CHAPTER 3 [Fundamental Theorems of Functional Analysis]

Definition (3.1): If $A$ is a subset of a topological space $X$, then the interior $\text{int} A$ of $A$ is defined as the union of all open sets of $X$ that are contained in $A$, i.e.,

$$A = \bigcup_{i=1}^{n} O_i,$$

where each $O_i$ is open and $O_i \subseteq A$.

We say that $A$ has an empty interior, i.e., $\text{int} A = \emptyset$, if and only if $A$ does not contain any non-empty open set.

Definition: A subset $M$ of a topological space $X$ is said to be:

(a) nowhere dense (or rare) in $X$ if $\overline{M}$ has empty interior, i.e., $M$ does not contain any non-empty open set.

(b) of the first category (or meager) in $X$ if $M$ is the countable union of nowhere dense sets.

(c) of the second category (non-meager) in $X$ if $M$ is not of the first category, i.e., $M$ cannot be expressed as a countable union of nowhere dense sets.

Theorem (3.2) [Baire's Category Theorem]

Every complete metric space is of the second category.

Hence, if $X \neq \emptyset$ is complete and $X = \bigcup_{k=1}^{n} A_k$ (Ak is closed), then at least one $A_k$ contains a non-empty open set.
Theorem 3.3 (The Principle of Uniform Boundedness).

Let \( \{T_n\} \) be a sequence of bounded linear operators \( T_n : X \rightarrow Y \) from a Banach space \( X \) into a normed linear space \( Y \) such that \( \{T_n x\} \) is bounded in \( Y \) for all \( x \in X \). Then \( \{\|T_n\|\} \) is a bounded subset of \( \mathbb{R} \); there is a constant \( C > 0 \) such that \( \|T_n\| \leq C \), \( n = 1, 2, \ldots \).

**Proof:** For each \( k \in \mathbb{N} \), we define
\[
A_k = \{x \in X : \|T_n x\| \leq k, \forall n\},
\]
we show that \( A_k \) is closed.

Let \( \{x_j\} \) be a sequence in \( A_k \) such that \( x_j \rightarrow x \).

By continuity of \( T_n \) (because \( T_n \) is bounded), we have \( T_n x_j \rightarrow T_n x \). Also by continuity of a norm, we can write: \( \|T_n x_j\| \rightarrow \|T_n x\| \)
\[
\text{ie } \|T_n x\| = \lim_{j \to \infty} \|T_n x_j\| \leq k. \quad (\ast x_j \in A_k).
\]

ie \( \|T_n x\| \leq k \), which shows that \( x \in A_k \) (by def. of \( A_k \)).

And hence \( A_k \) is closed (\( : x_j \rightarrow x \Rightarrow x \in \lim_{j \to \infty} x_j \)).

Since by hypothesis, each \( x \in X \) is in some \( A_k \), because \( \{\|T_n x\|\} \) is bounded for each \( x \in X \) ie
\[
\|T_n x\| \leq k \quad \text{for some constant } k. \quad \text{Hence } X = \bigcup_{k=1}^{\infty} A_k.
\]

Since \( X \) is complete, so by Baire’s Category Theorem, \( X \) must be of the second category. So there exists at least one \( A_k \), namely \( A_k \), which contains an open ball \( B_k(x_0) = \{x \in X : \|x - x_0\| < r\} \) ie \( B_k(x_0) \subseteq A_k \).

Let \( x \) be an arbitrary non-zero vector in \( X \). We set
\[ z = yx + x_0, \text{ where } y = \frac{y}{2\|x\|} \rightarrow 0 \]

since \( x - x_0 = yx \Rightarrow \|z - x_0\| = \|yx\| = \left\| \frac{y}{2\|x\|} \cdot x \right\| \]

\[ = \frac{y}{2\|x\|} \cdot \|x\| \]

\[ = \frac{y}{2} \]

\[ < y \]

So \( \|z - x_0\| < y \), which shows that \( z \in B(y) \).

But \( B(y) \subset A_k \), so that \( z \in A_k \). And from the definition of \( A_k \), we have: \( \|T_n x\| \leq k_0 ; \forall n \rightarrow (2) \)

Also since \( x_0 \in B(y) \subset A_k \), so that \( x_0 \in A_k \).

So \( \|T_n x_0\| \leq k_0 ; \forall n \rightarrow (3) \) [by def. of \( A_k \)].

By (1), we have \( z = yx + x_0 \), where \( y = \frac{y}{2\|x\|} \) so we can write: \( x = \frac{z - x_0}{y} \). This yields for all \( n \).

\[ T_n x = T_n \left( \frac{z - x_0}{y} \right) \]

\[ \Rightarrow \|T_n x\| = \left\| T_n \left( \frac{z - x_0}{y} \right) \right\| \]

\[ = \frac{1}{y} \left\| T_n (z - x_0) \right\| \leq \frac{1}{y} \left[ \|T_n z + T_n (-x_0)\| \right] \]

\[ \leq \frac{1}{y} \cdot \left[ \|T_n z\| + \|T_n x_0\| \right] \quad (|1| = 1) \]

\[ \leq \frac{1}{y} \left[ k_0 + k_0 \right] \quad (\text{use } \circ \in \Theta), \]

\[ = \frac{2k_0}{y} \]

\[ = \frac{\frac{2k_0}{\|x\|}} \]

\[ = \frac{\frac{4k_0 \|x\|}{y}} \]

Hence for all \( x \): \( \|T_n x\| \leq \frac{4\|x\| k_0}{y} \)
\[ \| T_n \| \leq \frac{U_k}{y} \quad (\text{by taking } \sup \text{ over both sides}) \]

Assume that \( \frac{U_k}{y} = c \), then

\[ \| T_n \| \leq c \quad \forall n. \]

which completes the required proof.

**Note:** The principle of uniform boundedness is often called the Banach-Steinhaus Theorem.

**Theorem (3.4):** Let \( \{ T_n \} \) be a sequence of bounded linear operators \( T_n : X \to Y \) from a Banach space \( X \) into a n.l.s. \( Y \) such that \( \lim_{n \to \infty} T_n x = T x \) exists for each \( x \in X \). Then \( T \) is a bounded linear operator.

**Proof:** Since \( \lim_{n \to \infty} T_n x = T x \) exists for each \( x \in X \), so \( \{ T_n x \} \)

is a convergent sequence and hence bounded, because every convergent sequence is bounded. Therefore by the Principle of Uniform Boundedness, \( \{ \| T_n \| \} \) is a bounded sequence of real number, that is there exist a constant \( K > 0 \) such that:

\[ \| T_n \| \leq K \quad \forall n. \]

we prove that \( T \) is a bounded linear operator. \( T \) is linear, because for \( x, y \in X \) and a scalar \( \alpha \), we have:

\[ T(\alpha x + y) = \lim_{n \to \infty} T_n (\alpha x + y) \quad \left[ = T x = \lim_{n \to \infty} T_n x \text{ from above} \right] \]

\[ = \lim_{n \to \infty} (T_n x + T_n y) \quad \left[ = T_n x + T_n y \text{ is linear} \right] \]

\[ = \lim_{n \to \infty} T_n x + \lim_{n \to \infty} T_n y = T x + T y \quad \left[ \text{" } \right] \]
Also, \( T(ax) = \lim_{n \to \infty} T_n(ax) \) \[ \Rightarrow T = \lim_{n \to \infty} T_n \] \[ = \lambda \lim_{n \to \infty} T_n \] \[ = \lambda T \] \[ \Rightarrow T = \lim_{n \to \infty} T_n \] Also, \( T \) is bounded, because since \( \{T_n\} \) is a bounded sequence, so from (i), we have:

\[ \|T_n\| \leq K \quad \forall n. \]

Now \( \|T_n x\| \leq \|T_n\| \|x\| \) \( (\Rightarrow T_n \) is bounded)\n
\[ \leq K \|x\| \] \( (\Rightarrow \|T_n\| \leq K) \); \( \forall n. \)

i.e., \( \|T_n x\| \leq K \|x\|. \)

Taking limit over both sides as \( n \to \infty \), we get:

\[ \lim_{n \to \infty} \|T_n x\| \leq \lim_{n \to \infty} K \|x\| \]

\[ \Rightarrow \lim_{n \to \infty} \|T_n x\| \leq \lim_{n \to \infty} K \|x\| \]

\[ \Rightarrow \|T x\| \leq K \|x\| \quad \forall n. \]

which shows that \( T \) is bounded.

Hence \( T \) is a bounded linear operator.

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**Definition (3.5):** Let \( X \) be a linear space. A real valued function \( p \) defined on \( X \) is called \textit{subadditive} if

\[ p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X. \]

and positive homogeneous if

\[ p(\alpha x) = \alpha p(x), \]

where \( \alpha > 0 \) in \( \mathbb{R} \) and \( x \in X. \)

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**Definition (3.6):** A real valued function \( p \) on a linear space \( X \) is called \textit{sublinear functional} if it is both subadditive and positive homogeneous.
Definition (3.7): Let \( X \) be a linear space (real or complex).

A semi-norm on \( X \) is a real valued function \( p \) defined on \( X \) such that

(i) \( p(x) \geq 0 \) \( \forall x \in X \) (ii) \( p(x+y) \leq p(x) + p(y) \) \( \forall x, y \in X \).

(iii) \( p(\alpha x) = |\alpha| p(x) \); for all scalars \( \alpha \) and \( x \in X \).

If \( p \) has the further property that \( p(x) = 0 \) iff \( x = 0 \) (or equivalently \( p(x) = 0 \) iff \( x = 0 \)), then \( p \) is a norm on \( X \).

Remark: As with a norm, the properties of \( p \) imply the further properties.

(i) \( p(0) = 0 \) (ii) \( |p(x) - p(y)| \leq p(x-y) \).

(iii) \( p(-x) = -p(x) \).

Proof: (i) we show that \( p(0) = 0 \).

\[
p(0) = p(0,0) = 0 \quad [p(0) + p(0)]
\]

(ii) since \( x = x - y + y \)

\[
\Rightarrow p(x) = p(x-y+y) \leq p(x-y) + p(y) \quad [p(x+y) \leq p(x) + p(y)]
\]

\[
\Rightarrow p(x) - p(y) \leq p(x-y) \quad \Box 1
\]

Now again \( y = x+y-x \) \( \Rightarrow p(y) = p(x+y-x) \leq p(x) + p(y-x) \)

\[
\Rightarrow p(0) = p(y-x) \leq p(x) + p(y-x) \quad \Box 2
\]

\[
\Rightarrow p(x) - p(y) \leq p(x-y) \Rightarrow p(x) - p(x-y) \leq |1| p(x-y) \]

\[
\Rightarrow p(x) - p(y) \leq p(x-y) \Rightarrow -p(x-y) \leq p(x) - p(x-y) \Rightarrow p(x) - p(x-y) \leq 0 \quad \Box 2
\]

\[
\Rightarrow p(x) - p(y) \leq 0 \quad \Box 2
\]

\[
|p(x) - p(y)| \leq p(x-y) \quad \Box 3
\]

We have: \( p(-x) = p(-x+x) = 2p(x) + p(x) \)

\[
\Rightarrow p(-x) - 2p(x) \leq p(x) \Rightarrow -p(x) \leq p(x) \Rightarrow p(x) \geq p(x) \neq \emptyset.
\]
**Definition (3.8):**

If \( X \) is a set, \( M \) a proper subset of \( X \) and \( f \) is a function defined on \( M \) (i.e. \( f : M \rightarrow M \)), then a function \( F \) defined on \( X \) (i.e. \( F : X \rightarrow X \)) is called an extension of \( f \) if \( F(x) = f(x) \) for all \( x \in M \)

and \( f \) is called the restriction of \( F \) to \( M \).

**Theorem (Hahn–Banach Theorem – Real Version):**

Let \( M \) be a proper subspace of a real linear space \( X \). Let \( p \) be a sublinear functional on \( X \) and let \( F \) be a linear functional defined on \( M \) such that

\[
p(x) \leq F(x) \quad \forall x \in M.
\]

Then there exists a linear functional \( \hat{F} \) on \( X \), which extends \( F \) and that

\[
p(x) \leq \hat{F}(x) \leq F(x) \quad \forall x \in X.
\]

(3.10) **Theorem [Hahn Banach Theorem – Complex Version]:**

Let \( X \) be a complex linear space and \( M \) a linear subspace of \( X \). Let \( p \) be a semi-norm defined on \( X \). Let \( F \) be a linear functional on \( M \) such that

\[
|F(x)| \leq p(x) \quad \forall x \in M.
\]

Then there exists a linear functional \( \hat{F} \) on \( X \) such that

\[
F(x) = \hat{F}(x) \quad \forall x \in X \quad \text{[Extension]} \quad \text{and that}
\]

\[
|\hat{F}(x)| \leq p(x) \quad \forall x \in X.
\]

(3.11) **Theorem [Hahn–Banach Thm for n.b.s.]:**

Statement: Let \( X \) be a n.b.s. over a field \( K \) and let \( M \) be a subspace of \( X \). If \( m \in M \) (i.e. \( m \) is a linear functional defined on \( M \)), then there exists \( x \in X \) such that

\[
\|x\| = \|m\| \quad \text{and} \quad m'(x) = F(x) \quad \forall x \in M \quad \text{[Extension]}
\]
Proof: Let \( p \) be a real valued function defined on \( X \) by:
\[
p(x) = \|m\| \|x\| \quad \forall x \in X \rightarrow 0
\]
First we show that \( p \) is a semi-norm on \( X \).

(i) For \( x \in X \), \( p(x) > 0 \) because \( p(x) = \|m\| \|x\| > 0 \).

(ii) For \( x, y \in X \), we have:
\[
p(x + y) = \|m\| \|x + y\| \quad \text{(by (i))}
\leq \|m\| (\|x\| + \|y\|) \quad \text{[\( \|x + y\| \leq \|x\| + \|y\| \)]}
= \|m\| \|x\| + \|m\| \|y\|.
= p(x) + p(y). \quad \text{[by (i)]}
\]
i.e. \( p(x + y) \leq p(x) + p(y) \).

(iii) For any scalar \( \lambda \) and \( x \in X \), we have:
\[
p(\lambda x) = \|m\| \|\lambda x\| \quad \text{[by (i)]}
= |\lambda| \|m\| \|x\| \quad \text{[by definition of norm]}
= |\lambda| p(x) \quad \text{[by (i)]}
\]
Hence \( p \) is a semi-norm on \( X \).

Now \( |m(x)| \leq \|m\| \|x\| \quad \text{(\( : |\lambda x| \leq \|x\| \|\lambda\| \))}
\]
\[
= p(x) \quad \text{[by (i)]}
\]
i.e. \( |m(x)| \leq p(x) \) for all \( x \in M \).

Hence by "Hahn–Banach theorem for complex space," there exists a linear functional \( \lambda x \) on \( X \) such that:

(i) \( \lambda x(x) = m(x) \forall x \in M \) (i.e. \( \lambda x \) is extension of \( m \))
and (ii) \( |\lambda x| \leq p(x) \) \( \forall x \in X \).

To prove the required result, we will prove that
\[
\|m\| = \|\lambda x\| \quad \text{and} \quad \lambda x(x) = m(x) \quad \forall x \in M.
\]
But \( \lambda x(x) = m(x) \forall x \in M \) is satisfied from (i) above:
Also from (1), we have:

$$|x(x)| \leq f(x) = \|m||x|| \quad \forall x \in X.$$  

Taking supremum over both sides with $\|x\|=1$, we obtain

$$\|x\| \leq \|m\| \to 0$$

Also

$$\|x\| = \sup_{x \to 0} \frac{|x(x)|}{\|x\|} \geq \sup_{x \to 0} \frac{|m(x)|}{\|x\|} = \|m\|$$

$$\Rightarrow \|m\| \leq \|x\|.$$  

Since $x$ is the extension of $x$, so that $\|m\|$ cannot be less than $\|x\|$ and so $\|x\| \geq \|m\| \to 0$

From (2) and (3), we get:

$$\|x\| = \|m\|.$$  

Thus completing the proof.

**Theorem (3.2):** Let $X$ be a normed linear space and let $x \to 0$ be any element of $X$. Then there exists $x \in X$ such that $\|x\| = 1$ and $x(x) = \|x\|$.

*Proof:* we consider the subspace $M$ of $X$ containing all elements $x = \alpha x_0$, where $\alpha$ is a scalar,

ie $M = \{x \in X: x = \alpha x_0, \text{where } \alpha \text{ is a scalar}\}$.

Define a linear functional $m$ on $M$ such that:

$$m(x) = m(\alpha x_0) = \alpha \|x_0\| \to 0.$$  

Then $m$ is bounded ie $m \in M$, because

$$|m(x)| = |m(\alpha x)| \leq \|m\||\alpha x_0|| = \|m\||x|| \quad \text{(by def. of } m)$$  

ie

$$|m(x)| \leq \|m\||x||$$

Also

$$|m(x)| = |m(\alpha x)| = |\alpha||x_0| \quad \text{(by (1))}.$$  

$$= |\alpha||x_0| = \|\alpha x_0\| = \|x\| \quad \text{(by def. of } m).$$
That is \( |m'(x)| = \|x\| ; \forall x \in M \).

Taking supremum over both sides, we have

\[
\sup_{x \to 0} \frac{|m'(x)|}{\|x\|} = 1 \quad \text{ie} \quad \|m'\| = 1. \quad \rightarrow (2)
\]

since \( m' \) is a bounded linear functional defined on \( M \). So by "Hahn–Banach Theorem for n.l.s."

\( m' \) has a linear extension from \( M \) to \( X \),

ie \( x'(x) = m'(x) \); \( \forall x \in M \rightarrow (3) \)

and \( \|x'\| = \|m'\| \rightarrow (4) \)

But by (2), \( \|m'\| = 1 \). So that (4) \( \Rightarrow \|x'\| = 1 \).

It remains to show that \( x'(x_0) = \|x_0\| \).

From (1), we have \( m'(x) = \alpha \|x_0\| \), where \( \alpha = \alpha(x_0) \).

and so by (3), we have \( x'(x) = m'(x) = \alpha \|x_0\| \), where \( x = x_0 \).

\[ \Rightarrow x'(x) = \alpha \|x_0\| , \text{ where } x = x_0 . \]

\[ \Rightarrow x'(x_0) = \alpha \|x_0\| \quad \rightarrow (5) \]

\[ \Rightarrow \alpha x'(x_0) = \alpha \|x_0\| \quad (\because x' \text{ is a bounded linear functional}) \]

\[ \Rightarrow x'(x_0) = \|x_0\| \]

Thus completing the proof of the theorem.

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Recall that a mapping \( f : X \rightarrow Y \), where \( X \) and \( Y \) are topological spaces, is said to be open mapping if \( f \) maps open subsets of \( X \) into open subsets of \( Y \).

In connection with subset \( A \) of \( X \), we define for scalar \( \alpha \) and \( x_0 \in X \), as follows:

\[
\alpha A = \{ x \in X : x = \alpha a, \text{ where } a \in A \}
\]

and \( x_0 + A = \{ x \in X : x = x_0 + a, \text{ where } a \in A \} \).

**Lemma:** Let \( X \) be a n.l.s and \( B(x_0; r) \) be an open ball in \( X \). Then \( B(x_0; r) = x_0 + rB(0,1) \).

**Proof:** By definition, \( B(x_0; r) = \{ x \in X : \| x - x_0 \| < r \} \)

\[
= \{ x \in X : \| x \| < r, \text{ where } x = x_0 + x \}
\]

\[
= \{ x \in X : \| x \| < r, \text{ where } x = x_0 + z \}
\]

\[
= \{ x \in X + z \in X : \| x \| < r \} \quad (\text{by recall 9})
\]

\[
= x_0 + \{ z \in X : \| z \| < r \}
\]

\[
= x_0 + \{ z \in X : \| z \| < 1, \text{ where } z = \frac{z}{r} \}
\]

\[
= x_0 + \{ z \in X : \| z \| < 1, \text{ where } z = \frac{x_0}{r} \}
\]

\[
= x_0 + \{ z \in X : \| z \| < 1 \}
\]

\[
= x_0 + rB(0,1) \quad (\text{by recall 9})
\]

\[
= x_0 + rB(0,1)
\]

\[
\text{ie} \quad B(x_0; r) = x_0 + rB(0,1)
\]

**Remark:** In particular, \( B(0; r) = rB(0,1) \).

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The proof of the "open mapping theorem" depends upon the following lemma, however, we omit the proof, because it is too lengthy.

**Lemma (3.15)**: Let \( T \) be a bounded linear operator from a Banach space \( X \) into a Banach space \( Y \). Then for each open ball \( B_0 = B(0,1) \subseteq X \), the image \( T(B_0) \) contains an open ball in \( Y \) with centre at the origin.

**Theorem (3.16)**: The open mapping theorem

**Statement**: A bounded linear operator \( T \) from a Banach space \( X \) into a Banach space \( Y \) is an open mapping.

**Proof**: Let \( T : X \to Y \) be a bounded linear operator from a Banach space \( X \) into a Banach space \( Y \). In order to show that \( T \) is an open mapping, we need to show that for any open set \( A \subseteq X \), the image of \( A \) under \( T \) is open in \( Y \) i.e. \( T(A) \) is open in \( Y \).

For this let \( y \in T(A) \); since \( T \) is an operator, so there exists \( x \in A \) such that \( y = Tx \in T(A) \).

It is enough to show that \( T(A) \) contains an open ball around \( y = Tx \).

Since \( A \) is open in \( X \); so by definition, it contains an open ball with centre \( x \) and radius \( r \) i.e. \( B(x;r) \subseteq A \).

We know by Lemma (2.14) that:

\[
B(x;r) = x + rB(0;1) \rightarrow 0
\]

By Lemma (3.15), for the open ball \( B(0;1) \subseteq X \), there is an open ball \( B(0;r) \), with centre at origin.
in $Y$ such that:

$$B'(x_0,y) \leq T(B(x_0;1)) \leq T(B(0;1)) = T(B(x_0;1)) \quad \text{by (1)}$$

Hence $B'(y; y') = \frac{y + y' B(x_0;1)}{y} \rightarrow [b_0\theta]$

$$y + T(B(x_0;1)) \quad [b_0\theta]$$

$$= T(x + T(B(x_0;1)) \quad [y = Tx]$$

$$= T(x + y B(x_0;1)) \quad [T \text{ is linear}]$$

$$\leq T(x) \quad [b_0\theta].$$

i.e. $B'(y; y') \leq T(y).$

This shows that $T(x)$ contains an open ball around $y = Tx.$ Consequently $T(x)$ is open in $y$ and hence $T$ is an open mapping.

**Corollary (3.17):** Let $T: X \rightarrow Y$ be a bijective bounded linear operator from a Banach space $X$ into a Banach space $Y.$ Then $T$ is a homeomorphism.

**Proof:** Recall that $T$ is a homeomorphism if

(i) $T$ is continuous

(ii) $T$ is bijective

(iii) $T^{-1}$ is continuous.

Since $T$ is continuous (i.e. $T$ is bounded) and bijective, so by the latter condition, $T^{-1}: Y \rightarrow X$ exists. To show that $T^{-1}$ is continuous, let $U$ be an open subset of $X$; then $(T^{-1})^{-1}U = T\overline{U}$, which is open in $Y$ because $T$ is open by above Thm. So that the image of any open set in $Y$ is open in $X$ under $T^{-1},$ showing that $T^{-1}$ is continuous.

Hence $T$ is a homeomorphism. 

...
**Definition (3.18):** Let $X$ and $Y$ be norm linear spaces,

Then $X \times Y$ is also a norm linear space, where the two algebraic operations of addition and scalar multiplication are defined by:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

and $\alpha (x, y) = (\alpha x, \alpha y)$

and the norm on $X \times Y$ is defined by: $\| (x, y) \| = \| x \| + \| y \|$. 

(Check it).

**Theorem (3.19):** Suppose that $X$ and $Y$ are Banach spaces, then $X \times Y$ is also a Banach space.

**Proof:** We show that the product space $X \times Y$ is complete.

Let $\{ z_n \}$ be a Cauchy sequence in $X \times Y$, where $z_n = (x_n, y_n)$ then by definition of Cauchy sequence, for every $\varepsilon > 0$ there exists an $N$ such that:

$$\| z_n - z_m \| < \varepsilon \text{ for } m, n \geq N \rightarrow 0$$

Now:

$$\| x_n - x_m \| + \| y_n - y_m \| = \| (x_n, y_n) - (x_m, y_m) \|$$

$$= \| (x_n, y_n) - (x_m, y_m) \| \quad \text{[by the operation of addition in $X \times Y$]}$$

$$= \| z_n - z_m \|$$

$$\leq \varepsilon \text{ for } m, n \geq N \quad \text{[by $\theta$]}.$$ 

$$\Rightarrow \| x_n - x_m \| < \varepsilon \text{ and } \| y_n - y_m \| < \varepsilon \text{ for } m, n \geq N \quad \text{[by $\theta$]}.$$ 

Therefore $\{ x_n \}$ and $\{ y_n \}$ are Cauchy sequence in $X$ and $Y$ respectively.
and since $X$ and $Y$ are Banach spaces, so that
\[ \{x_n\} \text{ and } \{y_n\} \text{ converges, say } x_n \to x \text{ and } y_n \to y. \]

Since norm is a continuous function, therefore
\[ ||x_n - x|| \to 0 \text{ and } ||y_n - y|| \to 0 \text{ as } n \to \infty \]
we show that $\{z_n\}$ is convergent in $X \times Y$

i.e. $z_n = (x_n, y_n) \to (x, y) = z \text{ (say).}$

Now $||z_n - z|| = ||(x_n, y_n) - (x, y)||$
\[ = ||(x_n - x, y_n - y)|| \quad \text{[by def. of operation of addition in } X \times Y \]
\[ = ||x_n - x|| + ||y_n - y|| \]
\[ \to 0 \text{ as } n \to \infty \quad \text{[by (3).]} \]

This shows that the Cauchy sequence $\{z_n\}$ in $X \times Y$ is

Convergent. Since $\{z_n\}$ was chosen arbitrary, therefore

$X \times Y$ is Complete. Consequently $X \times Y$ is a Banach space.

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**Definition (3.20):**

1. Let $X$ and $Y$ be normed-linear spaces and $T: D(X) \subset X \to Y$ be a linear operator from $D(X) \subset X$ into $Y$, then the set $G = \{(x, Tx) \in X \times Y : x \in D(X)\}$ is called the graph of $T$.

2. Let $X$ and $Y$ be normed-linear spaces and $T: D(X) \subset X \to Y$ be a linear operator from $D(X) \subset X$ into $Y$, then $T$ is called a closed linear operator if its graph $G = \{(x, Tx) \in X \times Y : x \in D(X)\}$ is a closed set in the normed-linear space $X \times Y$. 

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Theorem (3.21) The Closed Graph Theorem:

Statement: Let $X$ and $Y$ be Banach spaces. Let $T$ be a closed linear operator whose domain is all of $X$ and whose range is in $Y$, then $T$ is continuous.

Or

A closed linear operator from a Banach space $X$ into a Banach space $Y$ is continuous.

Proof: Let $T$ be a closed linear operator whose domain is all of $X$ and whose range is in $Y$ i.e. $D(T) = X$ and $R(T) = Y$.

Then by above definition, the graph $G$ of $T$ is closed subspace of the Banach space $X \times Y$, with the norm defined by: $\| (x,y) \| = \| x \| + \| y \|$. Since $X$ and $Y$ are Banach spaces and $G$ is a closed subspace of the Banach space $X \times Y$, so $G$ itself is complete (i.e. a closed subspace of a complete metric space is complete) and hence $G$ is a Banach space. Define a mapping $\delta: G \rightarrow X$ by:

$\delta(x,Tx) = x$ for all $x$ in $X$.

First we show that $\delta$ is a linear mapping:

Now $\delta[(x,Tx) + (y,Ty)] = \delta(x+Ty, Tx+Ty)$

$= \delta(x+Ty, Tx+Ty)$ [by def. of $\delta$]

$= x + y$ [by def. of $\delta$]

$= \delta(x,Tx) + \delta(y,Ty)$ [" + "].

and $\delta[\alpha(x,Tx)] = \delta(\alpha x, \alphaTx) = \alpha x$ [by def. of $\delta$]

$= \alpha \delta(x,Tx)$. 
which shows that $T$ is linear.

Clearly $\beta$ is one-one and onto.

Next we show that $\beta$ is bounded.

Now $\|\beta(x,Tx)\| = \|x\|$ (i.e. $\beta(x,Tx) = x$)

$\leq \|x\| + \|Tx\|

= \|(x,Tx)\|

i.e $\|\beta(x,Tx)\| \leq \|(x,Tx)\|

$\Rightarrow \beta$ is bounded and hence continuous.

Since $\beta$ is bijective, so the inverse mapping $\beta^{-1}: X \to G$

exists and is defined by $\beta^{-1}x = (x,Tx)$; $\forall x \in X$.

Now since $X$ and $G$ are complete and $\beta$ is bijective

bounded linear operator from $G$ to $X$, so by

Corrolary (3.17), $\beta^{-1}$ is continuous ($\beta^{-1}$ is homeomorphism).

and hence bounded.

Now $\|Tx\| \leq \|x\| + \|Tx\|$ (i.e. $\|\beta(x,Tx)\| = \|x\|$; $\forall x \in X$).

$\Rightarrow \|Tx\| = \|\beta^{-1}x\|

\leq \|\beta^{-1}\| \|x\|; \forall x \in X.$

This inequality shows that $T$ is bounded and hence continuous as required.

The end of Chapter 3

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