{|(x,y)|^2}{|(y,y)|^2}$ $\quad (\because z\bar{z} = |z|^2 \text{ for all } z \in \mathbb{C})$.

Substituting these values in (2), we obtain:

$$(x,x) - \frac{\overline{(x,y)}}{(y,y)}\cdot(x,y) - \frac{(x,y)}{(y,y)}\cdot(y,x) + \frac{|(x,y)|^2}{|(y,y)|^2}\cdot(y,y) \geq 0.$$

$$\Rightarrow (x,x) - \frac{|(x,y)|^2}{(y,y)} - \frac{|(x,y)|^2}{(y,y)} + \frac{|(x,y)|^2}{(y,y)} \geq 0 \quad \left[\because (x,y)(y,x) = (x,y)\overline{(x,y)} = |(x,y)|^2\right]$$

$$\Rightarrow (x,x) - \frac{|(x,y)|^2}{(y,y)} \geq 0 \Rightarrow \frac{|(x,y)|^2}{(y,y)} \leq (x,x). \Rightarrow |(x,y)|^2 \leq (x,x)(y,y)$$

$$\Rightarrow |(x,y)| \leq \sqrt{(x,x)}\cdot\sqrt{(y,y)}$$, which is required Inequality #.

Next we see that the equality in ① holds iff $y=0$ or equality holds in ②. But from ②, we have:

$$|(x,y)| = \sqrt{(x,x)} \cdot \sqrt{(y,y)} \text{ iff } (x-\alpha y, x-\alpha y)=0$$
$$\text{iff } x-\alpha y=0$$
$$\text{iff } x=\alpha y.$$
iff x & y are linearly dependent.

This completes the required proof.

Corollary (4.4): let X be an Inner product space, then for any x and y in X, $\sqrt{(x+y,x+y)} \le \sqrt{(x,x)} + \sqrt{(y,y)}$.

Proof: we can write:

$$\begin{aligned}(x+y,x+y) &= (x,x)+(x,y)+(y,x)+(y,y)\\ &= (x,x)+(x,y)+\overline{(x,y)}+(y,y).\\ &= (x,x)+2\,\mathrm{Re}\,(x,y)+(y,y) \quad [\because z+\bar{z}=2\,\mathrm{Re}\,z].\\ &\le (x,x)+2|(x,y)|+(y,y) \quad [\because \mathrm{Re}\,z \le |z|].\\ &\le (x,x)+2\left[\sqrt{(x,x)}\cdot\sqrt{(y,y)}\right]+(y,y) \quad [\text{by above inequality}]\\ &= \left[\sqrt{(x,x)}+\sqrt{(y,y)}\right]^2.\end{aligned}$$

$$\Rightarrow (x+y,x+y) \le \left[\sqrt{(x,x)}+\sqrt{(y,y)}\right]^2.$$

Taking square root on both sides, we get:

$$\sqrt{(x+y,x+y)} \le \sqrt{(x,x)}+\sqrt{(y,y)} \quad \# \text{ proved.}$$

Theorem (4.5): If X is an Inner product space, Then $\sqrt{(x,x)}$ has the properties of a norm OR

An Inner product space is a norm linear space.

Proof: let X be an Inner product space. Define a map $\|\cdot\|: X \to \mathbb{R}$ by $\|x\| = \sqrt{(x,x)}$; $\forall x \in X$.

In order to show that $\|x\| = \sqrt{(x,x)}$ defines a norm on the

Inner product space X, we need to show that it satisfies all the conditions of a norm.

Now (i) For any $x \in X$, $\|x\| = \sqrt{(x,x)}$ which gives

$$\|x\|^2 = (x,x) \geq 0 \quad \text{(by definition)}$$

$$\Rightarrow \|x\|^2 \geq 0 \Rightarrow \|x\| \geq 0.$$

Also $\|x\| = \sqrt{(x,x)} \Rightarrow \|x\|^2 = (x,x) = 0$ iff $x = 0$.

$$\Rightarrow \|x\|^2 = 0 \text{ iff } x = 0 \text{ ie } \|x\| = 0 \text{ iff } x = 0.$$

(ii) By def:, $\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} \Rightarrow \|\alpha x\|^2 = (\alpha x, \alpha x)$.

$$= \alpha\bar{\alpha}(x,x).$$

$$= |\alpha|^2 \|x\|^2 \quad (\because \|x\| = \sqrt{(x,x)})$$

$$\Rightarrow \|\alpha x\| = |\alpha| \|x\|$$

(III) For x, y in X, we have: $\|x\| = \sqrt{(x,x)}$ & $\|y\| = \sqrt{(y,y)}$

Now by Corollary (4.3), we have,

$$\sqrt{(x+y, x+y)} \leq \sqrt{(x,x)} + \sqrt{(y,y)}$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

We see that all the conditions of a norm are satisfied. Thus $\|x\| = \sqrt{(x,x)}$ is a norm on X and hence $(X, \|\cdot\|)$ is a norm linear space.

<u>Remark (4.6)</u>: The Schwarz's inequality can now be written in the form $|(x,y)| \leq \|x\| \|y\|$

when X is an Inner product space, we shall henceforth X as a norm linear space with norm $\|x\| = \sqrt{(x,x)}$.

Proposition (4.7) [Polarization identity].

Statement: If X is a inner product space, then for x, y in X, $4(x,y) = \sum_{r=0}^{3} i^r \|x+i^r y\|^2$

Proof: we have

$$\sum_{r=0}^{3} i^r \|x+i^r y\|^2 = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

$$= (x+y, x+y) - (x-y, x-y) + i(x+iy, x+iy) - i(x-iy, x-iy)$$

$$= (x,x) + (x,y) + (y,x) + (y,y) - [(x,x) - (x,y) - (y,x) + (y,y)] + i[(x,x) + (x,iy) + (iy,x) + (iy,iy)] - i[(x,x) - (x,iy) - (iy,x) + (iy,iy)].$$

$$= \cancel{(x,x)} + (x,y) + (y,x) + \cancel{(y,y)} - \cancel{(x,x)} + (x,y) + (y,x) - \cancel{(y,y)} + i\cancel{(x,x)} + i(x,iy) + i(iy,x) + i\cancel{(iy,iy)} - i\cancel{(x,x)} + i(x,iy) + i(iy,x) - i\cancel{(iy,iy)}.$$

$$= 2(x,y) + 2(y,x) + 2i(x,iy) + 2i(iy,x).$$

$$= 2(x,y) + 2(y,x) + 2i\bar{i}(x,y) \mp 2i\bar{i}(y,x)$$

$$= 2(x,y) + 2\cancel{(y,x)} + 2(x,y) - 2\cancel{(y,x)}$$

$$= 4(x,y).$$

Hence $\sum_{r=0}^{3} i^r \|x+i^r y\|^2 = 4(x,y)$.

Remark (4.8):- If X is a real inner product space, then the above polarization identity becomes:

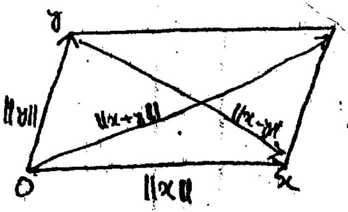
$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 \text{ for } x, y \text{ in } X.$$

Proposition (4.9) [Parallelogram law for Inner product spaces]

If X is an Inner product space, then
$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for x and y in X

OR In Inner product spaces, $\|$gm law holds.

Proof.



$$\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y)$$
$$= (x,x) + \cancel{(x,y)} + \cancel{(y,x)} + (y,y)$$
$$+ (x,x) - \cancel{(x,y)} - \cancel{(y,x)} + (y,y)$$
$$= 2(x,x) + 2(y,y)$$
$$= 2\|x\|^2 + 2\|y\|^2 \quad (\because \|x\| = \sqrt{(x,x)},\ \|y\| = \sqrt{(y,y)})$$
$$= 2[\|x\|^2 + \|y\|^2]$$

Hence $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$, as required.

Definition (4.10): Let X be an Inner product space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$, then we say that an Inner product (x,y) on $X \times X$ is a Continuous function or Jointly Continuous if $(x_n, y_n) \to (x,y)$.

Theorem: (4.11) The Inner product (x,y) is a Continuous function on $X \times X$.

Proof:- Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $x_n \to x$ and $y_n \to y$. We need to show that $(x_n, y_n) \to (x,y)$.

To prove this, it is enough to observe that

$$|(x_n, y_n) - (x,y)| = |(x_n, y_n) - (x_n, y) + (x_n, y) - (x,y)| \quad \left(\begin{matrix}\text{+ing and}\\ \text{subtracting}\end{matrix}\right)$$
$$\le |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x,y)|$$
$$= |(x_n, y_n - y)| + |(x_n - x, y)| \quad [\text{by def: of Inner Product}]$$
$$\le \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \cdot \|y\| \quad [\text{Schwartz's inequality}].$$

$\Rightarrow |(x_n, y_n) - (x, y)| \leq \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \|y\|.$

$\longrightarrow 0$ as $n \to \infty$, because

$y_n - y \to 0$ and $x_n - x \to 0$ as $n \to \infty$.

Thus $\lim\limits_{n\to\infty} |(x_n, y_n) - (x, y)| = 0$

which shows that Inner product is a continuous function.

<u>Exercise (4.12)</u>: ① Show that an Inner product space is a metric space with metric defined by $d(x,y) = \sqrt{(x-y, x-y)}$; $\forall x, y \in X$

② Give some examples of n.l.s which are not Inner Product spaces.

<u>Sol</u>:- ① Let X be an Inner product space, Then for any $x, y \in X$ we define the ~~mapping~~ metric as follows:

$$d(x,y) = \sqrt{(x-y, x-y)}$$

In order to show that X is a metric space, we have to show that $d(x,y) = \sqrt{(x-y, x-y)}$ satisfies all the properties of a metric that is:

(i) $d(x,y) \geq 0$ (ii) $d(x,y) = 0$ iff $x = y$ (iii) $d(x,y) = d(y,x)$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$.

Recall that every Inner product space is a n.l.s with norm defined by: $\|x\| = \sqrt{(x,x)}$.

Therefore (i) $d(x,y) = \sqrt{(x-y, x-y)} = \|x-y\| \geq 0$ $[\because \|x\| = \sqrt{(x,x)}]$

ie $d(x,y) \geq 0 \quad \forall x, y \in X$.

(ii) $d(x,y) = \sqrt{(x-y, x-y)} = \underbrace{\|x-y\|}_{=0 \text{ iff } x-y=0}$ $[\because \|x\| = \sqrt{(x,x)}]$

ie $d(x,y) = 0$ iff $x = y$.

(iii) $d(x,y) = \sqrt{(x-y,\, x-y)} = \|x-y\| \quad [\because \|x\| = \sqrt{(x,x)}]$.

$$= \|y-x\|$$
$$= \sqrt{(y-x,\, y-x)}$$
$$= d(y,x).$$

(iv) $d(x,y) = \sqrt{(x-y,\, x-y)} = \|x-y\| \quad [\because \|x\| = \sqrt{(x,x)}]$.

$$= \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = \sqrt{(x-z,\, x-z)} + \sqrt{(z-y,\, z-y)}$$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$; $\forall\, x, y, z \in X$.

Thus $d(x,y) = \sqrt{(x-y,\, x-y)}$ satisfies all the properties of a metric and hence X with the metric $d(x,y) = \sqrt{(x-y,\, x-y)}$ is a metric space. This completes the proof.

(2) <u>First example:</u> let $X = l_p$; $p>1$, $p \neq 2$, Then l_p is a n.l.s with norm defined by: $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$; $\forall\, x = (x_1, x_2, \ldots) \in l_p$.

We know that in an Inner product space X, Parallelogram law holds ie for any x, y in X: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

But in this case it does not hold, for let

$x = (1, 1, 0, 0, \ldots) \in l_p$ and $y = (1, -1, 0, 0, \ldots) \in l_p$.

Then $\|x+y\| = \left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p}$

$$= \left(|x_1+y_1|^p + |x_2+y_2|^p + \cdots\cdots\right)^{1/p}$$
$$= \left(|1+1|^p + |1-1|^p + |0+0|^p + \cdots\right)^{1/p}$$
$$= \left(|2|^p\right)^{1/p} = 2$$

and similarly we can show that $\|x-y\| = 2$.

Also $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = \left(|x_1|^p + |x_2|^p + |x_3|^p + \cdots\right)^{1/p}$

$$= (1+1+0+0+\cdots)^{1/2} = 2^{1/p}$$

Similarly $\|y\| = 2^{1/p}$; where $p>1$, $p \neq 2$.

Therefore $\|x+y\|^2+\|x-y\|^2 = (2)^2+(2)^2 = 8$

and $2\left(\|x\|^2+\|y\|^2\right) = 2\left(2^{2/p}+2^{2/p}\right) = 2\cdot 2\cdot 2^{2/p} = 4\cdot 2^{2/p}$; $p>1, p\neq 2$.

we see that $\|x+y\|^2+\|x-y\|^2 \neq 2\left(\|x\|^2+\|y\|^2\right)$; $\forall\, x,y \in l_p$.

and hence $X = l_p$; $p>1, p\neq 2$ is not an Inner product space.

<u>Second example:-</u> The space $C[a,b]$, the space of continuous real valued functions on $[a,b]$, with the norm defined by: $\|x\| = \max\limits_{t\in[a,b]} |x(t)|$ is a norm linear space.

we know that in an Inner product space, the Parallelogram law holds ie for any x,y in $C[a,b]$

$$\|x+y\|^2+\|x-y\|^2 = 2\left(\|x\|^2+\|y\|^2\right)$$

But it does not hold in this case, for if we take $x(t) = 1$ and $y(t) = \dfrac{t-a}{b-a}$; Then

$$\|x\| = \max_{t\in[a,b]} |x(t)| = \max_{t\in[a,b]} |1| = 1$$

$$\|y\| = \max_{t\in[a,b]} |y(t)| = \max_{t\in[a,b]} \left|\frac{t-a}{b-a}\right| = \left|\frac{b-a}{b-a}\right| = 1 \quad \left(\because \text{for } t=b, \text{ we get max: value}\right)$$

Also $\|x+y\| = \max\limits_{t\in[a,b]} \left|(x+y)(t)\right| = \max\limits_{t\in[a,b]} \left|x(t)+y(t)\right|$

$$= \max_{t\in[a,b]} \left|1+\frac{t-a}{b-a}\right| = \left|1+\frac{b-a}{b-a}\right| \quad \left(\because \text{for } t=b, \text{ we get max: value}\right)$$

$$= |1+1| = 2$$

and $\|x-y\| = \max\limits_{t\in[a,b]} \left|(x-y)(t)\right| = \max\limits_{t\in[a,b]} \left|x(t)-y(t)\right|$

$$= \max_{t\in[a,b]} \left|1-\frac{t-a}{b-a}\right| = \left|1-\frac{a-a}{b-a}\right| \quad \left(\because \text{for } t=a, \text{ we get max: value}\right)$$

$$= 1$$

Now $\|x+y\|^2 + \|x-y\|^2 = (2)^2 + (1)^2 = 5$

and $2\left(\|x\|^2 + \|y\|^2\right) = 2\,(1+1) = 4$

Thus $\|x+y\|^2 + \|x-y\|^2 \neq 2\left(\|x\|^2 + \|y\|^2\right)$

and hence $C[a,b]$ is not an Inner product space.

Orthogonal and orthonormal sets:

Definition: Let X be an Inner product space and let A, B be any two non-empty subsets of X, Then

(a) we say that $x, y \in X$ are orthogonal if $(x,y) = 0$. We express symbolically the orthogonal vectors x and y by $x \perp y$. (ie x perp y).

(b) we say that $x \in X$ is orthogonal to A and write $x \perp A$ if $x \perp y$ for every y in A. ie $(x,y) = 0 \;\forall y \in A$.

(c) we say that A is orthogonal to B and write $A \perp B$ if $a \perp b$ for every $a \in A$ and every $b \in B$ ie $(a,b) = 0 \;\forall a \in A, b \in B$.

(d) we say that A is orthogonal if for each pair of vectors x, y in A, $x \perp y$ $(x \neq y)$ ie $(x,y) = 0 \;\forall x, y \in A$, where $x \neq y$.

Remark: ① Every vector is orthogonal to zero ie $x \perp 0 \;\forall x \in X$.

② if $x \perp y$, Then $y \perp x$ ie $(x,y) = 0 \Rightarrow (y,x) = 0$.

③ if $x \perp y$ and $x \perp z$, Then $x \perp y+z$ and $x \perp \alpha y$ for every scalar α.

④ If $x \perp y_n$, where $y_n \to y$; Then $x \perp y$.

⑤ 0 is the only vector orthogonal to itself.

Proof: (1) Since $(x, 0) = 0$ [by def: of Inner product]; so that $x \perp 0$.

(2) Since $x \perp y$, so $(x,y) = 0$.

Now $(y,x) = \overline{(x,y)} = \bar{0} = 0$, so $y \perp x$.

(3) Since $x \perp y$ and $x \perp z \Rightarrow (x,y)=0$. and $(x,z)=0$.

Now $(x, y+z) = (x,y)+(x,z) = 0+0=0$

$\Rightarrow x \perp y+z$.

Also $(x,\alpha y) = \bar{\alpha}(x,y) = \bar{\alpha}\cdot 0 = 0 \Rightarrow x \perp \alpha y;\ \forall \alpha$.

(4) Since $y_n \to y \Rightarrow \lim_{n\to\infty} y_n = y$.

$(x,y) = \lim_{n\to\infty}(x,y_n)$ [$\because$ Inner product is a Continuous function]

$= 0$ [$\because x \perp y_n$].

So $x \perp y$.

(5). Since $(x,x)=\|x\|^2$ (by defi)

So $(0,0)=\|0\|^2=0 \Rightarrow 0$ is the only vector orthogonal to itself.

<u>Example:</u> ① $\mathbb{R}^n$ is an Inner product space with inner product defined by: $(x,y)=\sum_{i=1}^{n} x_i y_i$.

Then the vectors $(1,0,0,\ldots\ldots,0)$, $(0,1,0,\ldots\ldots 0)$, $\ldots\ldots$, $(0,0,\ldots,1)$ are orthogonal, because the Inner Product of any two of above vectors is 0.

(2) $\mathbb{R}^3$ is an Inner product space with Inner product defined by: $(x,y)=\sum_{i=1}^{3} x_i y_i$, then the set of vectors $\{(1,1,0),(0,-1,0)\}$ is not orthogonal.

(3) ℓ^2 is an Inner product space with Inner product defined by: $(x,y)=\sum_{i=1}^{\infty} x_i \overline{y_i}$; then the vectors

$e_1=(1,0,0,\ldots\ldots)$, $e_2=(0,1,0,\ldots\ldots)$, $e_3=(0,0,1,0,\ldots\ldots)$, $\ldots\ldots$,

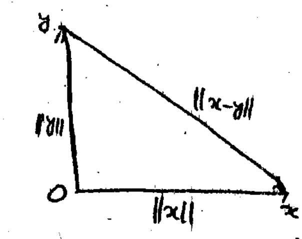
$e_i=(0,0,0,\ldots\ldots \underset{i\text{th place}}{1},0,0,\ldots\ldots)$, $\ldots\ldots$ in ℓ^2 are orthogonal

because $e_i \perp e_j$ for all i,j with $i \neq j$.

Remark: one of the simple geometric fact about orthogonal vectors is the pythagrean theorem, which is given as follows.

Theorem: If x and y are orthogonal vector in an Inner Product space X, Then $\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x-y\|^2$.

Proof: we have:



$$\|x+y\|^2 = (x+y, x+y) \quad [\because (x,x) = \|x\|^2]$$
$$= (x,x) + (x,y) + (y,x) + (y,y).$$
$$= (x,x) + 0 + 0 + (y,y) \quad [\because x \text{ and } y \text{ are orthogonal}]$$
$$= (x,x) + (y,y)$$
$$= \|x\|^2 + \|y\|^2$$

Similarly we can show that $\|x-y\|^2 = \|x\|^2 + \|y\|^2$

Hence $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$, which completes the proof.

Theorem [Generalization of Pythagorean Thm].

Let $x_1, x_2, \ldots, x_n$ be mutually orthogonal vectors in the inner Product space X, Then

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2$$

OR $\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$

Proof:- $\|x_1 + x_2 + \cdots + x_n\|^2 = (x_1+x_2+\cdots+x_n, x_1+x_2+x_3+\cdots+x_n) \quad [\because (x,x) = \|x\|^2]$

$$= \underline{(x_1,x_1)} + (x_1,x_2) + \cdots + (x_1,x_n) + (x_2,x_1) + \underline{(x_2,x_2)} + \cdots + (x_2,x_n) + \cdots$$
$$\cdots + (x_n,x_1) + (x_n,x_2) + \cdots \underline{(x_n,x_n)}$$
$$= (x_1,x_1) + (x_2,x_2) + \cdots + (x_n,x_n) \quad [\because x_1, x_2, \ldots, x_n \text{ are mutually orthogonal}].$$
$$= \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2$$

OR $\left\|\sum_{i=1}^{n} x_i\right\| = \sum_{i=1}^{n} \|x_i\|^2$, as required #.

Orthonormal sets:

Definition: A set $S=\{x_i : i\in I\}$ in an Inner product space X is said to be orthonormal if

$$(x_i, x_j) = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$$

For $i=j$, $(x_i, x_j) = (x_i, x_i) = \|x_i\|^2$.

ie $(x_i, x_j) = \delta_{ij}$, the standard Kronecker delta.

In other words the set S is said to be orthonormal if it is orthogonal and $\|x\|=1$ for every $x\in S$.

Example: let $\{x_i : i\in I\}$ be an orthogonal set in an inner product space X, Then the set,

$$A=\left\{\frac{x_i}{\|x_i\|} : i\in I\right\}$$ is orthonormal.

Sol: Let $\frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|} \in A$, Then

$$\left(\frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|}\right) = \frac{1}{\|x_i\|\,\|x_j\|}(x_i, x_j) \qquad [\because \|x_i\| \,\&\, \|x_j\| \in \mathbb{R}]$$

$$= \frac{1}{\|x_i\|\,\|y_i\|} \times (0) \qquad [\because (x_i, x_j)=0 \text{ being orthogonal}]$$

$$= 0.$$

Thus the inner product of two different element of A is zero. So that A is orthogonal.

Next we show that norm of every element of A is 1. For this let $\frac{x_i}{\|x_i\|}\in A$, Then

$$\left\|\frac{x_i}{\|x_i\|}\right\| = \frac{\|x_i\|}{\|x_i\|} \qquad (\because \|x_i\|\in\mathbb{R})$$

$$= 1$$

this shows that A is an orthonormal set.

Def: (Complete orthonormal set)

An orthonormal set S in an Inner product space X is said to be complete if there exists no orthonormal set in X of which S is a proper subset.

In other words, S is complete if it is maximal w.r.t the property of being orthonormal.

Note: If S is complete orthonormal set, then there does not exists any non-zero vector such that $x \perp S$ and $\|x\| = 1$.

Example: In the space ℓ^2, the orthonormal set composed of

$e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, $e_3 = (0, 0, 1, 0, \ldots)$, ...

is a complete orthonormal set.

The end of CH#4.

Written by: TAHIR HUSSAIN BANGASH
Parachinar.