CHAPTER # 3 { Fundamental Theorems of Functional Analysis }

Definition (3.1): ① If A is a subset of a topological space X, then the interior of A is defined as the union of all open sets of X that are contained in A ie

$$\mathring{A} = \bigcup_i O_i \text{ , where each } O_i \text{ is open and } O_i \subseteq A.$$

② we say that A has an empty interior ie $\mathring{A} = \phi$ if and only if A does not contain any non-empty open set.

Definition: A subset M of a topological space X is said to be:

(a) nowhere dense (or rare) in X if $\overline{M}$ has empty interior ie $\overline{M}$ does not contain any non-empty open set.

(b) of the first Catagory (or meager) in X if M is the Countable union of nowhere dense sets.

(c) of the second catagor (Non-meager) in X if M is not of the first Catagor. ie M cannot be expressed as a Countable union of nowhere dense sets.

Theorem (3.2) (Baire's Catagory Theorem)

Every complete metric ~~metric~~ space is of the second catagory.

Hence if $X \neq \phi$ is complete and $X = \bigcup_{k=1}^{\infty} A_k$ (A_k is closed) Then at least one A_k contains a non-empty open set.

Theorem 3.3 (The principal of uniform Boundedness).

Let $\{T_n\}$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed linear space Y such that $\{T_n x\}$ is bounded in Y for all $x \in X$. Then $\{\|T_n\|\}$ is a bounded subset of real nos ie There is a constant $c > 0$ such that

$\|T_n\| \leq c$; $n = 1, 2, \ldots\ldots$

Proof: For each $k \in \mathbb{N}$, we define

$A_k = \{x \in X : \|T_n x\| \leq k ;\ \forall n\}$. we show That A_k is closed in X.

let $\{x_j\}$ be a sequence in A_k such that $x_j \to x$.

By Continuity of T_n (because T_n is bounded), we have

$T_n x_j \longrightarrow T_n x$. Also by Continuity of a norm we can write: $\|T_n x_j\| \longrightarrow \|T_n x\|$

ie $\|T_n x\| = \lim\limits_{j \to \infty} \|T_n x_j\| \leq k$. ($\because x_j \in A_k$).

ie $\|T_n x\| \leq k$, which shows That $x \in A_k$ (by def: of A_k).

and hence A_k is closed ($\because x_j \to x \Rightarrow$ we proved $x \in X$)

Since by hypothesis, each $x \in X$ is in some A_k, be cause $\{\|T_n x\|\}$ is bounded for each $x \in X$ ie $\|T_n x\| \leq k$ for some constant k. Hence $X = \bigcup\limits_{k=1}^{\infty} A_k$

Since X is Complete, so by Baire's catagory Theorem X must be of the second catagory. so There exists at least one A_k, namely A_{k_0}, which contains an open ball $B_r(x_0) = \{x \in X : \|x - x_0\| < r\}$ ie $B_r(x_0) \subset A_{k_0}$.

let x be an arbitrary non-zero vector in X. we set

$z = \gamma' x + x_0$, where $\gamma' = \dfrac{\gamma}{2\|x\|} \longrightarrow ①$

Since $z - x_0 = \gamma' x \Rightarrow \|z - x_0\| = \|\gamma' x\| = \left\|\dfrac{\gamma}{2\|x\|} \cdot x\right\|$

$$= \frac{\gamma}{2\|x\|} \cdot \|x\|$$

$$= \frac{\gamma}{2}$$

$$< \gamma$$

So $\|z - x_0\| < \gamma$, which shows that $z \in B_\gamma(x_0)$.

But $B_\gamma(x_0) \subset A_{k_0}$, so that $z \in A_{k_0}$. And from the definition of A_{k_0}, we have: $\|T_n z\| \le k_0 ; \forall n \longrightarrow ②$

Also since $x_0 \in B_\gamma(x_0) \subset A_{k_0}$, so that $x_0 \in A_{k_0}$

so $\|T_n x_0\| \le k_0 ; \forall n \longrightarrow ③$ [by def. of A_{k_0}].

By ①, we have $z = \gamma' x + x_0$, where $\gamma' = \dfrac{\gamma}{2\|x\|}$, so we can write: $x = \dfrac{z - x_0}{\gamma'}$. This yields for all n.

So $T_n x = T_n \dfrac{(z - x_0)}{\gamma'}$

$$\Rightarrow \|T_n x\| = \left\|T_n \cdot \frac{(z - x_0)}{\gamma'}\right\|$$

$$= \frac{1}{\gamma'} \|T_n(z - x_0)\| \le \frac{1}{\gamma'} [T_n z + T_n(-x_0)]$$

$$\le \frac{1}{\gamma'} \cdot [\|T_n z\| + \|T_n x_0\|] \quad (\because |-1| = 1)$$

$$\le \frac{1}{\gamma'} [k_0 + k_0] \quad (\text{using ② \& ③}).$$

$$= \frac{2k_0}{\gamma'}$$

$$= \frac{2k_0}{\frac{\gamma}{2\|x\|}}$$

$$= \frac{4k_0\|x\|}{\gamma} = \frac{4\|x\| k_0}{\gamma}$$

Hence for all x, $\|T_n x\| \le \dfrac{4\|x\| k_0}{\gamma}$

$\Rightarrow \|T_n\| \leq \frac{4K_0}{\gamma}$ (by taking sup: over both sides with $\|x\|=1$)

Assume that $\frac{4K_0}{\gamma} = c$, then

$\|T_n\| \leq c \; ; \forall n.$

which completes the required proof.

Note: The principle of uniform boundedness is often called Banach-Steinhaus Theorem.

Theorem (3.4): Let $\{T_n\}$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a n.l.s Y such that $\lim_{n\to\infty} T_n x = Tx$ exists for each $x \in X$. Then T is a bounded linear operator.

Proof: Since $\lim_{n\to\infty} T_n x = Tx$ exists for each $x \in X$, so $\{T_n x\}$ is a convergent sequence and hence bounded, because every convergent sequence is bounded. Therefore by ~~[illegible]~~ the principle of uniform boundedness, $\{\|T_n\|\}$ is a bounded sequence of real number, that is there exists a constant $K>0$ such that:

$$\|T_n\| \leq K \; ; \forall n. \longrightarrow (1)$$

we prove that T is a bounded linear operator.

T is linear, because for $x, y \in X$ and a scaler α we have,

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) \quad [\because Tx = \lim_{n\to\infty} T_n x \text{ (from above)}]$$

$$= \lim_{n\to\infty} (T_n x + T_n y) \quad [\because T_n \text{ is linear}]$$

$$= \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty \quad [\text{'' ''}]$$

Also $T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x)$ $\quad [\because Tx = \lim_{n\to\infty} T_n x]$

$= \alpha \lim_{n\to\infty} T_n x$ $\quad [\because T_n \text{ is linear}]$

$= \alpha Tx.$ $\quad [\because Tx = \lim_{n\to\infty} T_n x]$.

Also T is bounded, because since $\{\|T_n\|\}$ is a bounded sequence; so from (1), we have:

$\|T_n\| \le K \; ; \; \forall n.$

Now $\|T_n x\| \le \|T_n\| \|x\|$ $\quad (\because T_n \text{ is bounded}).$

$\le K\|x\|$ $\quad (\because \|T_n\| \le K).$

if T is bounded Then $\|Tx\| \le \|T\|\|x\|$

i.e $\|T_n x\| \le K\|x\|.$

Taking limit over both sides as $n\to\infty$, we get.

$$\lim_{n\to\infty} \|T_n x\| \le \lim_{n\to\infty} K\|x\|$$

$$\Rightarrow \left\|\lim_{n\to\infty} T_n x\right\| \le \lim_{n\to\infty} K\|x\|$$

$$\Rightarrow \|Tx\| \le K\|x\| \; ; \; \forall n.$$

which shows that T is bounded.

Hence T is a bounded linear operator.

Definition (3.5): Let X be a linear space. A real valued function p defined on X (i.e $p: X \to \mathbb{R}$) is called subadditive if

$$p(x+y) \le p(x) + p(y) \; ; \; \forall x, y \in X.$$

and positive homogeneous if $p(\alpha x) = \alpha(px)$, where $\alpha \geqslant 0$ in $\mathbb{R}$ and $x \in X$.

Definition (3.6): A real valued function p on a linear space X is called sublinear functional, if it is both subadditive and positive homogeneous.

Definition (3.9): Let X be a linear space (real or complex). A semi-norm on X is a real valued function p defined on X such that

(i) $p(x) \geq 0$; $\forall x \in X$ (ii) $p(x+y) \leq p(x)+p(y)$; $\forall x, y \in X$.

(iii) $p(\alpha x) = |\alpha| (px)$; for all scalers α and $x \in X$.

If p has the further property that $p(x) \neq 0$ if $x \neq 0$. (or equivalently $p(x)=0$ iff $x=0$), then p is a norm on X.

Remark: As with a norm, the properties of p imply the further properties.

(i) $p(0)=0$ (ii) $|p(x)-p(y)| \leq p(x-y)$.

(iii) $p(-x) \geq -p(x)$.

Proof: (i) we show that $p(0)=0$.

$$p(0) = p(0.x) = 0.p(x) = 0 \qquad [\because p(\alpha x) = |\alpha| p(x)].$$

(ii) since $x = x-y+y$

$$\Rightarrow p(x) = p(x-y+y)$$
$$\leq p(x-y)+p(y) \qquad [\because p(x+y) \leq p(x)+p(y)].$$

$$\Rightarrow p(x)-p(y) \leq p(x-y) \longrightarrow ①$$

Now again $y = x+y-x \Rightarrow p(y) = p(x+y-x)$
$$\leq p(x)+p(y-x)$$

$$\Rightarrow p(y) \leq p(x)+p(y-x) \Rightarrow p(y)-p(x) \leq p(y-x)$$

$$\Rightarrow p(y)-p(x) \leq p(-(x-y)) \Rightarrow p(y)-p(x) \leq |-1| \, p(x-y)$$

$$\Rightarrow p(y)-p(x) \leq p(x-y) \Rightarrow -(p(x)-p(y)) \leq p(x-y)$$

$$\Rightarrow p(x)-p(y) \geq -p(x-y) \Rightarrow -p(x-y) \leq p(x)-p(y) \hookrightarrow ②$$

① & ② $\Rightarrow$ $|p(x)-p(y)| \leq p(x-y)$

(III) we have: $p(-x) = p(-2x+x) \leq p(-2x)+p(x)$
$$= 2\,p(-x)+p(x)$$

$$\Rightarrow p(-x)-2p(-x) \leq p(x) \Rightarrow -p(-x) \leq p(x)$$
$$\Rightarrow p(-x) \geq -p(x) \quad \#.$$

Definition (3.8):

If X is a set, M a proper subset of X and f is a function defined on M (ie $f: M \to M$), then a function F defined on X (ie $F: X \to X$) is called an extension of f if $F(x) = f(x)$ for all $x \in M$. and f is called the restriction of F to M.

(3.9) Theorem (Hahn – Banach Theorem – Real version)

Let M be a proper subspace of a real linear space X. Let p be a sublinear functional on X and let f be a linear functional defined on M such that

$$f(x) \leq p(x) \;;\; \forall x \in M.$$

Then there exists a linear functional $\hat{f}$ on X, which extends f and that $-p(-x) \leq \hat{f}(x) \leq p(x)$; $\forall x \in X$.

(3.10) Theorem [Hahn Banach Theorem – Complex version]:

Let X be a complex linear space and M a linear subspace of X. Let p be a semi-norm defined on X. Let f be a linear functional on M such that

$$|f(x)| \leq p(x) \;;\; \forall x \in M.$$

Then there exists a linear functional F on X such that $F(x) = f(x)$; $\forall x \in X$ [Extension] and that

$$|F(x)| \leq p(x) \;;\; \forall x \in X.$$

(3.11) Theorem [Hahn – Banach Thm for n.l.s].

Statement:- Let X be a n.l.s over a field K and let M be a subspace of X. If $m' \in M'$ (ie m' is a linear functional defined on M); then there exists a $x' \in X'$ such that $\|x'\| = \|m'\|$ and $m'(x) = x'(x)$; $\forall x \in M$ (extension).

Proof: Let p be a real valued function defined on X by: $p(x) = \|m'\| \|x\|$; $\forall x \in X \longrightarrow$ ①

First we show that p is a semi-norm on X.

(i) For $x \in X$, $p(x) \geq 0$ because $p(x) = \|m'\| \|x\| \geq 0$.

(ii) For $x, y \in X$, we have:

$$p(x+y) = \|m'\| \|x+y\| \quad (\text{by } ①).$$
$$\leq \|m'\| \{ \|x\| + \|y\| \} \quad [\because \|x+y\| \leq \|x\| + \|y\|].$$
$$= \|m'\| \|x\| + \|m'\| \|y\|.$$
$$= p(x) + p(y). \quad [\text{by } ①].$$

ie $p(x+y) \leq p(x) + p(y)$.

(iii) For any scaler α and $x \in X$, we have,

$$p(\alpha x) = \|m'\| \|\alpha x\| \quad [\text{by } ①].$$
$$= |\alpha| \|m'\| \|x\| \quad [\text{by definition of norm}].$$
$$= |\alpha| p(x) \quad [\text{by } ①].$$

Hence p is a semi-norm on X.

Now $|m'(x)| \leq \|m'\| \|x\|$ $\quad (\because |x'(x)| \leq \|x'\| \|x\|)$

$= p(x)$ (by ①).

ie $|m'(x)| \leq p(x)$ for all $x \in M$.

Hence by "Hahn-Banach theorem for complex space", There exists a linear functional x' on X such that:

(i) $x'(x) = m'(x)$ $\forall x \in M$ (ie x' is extension of m')

and (ii) $|x'(x)| \leq p(x)$; $\forall x \in X$.

To prove the required result, we will prove that $\|m'\| = \|x'\|$ and $x'(x) = m'(x)$; $\forall x \in M$.

But $x'(x) = m'(x)$ $\forall x \in M$ is satisfied from (i) above:

Also from (i), we have:

$|x'(x)| \leq p(x) = \|m'\| \|x\|$; $\forall x \in X$.

Taking supremum over both sides with $\|x\|=1$, we obtain

$\|x'\| \leq \|m'\| \longrightarrow ②$

Also $\|x'\| = \sup\limits_{\substack{x\neq 0\\ x\in X}} \dfrac{|x'(x)|}{\|x\|} \geqslant \sup\limits_{\substack{x\neq 0\\ x\in X}} \dfrac{|m'(x)|}{\|x\|} = \|m'\|$

$\Rightarrow \|m'\| \leq \|x'\|$.

Since x' is the extension of m', so that $\|m'\|$ cannot be ~~less~~ greater than $\|x'\|$ and so $\|x'\| \geqslant \|m'\| \longrightarrow ③$

From ② and ③, we get:

$\|x'\| = \|m'\|$.

Thus completing the proof.

<u>Theorem (3.12)</u>: let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Then there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'(x_0) = \|x_0\|$.

<u>Proof</u>:- we consider the subspace M of X containing of all elements $x = \alpha x_0$, where α is a scaler

ie $M = \{x \in X : x = \alpha x_0$, where α is a scaler$\}$.

Define a linear functional m' on M (ie $m' : M \to R$) such that:

$m'(x) = m'(\alpha x_0) = \alpha \|x_0\| \longrightarrow ①$.

Then m' is bounded ie $m' \in M'$, because

$|m'(x)| = |m'(\alpha x_0)| \leq \|m'\| \|\alpha x_0\| = \|m'\| \|x\|$ (by def: of M)

ie $|m'(x)| \leq \|m'\| \|x\|$

Also $|m'(x)| = |m'(\alpha x_0)| = |\alpha \|x_0\||$ (by ①).

$= |\alpha| \|x_0\| = \|\alpha x_0\| = \|x\|$ (by def: of M).

That is $|m'(x)| = \|x\|$; $\forall\, x \in M$.

Taking supremium over both sides, we have

$$\sup_{x\neq 0} \frac{|m'(x)|}{\|x\|} = 1 \quad \text{ie} \quad \|m'\| = 1 \longrightarrow (2)$$

since m' is a bounded linear functional defined on M. so by "Hahn–Banach Theorem for n.l.s",

m' has a linear extension x' from M to X

ie $x'(x) = m'(x)$; $\forall\, x \in M \longrightarrow (3)$

and $\|x'\| = \|m'\| \longrightarrow (4)$

But by (2), $\|m'\| = 1$. so that (4) $\Rightarrow \|x'\| = 1$.

It remains to show that $x'(x_0) = \|x_0\|$.

From (1), we have $m'(x) = \alpha\|x_0\|$, where $x = \alpha x_0$.

and so by (3), we have $x'(x) = m'(x) = \alpha\|x_0\|$, where $x = \alpha x_0$

$\Rightarrow x'(x) = \alpha\|x_0\|$, where $x = \alpha x_0$.

$\Rightarrow x'(\alpha x_0) = \alpha\|x_0\|$

$\Rightarrow \alpha x'(x_0) = \alpha\|x_0\|$ ($\because x'$ is a bounded linear functional)

$\Rightarrow x'(x_0) = \|x_0\|$

Thus completing the proof of the theorem.

Recall (3.13): ① A mapping $f: X \to Y$, where X and Y are topological spaces, is said to be open mapping if f maps open subsets of X into open subsets of Y.

② In connection with subset A of X, we define for scaler α and $x_0 \in X$, as follows:

$$\alpha A = \{x \in X : x = \alpha a, \text{ where } a \in A\}$$

and $x_0 + A = \{x \in X : x = x_0 + a, \text{ where } a \in A\}$.

Lemma (3.14): Let X be a n.l.s and $B(x_0; r)$ be an open ball in X. Then $B(x_0; r) = x_0 + r\,B(0,1)$.

Proof: By definition,
$$\begin{aligned}
B(x_0; r) &= \{x \in X : \|x - x_0\| < r\} \\
&= \{x \in X : \|z\| < r, \text{ where } z = x - x_0\} \\
&= \{x \in X : \|z\| < r, \text{ where } x = z + x_0\} \\
&= \{x_0 + z \in X : \|z\| < r\} \\
&= x_0 + \{z \in X : \|z\| < r\} \quad (\text{by Recall ②}) \\
&= x_0 + \{z \in X : \|\tfrac{z}{r}\| < 1\} \\
&= x_0 + \{z \in X : \|z'\| < 1, \text{ where } z' = \tfrac{z}{r}\} \\
&= x_0 + \{z \in X : \|z'\| < 1, \text{ where } z = z'r\} \\
&= x_0 + \{rz' \in X : \|z'\| < 1\} \\
&= x_0 + r\{z' \in X : \|z'\| < 1\} \quad (\text{by Recall ②}) \\
&= x_0 + r\{z' \in X : \|z' - 0\| < 1\} \\
&= x_0 + r\,B(0, 1).
\end{aligned}$$

ie $\boxed{B(x_0; r) = x_0 + r\,B(0,1)}$

Remark: In particular $B(0; r) = r\,B(0,1)$.

Ch-03

Note: The proof of the "open mapping theorem" depends upon the following lemma, however we omit the Proof, because it is too lengthy.

Lemma (3.15): let T be a bounded linear operator from a Banach space X into a Banach space Y. Then for each open ball $B_0 = B(0,1) \subset X$, the image $T(B_0)$ contains an open ball in Y with centre at the origin.

Theorem (3.16) [The open mapping Theorem]

Statement: A bounded linear operator T from a Banach space X into a Banach space Y is an open mapping.

Proof: let $T: X \longrightarrow Y$ be a bounded linear operator from a Banach space X into a Banach space Y.
In order to show that T is an open mapping, we need to show that for any open set $A \subseteq X$, the image of A under T is open in Y ie $T(A)$ is open in Y.
For this let $y \in T(A)$: since T is an operator, so there exists $x \in A$ such that $y = Tx \in T(A)$.
It is enough to show that $T(A)$ contains an open ball around $y = Tx$.
And $x \in A$
Since A is open in X; so by definition, it contains an open ball with centre x and radius r

ie $B(x;r) \subseteq A$.

we know by Lemma (3.14) that:

$$B(x;r) = x + r\,B(0;1) \longrightarrow (1)$$

By Lemma (3.15), for the open ball $B(0;1)$ in X, there is an open ball $B'(0;r')$, with centre at origin, in Y such that:

$$B'(0;r') \subseteq T(B(0;1)).$$
$$\subseteq rT(B(0;1))$$
$$= T(B(0;r)) \quad [\text{using ① Taking } x=0].$$

Hence
$$B'(y;r') = y + rB'(0,1) \quad (2)$$
$$= y + B'(0;r') \quad [\text{by ①}]$$
$$\subseteq y + T(B(0;r)) \quad [\text{by ②}].$$
$$= Tx + T(B(0;r)) \quad [\because y = Tx].$$
$$= T(x + B(0;r)) \quad [\because T \text{ is linear}].$$
$$= T(x + rB(0,1))$$
$$\subseteq T(A) \quad [\text{by ①}].$$

i.e. $B'(y;r') \subseteq T(A)$.

this shows that $T(A)$ contains an open ball around $y = Tx$. Consequently $T(A)$ is open in Y and hence T is an open mapping.

Corrolary (3.17): Let $T: X \to Y$ be a bijective bounded linear operator from a Banach space X into a Banach space Y, Then T is a homeomorphism.

Proof: Recall that T is a homeomorphism if
(I) T is continuous (II) T is bijective.
(III) T^{-1} is continuous.

Since T is continuous ($\because T$ is bounded). and bijective, so by the latter condition, $T^{-1}: Y \to X$ exists.
To show that T^{-1} is continuous, let U be an open set in X; Then $(T^{-1})^{-1}U = TU$, which is open in Y because T is open by above thm. So that the image of any open set in Y is open in X under T^{-1}, showing that T^{-1} is continuous.

Hence T is a homeomorphism. #proved

Definition (3.18): let X and Y be norm linear spaces, Then $X\times Y$ is also a norm linear space, where the two algebraic operations of addition and scaler multiplication are defined by,

$$(x_1, y_1)+(x_2, y_2)=(x_1+x_2,\ y_1+y_2).$$

and $\alpha(x,y) = (\alpha x, \alpha y)$

and the norm on $X\times Y$ is defined by: $\|(x,y)\| = \|x\|+\|y\|$.

(Check it).

Theorem (3.19): suppose that X and Y are Banach spaces, then $X\times Y$ is also a Banach space.

Proof:- We show that the product space $X\times Y$ with norm defined by $\|(x,y)\| = \|x\|+\|y\| \longrightarrow ①$ is Complete.

let $\{z_n\}$ be a Cauchy sequence in $X\times Y$, where $z_n=(x_n,y_n)$ then by definition of Cauchy sequence, for every $\varepsilon>0$ there exists an N such that:

$$\|z_n - z_m\| < \varepsilon \text{ for } m,n \geq N \longrightarrow ②$$

Now $\|x_n - x_m\| + \|y_n - y_m\| = \|(x_n-x_m, y_n-y_m)\|$ [using ①].

$= \|(x_n,y_n) - (x_m,y_m)\|$ [by The operation of add: in $X\times Y$]

$= \|z_n - z_m\|$ $(\because z_n=(x_n,y_n))$.

$< \varepsilon$ for $m,n \geq N$ [by ②].

$\Rightarrow \|x_n - x_m\| < \varepsilon$ and $\|y_n - y_m\| < \varepsilon$ for $n,m\geq N$ [$\because \|x_n-x_m\|$ & $\|y_n-y_m\|$ are +ve]

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X and Y respectively.

and since X and Y are Banach spaces. So that $\{x_n\}$ and $\{y_n\}$ converges, say $x_n \to x$ and $y_n \to y$.

Since norm is a continuous function, therefore

$$\|x_n - x\| \to 0 \text{ and } \|y_n - y\| \to 0 \text{ as } n \to \infty \qquad \to (3)$$

$\| \ \|$ is contin: if $x_n \to x \Rightarrow \|x_n\| \to \|x\|$.

we show that $\{z_n\}$ is convergent in $X \times Y$

i.e. $z_n = (x_n, y_n) \longrightarrow (x, y) = z$ (say).

Now $\|z_n - z\| = \|(x_n, y_n) - (x, y)\|$

$= \|(x_n - x, y_n - y)\|$ [by def. of operation of add: in $X \times Y$]

$= \|x_n - x\| + \|y_n - y\|$

$\longrightarrow 0$ as $n \to \infty$ [by (3)].

This shows that the Cauchy sequence $\{z_n\}$ in $X \times Y$ is convergent. Since $\{z_n\}$ was choosen arbitrary, therefore $X \times Y$ is complete. Consequently $X \times Y$ is a Banach space.

Definitions (3.20):

(1) Let X and Y be normed linear spaces and $T : D(T) \subset X \longrightarrow Y$ be a linear operator from $D(T) \subset X$ into Y, then the set $G = \{(x, Tx) \in X \times Y : x \in D(T)\}$ is called the graph of T.

(2) Let X and Y be normed linear spaces and $T : D(T) \subset X \longrightarrow Y$ be a linear operator from $D(T) \subset X$ into Y, then T is called a closed linear operator if its graph $G = \{(x, Tx) \in X \times Y : x \in D(T)\}$ is a closed set in the normed linear space $X \times Y$.

Ch-03

Theorem (3.21) The Closed Graph Theorem:

Statement: Let X and Y be Banach spaces. Let T be a closed linear operator whose domain is all of X and whose range is in Y, Then T is continuous.

OR

A closed linear operator from a Banach space X into a Banach space Y is continuous.

Proof:- Let T be a closed linear operator whose domain is all of X and whose range is in Y ie

$D(T) = X$ and $R(T) \subset Y$.

Then by above definition, The graph G of T is closed subspace of The Banach space $X \times Y$, with The norm defined by: $\|(x,y)\| = \|x\| + \|y\|$.

Since X and Y are Banach spaces and G is a closed subspace of The Banach space $X \times Y$, So G itself is Complete ($\because$ A closed subspace of a Complete metric space is Complete) and hence G is a Banach space. Define a mapping $S : G \longrightarrow X$ by:

$$S(x, Tx) = x \quad \text{for all } x \text{ in } X.$$

First we show that S is a linear mapping:

$$\begin{aligned}
\text{Now } S[(x,Tx)+(y,Ty)] &= S(x+y, Tx+Ty) \\
&= S(x+y, T(x+y)) \quad [\because T \text{ is linear}]. \\
&= x+y \quad [\text{by def: of } S]. \\
&= S(x,Tx) + S(y,Ty) \quad [\text{'' ''}].
\end{aligned}$$

$$\begin{aligned}
\text{and } S[\alpha(x,Tx)] &= S(\alpha x, \alpha Tx) = \alpha x \quad [\text{by def: of } S] \\
&= \alpha S(x, Tx).
\end{aligned}$$

which shows that T is linear.

clearly S is one-one and onto.

Next we show that S is bounded.

$$\text{Now } \|S(x,Tx)\| = \|x\| \qquad (\because S(x,Tx) = x)$$
$$\leq \|x\| + \|Tx\|$$
$$= \|(x,Tx)\|$$
$$\text{ie } \|S(x,Tx)\| \leq \|(x,Tx)\|.$$

T is bounded if $\|Tx\| \leq M\|x\|$. Here $M=1$ so S is bdd.

$\Rightarrow$ S is bounded and hence continuous.

Since S is bijective, so the inverse mapping $S^{-1}: X \to G$ exists and is defined by: $S^{-1}x = (x,Tx)$; $\forall x \in X$.

Now since X and G are complete and S is bijective bounded linear operator from G to X, so by Corrolary (3.17), S^{-1} is continuous ($\because S$ is homeomorphism).

and hence bounded.

$$\text{Now } \|Tx\| \leq \|x\| + \|Tx\| \qquad (\text{[illegible]})$$
$$= \|(x,Tx)\|$$
$$= \|S^{-1}x\|$$
$$\leq \|S^{-1}\|\,\|x\| \; ; \forall x \in X \quad (\because S^{-1} \text{ is bounded}).$$
$$\text{ie } \|Tx\| \leq \|S^{-1}\|\,\|x\| \; ; \forall x \in X.$$

This inequality shows that T is bounded and hence continuous as required.

The end of Chapter #3

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